On the analytic theory of non-commutative distributions in free probability

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Abstract

Non-commutative distributions constitute the backbone of non-commutative probability in general and free probability in particular. In the multivariate case, these objects are mostly treated by combinatorial means, because an analytic description in terms of measures – as one is used to in classical probability – fails due to the underlying non-commutativity. However, the rapidly growing field of "free analysis", which is developed as a counterpart of classical analysis at the highest degree of non-commutativity, offers already many powerful tools for an analytic treatment of non-commutative distributions.

Indeed, during the last years, some significant progress has been made on very fundamental questions in that context. This thesis reports on successful attempts which follow the common strategy that properties of the joint non-commutative distribution $\mu_{X_1,\dots,X_n}$ of non-commutative random variables $X_1, \dots, X_n$ can be understood by studying the single-variable distributions $\mu_{f(X_1,\dots,X_n)}$ of $f(X_1, \dots, X_n)$ for suitable "non-commutative test functions" f. More precisely, we will discuss here the following topics:

Computation of analytic distributions and of Brown measures: If $X_1, \dots, X_n$ are freely independent non-commutative random variables, then the single-variable distributions $\mu_{X_1}, \dots, \mu_{X_n}$ fully determine their joint non-commutative distribution $\mu_{X_1,\dots,X_n}$ and so $\mu_{P(X_1,\dots,X_n)}$ for any non-commutative polynomial P. Nevertheless, apart from a few special cases, there was for a long time no general machinery for making this relation explicit. We will explain how the so-called "linearization trick" in several refined versions gives in combination with operator-valued free probability an algorithmic solution to this problem, which applies even more to non-commutative rational expressions and is moreover easily accessible for numerical computations. Depending on the concrete situation, $\mu_{P(X_1,\dots,X_n)}$ can be encoded either by the analytic distribution or (at least partially) by the Brown measure of $P(X_1, \dots, X_n)$.

Regularity questions: Free probability has produced some deep quantities like free Fisher information, free entropy, and free entropy dimension that can be attached to families of non-commutative random variables. Despite the lack of a rigorous justification, it was believed that these quantities measure the regularity of the corresponding non-commutative distributions. Following the so-called non-microstates approach, we will give evidence to this by showing that maximality of the free entropy dimension excludes atoms in the distribution of any non-constant self-adjoint polynomial expressions in these variables. Furthermore, we will see that the method of this proof can be generalized such that it also applies to free stochastic calculus. We will use this to exclude atoms in the distribution of any non-constant and self-adjoint element in the finite Wigner chaos (which is the free counterpart of the Wiener-Itô chaos in classical probability theory).

Abstrakt

Nichtkommutative Verteilungen bilden eine tragende Säule in der nichtkommutativen Wahrscheinlichkeitstheorie im Allgemeinen und der freien Wahrscheinlichkeitstheorie im Besonderen. Im Fall mehrerer Variablen werden diese Objekte meist mit kombinatorischen Mitteln behandelt, da eine analytische Beschreibung durch Maße – in der Form, wie man es in der klassischen Wahrscheinlichkeitstheorie gewohnt ist – wegen der zugrundeliegenden Nichtkommutativität nicht möglich ist. Jedoch stellt das stark wachsende Gebiet der "freien Analysis", welche als ein Gegenstück zur klassischen Analysis auf der Ebene maximaler Nichtkommutativität entwickelt wird, bereits eine Vielzahl mächtiger Werkzeuge zur analytischen Behandlung nichtkommutativer Verteilungen bereit.

In der Tat wurden in den letzten Jahren einige wesentliche Fortschritte bei sehr fundamentalen Fragestellungen in diesem Zusammenhang erzielt. Die vorliegende Arbeit berichtet über erfolgreiche Ansätze, welche der gemeinsamen Strategie folgen, dass Eigenschaften nichtkommutativer Verteilungen $\mu_{X_1,\dots,X_n}$ von nichtkommutativen Zufallsvariablen $X_1, \dots, X_n$ dadurch verstanden werden können, dass man die einvariabligen Verteilungen $\mu_{f(X_1,\dots,X_n)}$ von $f(X_1, \dots, X_n)$ für passende "nichtkommutative Testfunktionen" f untersucht. Genauer werden wir hier die folgenden Themen diskutieren:

Berechnung von analytischen Verteilungen und Brown-Maßen: Sind $X_1, \dots, X_n$ frei unabhängige nichtkommutative Zufallsvariablen, dann bestimmen die einvariabligen Verteilungen $\mu_{X_1}, \dots, \mu_{X_n}$ vollständig ihre gemeinsame Verteilung $\mu_{X_1,\dots,X_n}$ und somit $\mu_{P(X_1,\dots,X_n)}$ für jedes nichtkommutative Polynom P. Dennoch gab es lange Zeit, außer in ein paar Spezialfällen, keinen allgemeinen Apparat, mit dem man diese Beziehung explizit machen konnte. Wir werden erklären, wie der sogenannte "Linearisierungstrick" in verschiedenen verfeinerten Versionen in Kombination mit operator-wertiger freier Wahrscheinlichkeitstheorie eine algorithmische Lösung dieses Problems liefert, welche sogar auf rationale Ausdrücke angewendet werden kann und die darüber hinaus auch für numerische Berechnungen sehr gut geeignet ist. Je nach konkret gegebener Situation wird $\mu_{P(X_1,\dots,X_n)}$ hierbei entweder durch die analytische Verteilung oder (zumindest teilweise) durch das Brown-Maß von $P(X_1, \dots, X_n)$ erfasst werden.

Regularitätsfragen: Die freie Wahrscheinlichkeitstheorie hat etwa mit der freien Fisher Information, der freien Entropie und der freien Entropiedimension einige tiefliegende Größen hervorgebracht, die Familien von nichtkommutativen Zufallsvariablen zugeordnet werden können. Obwohl eine formale Bestätigung bisher fehlte, ging man davon aus, dass diese Größen ein Maß für die Regularität der zughehörigen Verteilungen darstellen. Dem sogenannten non-microstates Zugang folgend, werden wir dies belegen, indem wir zeigen, dass die Maximalität der freien Entropiedimension Atome in den Verteilungen nicht-konstanter, selbstadjungierter Polynome in diesen Variablen ausschließt. Darüber hinaus werden wir sehen, dass diese Beweismethode derart verallgemeinert werden kann, dass sie auch im Rahmen des freien stochastischen Kalküls angewendet werden kann. Wir verwenden dies, um Atome in den Verteilungen nicht-konstanter, selbstadjungierter Elemente im endlichen Wigner Chaos (welches das freie Gegenstück zum Wiener-Itô Chaos in der klassischen Wahrscheinlichkeitstheorie darstellt) auszuschließen.

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Introduction

The main objects which are treated in this thesis are non-commutative distributions. They constitute the combinatorial backbone of non-commutative probability theory in general and are therefore of particular interest in free probability theory. Our emphasis is on the analytic aspects of their theory.

Free probability was invented around 1985 by Dan-Virgil Voiculescu; see [**Voi85**]. This theory was originally intended to serve as a tool for attacking one of the most influential open question in the theory of von Neumann algebras, namely the isomorphism problem for the so-called free group factors $L(\mathbb{F}_n)$ with $n \geq 2$. Recall that the free group factor $L(\mathbb{F}_n)$ arises as the group von Neumann algebra associated to the free group $\mathbb{F}_n$ with n generators. The problem is to decide whether this operator-algebraic object $L(\mathbb{F}_n)$ memorizes the group $\mathbb{F}_n$ from which it was constructed, or in other words, the number n of generators. More formally, the question is whether for integers $n, m \geq 2$ the following implication holds true:

$$L(\mathbb{F}_n) \cong L(\mathbb{F}_m) \qquad \Longrightarrow \qquad n = m$$

The starting point for Voiculescu's ingenious considerations around this important and intricate question was the observation that, loosely speaking, the free product $\mathbb{F}_n * \mathbb{F}_m \cong \mathbb{F}_{n+m}$ on the group level is reflected by the relative position of $L(\mathbb{F}_n)$ and $L(\mathbb{F}_m)$ inside $L(\mathbb{F}_{n+m})$. In order to make this statement precise, we must involve the unique normal tracial state τ on the type II_1-factor $L(\mathbb{F}_{n+m})$. With a bit of work, one can show then that $\mathbb{F}_n * \mathbb{F}_m \cong \mathbb{F}_{n+m}$ yields some kind of factorization property for the values of τ. This relation was ingeniously interpreted by Voiculescu in a probabilistic manner as some kind of independence between $L(\mathbb{F}_n)$ and $L(\mathbb{F}_m)$, regarded as unital subalgebras of the non-commutative probability space $(L(\mathbb{F}_{n+m}), \tau)$. This notion of independence, which is called *free independence*, makes perfectly sense not only in $(L(\mathbb{F}_{n+m}), \tau)$ but in any non-commutative probability spaces.

By definition, a *non-commutative probability space* is a tuple $(\mathcal{A}, \phi)$, which consists of a unital complex algebra $\mathcal{A}$ and some linear functional $\phi : \mathcal{A} \to \mathbb{C}$ that satisfies the condition $\phi(1) = 1$. This nomenclature is justified by the observation that any classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ yields by $\mathcal{A} = L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ and $\phi(X) = \int_\Omega X(\omega) d\mathbb{P}(\omega)$ a canonical example of a non-commutative probability space. It might be confusing that this example is actually commutative, but the fact that classical probability spaces fit into this frame simply highlights that the definition of non-commutative probability spaces is designed in such a way that it properly imitates the setting of classical probability theory as far as this is possible without referring to the underlying structure of classical probability spaces. This is in accordance with some common strategy by which commutative concepts are often transferred into the non-commutative world: passing from the commutative object $(\Omega, \mathcal{F}, \mathbb{P})$ to some suitable algebra of functions over it leads us to $(L^\infty(\Omega, \mathbb{P}), \mathbb{E})$ and finally to the more general framework of non-commutative probability spaces $(\mathcal{A}, \phi)$, where the inherent commutativity of the algebra $L^\infty(\Omega, \mathbb{P})$ can easily be dropped, but to the price that the underlying structure $(\Omega, \mathcal{F}, \mathbb{P})$ disappears.

With this artifice, Voiculescu put the original problem into the setting of non-commutative probability theory, where free independence can be studied in much wider generality and can be treated detached from the concrete case of free group factors. This opened a completely new and very promising perspective on the original operator algebraic question and marked the birth of free probability theory, which Voiculescu started to develop as a counterpart of classical probability theory at the highest level of non-commutativity.

Although we must admit that free probability was not yet able to solve the isomorphism problem for the free group factors, it gave nonetheless incredibly deep insights and has produced striking results about the structure of von Neumann algebras in general and the free group factors in particular.

Despite this undoubtedly great success, free probability would have probably become known only among experts in the field of operator algebras. The reason is that for people from outside this area, free independence might appear at first sight as a rather artificial concept with seemingly no contact to the "real world", in contrast to the much more intuitive notion of independence in classical probability. However, the situation totally changed when Voiculescu discovered some deep connections to random matrix theory. By what later became known as the phenomenon of asymptotic freeness, he explained that free independence governs the asymptotic behavior of many types of classically independent random matrices as their dimension tends to infinity. This attracted the attention of many people, both in pure mathematics and in the more applied disciplines, and started in particular an extremely fruitful interaction between the theory of operator algebras and random matrix theory. In the course of this active exchange of ideas, random matrix methods found their way into the field of operator algebras, while free probability became a very powerful tool in analyzing the asymptotic behavior of large random matrices.

It is therefore not surprising that non-commutative distributions, which arise from considerations in free probability, have attracted a lot of attention since the early days of this theory. Conversely, the desired applications have also raised their own questions.

But what actually are non-commutative distributions? Non-commutative distributions are by definition purely combinatorial objects, which can be introduced in the generality of non-commutative probability spaces: given any family $X = (X_i)_{i\in I}$ of non-commutative random variables X_i in some non-commutative probability space $(\mathcal{A}, \phi)$ over some (mostly but not necessarily finite) index set I, then the *non-commutative (joint) distribution* μ_X *of* $X = (X_i)_{i\in I}$ is defined as the collection

$$\Big(\phi\big(X_{i_1}X_{i_2}\cdots X_{i_k}\big)\Big)_{\substack{k\geq 0\\ i_1,\dots,i_k\in I}}$$

of all *(joint) moments* $\phi(X_{i_1}X_{i_2}\cdots X_{i_k})$ *with* $i_1,\dots,i_k \in I$ *of any order* k, including $\phi(1) = 1$ as the moment of order $k = 0$.

Like distributions in classical probability theory, non-commutative distributions are meant for describing the family of non-commutative random variables to which they correspond. Indeed, this strategy works well under a few natural assumptions on the underlying non-commutative probability space. This culminates in the fascinating idea that properties of operators, viewed as non-commutative random variables, can be encoded by their joint non-commutative distribution and hence by purely combinatorial data – although reading out this information can be a highly non-trivial task.

For some purposes, one might prefer to add a more analytic component to the combinatorial picture of non-commutative distributions, comparable to the measure theoretic description of distributions in classical probability theory. However, due to the underlying non-commutativity, these classical methods fail in general and a description in terms of measures is limited typically to the case of distributions of a single non-commutative random variable. This was the reason why the idea came up that multivariate non-commutative distributions should be understood via suitable "non-commutative test functions", which are evaluated in the given family of non-commutative random variables. If the result of this evaluation is such that its distribution allows an analytic description, one gains at least some partial information about the original non-commutative distribution, which becomes the more accurate the larger the considered class of non-commutative test functions is.

The still undefined term "non-commutative test functions", which we have used above in order to outline our strategy, raises actually two different questions, which we should better separate now: firstly, what are non-commutative functions, and secondly, how can we evaluate them? The first question is answered by "free analysis", which provides an analogue of (complex) analysis at the highest degree of non-commutativity. It originates in the work of J. L. Taylor [**Tay72, Tay73**], whose ideas later were taken up and developed further by Voiculescu for applications in free probability theory; see the survey [**Voi08**]. Many authors, like D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov, worked out free analysis as a theory in its own right with numerous applications in different fields of mathematics; their beautiful book [**KV14**] presents the current state of the art. The second question concerns the existence of some sort of functional calculus for non-commuting operators. Indeed, non-commutative functions as considered in free analysis are natural candidates that could lead us to an non-commutative analogue of the well-known holomorphic functional calculus in the commutative setting. This again goes back to the work of Taylor [**Tay72, Tay73**], but is still far from its final stage. It therefore constitutes a very active field of current research (see for instance [**AM16b, AM16a**]) and we leave it for further investigation, to which extend these recent achievements could be used in studying non-commutative distributions in a similar fashion like here. We will focus here on such non-commutative functions that are induced by some "universal expression", i.e., by some combination of formal variables and arithmetic operations, which make sense on every complex unital algebra, so that evaluation works here in a straightforward way. What we have in mind are more precisely non-commutative polynomials and non-commutative rational expressions. They will play the leading role in what follows. Let us note that also non-commutative power series work to some extend if we assume in addition that the algebra, on which evaluations are considered, carries a norm with respect to which it becomes a Banach algebra.

This approach to non-commutative joint distributions was already used in [**MS13**], where the speed of convergence in the multivariate free central limit theorem was measured in terms of polynomial evaluations and their Cauchy transforms. Using results presented by the author in the appendix of [**SV12**], it was possible to control this convergence even in terms of the Kolmogorov distance. It is work in progress to strengthen the statements presented in [**MS13**] by inventing some non-commutative analogue of the Lindeberg method for the setting of operator-valued free probability theory.

In the following two sections, we will outline the precise questions that are treated in this thesis and which are spread out over Chapters III, IV, V, VI and VII.

The rest of the thesis is organized as follows. In Chapter I, we will give a streamlined introduction to free probability theory. For convenience of the readers, we collect here all results which are needed in the subsequent chapters. Note that our exposition conveys a more analytic perspective on that theory and correspondingly leaves out most of its combinatorial aspects. It covers both the scalar-valued and the operator-valued case, as well as a brief discussion of the Brown measure. In Chapter II, we will provide some basic terminology from random matrix theory, where our main focus is on its connections to free probability theory. In the appendix, Chapter A is devoted to the Schur complement formula, Chapter B collects some facts about analytic functions between Banach spaces, and Chapter C aims at giving an overview over different topologies on sets of Radon measures, such as the vague or the weak topology.

Computation of analytic distributions and of Brown measures

It is a basic fact that free independence among a collection $X_1, \dots, X_n$ of non-commutative random variables fully determines their joint distribution $\mu_{X_1,\dots,X_n}$ in terms of the individual distributions $\mu_{X_1}, \dots, \mu_{X_n}$. This particularly means that the distribution $\mu_{P(X_1,\dots,X_n)}$ for each non-commutative polynomial P, evaluated in $X_1, \dots, X_n$, is completely determined by $\mu_{X_1}, \dots, \mu_{X_n}$. Accordingly, we can write

$$\mu_{P(X_1,\dots,X_n)} = P^{\square}(\mu_{X_1}, \dots, \mu_{X_n}),$$

where $P^{\square}(\mu_{X_1}, \dots, \mu_{X_n})$ stands for the free polynomial convolution as introduced in [**BV93**]. However, apart from the basic cases $P(x_1, x_2) = x_1 + x_2$ and $P(x_1, x_2) = x_1 \cdot x_2$, which can be computed by means of the free additive convolution $\mu_{X_1} \boxplus \mu_{X_2}$ and the free multiplicative convolution $\mu_{X_1} \boxtimes \mu_{X_2}$, respectively, only the commutator $P(x_1, x_2) = i(x_1x_2 - x_2x_1)$ and the anti-commutator $P(x_1, x_2) = x_1x_2 + x_2x_1$ were treated in detail (see [**NS98, Vas03**]). Computing $\mu_{P(X_1,\dots,X_n)}$ for more general polynomials P was out of reach for a long time – not to mention the analogous question with non-commutative polynomials replaced by non-commutative rational expressions.

Clearly, we can only hope for some deeper insights going beyond the combinatorial tools when working in some analytic setting. The precise problems thus read as follows.

Problem 1. *Given a self-adjoint non-commutative rational expression r in the formal variables $x = (x_1, \dots, x_n)$. Let $X_1, \dots, X_n$ be freely independent self-adjoint elements in some non-commutative C^*-probability space $(\mathcal{A}, \phi)$, for which the evaluation $r(X_1, \dots, X_n)$ is well-defined. If the distribution of each of the X_j's is known, how can we compute the distribution of $r(X_1, \dots, X_n)$?*

Problem 2. *Given an arbitrary non-commutative rational expression r in the formal variables $x = (x_1, \dots, x_n)$. Let $X_1, \dots, X_n$ be freely independent self-adjoint elements in some tracial W^*-probability space $(\mathcal{A}, \phi)$ for which the evaluation $r(X_1, \dots, X_n)$ is well-defined. If the distribution of each of the X_j's is known, how can we compute the Brown-measure of $r(X_1, \dots, X_n)$?*

In Chapter IV, we will present a systematic approach to these problems, resulting in Algorithm IV.4.1 and Algorithm IV.4.2, which provide their complete solutions and also some very efficient machinery for carrying out numerical computations. Lemma II.4.1 and Lemma II.4.4 will show that our results even apply to questions in random matrix theory.

Let us point out that both of these algorithms are based essentially on two pillars, namely

- on *operator-valued free probability theory*, especially on Theorem I.2.18, which allows an effective analytic treatment of the operator-valued free additive convolution by means of *subordination functions* for general operator-valued C^*-probability spaces. Theorem I.2.18, which was obtained in [**BMS13**], finalized several previous attempts [**Bia98a, Voi00b, Voi02a**], which were accomplished in an operator-valued frame but under more restrictive assumptions. In addition, it provides a fixed point iteration scheme for the involved subordination functions, which is analogous to the scalar-valued case as treated in [**BB07**].
- on the *method of linearization*, which allows us to transfer by purely algebraic means any well-defined non-commutative rational expression $r(X_1, \dots, X_n)$ into some linear but matrix-valued expression

$$L(X_1, \dots, X_n) = L^{(0)} + L^{(1)} X_1 + \cdots + L^{(n)} X_n,$$

whose coefficients $L^{(0)}, L^{(1)}, \dots, L^{(n)}$ are complex matrices. This translates the scalar-valued problem concerning $r(X_1, \dots, X_n)$ into an operator-valued problem about $L(X_1, \dots, X_n)$. The method of linearization, which is inside the free probability community also known under the name "linearization trick", will be presented in detail in Chapter III. Our exposition relies on [**BMS13**] and [**HMS15**], but provides in addition several refinements and generalizations of the results that were presented therein.

Regularity questions

Non-commutative distributions are by definition purely combinatorial objects and apart from the commutative case (including especially the case of one variable), no analytic description in terms of measures is possible. Accordingly, there is no natural notion of regularity for non-commutative distributions.

However, based on the analogy to the classical situation, it was commonly believed that conditions on quantities like the free Fisher information, the free entropy, or the free entropy dimension, which were introduced by Voiculescu in his famous series of papers [**Voi93, Voi94, Voi96, Voi97, Voi98, Voi99**] (see also the survey [**Voi02b**]), imply strong regularity properties of the considered distributions. But how should one make such statements precise without having an absolute notion of regularity?

A very natural and also quite promising approach is that regularity of non-commutative distributions $\mu_{X_1,\dots,X_n}$ for non-commutative random variables should be understood as regularity – in the usual measure theoretic meaning – of those one-variable distributions $\mu_{f(X_1,\dots,X_n)}$, which arise under evaluation of sufficiently many self-adjoint test functions f in the given variables $X_1, \dots, X_n$. Following this strategy, one typically imposes some of the previously mentioned more abstract conditions on non-commutative distributions and tries then to detect regularity in the latter sense via evaluations.

All of the papers [**MSW14, MSW17, Mai15**], which underlie our approach to these regularity questions, are based on the theory of non-commutative derivatives, which arises from the work of Voiculescu [**Voi98, Voi99**] and of Dabrowski [**Dab10, Dab14**]. This will be the topic of Chapter V. It aims at a uniform exposition of their results on a general level like in [**Mai15**], culminating in Proposition V.6.1, which is taken in this form from [**Mai15**] and which appeared before in a more specialized version in [**MSW14, MSW17**]. This statement is at the core of some general reduction argument, which will be essential in Chapter VI and Chapter VII.

In Chapter VI, which is based on [**MSW14, MSW17**], we show that in a tracial and finitely generated W^*-probability space existence of conjugate variables excludes algebraic relations for the generators; see Theorem VI.1.5. Moreover, under the assumption of maximal non-microstates free entropy dimension, i.e. $\delta^*(X_1, \dots, X_n) = n$, we prove that there are no zero divisors in the sense that the product of any non-commutative polynomial in $X_1, \dots, X_n$ with any element from the generated von Neumann algebra is zero if and only if at least one of those factors is zero; see Theorem VI.2.1. On the one hand, this gives an interesting connection to the work of Linnell [**Lin91, Lin92, Lin93, Lin98**] on analytic versions of the zero divisor conjecture, especially in the case of the free group. On the other hand, it shows that under the assumption $\delta^*(X_1, \dots, X_n) = n$ the distribution of any non-constant self-adjoint non-commutative polynomial $P(X_1, \dots, X_n)$ in $X_1, \dots, X_n$ does not have atoms; see Corollary VI.2.2. Questions on the absence of atoms for polynomials in non-commuting random variables (or for polynomials in random matrices) have been an open problem for quite a while. We solve this general problem by showing that maximality of free entropy dimension excludes atoms. This continued and generalized the previous work [**SS15**] on regularity questions for polynomial evaluations and the methods investigated in [**MSW14, MSW17**] have already initiated an impressive variety of follow-up research; see [**CS16, Dab15, Har15**].

Following [**Mai15**], we discuss in Chapter VII another extension of the methods of Chapter VI, namely to the continuous setting of free stochastic calculus. Wigner integrals

$$I_n^S(f) = \int_{\mathbb{R}_+^n} f(t_1, \dots, t_n)\; dS_{t_1} \cdots dS_{t_n}$$

for $f \in L^2(\mathbb{R}_+^n)$ on $\mathbb{R}_+ = [0, \infty)$ and the corresponding Wigner chaos were introduced by P. Biane and R. Speicher in 1998 as a non-commutative counterpart of classical Wiener-Itô integrals and the corresponding Wiener-Itô chaos, respectively, in free probability; see [**BS98**]. In the classical case, a famous result of I. Shigekawa [**Shi78, Shi80**] states that non-trivial elements in the finite Wiener-Itô chaos have an absolutely continuous distribution. We provide here a first contribution to such regularity questions for Wigner integrals by showing that the distribution of non-trivial elements in the finite Wigner chaos of the form

$$I_1^S(f_1) + I_2^S(f_2) + \cdots + I_N^S(f_N)$$

with mirror-symmetric $f_n \in L^2(\mathbb{R}_+^n)$ for $n = 1, \dots, N$ and $f_N \neq 0$ cannot have atoms. The corresponding Theorem VII.1.4 answers a question of I. Nourdin and G. Peccati [**NP13**]. Similar to the discrete case, we will deduce Theorem VII.1.4 from a more general statement, Theorem VII.3.12, by which we exclude zero-divisors in the finite Wigner chaos.

For doing so, we establish in Subsection VII.3.1 the notion of directional gradients in the context of the free Malliavin calculus. These directional gradients bridge between free Malliavin calculus and the theory of non-commutative derivations presented in Chapter V. The methods of [**MSW14, MSW17**], which were used for treating similar questions in the case of finitely many variables as outlined in Chapter VI, will be extended in such a way that they even apply to directional gradients.

CHAPTER I

Some basics of free probability theory

Free probability theory can be seen as a counterpart of classical probability at the highest degree of non-commutativity. It gives a special instance of non-commutative probability theory and is therefore formulated in this general language, but it enjoys the characteristic feature that it comes with its own notion of independence, called "free independence". This independence totally differs from the classical notion of independence, but it shows many conceptional similarities, so that free probability evolves excitingly far in parallel to the classical theory.

This theory was invented around 1985 by Voiculescu and it was intended originally to serve as a tool for attacking the isomorphism problem for the free group factors $L(\mathbb{F}_n)$. This is based on the fascinating observation that free independence reflects on the operator algebraic side the structure that is induced by free products on the group side.

Later, Voiculescu also found a quite surprising and extremely fruitful connection to random matrix theory. He noticed that free independence shows up very naturally for many classes of independent random matrices in the limit when their dimension tends to infinity. Based on this so-called "asymptotic freeness" phenomenon, free probability nowadays provides a powerful machinery to understand limiting eigenvalue distributions for many types of random matrices. This replaced, systematized, and generalized previously given ad hoc arguments and has even more produced an impressive amount of totally new results.

However, apart from the free product construction and the limiting behavior of certain random matrices, there is still another important source of free independence. In fact, it shows up among creation and annihilation operators for orthogonal vectors on the full Fock space. We admit that this might sound like a rather artificial approach to free independence, but since this construction is easily accessible to concrete computations, it is of great theoretical importance. In particular, it underlies the free Malliavin calculus as we will outline in Chapter VII.

For the seek of completeness but without going into details, we mention that free probability found applications also in the study of some asymptotic phenomena in the representation theory of the symmetric group; see, for instance, [**Bia98b**] or the recent survey paper [**Spe16**].

In this chapter, we will provide a brief introduction to the field of free probability. Of course, we can only discuss here some of its basic aspects, and we will do this in a way which is mostly streamlined to our needs. For a more detailed introduction, we refer the interested reader to the monographs [**VDN92, Voi00a, HP00b, NS06**].

In Section I.1, we will first present the scalar-valued theory of free probability, and in Section I.2, we will turn our attention to its operator-valued generalization. In each of these cases, we will focus on the analytic aspects of that theory. Nonetheless, our discussions will begin in most cases on a purely algebraic level and we also try to highlight whenever we hit an important link to the combinatorial side.

I.1. Scalar-valued free probability theory

We first present the scalar-valued part of free probability theory. As we will see in Section I.2, there exists also an operator-valued extension of free probability, which generalizes the scalar-valued theory in the same vein as conditional expectations generalize expectations in classical probability. For the seek of clarity, we prefer to discuss these topics separately.

I.1.1. Non-commutative probability spaces. At the basis of non-commutative probability and free probability in particular is the notion of non-commutative probability spaces.

I.1.1.1. *The basic terminology.* We are going to present the most general and purely algebraic definition first.

DEFINITION I.1.1. A *non-commutative probability space* is a pair $(\mathcal{A}, \phi)$ consisting of a unital complex algebra[1] $\mathcal{A}$ and a linear functional $\phi : \mathcal{A} \to \mathbb{C}$ that satisfies $\phi(1) = 1$. Elements of $\mathcal{A}$ are called *non-commutative random variables* and ϕ is called *expectation* on $\mathcal{A}$.

In order to convince the reader that this is indeed some reasonable terminology, we should first check that the classical notion of probability spaces fits into this general frame. This will be done in the following example.

EXAMPLE I.1.2. Let us take any *classical probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. Recall that this means that Ω is some set, $\mathcal{F}$ some σ-algebra consisting of subsets of Ω, and $\mathbb{P}$ some probability measure, which is defined on all sets belonging to $\mathcal{F}$. The complex unital algebra $L^\infty(\Omega, \mathbb{P})$ of bounded random variables is then naturally endowed with the expectation $\mathbb{E} : L^\infty(\Omega, \mathbb{P}) \to \mathbb{C}$, which is the unital linear functional that is defined by

$$\mathbb{E}[X] := \int_\Omega X(\omega)\, d\mathbb{P}(\omega) \qquad \text{for any } X \in L^\infty(\Omega, \mathbb{P}).$$

We thus see that $(L^\infty(\Omega, \mathbb{P}), \mathbb{E})$ indeed provides an example of a non-commutative probability space.

Matrix algebras constitute another basic example of non-commutative probability spaces. They will be discussed next.

EXAMPLE I.1.3. For any positive integer N, let $M_N(\mathbb{C})$ denote the algebra of $N \times N$ matrices over $\mathbb{C}$. If we endow $M_N(\mathbb{C})$ with the normalized trace

$$\operatorname{tr}_N(X) := \frac{1}{N} \sum_{i=1}^N X_{i,i} \qquad \text{for all } X = (X_{i,j})_{i,j=1}^N \in M_N(\mathbb{C}).$$

[1]The term "algebra" is – as experience teaches us – a constant source of confusion, since different communities typically agree on slightly different properties. Thus, let us stipulate that "a complex algebra $\mathcal{A}$" for us always means "an associative complex algebra $\mathcal{A}$", i.e., a complex vector space $\mathcal{A}$ that is endowed with a $\mathbb{C}$-bilinear mapping $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, called the *multiplication*, which is associative in the sense that $X \cdot (Y \cdot Z) = (X \cdot Y) \cdot Z$ holds for all $X, Y, Z \in \mathcal{A}$. By the additional term "unital", we indicate that $\mathcal{A}$ contains a unique element $1 = 1_\mathcal{A}$, called the *identity element of* $\mathcal{A}$, which satisfies $1 \cdot X = X = X \cdot 1$ for all $X \in \mathcal{A}$.

we obtain the non-commutative probability space $(M_N(\mathbb{C}), \mathrm{tr}_N)$.

We point out that a suitable combination of the previously mentioned examples will provide us in Chapter II some appropriate framework for dealing with random matrices; see Definition II.1.1.

Example I.1.3 motivates the following definition.

DEFINITION I.1.4. Let $\mathcal{A}$ be a algebra and $\phi : \mathcal{A} \to \mathbb{C}$ be a linear map. We call ϕ *tracial* (or *trace*), if it satisfies

$$\phi(XY) = \phi(YX) \qquad \text{for all } X, Y \in \mathcal{A}.$$

Well-known facts from linear algebra say that the non-commutative probability space $(M_N(\mathbb{C}), \mathrm{tr}_N)$, which was presented in Example I.1.3, indeed comes with a tracial expectation. For more trivial reasons, namely due to the commutativity of the underlying algebra of random variables, the classical expectation appearing in Example I.1.2 is also tracial. Furthermore, as we will see in Chapter II, the traciality of these two examples passes to the non-commutative probability space of random matrices.

I.1.1.2. *$*$-probability spaces.* All non-commutative probability spaces that we discussed in the examples above have in common that their corresponding algebras carries some additional $*$-structure. Recall that a complex unital algebra $\mathcal{A}$ is called *$*$-algebra*, if it is endowed with a complex anti-linear mapping $* : \mathcal{A} \to \mathcal{A}$, which satisfies $(X^*)^* = X$ for all $X \in \mathcal{A}$ and $(XY)^* = Y^*X^*$ for all $X, Y \in \mathcal{A}$. We will now focus on such situations.

DEFINITION I.1.5. Let $(\mathcal{A}, \phi)$ be a non-commutative probability space. We call $(\mathcal{A}, \phi)$ a *$*$-probability space*, if $\mathcal{A}$ is a $*$-algebra and the expectation ϕ is *positive*, i.e., if it satisfies $\phi(X^*X) \geq 0$ for all $X \in \mathcal{A}$.

Of particular interest are $*$-probability spaces $(\mathcal{A}, \phi)$, whose expectation ϕ is moreover faithful. Loosely speaking, this means that ϕ can "see" all *positive* elements in $\mathcal{A}$, which are elements of the form X^*X for some $X \in \mathcal{A}$.

DEFINITION I.1.6. Let $\mathcal{A}$ be a $*$-algebra and $\phi : \mathcal{A} \to \mathbb{C}$ be a linear map. We call ϕ *faithful*, if for any $X \in \mathcal{A}$ the condition $\phi(X^*X) = 0$ implies that $X = 0$.

It is easy to check that the non-commutative probability spaces introduced in Example I.1.2 and in Example I.1.3 are indeed $*$-probability spaces with faithful expectations.

Although this definition is still of algebraic nature, the positivity constraint imposed on the expectation brings us already close to the analytic setting. Indeed, the assumed positivity of ϕ enforces many strong properties, such as (see [**NS06**, Remark 1.2])

$$\phi(X^*) = \overline{\phi(X)} \qquad \text{for all } X \in \mathcal{A}$$

and the *Cauchy-Schwarz inequality*

$$|\phi(Y^*X)|^2 \leq \phi(X^*X)\phi(Y^*Y) \qquad \text{for all } X, Y \in \mathcal{A}.$$

I.1.1.3. *C^*- and W^*-probability spaces.* The actual analytic setting is reached by putting some topological structure on $\mathcal{A}$, with which the expectation ϕ must be compatible. We will focus here on two important examples, namely on C^*- and W^*-probability spaces.

Let us first consider C^*-probability spaces.

Definition I.1.7. A *C^*-probability space* is a pair $(\mathcal{A}, \phi)$ consisting of a unital C^*-algebra $\mathcal{A}$ and a state ϕ on $\mathcal{A}$.

Recall that a *state* ϕ on a unital C^*-algebra $\mathcal{A}$ is a linear functional $\phi : \mathcal{A} \to \mathbb{C}$, which is , i.e. satisfies $\phi(X^*X) \geq 0$ for all $X \in \mathcal{A}$, and for which $\phi(1) = 1$ holds. Therefore, a C^*-probability space is clearly a non-commutative probability space and, with respect to its involution $*$, it is in fact a $*$-probability space. Depending on the intended application, we will sometimes assume in addition that the corresponding state is faithful, tracial, or even both.

Let us continue with W^*-probability spaces. Each von Neumann algebra is of course a C^*-algebra and W^*-probability spaces are correspondingly a special instance of C^*-probability spaces, but this description does by no means meet the actual truth. Therefore, W^*-probability spaces should like von Neumann algebras be considered as objects of their own right. Among several possible definitions, we choose here the following one which perfectly suits our needs.

Definition I.1.8. A *tracial W^*-probability space* is a pair (M, τ), where M is a von Neumann algebra and τ a faithful normal tracial state on M.

Recall that a state τ on M is called *normal*, if $\lim_{\lambda\in\Lambda} \tau(T_\lambda) = \tau(T)$ holds for each monotone increasing net $(T_\lambda)_{\lambda\in\Lambda}$ of positive operators in M with least upper bound $T \in M$. We point out that normality is in fact equivalent to the statement that τ is continuous with respect to the weak (or the strong) operator topology if it is restricted to sets in M of bounded operator-norm; see [**Bla06**, Theorem III.2.1.4].

We note that the non-commutative probability space $(M_N(\mathbb{C}), \mathrm{tr}_N)$, which we know from Example I.1.3 and which was already identified as a $*$-probability space, is in fact a C^*-probability space (with tr_N being a faithful tracial state) and even more a tracial W^*-probability space.

It is important to note that there is a canonical tensor product for von Neumann algebras that respects even the notion of W^*-probability spaces.

Remark I.1.9. If (M_1, τ_1) and (M_2, τ_2) are two tracial W^*-probability spaces, then also their von Neumann algebra tensor product $M_1 \otimes M_2$ becomes, endowed with the tensor product state $\tau_1 \otimes \tau_2$, a tracial W^*-probability space.

Another construction that will be used repeatedly in the subsequent considerations are the non-commutative L^p-spaces.

Remark I.1.10. Given any tracial W^*-probability space (M, τ), we may introduce the non-commutative L^p-spaces $L^p(M, \tau)$ for $1 \leq p \leq \infty$ as the completion of M with respect to the norm $\|x\|_{L^p(M,\tau)} := \tau\big((x^*x)^{\frac{p}{2}}\big)^{\frac{1}{p}}$, and for $p = \infty$ simply by $L^\infty(M, \tau) := M$ where we put $\|x\|_{L^\infty(M,\tau)} := \|x\|$. Whenever it is not necessary to indicate explicitly the underlying von Neumann algebra, we will abbreviate $\|\cdot\|_p := \|\cdot\|_{L^p(M,\tau)}$.

I.1.2. Non-commutative distributions. Non-commutative distributions transfer the well-established notion of joint distributions known from classical probability theory to the realm of non-commutative probability. It is therefore instructive to recall the classical situation first.

EXAMPLE I.1.11. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a classical probability space. For any given collection $X = (X_1, \dots, X_n)$ of finitely many random variables $X_1, \dots, X_n \in L^\infty(\Omega, \mathbb{P})$, the *(joint) distribution of X* is given as the probability measure μ_X, which is defined by

$$\mu_X(B) = \mathbb{P}\big(\{\omega \in \Omega |\ (X_1(\omega), \dots, X_n(\omega)) \in B\}\big)$$

for any Borel subset B of $\mathbb{R}^n$. In other words, μ_X is nothing but the push forward measure of $\mathbb{P}$ under the measurable map

$$X:\ \Omega \to \mathbb{R}^n,\ \omega \mapsto (X_1(\omega), \dots, X_n(\omega)).$$

With a bit of work, one can show that the probability measure μ_X is compactly supported and that

$$\begin{aligned} \mathbb{E}\big[P(X_1, \dots, X_n)\big] &= \int_\Omega P(X_1(\omega), \dots, X_n(\omega))\, d\mathbb{P}(\omega) \\ &= \int_{\mathbb{R}^n} P(x_1, \dots, x_n)\, d\mu_X(x_1, \dots, x_n) \end{aligned} \tag{I.1}$$

holds for each commutative polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$; see [**NS06**, Example 4.4 (1)].

The latter formula (I.1) turns the classical joint distribution into the linear functional

$$\mathbb{C}[x_1, \dots, x_n] \to \mathbb{C},\ P \mapsto \mathbb{E}\big[P(X_1, \dots, X_n)\big].$$

It is a very important feature of this formula that it gives meaning to the classical joint distribution μ_X only in terms of $(L^\infty(\Omega, \mathbb{P}), \mathbb{E})$ and hence without referring to the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The linear functional

$$\mu_X:\ \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C},\ P \mapsto \phi\big(P(X_1, \dots, X_n)\big)$$

gives therefore a natural generalization of the classical joint distribution to the case of non-commutative random variables $(X_1, \dots, X_n)$ in some non-commutative probability space $(\mathcal{A}, \phi)$, for which typically no underlying structure such as $(\Omega, \mathcal{F}, \mathbb{P})$ exists. Of course, for this purpose, we should also replace the much too restrictive algebra $\mathbb{C}[x_1, \dots, x_n]$ of commutative polynomials by the algebra $\mathbb{C}\langle x_1, \dots, x_n\rangle$ of non-commutative polynomials.

I.1.2.1. *Non-commutative distributions in general non-commutative probability spaces.* The motivating discussion about the classical case gives justification for the terminology of non-commutative joint distributions, which we are going to introduce now in the generality of non-commutative probability spaces.

DEFINITION I.1.12. Let I be some non-empty index set.

(i) By $\mathbb{C}\langle x_i |\ i \in I\rangle$, we denote the free algebra over $\mathbb{C}$ with generators $\{x_i |\ i \in I\}$. In the following, we will refer to $\mathbb{C}\langle x_i |\ i \in I\rangle$ as the algebra of *non-commutative polynomials* in the formal non-commuting variables $\{x_i |\ i \in I\}$.

(ii) Let $X = (X_i)_{i \in I}$ be a family of non-commutative random variables, over the index set I, living in a non-commutative probability space $(\mathcal{A}, \phi)$. Denote by ev_X the *evaluation homomorphism*

$$\mathrm{ev}_X:\ \mathbb{C}\langle x_i |\ i \in I\rangle \to \mathcal{A},$$

which is, as a homomorphism, uniquely determined by $1 \mapsto 1_\mathcal{A}$ and $x_i \mapsto X_i$ for all $i \in I$. For any given $P \in \mathbb{C}\langle x_i |\ i \in I\rangle$, we mostly abbreviate $P(X) := \mathrm{ev}_X(P)$. The *non-commutative (joint) distribution μ_X of $X = (X_i)_{i \in I}$* is defined as the linear functional $\mu_X := \phi \circ \mathrm{ev}_X$, i.e.

$$\mu_X:\ \mathbb{C}\langle x_i |\ i \in I\rangle \to \mathbb{C},\ P \mapsto \phi(P(X)).$$

REMARK I.1.13. By linearity and due to the normalization $\mu_X(1) = \phi(1) = 1$, the non-commutative distribution μ_X of any family $X = (X_i)_{i\in I}$ is fully determined by its values

$$\mu_X(x_{i_1}x_{i_2}\cdots x_{i_k}) = \phi\big(X_{i_1}X_{i_2}\cdots X_{i_k}\big)$$

on all *monomials* $x_{i_1}x_{i_2}\cdots x_{i_k}$ with $k \geq 1$ and $i_1, i_2, \ldots, i_k \in I$. An expression of the form

$$\phi\big(X_{i_1}X_{i_2}\cdots X_{i_k}\big) \qquad \text{with } i_1, \ldots, i_k \in I,$$

will be called *moment of order* $k \geq 0$, where we include $\phi(1) = 1$ as the moment of order $k = 0$. Thus, μ_X can be seen as the collection of all *(joint) moments of* $(X_i)_{i\in I}$.

I.1.2.2. *Non-commutative distributions in $*$-probability spaces.* When working in $*$-probability spaces, it is natural to consider non-commutative random variables at once with their adjoints. This leads to the notion of $*$-distributions.

DEFINITION I.1.14. Let I be some non-empty index set.

(i) By $\mathbb{C}\langle x_i, x_i^*|\ i \in I\rangle$, we denote the $*$-algebra of all *non-commutative $*$-polynomials* in the formal variables $\{x_i|\ i \in I\}$. Formally, $\mathbb{C}\langle x_i, x_i^*|\ i \in I\rangle$ is nothing but the free algebra with formal generators $\{x_i|\ i \in I\} \cup \{x_i^*|\ i \in I\}$. It carries naturally an involution $*$ by declaring $1^* = 1$ and $(x_i)^* = x_i^*$ for all $i \in I$.

(ii) Let $(\mathcal{A}, \phi)$ be a $*$-probability space and let $X = (X_i)_{i\in I}$ be some family of non-commutative random variables in $\mathcal{A}$. We denote by ev_{X,X^*} the *evaluation $*$-homomorphism*

$$\mathrm{ev}_{X,X^*}:\ \mathbb{C}\langle x_i, x_i^*|\ i \in I\rangle \to \mathcal{A},$$

which is, as a $*$-homomorphism, uniquely determined by $1 \mapsto 1_{\mathcal{A}}$ and $x_i \mapsto X_i$ for all $i \in I$. For any given $P \in \mathbb{C}\langle x_i, x_i^*|\ i \in I\rangle$, we mostly abbreviate $P(X, X^*) := \mathrm{ev}_{X,X^*}(P)$. The *non-commutative (joint) $*$-distribution of* $X = (X_i)_{i\in I}$ is given as the linear functional $\mu_{X,X^*} := \phi \circ \mathrm{ev}_{X,X^*}$, i.e.

$$\mu_{X,X^*}:\ \mathbb{C}\langle x_i, x_i^*|\ i \in I\rangle \to \mathbb{C},\ \ P \mapsto \phi(P(X, X^*)).$$

Clearly, we can view μ_{X,X^*} as the non-commutative joint distribution of the family $(X_i^\varepsilon)_{(i,\varepsilon)\in I\times\{1,*\}}$.

REMARK I.1.15. In the same way as each non-commutative distributions μ_X is uniquely determined by the collection of all joint moments of X, as it was observed in Remark I.1.13, non-commutative $*$-distributions μ_{X,X^*} are determined by their values

$$\mu_{X,X^*}(x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}\cdots x_{i_k}^{\varepsilon_k}) = \phi\big(X_{i_1}^{\varepsilon_1}X_{i_2}^{\varepsilon_2}\cdots X_{i_k}^{\varepsilon_k}\big)$$

on all *$*$-monomials* $x_{i_1}^{\varepsilon_1}x_{i_2}^{\varepsilon_2}\cdots x_{i_k}^{\varepsilon_k}$ with $k \geq 1$, $i_1, i_2, \ldots, i_k \in I$, and $\varepsilon_1, \ldots, \varepsilon_k \in \{1, *\}$. An expression of the form

$$\phi\big(X_{i_1}^{\varepsilon_1}X_{i_2}^{\varepsilon_2}\cdots X_{i_k}^{\varepsilon_k}\big) \qquad \text{with } i_1, \ldots, i_k \in I \text{ and } \varepsilon_1, \ldots, \varepsilon_k \in \{1, *\},$$

is called *$*$-moment of order* $k \geq 0$, where we include $\phi(1) = 1$ as the $*$-moment of order $k = 0$. Again, μ_{X,X^*} can be seen as the collection of all *(joint) $*$-moments of* $(X_i)_{i\in I}$.

Whenever we work with families $(X_i)_{i\in I}$ of self-adjoint non-commutative random variables in some $*$-probability space $(\mathcal{A}, \phi)$, then its non-commutative $*$-distribution μ_{X,X^*} : $\mathbb{C}\langle x_i, x_i^*|\ i \in I\rangle \to \mathbb{C}$ clearly does not contain more information than just the non-commutative distribution $\mu_X :\ \mathbb{C}\langle x_i|\ i \in I\rangle \to \mathbb{C}$. We will come back to this point when talking about analytic distributions in the case of normal and self-adjoint operators; see Definition I.1.17 and Definition I.1.18 below. There, we will make the distinction between measures on $\mathbb{C}$ and measures on $\mathbb{R}$, which in our current combinatorial setting indicates that, for self-adjoint $X = (X_i)_{i\in I}$, we should think of the formal variables $(x_i)_{i\in I}$ appearing in $\mu_X : \mathbb{C}\langle x_i | i \in I\rangle \to \mathbb{C}$ as being self-adjoint in some appropriate sense. Let us make this more precise.

Definition I.1.16. Let I be some non-empty index set. Then the algebra $\mathbb{C}\langle x_i | i \in I\rangle$ carries a natural involution $*$, with respect to which it becomes a $*$-algebra. This involution $*$ is uniquely determined by the conditions that $1^* = 1$ and $x_i^* = x_i$ for all $i \in I$ holds.

I.1.2.3. *Non-commutative distributions in C^*- and W^*-probability spaces.* The advantage of working in C^*- or W^*-probability spaces is that their underlying topological structure allows to treat non-commutative distributions by more analytic tools – at least in the case of one variable.

Let us take any non-commutative random variable X in some C^*-probability space $(\mathcal{A}, \phi)$ and suppose that X is *normal*, i.e., that X commutes with its adjoint X^*. Then the non-commutative $*$-distribution μ_{X,X^*} of X, i.e. the collection of all $*$-moments of X, can be encoded by some compactly supported Borel probability measure on the complex plane $\mathbb{C}$. Its construction proceeds as follows: the functional calculus for X yields an isometric $*$-homomorphism $f \mapsto f(X)$ from $C(\sigma(X))$, the space of all complex-valued continuous functions on the spectrum $\sigma(X)$ of X, into the C^*-algebra $\mathcal{A}$. Accordingly, we obtain a bounded linear functional

$$I : C(\sigma(X)) \to \mathbb{C}, \ f \mapsto \phi(f(X))$$

and the positivity of ϕ gives that this linear functional is also positive. Thus, the Riesz representation theorem (see Corollary C.5) tells us that I can be written as $I = I_{\mu_X}$, i.e.

$$\phi(f(X)) = \int_{\mathbb{C}} f(z)\, d\mu_{X,X^*}(z) \qquad \text{for all } f \in C(\sigma(X)), \tag{I.2}$$

for some unique Borel measure μ_{X,X^*} on $\mathbb{C}$, whose support is contained in the compact set $\sigma(X)$ and which is due to $\phi(1) = 1$ in fact a probability measure. Hence, we can record:

Definition I.1.17. Let X be a normal non-commutative random variable in a C^*-probability space $(\mathcal{A}, \phi)$. The *(analytic) $*$-distribution μ_{X,X^*} of X* is the compactly supported Borel probability measure on $\mathbb{C}$, which is uniquely determined by the condition

$$\phi(X^k(X^*)^l) = \int_{\mathbb{C}} z^k \overline{z}^l\, d\mu_{X,X^*}(z) \qquad \text{for } k, l \in \mathbb{N}_0. \tag{I.3}$$

Note that we have replaced in Definition I.1.17 the determining condition (I.2) for the analytic distribution μ_{X,X^*} by (I.3); this is possible since the $\mathbb{C}$-linear span of all functions $z \mapsto z^k\overline{z}^l$ with $k, l \in \mathbb{N}_0$ is dense in $C(\sigma(X))$ by the Stone-Weierstraß theorem.

Moreover, let us point out that one can show (see [**NS06**, Proposition 3.15]) that in fact $\operatorname{supp}(\mu_X) = \sigma(X)$ holds.

In the special case of a self-adjoint non-commutative random variable X, we can specify the construction in such a way that its representing measure becomes a Borel probability measure on $\mathbb{R}$. Like in the normal case above, the analytic distribution μ_X of X is based on the functional calculus for X and on the Riesz representation theorem (see Corollary C.5). This identifies μ_X as the unique Borel probability measure on $\mathbb{R}$, which satisfies

$$\phi(f(X)) = \int_{\mathbb{R}} f(t)\, d\mu_X(t) \qquad \text{for all } f \in C(\sigma(X)). \tag{I.4}$$

DEFINITION I.1.18. Let X be a self-adjoint non-commutative random variable in a C^*-probability space $(\mathcal{A}, \phi)$. The *(analytic) distribution* μ_X *of* X is the compactly supported Borel probability measure on $\mathbb{R}$, which is uniquely determined by the condition

$$\phi(X^k) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \text{for } k \in \mathbb{N}_0. \tag{I.5}$$

Definition I.1.18 characterizes the analytic distribution μ_X by the condition (I.5), whereas the stronger condition (I.4) is required in order to deduce uniqueness from the Riesz representation theorem. That these conditions are indeed equivalent follows from the observation that the $\mathbb{C}$-linear span of all functions $t \mapsto t^k$ with $k \in \mathbb{N}_0$ is dense in $C(\sigma(X))$ according to the Stone-Weierstraß theorem.

Clearly, for a self-adjoint X we can also consider μ_{X,X^*} besides the more appropriate μ_X, but since the $*$-distribution of X contains exactly the same amount of information as its distribution, it is not surprising that μ_{X,X^*} is related with μ_X via $\mu_{X,X^*}(B) = \mu_X(B \cap \mathbb{R})$ for each Borel subset B of $\mathbb{C}$.

Finally, a few words on the notation are in order. Note that the analytic distribution μ_{X,X^*} (respectively μ_X) encodes all $*$-moments (respectively moments) of any normal (respectively self-adjoint) non-commutative random variable X and can therefore be uniquely identified with the distribution of X in the previous sense of Definition I.1.14 (respectively Definition I.1.12). This excuses that we use the same symbol both for the combinatorial and the analytic distribution.

So far, we were concerned with the case of C^*-probability spaces. Now, let us turn our attention to W^*-probability spaces. Here, analytic $*$-distributions for non-commutative random variables can be connected with the spectral distribution measure. We first remind ourselves of some background details.

REMARK I.1.19. The spectral theorem for normal operators (see, for instance, [**Bla06**, Theorem I.6.2.2]) tells us that for any normal operator X on some Hilbert space H, we can find a projection-valued measure E_X on the Borel subsets of the spectrum $\sigma(X)$ of X, the so-called *spectral measure of* X, such that

$$X = \int_{\sigma(X)} z \, dE_X(z)$$

holds. Furthermore (see [**Bla06**, I.6.2.4]), we can use this representation to define a functional calculus: for each bounded Borel measurable function $f : \sigma(X) \to \mathbb{C}$, we put

$$f(X) := \int_{\sigma(X)} f(z) \, dE_X(z).$$

It is known that this functional calculus extends the polynomial functional calculus, i.e. we have

$$X^k (X^*)^l = \int_{\sigma(X)} f(z) \, dE_X(z) \qquad \text{for } k, l \in \mathbb{N}_0. \tag{I.6}$$

Now, if we take any normal non-commutative random variable in some tracial W^*-probability space (M, τ), the spectral measure E_X takes its values in the von Neumann subalgebra $\mathrm{vN}(X) \subseteq M$ generated by X. Thus, we can apply τ to (I.6) in order to deduce that

$$\tau(X^k (X^*)^l) = \int_{\sigma(X)} f(z) \, d(\tau \circ E_X)(z) \qquad \text{for } k, l \in \mathbb{N}_0.$$

Comparing this with (I.3), we see that the compactly supported measure given by $\tau \circ E_X$ must agree with the analytic distribution μ_{X,X^*}. We record this important observation for later references.

LEMMA I.1.20. *Let X be a normal non-commutative random variable in some tracial W^*-probability space (M, τ). If E_X denotes its spectral measure on $\sigma(X) \subset \mathbb{C}$, then the non-commutative $*$-distribution μ_{X,X^*} of X is given by*

$$\mu_{X,X^*} = \tau \circ E_X.$$

Analogously, if X is even self-adjoint, then the analytic distribution μ_X of X is given by

$$\mu_X = \tau \circ E_X,$$

where E_X now stands for spectral measure of X on $\sigma(X) \subset \mathbb{R}$.

Consequently, an atom α of μ_{X,X^*} (respectively of μ_X) implies by the spectral theorem, which we have recalled in Remark I.1.19, the existence a non-zero-projection $u := E_X(\{\alpha\}) \in M$, such that $(X - \alpha 1)u = 0$ holds. Note that $\alpha \in \mathbb{C}$ (respectively $\alpha \in \mathbb{R}$) is said to be an *atom* of a Borel probability measure μ on $\mathbb{C}$ (respectively on $\mathbb{R}$), if $\mu(\{\alpha\}) \neq 0$.

We conclude our discussion with the following example.

EXAMPLE I.1.21. An easy but very enlightening task is the computation of analytic distributions in the C^*-probability space $(M_N(\mathbb{C}), \mathrm{tr}_N)$, which we already know from Example I.1.3. Indeed, if we take any matrix $X \in M_N(\mathbb{C})$, which is self-adjoint, then basic linear algebra tells us that we can find a unitary matrix $U \in M_N(\mathbb{C})$, such that X can be written as

$$X = U\Lambda U^* \qquad \text{with} \qquad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_N$ are the eigenvalues of X, listed according to their multiplicity. The analytic distribution μ_X of X is then given as

$$\mu_X = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i},$$

where δ_λ denotes the *Dirac measure with atom* λ; hence, we see that μ_X is just the *eigenvalue distribution of* X. Indeed, we can check (by using the trace property of tr_N) that

$$\mathrm{tr}_N(X^k) = \mathrm{tr}_N(U\Lambda^k U^*) = \mathrm{tr}_N(\Lambda^k) = \frac{1}{N} \sum_{i=1}^{N} \lambda_i^k = \int_{\mathbb{R}} t^k \, d\mu_X(t)$$

holds for all $k \in \mathbb{N}_0$. In particular, we obtain the Cauchy transform of μ_X as

$$G_{\mu_X}(z) = \int_{\mathbb{R}} \frac{1}{z - t} \, d\mu_X(t) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z - \lambda_i}.$$

I.1.2.4. *Non-commutative $*$-distributions and isomorphisms.* Non-commutative $*$-distributions that are built with respect to faithful expectation functionals have the very nice feature that, up to isomorphism, they determine uniquely the $*$-algebra, which is generated by their corresponding family of non-commutative random variables. The following theorem is taken from [**NS06**, Theorem 4.10] and explains this phenomenon.

THEOREM I.1.22. *Let $(\mathcal{A}, \phi)$ and $(\mathcal{B}, \psi)$ be $*$-probability spaces, such that ϕ and ψ are both faithful. Denote by $1_\mathcal{A}$ and $1_\mathcal{B}$ the units of $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $X = (X_i)_{i\in I}$ and $Y = (Y_i)_{i\in I}$ families of non-commutative random variables in $\mathcal{A}$ and $\mathcal{B}$, respectively, which are indexed by the same index set I. Assume that*

(i) *$\mathcal{A}$ is generated as a $*$-algebra by $\{X_i |\ i \in I\} \cup \{1_\mathcal{A}\}$,*
(ii) *$\mathcal{B}$ is generated as a $*$-algebra by $\{Y_i |\ i \in I\} \cup \{1_\mathcal{B}\}$, and*
(iii) *we have $\mu_{X,X^*} = \mu_{Y,Y^*}$.*

Then there exists a unique $*$-isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$, such that $\Phi(1_\mathcal{A}) = 1_\mathcal{B}$ and $\Phi(X_i) = Y_i$ for all $i \in I$ and $\psi \circ \Phi = \phi$ holds.

Since the proof of this important theorem is both simple and instructive, we do not want to withhold it completely from the reader. So let us briefly sketch the beautiful ideas on which the proof is based: given any $*$-isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$, which satisfies $\Phi(X_i) = Y_i$ for all $i \in I$, it is clear that we must have $\Phi(P(X)) = P(Y)$ for any $P \in \mathbb{C}\langle x_i, x_i^* |\ i \in I\rangle$, and because $\mathcal{A}$ and $\mathcal{B}$ are generated by the variables X and Y, respectively, we see that Φ is uniquely determined by these conditions. The following commuting diagram illustrates this situation:

$$\begin{array}{ccc} \mathcal{A} & \overset{\Phi}{\dashrightarrow} & \mathcal{B} \\ {\scriptstyle \mathrm{ev}_X}\nwarrow & & \nearrow{\scriptstyle \mathrm{ev}_Y} \\ & \mathbb{C}\langle x_i, x_i^* |\ i \in I\rangle & \end{array}$$

Hence, if we want to define such Φ, the only choice that we have is to put $\Phi(A) := P(Y)$ for each given $A \in \mathcal{A}$, which can be written as $A = P(X)$ for some $P \in \mathbb{C}\langle x_i, x_i^* |\ i \in I\rangle$. Of course, if $A \in \mathcal{A}$ is given, one can always find a non-commutative polynomial P satisfying the condition $A = P(X)$, but this P might not be unique. This definition of Φ thus requires to check that the assigned value $P(Y)$ does not depend on the actual choice of P. For seeing this, we must involve the condition $\mu_{X,X^*} = \mu_{Y,Y^*}$ and the faithfulness of ψ. Indeed, if $P_1, P_2 \in \mathbb{C}\langle x_i, x_i^* |\ i \in I\rangle$ are given such that $P_1(X) = P_2(X)$ holds, we clearly have $P(X) = 0$, where $P \in \mathbb{C}\langle x_i, x_i^* |\ i \in I\rangle$ is given by $P := (P_1 - P_2)(P_1 - P_2)^*$. This implies

$$\psi\big((P_1(Y)-P_2(Y))(P_1(Y)-P_2(Y))^*\big) = \psi(P(Y)) = \mu_{Y,Y^*}(P) = \mu_{X,X^*}(P) = \phi(P(X)) = 0$$

and finally $P_1(Y) - P_2(Y) = 0$, since ψ was assumed to be faithful. This proves the existence of a $*$-homomorphism $\Phi : \mathcal{A} \to \mathcal{B}$ satisfying $\Phi(X_i) = Y_i$ for all $i \in I$ and $\psi \circ \Phi = \phi$. Switching now the roles of X and Y and repeating the above argument produces another $*$-homomorphism $\Psi : \mathcal{B} \to \mathcal{A}$ satisfying $\Psi(Y_i) = X_i$ for all $i \in I$ and $\phi \circ \Psi = \psi$. This yields the assertion since Φ and Ψ are clearly inverses of each other.

Amazingly, in the setting of C^*- and W^*-probability spaces, these isomorphisms even extend to the corresponding C^*-and W^*-algebras, respectively. This means that all inherent operator-algebraic properties are encoded by the non-commutative distribution of their generators and hence in purely combinatorial terms.

The following theorem, which is taken from [**NS06**, Exercise 4.20], gives the precise statement in the case of C^*-probability spaces.

THEOREM I.1.23. *Let $(\mathcal{A},\phi)$ and $(\mathcal{B},\psi)$ be C^*-probability spaces, such that ϕ and ψ are both faithful. Denote by $1_\mathcal{A}$ and $1_\mathcal{B}$ the units of $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $X = (X_i)_{i\in I}$ and $Y = (Y_i)_{i\in I}$ families of non-commutative random variables in $\mathcal{A}$ and $\mathcal{B}$, respectively, which are indexed by the same index set I. Assume that*

(i) *$\mathcal{A}$ is generated as a C^*-algebra by $\{X_i|\ i \in I\} \cup \{1_\mathcal{A}\}$,*
(ii) *$\mathcal{B}$ is generated as a C^*-algebra by $\{Y_i|\ i \in I\} \cup \{1_\mathcal{B}\}$, and*
(iii) *we have $\mu_{X,X^*} = \mu_{Y,Y^*}$.*

Then there exists a unique isometric $*$-isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$, such that $\Phi(1_\mathcal{A}) = 1_\mathcal{B}$ and $\Phi(X_i) = Y_i$ for all $i \in I$ and $\psi \circ \Phi = \phi$ holds.

In other words, the unital C^*-algebra generated by a family of non-commutative random variables $X = (X_i)_{i\in I}$ is determined, up to isomorphism, by the non-commutative $*$-distribution μ_{X,X^*}. This opens a completely new point of view, since it means for example that properties of an operator of the form $P(X)$ with $P \in \mathbb{C}\langle x_i, x_i^*|\ i \in I\rangle$, which are invariant under isometric $*$-isomorphisms (such as norm and spectrum), only depend on μ_{X,X^*} and P, and hence on purely combinatorial data. Let us illustrate this by the following proposition.

PROPOSITION I.1.24 (Proposition 3.17 in [**NS06**]). *Let $(\mathcal{A},\phi)$ be a C^*-probability space with ϕ being faithful. Then, for any $X \in \mathcal{A}$, we have*

$$\|X\| = \lim_{n\to\infty} \phi\big((X^*X)^n\big)^{\frac{1}{2n}} \tag{I.7}$$

Note that (I.7), if written in the alternative form

$$\|X\| = \lim_{n\to\infty} \mu_{X,X^*}\big((x^*x)^n\big)^{\frac{1}{2n}}, \tag{I.8}$$

perfectly fits the strategy explained above: according to Theorem I.1.23, the non-commutative $*$-distribution $\mu_{X,X^*} : \mathbb{C}\langle x, x^*\rangle \to \mathbb{C}$ determines the C^*-algebra generated by X, up to isometric $*$-isomorphisms, and so, since the quantity $\|X\|$ is invariant under isometric $*$-isomorphisms, its precise value must be contained in μ_{X,X^*}; this is what is confirmed and made explicit by Formula (I.8).

For the seek of completeness, we point out that there is an analogous statement in the context of von Neumann algebras; see [**MS16**, Theorem 6.2].

THEOREM I.1.25. *Let $(\mathcal{A},\phi)$ and $(\mathcal{B},\psi)$ be W^*-probability spaces. (Recall that this means in our terminology that ϕ and ψ are both faithful normal tracial states; see Definition I.1.8.) Let $X = (X_i)_{i\in I}$ and $Y = (Y_i)_{i\in I}$ families of non-commutative random variables in $\mathcal{A}$ and $\mathcal{B}$, respectively, which are indexed by the same index set I. Assume that*

(i) *$\mathcal{A}$ is generated as a W^*-algebra by $\{X_i|\ i \in I\}$,*
(ii) *$\mathcal{B}$ is generated as a W^*-algebra by $\{Y_i|\ i \in I\}$, and*
(iii) *we have $\mu_{X,X^*} = \mu_{Y,Y^*}$.*

Then there exists a unique isometric $*$-isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$, such that $\Phi(1_\mathcal{A}) = 1_\mathcal{B}$ and $\Phi(X_i) = Y_i$ for all $i \in I$ and $\psi \circ \Phi = \phi$ holds.

I.1.3. Cauchy-Stieltjes transform. This subsection is devoted to Cauchy-Stieltjes transforms. This kind of transform plays a similarly important role in free probability as the Fourier transform does in classical probability theory. However, Cauchy-Stieltjes transforms appeared long before, mainly in the context of moment problems, and they were also used in random matrix theory – a surprising fact, which one can see as a first hint on some deeper connections between free probability and random matrix theory.

The term "Cauchy-Stieltjes transform" actually subsumes two closely related concepts, namely Cauchy transforms and Stieltjes transforms. Let us begin with Cauchy transforms.

Definition I.1.26. Let $\mathbb{C}^+$ and $\mathbb{C}^-$ denote the upper and the lower half-plane in $\mathbb{C}$, respectively, i.e.

$$\mathbb{C}^+ := \{z \in \mathbb{C} \mid \Im(z) > 0\} \qquad \text{and} \qquad \mathbb{C}^- := \{z \in \mathbb{C} \mid \Im(z) < 0\}.$$

The *Cauchy transform* G_μ of a Borel probability measure μ on $\mathbb{R}$ is the holomorphic function

$$G_\mu :\ \mathbb{C}^+ \to \mathbb{C}^-,\ z \mapsto \int_{\mathbb{R}} \frac{1}{z-t}\, d\mu(t).$$

The Stieltjes transform only differs by a minus sign from the aforementioned Cauchy transform. More explicitly, for any Borel probability measure μ on $\mathbb{R}$, the *Stieltjes transform* S_μ is the holomorphic function $S_\mu : \mathbb{C}^+ \to \mathbb{C}^+$, which is defined by

$$S_\mu(z) := \int_{\mathbb{R}} \frac{1}{t-z}\, d\mu(t) \qquad \text{for all } z \in \mathbb{C}^+.$$

Following the tradition of free probability, we will work here with Cauchy transforms rather than with Stieltjes transforms.

Cauchy transforms attach to each Borel probability measure μ on $\mathbb{R}$ a certain holomorphic function on $\mathbb{C}^+$. Surprisingly, one can write down a very short list of properties, which all functions arising in this way have in common and by which they are characterized among all holomorphic functions on $\mathbb{C}^+$. This is the content of the following theorem; see for instance [**GH03**, Lemma 2].

Theorem I.1.27. *Let G be a holomorphic function on the upper half-plane $\mathbb{C}^+$. Then G is the Cauchy transform of a Borel probability measure μ on $\mathbb{R}$, if and only if the following two conditions are satisfied:*

(i) *All values of G lie in the lower half-plane, i.e. $\Im(G(z)) < 0$ holds for all $z \in \mathbb{C}^+$.*
(ii) *It holds true that $\lim_{y\to\infty} iyG(iy) = 1$.*

Viewing the Cauchy transform abstractly as a map from the space of all Borel probability measures on $\mathbb{R}$ to the space of holomorphic functions living on $\mathbb{C}^+$, it is natural to study their "continuity" with respect to different topologies. The following theorem, which can be found for instance in [**GH03**], says that weak convergence can be detected easily with the help of Cauchy transforms.

Theorem I.1.28. *If $(\mu_n)_{n\in\mathbb{N}}$ is a sequence of Borel probability measures on $\mathbb{R}$ and μ another Borel probability measure on $\mathbb{R}$, then $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ if and only if $(G_{\mu_n})_{n\in\mathbb{N}}$ converges pointwise on $\mathbb{C}^+$ to G_μ.*

Surprisingly, one does not even need the convergence $(G_{\mu_n}(z))_{n\in\mathbb{N}}$ to $G_\mu(z)$ at all points $z \in \mathbb{C}^+$ in order to conclude that $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to μ. It is a very nice application of the well-known Vitali-Porter Theorem that pointwise convergence on any infinite subset K of $\mathbb{C}^+$, which has some accumulation point in $\mathbb{C}^+$, is already enough; see [**GH03**].

The content of the next theorem is the well-known *Stieltjes inversion formula*. This important theorem tells us that the measure μ can be recovered from its Cauchy transform G_μ by a certain limit procedure.

Theorem I.1.29 (Stieltjes inversion formula). *Consider the Cauchy transform $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ of a Borel probability measure μ on $\mathbb{R}$. Then*

$$d\mu_\varepsilon(t) = \frac{-1}{\pi}\Im(G_\mu(t+i\varepsilon))\,dt$$

defines for any $\varepsilon > 0$ an absolutely continuous probability measure μ_ε on $\mathbb{R}$. These measures μ_ε converge weakly to μ as $\varepsilon \searrow 0$, i.e. we have

$$\lim_{\varepsilon\searrow 0}\int_\mathbb{R} f(t)\,d\mu_\varepsilon(t) = \int_\mathbb{R} f(t)\,d\mu(t)$$

for all bounded continuous functions $f : \mathbb{R} \to \mathbb{C}$.

Cauchy transforms encode in a very nice way the moments of the corresponding probability measure μ, supposed that moments up to so some order exist. For making this more precise, let us consider first the simplest case of compactly supported probability measures. If μ has compact support, it is easy to see that its Cauchy transform $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$ extends (uniquely) to a holomorphic function $G_\mu : \mathbb{C}\setminus\operatorname{supp}(\mu) \to \mathbb{C}$ and hence admits a Laurent expansion around infinity. More precisely, for any $R > 0$ satisfying $\operatorname{supp}(\mu) \subseteq [-R, R]$, we have that

$$G_\mu(z) = \sum_{k=0}^{\infty}\frac{m_k(\mu)}{z^{k+1}} \qquad \text{for all } z \in \mathbb{C} \text{ with } |z| > R, \tag{I.9}$$

where $m_k(\mu)$ denotes the *k-th moment of* μ, i.e.

$$m_k(\mu) := \int_\mathbb{R} t^k\,d\mu(t).$$

In particular, we see that the moment sequence $(m_k(\mu))_{k\geq 0}$ of a compactly supported probability measure μ on $\mathbb{R}$ uniquely determines μ among all compactly supported probability measures on $\mathbb{R}$. Indeed, if there would be another compactly supported probability measure μ', which has the same moments as μ, its Cauchy transform $G_{\mu'}$ would also admit a Laurent expansion around infinity and would by assumption agree with the Laurent expansion of G_μ. Thus, by the identity principle for holomorphic functions, it follows that $G_\mu = G_{\mu'}$, which tells us due to the Stieltjes inversion formula Theorem I.1.29 that $\mu = \mu'$, as claimed. Alternatively, we could conclude by Corollary C.5. Indeed, since complex polynomials are known to be dense in $C([-R, R])$ with respect to the uniform norm, where $R > 0$ is chosen such that the supports of μ and μ' are both contained in $[-R, R]$, it follows that the two positive and continuous linear functionals I_μ and $I_{\mu'}$ on $C([-R, R])$ must agree on $C([-R, R])$. Hence, by uniqueness, $\mu = \mu'$.

Surprisingly, the moment sequence $(m_k(\mu))_{k\geq 0}$ of a compactly supported probability measure μ on $\mathbb{R}$ determines μ even among all probability measures on $\mathbb{R}$, which have moments of each order. This statement is less obvious and its validity is guaranteed by the assumption that μ has compact support. However, the condition of having compact support can be significantly relaxed, such that the same conclusion holds for a larger class of measures μ, namely those, which are determined by their moments.

DEFINITION I.1.30. A probability measure μ on $\mathbb{R}$, not necessarily compactly supported, but for which all moments

$$m_k(\mu) = \int_{\mathbb{R}} t^k \, d\mu(t) \qquad \text{with } k \in \mathbb{N}_0$$

exist, is said to be *determined by its moments*, if for any other Borel probability measure μ' on $\mathbb{R}$ the condition $m_k(\mu) = m_k(\mu')$ for all $k \geq 0$ implies that $\mu = \mu'$.

Let us point out the following.

REMARK I.1.31. In order to detect weak convergence of a sequence $(\mu_n)_{n\in\mathbb{N}}$ of Borel probability measures on $\mathbb{R}$, which all have moments of all orders, towards a Borel probability measure μ on $\mathbb{R}$, which is determined by its moments, it is sufficient to check *convergence of all moments*, i.e.

$$\lim_{n\to\infty} \int_{\mathbb{R}} t^k \, d\mu_n(t) = \int_{\mathbb{R}} t^k \, d\mu(t) \qquad \text{for all } k \in \mathbb{N}_0.$$

There are powerful results providing sufficient conditions for a measure μ to be determined by its moments. Typically, they formulate certain constraints on the growth of the moment sequence $(m_k(\mu))_{k\geq 0}$. As a particularly important example, let us mention here *Carleman's condition*, which says that a Borel probability measure μ on $\mathbb{R}$ having moments of all orders is determined by its moments, if the series

$$\sum_{k=1}^{\infty} m_{2k}(\mu)^{-\frac{1}{2k}}$$

is divergent. This allows the announced conclusion that each compactly supported Borel probability measure on $\mathbb{R}$ is determined by its moments.

REMARK I.1.32. Consider a Borel probability measure μ on $\mathbb{R}$ and suppose that μ has compact support. In analogy to (I.7), we have that

$$\lim_{k\to\infty} m_{2k}(\mu)^{\frac{1}{2k}} = \lim_{k\to\infty} \Big(\int_{\mathbb{R}} t^{2k} \, d\mu(t) \Big)^{\frac{1}{2k}} = \max_{t\in \operatorname{supp}(\mu)} |t| < \infty$$

and hence $\lim_{k\to\infty} m_{2k}(\mu)^{-\frac{1}{2k}} \in (0,\infty]$, which forces $\sum_{k=1}^{\infty} m_{2k}(\mu)^{-\frac{1}{2k}}$ to be divergent. By Carleman's condition, we may conclude that μ is indeed determined by its moments.

In most cases, we will work with measures that arise as analytic distributions of non-commutative random variables in C^*- or W^*-probability spaces. Such measures clearly have compact support.

REMARK I.1.33. If μ_X is the distribution of any non-commutative random variable $X = X^* \in \mathcal{A}$ in a C^*-probability space $(\mathcal{A}, \phi)$, then (I.4) tells us that we have

$$G_{\mu_X}(z) = \int_{\mathbb{R}} \frac{1}{z-t} \, d\mu_X(t) = \phi\big((z-X)^{-1}\big) \qquad \text{for } z \in \mathbb{C}^+.$$

In such cases, we will therefore often write G_X instead of G_{μ_X}. This holomorphic function enjoys an analytic extension to $\mathbb{C}\backslash \operatorname{supp}(\mu_X) = \mathbb{C}\backslash\sigma(X) = \rho(X)$, where $\sigma(X) = \sigma_{\mathcal{A}}(X)$ and $\rho(X) = \rho_{\mathcal{A}}(X)$ denote the spectrum and the resolvent set of X in $\mathcal{A}$, respectively. On the set $\{z \in \mathbb{C} |\ |z| > R\}$, where R is chosen such that $\operatorname{supp}(\mu_X) = \sigma(X) \subseteq [-R, R]$, the resolvent $(z - X)^{-1}$ can be written as a convergent series

$$(z - X)^{-1} = \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} X^n.$$

This yields the Laurent expansion

$$G_X(z) = \sum_{k=0}^{\infty} \frac{\phi(X^n)}{z^{k+1}}.$$

Compared with the Laurent expansion (I.9) of G_{μ_X}, this gives

$$\phi(X^k) = m_k(\mu_X) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \text{for } k \in \mathbb{N}_0,$$

which simply reflects the determining condition (I.5) of μ_X, which was given above in Definition I.1.18.

I.1.4. Free independence. The setting that we have introduced so far is of general nature and not at all specific for free probability theory. The actual starting point of free probability is Voiculescu's notion of free independence, which we will introduce next.

Definition I.1.34 (Free independence). Let $(\mathcal{A}, \phi)$ be a non-commutative probability space.

(i) Let $(\mathcal{A}_i)_{i\in I}$ be a family of unital subalgebras[2] of $\mathcal{A}$ with an arbitrary index set $I \neq \emptyset$. We call $(\mathcal{A})_{i\in I}$ *freely independent* (or just *free*), if

$$\phi(X_1 \cdots X_n) = 0$$

holds whenever the following conditions are fulfilled:
- We have $n \geq 1$ and there are indices $i_1, \dots, i_n \in I$ satisfying
$$i_1 \neq i_2, \dots, i_{n-1} \neq i_n.$$
- For $j = 1, \dots, n$, we have $X_j \in \mathcal{A}_{i_j}$ and it holds true that $\phi(X_j) = 0$.

(ii) Let $(\mathcal{X}_i)_{i\in I}$ a family of subsets of $\mathcal{A}$ with an arbitrary index set $I \neq \emptyset$. We call $(\mathcal{X}_i)_{i\in I}$ *freely independent* (or just *free*), if $(\mathcal{A}_i)_{i\in I}$ are freely independent in the sense of (i), where $\mathcal{A}_i$ denotes for each $i \in I$ the unital subalgebra of $\mathcal{A}$ that is generated by the elements of $\mathcal{X}_i$.

(iii) Elements $(X_i)_{i\in I}$ are called *freely independent* (or just *free*), if $(\mathcal{A}_i)_{i\in I}$ are freely independent in the sense of (i), where $\mathcal{A}_i$ denotes for each $i \in I$ the unital subalgebra of $\mathcal{A}$ that is generated by X_i.

Roughly speaking, free independence gives some kind of "universal rule" to compute expectations of mixed products. The following theorem gives the precise statement.

Theorem I.1.35 (Lemma 5.13 in [**NS06**]). *Let $(\mathcal{A}, \phi)$ be a non-commutative probability space and let $(A_i)_{i\in I}$ be a family of unital subalgebras $\mathcal{A}_i$ of $\mathcal{A}$, which are freely independent. Denote by $\mathcal{B}$ the subalgebra of $\mathcal{A}$, which is generated by $\bigcup_{i\in I} \mathcal{A}_i$. Then the restriction $\phi|_{\mathcal{B}}$ of ϕ to $\mathcal{B}$ is fully determined by the family of restrictions $(\phi|_{\mathcal{A}_i})_{i\in I}$.*

[2]We note that for us "unital subalgebra" always means "unitally embedded subalgebra", i.e. the unit of the smaller algebra is the unit of the larger algebra.

Since the proof of this statement is quite constructive, it might convey a better feeling for the concept of free independence, and so we want to discuss briefly its idea. Clearly, any element in $\mathcal{B}$ is a linear combination of elements of the form

$$X_1 \cdots X_n$$

with $X_j \in \mathcal{A}_{i_j}$ for $j = 1, \dots, n$, where $n \geq 1$ and $i_1, \dots, i_n \in I$ are such that $i_1 \neq i_2, \dots, i_{n-1} \neq i_n$ holds. Hence, by linearity of ϕ, it suffices to show that the expectation of such elements is determined by $(\phi|_{\mathcal{A}_i})_{i \in I}$. For proving this, we proceed by (strong) induction on the length n of the product. For $n = 1$, the statement is trivially true. If we assume that the statement is already proven for any length $< n$ for some $n \geq 2$, we may involve the freeness condition, which tells us that

$$\phi\big((X_1 - \phi(X_1)) \cdots (X_n - \phi(X_n))\big) = 0.$$

The latter yields after expanding

$$\phi(X_1 \cdots X_n) + \sum_{k=1}^{n} \sum_{1 \leq j_1 < \cdots < j_k \leq n} (-1)^k \phi(X_{j_1}) \cdots \phi(X_{j_k}) \phi(\Pi_{j_1, \dots, j_k}) = 0$$

where $\Pi_{j_1,\dots,j_k}$ stands for the ordered product of all $X_1, \dots, X_n$ but with the factors $X_{j_1}, \dots, X_{j_k}$ omitted, i.e.

$$\Pi_{j_1,\dots,j_k} = X_1 \cdots X_{j_1-1} X_{j_1+1} \cdots X_{j_k-1} X_{j_k+1} \cdots X_n.$$

This implies that $\phi(X_1 \cdots X_n)$ is determined by expectations of products of length $< n$. By the induction hypothesis, we conclude that $\phi(X_1 \cdots X_n)$ is determined by $(\phi|_{\mathcal{A}_i})_{i \in I}$.

Example I.1.36. Consider a non-commutative probability space $(\mathcal{A}, \phi)$ and two unital subalgebras $\mathcal{A}_1, \mathcal{A}_2$ of $\mathcal{A}$, which are freely independent. If we chose $X_1, X_2, X_3 \in \mathcal{A}_1$ and $Y \in \mathcal{A}_2$ with $\phi(Y) = 0$, then

$$\phi(X_1 Y X_2 Y X_3) = \phi(X_1 X_3) \phi(X_2) \phi(Y^2).$$

Indeed, the freeness condition yields

$$\phi\big((X_1 - \phi(X_1)) Y (X_2 - \phi(X_2)) Y (X_3 - \phi(X_3))\big) = 0.$$

An expansion of the left hand side gives

$$\begin{aligned}
&\phi\big((X_1 - \phi(X_1)) Y (X_2 - \phi(X_2)) Y (X_3 - \phi(X_3))\big) \\
&= \phi(X_1 Y X_2 Y X_3) - \phi(X_1) \phi(Y X_2 Y X_3) - \phi(X_2) \phi(X_1 Y^2 X_3) - \phi(X_3) \phi(X_1 Y X_2 Y) \\
&\quad + \phi(X_1) \phi(X_2) \phi(Y^2 X_3) + \phi(X_1) \phi(X_3) \phi(Y X_2 Y) + \phi(X_2) \phi(X_3) \phi(X_1 Y^2) \\
&\quad - \phi(X_1) \phi(X_2) \phi(X_1) \phi(Y^2).
\end{aligned}$$

Next, we study the expressions $\phi(Y X_2 Y X_3)$, $\phi(X_1 Y^2 X_3)$ and $\phi(X_1 Y X_2 Y)$. For the first, we have

$$\begin{aligned}
0 &= \phi\big(Y (X_2 - \phi(X_2)) Y (X_3 - \phi(X_3))\big) \\
&= \phi(Y X_2 Y X_3) - \phi(X_2) \phi(Y^2 X_3) - \phi(X_3) \phi(Y X_2 Y) + \phi(X_2) \phi(X_3) \phi(Y^2),
\end{aligned}$$

for the second

$$\begin{aligned}
0 &= \phi\big((X_1 - \phi(X_1))(Y^2 - \phi(Y^2))(X_3 - \phi(X_3))\big) \\
&= \phi(X_1 Y^2 X_3) - \phi(X_1) \phi(Y^2 X_3) - \phi(X_1 X_3) \phi(Y^2) - \phi(X_3) \phi(X_1 Y^2) \\
&\quad + 2\phi(X_1) \phi(X_3) \phi(Y^2),
\end{aligned}$$

and finally for the third one

$$\begin{aligned}0 &= \phi\big((X_1 - \phi(X_1))Y(X_2 - \phi(X_2))Y\big)\\ &= \phi(X_1YX_2Y) - \phi(X_1)\phi(YX_2Y) - \phi(X_2)\phi(X_1Y^2) + \phi(X_1)\phi(X_2)\phi(Y^2).\end{aligned}$$

In the next step, we compute in the same way

$$\phi(Y^2X_3) = \phi(X_3)\phi(Y^2), \quad \phi(YX_2Y) = \phi(X_2)\phi(Y^2), \quad \text{and} \quad \phi(X_1Y^2) = \phi(X_1)\phi(Y^2),$$

such that the previously found relations reduce to

$$\begin{aligned}\phi(YX_2YX_3) &= \phi(X_2)\phi(X_3)\phi(Y^2)\\ \phi(X_1Y^2X_3) &= \phi(X_1X_3)\phi(Y^2)\\ \phi(X_1YX_2Y) &= \phi(X_1)\phi(X_2)\phi(Y^2)\end{aligned}$$

Combining this with the very first result, we obtain

$$\begin{aligned}0 &= \phi(X_1YX_2YX_3) - \phi(X_1)\phi(YX_2YX_3) - \phi(X_2)\phi(X_1Y^2X_3) - \phi(X_3)\phi(X_1YX_2Y)\\ &\quad + \phi(X_1)\phi(X_2)\phi(Y^2X_3) + \phi(X_1)\phi(X_3)\phi(YX_2Y) + \phi(X_2)\phi(X_3)\phi(X_1Y^2)\\ &\quad - \phi(X_1)\phi(X_2)\phi(X_1)\phi(Y^2)\\ &= \phi(X_1YX_2YX_3) - \phi(X_2)\phi(X_1X_3)\phi(Y^2),\end{aligned}$$

from which the stated formula follows.

We point out that one can significantly simplify such computations by using the powerful combinatorial machinery of *free cumulants*, which was introduced to free probability by Speicher [**Spe90, Spe94**]; see also[**Nic96**] and [**NS06**].

We conclude by recording an important consequence of Theorem I.1.35 for later reference.

Remark I.1.37. A direct consequence of the previous Theorem I.1.35 is, that the non-commutative distribution μ_X for any family $(X_i)_{i\in I}$ of freely independent non-commutative random variables in a non-commutative probability space $(\mathcal{A}, \phi)$ is completely determined by the family of single variable distributions $(\mu_{X_i})_{i\in I}$.

I.1.5. Free additive convolution. From the observation recorded in Remark I.1.37 it follows in particular that the distribution $\mu_{X_1+X_2}$ of the sum of two freely independent non-commutative random variables X_1 and X_2 only depends on the distributions μ_{X_1} and μ_{X_2} of X_1 and X_2, respectively, and not on the concrete realization of X_1 and X_2 in a non-commutative probability space. Indeed, any moment $\phi((X_1 + X_2)^n)$ of $X_1 + X_2$ expands as

$$\phi((X_1 + X_2)^n) = \sum_{(i_1,\dots,i_n)\in\{1,2\}^n} \phi(X_{i_1}X_{i_2}\cdots X_{i_n}),$$

where now each mixed moment $\phi(X_{i_1}X_{i_2}\cdots X_{i_n})$ can be computed in a recursive but universal way (which only relies on the structure of the freeness condition) out of the moments of X_1 and X_2. We only mention here that these universal formulas take a much more explicit form in terms of free cumulants. Accordingly, Voiculescu's *free additive convolution* $\boxplus$ can be defined by as a binary operation on the set of all abstract distributions (i.e., the set of all linear functionals $\mu : \mathbb{C}\langle x\rangle \to \mathbb{C}$ satisfying $\mu(1) = 1$), such that $\mu_{X_1} \boxplus \mu_{X_2} = \mu_{X_1+X_2}$.

Driven by Definition I.1.18, which identifies the combinatorial distribution of a single self-adjoint element with a compactly supported Borel probability measure on $\mathbb{R}$, we want to extend now the free additive convolution $\boxplus$ to a binary operation on all compactly supported Borel probability measures on $\mathbb{R}$. This, however, requires some additional arguments, which we collect in the following remark.

REMARK I.1.38. The main issue here is that the free additive convolution is defined originally in terms of operators. Thus, we have to convince ourselves that for two given compactly supported probability measures μ_1, μ_2 on $\mathbb{R}$, we can always find operators X_1 and X_2 in some C^*-probability space, which are freely independent and whose distributions are given by μ_1 and μ_2, respectively.

(i) If μ is any compactly supported probability measure on $\mathbb{R}$, we need to find some self-adjoint non-commutative random variable X that lives in some non-commutative C^*-probability space $(\mathcal{A}, \phi)$, such that $\mu = \mu_X$. Let us consider $\mathcal{A} = C(\operatorname{supp}(\mu))$, the unital C^*-algebra of all $\mathbb{C}$-valued continuous functions on the compact support $\operatorname{supp}(\mu)$ of μ, endowed with the uniform norm $\|\cdot\|_\infty$. Elements from $\mathcal{A}$ act naturally as multiplication operators on the Hilbert space $L^2(\mathbb{R}, \mu)$. This action leads us directly to the unital $*$-homomorphism
$$\pi:\ \mathcal{A} \to B(L^2(\mathbb{R}, \mu)),\ g \mapsto M_g,$$
where $M_g \in B(L^2(\mathbb{R}, \mu))$ is defined by $M_g f := g \cdot f$ for all $f \in L^2(\mathbb{R}, \mu)$ and satisfies $\|M_g\| = \|g\|_\infty$. Thus, we see that the $*$-homomorphism π is in fact isometric. On $\mathcal{A}$, we can introduce a state ϕ by
$$\phi(g) = \langle \pi(g)\,\mathbf{1}, \mathbf{1}\rangle = \int_{\mathbb{R}} g(x)\,d\mu(x) \qquad \text{for all } g \in \mathcal{A},$$
where $\mathbf{1}$ denotes the function in $L^2(\mathbb{R}, \mu)$, that takes constantly the value 1. Due to the commutativity of $\mathcal{A}$, the state ϕ is trivially a trace, and since π is faithful, we conclude that ϕ is also faithful. Now, if we consider $X := \mathrm{id}_{\operatorname{supp}(\mu)} \in \mathcal{A}$, we can readily check that $\mu_X = \mu$. Indeed, we have
$$\phi(X^k) = \langle \pi(X^k)\,\mathbf{1}, \mathbf{1}\rangle = \langle X^k, \mathbf{1}\rangle = \int_{\mathbb{R}} x^k\,d\mu(x) \qquad \text{for } k \in \mathbb{N}_0,$$
as Definition I.1.18 requires.

(ii) There is a general construction that produces out of any given family $((\mathcal{A}_i, \phi_i))_{i\in I}$ of C^*-probability spaces $(\mathcal{A}_i, \phi_i)$, which are endowed with faithful and tracial states ϕ_i, some C^*-probability space $(\mathcal{A}, \phi)$ with ϕ being faithful and tracial, such that
- each $\mathcal{A}_i$ is unitally and isometrically embedded into $\mathcal{A}$,
- $\phi|_{\mathcal{A}_i} = \phi_i$ holds for all $i \in I$, and
- the subalgebras $(\mathcal{A}_i)_{i\in I}$ are freely independent in $(\mathcal{A}, \phi)$.

This C^*-probability space is called the *reduced free product* of the C^*-algebras $(\mathcal{A}_i)_{i\in I}$ with respect to $(\phi_i)_{i\in I}$, and it is denoted by
$$(\mathcal{A}, \phi) = \mathop{*}_{i\in I}(\mathcal{A}_i, \phi_i).$$

The arguments given above even show that for arbitrarily many compactly supported Borel probability measures $\mu_1, \dots, \mu_N$ on $\mathbb{R}$ one can find freely independent self-adjoint non-commutative random variables $X_1, \dots, X_N$ in some C^*-probability space $(\mathcal{A}, \phi)$, such that the analytic distribution of X_i is given by μ_i for $i = 1, \dots, N$.

The promised definition of $\boxplus$ as a binary operation on the set of all compactly supported Borel probability measures on $\mathbb{R}$ reads then as follows.

Definition I.1.39. Let μ_1, μ_2 be two compactly supported Borel probability measures on the real line $\mathbb{R}$. The free additive convolution $\mu_1 \boxplus \mu_2$ is defined as analytic distribution of $X_1 + X_2$, where X_1 and X_2 are self-adjoint elements in any C^*-probability space $(\mathcal{A}, \phi)$, which are freely independent and whose analytic distributions are given by $\mu_{X_1} = \mu_1$ and $\mu_{X_2} = \mu_2$.

We point out that the free additive convolution $\boxplus$ can also be defined for Borel probability measures, which are not compactly supported. The assumption of compact support was dropped in [**BV93**] with the help of affiliated operators. We leave out the details here.

Of course, Definition I.1.17 is tempting to believe that $\boxplus$ would even extend to some binary operation on the set all compactly supported Borel probability measures on the complex plane $\mathbb{C}$, but since $X_1 + X_2$ is not necessarily normal if X_1 and X_2 are so, the construction simply fails in this generality unlike the self-adjoint case.

Remark I.1.40. Without going into details, let us note that in a similar way, the *free multiplicative convolution* $\boxtimes$ can be defined; see [**Voi87**]. It is given as a binary operation on the space of all abstract distributions, such that $\mu_{X_1} \boxtimes \mu_{X_2} = \mu_{X_1 X_2}$ holds. Like the free additive convolution $\boxplus$, also $\boxtimes$ extends to the level of compactly supported Borel probability measures, but in order to stay inside the class of Borel probability measures on $\mathbb{R}$, we need to impose the additional condition that at least one of the measures μ_1 and μ_2 is supported on $\mathbb{R}_+ = [0, \infty)$ in order to define $\mu_1 \boxtimes \mu_2$.

I.1.5.1. *The R-transform.* The free additive convolution $\boxplus$ is clearly the free analogue of the *classical convolution* $*$ of probability measures. In classical probability, the *Fourier transform* $\mu \mapsto \widehat{\mu}$ can be used to compute this kind of convolution, since its logarithm linearizes $*$ in the sense that $\log(\widehat{\mu_1 * \mu_2}) = \log(\widehat{\mu_1}) + \log(\widehat{\mu_2})$. In free probability, the role of the linearizing transform is played by the so-called *R-transform* $\mu \mapsto R_\mu$, which was introduced by Voiculescu [**Voi86**]. It is determined by the equation

$$G_\mu\Big(\frac{1}{z} + R_\mu(z)\Big) = z \qquad \text{for all } z \in \Omega', \tag{I.10}$$

for some domain $\emptyset \neq \Omega' \subseteq \mathbb{C}^-$; see also [**Haa97**]. We want to give a more detailed explanation of this important equation. For this purpose, let us recall some results from [**BV93**], which notably even apply in the case of measures having unbounded support.

To begin with, let μ be any Borel probability measure on $\mathbb{R}$. Let us introduce the so-called *F-transform F_μ of μ*, which is the holomorphic function

$$F_\mu : \mathbb{C}^+ \to \mathbb{C}^+,\ z \mapsto \frac{1}{G_\mu(z)}.$$

Furthermore, for any $\alpha, \beta > 0$, we will consider the so-called *Stolz angle* $\Gamma_{\alpha,\beta}$ as the set

$$\Gamma_{\alpha,\beta} := \{z \in \Gamma_\alpha |\ |z| > \beta\}, \qquad \text{where} \qquad \Gamma_\alpha := \{z \in \mathbb{C}^+ |\ |\Re(z)| < \alpha \Im(z)\}.$$

If we chose now $0 < \varepsilon < \alpha$, then according to [**BV93**, Proposition 5.4], we can find $\beta > 0$, such that F_μ is injective on the Stolz angle $\Gamma_{\alpha,\beta}$ and such that $F_\mu(\Gamma_{\alpha,\beta}) \supseteq \Gamma_{\alpha-\varepsilon,\beta(1+\varepsilon)}$ holds. By gluing together different Stolz angles (see [**BV93**, Corollary 5.5]), we find a domain Ω of the form $\Omega = \bigcup_{\alpha>0} \Gamma_{\alpha,\beta_\alpha}$, such that F_μ admits a holomorphic right inverse F_μ^{-1} defined on Ω, i.e. $F_\mu^{-1} : \Omega \to \mathbb{C}^+$ and

$$F_\mu(F_\mu^{-1}(z)) = z \qquad \text{for all } z \in \Omega.$$

Properties of F_μ (see [**BV93**, Proposition 5.2] and [**BV93**, Corollary 5.3]) translate to statements about F_μ^{-1}, namely

$$\Im(F_\mu^{-1}(z)) \leq \Im(z) \quad \text{for all } z \in \Omega \qquad \text{and} \qquad \lim_{\substack{|z|\to\infty \\ z\in\Gamma_\alpha}} \frac{F_\mu^{-1}(z)}{z} = 1 \quad \text{for all } \alpha > 0.$$

Finally, if we consider the *Voiculescu transform* ϕ_μ, which lives on the same domain Ω and is given by

$$\phi_\mu(z) := F_\mu^{-1}(z) - z \qquad \text{for all } z \in \Omega,$$

we immediately get that ϕ_μ satisfies

$$\Im(\phi_\mu(z)) \leq 0 \quad \text{for all } z \in \Omega \qquad \text{and} \qquad \lim_{\substack{|z|\to\infty \\ z\in\Gamma_\alpha}} \frac{\phi_\mu(z)}{z} = 0 \quad \text{for all } \alpha > 0.$$

This is a good point to take a short rest in order to summarize what we have done so far. Ignoring for the moment all technical details, we can summarize that we have introduced some special function ϕ_μ, which can be attached to any Borel probability measure μ on the real line. Although there is no universal domain on which all these functions ϕ_μ can be defined, we know at least that each individual domain is a union of certain Stolz angles. It might happen (and it actually happens in some important cases) that ϕ_μ enjoys an analytic extension beyond these domains and even to the entire upper half-plane. This feature (see [**BV93**, Theorem 5.10]) in fact characterizes the so-called $\boxplus$-*infinitely divisible* probability measures. Note, if ϕ_{μ_1} and ϕ_{μ_2} for two probability measures μ_1, μ_2 on $\mathbb{R}$ are given, we can always find some Stolz angle that belongs to both of their domains, so that we can always compare ϕ_{μ_1} and ϕ_{μ_2}. Assume now that these functions agree on their joint domain. Following the construction backwards, we see that in this case also $F_{\mu_1}^{-1}$ and $F_{\mu_2}^{-1}$ must coincide there, such that by the identity theorem their F-transforms and finally their Cauchy transforms must agree. Using Stieltjes inversion, we may conclude that $\mu_1 = \mu_2$. This means that ϕ_μ determines μ uniquely. However, it remains unclear which holomorphic functions ϕ arise as the Voiculescu transform of some probability measure on $\mathbb{R}$. The following theorem provides such a characterization.

Theorem I.1.41 ([**BV93**, Proposition 5.6]). *Let ϕ be a holomorphic function, which is defined on some Stolz angle $\Gamma_{\alpha,\beta}$. Then the following statements are equivalent:*

(i) *There exists a probability measure μ on $\mathbb{R}$ and some $\beta' \geq \beta$, such that $\phi(z) = \phi_\mu(z)$ for all $z \in \Gamma_{\alpha,\beta'}$.*
(ii) *There exists $\beta' \geq \beta$, such that*
  - $\Im(\phi(z)) \leq 0$ *for all* $z \in \Gamma_{\alpha,\beta'}$,
  - $\lim_{\substack{|z|\to\infty \\ z\in\Gamma_{\alpha,\beta'}}} \frac{\phi(z)}{z} = 0$, *and*
  - *for any choice of finitely but arbitrarily many points $z_1, \dots, z_n \in \Gamma_{\alpha,\beta'}$, the matrix*

$$\left(\frac{z_k - \overline{z_l}}{(z_k + \phi(z_k)) - \overline{(z_l + \phi(z_l))}} \right)_{k,l=1}^n$$

  is positive

But what is the actual use of these functions? The answer to this question can be found in [**BV93**, Corollary 5.8]. This fundamental theorem connects the previously explained complex analysis construction with the operator-algebraic world of free independence by establishing that $\mu \mapsto \phi_\mu$ yields a linearizing transform for $\boxplus$. More precisely, it tells us that we have for each $\alpha > 0$

$$\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z) \qquad \text{for all } z \in \Gamma_{\alpha,\beta},$$

if $\beta > 0$ is chosen large enough. Note that in contrast to the R-transform, which appears in (I.10), the Voiculescu transform is determined by

$$G_\mu\big(z + \phi_\mu(z)\big) = \frac{1}{z} \qquad \text{for all } z \in \Omega.$$

The desired R-transform, which satisfies an equation of the form (I.10), is thus obtained by putting $R_\mu(z) := \phi_\mu\big(\frac{1}{z}\big)$ for all z belonging to $\Omega' := \big\{\frac{1}{z} \mid z \in \Omega\big\}$. Put $\Gamma'_{\alpha,\beta} := \{\frac{1}{z} \mid z \in \Gamma_{\alpha,\beta}\}$. Then the above addition formula for ϕ_μ rephrases as

$$R_{\mu_1 \boxplus \mu_2}(z) = R_{\mu_1}(z) + R_{\mu_2}(z) \qquad \text{for all } z \in \Gamma'_{\alpha,\beta}.$$

Thus, if we want to compute $\mu_1 \boxplus \mu_2$ for two given Borel probability measures μ_1, μ_2 on $\mathbb{R}$, we can proceed now as follows.

(i) Compute the Cauchy transforms G_{μ_1} and G_{μ_2}.
(ii) Solve equation (I.10) for μ_1 and μ_2 separately in order to obtain the R-transforms R_{μ_1} and R_{μ_2}.
(iii) Compute the R-transform of $\mu_1 \boxplus \mu_2$ by $R_{\mu_1 \boxplus \mu_2} = R_{\mu_1} + R_{\mu_2}$ on their joint domain.
(iv) Solve (I.10) in order to obtain an expression for $G_{\mu_1 \boxplus \mu_2}$ (at least locally and if necessary extend to $\mathbb{C}^+$).
(v) Apply Stieltjes inversion to $G_{\mu_1 \boxplus \mu_2}$ in order to get $\mu_1 \boxplus \mu_2$.

This algorithm sounds quite simple, but actually it is not, since it requires to deal with equation (I.10), which is in general not an easy task. Even worse, one often arrives at equations for which no analytic solution is known. Therefore, one would surely appreciate an alternative approach. This will be addressed next.

I.1.5.2. *Subordination.* Indeed, there is the powerful concept of *subordination*, which is both of great practical and theoretical use. These ideas were developed by many authors, starting from Voiculescu [**Voi93**], and brought into its final form in [**Bia98a**]; see also [**CG11**]. We refer to [**BB07**] for a beautiful proof based on the theory of Denjoy-Wolff points, which in addition provides a fixed point iteration scheme for the desired subordination functions.

Before giving the statement, let us introduce the so-called *h-transform*

$$h_\mu: \ \mathbb{C}^+ \to \mathbb{C}^+, \ z \mapsto F_\mu(z) - z$$

for any Borel probability measure μ on $\mathbb{R}$.

Theorem I.1.42 (see [**BB07**, Theorem 4.1]). *Given Borel probability measures μ_1 and μ_2 on $\mathbb{R}$, there exist unique holomorphic functions $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$ such that*

- *For $j \in \{1, 2\}$, we have $\Im(\omega_j(z)) \geq \Im(z)$ for all $z \in \mathbb{C}^+$ and*
$$\lim_{y \to \infty} \frac{\omega_j(iy)}{iy} = 1;$$
- *$F_{\mu_1 \boxplus \mu_2}(z) = F_{\mu_1}(\omega_1(z)) = F_{\mu_2}(\omega_2(z))$ for all $z \in \mathbb{C}^+$;*
- *$\omega_1(z) + \omega_2(z) = z + F_{\mu_1 \boxplus \mu_2}(z)$ for all $z \in \mathbb{C}^+$.*

Moreover, if $z \in \mathbb{C}^+$ is given, then $\omega_1(z)$ is the unique fixed point of the map

$$f_z:\ \mathbb{C}^+ \to \mathbb{C}^+,\ w \mapsto h_2(h_1(w)+z)+z,$$

and $\omega_1(z) = \lim_{n\to\infty} f_z^{\circ n}(w)$ for any $w \in \mathbb{C}^+$, where $f_z^{\circ n}$ means the n-fold composition of f_z with itself. Same statements hold for ω_2, with f_z replaced by $w \mapsto h_1(h_2(w)+z)+z$.

I.1.6. The semicircular and the Marchenko-Pastur distribution. The most important (analytic) distribution in free probability is the *semicircular distribution*, which plays here the same role as the Gaussian distribution does in classical probability.

Definition I.1.43. The *semicircular distribution with mean* 0 *and variance* $t > 0$ is the compactly supported Borel probability measure σ_t on the real line $\mathbb{R}$ that is given by

$$d\sigma_t(x) = \frac{1}{2\pi t}\sqrt{4t-x^2}\,1_{[-2\sqrt{t},2\sqrt{t}]}(x)\,dx.$$

A non-commutative random variable living in some non-commutative C^*-probability space is called *semicircular element (of mean* 0 *and variance* t*)*, if its analytic distribution is given by the semicircular distribution (with mean 0 and variance t), i.e., if we have that $\mu_{S_t} = \sigma_t$. The following example shows that semicircular operators arise in a very natural way.

Example I.1.44. Let H be a separable complex Hilbert space with fixed orthonormal basis $(e_n)_{n\in\mathbb{N}_0}$. Consider the C^*-probability space $(B(H),\phi)$ with ϕ being the *vector state* with respect to e_0, i.e.

$$\phi:\ B(H) \to \mathbb{C},\ X \mapsto \langle Xe_0, e_0\rangle.$$

Let us denote by l the right shift on H with respect to the given orthonormal basis $(e_n)_{n\in\mathbb{N}_0}$ and by l^* its adjoint, i.e. the left shift on H with respect to $(e_n)_{n\in\mathbb{N}_0}$. The analytic distribution of the non-commutative random variable $S_t = \sqrt{t}(l+l^*)$ is then given by the semicircular distribution σ_t. This can be shown as follows:

- It clearly suffices to prove that $\mu_{S_1} = \sigma_1$ holds. Indeed, as soon as $\mu_{S_1} = \sigma_1$ is established, the obvious relation $S_t = \sqrt{t}S_1$ gives us for all $k \in \mathbb{N}_0$

$$\phi(S_t^k) = t^{\frac{k}{2}}\phi(S_1^k) = \frac{t^{\frac{k}{2}}}{2\pi}\int_{-2}^{2} x^k\sqrt{4-x^2}\,dx = \frac{1}{2\pi t}\int_{-2\sqrt{t}}^{2\sqrt{t}} y^k\sqrt{4t-y^2}\,dy = \int_{\mathbb{R}} y^k\,d\sigma_t(y),$$

  where we used the substitution $y = \sqrt{t}x$. Having this, Definition I.1.18 tells us that $\mu_{S_t} = \sigma_t$.
- According to Definition I.1.18, we have to show that $\phi(S_1^k) = m_k(\sigma_1)$ holds for each $k \in \mathbb{N}_0$. This can be done by combinatorial methods: on the one hand, if we put formally $l^{-1} := l^*$, then the moments of S_1 turn out to be

$$\phi(S_1^k) = \sum_{\varepsilon(1),\dots,\varepsilon(k)\in\{-1,1\}} \langle l^{\varepsilon(k)}\cdots l^{\varepsilon(1)}e_0, e_0\rangle = \sum_{\substack{\varepsilon(1),\dots,\varepsilon(k)\in\{-1,1\}\\ \forall 1\leq p<k:\ \varepsilon(1)+\dots+\varepsilon(p)\geq 0\\ \varepsilon(1)+\dots+\varepsilon(k)=0}} 1,$$

  which means that $\phi(S_1^k)$ counts the number of *Dyck paths* of length k. Their number is clearly 0, if k is odd, and is known to be the Catalan number $C_{k/2}$, if k is even; recall that the *Catalan numbers* are defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$. On the other hand, due to the symmetry of σ_1, we have that the moment $m_k(\sigma_1)$ is 0, if k is odd, and one can prove that $m_k(\sigma_1) = C_{k/2}$ holds, if k is even.

A generalization of this construction will play an important role in Chapter VII.

We now collect a few important properties of the semicircular distribution.

REMARK I.1.45.

(i) The Cauchy transform of σ_t satisfies the equation
$$tG_{\sigma_t}(z)^2 - zG_{\sigma_t}(z) + 1 = 0 \qquad \text{for all } z \in \mathbb{C}^+$$
and is therefore given by
$$G_{\sigma_t}(z) = \frac{z}{2t}\Big(1 - \sqrt{1 - \frac{4t}{z^2}}\Big) \qquad \text{for all } z \in \mathbb{C}^+,$$
where the branch of the square root is chosen such that the necessary condition $\lim_{y\to\infty} iyG_{\sigma_t}(iy) = 1$ is satisfied (see condition (ii) of Theorem I.1.27). The R-transform takes the very simple form $R_{\sigma_t}(z) = tz$.

(ii) We note that $(\sigma_t)_{t\geq 0}$ forms a semi-group with respect to the free additive convolution, i.e. we have that
$$\sigma_s \boxplus \sigma_t = \sigma_{s+t} \qquad \text{for all } s, t \geq 0.$$
This can be checked directly by using the additivity of the R-transforms.

(iii) One can formulate a free analogue of the classical central limit theorem, where the semicircular distribution arises as the limiting distribution; see [**Voi85**] or [**NS06**, Theorem 8.10].

Another very important distribution, which takes over in free probability the role of the classical Poisson distribution, is the so-called free Poisson distribution. Due to its first appearance in random matrix theory, see [**MP68**], the free Poisson distribution also goes under the name of *Marchenko-Pastur distribution*.

DEFINITION I.1.46. The *free Poisson distribution $\mu_{\lambda,\alpha}$ with rate $\lambda \geq 0$ and jump size* $\alpha \in \mathbb{R}$ is defined by
$$\mu_{\lambda,\alpha} = \begin{cases} (1-\lambda)\delta_0 + \nu_{\lambda,\alpha}, & 0 \leq \lambda < 1 \\ \nu_{\lambda,\alpha}, & \lambda \geq 1 \end{cases},$$
where $\nu_{\lambda,\alpha}$ is an absolutely continuous measure (with respect to the Lebesgue measure on $\mathbb{R}$), which is given by
$$d\nu_{\lambda,\alpha}(t) = \frac{1}{2\pi\alpha t}\sqrt{(t-\rho_{\min})(\rho_{\max}-t)}\; \mathbf{1}_{[\rho_{\min},\rho_{\max}]}(t)\, dt.$$
where $\rho_{\min} \leq \rho_{\max}$ are the two solutions t of $4\lambda\alpha^2 - (t - \alpha(1+\lambda))^2 = 0$.

It is a very special feature of the free world that the free Poisson distribution $\mu_{1,t}$ can be realized as the distribution of an operator $W_t := S_t^2$, where S_t is a semicircular element of mean 0 and variance t, i.e. $\mu_{S_t} = \sigma_t$.

REMARK I.1.47.

(i) We recall that its Cauchy transform is given by
$$G_{\mu_{\lambda,\alpha}}(z) = \frac{z + \alpha - \lambda\alpha - \sqrt{(z-\alpha(1+\lambda))^2 - 4\lambda\alpha^2}}{2\alpha z} \qquad \text{for all } z \in \mathbb{C}^+$$
and that its R-transform is of the form $R_{\mu_{\lambda,\alpha}}(z) = \frac{\lambda\alpha}{1-\alpha z}$.

(ii) One easily sees with the help of the R-transforms, that $(\mu_{\lambda,\alpha})_{\lambda\geq 0}$ forms for any fixed $\alpha \in \mathbb{R}$ a semigroup with respect to the free additive convolution $\boxplus$.
(iii) One can formulate a free analogue of the Poisson limit theorem, where $\mu_{\lambda,\alpha}$ shows up in the limit; see [**NS06**, Proposition 12.11].

I.2. Operator-valued free probability theory

Operator-valued free probability theory generalizes the setting of free probability theory, which we have presented in the previous section, in the sense that the role of the complex numbers is taken over by any another unital subalgebra of the given algebra of non-commutative random variables. This theory was initiated by Voiculescu in [**Voi95**] and the combinatorial approach was developed by Speicher in [**Spe98**].

I.2.1. Operator-valued non-commutative probability spaces. Strictly following the rule that the complex numbers are replaced by more general algebras, it is natural to adapt also the definition of non-commutative probability spaces. The main difference between the usual setting of scalar-valued free probability theory and operator-valued free probability theory is that expectations are replaced by conditional expectations. These objects can be seen as the natural non-commutative analogues of conditional expectations, which we know from classical probability.

I.2.1.1. *The basic terminology.* Let us begin with the purely algebraic setting, which generalizes Definition I.1.1.

Definition I.2.1. An *operator-valued non-commutative probability space* $(\mathcal{A}, E, \mathcal{B})$ consists of a unital complex algebra $\mathcal{A}$, a unital subalgebra $\mathcal{B}$ of $\mathcal{A}$, which is unitally embedded in $\mathcal{A}$, and a *conditional expectation* $E : \mathcal{A} \to \mathcal{B}$, i.e. a unital map $E : \mathcal{A} \to \mathcal{B}$ satisfying

- $E[b] = b$ for all $b \in \mathcal{B}$ and
- $E[b_1 X b_2] = b_1 E[X] b_2$ for all $X \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$.

Example I.2.2. Like a non-commutative probability space $(\mathcal{A}, \phi)$ can be seen as a non-commutative analogue of $(L^\infty(\Omega,\mathbb{P}), \mathbb{E})$ for classical probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$, also operator-valued non-commutative probability spaces $(\mathcal{A}, E, \mathcal{B})$ have their classical ancestor. If we take any classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub-σ-algebra $\mathcal{F}'$ of $\mathcal{F}$, then the classical conditional expectation $\mathbb{E}[X, \mathcal{F}'] : \Omega \to \mathbb{C}$ of any random variable $X \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ is a $\mathcal{F}'$-measurable function that belongs to $L^\infty(\Omega, \mathcal{F}', \mathbb{P}|_{\mathcal{F}'})$ and it has the property that

$$\mathbb{E}[XY, \mathcal{F}'] = \mathbb{E}[X, \mathcal{F}']Y \qquad \text{for all } Y \in L^\infty(\Omega, \mathcal{F}', \mathbb{P}|_{\mathcal{F}'}).$$

Thus, the conditional expectation gives rise to a mapping

$$\mathbb{E}[\cdot, \mathcal{F}'] : L^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to L^\infty(\Omega, \mathcal{F}', \mathbb{P}|_{\mathcal{F}'}),$$

for which $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E}[\cdot, \mathcal{F}'], L^\infty(\Omega, \mathcal{F}', \mathbb{P}|_{\mathcal{F}'}))$ forms an operator-valued non-commutative probability space in the sense of Definition I.2.1.

Example I.2.3. Let $(\mathcal{C}, \phi)$ be any non-commutative probability space. Then

$$\mathcal{A} := M_N(\mathbb{C}) \otimes \mathcal{C}, \qquad \mathcal{B} := M_N(\mathbb{C}), \qquad \text{and} \qquad E := \mathrm{id}_{M_N(\mathbb{C})} \otimes \phi,$$

where $\otimes$ stands for the algebraic tensor product over $\mathbb{C}$, defines an operator-valued non-commutative probability space $(\mathcal{A}, E, \mathcal{B})$. In the following, whenever we want to distinguish between several dimensions N, we write more precisely E_N instead of E.

I.2.1.2. *Operator-valued C^*- and W^*-probability spaces.* Since we are mostly interested in analytic aspects of operator-valued free probability theory, we need to add some analytic structure to the purely algebraic framework of operator-valued non-commutative probability spaces. In analogy to the scalar-valued case, which was presented in Paragraph I.1.2.3, we will mention here operator-valued C^*- and W^*-probability spaces.

DEFINITION I.2.4. An *operator-valued C^*-probability space* $(\mathcal{A}, E, \mathcal{B})$ is an operator-valued non-commutative probability space, where $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{B}$ a C^*-subalgebra of $\mathcal{A}$, which contains the unit, and where the conditional expectation $E : \mathcal{A} \to \mathcal{B}$ is a positive.

REMARK I.2.5.

(i) Note that $E : \mathcal{A} \to \mathcal{B}$ being *positive* simply means that E is positive as a linear map between the unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. Recall that a linear map $\Phi : \mathcal{A} \to \mathcal{B}$ between arbitrary C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is said to be *positive*, if it maps positive elements in $\mathcal{A}$ to positive elements in $\mathcal{B}$, i.e., if it satisfies $\Phi(a^*a) \geq 0$ for all $a \in \mathcal{A}$.
(ii) Since the linear map E is positive and satisfies $E[1] = 1$, it follows that E is bounded with norm 1.
(iii) As a conditional expectation, the positivity of E already implies that E is even *completely positive*. Recall that a map $\Phi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras is called *completely positive*, if for each $n \in \mathbb{N}$ the induced map
$$\Phi^{(n)} : \ M_n(\mathcal{A}) \to M_n(\mathcal{B}), \ (a_{k,l})_{k,l=1}^n \mapsto \big(\Phi(a_{k,l})\big)_{k,l=1}^n$$
is positive as a linear map between the C^*-algebras $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$.

DEFINITION I.2.6. An *operator-valued W^*-probability space* (M, E, N) is an operator-valued non-commutative probability space, where M is a von Neumann algebra and N a unitally embedded von Neumann subalgebra of M, and where the conditional expectation $E : M \to N$ is positive, weakly continuous, and faithful.

REMARK I.2.7. It follows from results of [**Ume54**] (see also [**Tak72**] for generalizations) that whenever (M, τ) is a W^*-probability space and N is any (unitally embedded) von Neumann subalgebra of M, then there exists a unique conditional expectation $E : M \to N$, such that (M, E, N) is an operator-valued W^*-probability space and such that $\tau|_N \circ E = \tau$ is satisfied.

I.2.2. Operator-valued non-commutative distributions. There is also an operator-valued generalization of non-commutative distributions.

DEFINITION I.2.8. Let I be some non-empty index set and $\mathcal{B}$ be a unital complex algebra.

(i) By $\mathcal{B}\langle x_i |\ i \in I\rangle$, we denote the algebra of *non-commutative polynomials over $\mathcal{B}$* in the formal variables $\{x_i |\ i \in I\}$. Formally, as a vector space, it is given by
$$\mathcal{B}\langle x_i |\ i \in I\rangle = \bigoplus_{n=0}^{\infty} (\mathcal{B} \otimes \mathcal{X})^{\otimes n} \otimes \mathcal{B},$$
where $\mathcal{X}$ is the vector space with basis $\{x_i |\ i \in I\}$ and with the multiplication induced by the tensor product $\otimes_{\mathcal{B}}$ with amalgamation over $\mathcal{B}$.

(ii) Let $(\mathcal{A}, E, \mathcal{B})$ be an operator-valued non-commutative probability space and consider a family $X = (X_i)_{i\in I}$ of non-commutative random variables in $\mathcal{A}$. We denote by ev_X the *evaluation homomorphism*

$$\mathrm{ev}_X : \mathcal{B}\langle x_i |\ i \in I\rangle \to \mathcal{A},$$

which is, as a homomorphism, uniquely determined by $b \mapsto b$ for all $b \in \mathcal{B}$ and $x_i \mapsto X_i$ for all $i \in I$. For any given $P \in \mathcal{B}\langle x_i |\ i \in I\rangle$, we mostly abbreviate $P(X) := \mathrm{ev}_X(P)$. The *operator-valued (joint) distribution* μ_X *of* X means the $\mathcal{B}$-linear functional given by $\mu_X := E \circ \mathrm{ev}_X$, i.e.

$$\mu_X : \mathcal{B}\langle x_i |\ i \in I\rangle \to \mathcal{B},\ P \mapsto E[P(X)].$$

I.2.3. Operator-valued Cauchy transform. Similar to the scalar-valued case, Cauchy transforms play an important role in the analytic description of free independence with amalgamation. Their generalizations to the operator-valued setting are defined as follows.

Definition I.2.9. Let $(\mathcal{A}, E, \mathcal{B})$ be an operator-valued C^*-probability space. We call

$$\mathbb{H}^+(\mathcal{B}) := \{b \in \mathcal{B} |\ \exists \varepsilon > 0 :\ \Im(b) \geq \varepsilon 1\} \quad \text{and} \quad \mathbb{H}^-(\mathcal{B}) := \{b \in \mathcal{B} |\ \exists \varepsilon > 0 :\ -\Im(b) \geq \varepsilon 1\}$$

the upper and lower half-plane of $\mathcal{B}$, respectively, where we use the notation $\Im(b) := \frac{1}{2i}(b - b^*)$. The *$\mathcal{B}$-valued Cauchy transform* G_X of any $X = X^* \in \mathcal{A}$ is the Fréchet analytic function

$$G_X : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^-(\mathcal{B}),\ b \mapsto E[(b - X)^{-1}]$$

A few comments on the analyticity of operator-valued Cauchy transforms (using the terminology of Appendix B) are in order. It is not hard to check that

$$\delta G_X(b; h) = \lim_{\substack{z\to 0\\ z\in U(b;h)\setminus\{0\}}} \frac{1}{z}(G_X(b + zh) - G_X(b)) = -E\big[(b - X)^{-1}h(b - X)^{-1}\big]$$

holds for each $b \in \mathbb{H}^+(\mathcal{B})$ and all $h \in \mathcal{B}$, where we put $U(b; h) := \{z \in \mathbb{C} |\ b+zh \in \mathbb{H}^+(\mathcal{B})\}$. Indeed, for each $z \in U(b; h)$, we have

$$\frac{1}{z}\big(((b + zh) - X)^{-1} - (b - X)^{-1}\big) = -((b + zh) - X)^{-1}h(b - X)^{-1}.$$

Correspondingly, Definition B.1 tells us that $G_X : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^-(\mathcal{B})$ is Gâteaux analytic on $\mathbb{H}^+(\mathcal{B})$. Furthermore, apart from the invertibility of $b - X$ for $b \in \mathbb{H}^+(\mathcal{B})$, it is known (see [**BPV12**], for instance) that $\|(b - X)^{-1}\| \leq \|\Im(b)^{-1}\|$ holds. Therefore, we see that

$$\delta G_X(b; \cdot) :\ \mathcal{B} \to \mathcal{B},\ h \mapsto -E\big[(b - X)^{-1}h(b - X)^{-1}\big]$$

is a bounded linear map with $\|\delta G_X(b; \cdot)\| \leq \|\Im(b)^{-1}\|^2$. Definition B.1, together with Remark B.2, yields that G_X is in fact Fréchet analytic on $\mathbb{H}^+(\mathcal{B})$. Even better, the inequality $\|(b - X)^{-1}\| \leq \|\Im(b)^{-1}\|$ tells us that $\|G_X(b)\| \leq \|\Im(b)^{-1}\|$ holds at any point $b \in \mathbb{H}^+(\mathcal{B})$, from which it follows then that G_X is locally bounded. According to Definition B.3, this means that G_X must in fact be analytic.

Let us point out that operator-valued Cauchy transforms in the sense of the previous Definition I.2.9 always enjoy an analytic extension like in the scalar-valued case; see Subsection I.1.3, in particular Remark I.1.33. More precisely, the $\mathcal{B}$-valued Cauchy transform $G_X : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^-(\mathcal{B})$ of any non-commutative random variable $X = X^*$ living in some operator-valued C^*-probability space $(\mathcal{A}, E, \mathcal{B})$ can be extended uniquely to an analytic function

$$G_X:\ \rho_{\mathcal{A}/\mathcal{B}}(X) \to \mathbb{H}^-(\mathcal{B}),\ b \mapsto E[(b-X)^{-1}],$$

where we denote by $\rho_{\mathcal{A}/\mathcal{B}}(X)$ the *$\mathcal{B}$-valued resolvent set of X in $\mathcal{A}$*. It is defined as the set of all $b \in \mathcal{B}$, for which $b - X$ is invertible in $\mathcal{A}$. Note that $\mathbb{H}^+(\mathcal{B}) \subseteq \rho_{\mathcal{A}/\mathcal{B}}(X)$. The analyticity can be checked by using the Taylor expansion

$$G_X(b+h) = \sum_{k=0}^{\infty} (-1)^k E\big[(b-X)^{-1}\big(h(b-X)^{-1}\big)^k\big],$$

which holds due to

$$((b+h)-X)^{-1} = (b-X)^{-1}(1+h(b-X)^{-1})^{-1} = \sum_{k=0}^{\infty}(b-X)^{-1}\big(h(b-X)^{-1}\big)^k$$

at any fixed point $b \in \rho_{\mathcal{A}/\mathcal{B}}(X)$ for each $h \in \mathcal{B}$ satisfying $\|h\| < \|(b-X)^{-1}\|^{-1}$.

For the seek of completeness, let us mention that correspondingly the *$\mathcal{B}$-valued spectrum of X in $\mathcal{A}$* is defined as the complement $\sigma_{\mathcal{A}/\mathcal{B}}(X) = \mathcal{B}\backslash\rho_{\mathcal{A}/\mathcal{B}}(X)$. It is easy to see that, in analogy to the more familiar case $\mathcal{B} = \mathbb{C}$, the $\mathcal{B}$-valued resolvent set $\rho_{\mathcal{A}/\mathcal{B}}(X)$ is an open subset of $\mathcal{B}$ and the $\mathcal{B}$-valued spectrum $\sigma_{\mathcal{A}/\mathcal{B}}(X)$ is a closed subset of $\mathcal{B}$ for any $X \in \mathcal{A}$. Moreover, since $\{z1|\ z \in \sigma_{\mathcal{A}}(X)\} \subseteq \sigma_{\mathcal{A}/\mathcal{B}}(X)$, we have that $\sigma_{\mathcal{A}/\mathcal{B}}(X)$ is non-empty, but it fails in general to be a compact or a bounded set.

The following example will be of great importance in Chapter IV.

Example I.2.10. We have seen in Example I.2.3 that each non-commutative probability space $(\mathcal{C}, \phi)$ induces an operator-valued non-commutative probability space $(M_N(\mathcal{C}), E_N, M_N(\mathbb{C}))$ for each $N \in \mathbb{N}$ (where we used the isomorphism $M_N(\mathcal{C}) \cong M_N(\mathbb{C})\otimes\mathcal{C}$). It is easy to check that $(M_N(\mathcal{C}), E_N, M_N(\mathbb{C}))$ gives an example of an operator-valued C^*-probability space, if we start from a C^*-probability space $(\mathcal{C}, \phi)$.

Now, if we take any non-commutative random variable $X = X^* \in \mathcal{C}$ and any matrix $L = L^* \in M_N(\mathbb{C})$, the distribution of X with respect to ϕ determines the $M_N(\mathbb{C})$-valued distribution of LX with respect to E_N. This is easy to see on the combinatorial level, but of course, we should also be able to compute the $M_N(\mathbb{C})$-valued Cauchy transform of LX in terms of the matrix L and the scalar-valued Cauchy transform of X.

Indeed, we have the following relation

$$G_{LX}(b) = \int_{\mathbb{R}} (b-tL)^{-1}\, d\mu_X(t)$$

with the matrix-valued integral understood in the Bochner sense, from which we can deduce with the help of Stieltjes inversion formula, Theorem I.1.29, that

$$G_{LX}(b) = \lim_{\varepsilon \searrow 0} \frac{-1}{\pi} \int_{\mathbb{R}} (b-tL)^{-1}\Im(G_X(t+i\varepsilon))\, dt. \tag{I.11}$$

However, from a computational point of view, the formula (I.11) given above in Example I.2.10 is not satisfying since many expensive matrix inversions are needed in order to reach a sufficiently accurate approximation of the integral, for instance by Riemann sums. Several very inspiring discussions that the author had with J. W. Helton led to the following algorithm, which significantly increases the calculation speed compared to the former approach based on formula (I.11).

ALGORITHM I.2.11. *Let $(\mathcal{C},\phi)$ be a C^*-probability space and consider $X = X^* \in \mathcal{C}$ with given scalar-valued Cauchy transform G_X, analytically extended to the resolvent set of X. For any matrix $L = L^* \in M_N(\mathbb{C})$, the matrix-valued Cauchy transform G_{LX} of $LX \in M_N(\mathcal{C})$ at any point $b \in \mathbb{H}^+(M_N(\mathbb{C}))$ can be obtained as follows:*

(i) *Since the matrix L is supposed to be self-adjoint, we can find a unitary matrix $U \in M_N(\mathbb{C})$ such that*

$$U^*LU = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{with} \qquad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{pmatrix},$$

where $\lambda_1,\dots,\lambda_d$ are the non-zero eigenvalues of b_0, listed with multiplicities.

(ii) *For the given point $b \in \mathbb{H}^+(M_N(\mathbb{C}))$, we decompose*

$$U^*bU = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix},$$

such that $b_{1,1}$ belongs to $M_d(\mathbb{C})$ and all other blocks are of appropriate size.

(iii) *Since $\Im(b_{2,2}) > 0$, we know that $b_{2,2}$ must invertible. Thus, we may introduce*

$$S = b_{1,1} - b_{1,2}b_{2,2}^{-1}b_{2,1}.$$

The Schur complement formula, Lemma A.1, tells us that $S - \Lambda X$ is invertible and that

$$\begin{pmatrix} b_{1,1} - \Lambda X & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -b_{2,2}^{-1}b_{2,1} & 1 \end{pmatrix} \begin{pmatrix} (S-\Lambda X)^{-1} & 0 \\ 0 & b_{2,2}^{-1} \end{pmatrix} \begin{pmatrix} 1 & -b_{1,2}b_{2,2}^{-1} \\ 0 & 1 \end{pmatrix}.$$

(iv) *Combining the previous observations, we deduce*

$$\begin{aligned} G_{LX}(b) &= E_N\big[(b - LX)^{-1}\big] \\ &= UE_N\big[\big(U^*bU - (U^*LU)X\big)^{-1}\big]U^* \\ &= UE_N\left[\begin{pmatrix} b_{1,1} - \Lambda X & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}^{-1}\right]U^* \\ &= U\begin{pmatrix} 1 & 0 \\ -b_{2,2}^{-1}b_{2,1} & 1 \end{pmatrix} E_N\left[\begin{pmatrix} (S-\Lambda X)^{-1} & 0 \\ 0 & b_{2,2}^{-1} \end{pmatrix}\right] \begin{pmatrix} 1 & -b_{1,2}b_{2,2}^{-1} \\ 0 & 1 \end{pmatrix} U^* \\ &= U\begin{pmatrix} 1 & 0 \\ -b_{2,2}^{-1}b_{2,1} & 1 \end{pmatrix} \begin{pmatrix} E_d[(S-\Lambda X)^{-1}] & 0 \\ 0 & b_{2,2}^{-1} \end{pmatrix} \begin{pmatrix} 1 & -b_{1,2}b_{2,2}^{-1} \\ 0 & 1 \end{pmatrix} U^*. \end{aligned}$$

(v) *Therefore, the initial problem is now reduced to the calculation of $E_d[(S-\Lambda X)^{-1}]$. Here, we proceed as follows. First, let us assume that $\Lambda^{-1}S$ can be diagonalized, i.e. there exists an invertible matrix $V \in M_d(\mathbb{C})$ such that*

$$\Lambda^{-1}S = V\begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_d \end{pmatrix}V^{-1},$$

where $\mu_1,\dots,\mu_d$ are the eigenvalues of $\Lambda^{-1}S$, listed according to multiplicity. Notice that, as the invertibility of Λ and $S - \Lambda X$ is guaranteed, each of the complex numbers $\mu_1,\dots,\mu_d$ must belong to the resolvent set of X, because $S -$

$\Lambda X = \Lambda(\Lambda^{-1}S - 1_d X)$. Hence, the points $\mu_1, \dots, \mu_d$ belong to the domain of the unique analytic extension of G_X and we obtain

$$E_d\big[(S - \Lambda X)^{-1}\big] = E_d[(\Lambda^{-1}S - 1_d X)^{-1}]\Lambda^{-1} = V \begin{pmatrix} G_X(\mu_1) & & \\ & \ddots & \\ & & G_X(\mu_d) \end{pmatrix} V^{-1}\Lambda^{-1}.$$

Otherwise, if $\Lambda^{-1}S$ fails to be diagonalizable, we can use instead its Jordan normal form

$$\Lambda^{-1}S = V \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{pmatrix} V^{-1}$$

with an invertible matrix $V \in M_d(\mathbb{C})$ and a block diagonal matrix consisting of the Jordan blocks $J_1, \dots, J_p$ associated to the eigenvalues $\mu_1, \dots, \mu_p$, respectively. Proceeding like above, one easily sees that $J_1 - X1_{d_1}, \dots, J_p - X1_{d_p}$ are invertible and that

$$E_d\big[(S - \Lambda X)^{-1}\big] = V \begin{pmatrix} E_{d_1}[(J_1 - X1_{d_1})^{-1}] & & \\ & \ddots & \\ & & E_{d_p}[(J_p - X1_{d_p})^{-1}] \end{pmatrix} V^{-1}\Lambda^{-1}.$$

Thus, we are done after involving Lemma I.2.12 below.

LEMMA I.2.12. *Let X be a self-adjoint non-commutative random variable living in some C^*-probability space $(\mathcal{C}, \phi)$. Fix $d \in \mathbb{N}$ and consider any matrix $J \in M_d(\mathbb{C})$ of the form*

$$J := \mu 1_d + N = \begin{pmatrix} \mu & 1 & 0 & \dots & 0 \\ 0 & \mu & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \mu & 1 \\ 0 & \dots & 0 & 0 & \mu \end{pmatrix} \qquad \text{with} \qquad N = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix},$$

where μ is any complex number. Then $J - X1_d$ is invertible in $M_d(\mathcal{C})$ if and only if μ belongs to the resolvent set of X. In this case, the conditional expectation of $(J - X1_d)^{-1}$ can be computed, by involving the values of the extended Cauchy transform G_X and of its derivatives $G^{(1)}, \dots, G^{(d-1)}$ at the point μ, via the formula

$$E_d[(J - X1_d)^{-1}] = \sum_{k=0}^{d-1} \frac{1}{k!} G^{(k)}(\mu) N^k = \begin{pmatrix} G_X(\mu) & G_X^{(1)}(\mu) & \dots & \frac{1}{(d-1)!} G_X^{(d-1)}(\mu) \\ 0 & G_X(\mu) & \ddots & \vdots \\ \vdots & \ddots & \ddots & G_X^{(1)}(\mu) \\ 0 & \dots & 0 & G_X(\mu) \end{pmatrix}.$$

PROOF. Consider the decomposition $J = \mu 1_d + N$ and note that N is a nilpotent matrix, which satisfies $N^d = 0$.

Assume first that μ belongs to the resolvent set of X. In this case, we can write

$$J - X1_d = (\mu - X)1_d + N = (\mu - X)\big(1_d + (\mu - X)^{-1}N\big),$$

where $1_d + (\mu - X)^{-1}N$ is invertible and has the inverse

$$\big(1_d + (\mu - X)^{-1}N\big)^{-1} = \sum_{k=0}^{d-1}(-1)^k(\mu - X)^{-k}N^k,$$

since the matrices $(\mu - X)1_d$ and N commute. Therefore, also $J - X1_d$ must be invertible and we have that

$$(J - X1_d)^{-1} = \big(1_d + (\mu - X)^{-1}N\big)^{-1}(\mu - X)^{-1} = \sum_{k=0}^{d-1}(-1)^k(\mu - X)^{-(k+1)}N^k. \tag{I.12}$$

Conversely, suppose now that $J - X1_d$ is invertible in $M_d(\mathcal{A})$. Since in fact $J - X1_d \in M_d(\mathcal{A}_0)$, where $\mathcal{A}_0$ denotes the commutative C^*-subalgebra of $\mathcal{A}$ generated by $X = X^*$, we infer that $J - X1_d$ must be invertible also in $M_d(\mathcal{A}_0)$. Put $R := (J - X1_d)^{-1}$ and write $R = (R_{i,j})_{i,j=1}^d$. From the relation $R(J - X1_d) = 1_d$, we easily obtain $R_{1,1}(\mu - X) = 1$ and from this $(\mu - X)R_{1,1} = 1$, by the commutativity of $\mathcal{A}_0$. In summary, this yields the invertibility of $\mu - X$ with $(\mu - X)^{-1} = R_{1,1}$.

In the case where $J - X1_d$ and $\mu - X$ are both invertible, we use

$$G_X^{(k)}(\mu) = (-1)^k k!\phi\big((\mu - X)^{-(k+1)}\big) \qquad \text{for each } k \in \mathbb{N}_0$$

in order to obtain with the help of (I.12) that

$$E_d[(J - X1_d)^{-1}] = \sum_{k=0}^{d-1}(-1)^k\phi\big((\mu - X)^{-(k+1)}\big)N^k = \sum_{k=0}^{d-1}\frac{1}{k!}G^{(k)}(\mu)N^k.$$

This yields the stated formula and hence concludes the proof. □

Note that in the scalar-valued setting Cauchy transforms were defined first for Borel probability measures on $\mathbb{R}$ and after that for self-adjoint non-commutative random variables via their analytic distributions. In the operator-valued setting, we have to be content with Definition I.2.9, since there is no measure theoretic description of $\mathcal{B}$-valued distributions similar to the classical case behind the scenes. However, there is an intermediate level, on which we can come closer to the scalar-valued situation: for any $b \in \mathbb{H}^+(\mathcal{B})$ with $\|b^{-1}\| < \|X\|$, we see that the $\mathcal{B}$-valued Cauchy transform G_X admits a series expansion

$$G_X(b) = \sum_{k=0}^{\infty} E[(b^{-1}X)^k b^{-1}] = \sum_{k=0}^{\infty}\mu_X\big((b^{-1}x)^k b^{-1}\big),$$

where only the $\mathcal{B}$-valued distribution $\mu_X : \mathcal{B}\langle x\rangle \to \mathcal{B}$ is involved. This is the $\mathcal{B}$-valued analogue of (I.9). In this sense μ_X determines G_X uniquely and we are thus allowed to write G_{μ_X} instead of G_X. Accordingly, we can talk about the $\mathcal{B}$-valued Cauchy transform G_μ of μ, whenever we start with $\mu : \mathcal{B}\langle x\rangle \to \mathcal{B}$, which arises as the $\mathcal{B}$-valued distribution of a non-commutative random variable in some C^*-probability space over $\mathcal{B}$. Amazingly, these "abstract distributions" can be characterized. This works as follows.

We denote by $\Sigma_{\mathcal{B}}$ the set of all linear mappings $\mu : \mathcal{B}\langle x\rangle \to \mathcal{B}$ satisfying $\mu(b) = b$ for all $b \in \mathcal{B}$, which are moreover positive (i.e. they satisfy $\mu(PP^*) \geq 0$ in $\mathcal{B}$ for all $P \in \mathcal{B}\langle x\rangle$) and which have the property that

$$\mu(b_1 P b_2) = b_1\mu(P)b_2 \qquad \text{for all } b_1, b_2 \in \mathcal{B} \text{ and } P \in \mathcal{B}\langle x\rangle.$$

Furthermore, we denote by $\Sigma_{\mathcal{B}}^0$ the subset of $\Sigma_{\mathcal{B}}$, consisting of those $\mu:\ \mathcal{B}\langle x\rangle \to \mathcal{B}$, for which some constant $M>0$ exists, such that

$$\|\mu(xb_1x\cdots xb_kx)\| < M^{k+1}\|b_1\|\cdots\|b_k\|$$

for all $k\in\mathbb{N}$ and $b_1,\dots,b_k\in\mathcal{B}$ holds. We can formulate now the following result.

Theorem I.2.13 (see [**PV13**, Proposition 2.2]). *Let $\mu\in\Sigma_{\mathcal{B}}$ be given. Then $\mu\in\Sigma_{\mathcal{B}}^0$ if and only if there exists a C^*-probability space $(\mathcal{A},E,\mathcal{B})$ over $\mathcal{B}$ and a non-commutative random variable $X=X^*\in\mathcal{A}$, such that $\mu=\mu_X$.*

Any operator-valued C^*-probability space $(\mathcal{A},E,\mathcal{B})$ induces naturally a family of operator-valued C^*-probability spaces. They are given by $(M_n(\mathcal{A}),E^{(n)},M_n(\mathcal{B}))$ for each $n\in\mathbb{N}$, where $E^{(n)}$ denotes the amplification of E (see part (iii) of Remark I.2.5) defined by

$$E^{(n)}:\ M_n(\mathcal{A})\to M_n(\mathcal{B}),\ (X_{k,l})_{k,l=1}^n \mapsto (E[X_{k,l}])_{k,l=1}^n.$$

Accordingly, for any $X=X^*\in\mathcal{A}$, we have a whole family of Cauchy transforms

$$G_X^{(n)}:\ \mathbb{H}^+(M_n(\mathcal{B}))\to\mathbb{H}^-(M_n(\mathcal{B})),\ b\mapsto E^{(n)}\big[(b-X1_n)^{-1}\big].$$

This observation was at the base of Voiculescu's "free analysis" [**Voi08**] and it allows to treat operator-valued Cauchy transforms as non-commutative functions in the sense of [**KV14**]. The motivation comes from the fact that knowledge of G_X is not enough to recover the full $\mathcal{B}$-valued distribution of X, whereas the tower $(G_X^{(n)})_{n\in\mathbb{N}}$ contains all this information. Although we cannot formulate a precise analogue of Stieltjes inversion formula, the following theorem should give some justification to our claim that $(G_X^{(n)})_{n\in\mathbb{N}}$ fully controls the $\mathcal{B}$-valued distribution of X.

Theorem I.2.14 ([**BPV12**, Proposition 2.11]). *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence in $\Sigma_{\mathcal{B}}^0$, which is uniformly bounded in the sense that there exists a constant $M>0$, such that we have*

$$\|\mu_n(xb_1x\cdots xb_kx)\| < M^{k+1}\|b_1\|\cdots\|b_k\|$$

for all $n,k\in\mathbb{N}$ and all $b_1,\dots,b_k\in\mathcal{B}$. Then the following statements are equivalent:

(i) $(\mu_n)_{n\in\mathbb{N}}$ *norm-converges to some $\mu\in\Sigma_{\mathcal{B}}^0$, i.e., we have for all $k\in\mathbb{N}$*

$$\lim_{n\to\infty}\sup_{\|b_1\|=1,\dots,\|b_k\|=1}\|\mu_n(xb_1x\cdots xb_kx)-\mu(xb_1x\cdots xb_kx)\|=0.$$

(ii) *For all $m\in\mathbb{N}$, the sequence $(G_{\mu_n}^{(m)})_{n\in\mathbb{N}}$ converges uniformly to $G_\mu^{(m)}$ on balls in $\mathbb{H}^+(M_m(\mathcal{B}))$, which lay at positive distance from the boundary $\partial\mathbb{H}^+(M_m(\mathcal{B}))$.*

The above given formulations are adjusted to our needs and so they do not present [**PV13**, Proposition 2.2] and [**BPV12**, Proposition 2.11] in full strength and generality.

Example I.2.15.

(i) There is an important example of an operator-valued distribution, namely operator-valued semicircular elements. Since a description of the $\mathcal{B}$-valued distribution of $\mathcal{B}$-valued semicircular elements would require some combinatorial terminology, which we did not introduce here, we stick to the following definition, which is inspired by [**Spe98**, Theorem 4.1.12.]: a self-adjoint element S in an operator-valued C^*-probability space $(\mathcal{A}, E, \mathcal{B})$ is called *$\mathcal{B}$-valued semicircular element (with zero mean and covariance map $\eta : \mathcal{B} \to \mathcal{B}$)*, if η is completely positive and if the $\mathcal{B}$-valued Cauchy transform G_S of S solves the equation

$$\eta(G_S(b))G_S(b) - bG_S(b) + 1 = 0 \qquad \text{for all } b \in \mathbb{H}^+(\mathcal{B}).$$

In fact, it was shown in [**HFS07**] as part of a more general statement that this equation has for each completely positive map η a unique solution $G : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^-(\mathcal{B})$. It is an additional feature of the proof given in [**HFS07**] that the pointwise inverse of this solution G, i.e. the operator-valued F-transform

$$F : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B}), b \mapsto G(b)^{-1},$$

can be obtained by a fixed point iteration. From this, it can be deduced that G is in fact a locally bounded Fréchet holomorphic function and hence analytic.

(ii) Operator-valued semicircular elements arise naturally by some construction based on Remark I.2.17: if $s_1, \dots, s_n$ are scalar-valued semicircular elements (not necessarily freely independent) in some C^*-probability space $(\mathcal{C}, \phi)$ and if $b_1, \dots, b_n$ are any self-adjoint matrices in $M_N(\mathbb{C})$, then

$$S := b_1 \otimes s_1 + \dots + b_n \otimes s_n$$

gives an operator-valued semicircular element in $(M_N(\mathbb{C}) \otimes \mathcal{C}, E, M_N(\mathbb{C}))$ with mean zero and covariance map $\eta : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ given by

$$\eta(b) = E[SbS] = \sum_{i,j=1}^{n} \phi(s_i s_j)\, b_i b b_j.$$

I.2.4. Free independence with amalgamation. The definition of free independence in the general setting of operator-valued non-commutative probability spaces (see Definition I.1.34) reads as follows.

Definition I.2.16 (Free independence with amalgamation). Let $(\mathcal{A}, E, \mathcal{B})$ be an operator-valued non-commutative probability space

(i) Let $(\mathcal{A}_i)_{i\in I}$ be a family of subalgebras $\mathcal{B} \subseteq \mathcal{A}_i \subseteq \mathcal{A}$ with an arbitrary index set $I \neq \emptyset$. We call $(\mathcal{A})_{i\in I}$ *freely independent with amalgamation over* $\mathcal{B}$ (or just *free over* $\mathcal{B}$), if

$$E[X_1 \cdots X_n] = 0$$

holds whenever the following conditions are fulfilled:
- We have $n \geq 1$ and there are indices $i_1, \dots, i_n \in I$ satisfying

$$i_1 \neq i_2, \dots, i_{n-1} \neq i_n.$$

- For $j = 1, \dots, n$, we have $X_j \in \mathcal{A}_{i_j}$ and it holds true that $E[X_j] = 0$.

(ii) Let $(\mathcal{X}_i)_{i\in I}$ a family of subsets of $\mathcal{A}$ with an arbitrary index set $I \neq \emptyset$. We call $(\mathcal{X}_i)_{i\in I}$ *freely independent* (or just *free*), if $(\mathcal{A}_i)_{i\in I}$ are freely independent in the sense of (i), where $\mathcal{A}_i$ denotes for each $i \in I$ the unital subalgebra of $\mathcal{A}$ that is generated by $\mathcal{B}$ and the elements of $\mathcal{X}_i$.

(iii) Elements $(X_i)_{i\in I}$ are called *freely independent with amalgamation over* $\mathcal{B}$ (or just *free with amalgamation over* $\mathcal{B}$), if $(\mathcal{A}_i)_{i\in I}$ are freely independent with amalgamation over $\mathcal{B}$ in the sense of (i), where $\mathcal{A}_i$ denotes for each $i \in I$ the subalgebra of $\mathcal{A}$ that is generated by $\mathcal{B}$ and X_i.

For our purposes, it is important to note that operator-valued non-commutative probability spaces can easily be constructed by passing to matrices over scalar-valued non-commutative probability spaces.

LEMMA I.2.17. *Let* $(\mathcal{C},\phi)$ *be any non-commutative probability space. Then*

$$\mathcal{A} := M_N(\mathbb{C})\otimes\mathcal{C}, \qquad \mathcal{B} := M_N(\mathbb{C}), \qquad \text{and} \qquad E := \mathrm{id}_{M_N(\mathbb{C})}\otimes\phi$$

defines, as we noticed in Example I.2.3, an operator-valued non-commutative probability space $(\mathcal{A},E,\mathcal{B})$. *If* $(\mathcal{C}_i)_{i\in I}$ *is any family of freely independent subalgebras of* $\mathcal{C}$, *then* $\mathcal{A}_i := M_N(\mathbb{C})\otimes\mathcal{C}_i$ *for* $i\in I$ *defines a family* $(\mathcal{A}_i)_{i\in I}$ *of subalgebras of* $\mathcal{A}$, *which is freely independent with amalgamation over* $\mathcal{B}$.

Before proceeding to the proof of Lemma I.2.17, let us introduce first the following terminology: for $k,l=1,\dots,N$, we denote by $e_{k,l}$ the (k,l)-*matrix unit in* $M_N(\mathbb{C})$, i.e., the matrix $e_{k,l}\in M_N(\mathbb{C})$ whose entries are all zero, except for the (k,l)-entry, which is set to be 1.

PROOF OF LEMMA I.2.17. It is easy to check that $(\mathcal{A},E,\mathcal{B})$ satisfies the conditions given in Definition I.2.16. It thus only remains to show that operator-valued free independence arises from scalar-valued free independence in the described way. For seeing this, we take $n\geq 1$ and indices $i_1,\dots,i_n\in I$ satisfying $i_1\neq i_2,\dots,i_{n-1}\neq i_n$, as well as non-commutative random variables $X_1,\dots,X_n$ satisfying $X_j\in\mathcal{A}_{i_j}$ and $E[X_j]=0$ for $j=1,\dots,n$. With respect to the matrix units $e_{k,l}$ of $M_n(\mathbb{C})$, we can write

$$X_j = \sum_{k,l=1}^N e_{k,l}\otimes X_{k,l}^{(j)}, \qquad \text{where } X_{k,l}^{(j)}\in\mathcal{C}_{i_j} \text{ for } k,l=1,\dots,N.$$

Since we have by assumption

$$0 = E[X^{(j)}] = \sum_{k,l=1}^N e_{k,l}\phi(X_{k,l}^{(j)}),$$

we can rewrite this as

$$X_j = \sum_{k,l=1}^N e_{k,l}\otimes\big(X_{k,l}^{(j)}-\phi(X_{k,l}^{(j)})\big).$$

Thus, ignoring some obvious cancellations coming from matrix products of the form $e_{k_1,l_1}\cdots e_{k_n,l_n}$, we obtain

$$X_1\cdots X_n = \sum_{k_1,l_1=1}^N\cdots\sum_{k_n,l_n=1}^N (e_{k_1,l_1}\cdots e_{k_n,l_n})\otimes\Big(\big(X_{k_1,l_1}^{(1)}-\phi(X_{k_1,l_1}^{(1)})\big)\cdots\big(X_{k_n,l_n}^{(n)}-\phi(X_{k_n,l_n}^{(n)})\big)\Big)$$

and after applying E on both sides

$$E[X_1\cdots X_n] = \sum_{k_1,l_1=1}^N\cdots\sum_{k_n,l_n=1}^N \phi\Big(\big(X_{k_1,l_1}^{(1)}-\phi(X_{k_1,l_1}^{(1)})\big)\cdots\big(X_{k_n,l_n}^{(n)}-\phi(X_{k_n,l_n}^{(n)})\big)\Big)\,(e_{k_1,l_1}\cdots e_{k_n,l_n}).$$

The assumed free independence for $(\mathcal{C}_i)_{i\in I}$ gives that

$$\phi\Big(\big(X_{k_1,l_1}^{(1)}-\phi(X_{k_1,l_1}^{(1)})\big)\cdots\big(X_{k_n,l_n}^{(n)}-\phi(X_{k_n,l_n}^{(n)})\big)\Big)=0$$

for any choice of $k_1,l_1,\dots,k_n,l_n\in\{1,\dots,n\}$ and thus

$$E[X_1\cdots X_n]=0,$$

as we had to show. □

I.2.5. Operator-valued free additive convolution. Like in the scalar-valued case, also the $\mathcal{B}$-valued non-commutative distribution $\mu_{X_1+X_2}$ for elements X_1 and X_2 living in some operator-valued non-commutative probability space $(\mathcal{A}, E, \mathcal{B})$, which are freely independent with amalgamation over $\mathcal{B}$, only depends on μ_{X_1} and μ_{X_2}, and not on the concrete choice of the operators X_1 and X_2. With respect to the universal formulas provided by the freeness condition, we can introduce the *free additive convolution* $\boxplus$ as a binary operation on the set of all abstract $\mathcal{B}$-valued distributions (i.e., the set of all linear functionals $\mu : \mathcal{B}\langle x\rangle \to \mathcal{B}$ satisfying the conditions $\mu(b) = b$ for all $b \in \mathcal{B}$ and $\mu(b_1 P b_2) = b_1\mu(P)b_2$ for all $b_1, b_2 \in \mathcal{B}$ and $P \in \mathcal{B}\langle x\rangle$.), such that we can write $\mu_{X_1+X_2} = \mu_{X_1} \boxplus \mu_{X_2}$.

In analogy to the scalar-valued case, which was treated in [**Voi86**], Voiculescu introduced in [**Voi95**] a linearizing transform for the $\mathcal{B}$-valued free additive convolution $\boxplus$, the so-called *operator-valued R-transform*; see [**Dyk06**] for an alternative description. We have already seen in the scalar-valued case that the most convenient way to deal with the free additive convolution is the subordination formalism. This approach is appropriate also in the operator-valued case, but technically even more demanding. Preliminary versions were obtained under more restrictive assumptions in [**Bia98a, Voi00b, Voi02a**], and it was shown in [**BMS13**] that subordination even works in the more general situation of operator-valued C^*-probability spaces. Let us point out that this approach enjoys the additional feature that it is easily accessible for numerical computations, as it provides a fixed point iteration scheme similar to [**BB07**]. This will be of great importance in Chapter IV.

Before we give the precise statement, let us introduce the following transforms, which are both related to Cauchy transforms, namely

- the reciprocal Cauchy transform, called *F-transform*, $F_X : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$ by
$$F_X(b) := E\left[(b - X)^{-1}\right]^{-1} = G_X(b)^{-1},$$
- and the *h-transform* $h_X : \mathbb{H}^+(\mathcal{B}) \to \overline{\mathbb{H}^+(\mathcal{B})}$ by
$$h_X(B) := E\left[(b - X)^{-1}\right]^{-1} - B = F_X(b) - b.$$

Note, that these mappings are indeed well-defined since it has been shown in [**BPV12**] that $\Im(F_X(b)) \geq \Im(b)$ for all $b \in \mathbb{H}^+(\mathcal{B})$, which implies $\Im(h_X(b)) \geq 0$ for all $b \in \mathbb{H}^+(\mathcal{B})$.

For the relevant definitions and more details about holomorphic functions in Banach spaces, we refer to Appendix B.

THEOREM I.2.18 ([**BMS13**]). *Assume that $(\mathcal{A}, E, \mathcal{B})$ is a C^*-operator-valued non-commutative probability space and $X_1, X_2 \in \mathcal{A}$ are two self-adjoint operator-valued random variables, which are free with amalgamation over $\mathcal{B}$. Then there exists a unique pair of Fréchet (and thus also Gâteaux) holomorphic maps $\omega_1, \omega_2 : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B})$ so that*

(i) $\Im(\omega_j(b)) \geq \Im(b)$ *for all* $b \in \mathbb{H}^+(\mathcal{B})$ *and* $j \in \{1, 2\}$,
(ii) $F_{X_1}(\omega_1(b)) + b = F_{X_2}(\omega_2(b)) + b = \omega_1(b) + \omega_2(b)$ *for all* $b \in \mathbb{H}^+(\mathcal{B})$,
(iii) $G_{X_1}(\omega_1(b)) = G_{X_2}(\omega_2(b)) = G_{X_1+X_2}(b)$ *for all* $b \in \mathbb{H}^+(\mathcal{B})$.

Moreover, if $b \in \mathbb{H}^+(\mathcal{B})$ is given, then $\omega_1(b)$ is the unique fixed point of the map
$$f_b : \mathbb{H}^+(\mathcal{B}) \to \mathbb{H}^+(\mathcal{B}), \; w \mapsto h_{X_2}(h_{X_1}(w) + b) + b,$$

and $\omega_1(b) = \lim_{n\to\infty} f_b^{\circ n}(w)$ *for any* $w \in \mathbb{H}^+(\mathcal{B})$*, where* $f_b^{\circ n}$ *means the* n*-fold composition of* f_b *with itself. Same statements hold for* ω_2*, with* f_b *replaced by* $w \mapsto h_{X_1}(h_{X_2}(w)+b)+b$*.*

We do not want to present a detailed proof of this statement, but we want to mention that the proof given in [**BMS13**] is based on the Earle-Hamilton Theorem B.5. This theorem replaces in the present operator-valued setting the Denjoy-Wolff theory, which was crucially used in the proof of the scalar-valued version, Theorem I.1.42, as given in [**BB07**].

I.3. Brown measures

The so-called Brown measure has its origins outside the area of free probability. It was invented in 1986 by L. G. Brown [**Bro86**] in order to generalize Lidskii's theorem. This famous theorem states that the trace of any trace class operator on a separable Hilbert space is the sum of its eigenvalues, where multiplicities are counted. In the setting of a von Neumann algebra M, which is endowed with a faithful, normal, semi-finite trace[3], Brown proved that for any operator $X \in L^1(M,\tau)$, there exists a unique measure ν_X on $\sigma(X)\backslash\{0\}$, such that

$$\tau(\log|1-zX|) = \int_{\sigma(X)\backslash\{0\}} \log|1-zw|\, d\nu_X(w) \qquad \text{for all } z \in \mathbb{C}$$

holds true. He was able to show that this measure satisfies

$$\int_{\sigma(X)\backslash\{0\}} |w|^p\, d\nu_X(w) \leq \tau(|X|^p) \qquad \text{for all } 0 < p < \infty$$

and even

$$\tau(X) = \int_{\sigma(X)\backslash\{0\}} w\, d\nu_X(w),$$

which completes the analogy to Lidskii's theorem. Much later, namely in 2000, the Brown measure was introduced to the free probability community by U. Haagerup and F. Larsen. They took up Brown's ingenious work in their influential paper [**HL00**], where they used the Brown measure as some replacement of the analytic distribution for more general operators beyond the self-adjoint or the normal case; see Definition I.1.18 and Definition I.1.17. This revived Brown's beautiful theory and attracted attention of many people, not only from free probability but also from other areas of mathematics; see [**BL01, Sni02, Sni03, GKZ11**].

Of particular interest for us is the paper [**GKZ11**], as it shows with the help of free probability tools (in particular those coming from [**HL00**]) that the Brown measure can be used to describe the limiting eigenvalue distribution of certain non-self-adjoint random matrices. This kind of phenomenon is conjectured in many other cases. We will come back to this in Chapter IV, where we will present an algorithm that allows us to compute numerically the Brown measures for non-commutative rational expressions evaluated in freely independent variables.

Although the Brown measure is of totally different nature than the analytic distributions, which we discussed so far, the familiar machinery of Cauchy transforms can still be

[3]These terms are understood in the sense of weights. Since we will only work in tracial W^*-probability spaces (M,τ), we omit the general definitions here. Instead, we refer the interested reader to [**Bla06**, Section II.6.7]

used to deal with this technically difficult object. For that purpose, however, the scalar-valued theory is not sufficient and we must go over to setting of the operator-valued free probability. This was worked out in [**BSS15**].

I.3.1. Background and definition. Since we are only interested in applications to free probability, we will not present the theory in full generality. However, we point out that in order to use this theory, we have to stay in the setting of finite von Neumann algebras. Hence, we will in the following discussions around the Brown measure always work in a tracial W^*-probability space (M, τ).

Given an arbitrary element X in any tracial W^*-probability space (M, τ), we may define its *Fuglede-Kadison determinant* $\Delta(X)$ by the equation

$$\log(\Delta(X)) = \lim_{\varepsilon \searrow 0} \frac{1}{2}\tau(\log(XX^* + \varepsilon^2)).$$

This quantity was introduced in [**FK51, FK52**].

It was shown in [**Bro86**], that the function $z \mapsto \frac{1}{2\pi}\log(\Delta(X - z))$ is subharmonic on $\mathbb{C}$ and harmonic outside the spectrum of X. Thus, we may consider the associated *Riesz measure* (see also the Riesz Decomposition Theorem [**Ran95**, Theorem 3.7.9]), which is a Radon measure ν_X on $\mathbb{C}$ such that

$$\int_{\mathbb{C}} \psi(z)\, d\nu_X(z) = \frac{1}{2\pi}\int_{\mathbb{C}} \Big(\frac{\partial^2 \psi}{\partial x^2}(z) + \frac{\partial^2 \psi}{\partial y^2}(z)\Big) \log(\Delta(X - z))\, d\lambda^2(z)$$

holds for all functions $\psi \in C_c^\infty(\mathbb{C})$. There, we denote by λ^2 the Lebesgue measure on $\mathbb{C}$, which is induced under the usual identification of $\mathbb{C}$ with $\mathbb{R}^2$. In other words, the *Brown measure* ν_X is the *generalized Laplacian* of $z \mapsto \frac{1}{2\pi}\log(\Delta(X - z))$, which means that ν_X of X is determined (in the distributional sense) by

$$\nu_X = \frac{2}{\pi}\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\log(\Delta(X - z)) \tag{I.13}$$

Note that we made use of the fact that, on C^2-functions, the usual *Laplacian* $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ can be rewritten as

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}$$

in terms of the Pompeiu-Wirtinger derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \qquad \text{and} \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big).$$

I.3.2. The hermitization method. Following [**BSS15**] (see also [**Lar99**]), we will now discuss how tools from operator-valued free probability can be used to compute Brown measures.

Conceptually, this approach is similar to the way, how we usually deal with distributions for self-adjoint operators: given an operator X, we must find the Cauchy transform G_X, from which we can recover the analytic distribution μ_X of X by means of the Stieltjes inversion formula as explained in Theorem I.1.29. This means more precisely that we approximate μ_X by certain absolutely continuous regularizations $\mu_{X,\varepsilon}$, which converge weakly to μ_X as $\varepsilon \searrow 0$. In order to compute the Brown measure ν_X of some operator X, it is also convenient to approximate ν_X by certain regularizations $\nu_{X,\varepsilon}$. But at first sight, it is absolutely not clear, by which object we should replace the Cauchy transform in order to construct the desired regularizations. It is therefore much more intuitive to proceed the other way around, namely, we first give a reasonable candidate for such regularizations and then we check if there some sort of transform behind, which could replace the Cauchy transform.

To begin with, let us take a look at the construction of the Brown measure. It crucially relies on the Fuglede-Kadison determinant Δ, which itself involves some limit procedure. It is therefore a very natural starting point to replace Δ by the so-called *regularized Fuglede-Kadison determinant* Δ_ε, which is determined by the following equation

$$\log(\Delta_\varepsilon(X)) := \frac{1}{2}\tau(\log(XX^* + \varepsilon^2)).$$

Thus, natural regularizations $\nu_{X,\varepsilon}$ of the Brown measure ν_X can be obtained as follows.

Definition I.3.1. The *regularized Brown measures* $\nu_{X,\varepsilon}$ *of* X are obtained by replacing in its defining equation (I.13) the Fuglede-Kadison determinant Δ by the regularization Δ_ε. Explicitly and again in the distributional sense, this means that

$$\nu_{X,\varepsilon}(z) = \frac{2}{\pi}\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} \log(\Delta_\varepsilon(X - z)). \tag{I.14}$$

One can show, by using for example [**Ran95**, Exercise 3.7 (4)], that $\nu_{X,\varepsilon}$ converges weakly to ν_X as $\varepsilon \searrow 0$.

It remains to find some kind of Cauchy transform, which will hopefully allow us to compute these regularizations in a slightly simpler way.

Lemma I.3.2. *If we consider the* regularized Cauchy transform of X

$$G_{X,\varepsilon}(z) = \tau\big((z - X)^*\big((z - X)(z - X)^* + \varepsilon^2\big)^{-1}\big),$$

which is a C^∞*-function on* $\mathbb{C}$ *(but obviously not holomorphic on* $\mathbb{C}$*), we have that*

$$d\nu_{X,\varepsilon}(z) = \frac{1}{\pi}\frac{\partial}{\partial \overline{z}} G_{X,\varepsilon}(z)\, d\lambda^2(z). \tag{I.15}$$

The answer provided by the lemma above is surprisingly simple and looks quite appealing: the formula for $\nu_{X,\varepsilon}$ given in (I.15) yields indeed a satisfying replacement for the Stieltjes inversion formula, Theorem I.1.29, and the obtained regularized Cauchy transform $G_{X,\varepsilon}$ (which can be seen as an analogue for the function $z \mapsto G_\mu(z+i\varepsilon)$ appearing in the Stieltjes inversion formula) seems to be fairly close to being a Cauchy transform. But unfortunately, on closer inspection, one realizes that $G_{X,\varepsilon}$ is – though its striking similarity to Cauchy transforms – still a different object. Since free probability theory provides powerful tools to deal with Cauchy transforms, this conclusion is clearly quite disappointing. However, operator-valued free probability comes to our rescue. It turns out that $G_{X,\varepsilon}$ can indeed be related to Cauchy transforms, but for this purpose, we must leave the scalar-valued setting and work on an operator-valued level. More precisely, we must go over to the $M_2(\mathbb{C})$-valued C^*-probability space $(M_2(M), E, M_2(\mathbb{C}))$ as introduced in Lemma I.2.17 and use the so-called *hermitian reduction method*. This trick, which originally comes from random matrix theory [**JNPZ97**], is explained in the following lemma.

Lemma I.3.3. *Consider the self-adjoint element*

$$\underline{X} := \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix} \in M_2(M).$$

The value of the regularized Cauchy transform $G_{X,\varepsilon}$ *at the point* $z \in \mathbb{C}$ *can then be obtained as the* $(2,1)$*-entry of the* $M_2(\mathbb{C})$*-valued Cauchy transform of* $\underline{X}$*, if it is evaluated at the point*

$$\Lambda_\varepsilon(z) := \begin{pmatrix} i\varepsilon & z \\ \overline{z} & i\varepsilon \end{pmatrix} \in \mathbb{H}^+(M_2(\mathbb{C})).$$

More precisely, we have for each $z \in \mathbb{C}$ *that*

$$G_{X,\varepsilon}(z) = [G_{\underline{X}}(\Lambda_\varepsilon(z))]_{2,1}. \tag{I.16}$$

Collecting our observations, we see that the regularized Brown measures $\nu_{X,\varepsilon}$ defined by (I.14) (and thus the Brown measure ν_X in the limit $\varepsilon \searrow 0$) can be computed via (I.15) from its regularized Cauchy transforms $G_{X,\varepsilon}$, whereas the regularized Cauchy transform $G_{X,\varepsilon}$ itself can be deduced by (I.16) from the $M_2(\mathbb{C})$-valued Cauchy transform of the self-adjoint element $\underline{X}$. All this puts the Brown measure into the more familiar setting of self-adjoint operators, where the analytic machinery of operator-valued free probability theory applies. This approach will be used in Chapter IV.

CHAPTER II

Random matrices and asymptotic freeness

While free probability theory was invented with the intention to used it as a tool for operator algebraic questions, many surprising and fascinating connections to completely different areas of mathematics were found later on. Most of these connections have in common that contact to free probability is made by certain limit processes, in which some free structure shows up.

One of the first and certainly also one of the most exciting connections of this kind was found by Voiculescu [**Voi91**], namely to random matrix theory. Here, free independence surprisingly turned out to describe the relations among many types of classically independent random matrices in the limit when their dimension tends to infinity. In turn, based on the free convolutions $\boxplus$ and $\boxtimes$, free probability allowed to understand the asymptotic behavior of sums and products of independent random matrices of this type and provided by its powerful analytic machinery an effective way to compute these limiting distributions. In this sense, free probability can be seen as a very natural limit of classical probability theory, so that it unquestionably leaves what some people might consider as the “ivory tower of operator algebras”.

Nowadays, many generalizations of Voiculescu’s groundbreaking results to a wide range of random matrices are known and beyond the case of sums and products it is even possible to describe the limiting behavior of general polynomial and rational expressions in independent random matrices. The latter will be outlined in Chapter IV.

This exciting applicability of free probability methods to questions of random matrix theory – although they are located originally in the realm of classical probability theory – is essentially due to the following two observations:

(i) The eigenvalue distributions of many types of random matrices, like Wigner or Wishart random matrices, show a nice asymptotic behavior when the dimension tends to infinity. In fact, the randomness disappears in the limit and the resulting deterministic distributions can be described.
(ii) Classical independence among collections of independent random matrices often produces free independence in the limit when the dimension of the involved matrices goes to infinity.

The first mentioned phenomenon was well-known in random matrix theory long before the birth of free probability theory. For instance, Wigner’s famous semicircular law (see Theorem II.3.4 below) states that the eigenvalue distribution of self-adjoint Gaussian random matrices, which are a special instance of Wigner matrices, converges almost surely to the semicircular distribution as their dimension grows to infinity.

However, what might be surprising at first sight is that those limiting eigenvalue distributions agree with some prominent distributions in free probability, such as the semicircular distribution (which is the free analogue of the normal distribution) or the free Poisson distribution. One could think about this just as a curious coincidence but it rather indicates a deeper link between these two fields, as explained by the second phenomenon explains. It became now known under the name "asymptotic freeness" and was one of the big discoveries of Voiculescu, by which he opened the door for an extremely fruitful interaction between random matrix theory and the theory of operator algebras, with a strong impetus in both directions.

This chapter, which is devoted to the connections between free probability and random matrix theory, is organized as follows. In Section II.1, we will introduce the terminology and provide some basic knowledge about random matrices. We will introduce them as elements in certain non-commutative probability spaces. This approach has the advantage that it fits nicely to the general setting of asymptotic freeness, which will be presented in Section II.2. Finally, in Section II.3, we turn our attention to the important case of Wigner and Wishart random matrices.

Nevertheless, this chapter can of course only be a very tiny scratch on the surface of random matrix theory. For a more detailed introduction, we refer the reader to the excellent lecture notes [**Kem13**] and to [**AGZ10**].

II.1. Some basic facts about random matrices

In this section we will see that random matrices fit nicely into the general frame of non-commutative probability spaces. But what actually are random matrices? There are essentially two point of views: either we could say that random matrices are matrices which are randomly chosen according to some given distribution on the space of all (mostly self-adjoint) matrices of some fixed size, or we could think of random matrices as (again, mostly self-adjoint) matrices whose entries are random variables. On closer inspection, these pictures turn out to be equivalent, but both of them have advantages and disadvantages, depending on the intended application. For our purposes, we prefer the second named approach.

II.1.1. Non-commutative probability space of random matrices. To begin with, let us fix some classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Our point of view is that classical random variables over $(\Omega, \mathcal{F}, \mathbb{P})$ should constitute the entries of random matrices. If we want to deal with random variables in such a way that the corresponding random matrices fit into the frame of non-commutative probability theory, then Example I.1.2 proposes the non-commutative probability space $(L^\infty(\Omega, \mathbb{P}), \mathbb{E})$ as some reasonable choice. The drawback, however, is that $L^\infty(\Omega, \mathbb{P})$ contains by definition only bounded random variables, whereas the most prominent examples of random matrices are built on unbounded ones, such as Gaussian random variables. We thus need a slight modification of Example I.1.2 in order to bring at least some unbounded random variables in the range of non-commutative probability theory. For this purpose, let us define the following variant of $L^\infty(\Omega, \mathbb{P})$, namely

$$L^{\infty-}(\Omega, \mathbb{P}) := \bigcap_{1 \leq p < \infty} L^p(\Omega, \mathbb{P}),$$

which is the complex unital algebra of all random variables that have finite moments of any order. It is not totally obvious that $L^{\infty-}(\Omega, \mathbb{P})$ is indeed closed under multiplication, but this can be shown with the help of Hölder's inequality. On $L^{\infty-}(\Omega, \mathbb{P})$, we finally introduce the expectation functional $\mathbb{E}$ by

$$\mathbb{E}[X] := \int_\Omega X(\omega)\, d\mathbb{P}(\omega) \qquad \text{for all } X \in L^{\infty-}(\Omega, \mathbb{P}).$$

With this underlying non-commutative probability space $(L^{\infty-}(\Omega,\mathbb{P}),\mathbb{E})$, our definition of random matrices reads as follows.

Definition II.1.1. A *random matrix (of size $n\times n$)* is an element of the non-commutative probability space $(\mathfrak{M}_n, \tau_n)$ given by

$$\mathfrak{M}_n := M_n(L^{\infty-}(\Omega,\mathbb{P})) \qquad \text{and} \qquad \tau_n := \operatorname{tr}_n \circ \mathbb{E}^{(n)},$$

where tr_n denotes the normalized trace on $M_n(\mathbb{C})$ and $\mathbb{E}^{(n)} : M_n(L^{\infty-}(\Omega,\mathbb{P})) \to M_n(\mathbb{C})$ the linear functional that is given as the natural amplification of $\mathbb{E}$, i.e.

$$\mathbb{E}^{(n)}\big[(X_{k,l})_{k,l=1}^n\big] := \big(\mathbb{E}[X_{k,l}]\big)_{k,l=1}^n \qquad \text{for all } X = (X_{k,l})_{k,l=1}^n \in \mathfrak{M}_n.$$

Note that we have the relation $\tau_n = \operatorname{tr}_n \circ\mathbb{E}^{(n)}$ in the first and $\tau_n = \mathbb{E}\circ \operatorname{tr}_n$ in the second case.

Remark II.1.2. Note that under the natural (algebraic) isomorphism

$$\mathfrak{M}_n = M_n(L^{\infty-}(\Omega,\mathbb{P})) \cong M_n(\mathbb{C}) \otimes L^{\infty-}(\Omega,\mathbb{P}),$$

the expectation τ_n on $\mathfrak{M}_n = M_n(L^{\infty-}(\Omega,\mathbb{P}))$ is identified with $\operatorname{tr}_n \otimes \mathbb{E}$ on $M_n(\mathbb{C}) \otimes L^{\infty-}(\Omega,\mathbb{P})$. In particular, we see that $\mathfrak{M}_n$ induces naturally two operator-valued non-commutative probability spaces by

$$(\mathfrak{M}_n, \mathbb{E}^{(n)}, M_n(\mathbb{C})) \qquad \text{and} \qquad (\mathfrak{M}_n, \operatorname{tr}_n, L^{\infty-}(\Omega,\mathbb{P})),$$

where in the latter one tr_n is understood as a linear functional $\operatorname{tr}_n : \mathfrak{M}_n \to L^{\infty-}(\Omega,\mathbb{P})$ by

$$\operatorname{tr}_n\big((X_{k,l})_{k,l=1}^n\big) := \frac{1}{n}\sum_{k=1}^n X_{k,k} \qquad \text{for all } X = (X_{k,l})_{k,l=1}^n \in \mathfrak{M}_n.$$

We have introduced above $(\mathfrak{M}_n, \tau_n)$ as a non-commutative probability space, ignoring that there is a natural involution $*$ on $L^{\infty-}(\Omega,\mathbb{P})$. This $*$-structure goes over to $\mathfrak{M}_n$, which turns both $L^{\infty-}(\Omega,\mathbb{P})$ and $\mathfrak{M}_n$ into $*$-probability spaces. Indeed, for $X \in L^{\infty-}(\Omega,\mathbb{P})$, we denote by X^* the random variable in $L^{\infty-}(\Omega,\mathbb{P})$, which is determined by the condition that $X^*(\omega) = \overline{X(\omega)}$ for $\mathbb{P}$-almost all $\omega\in\Omega$. It is easy to check that $(L^{\infty-}(\Omega,\mathbb{P}),\mathbb{E})$ becomes with respect to $*$ a $*$-probability space in the sense of Definition I.1.5. Corresponding to $*$ on $L^{\infty-}(\Omega,\mathbb{P})$, we have a natural involution on $\mathfrak{M}_n$; since it extends $*$, we will denote it by the same symbol. Given a random matrix $X = (X_{k,l})_{k,l=1}^n \in \mathfrak{M}_n$, the random matrix $X^* \in \mathfrak{M}_n$ is defined by $X^* := (X_{l,k}^*)_{k,l=1}^n$. It is easy to see that $(\mathfrak{M}_n, \tau_n)$ becomes in this way a $*$-probability space.

Remark II.1.3. Having this underlying $*$-structure, it is very natural to call a random matrix $X = (X_{k,l})_{k,l=1}^n \in \mathfrak{M}_n$ *self-adjoint*, if $X = X^*$ holds. This means explicitly, that matrix $X(\omega) = (X_{k,l}(\omega))_{k,l=1}^n \in M_n(\mathbb{C})$ is self-adjoint with respect to the usual conjugate transpose on $M_n(\mathbb{C})$ for $\mathbb{P}$-almost all $\omega \in \Omega$.

II.1.2. The empirical eigenvalue distribution. We consider now a self-adjoint random matrix $X \in \mathfrak{M}_n$. As we have expounded in Remark II.1.3, X being self-adjoint means that the matrix $X(\omega)$ is a self-adjoint non-commutative random variable in the non-commutative C^*-probability space $(M_n(\mathbb{C}), \mathrm{tr}_n)$ for $\mathbb{P}$-almost all $\omega \in \Omega$. Furthermore, we have seen in Example I.1.21 that the analytic distribution $\mu_{X(\omega)}$ of $X(\omega)$ is given by its normalized eigenvalue distribution, i.e.

$$\mu_{X(\omega)} = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j(\omega)},$$

where $\lambda_1(\omega) \leq \cdots \leq \lambda_n(\omega)$ denote the eigenvalues of $X(\omega)$, listed according to their multiplicity.

Since the measure $\mu_{X(\omega)}$ depends on the outcome $\omega \in \Omega$, one is tempted to call $\omega \mapsto \mu_{X(\omega)}$ a *random probability measure*. However, this does not come for free, since talking in a formally correct way about measure-valued random variables forces us to impose some measurability condition on $\omega \mapsto \mu_{X(\omega)}$, which itself requires to have identified some σ-algebra on the corresponding set of probability measures. This is usually done in such a way that being a random probability measure means for $\omega \mapsto \mu_{X(\omega)}$ that the function $\omega \mapsto \mu_{X(\omega)}(B)$ is measurable for each fixed Borel subset B of $\mathbb{R}$.

Definition II.1.4. Let $X \in \mathfrak{M}_n$ be a self-adjoint random matrix, such that the function $\omega \mapsto \mu_{X(\omega)}(B)$ is measurable for each Borel subset B of $\mathbb{R}$.

(i) The *empirical eigenvalue distribution of* X is the random probability measure

$$\mu_{X(\cdot)}:\ \omega \mapsto \mu_{X(\omega)}$$

given by the eigenvalue distribution of $X(\omega)$ for $\omega \in \Omega$.

(ii) The *mean empirical eigenvalue distribution of* X is the Borel probability measure $\bar{\mu}_X$ on $\mathbb{R}$, which is defined by

$$\bar{\mu}_X(B) := \mathbb{E}\big[\mu_{X(\cdot)}(B)\big] = \int_\Omega \mu_{X(\omega)}(B)\, d\mathbb{P}(\omega)$$

for each Borel subset B of $\mathbb{R}$.

One easily sees that under the assumptions of the previous definition

$$\int_{\mathbb{R}} f(t)\, d\bar{\mu}_X(t) = \mathbb{E}\Big[\int_{\mathbb{R}} f(t)\, d\mu_{X(\cdot)}(t)\Big] \tag{II.1}$$

holds for each simple function $f : \mathbb{R} \to \mathbb{C}$. By some standard approximation argument (with respect to the uniform norm $\|\cdot\|_\infty$ on $\mathbb{R}$) we may deduce that (II.1) even holds for each function $f \in C_0(\mathbb{R})$. The Riesz-representations theorem C.6 guarantees that $\bar{\mu}_X$ is uniquely determined by this condition among all positive Radon measures on $\mathbb{R}$.

In particular, given $z \in \mathbb{C}^+$, we may apply (II.1) to the resolvent function $f_z \in C_0(\mathbb{R})$, which is defined by $f_z(t) := \frac{1}{z-t}$ for $t \in \mathbb{R}$. This gives

$$G_{\bar{\mu}_X}(z) = \mathbb{E}\big[G_{\mu_{X(\cdot)}}(z)\big],$$

which says that the Cauchy transform $\bar{\mu}_X$ can be obtained pointwise as the expectation of the random variable $\omega \mapsto G_{\mu_{X(\omega)}}(z)$. The measurability of $\omega \mapsto G_{\mu_{X(\omega)}}(z)$ for fixed $z \in \mathbb{C}^+$ follows, since $G_{\mu_{X(\omega)}}$ can be expressed in the following way

$$G_{\mu_{X(\omega)}}(z) = \mathrm{tr}_n\big((z - X(\omega))^{-1}\big) \qquad \text{for all } z \in \mathbb{C}^+$$

and in particular without referring to the eigenvalues $\lambda_1(\omega) \leq \cdots \leq \lambda_n(\omega)$ of $X(\omega)$.

Thus, alternatively, we could introduce $\bar{\mu}_X$ by means of Theorem I.1.27, which guarantees that the holomorphic function $G_{\bar{\mu}_X}$ defined by

$$G_{\bar{\mu}_X}(z) := \int_\Omega \operatorname{tr}_n\left((z - X(\omega))^{-1}\right) d\mathbb{P}(\omega) \qquad \text{for all } z \in \mathbb{C}^+$$

is indeed the Cauchy transform of some Borel probability measure $\bar{\mu}_X$ on $\mathbb{R}$. Since the Stone-Weierstraß theorem shows that the linear span of $\{f_z | \, z \in \mathbb{C}\backslash\mathbb{R}\}$ is dense in $C_0(\mathbb{R})$, we can conclude backwards that $\bar{\mu}_X$ satisfies the characterizing condition (II.1) for each $f \in C_0(\mathbb{R})$.

For general random matrices X, the mean empirical eigenvalue distribution $\bar{\mu}_X$ is hard to compute, but in some particular cases explicit formulas are known; see Remark II.3.3.

II.2. Asymptotic freeness

We will now present Voiculescu's definition of asymptotic freeness as given in [**Voi91**]. Though it will be formulated in general terms, we should keep in mind that random matrices constitute the most prominent example, according to which this definition is actually modeled.

Definition II.2.1. Fix some non-empty index set I. For each $n \in \mathbb{N}$, let $(\mathcal{A}_n, \phi_n)$ be a non-commutative probability space and let $X^{(n)} = (X_i^{(n)})_{i \in I}$ be a family of non-commutative random variables in $\mathcal{A}_n$.

(i) The sequence $(X^{(n)})_{n\in\mathbb{N}}$ is said to be *convergent in distribution to* $X = (X_i)_{i\in I}$, for some family $X = (X_i)_{i\in I}$ of non-commutative random variables living in some non-commutative probability space $(\mathcal{A}, \phi)$, if the sequence $(\mu_{X^{(n)}})_{n\in\mathbb{N}}$ converges pointwise to the non-commutative distribution μ_X, i.e., if we have
$$\lim_{n\to\infty} \phi_n(P(X^{(n)})) = \phi(P(X)) \qquad \text{for all } P \in \mathbb{C}\langle x_i |\; i \in I\rangle.$$

(ii) The sequence $(X^{(n)})_{n\in\mathbb{N}}$ is said to be *asymptotically free*, if $(X^{(n)})_{n\in\mathbb{N}}$ converges in distribution to a family $X = (X_i)_{i\in I}$ in some non-commutative probability space $(\mathcal{A}, \phi)$, such that $(X_i)_{i\in I}$ are in addition freely independent in $(\mathcal{A}, \phi)$.

In the case of random matrices, the family $((\mathcal{A}_n, \phi_n))_{n\in\mathbb{N}}$ of non-commutative probability spaces is given by $((\mathfrak{M}_n, \tau_n))_{n\in\mathbb{N}}$. As explained in Remark II.1.2, the expectation τ_n can be written as a composition $\tau_n = \mathbb{E} \circ \operatorname{tr}_n$, where the trace tr_n is understood as a linear functional $\operatorname{tr}_n : \mathfrak{M}_n \to L^{\infty-}(\Omega, \mathbb{P})$. Therefore, it becomes possible to separate its random part and its deterministic part. This leads us to the notion of almost sure convergence in distribution and almost sure asymptotic freeness; see [**HP00b**, Section 4.3].

Definition II.2.2. Fix a classical probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and some non-empty index set I. For each $n \in \mathbb{N}$, let $X^{(n)} = (X_i^{(n)})_{i\in I}$ be a family of non-commutative random variables $X^{(n)}$, living in the non-commutative probability space $(\mathfrak{M}_n, \tau_n)$ build over $(\Omega, \mathcal{F}, \mathbb{P})$.

(i) The sequence $(X^{(n)})_{n\in\mathbb{N}}$ is said to be *almost surely convergent in distribution to* $X = (X_i)_{i\in I}$, for some family $X = (X_i)_{i\in I}$ of non-commutative random variables living in some non-commutative probability space $(\mathfrak{M}, \tau)$, if we have
$$\lim_{n\to\infty} \operatorname{tr}_n(P(X^{(n)}(\omega))) = \tau(P(X)) \qquad \text{for } \mathbb{P}\text{-almost all } \omega \in \Omega$$

for each $P \in \mathbb{C}\langle x_i |\ i \in I\rangle$.

(ii) The sequence $(X^{(n)})_{n\in\mathbb{N}}$ is said to be *almost surely asymptotically free*, if $(X^{(n)})_{n\in\mathbb{N}}$ converges almost surely in distribution to a family $X = (X_i)_{i\in I}$ in some non-commutative probability space $(\mathfrak{M}, \tau)$, such that $(X_i)_{i\in I}$ are in addition freely independent in $(\mathfrak{M}, \tau)$.

Note that the expectation τ of the limiting space $(\mathfrak{M}, \tau)$ can be chosen to be tracial, since we can always restrict to the subalgebra $\mathfrak{M}_0$ of $\mathfrak{M}$ generated by $\{X_i |\ i \in I\}$, on which $\tau_0 := \tau|_{\mathfrak{M}_0}$ must be tracial since each tr_n is so.

Now, let us have a look at the case of a single random matrix.

Remark II.2.3. For each $n \in \mathbb{N}$, let $X^{(n)}$ be a random matrix in $(\mathfrak{M}_n, \tau_n)$, which is defined over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, let X be a self-adjoint non-commutative random variable living in some C^*-probability space $(\mathfrak{M}, \tau)$. Consider its analytic distribution μ_X, which is a compactly supported Borel probability measure on $\mathbb{R}$ and thus, in particular, determined by its moments; see Definition I.1.30 and Remark I.1.32. We record the following useful observations:

(i) The sequence $(X^{(n)})_{n\in\mathbb{N}}$ converges in distribution to X in the sense of Definition II.2.1, if and only if
$$\lim_{n\to\infty} \mathbb{E}\big[\,\mathrm{tr}_n\big((X^{(n)})^k\big)\big] = \tau(X^k) \qquad \text{for all } k \geq 0,$$
or equivalently, in terms of the mean empirical eigenvalue distributions $\bar{\mu}_{X^{(n)}}$, if
$$\lim_{n\to\infty} \int_{\mathbb{R}} t^k \, d\bar{\mu}_{X^{(n)}}(t) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \text{for all } k \geq 0.$$
With the help of Remark I.1.31, we conclude that $(\bar{\mu}_{X^{(n)}})_{n\in\mathbb{N}}$ converges even weakly to μ_X.

(ii) The sequence $(X^{(n)})_{n\in\mathbb{N}}$ converges almost surely in distribution to X in the sense of part (i) of Definition II.2.2, if and only if for each fixed integer $k \geq 0$
$$\lim_{n\to\infty} \mathrm{tr}_n\big(X^{(n)}(\omega)^k\big) = \tau(X^k) \qquad \text{for } \mathbb{P}\text{-almost all } \omega \in \Omega,$$
or equivalently, in terms of the random probability measures $\mu_{X^{(n)}}$, if for each fixed integer $k \geq 0$
$$\lim_{n\to\infty} \int_{\mathbb{R}} t^k \, d\mu_{X^{(n)}(\omega)}(t) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \text{for } \mathbb{P}\text{-almost all } \omega \in \Omega.$$
The latter means more explicitly, that we can find or each $k \geq 0$ a set $A_k \in \mathcal{F}$ with $\mathbb{P}(A_k) = 0$, such that
$$\lim_{n\to\infty} \int_{\mathbb{R}} t^k \, d\mu_{X^{(n)}(\omega)}(t) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \text{for all } \omega \in \Omega\backslash A_k.$$
If we put $A := \bigcup_{k\geq 0} A_k$, we obtain another set $A \in \mathcal{F}$ with the property $\mathbb{P}(A) = 0$, which is such that for all $\omega \in \Omega\backslash A$
$$\lim_{n\to\infty} \int_{\mathbb{R}} t^k \, d\mu_{X^{(n)}(\omega)}(t) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \text{for all } k \geq 0.$$
Hence, we see that for $\mathbb{P}$-almost all $\omega \in \Omega$ the moments of the sequence $(\mu_{X^{(n)}(\omega)})_{n\in\mathbb{N}}$ converge to the respective moments of μ_X. With the help of Remark I.1.31, we conclude that $(\mu_{X^{(n)}(\omega)})_{n\in\mathbb{N}}$ converges even weakly to μ_X for $\mathbb{P}$-almost all $\omega \in \Omega$.

In Chapter IV, we will be interested in certain non-commutative random variables X, for which the analytic distribution μ_X can be computed numerically – more precisely, we will explain how to find Borel probability measures $\mu_{X,\varepsilon}$, which are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ and which converge weakly to μ_X as $\varepsilon \searrow 0$. At the same time, random matrix models $X^{(n)}$ for X are available, so that we can use their eigenvalue distribution as an alternative approximation of the limiting distribution μ_X. The previous observations made in (i) and (ii) now explain, why we see in most case a striking similarity between the shape of the normalized eigenvalue histograms for $X^{(n)}$ for sufficiently large n and the approximating densities of μ_X. In fact, while the setting of (i) requires for this purpose to average over independent realizations of $X^{(n)}$, the stronger conditions imposed in (ii) guarantee that already one "generic" realization of $X^{(n)}$ will be sufficient.

II.3. Gaussian and Wishart random matrices

In Chapter IV, we will use random matrices as models for non-commutative distributions. Our models rely mostly on two, very prominent types of random matrices, namely self-adjoint Gaussian and Wishart random matrices. These random matrices are build out of Gaussian random variables and accordingly on the normal distribution. Recall that the *normal distribution* γ_{σ^2} *of variance* $\sigma^2 > 0$ is the absolutely continuous Borel probability measure on $\mathbb{R}$, which is given by

$$d\gamma_{\sigma^2}(x) = \frac{1}{\sqrt{2\sigma^2\pi}} \exp\Big(-\frac{x^2}{2\sigma^2}\Big)\, dx.$$

Let us agree on the following terminology.

Definition II.3.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be any probability space.

(i) A random variable $X \in L^{\infty-}(\Omega, \mathbb{P})$ is called a *real Gaussian random variable (of mean* 0 *and variance* σ^2*)*, if $X(\omega)$ is real for $\mathbb{P}$-almost all $\omega \in \Omega$ and if

$$\mathbb{E}[X^k] = \int_{\mathbb{R}} x^k \, d\gamma_{\sigma^2}(x) \qquad \text{for all } k \in \mathbb{N}_0.$$

(ii) A random variable $X \in L^{\infty-}(\Omega, \mathbb{P})$ is called a *complex Gaussian random variable*, if $\Re(X)$ and $\Im(X)$ are independent real Gaussian random variables.

Note that $\sigma^2 = \mathbb{E}[X^2]$ holds for each real Gaussian random variable X with mean 0 and variance σ^2. Thus, a Gaussian random variable with mean 0 is fully characterized by prescribing its second moment. This fact will be used below in Definition II.3.2 and Definition II.3.5.

Furthermore, let us recall some well-known fact from classical probability theory, saying that the normal distribution γ_{σ^2} of each variance σ^2 is determined by its moments in the sense of Definition I.1.30. This explains why we have introduced Gaussian random variables in terms of their moments.

II.3.1. Self-adjoint Gaussian random matrices and their asymptotic eigenvalue distribution.

Let us begin with the definition of self-adjoint Gaussian random matrices. More formally, we should call them standard self-adjoint complex Gaussian random matrices, but for the seek of simplicity and since there is no risk of confusion, we will suppress the terms "standard" and "complex" in the following.

Definition II.3.2. A random matrix $X = (X_{k,l})_{k,l=1}^n \in \mathfrak{M}_n$ is called a *self-adjoint Gaussian random matrix*, if X is self-adjoint and if

$$\{\Re(X_{k,l}) \mid 1 \le k \le l \le n\} \cup \{\Im(X_{k,l}) \mid 1 \le k < l \le n\}$$

are independent Gaussian random variables satisfying the following conditions:

- $\mathbb{E}[X_{k,l}] = 0$ for $k, l = 1, \dots, n$.
- $\mathbb{E}[(\Re(X_{k,l}))^2] = \frac{1}{2n}$ and $\mathbb{E}[(\Im(X_{k,l}))^2] = \frac{1}{2n}$ for $1 \le k < l \le n$.
- $\mathbb{E}[(X_{k,k})^2] = \frac{1}{n}$ for $1 \le k \le n$.

Note that this definition does not depend on the underlying probability space, since the properties of these random matrices are influenced only by the distribution of their entries and not by the concrete space on which they are realized.

As announced before, we have introduced here self-adjoint Gaussian random matrices as matrices that are build out of classical random variables. At this point, it is worth to take a look at the alternative picture, which describes self-adjoint Gaussian random matrices by introducing some probability measure on the space $M_n(\mathbb{C})_{\text{sa}}$ of all self-adjoint matrices of size $n \times n$ over $\mathbb{C}$. As a real vector space, $M_n(\mathbb{C})_{\text{sa}}$ is naturally isomorphic to $\mathbb{R}^{n^2}$ (by counting n degrees of freedom for the diagonal and $\frac{n(n-1)}{2}$ both for the real and the imaginary part of all entries above the diagonal, which gives in total the real dimension n^2). With respect to this fixed real basis, the Lebesgue measure λ^{n^2} on $\mathbb{R}^{n^2}$ can be transferred to $M_n(\mathbb{C})_{\text{sa}}$, yielding the measure

$$dX = \prod_{1 \le i \le n} dX_{i,i} \prod_{1 \le i < j \le n} d\Re(X_{i,j})\, d\Im(X_{i,j}).$$

Sometimes, it is more appropriate to rescale the chosen basis on $M_n(\mathbb{C})_{\text{sa}}$ in such a way that the isomorphism between $M_n(\mathbb{C})_{\text{sa}}$ and $\mathbb{R}^{n^2}$ becomes isometric if $M_n(\mathbb{C})_{\text{sa}}$ is endowed with the Hilbert-Schmidt norm and $\mathbb{R}^{n^2}$ with the usual Euclidean norm; this results then in the measure Λ_n on $M_n(\mathbb{C})_{\text{sa}}$, which is given by (see [**HP00b**])

$$d\Lambda_n(X) = 2^{\frac{n(n-1)}{2}} \prod_{1 \le i \le n} dX_{i,i} \prod_{1 \le i < j \le n} d\Re(X_{i,j})\, d\Im(X_{i,j}).$$

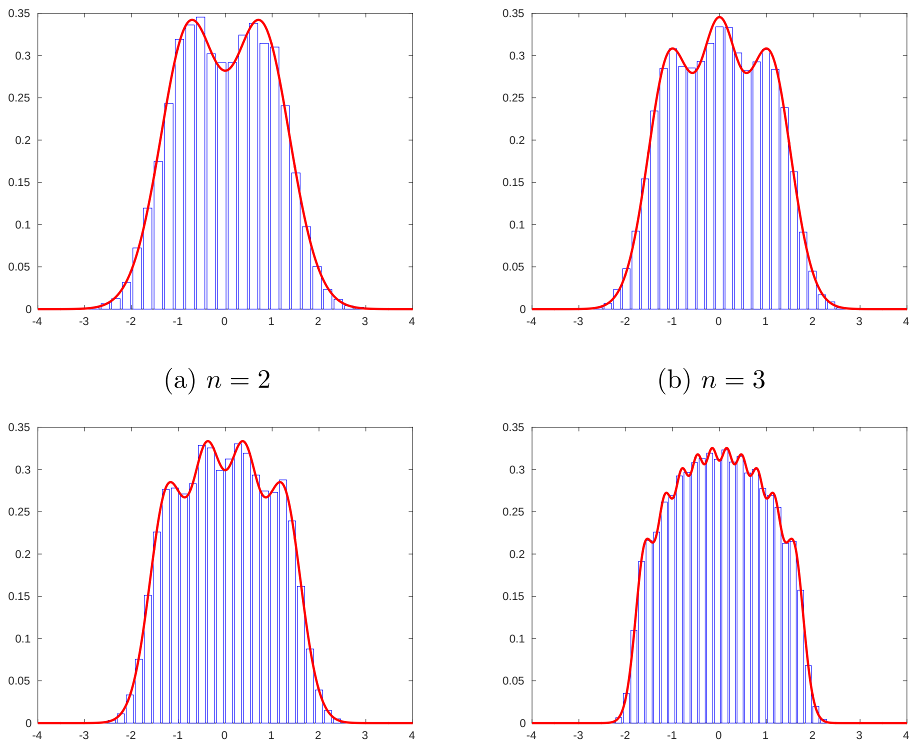
The self-adjoint Gaussian random matrices are distributed according to

$$C_n \exp\Big(-\frac{n}{2} \operatorname{Tr}_n(X^2)\Big)\, dX,$$

where Tr_n denotes the unnormalized trace on $M_n(\mathbb{C})$ and where we abbreviate $C_n := 2^{-n/2}\big(\frac{\pi}{n}\big)^{-n^2/2}$. The important fact that with X also UXU^* forms a self-adjoint Gaussian random matrix for any unitary matrix $U \in M_n(\mathbb{C})$ is reflected by the invariance of this measures under the mapping $X \mapsto UXU^*$.

We already mentioned earlier that computing the mean empirical eigenvalue distribution is a challenging task in general and that explicit formulas are known only in very few cases. One such case are self-adjoint Gaussian random matrices.

Remark II.3.3. If $X^{(n)} \in \mathfrak{M}_n$ is a self-adjoint Gaussian random matrix, it can be shown that $\bar{\mu}_{X^{(n)}}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, i.e. we have $d\bar{\mu}_{X^{(n)}}(t) = \rho_n(t)\, dt$. The density ρ_n can be obtained in the following way:



(a) $n = 2$ (b) $n = 3$

(c) $n = 4$ (d) $n = 10$

FIGURE II.1. Plot of ρ_n for different values of n, compared to the normalized histogram of eigenvalues for a self-adjoint Gaussian random matrix of size $n \times n$, averaged over 10000 independent realizations; see Remark II.3.3.

- For $n \in \mathbb{N}_0$, let $H_n : \mathbb{R} \to \mathbb{R}$ denote the *n-th Hermite polynomial*. The Hermite polynomials are defined recursively by $H_0(x) = 1$, $H_1(x) = x$ and
$$xH_n(x) = H_{n+1}(x) + nH_{n-1}(x) \qquad \text{for all } n \geq 1.$$
As this three term recurrence relation already suggests, these polynomials are orthogonal polynomials for some measure on $\mathbb{R}$. Indeed, the Hermite polynomials are orthogonal with respect to the Gaussian distribution $\gamma := \gamma_1$ and more precisely they satisfy
$$\int_{\mathbb{R}} H_n(x) H_m(x)\, d\gamma(x) = \delta_{n,m} n!.$$
- For $n \in \mathbb{N}_0$, we denote by $\psi_n : \mathbb{R} \to \mathbb{R}$ the *n-th Hermite function*. The Hermite functions are given by
$$\psi_n(x) := (2\pi)^{-\frac{1}{4}} (n!)^{-\frac{1}{2}} e^{-\frac{1}{4}x^2} H_n(x) \qquad \text{for all } n \geq 0.$$
The normalization is such that the Hermite functions $(\psi_n)_{n\geq 0}$ form an orthonormal basis of $L^2(\mathbb{R}, dx)$ with respect to the Lebesgue measure.
- For $n \in \mathbb{N}$, we consider the *n-th Hermite kernel* $K_n : \mathbb{R}^2 \to \mathbb{R}$, which is defined by
$$K_n(x, y) := \sum_{k=0}^{n-1} \psi_k(x)\psi_k(y).$$

- For $n \in \mathbb{N}$, the density ρ_n is given by

$$\rho_n(x) := n^{-\frac{1}{2}} K_n\big(n^{\frac{1}{2}}x, n^{\frac{1}{2}}x\big).$$

The densities ρ_n for the values $n = 2, 3, 4, 10$ are shown in Figure II.1.

The pictures shown in Figure II.1 suggest that the densities ρ_n of the mean empirical eigenvalue distributions converge to some limiting function as the size n of the corresponding random matrices tends to infinity. This is indeed true and one can prove this fact for instance by examining the asymptotic behavior of the involved Hermite kernels. Doing this, the limiting eigenvalue distribution turns out to be the semicircular distribution.

Let us now formulate the precise statement, which goes back to Wigner [**Wig55, Wig58**], but was improved to almost sure convergence in the work of Arnold [**Arn67**].

THEOREM II.3.4 (Wigner's semicircle law). *Let $(X^{(n)})_{n\in\mathbb{N}}$ be a sequence of self-adjoint Gaussian random matrices $X^{(n)} \in \mathfrak{M}_n$ over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, let S be a semicircular element in some C^*-probability space $(\mathfrak{M}, \tau)$, meaning that S is a self-adjoint non-commutative random variable in $\mathfrak{M}$, whose analytic distribution μ_S is given by the semicircular distribution σ_1 of mean 0 and variance 1 (see Definition I.1.43). Then the following holds true:*

(i) *The sequence $(X^{(n)})_{n\in\mathbb{N}}$ converges in distribution to S. In the sense of part (i) of Definition II.2.1, this means that*

$$\lim_{n\to\infty} \mathbb{E}[\mathrm{tr}_n((X^{(n)})^k)] = \tau(S^k) \qquad \text{for all } k \geq 0.$$

According to part (i) of Remark II.2.3, this means that the mean empirical eigenvalue distribution $\bar{\mu}_{X^{(n)}}$ converges weakly to the semicircular distribution σ_1 as $n \to \infty$.

(ii) *The sequence $(X^{(n)})_{n\in\mathbb{N}}$ almost surely converges in distribution to S. In the sense of part (i) of Definition II.2.2, this means that for each fixed integer $k \geq 0$*

$$\lim_{n\to\infty} \mathrm{tr}_n(X^{(n)}(\omega)^k) = \tau(S^k) \qquad \text{for } \mathbb{P}\text{-almost all } \omega \in \Omega.$$

According to part (ii) Remark II.2.3, this means that for $\mathbb{P}$-almost all $\omega \in \Omega$, the empirical eigenvalue distributions $\mu_{X^{(n)}(\omega)}$ converges weakly to the semicircular distribution σ_1 as $n \to \infty$.

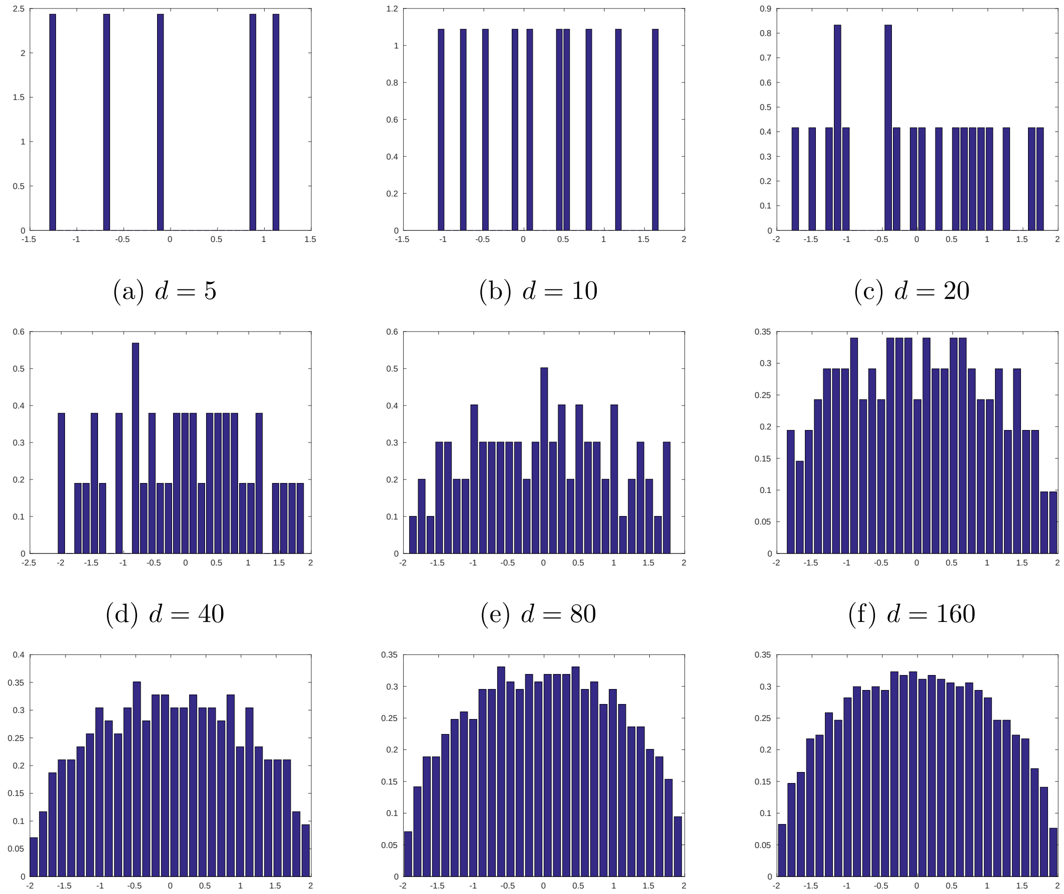
II.3.2. Wishart random matrices and their asymptotic eigenvalue distribution. Another important class of random matrices are self-adjoint Wishart matrices. A more accurate name for them would be standard self-adjoint complex Wishart matrix, but again for the seek of simplicity, we prefer to shorten this clumsy nomenclature.

DEFINITION II.3.5. Let $\{v_{k,l} \mid 1 \leq k \leq p,\ 1 \leq l \leq n\} \subset L^{\infty-}(\Omega, \mathbb{P})$ be given, such that

$$\{\Re(v_{k,l}) \mid 1 \leq k \leq p,\ 1 \leq l \leq n\} \cup \{\Im(v_{k,l}) \mid 1 \leq k \leq p,\ 1 \leq l \leq n\}$$

are independent Gaussian random variables satisfying

- $\mathbb{E}[v_{k,l}] = 0$
- $\mathbb{E}[(\Re(v_{k,l}))^2] = \dfrac{1}{2n}$ and $\mathbb{E}[(\Im(v_{k,l}))^2] = \dfrac{1}{2n}$



(a) $d = 5$ (b) $d = 10$ (c) $d = 20$

(d) $d = 40$ (e) $d = 80$ (f) $d = 160$

(g) $d = 320$ (h) $d = 640$ (i) $d = 1280$

FIGURE II.2. Normalized histograms of the eigenvalues of one realization of a self-adjoint Gaussian random matrix (see Definition II.3.2) for several dimensions d.

for $k = 1, \ldots, p$ and $l = 1, \ldots, n$. We put

$$V = (v_{k,l})_{\substack{k=1,\ldots,p \\ l=1,\ldots,n}}.$$

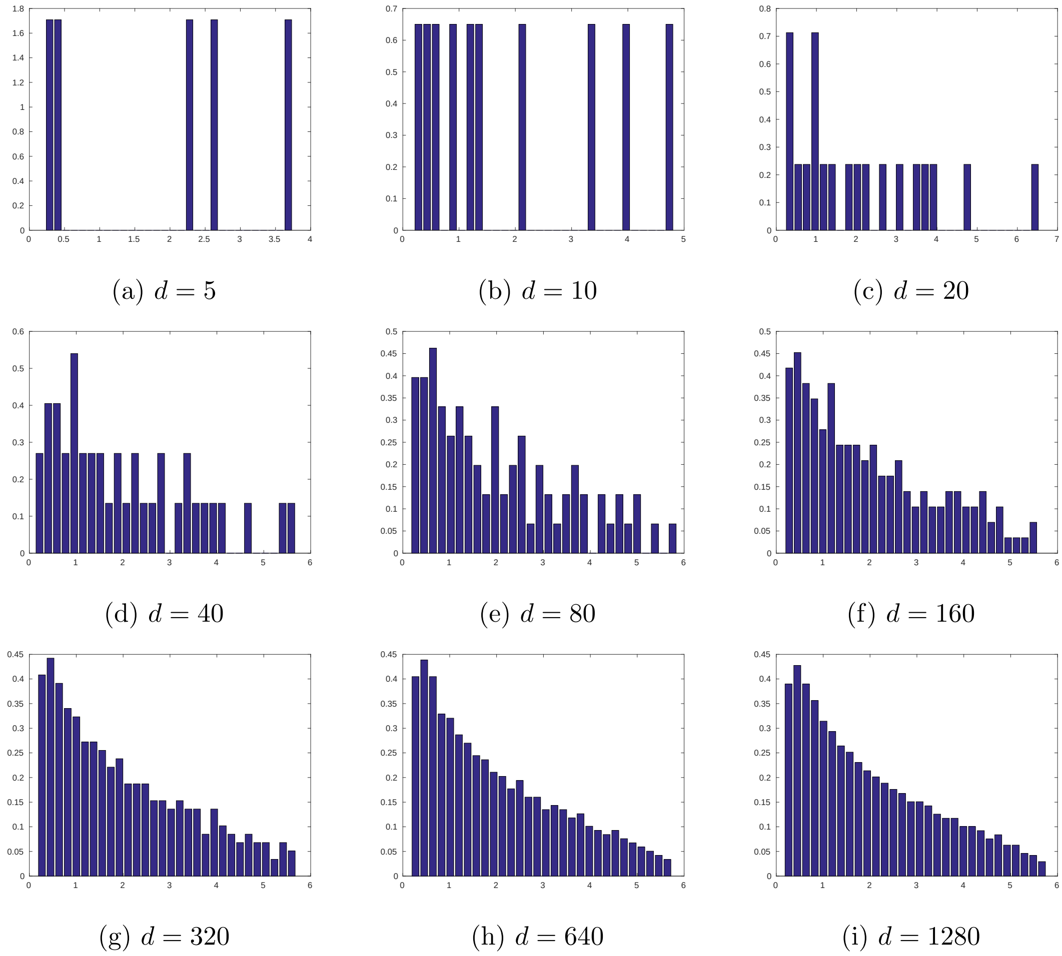
Then the matrix $X := V^*V \in \mathfrak{M}_n$ is called a *standard self-adjoint (complex) Wishart matrix*.

The eigenvalue distribution of Wishart random matrices has like in the case of Gaussian random matrices a deterministic limit. The following theorem, which goes back to [**MP68**], gives the precise statement.

THEOREM II.3.6 (Marchenko-Pastur law). *Let $(X^{(n)})_{n\in\mathbb{N}}$ be a sequence of standard Wishart matrices $X^{(n)} \in \mathfrak{M}_n$, which are given as $X^{(n)} = V_n^* V_n$, where V_n is a $p(n) \times n$ matrix of elements in $L^{\infty-}(\Omega, \mathbb{P})$. Assume additionally that the limit*

$$\lambda := \lim_{n\to\infty} \frac{p(n)}{n}$$

exists. Moreover, let W be a free Poisson element with rate λ and jump size $\alpha = 1$ in some C^*-probability space $(\mathfrak{M}, \tau)$, i.e., W is a self-adjoint non-commutative random variable in



(a) $d = 5$ (b) $d = 10$ (c) $d = 20$

(d) $d = 40$ (e) $d = 80$ (f) $d = 160$

(g) $d = 320$ (h) $d = 640$ (i) $d = 1280$

FIGURE II.3. Normalized histograms of the eigenvalues of one realization of a self-adjoint Wishart random matrix with rate $\lambda = 2$ (see Definition II.3.5) for several dimensions d.

$\mathfrak{M}$, whose analytic distribution μ_W is given by the Marchenko-Pastur distribution $\mu_{\lambda,1}$ (see Definition I.1.46). Then the following holds true:

(i) *The sequence $(X^{(n)})_{n\in\mathbb{N}}$ converges in distribution to W. In the sense of part (i) of Definition II.2.1, this means that*

$$\lim_{n\to\infty} \mathbb{E}[\mathrm{tr}_n((X^{(n)})^k)] = \tau(W^k) \qquad \text{for all } k \geq 0,$$

According to part (i) of Remark II.2.3, this means that the mean empirical eigenvalue distribution $\bar{\mu}_{X^{(n)}}$ converges weakly to the free Poisson distribution $\mu_{\lambda,1}$ as $n \to \infty$.

(ii) *The sequence $(X^{(n)})_{n\in\mathbb{N}}$ almost surely converges in distribution to W. In the sense of part (i) of Definition II.2.2, this means that for each fixed integer $k \geq 0$*

$$\lim_{n\to\infty} \mathrm{tr}_n(X^{(n)}(\omega)^k) = \tau(W^k) \qquad \text{for } \mathbb{P}\text{-almost all } \omega \in \Omega.$$

According to part (ii) of Remark II.2.3, this means that for $\mathbb{P}$-almost all $\omega \in \Omega$, the empirical eigenvalue distributions $\mu_{X^{(n)}(\omega)}$ converges weakly to the free Poisson distribution $\mu_{\lambda,1}$ as $n \to \infty$.

II.3.3. Asymptotic freeness of Gaussian and Wishart random matrices. So far, we have seen that the eigenvalue distribution of a single Gaussian or a single Wishart random matrix shows a nice asymptotic behavior if ifs dimension tends to infinity. However, typical questions in random matrix theory are concerned with more than only one random matrix. For instance, we could take N independent series of Gaussian random matrices, say $(X_1^{(n)})_{n\in\mathbb{N}},\dots,(X_N^{(n)})_{n\in\mathbb{N}}$, where *independence for random matrices* simply means that the their entries form independent sets of classical random variables. We know then from Wigner's semicircle law (see Theorem II.3.4) that the semicircular distributions shows up almost surely in the limit of each of these sequences, separately. But what happens, if we look instead at the sequence

$$(P(X_1^{(n)},\dots,X_N^{(n)}))_{n\in\mathbb{N}}$$

for any fixed non-commutative polynomial $P\in\mathbb{C}\langle x_1,\dots,x_N\rangle$, which is supposed to be self-adjoint? It is clear that this gives just another sequence of random matrices, but it is rather questionable whether its empirical eigenvalue distribution still shows a controllable behavior as $n\to\infty$. Surprisingly, it does, and the deterministic distribution that arises in the limit turns out to be the analytic distribution of

$$P(S_1,\dots,S_N),$$

for freely independent semicircular elements $S_1,\dots,S_N$. This phenomenon is explained by the following theorem due to Voiculescu [**Voi91**], which complements Wigner's semicircle law, Theorem II.3.4.

Theorem II.3.7 (Asymptotic freeness for self-adjoint Gaussian random matrices). *Let $(X^{(n)})_{n\in\mathbb{N}}$ be a sequence of N-tuples $X^{(n)}=(X_i^{(n)})_{1\leq i\leq N}$ of independent self-adjoint Gaussian random matrices $X_1^{(n)},\dots,X_N^{(n)}\in\mathfrak{M}_n$. Then $(X^{(n)})_{n\in\mathbb{N}}$ converges in distribution to an N-tuple $S=(S_1,\dots,S_N)$ of freely independent semicircular elements $S_1,\dots,S_N$, living in some non-commutative C^*-probability space $(\mathfrak{M},\phi)$.*

The phenomenon of asymptotic freeness is by no means limited to the case of self-adjoint Gaussian random matrices. In fact, a similar statement is also true for self-adjoint Wishart random matrices. This is the content of the next theorem; see [**HP00b, HP00a**] and [**Tho00**].

Theorem II.3.8 (Asymptotic freeness for self-adjoint Wishart random matrices). *Let $(X^{(n)})_{n\in\mathbb{N}}$ be a sequence of N-tuples $X^{(n)}=(X_i^{(n)})_{1\leq i\leq N}$ of independent self-adjoint Wishart matrices $X_1^{(n)},\dots,X_N^{(n)}\in\mathfrak{M}_n$. Then $(X^{(n)})_{n\in\mathbb{N}}$ converges in distribution to an N-tuple $(W_1,\dots,W_N)$ of freely independent Poisson elements $W_1,\dots,W_N$, living in some non-commutative C^*-probability space $(\mathfrak{M},\phi)$.*

II.3.4. Asymptotic freeness for unitarily invariant random matrices. The previously quoted theorems state that independent self-adjoint Gaussian random matrices and independent self-adjoint Wishart random matrices are asymptotically free separately. But what happens if we mix both types of matrices? Amazingly, asymptotic freeness even shows up in this generality. The reason is that matrices of both types are *unitarily invariant*. The latter means that the considered class of random matrices is stable under unitary conjugation $X\mapsto UXU^*$ for an arbitrary unitary matrix $U\in M_n(\mathbb{C})$. The following theorem, which is taken from [**HP00b**, Theorem 4.3.5], gives the precise statement. Notably, it supersedes both Theorem II.3.7 and Theorem II.3.8, and it covers also the mixed case.

THEOREM II.3.9 (Asymptotic freeness for unitarily invariant self-adjoint random matrices). *Let $(X^{(n)})_{n\in\mathbb{N}}$ be a sequence of families $X^{(n)} = (X_i^{(n)})_{i\in I}$ of independent self-adjoint random matrices $X_i^{(n)} \in \mathfrak{M}_n$ over some fixed index set $I \neq \emptyset$, which are unitarily invariant. Assume that for each $i \in I$ a compactly supported Borel probability measure μ_i on $\mathbb{R}$ exists, such that for each integer $k \geq 0$*

$$\lim_{n\to\infty} \operatorname{tr}_n((X_i^{(n)}(\omega))^k) = \int_{\mathbb{R}} t^k \, d\mu_i(t) \qquad \text{for } \mathbb{P}\text{-almost all } \omega \in \Omega.$$

Then $(X^{(n)})_{n\in\mathbb{N}}$ is almost surely asymptotically free.

II.4. Non-commutative functions in asymptotically free random matrices

Roughly speaking, all results collected in the previous subsections tell us that free independence arises in the limit out of classical independence for many interesting classes of random matrices. In this sense, the concept of asymptotic freeness completes the picture of Theorem II.3.4 and Theorem II.3.6. It bridges between random matrix theory and free probability as it puts questions concerning the limiting behavior of random matrices of the form

$$(f(X_1^{(n)}, \ldots, X_N^{(n)}))_{n\in\mathbb{N}},$$

with independent random matrices $X_1^{(n)}, \ldots, X_N^{(n)}$ and some "non-commutative function" f, into the free probability problem concerning the distribution of operators of the form

$$f(X_1, \ldots, X_N),$$

with freely independent non-commutative random variables $X_1, \ldots, X_N$. Depending on f, we need to impose different conditions on the convergence of the random matrix ensemble $(X_1^{(n)}, \ldots, X_N^{(n)})$ towards $(X_1, \ldots, X_N)$. These issues will be discussed in the next subsections.

II.4.1. Non-commutative polynomials. Let us consider first treat the case of non-commutative polynomials. With the following lemma, we make for f being a non-commutative polynomial the aforementioned relationship between $f(X_1^{(n)}, \ldots, X_N^{(n)})$ and $f(X_1, \ldots, X_N)$ more explicit.

LEMMA II.4.1. *For each $n \in \mathbb{N}$, let $X^{(n)} = (X_1^{(n)}, \ldots, X_N^{(n)})$ be an N-tuple of independent self-adjoint random matrices $X_1^{(n)}, \ldots, X_N^{(n)} \in \mathfrak{M}_n$ over $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that for each $i = 1, \ldots, N$ the empirical eigenvalue distribution of the sequence $(X_i^{(n)})_{n\in\mathbb{N}}$ converges almost surely to some limiting distribution in the sense that a compactly supported Borel probability measure μ_i on $\mathbb{R}$ exists, such that for each integer $k \geq 0$*

$$\lim_{n\to\infty} \operatorname{tr}_n((X_i^{(n)}(\omega))^k) = \int_{\mathbb{R}} t^k \, d\mu_i(t) \qquad \text{for } \mathbb{P}\text{-almost all } \omega \in \Omega.$$

Consider now any self-adjoint non-commutative polynomial $P \in \mathbb{C}\langle x_1, \ldots, x_N\rangle$. If $(X_1^{(n)}, \ldots, X_N^{(n)})_{n\in\mathbb{N}}$ are almost surely asymptotically free, then for $\mathbb{P}$-almost all $\omega \in \Omega$, the eigenvalue distribution of

$$P(X_1^{(n)}(\omega), \ldots, X_N^{(n)}(\omega))$$

converges weakly to $\mu = P^{\Box}(\mu_1, \ldots, \mu_N)$ as $n \to \infty$.

Proof. The assumption that the random matrices $(X_1^{(n)}, \dots, X_N^{(n)})$ are almost surely asymptotically free tells us according to Definition II.2.2 that we can find some tuple $X = (X_1, \dots, X_N)$ of freely independent elements $X_1, \dots, X_N$ in a non-commutative probability space $(\mathfrak{M}, \tau)$, such that for each $P \in \mathbb{C}\langle x_1, \dots, x_N \rangle$

$$\lim_{n\to\infty} \operatorname{tr}_n \left(P(X_1^{(n)}(\omega), \dots, X_N^{(n)}(\omega)) \right) = \tau(P(X_1, \dots, X_N)) \qquad \text{for } \mathbb{P}\text{-almost all } \omega \in \Omega$$

holds.

Consider now the non-commutative distribution $\mu_X : \mathbb{C}\langle x_1, \dots, x_N \rangle \to \mathbb{C}$ of X. The freeness condition gives us (see Remark I.1.37) that μ_X is completely determined by the individual distributions $\mu_{X_1}, \dots, \mu_{X_N} : \mathbb{C}\langle x \rangle \to \mathbb{C}$. Furthermore, for each $i = 1, \dots, N$,

$$\mu_{X_i}(x^k) = \int_{\mathbb{R}} t^k \, d\mu_i(t) \qquad \text{for each integer } k \geq 0,$$

since we have for $\mathbb{P}$-almost all $\omega \in \Omega$ that

$$\mu_{X_i}(x^k) = \tau(X_i^k) = \lim_{n\to\infty} \operatorname{tr}_n((X_i^{(n)}(\omega))^k) = \int_{\mathbb{R}} t^k \, d\mu_i(t).$$

This has the important consequence (see Remark I.1.38) that the non-commutative distribution μ_X can be realized on a C^*-probability space. More precisely, we can assume with no loss of generality that $(\mathfrak{M}, \tau)$ is a C^*-probability space and that $X_1, \dots, X_n$ are freely independent self-adjoint operators in $\mathfrak{M}$, whose analytic distributions are given by $\mu_1, \dots, \mu_N$.

Fix now a self-adjoint $P \in \mathbb{C}\langle x_1, \dots, x_N \rangle$ and consider the corresponding random matrices $Y^{(n)} := P(X_1^{(n)}, \dots, X_N^{(n)}) \in \mathfrak{M}_n$ and the operator $Y := P(X_1, \dots, X_N) \in \mathfrak{M}$. Given any integer $k \geq 0$, we have $Y^{(n)}(\omega)^k = P^k(X_1^{(n)}(\omega), \dots, X_N^{(n)}(\omega))$ for all $\omega \in \Omega$ and hence

$$\begin{aligned} \lim_{n\to\infty} \operatorname{tr}_n \left(Y^{(n)}(\omega)^k \right) &= \lim_{n\to\infty} \operatorname{tr}_n \left(P^k(X_1^{(n)}(\omega), \dots, X_N^{(n)}(\omega)) \right) \\ &= \tau(P^k(X_1, \dots, X_N)) \\ &= \tau(Y^k) \end{aligned}$$

for $\mathbb{P}$-almost all $\omega \in \Omega$. As the analytic distribution μ_Y of Y has compact support and is thus in particular determined by its moments (see Remark I.1.32), it follows by the observations made in Remark II.2.3 that the sequence of eigenvalue distributions $(\mu_{Y^{(n)}(\omega)})_{n\in\mathbb{N}}$ converges weakly to μ_Y for $\mathbb{P}$-almost all $\omega \in \Omega$. Since we have

$$\mu_Y = \mu_{P(X_1,\dots,X_N)} = P^{\square}(\mu_1, \dots, \mu_N),$$

this concludes the proof. ◻

This is clearly both of great theoretical and practical importance. If we take for instance the polynomial given by $P = x_1 + x_2$, then the limiting distribution can be computed by means of the free additive convolution $\boxplus$, and a polynomial like $P = x_1 x_2 x_1$ can be treated likewise by the free multiplicative convolution $\boxtimes$. Even for more general polynomials P, the abstract theory tells us that the distribution of $P(X_1, \dots, X_N)$ must be determined by P and the individual distributions of the variables $X_1, \dots, X_N$. However, there was for a long time no general machinery available that allows to make this relation explicit and finally computable. We will present a complete algorithmic solution with in Chapter IV.

However, while in the special case of a non-commutative polynomial P the convergence of the eigenvalue distribution of $P(X_1^{(n)}, \dots, X_N^{(n)})$ to $P(X_1, \dots, X_N)$ is an easy consequence of asymptotic freeness, the situation for more general test functions f, is much more intricate, as the next subsection will show.

II.4.2. Non-commutative rational expressions. Let us now take a look at the case of non-commutative rational expressions. We warn the reader that non-commutative rational expressions and related notions, such as domain and evaluation, are not yet defined. This will be done first in Subsection III.2.1 of the next Chapter III. We apologize for this inconsistency, but by postponing our subsequent discussion until then, we would take them out of their actual context. Any reader, who suspects a hidden circular reasoning, may skip this paragraph at first reading and may return after having worked through Subsection III.2.1, which should finally convince him of the contrary.

One of the main difficulties when trying to evaluate any non-commutative rational expression r at a given N-tuple $(X_1^{(n)},\dots,X_N^{(n)})$ of random matrices is that $(X_1^{(n)},\dots,X_N^{(n)})$ should almost surely belong to the domain of r if their dimension n is sufficiently large. More precisely, if $(X_1,\dots,X_N)$ belongs to the $\mathcal{A}$-domain of the non-commutative rational expression r, we want that $(X_1^{(n)},\dots,X_N^{(n)})$ *lies in the domain of* r *eventually*, that is, for $\mathbb{P}$-almost all $\omega\in\Omega$, we find some $n_\omega\in\mathbb{N}$, such that $(X_1^{(n)}(\omega),\dots,X_N^{(n)}(\omega))$ falls into the $M_n(\mathbb{C})$-domain of r for all $n\geq n_\omega$. Recent results due to Yin [**Yin16**] constitute a sufficient criterion, which is based on the notion of strong convergence of $((X_1^{(n)},\dots,X_N^{(n)}))_{n\in\mathbb{N}}$ to $(X_1,\dots,X_N)$. Though it applies equally well to random matrices, which are not self-adjoint, we restrict our attention to the self-adjoint case. The corresponding definition reads as follows.

Definition II.4.2. For each $n\in\mathbb{N}$, let $(X_1^{(n)},\dots,X_N^{(n)})$ be some N-tuple of self-adjoint random matrices $X_1^{(n)},\dots,X_N^{(n)}$ in $(\mathfrak{M}_n,\tau_n)$, constructed over some classical probability space $(\Omega,\mathcal{F},\mathbb{P})$. Given a C^*-probability space $(\mathfrak{M},\tau)$ with a faithful state τ and self-adjoint non-commutative random variables $X_1,\dots,X_N$, we say that $((X_1^{(n)},\dots,X_N^{(n)}))_{n\in\mathbb{N}}$ *converges strongly to* $(X_1,\dots,X_N)$, if the two following condition holds true:

(i) The sequence $((X_1^{(n)},\dots,X_N^{(n)}))_{n\in\mathbb{N}}$ almost surely converges in distribution to $(X_1,\dots,X_N)$ in the sense of Definition II.2.2.
(ii) For each $P\in\mathbb{C}\langle x_1,\dots,x_N\rangle$ (not necessarily self-adjoint), we have that

$$\lim_{n\to\infty}\|P(X_1^{(n)}(\omega),\dots,X_N^{(n)}(\omega))\|=\|P(X_1,\dots,X_N)\|\qquad\text{for }\mathbb{P}\text{-almost all }\omega\in\Omega.$$

The class of tuples of random matrices, for which strong convergence is known to hold true, contains tuples of self-adjoint Gaussian random matrices [**HT05**] and tuples of Wishart random matrices [**CD07**].

Let us now give the precise formulation of the beautiful results of [**Yin16**]. For reasons of simplicity, we decided to present the statement only in the self-adjoint case, which is sufficient for our purposes but does not exploit the full strength of [**Yin16**].

Theorem II.4.3. *For each $n\in\mathbb{N}$, let self-adjoint random matrices $X_1^{(n)},\dots,X_N^{(n)}$ in $(\mathfrak{M}_n,\tau_n)$ be given, where the non-commutative probability space $(\mathfrak{M}_n,\tau_n)$ is constructed over $(\Omega,\mathcal{F},\mathbb{P})$. Moreover, let $X_1,\dots,X_N$ be self-adjoint non-commutative random variables in some C^*-probability space $(\mathfrak{M},\tau)$ with a faithful state τ. Suppose that $((X_1^{(n)},\dots,X_N^{(n)}))_{n\in\mathbb{N}}$ converges strongly to $(X_1,\dots,X_N)$. If r is any non-commutative rational expression in the formal variables $x_1,\dots,x_N$, then the following statements holds true:*

(i) *As* $n \to \infty$, $(X_1^{(n)}, \dots, X_N^{(n)})$ *belongs eventually to the* $M_n(\mathbb{C})$*-domain of* r, *i.e., for* $\mathbb{P}$*-almost all* $\omega \in \Omega$, *we find some* $n_\omega \in \mathbb{N}$, *such that*

$$(X_1^{(n)}(\omega), \dots, X_N^{(n)}(\omega)) \in \operatorname{dom}_{M_n(\mathbb{C})}(r) \qquad \text{for all } n \geq n_\omega.$$

(ii) *For* $\mathbb{P}$*-almost all* $\omega \in \Omega$, *we have that*

$$\lim_{n\to\infty} \operatorname{tr}_n(r(X_1^{(n)}(\omega), \dots, X_N^{(n)}(\omega))) = \tau(r(X_1, \dots, X_N)) \qquad \text{and}$$

$$\lim_{n\to\infty} \|r(X_1^{(n)}(\omega), \dots, X_N^{(n)}(\omega))\| = \|r(X_1, \dots, X_N)\|.$$

Correspondingly, we have the following analogue of Lemma II.4.1.

LEMMA II.4.4. *For each* $n \in \mathbb{N}$, *let* $(X_1^{(n)}, \dots, X_N^{(n)})$ *be an* N*-tuple of independent self-adjoint random matrices* $X_1^{(n)}, \dots, X_N^{(n)} \in \mathfrak{M}_n$ *over* $(\Omega, \mathcal{F}, \mathbb{P})$. *Assume that* $((X_1^{(n)}, \dots, X_N^{(n)}))_{n\in\mathbb{N}}$ *converges strongly to* $(X_1, \dots, X_N)$ *in the sense of Definition II.4.2, where* $X_1, \dots, X_N$ *are self-adjoint non-commutative random variables in some* C^**-probability space* $(\mathfrak{M}, \tau)$ *that comes with a faithful state* τ. *Consider any non-commutative rational expression* r *in the formal variables* $x = (x_1, \dots, x_N)$, *such that the condition*

$$(X_1, \dots, X_N) \in \operatorname{dom}_{\mathfrak{M}}(r)$$

is satisfied. Then, for $\mathbb{P}$*-almost all* $\omega \in \Omega$, *there exists some* $n_\omega \in \mathbb{N}$, *such that*

$$(X_1^{(n)}(\omega), \dots, X_N^{(n)}(\omega)) \in \operatorname{dom}_{M_n(\mathbb{C})}(r) \qquad \text{for all } n \in \mathbb{N} \text{ with } n \geq n_\omega$$

holds, and the eigenvalue distribution of

$$r(X_1^{(n)}(\omega), \dots, X_N^{(n)}(\omega))$$

converges weakly to the analytic distribution μ *of* $r(X_1, \dots, X_N) \in \mathfrak{M}$ *as* $n \to \infty$.

PROOF. The above mentioned results of [**Yin16**] show that the strong convergence of $(X_1^{(n)}, \dots, X_N^{(n)})$ to the point $(X_1, \dots, X_N)$ in the $\mathfrak{M}$-domain of r implies, under the assumption that $(\mathfrak{M}, \tau)$ is a C^*-probability space with some faithful state τ, that $(X_1^{(n)}, \dots, X_N^{(n)})$ lies in the domain of r eventually. Recall that this means that we can find for $\mathbb{P}$-almost all $\omega \in \Omega$ some $n_\omega \in \mathbb{N}$, such that

$$(X_1^{(n)}(\omega), \dots, X_N^{(n)}(\omega)) \in \operatorname{dom}_{M_n(\mathbb{C})}(r) \qquad \text{for all } n \in \mathbb{N} \text{ with } n \geq n_\omega.$$

For such $\omega \in \Omega$ and $n \geq n_\omega$, we put $Y^{(n)}(\omega) := r(X_1^{(n)}(\omega), \dots, X_N^{(n)}(\omega))$. Now, if $k \geq 1$ is any integer, we clearly have $Y^{(n)}(\omega)^k = r^k(X_1^{(n)}(\omega), \dots, X_N^{(n)}(\omega))$, where r^k denotes the non-commutative rational expression, which is defined recursively by $r^1 := r$ and $r^k := r^{k-1} \cdot r$ for $k \geq 2$. If we put $Y := r(X_1, \dots, X_N) \in \mathfrak{M}$, we get

$$\begin{aligned}\lim_{n\to\infty} \operatorname{tr}_n\left(Y^{(n)}(\omega)^k\right) &= \lim_{n\to\infty} \operatorname{tr}_n\left(r^k(X_1^{(n)}(\omega), \dots, X_N^{(n)}(\omega))\right)\\ &= \tau(r^k(X_1, \dots, X_N))\\ &= \tau(Y^k)\end{aligned}$$

for $\mathbb{P}$-almost all $\omega \in \Omega$. As the analytic distribution μ_Y of Y has compact support and is thus in particular determined by its moments (see Remark I.1.32), it follows by the observations made in Remark II.2.3 that the sequence of eigenvalue distributions $(\mu_{Y^{(n)}(\omega)})_{n\in\mathbb{N}}$ converges weakly to $\mu = \mu_Y$ for $\mathbb{P}$-almost all $\omega \in \Omega$. This is exactly what we had to show. □

In the Brown measure case, despite the amazing similarity between the output of our algorithm and of the random matrix simulation (see Section IV.5), there is up to now no general statement, which would give a rigorous justification of this phenomenon, neither for non-commutative rational expression nor for the more basic case of non-commutative polynomials. However, it is conjectured to be true at least for non-commutative polynomials.

CHAPTER III

Linearization

This chapter is devoted some powerful technique of purely algebraic nature, which has become recently a very important tool in free probability theory. Here, it goes under the name "linearization trick" and it was introduced to this community by the work of Haagerup and Thorbjørnsen [**HT05**] and Haagerup, Schultz, and Thorbjørnsen [**HST06**], but some ideas can be traced back already to the early work of Voiculescu. Later, it got a fresh impetus by Anderson [**And12, And13, And15**].

All these techniques are strongly related and they have in common that they allow an effective treatment of non-commutative polynomials in terms of matrices whose entries are linear polynomials. More precisely, if p is any non-commutative polynomial in g formal non-commuting variables $x_1, \dots, x_g$, the method of linearization allows us to construct in an explicit way some linear expression of the form

$$L = L^{(0)} + L^{(1)} x_1 + \dots + L^{(g)} x_g,$$

where $L^{(0)}, L^{(1)}, \dots, L^{(g)}$ are complex matrices of some dimension N depending on p, such that, after evaluation of p in non-commutative random variables $X_1, \dots, X_g$, all "relevant information" about $p(X_1, \dots, X_g)$ is encoded in $L(X_1, \dots, X_g)$ and can be easily recovered from it.

From the viewpoint of free probability theory, this has the important consequence that we can reformulate questions about polynomial expressions $p(X_1, \dots, X_g)$, build in non-commutative random variables $X_1, \dots, X_g$, to questions about some linear expression of the form

$$L(X_1, \dots, X_g) = L^{(0)} + L^{(1)} X_1 + \dots + L^{(g)} X_g,$$

but to the price that the obtained linear expression has matricial coefficients. Therefore, at first sight, it is not clear that passing from $p(X_1, \dots, X_g)$ to the linearization $L(X_1, \dots, X_g)$ should cause a significant simplification in the treatment of $p(X_1, \dots, X_g)$. In order to see why this is indeed the case, one needs to take a closer look at the way how p relates to its linearization L. In fact, the "linearization trick" is built such that resolvents of $p(X_1, \dots, X_g)$ and $L(X_1, \dots, X_g)$ are connected in a very explicit way. This translates easily to some relation between the scalar-valued Cauchy transform of $p(X_1, \dots, X_g)$ and the operator-valued (in fact, matrix-valued) Cauchy transform of $L(X_1, \dots, X_g)$, where the latter puts the original scalar-valued problem in the setting of operator-valued free probability. Since operator-valued free probability theory evolves quite far in parallel to the scalar-valued theory and thus provides a similarly rich analytic toolbox, there are good reasons to hope that the linear expression $L(X_1, \dots, X_g)$ and thus its Cauchy transform can be understood within the operator-valued framework much easier than the initial polynomial expression $p(X_1, \dots, X_g)$ merely by means of scalar-valued free probability.

However, for some intended applications in free probability theory, the "linearization trick" of [**HT05, HST06**] showed the disadvantage that it was not able to preserve self-adjointness. By this, we mean that applying the "linearization trick" to a non-commutative polynomial $p \in \mathbb{C}\langle x_1, \ldots, x_g \rangle$, which is self-adjoint with respect to the canonical involution on $\mathbb{C}\langle x_1, \ldots, x_g \rangle$, leads in general to a linearization

$$L = L^{(0)} + L^{(1)} x_1 + \cdots + L^{(g)} x_g,$$

where the matrices $L^{(0)}, L^{(1)}, \ldots, L^{(g)}$ are not self-adjoint. In [**And12, And13, And15**], Anderson finally presented an improved version of the "linearization trick", which enjoys this additional feature. Beyond the fascinating applications, for which Anderson originally created this new framework, his "self-adjoint version of the linearization trick" was of great use also in the context of [**BMS13**]; this will be outlined in Chapter IV.

Nevertheless, as people in free probability became aware of recently, the methods, which are summarized under the name "linearization trick", are in fact not new, but were known outside their community since more than fifty years ago. Moreover, it became apparent that the method of linearization is by no means limited to non-commutative polynomials, but works equally well for another class of expressions, namely non-commutative rational expressions. Actually, it is not an easy task to locate the first appearance of these techniques, because they were rediscovered several times in other branches of mathematics, computer sciences, and engineering. Nevertheless, most people would arguably agree that the foundations were laid in the work of Schützenberger [**Sch61**] on automaton theory and non-commutative rational series, where the idea of linearizations appears in the context of *recognizable rational series*. Linearizations also show up in the work of Cohn [**Coh85, Coh06**], Cohn and Reutenauer [**CR94, CR99**], and Malcolmson [**Mal78, Mal80, Mal82**], on the skew-field of non-commutative rational functions. In their context, one is typically interested in *pure and linear representations*, but these are often composed to so-called *displays* and in this form they are pretty close to the linearizations as they appear in [**And12, And13, And15**]. Another near relative of linearizations are *non-commutative descriptor realizations* for (matrix-valued) non-commutative rational expressions, which are regular at zero. We refer to [**Kal63, Kal76, HMV06, KV09, KV12**].

Large parts of this chapter will follow the exposition of [**HMS15**], but it also continuous the work started therein, as far as its relations to the theory of non-commutative rational functions are concerned. This approach, which goes under the name of *formal linear representations*, is very much inspired by the language of linearizations [**And12, And13, And15**] and descriptor realizations [**HMV06**], but also goes beyond the case of non-commutative polynomials or regular non-commutative rational expressions by adapting constructions of Cohn and Reutenauer [**CR94, CR99**]. This will even cover operator-valued non-commutative rational expressions, as well as matrices of non-commutative rational expressions.

III.1. The linearization trick of Haagerup et al.

Let us start our excursion into the history of linearization with some of its great ancestors. Inside the free probability community, linearization found its initial spark in the work of Haagerup and Thorbjørnsen [**HT05**] and Haagerup, Schultz, and Thorbjørnsen [**HST06**]. Their "linearization trick", as it was presented in [**HST06**, Section 2], rests mainly on two pillars.

The first one provides a multiplicative decomposition of (matrix-valued) non-commutative polynomials into linear factors

Theorem III.1.1 (Proposition 2.1 in [**HST06**]). *For any matrix $p \in M_{n\times n'}(\mathbb{C}\langle x_1 \dots, x_g\rangle)$ of non-commutative polynomials of degree at most d, there exists a factorization*

$$p = u_1 u_2 \cdots u_d$$

with

- *matrices $u_j \in M_{m_j \times m_{j+1}}(\mathbb{C}\langle x_1, \dots x_g\rangle)$, $j = 1, \dots, d$, whose entries are polynomials of degree at most 1, and*
- *appropriate dimensions $m_1, m_2 \dots, \dots, m_d, m_{d+1} \geq 1$, subjected to $m_1 = n$ and $m_{d+1} = n'$.*

The second pillar is the following result, which uses a decomposition as found in the previous theorem (in the quadratic case $n = n'$) in order to construct the final linearization. Note that we will take the freedom to present a slight reformulation of their result in order to bring it in accordance with our terminology.

Theorem III.1.2 (Proposition 2.3 in [**HST06**]). *Let $p \in M_n(\mathbb{C}\langle x_1 \dots, x_g\rangle)$ be any square matrix of non-commutative polynomials of degree at most d and consider any factorization*

$$p = u_1 u_2 \cdots u_d$$

in the sense of Theorem III.1.1. Put $m := m_1 + \dots + m_d$. For each $\lambda \in \mathbb{C}$, we consider the matrix

$$A_\lambda := \begin{pmatrix} \lambda 1_{m_1} & -u_1 & 0 & 0 & \dots & 0 \\ 0 & 1_{m_2} & -u_2 & 0 & \dots & 0 \\ 0 & 0 & 1_{n_3} & -u_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1_{m_{d-1}} & -u_{d-1} \\ -u_d & 0 & 0 & \dots & 0 & 1_{m_d} \end{pmatrix}$$

in $M_m(\mathbb{C}\langle x_1, \dots, x_g\rangle)$. Now, if $X_1, \dots, X_g$ are arbitrary elements in some unital complex algebra $\mathcal{A}$, then $A_\lambda(X_1, \dots, X_g)$ is invertible in $M_m(\mathcal{A})$, if and only if $\lambda - p(X_1, \dots, X_g)$ is invertible in $\mathcal{A}$. In this case, the upper left $n \times n$ block of A_λ^{-1} is the resolvent $(\lambda - p)^{-1}$.

In the formulation of the previous theorem, we have used the natural evaluations

$$A_\lambda(X_1, \dots, X_g) := \mathrm{ev}_X^{(m)}(A_\lambda) \qquad \text{and} \qquad p(X_1, \dots, X_g) := \mathrm{ev}_X^{(n)}(p),$$

where $\mathrm{ev}_X : \mathbb{C}\langle x_1, \dots, x_g\rangle \to \mathcal{A}$ denotes the evaluation homomorphism at the point $X = (X_1, \dots, X_g)$, as it was introduced in part (i) of Definition I.1.12, and $\mathrm{ev}_X^{(k)} : M_k(\mathbb{C}\langle x_1, \dots, x_g\rangle) \to M_k(\mathcal{A})$ for any $k \in \mathbb{N}$ its canonical amplification in the sense of part (iii) of Remark I.2.5.

The statements, which we have collected in Theorem III.1.2, incarnate the brilliant linearization trick: it allows us to construct for each given square matrix p of non-commutative polynomials another square matrix A_λ, which is typically of larger size but has now the advantage that its entries are all (affine) linear polynomials; and this matrix A_λ is constructed in such a way, that the inverse of its evaluation $A_\lambda(X_1, \dots, X_N)$ at $(X_1, \dots, X_g)$ contains – if it exists – the resolvent of $p(X_1, \dots, X_g)$ at λ.

Theorem III.1.2 has laid the ground to spectacular results about random matrices and operator-algebras. Its proof, however, was based on some direct but rather tedious computations. Although it was not mentioned there explicitly by the authors, the proof of Theorem III.1.2 as given in [**HST06**] follows a much more general concept, the so-called

Schur complement formula, Lemma A.1; see Appendix A for the precise statement and its proof. Using this tool, we can give a simplified proof of Theorem III.1.2. Indeed, decomposing A_λ as

$$A_\lambda = \begin{pmatrix} \lambda 1_{m_1} & u \\ v & Q \end{pmatrix} \tag{III.1}$$

with

$$u := \begin{pmatrix} -u_1 & 0 & \dots & 0 \end{pmatrix}, \quad Q := \begin{pmatrix} 1_{m_2} & -u_2 & 0 & \dots & 0 \\ 0 & 1_{m_3} & -u_3 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1_{m_{d-1}} & -u_{d-1} \\ 0 & 0 & \dots & 0 & 1_{m_d} \end{pmatrix}, \quad \text{and} \quad v := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -u_d \end{pmatrix}$$

allows us to check inductively and with the help of Lemma A.1 that Q is invertible and that Q^{-1} is again an upper triangular matrix, whose $(1, d-1)$-block is of the form $u_2 \cdots u_{d-1}$. Thus, we have $uQ^{-1}v = u_1 u_2 \cdots u_{d-1} u_d = p$, and applying the Schur complement formula (A.1) gives the assertion.

In the light of both Lemma A.1 and the decomposition (III.1), it becomes apparent that the decomposition $p = uQ^{-1}v$ is at the heart of the linearization trick. Indeed, due to the Schur complement formula, Lemma A.1, it is exactly this relation $p = uQ^{-1}v$, which allows us to connect the resolvent $(\lambda - p(X_1, \dots, X_g))^{-1}$ with the inverse of $A_\lambda(X_1, \dots, X_g)$. Note that, since we prefer to write $A_\lambda(X_1, \dots, X_g)$ also as some kind of resolvent, we will change below the convention on the signs, such that the decomposition of p will rather look like $p = -uQ^{-1}v$.

It was kindly brought to our attention by J. W. Helton and V. Vinnikov that even non-commutative rational functions admit the same kind of representation. Therefore, the natural question came up whether results from [**BMS13**] can be extended to non-commutative rational functions. This was the initial ignition for some of the investigations that will be explained in the remaining part of this chapter and also in Chapter IV.

Notably, our discussion here is devoted only to the decomposition $r = -uQ^{-1}v$ with an affine linear pencil $Q = Q^{(0)} + Q^{(1)}x_1 + \dots + Q^{(g)}x_g$. The actual linearization L, which is itself an affine linear pencil $L = L^{(0)} + L^{(1)}x_1 + \dots + L^{(g)}x_g$ given by

$$L := \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix} = \begin{pmatrix} 0 & u \\ v & Q^{(0)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & Q^{(1)} \end{pmatrix} x_1 + \dots + \begin{pmatrix} 0 & 0 \\ 0 & Q^{(g)} \end{pmatrix} x_g,$$

will be discussed in Chapter IV.

III.2. Non-commutative rational expressions and functions

Inside the field of free probability theory, the story of the "linearizations trick" was after [**HT05, HST06**] continued by the work of Anderson [**And12, And13, And15**], who noticed the Schur complement formula, Lemma A.1, behind the scenes of [**HT05, HST06**] and the relevance of the decomposition $p = -uQ^{-1}v$. He used this insight to build up a general framework for linearizations, which was flexible enough to construct even self-adjoint linearizations for self-adjoint non-commutative polynomials. Anderson's ideas were taken up later in [**BMS13**] and developed further in [**HMS15**], with an eye towards applications of these methods in the realm of free probability. The aim of this section is to present the idea of linearization in the special language of [**HMS15**]. This allows us to treat not only non-commutative polynomials but also non-commutative rational expressions without any additional effort. Since we restrict ourselves here first to the case of single scalar-valued rational expressions – the more general case of matrices of non-commutative rational expressions and operator-valued non-commutative rational expressions will be discussed in Section III.3 – our approach will show strong relations to some similar constructions, which appeared before in the literature on non-commutative rational functions.

III.2.1. Non-commutative rational expressions and their evaluations. A rational expression is loosely speaking obtained by taking repeatedly sums, products and inverses, starting from scalars and some formal variables, without taking care about possible cancellations or resulting mathematical inconsistencies. More formally, the definitions reads as follows; see for instance [**Vol15**, Section 2].

Definition III.2.1. Let $x = (x_1, \dots, x_g)$ be a g-tuple of formal variables. A *(non-commutative) rational expression in* x is a syntactically valid combination of

- scalars $\lambda \in \mathbb{C}$ and the variables $x_1, \dots, x_g$,
- the arithmetic operations $+, \cdot, ^{-1}$, and
- parentheses $(,)$.

In the following, the set of all non-commutative rational expressions in x will be denoted by $\mathfrak{R}_\mathbb{C}(x)$.

We expect that some readers are not completely satisfied with this definition, as it relies on a presumed understanding of the term "syntactically valid". Thus, without going into details, we refer here also to [**GGOW15**, Section 3.3.1] and to the references collected therein (especially to [**HW15**]), where an alternative approach based on the graph theoretical notion of *circuits* is presented.

Indeed, there is one subtlety hidden behind the term "syntactically valid", which turns out to be crucial in what follows: we will always assume that parentheses are placed in such a way that sums and products of more than two rational expressions without a prescribed order are avoided. For example, we prefer to exclude $x_1 \cdot x_2 \cdot x_2$, because it might stand both for $(x_1 \cdot x_2) \cdot x_2$ and for $x_1 \cdot (x_2 \cdot x_2)$. This has the following easy but important consequence.

Remark III.2.2. Let $\mathfrak{R}$ be any subset of $\mathfrak{R}_\mathbb{C}(x)$, which has the following properties:

(i) $\mathfrak{R}$ contains all scalars $\lambda \in \mathbb{C}$ and the variables $x_1, \dots, x_g$.
(ii) $\mathfrak{R}$ is closed under the operation $^{-1}$ in the sense that $r \in \mathfrak{R}$ implies $r^{-1} \in \mathfrak{R}$.
(iii) $\mathfrak{R}$ is closed under the binary operations $+, \cdot$ in the sense that $r_1, r_2 \in \mathfrak{R}$ implies both $r_1 + r_2 \in \mathfrak{R}$ and $r_1 \cdot r_2 \in \mathfrak{R}$.

In this case, we must necessarily have that $\mathfrak{R} = \mathfrak{R}_\mathbb{C}(x)$.

Note that even defining $+$ and $\cdot$ as binary operations on $\mathfrak{R}_\mathbb{C}(x)$ requires some care, since one sometimes needs to insert additional parentheses in order to avoid ambiguities in the resulting rational expressions. For example, if we consider the non-commutative rational expressions $r_1 := x_1 \cdot x_2$ and $r_2 := x_2$, then $r_1 \cdot r_2$ results in $(x_1 \cdot x_2) \cdot x_2$, while $r_2 \cdot r_2$ gives $x_2 \cdot x_2$. It is certainly not surprising that brackets should be used when adding and multiplying arithmetic expressions, but the difference here is that, since elements of the underlying object $\mathfrak{R}_{\mathbb{C}}(x)$ are itself arithmetic expressions, brackets can appear in two different meanings – and the ones that we must insert if necessary are not part of the arithmetic over $\mathfrak{R}_{\mathbb{C}}(x)$ but are rather those that are intrinsic in $\mathfrak{R}_{\mathbb{C}}(x)$. Let us point out that these operations become more transparent in the graph theoretical approach to non-commutative rational expressions based on the aforementioned notion of circuits.

In any case, Definition III.2.1 explicitly includes rational expressions of the form 0^{-1} and $(x_1 + (-1) \cdot x_1)^{-1}$. This might appear strange at the first sight, since 0^{-1} should not be defined in any reasonable way. However, it simply highlights the crucial difference between formal operations on the side of rational expressions and the corresponding operations in the range of their evaluations, where inverses appearing in a rational expression may cause an restriction of its domain. The following definition seeks to clarify this difference by introducing domains and evaluations of rational expressions. As we will see, it is appropriate to introduce both notions at once. Thus, conceptually, our definition is very close to [**KV09**, Definition 2.1], which treats especially evaluations on matrix algebras. Similar notions – though not formulated in this generality – were surveyed in [**GR92**].

Definition III.2.3. Let $\mathcal{A}$ be any unital complex algebra with unit $1_{\mathcal{A}}$. For any non-commutative rational expression r in the formal variables $x = (x_1, \dots, x_g)$, we define its $\mathcal{A}$-*domain* $\operatorname{dom}_{\mathcal{A}}(r)$ together with its *evaluation* $\operatorname{ev}_X(r)$ *for any* $X = (X_1, \dots, X_g) \in \operatorname{dom}_{\mathcal{A}}(r)$ by the following rules:

(i) For any $\lambda \in \mathbb{C}$, we put $\operatorname{dom}_{\mathcal{A}}(\lambda) = \mathcal{A}^g$ and $\operatorname{ev}_X(\lambda) = \lambda 1_{\mathcal{A}}$.
(ii) For $i = 1, \dots, g$, we put $\operatorname{dom}_{\mathcal{A}}(x_i) = \mathcal{A}^g$ and $\operatorname{ev}_X(x_i) = X_i$.
(iii) If r_1, r_2 are rational expressions in x, we have
$$\operatorname{dom}_{\mathcal{A}}(r_1 \cdot r_2) = \operatorname{dom}_{\mathcal{A}}(r_1) \cap \operatorname{dom}_{\mathcal{A}}(r_2)$$
and
$$\operatorname{ev}_X(r_1 \cdot r_2) = \operatorname{ev}_X(r_1) \cdot \operatorname{ev}_X(r_2).$$
(iv) If r_1, r_2 are rational expressions in x, we have
$$\operatorname{dom}_{\mathcal{A}}(r_1 + r_2) = \operatorname{dom}_{\mathcal{A}}(r_1) \cap \operatorname{dom}_{\mathcal{A}}(r_2)$$
and
$$\operatorname{ev}_X(r_1 + r_2) = \operatorname{ev}_X(r_1) + \operatorname{ev}_X(r_2).$$
(v) If r is a rational expression in x, we have
$$\operatorname{dom}_{\mathcal{A}}(r^{-1}) = \{X \in \operatorname{dom}_{\mathcal{A}}(r) \mid \operatorname{ev}_X(r) \text{ is invertible in } \mathcal{A}\}$$
and
$$\operatorname{ev}_X(r^{-1}) = \operatorname{ev}_X(r)^{-1}.$$

In the following, for any given rational expression r and each $X \in \operatorname{dom}_{\mathcal{A}}(r)$, we will mostly abbreviate $r(X) := \operatorname{ev}_X(r)$.

At this point, it is worth to take a look at some important subclass of non-commutative rational expressions, namely non-commutative polynomial expressions.

Definition III.2.4. Let $x = (x_1, \dots, x_g)$ be a g-tuple of formal variables. A *(non-commutative) polynomial expression in* x is a syntactically valid combination of

- scalars $\lambda \in \mathbb{C}$ and the variables $x_1, \dots, x_g$,
- the arithmetic operations $+, \cdot$, and
- parentheses $($, $)$.

In the following, the set of all non-commutative polynomial expressions in x will be denoted by $\mathfrak{P}_{\mathbb{C}}(x)$.

Note that clearly $\mathfrak{P}_{\mathbb{C}}(x) \subset \mathfrak{R}_{\mathbb{C}}(x)$ and that $\mathrm{dom}_{\mathcal{A}}(p) = \mathcal{A}^g$ for each $p \in \mathfrak{P}_{\mathbb{C}}(x)$ and for any unital complex algebra $\mathcal{A}$.

We already know from Definition I.1.12 the unital complex algebra $\mathbb{C}\langle x_1, \dots, x_g\rangle$ of non-commutative polynomials in the variables $x_1, \dots, x_g$, but there is a slight difference between non-commutative polynomials and non-commutative polynomial expressions, namely that all arithmetic rules of $\mathbb{C}\langle x_1, \dots, x_g\rangle$ are ignored in $\mathfrak{P}_{\mathbb{C}}(x)$. For example, $p_1 = x_1 + x_2$ and $p_2 = x_2 + x_1$ are two different non-commutative polynomial expressions, though they "represent" the same non-commutative polynomial in $\mathfrak{p} \in \mathbb{C}\langle x_1, \dots, x_g\rangle$. However, the usage of the term "represent" here is – though quite intuitive – not yet justified by a rigorous definition. Formally, we say that $\mathfrak{p} \in \mathbb{C}\langle x_1, \dots, x_g\rangle$ *is represented by* $p \in \mathfrak{P}_{\mathbb{C}}(z)$ (where we rename for the moment the variables of our non-commutative polynomial expressions by $z = (z_1, \dots, z_g)$, for reasons of clarity), if the evaluation of p at the point $x = (x_1, \dots, x_g)$ over the complex unital algebra $\mathbb{C}\langle x_1, \dots, x_g\rangle$ gives $\mathfrak{p} = p(x)$. We will come back to this in Definition III.2.46.

Due to $\mathfrak{P}_{\mathbb{C}}(x) \subset \mathfrak{R}_{\mathbb{C}}(x)$, the concept of (scalar-valued) formal linear representations, which we are going to present in Subsection III.2.3 and which was originally developed in [**HMS15**], reformulates and generalizes the linearization trick for non-commutative polynomials that was used in [**BMS13**]. Even more, it highlights the therein disregarded subtlety that building a concrete linearization L in the sense of [**BMS13**] for some given $\mathfrak{p} \in \mathbb{C}\langle x_1, \dots, x_g\rangle$ actually requires to choose at first – as a kind of construction plan for L – some non-commutative polynomial expressions $p \in \mathfrak{P}_{\mathbb{C}}(x)$ that represents $\mathfrak{p}$.

For later use, let us record the following observation.

REMARK III.2.5. There is a natural involution $*$ on $\mathfrak{R}_{\mathbb{C}}(x)$, which is (according to Remark III.2.2) uniquely determined by the assumptions that

- $\lambda^* = \overline{\lambda}$ for all $\lambda \in \mathbb{C}$,
- $x_i^* = x_i$ for $i = 1, \dots, g$,
- $(r_1 + r_2)^* = r_1^* + r_2^*$ and $(r_1 \cdot r_2)^* = r_2^* \cdot r_1^*$,
- $(r^{-1})^* = (r^*)^{-1}$.

Accordingly, we could agree to call a non-commutative rational expression r self-adjoint, if $r = r^*$ holds in $\mathfrak{R}_{\mathbb{C}}(x)$ with respect to the involution $*$ introduced in Remark III.2.5. This notion, however, turns out to be much too restrictive as the following examples show.

EXAMPLE III.2.6.

(i) Consider the non-commutative rational expression $r = (x_1 \cdot x_2) \cdot (x_2 \cdot x_1)$. Because
$$r^* = (x_2 \cdot x_1)^* \cdot (x_1 \cdot x_2)^* = (x_1^* \cdot x_2^*) \cdot (x_2^* \cdot x_1^*) = (x_1 \cdot x_2) \cdot (x_2 \cdot x_1) = r,$$
we see that r would be self-adjoint in the sense that $r = r^*$.

(ii) For the non-commutative rational expression $r = x_1 \cdot x_2 + x_2 \cdot x_1$, we can check that
$$r^* = (x_1 \cdot x_2)^* + (x_2 \cdot x_1)^* = x_2^* \cdot x_1^* + x_1^* \cdot x_2^* = x_2 \cdot x_1 + x_1 \cdot x_2,$$
which is different from r on the purely formal level of rational expressions. Hence, r would not be self-adjoint in the sense that $r \neq r^*$.

Since our intuition says that any reasonable definition should identify $r = x_1 \cdot x_2 + x_2 \cdot x_1$ as a self-adjoint non-commutative rational expression, we prefer instead the following, slightly more sophisticated approach.

Definition III.2.7. Let r be a non-commutative rational expression in the formal variables $x = (x_1, \dots, x_g)$.

(i) If $\mathcal{A}$ is a unital complex $*$-algebra, we denote by $\operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r)$ the subset of $\operatorname{dom}_{\mathcal{A}}(r)$, which consists of all points $X = (X_1, \dots, X_g) \in \operatorname{dom}_{\mathcal{A}}(r)$ satisfying $X = X^*$, where we put $X^* := (X_1^*, \dots, X_g^*)$.
(ii) We say that the rational expression r is *self-adjoint*, if for any unital complex $*$-algebra $\mathcal{A}$ and for any $X \in \operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r)$ it holds true that $r(X)^* = r(X)$.

One easily sees that $r = x_1 \cdot x_2 + x_2 \cdot x_1$ becomes self-adjoint on the basis of this improved definition. Indeed, for any unital complex $*$-algebra $\mathcal{A}$ and each point $X = (X_1, X_2) \in \operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r)$, we have that $r(X)^* = (X_1X_2 + X_2X_1) = X_1X_2 + X_2X_1 = r(X)$.

Note that Definition III.2.7 is built according to our needs, in particular in Section III.2.4. It therefore slightly differs from the usual terminology of symmetric rational expressions in the real case, as used for instance in [**HMV06**]. We will see in the next lemma that any non-commutative rational expression r, which is self-adjoint in the naive sense that $r = r^*$, remains self-adjoint in the new and final terminology of Definition III.2.7.

Lemma III.2.8. *Let r be a non-commutative rational expression in the formal variables $x = (x_1, \dots, x_g)$ and consider the non-commutative rational expression r^* obtained according to Remark III.2.5. Then, for each unital complex $*$-algebra $\mathcal{A}$, we have that*

$$\operatorname{dom}_{\mathcal{A}}(r^*) = \{X^* |\ X \in \operatorname{dom}_{\mathcal{A}}(r)\}$$

and $r^*(X^*) = r(X)^*$ for any $X \in \operatorname{dom}_{\mathcal{A}}(r)$. In particular, for any unital complex $*$-algebra $\mathcal{A}$, $\operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r) = \operatorname{dom}_{\mathcal{A}}(r) \cap \operatorname{dom}_{\mathcal{A}}(r^*) = \operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r^*)$ is satisfied and the equality $r^*(X) = r(X)^*$ holds true at any point X in this joint domain.

Proof. Denote by $\mathfrak{R}$ the subset of $\mathfrak{R}_{\mathbb{C}}(x)$ consisting of all non-commutative rational expressions in $\mathfrak{R}_{\mathbb{C}}(x)$, which have the property that for each unital complex $*$-algebra $\mathcal{A}$

- $\operatorname{dom}_{\mathcal{A}}(r^*) = \{X^* |\ X \in \operatorname{dom}_{\mathcal{A}}(r)\}$ holds and
- $r^*(X^*) = r(X)^*$ at any point $X \in \operatorname{dom}_{\mathcal{A}}(r)$.

We want to show that $\mathfrak{R} = \mathfrak{R}_{\mathbb{C}}(x)$. Since obviously all scalars $\lambda \in \mathbb{C}$ and each of the variables $x_1, \dots, x_g$ belongs to $\mathfrak{R}$, it is sufficient to prove that the set $\mathfrak{R}$ is closed under the arithmetic operations $+$, $\cdot$, and $^{-1}$. Indeed, according to the properties of ev_X and $\operatorname{dom}_{\mathcal{A}}$ collected in Definition III.2.3 and the definition of $*$ in Remark III.2.5, we see that

- if $r_1, r_2 \in \mathfrak{R}$ are given, then $r_1 + r_2 \in \mathfrak{R}$ holds: since we have $(r_1 + r_2)^* = r_1^* + r_2^*$, it follows that

$$\begin{aligned}\operatorname{dom}_{\mathcal{A}}\big((r_1 + r_2)^*\big) &= \operatorname{dom}_{\mathcal{A}}(r_1^*) \cap \operatorname{dom}_{\mathcal{A}}(r_2^*)\\ &= \{X_1^* |\ X_1 \in \operatorname{dom}_{\mathcal{A}}(r_1)\} \cap \{X_2^* |\ X_2 \in \operatorname{dom}_{\mathcal{A}}(r_2)\}\\ &= \{X^* |\ X \in \operatorname{dom}_{\mathcal{A}}(r_1) \cap \operatorname{dom}_{\mathcal{A}}(r_2)\}\\ &= \{X^* |\ X \in \operatorname{dom}_{\mathcal{A}}(r_1 + r_2)\},\end{aligned}$$

and hence for all $X \in \operatorname{dom}_{\mathcal{A}}(r_1 + r_2)$

$$(r_1 + r_2)^*(X^*) = r_1^*(X^*) + r_2^*(X^*) = r_1(X)^* + r_2(X)^* = (r_1(X) + r_2(X))^* = (r_1 + r_2)(X)^*.$$

- if $r_1, r_2 \in \mathfrak{R}$ are given, then $r_1 \cdot r_2 \in \mathfrak{R}$ holds: since we have $(r_1 \cdot r_2)^* = r_2^* \cdot r_1^*$, it follows that

$$\begin{aligned}\operatorname{dom}_{\mathcal{A}}\big((r_1 \cdot r_2)^*\big) &= \operatorname{dom}_{\mathcal{A}}(r_2^*) \cap \operatorname{dom}_{\mathcal{A}}(r_1^*)\\ &= \{X_1^* \mid X_1 \in \operatorname{dom}_{\mathcal{A}}(r_1)\} \cap \{X_2^* \mid X_2 \in \operatorname{dom}_{\mathcal{A}}(r_2)\}\\ &= \{X^* \mid X \in \operatorname{dom}_{\mathcal{A}}(r_1) \cap \operatorname{dom}_{\mathcal{A}}(r_2)\}\\ &= \{X^* \mid X \in \operatorname{dom}_{\mathcal{A}}(r_1 \cdot r_2)\},\end{aligned}$$

and hence for all $X \in \operatorname{dom}_{\mathcal{A}}(r_1 \cdot r_2)$

$$(r_1 \cdot r_2)^*(X^*) = r_2^*(X^*) r_1^*(X^*) = r_2(X)^* r_1(X)^* = (r_1(X) r_2(X))^* = (r_1 \cdot r_2)(X)^*.$$

- if $r \in \mathfrak{R}$ is given, then $r^{-1} \in \mathfrak{R}$ holds: by definition, we have $(r^{-1})^* = (r^*)^{-1}$, and so it follows that

$$\begin{aligned}\operatorname{dom}_{\mathcal{A}}\big((r^{-1})^*\big) &= \operatorname{dom}_{\mathcal{A}}\big((r^*)^{-1}\big)\\ &= \{X \in \operatorname{dom}_{\mathcal{A}}(r^*) \mid r^*(X) \text{ is invertible}\}\\ &= \{X^* \mid X \in \operatorname{dom}_{\mathcal{A}}(r), \text{ such that } r^*(X^*) = r(X)^* \text{ is invertible}\}\\ &= \{X^* \mid X \in \operatorname{dom}_{\mathcal{A}}(r), \text{ such that } r(X) \text{ is invertible}\}\\ &= \{X^* \mid X \in \operatorname{dom}_{\mathcal{A}}(r^{-1})\},\end{aligned}$$

and thus for all $X \in \operatorname{dom}_{\mathcal{A}}(r^{-1})$

$$(r^{-1})^*(X^*) = r^*(X^*)^{-1} = (r(X)^*)^{-1} = (r(X)^{-1})^* = r^{-1}(X)^*.$$

The second assertion is now an immediate consequence. □

When dealing with evaluations of non-commutative rational expressions, the terminology of the following definition turns out to be helpful.

Definition III.2.9. Let r_1, r_2 be two non-commutative rational expressions in the formal variables $x = (x_1, \dots, x_g)$. Given any unital complex algebra $\mathcal{A}$, we say that *r_1 and r_2 are $\mathcal{A}$-evaluation equivalent*, written as $r_1 \sim_{\mathcal{A}} r_2$, if we have $\operatorname{dom}_{\mathcal{A}}(r_1) \cap \operatorname{dom}_{\mathcal{A}}(r_2) \neq \emptyset$ and

$$r_1(X) = r_2(X) \qquad \text{for all } X \in \operatorname{dom}_{\mathcal{A}}(r_1) \cap \operatorname{dom}_{\mathcal{A}}(r_2).$$

Remark III.2.10. If $\mathcal{A}$ carries some additional analytic structure, for instance, if $\mathcal{A}$ is a Banach algebra with respect to a norm $\|\cdot\|$, it can be checked for any rational expression r in g variables $x = (x_1, \dots, x_g)$ that

(i) $\operatorname{dom}_{\mathcal{A}}(r)$ is an open subset of $\mathcal{A}^g$ and
(ii) evaluation induces a continuous mapping

$$r|_{\mathcal{A}}:\ \mathcal{A}^g \supseteq \operatorname{dom}_{\mathcal{A}}(r) \to \mathcal{A},\ (X_1, \dots, X_g) \mapsto r(X_1, \dots, X_g),$$

where we suppose that $\mathcal{A}^g$ is endowed with the norm

$$\|(X_1, \dots, X_g)\| := \max_{j=1,\dots,g} \|X_j\|.$$

This was observed in recent discussions that the author had with Guillaume Cébron and Sheng Yin. It can be checked as follows:

Consider the subset $\mathfrak{R}$ of $\mathfrak{R}_{\mathbb{C}}(x)$, consisting of all non-commutative rational expressions r, which satisfy the above conditions (i) and (ii) for the fixed Banach algebra $\mathcal{A}$. We claim that $\mathfrak{R} = \mathfrak{R}_{\mathbb{C}}(x)$ holds true. Since we clearly have $\mathbb{C} \subseteq \mathfrak{R}$ and $x_1, \dots, x_g \in \mathfrak{R}$, it remains to prove according to Remark III.2.2 that $\mathfrak{R}$ is closed under the arithmetic operations $+$, $\cdot$, and $^{-1}$. While this is obvious for $+$ and $\cdot$, the statement for $^{-1}$ requires some explanation.

Take any $r \in \mathfrak{R}$. According to Item (v) of Definition III.2.3, the domain $\operatorname{dom}_{\mathcal{A}}(r^{-1})$ of r^{-1} is nothing but the pre-image of $\mathrm{GL}(\mathcal{A})$ under the continuous mapping $r|_{\mathcal{A}}$, where we denote by $\mathrm{GL}(\mathcal{A})$ the set of all invertible elements in $\mathcal{A}$. Since $\mathrm{GL}(\mathcal{A})$ is open in $\mathcal{A}$ with respect to $\|\cdot\|$, we may conclude that $\operatorname{dom}_{\mathcal{A}}(r^{-1})$ forms an open subset of $\operatorname{dom}_{\mathcal{A}}(r)$ and hence of $\mathcal{A}^g$. This proves (i) for the non-commutative rational expression r^{-1}. Let us now check the validity of (ii) for r^{-1}. It is well-known that $a \mapsto a^{-1}$ induces a continuous mapping $\operatorname{inv}_{\mathcal{A}}: \ \mathrm{GL}(\mathcal{A}) \to \mathcal{A}$. Thus, we may conclude that the composed mapping

$$r^{-1}|_{\mathcal{A}} = \operatorname{inv}_{\mathcal{A}} \circ (r|_{\mathcal{A}})|_{\operatorname{dom}_{\mathcal{A}}(r^{-1})}$$

is continuous as well. In summary, we obtain $r^{-1} \in \mathfrak{R}$, as desired.

III.2.2. Non-commutative rational functions. In this subsection, we turn our attention to non-commutative rational functions. Although we will mostly work here with non-commutative rational expressions, some pieces of the rich and fascinating theory of non-commutative rational functions will always be in the background. Thus, it is worth to give some brief overview over these important results.

The first and certainly also one of the most important questions that arises is: what actually do we mean by non-commutative rational functions? Surprisingly, giving an answer to this very fundamental question is much more complicated than one would expect and presenting it in full detail would lead us much too far away from our actual topic. However, for the seek of completeness, we want to give at least the basic ideas, which are essential in what follows.

III.2.2.1. *The free field.* Let us first have a look on the classical (commutative) situation. The algebraic definition of a rational functions is that they are given as elements in the quotient field $\mathbb{C}(x_1, \dots, x_g)$ of the commutative ring $\mathbb{C}[x_1, \dots, x_g]$, which consists of polynomials in g formal but commuting indeterminates $x_1, \dots, x_g$. It follows that we can write each rational function $\mathfrak{r}$ as a quotient $\mathfrak{r} = \frac{p}{q} = pq^{-1}$ for certain polynomials $p, q \in \mathbb{C}[x_1, \dots, x_g]$ where $q \neq 0$.

In the non-commutative situation, one would probably try first to repeat this appealing construction. Accordingly, one could expect that a non-commutative rational function is given as an element in the quotient field of the ring $\mathbb{C}\langle x_1, \dots, x_g \rangle$ of non-commutative polynomials – and of course, we should talk here more precisely about the "quotient skew field", since the commutativity condition clearly must to be dropped.

But unfortunately, because the classical construction heavily depends on the underlying commutativity, it simply fails in this generality. We note that at least for special non-commutative rings, namely those rings R, which are *integral domains* (i.e., the set $R \backslash \{0\}$ contains 1 and is closed under multiplication) and satisfy the so-called *Ore condition*

$$\forall\, a, b \in R,\ b \neq 0 \quad \exists\, c, d \in R,\ d \neq 0: \qquad ad = bc,$$

the classical approach can be saved to some extend. The idea behind is that in the desired quotient field one would need to relate expressions of the form $b^{-1}a$ with expressions of the form cd^{-1}, in order to get a well-defined multiplication. But anyway, this does not help us in our situation, since $\mathbb{C}\langle x_1, \dots, x_g \rangle$ does not satisfy the Ore condition.

So let us have another try. If we cannot construct our desired "quotient skew field" for $\mathbb{C}\langle x_1, \dots, x_g \rangle$ in the usual way, we have to accept a more abstract construction. In

[**Coh85, Coh06**], we can find the term "skew field of fractions", which sounds quite promising. This requires some terminology.

Definition III.2.11.

(i) Given a ring R, a *R-ring* (respectively, *R-field*) is a ring (respectively, field) K that comes together with a homomorphism $\phi : R \to K$.
(ii) An R-field K, which is generated by the image $\phi(R)$, is called *epic*.
(iii) An epic R-field K, for which ϕ is injective, is called *skew field of fractions of R*.

Note that a skew field of fractions K for a given ring R contains by $\phi(R)$ a ring that is (due to the injectivity of ϕ) isomorphic to R and the assumption of being epic implies that K is as small as possible in the sense that there is no proper skew sub-field of K that contains $\phi(R)$.

However, the problem is that skew fields of fractions do not need to exist and if they exist, they might not be unique – and the uniqueness even fails in the particular case $\mathbb{C}\langle x_1, \dots, x_g\rangle$. For example, it is possible to construct infinitely many pairwise non-isomorphic skew fields of fractions for $\mathbb{C}\langle x_1, x_2\rangle$; see [**KV12**].

Thus, if we want to find a distinguished "quotient skew field" for $\mathbb{C}\langle x_1, \dots, x_g\rangle$, we need to go a bit further. As it turns out, the right condition for these purposes is "universality". The following definitions are taken from [**Coh06**, Section 7.2].

Roughly speaking, an epic R-field U is universal, if for any other epic R-field K, the corresponding mapping ϕ_K factorizes over U, i.e.

$$\begin{array}{ccc} R & \xrightarrow{\phi_U} & U \\ & \overset{\phi_K}{\searrow} & \downarrow \\ & & K \end{array} .$$

But here we immediately run into trouble: it is not hard to show that, given two epic R-fields K and L, any *R-ring homomorphism* $f : K \to L$ (namely, a ring homomorphism f respecting the given homomorphisms $\phi_K : R \to K$ and $\phi_L : R \to L$ in the sense that $f \circ \phi_K = \phi_L$ holds), must be automatically an isomorphism. Hence, we must replace this rather canonical notion of homomorphisms by a more abstract one: a *subhomomorphism* between two R-fields K and L means an R-ring homomorphism $f : K_f \to L$, which is defined on some R-subring K_f of K and for which all elements of $\{x \in K_f \mid f(x) \neq 0\}$ are invertible in K_f. The obvious ambiguity in choosing the domain K_f of subhomomorphisms is resolved in the following way: two subhomomorphisms are declared to be equivalent, if there exists an R-ring K_0 of K, on which they agree and for which their common restriction yields again a subhomomorphism from K to L. An equivalence class of subhomomorphisms from K to L with respect to this equivalence relation will be called a *specialization from K to L*.

Definition III.2.12.

(i) An epic R-field U, satisfying the property that for any epic R-field K a unique specialization from U to K exists, is called *universal R-field*.
(ii) If U is in addition a field of fractions of R, then we call U the *universal skew field of fractions of R*.

In general, universal R-fields do not need to exists, but if they exist, its defining universal property forces it to be unique up to isomorphism.

It is an important (but highly non-trivial) fact that the universal skew field of fractions for $\mathbb{C}\langle x_1, \dots, x_g\rangle$, sometimes simply called the *free (skew) field*, exists. In the following, it will be denoted by

$$\mathbb{C}\langle\!\!\!<x_1, \dots, x_g>\!\!\!\rangle.$$

Its first construction is due to Amitsur [**Ami66**], but later on, several generalizations and simplifications of his arguments and also some very different constructions were found; see [**Ber70**], for instance. Some of them underly our subsequent investigations and are thus outlined in the next paragraphs.

III.2.2.2. *Amitsur's construction of the free field.* Amitsur's original construction of the free field in [**Ami66**] as well as its generalization in [**Ber70**] build on non-commutative rational expressions. In their approach, non-commutative rational functions are introduced as equivalence classes of non-commutative rational expressions, where the corresponding equivalence relation is induced by $\mathcal{A}$-evaluation equivalence for some large auxiliary skew field $\mathcal{A}$. In this sense, non-commutative rational functions in g variables form actual functions from their domains in $\mathcal{A}^g$ to $\mathcal{A}$. The drawback of this approach, however, is that such $\mathcal{A}$'s are rather difficult to treat.

But what could be an alternative choice for $\mathcal{A}$? While $\mathcal{A}$ in [**Ami66, Ber70**] was supposed to be a skew field, we can simplify matters, if we drop this condition and allow $\mathcal{A}$ to be an algebra. The most familiar non-commutative algebras are certainly matrix algebras $M_n(\mathbb{C})$. However, matrices of some fixed size are not sufficient to capture the high non-commutativity that we expect among the variables $x_1, \dots, x_g$ in the free field. Thus, instead of using matrices of fixed size, we should rather work with matrices of all sizes at once. Accordingly, the right object for this purpose – even though it does not form an algebra itself – is

$$M(\mathbb{C}) := \coprod_{n=1}^{\infty} M_n(\mathbb{C}).$$

Following [**KV12**], we outline in this paragraph how the free field can be constructed with the help of evaluations on $M(\mathbb{C})$. We begin with the following definition.

Definition III.2.13. Consider a tuple $x = (x_1, \dots, x_g)$ of formal variables.

(i) Given a non-commutative rational expression $r \in \mathfrak{R}_{\mathbb{C}}(x)$, we put

$$\mathrm{dom}_{M(\mathbb{C})}(r) := \coprod_{n=1}^{\infty} \mathrm{dom}_{M_n(\mathbb{C})}(r) \subseteq \coprod_{n=1}^{\infty} M_n(\mathbb{C})^g.$$

(ii) A non-commutative rational expression $r \in \mathfrak{R}_{\mathbb{C}}(x)$ is called *non-degenerate*, if it satisfies the condition that $\mathrm{dom}_{M(\mathbb{C})}(r) \neq \emptyset$. The subset of $\mathfrak{R}_{\mathbb{C}}(x)$ consisting of all non-commutative rational expressions that are non-degenerate will be denoted by $\mathfrak{R}^0_{\mathbb{C}}(x)$.

(iii) Let r_1, r_2 be any two non-degenerate non-commutative rational expressions in the formal variables $x = (x_1, \dots, x_g)$. We call r_1 and r_2 *$M(\mathbb{C})$-evaluation equivalent*, written as $r_1 \sim r_2$, if we have

$$\mathrm{dom}_{M(\mathbb{C})}(r_1) \cap \mathrm{dom}_{M(\mathbb{C})}(r_2) \neq \emptyset \qquad \text{(III.2)}$$

and

(III.3) $\quad r_1(X) = r_2(X) \qquad$ for all $X = (X_1, \ldots, X_g) \in \mathrm{dom}_{M(\mathbb{C})}(r_1) \cap \mathrm{dom}_{M(\mathbb{C})}(r_2)$.

The equivalence class in $\mathfrak{R}^0_{\mathbb{C}}(x)$ of any non-commutative rational expression $r \in \mathfrak{R}^0_{\mathbb{C}}(x)$ with respect to $\sim$ will be denoted by $[r]$.

Our candidate for the free field is the set $\mathfrak{R}^0_{\mathbb{C}}(x)/\sim$ of all equivalence classes $[r]$ for $r \in \mathfrak{R}^0_{\mathbb{C}}(x)$ with respect to $M(\mathbb{C})$-evaluation equivalence. The arithmetic operations $+$ and $\cdot$ on $\mathfrak{R}^0_{\mathbb{C}}(x)/\sim$ should be defined via the corresponding operations on representatives, i.e.

(III.4) $\quad [r_1] + [r_2] = [r_1 + r_2] \qquad$ and $\qquad [r_1] \cdot [r_2] = [r_1 \cdot r_2] \qquad$ for $r_1, r_2 \in \mathfrak{R}^0_{\mathbb{C}}(x)$.

This, however, can only be well-defined if both $r_1 + r_2$ and $r_1 \cdot r_2$ belong to $\mathfrak{R}^0_{\mathbb{C}}(x)$ for any given $r_1, r_2 \in \mathfrak{R}^0_{\mathbb{C}}(x)$. In order to see, why this is the case, we need some preparation.

The definition of $M(\mathbb{C})$-evaluation equivalence as given in Item (iii) of the previous Definition III.2.13 requires besides (III.3) the as such natural comparability condition (III.2). Surprisingly, on closer inspection, this turns out to be superfluous, since (III.2) is automatically satisfied, whenever non-degenerate rational expressions are considered. This was noted in a footnote of [**KV12**] and we will record this important observation in Corollary III.2.15 below. Corollary III.2.15 builds essentially on the following result (see [**KV09**, Remark 2.3]), but it requires some tricky argument, which can be found in the same footnote of [**KV12**], in order to prove that it is indeed a consequence of this Theorem III.2.14.

THEOREM III.2.14. *Let r be any non-degenerate non-commutative rational expression in the formal variables $x = (x_1, \ldots, x_g)$. Then there exists $n_0 = n_0(r) \in \mathbb{N}$, such that*

$$\mathrm{dom}_{M_n(\mathbb{C})}(r) \neq \emptyset$$

holds for all integers $n \geq n_0$.

A proof of this interesting statement is outlined in [**KV09**, Remark 2.15]. Let us formulate now the announced corollary.

COROLLARY III.2.15. *Let r_1 and r_2 be non-degenerate non-commutative rational expressions in the formal variables $x = (x_1, \ldots, x_g)$. Then there exists $n_0 = n_0(r_1, r_2) \in \mathbb{N}$, such that*

$$\mathrm{dom}_{M_n(\mathbb{C})}(r_1) \cap \mathrm{dom}_{M_n(\mathbb{C})}(r_2) \neq \emptyset$$

holds for all integers $n \geq n_0$. In particular, we have that

$$\mathrm{dom}_{M(\mathbb{C})}(r_1) \cap \mathrm{dom}_{M(\mathbb{C})}(r_2) \neq \emptyset.$$

An analogue of Corollary III.2.15 for $\mathcal{A}$-evaluation equivalence with respect to skew fields $\mathcal{A}$, which are infinite dimensional over their centers, can be found in [**Ber70**, Lemma 2].

REMARK III.2.16. With the help of Corollary III.2.15, we can now verify that (III.4) gives rise to well-defined arithmetic operations on $\mathfrak{R}^0_{\mathbb{C}}(x)/\sim$. Indeed, given non-degenerate non-commutative rational expressions r_1 and r_2, we have

$$\begin{aligned} \mathrm{dom}_{M_n(\mathbb{C})}(r_1 \cdot r_2) &= \mathrm{dom}_{M_n(\mathbb{C})}(r_1) \cap \mathrm{dom}_{M_n(\mathbb{C})}(r_2) \qquad \text{and} \\ \mathrm{dom}_{M_n(\mathbb{C})}(r_1 + r_2) &= \mathrm{dom}_{M_n(\mathbb{C})}(r_1) \cap \mathrm{dom}_{M_n(\mathbb{C})}(r_2) \end{aligned}$$

according to Item (iii) and Item (iii) in Definition III.2.3 for all $n \in \mathbb{N}$, from which it follows that

$$\operatorname{dom}_{M(\mathbb{C})}(r_1 \cdot r_2) = \operatorname{dom}_{M(\mathbb{C})}(r_1) \cap \operatorname{dom}_{M(\mathbb{C})}(r_2) \qquad \text{and}$$
$$\operatorname{dom}_{M(\mathbb{C})}(r_1 + r_2) = \operatorname{dom}_{M(\mathbb{C})}(r_1) \cap \operatorname{dom}_{M(\mathbb{C})}(r_2).$$

Thus, we may deduce from Corollary III.2.15 that both $r_1 \cdot r_2$ and $r_1 + r_2$ are non-degenerate non-commutative rational expressions. Having established this, it is now easy to see that the classes $[r_1 \cdot r_2]$ and $[r_1 + r_2]$ do not depend on the choice of representatives r_1 and r_2 of the classes $[r_1]$ and $[r_2]$, respectively.

Since $\mathfrak{R}^0_{\mathbb{C}}(x)/{\sim}$ is supposed to form even a skew field, we need that each $[r] \neq 0$ is invertible in $\mathfrak{R}^0_{\mathbb{C}}(x)/{\sim}$. A natural candidate for the inverse of $[r]$ is $[r^{-1}]$, but for seeing that this is indeed the case we have to check that $r^{-1} \in \mathfrak{R}^0_{\mathbb{C}}(x)$ holds for each $r \in \mathfrak{R}^0_{\mathbb{C}}(x)$ under the condition that r is not $M(\mathbb{C})$-evaluation equivalent to 0; if this is shown, then $[r]^{-1} = [r^{-1}]$ follows, since then $[r]$ and $[r^{-1}]$ form well-defined classes in $\mathfrak{R}^0_{\mathbb{C}}(x)/{\sim}$ with the property that both $[r] \cdot [r^{-1}] = [r \cdot r^{-1}] = [1]$ and $[r^{-1}] \cdot [r] = [r^{-1} \cdot r] = [1]$. For this purpose, let us include first the following result, which reflects a particular case of [**KV12**, Proposition 2.1].

Theorem III.2.17. *Let $r \in \mathfrak{R}^0_{\mathbb{C}}(x_1, \ldots, x_g)$ be given and suppose that*

$$\det(r(X)) = 0 \qquad \text{for all } X = (X_1, \ldots, X_g) \in \operatorname{dom}_{M(\mathbb{C})}(r), \tag{III.5}$$

where $\det : M(\mathbb{C}) \to \mathbb{C}$ denotes the mapping obtained by gluing together the usual determinants $\det : M_n(\mathbb{C}) \to \mathbb{C}$. Then r is $M(\mathbb{C})$-evaluation equivalent to 0.

Remark III.2.18. Using Theorem III.2.17, we immediately get that $\operatorname{dom}_{M(\mathbb{C})}(r^{-1}) \neq \emptyset$ is satisfied for each $r \in \mathfrak{R}^0_{\mathbb{C}}(x)$, which is not $M(\mathbb{C})$-evaluation equivalent to 0. Indeed, if we assume to the contrary that $\operatorname{dom}_{M(\mathbb{C})}(r^{-1}) = \emptyset$ holds, then Item (v) of Definition III.2.3 tells us that no point $X \in \operatorname{dom}_{M(\mathbb{C})}(r)$ exists, for which $r(X)$ is invertible, and we infer that r satisfies condition (III.5). The conclusion of Theorem III.2.17 then contradicts the assumption that r is not $M(\mathbb{C})$-evaluation equivalent to 0.

All our previous discussions merge now into the following remarkable result; see [**KV12**, Proposition 2.2]. For the seek of clarity, let us reserve $x = (x_1, \ldots, x_g)$ for the variables in $\mathbb{C}\langle x_1, \ldots, x_g\rangle \subset \mathbb{C}\langle\!\langle x_1, \ldots, x_g\rangle\!\rangle$, while non-commutative rational expressions are built in the formal variables $z = (z_1, \ldots, z_g)$.

Theorem III.2.19. *Consider the set*

$$\mathfrak{R}^0_{\mathbb{C}}(z)/{\sim} = \big\{[r] \big|\ r \in \mathfrak{R}^0_{\mathbb{C}}(z)\big\}$$

of equivalence classes of non-commutative rational expressions in the formal variables $z = (z_1, \ldots, z_g)$ with respect to the equivalence relation $\sim$ induced by $M(\mathbb{C})$-evaluation equivalence. If endowed with the induced operations $+$ and $\cdot$, i.e.

$$[r_1] + [r_2] = [r_1 + r_2] \qquad \text{and} \qquad [r_1] \cdot [r_2] = [r_1 \cdot r_2] \qquad \text{for } r_1, r_2 \in \mathfrak{R}^0_{\mathbb{C}}(x),$$

it forms the universal skew field of fractions for $\mathbb{C}\langle x_1, \ldots, x_g\rangle$ with respect to the embedding

$$\phi :\ \mathbb{C}\langle x_1, \ldots, x_g\rangle \to \mathfrak{R}^0_{\mathbb{C}}(z)/{\sim}$$

determined by $x_i \mapsto [z_i]$ for $i = 1, \ldots, g$.

For a proof of this theorem, we refer the interested reader to [**KV12**].

III.2.2.3. *A glimpse on Cohn's approach.* The construction of Cohn in [**Coh85, Coh06**] generalizes the idea of *localization* from the commutative to the non-commutative case. Classically, for a commutative ring R and any given set $S \subseteq R\backslash\{0\}$, which is closed under multiplication (i.e. $s, t \in S$ implies $st \in S$) and satisfies $1 \in S$, localization allows to construct another ring R_S together with a homomorphism $\phi : R \to R_S$, such that all elements in the image $\phi(R)$ are invertible in R_S. For an integral domain R, the corresponding quotient field is then obtained by applying this general construction to the particular multiplicative set S given by $S = R\backslash\{0\}$. Cohn discovered that, in order to adapt this construction to the non-commutative setting, one has to replace the set S of elements in R by a set Σ of matrices over R. This is particularly remarkable since it also fits nicely with the basic ideas of free analysis.

Let us abbreviate by $M(R) := \coprod_{n=1}^{\infty} M_n(R)$ the set of all matrices of all sizes over R. The relevant subsets $\Sigma \subseteq M(R)$ are those which satisfy the condition of the following definition.

Definition III.2.20. A subset $\Sigma \subseteq M(R)$ is called

(i) *upper multiplicative*, if $1 \in \Sigma$ holds and
$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \in \Sigma$$
for all $A, B \in \Sigma$ and each (rectangular) matrix C over R of appropriate size.

(ii) *lower multiplicative*, if $1 \in \Sigma$ holds and
$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \in \Sigma$$
for all $A, B \in \Sigma$ and each (rectangular) matrix C over R of appropriate size.

Given a homomorphism $f : R \to S$ between rings R and S, then the ensemble $(f^{(n)})_{n\in\mathbb{N}}$ of all its amplifications
$$f^{(n)} :\ M_n(R) \to M_n(S),\ (x_{i,j})_{i,j=1}^n \mapsto \big(f(x_{i,j})\big)_{i,j=1}^n$$
induces naturally a mapping from $M(R)$ to $M(S)$. For the seek of simplicity, we denote this mapping again by f.

Definition III.2.21. Let $f : R \to S$ be a homomorphism between rings R and S and let $\Sigma \subseteq M(R)$ be any subset.

(i) The homomorphism f is called Σ-*inverting*, if the image $\Sigma^f \subseteq M(S)$ of Σ under the ensemble of amplifications of f consists only of invertible matrices.
(ii) If f is Σ-inverting, then the Σ-*rational closure of* R *in* S, usually denoted by $R_\Sigma(S)$, is defined as the set of all entries appearing in matrices Q^{-1} for $Q \in \Sigma^f$.
(iii) If Σ is the set of all matrices in $M(R)$, which are mapped by f to invertible matrices over S, we call $R_\Sigma(S)$ the f-*rational closure of* R (or just the *rational closure of* R) and we write $R^f(S)$ instead of $R_\Sigma(S)$.

The following important theorem, which is an excerpt of [**Coh06**, Theorem 7.1.2], provides an interesting characterization of elements belonging to the Σ-rational closure. In particular, it shows that the representation $x = -uQ^{-1}v$, which underlies the "linearization trick" in free probability [**HT05, HST06**], especially in its self-adjoint variant [**And12, And13, And15**] (see also [**BMS13**]), is very natural from the algebraic point of view presented here. In the light of this, the question whether the methods invented in [**BMS13**] for treating the case of non-commutative polynomials could be generalized to non-commutative rational expressions imposes itself. Indeed, this was carried out in [**HMS15**] and the corresponding results will be explained in detail in Chapter IV.

THEOREM III.2.22 (Theorem 7.1.2 in [**Coh06**]). *Let R and S be rings and let $\Sigma \in M(R)$ be an upper multiplicative set. For any given Σ-inverting homomorphism $f : R \to S$, the Σ-rational closure $R_\Sigma(S)$ of R in S forms a subring of S that contains $f(R)$. Moreover, for any $x \in S$ the following statements are equivalent:*

(i) *$x \in R_\Sigma(S)$.*
(ii) *$x = -uQ^{-1}v$, where $Q \in \Sigma^f$ and where u and v are a row and column vector, respectively, with entries in $f(R)$.*

It is shown in [**Coh06**, Theorem 7.2.4] that for each ring R and any set $\Sigma \subseteq M(R)$, there exists a ring R_Σ, which is unique up to isomorphism, with a *universal Σ-inverting homomorphism* $\lambda : R \to R_\Sigma$ in the following sense: the homomorphism $\lambda : R \to R_\Sigma$ is Σ-inverting and whenever $f : R \to S$ is any other Σ-inverting homomorphism, then there exists a unique homomorphism $\tilde{f} : R_\Sigma \to S$, such that $f = \tilde{f} \circ \lambda$ holds. We call R_Σ the *universal localization of R with respect to Σ.*

A particularly important instance of $\Sigma \subseteq M(R)$ is the set that consists of all full matrices over R.

DEFINITION III.2.23. Let R be a ring. A matrix $Q \in M_n(R)$ is called *full*, if it cannot be written as a product
$$Q = R_1 R_2$$
of rectangular matrices
$$R_1 \in M_{n\times(n-1)}(R) \qquad \text{and} \qquad R_2 \in M_{(n-1)\times n}(R).$$
The subset of $M(R)$ consisting of all full matrices will be denoted by $\Phi(R)$.

The brilliant idea behind the construction of the free field $\mathbb{C}\langle\!\langle x_1, \ldots, x_g \rangle\!\rangle$ in [**Coh85, Coh06**] is that the full matrices in $M(\mathbb{C}\langle x_1, \ldots, x_g\rangle)$ are exactly those that should become invertible over $\mathbb{C}\langle\!\langle x_1, \ldots, x_g \rangle\!\rangle$. Since this criterion allows us to decide whether a matrix in $M(\mathbb{C}\langle x_1, \ldots, x_g\rangle)$ is invertible over $\mathbb{C}\langle\!\langle x_1, \ldots, x_g \rangle\!\rangle$ or not, even before knowing what $\mathbb{C}\langle\!\langle x_1, \ldots, x_g \rangle\!\rangle$ actually is, one can hope to turn this into a formal construction of $\mathbb{C}\langle\!\langle x_1, \ldots, x_g \rangle\!\rangle$. For more general R, this is addressed in the following theorem.

THEOREM III.2.24 (Theorem 7.5.13 in [**Coh06**]). *Let R a non-zero ring. If the set $\Phi = \Phi(R)$ of all full matrices over R is lower multiplicative, then R has a universal skew field of fractions, given by its universal localization R_Φ.*

It remains now to check that Theorem III.2.24 applies in the for us relevant case $R = \mathbb{C}\langle x_1, \ldots, x_g\rangle$. Unfortunately, we cannot go into details here, since this would go beyond the scope of our exposition, but let us highlight at least the following points:

- Theorem III.2.24 is only a small excerpt of [**Coh06**, Theorem 7.5.13]. Among several other things, it is proven there that our initial assumption on R, namely the one that asks $\Phi(R)$ to be lower multiplicative, is in fact equivalent to R being a so-called *Sylvester domain*. Sylvester domains are characterized by a certain rank condition in terms of the so-called *inner rank*; see [**Coh06**, Section 5.5] for the precise definitions.

- With this reformulation, we are now faced with the question, whether $\mathbb{C}\langle x_1, \ldots, x_g\rangle$ is a Sylvester domain. In order to find a confirmation, one needs to delve into the copious work of Cohn [**Coh06**] and so we prefer to guide the reader by a few notes. The statement, which seems to come the closest to what we need here, is [**Coh06**, Theorem 5.5.4]. The setting considered there is however much too general and, as a quick look into its proof shows, the actual argument for $\mathbb{C}\langle x_1, \ldots, x_g\rangle$ must be hidden somewhere else. Indeed, one finds that the first ingredient is the *law of nullity* given in [**Coh06**, Proposition 5.5.1]. This requires $\mathbb{C}\langle x_1, \ldots, x_g\rangle$ to be an *n-fir* for each positive integer n or, equivalently, to be a *semifir*. That $\mathbb{C}\langle x_1, \ldots, x_g\rangle$ forms even a *fir*, is explained in [**Coh06**, Corollary 2.5.2] and the comments made thereafter. This second ingredient relies on a more general construction, the so-called *weak algorithm*; see [**Coh06**, Section 2.4].

Remark III.2.25. This particular construction of the free field $\mathbb{C}\langle\!\!\!\langle x_1, \ldots, x_g\rangle\!\!\!\rangle$ immediately implies the following crucial properties:

- Any non-commutative rational function $\mathfrak{r}$ in $\mathbb{C}\langle\!\!\!\langle x_1, \ldots, x_g\rangle\!\!\!\rangle$ can be written in the form $\mathfrak{r} = -uQ^{-1}v$ with row and column vectors u and v, respectively, of some dimension n over $\mathbb{C}\langle x_1, \ldots, x_g\rangle$, and a full matrix $Q \in M_n(\mathbb{C}\langle x_1, \ldots, x_g\rangle)$. Sometimes, we will also work with representations of the more general form $\mathfrak{r} = c - uQ^{-1}v$, where we suppose in addition that $c \in \mathbb{C}\langle x_1, \ldots, x_g\rangle$.
- Conversely, any full matrix $Q \in M_n(\mathbb{C}\langle x_1, \ldots, x_g\rangle)$ is invertible over $\mathbb{C}\langle\!\!\!\langle x_1, \ldots, x_g\rangle\!\!\!\rangle$ and so all entries of Q^{-1} are non-commutative rational functions in the variables $x_1, \ldots, x_g$.

Despite its high non-commutativity, the free field $\mathbb{C}\langle\!\!\!\langle x_1, \ldots, x_g\rangle\!\!\!\rangle$ contains surprisingly many non-trivial identities, which are not easy to identify; in $\mathbb{C}\langle\!\!\!\langle x_1, x_2, x_3\rangle\!\!\!\rangle$, for example, we have that (see [**CR99**])

$$x_2^{-1} + x_2^{-1}(x_3^{-1}x_1^{-1} - x_2^{-1})^{-1}x_2^{-1} = (x_2 - x_3x_1)^{-1}.$$

It is the content of [**CR99**, Theorem 3.2] that all such identities can in fact be deduced by purely algebraic means, only using that $\mathfrak{r}^{-1}$ is the inverse of $\mathfrak{r}$ for any $0 \neq \mathfrak{r} \in \mathbb{C}\langle\!\!\!\langle x_1, \ldots, x_g\rangle\!\!\!\rangle$. Since it is typically quite expensive in labor to find the right algebraic manipulations, one is interested in more handy tools for deciding whether a given non-commutative rational function $\mathfrak{r}$ is 0 in $\mathbb{C}\langle\!\!\!\langle x_1, \ldots, x_g\rangle\!\!\!\rangle$. Here, we quote the following result, which shows how to use the representation $\mathfrak{r} = c - uQ^{-1}v$ for this purpose.

Theorem III.2.26 (Proposition 7.8.1, [**Coh06**]). *Let $\mathfrak{r} \in \mathbb{C}\langle\!\!\!\langle x_1, \ldots, x_g\rangle\!\!\!\rangle$ be a non-commutative rational function, which is written in the form $\mathfrak{r} = c - uQ^{-1}v$, where $Q \in M_n(\mathbb{C}\langle x_1, \ldots, x_g\rangle)$ is a full matrix, u and v are a row and column vector, respectively, of dimension n with entries in $\mathbb{C}\langle x_1, \ldots, x_g\rangle$, and $c \in \mathbb{C}\langle x_1, \ldots, x_g\rangle$. Then $\mathfrak{r} \neq 0$ if and only if the* display

$$\begin{pmatrix} c & u \\ v & Q \end{pmatrix}$$

is full in $M_{n+1}(\mathbb{C}\langle x_1, \ldots, x_g\rangle)$.

III.2.2.4. *The construction of Cohn and Reutenauer.* We take now a closer look at some alternative construction of the free field, namely the one that appeared in [**CR99**]. While the construction of [**CR99**] works more generally for the universal skew field of fractions $D_K \langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ of the tensor ring $R = D_K\langle x_1, \dots, x_g\rangle$ for a skew field D with central subfield K, we restrict ourselves to the case $D = K = \mathbb{C}$ yielding $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$. Note that we will slightly modify the notation of [**CR99**], such that it suits our purposes.

The construction of [**CR99**] relies like the previously discussed approach of [**Coh85, Coh06**] on the notion of full matrices as introduced in Definition III.2.23, but it takes a slightly different starting point. While the construction of [**Coh85, Coh06**], which was presented in Paragraph III.2.2.3, designs $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ in such a way that all full matrices in $M(\mathbb{C}\langle x_1, \dots, x_g\rangle)$ become invertible over $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ and then obtains as a consequence the existence of representations $\mathfrak{r} = c - uQ^{-1}v$ for each of its elements $\mathfrak{r}$ (see Remark III.2.25), the approach of [**CR99**] reverses the order of arguments by modeling $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ according to the desired representations.

It is appropriate to clarify first the terminology of representations as motivated by Paragraph III.2.2.3.

Definition III.2.27. A tuple $\rho = (c; u, Q, v)$ with

- a full matrix $Q \in M_n(\mathbb{C}\langle x_1, \dots, x_g\rangle)$ for some n,
- row and column vectors u and v, respectively, of dimension n over $\mathbb{C}\langle x_1, \dots, x_g\rangle$,
- and a non-commutative polynomial $c \in \mathbb{C}\langle x_1, \dots, x_g\rangle$,

is called a *representation*. If $\mathfrak{r} \in \mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ is the rational function given by $\mathfrak{r} = c - uQ^{-1}v$, we say that ρ *is a representation of* $\mathfrak{r}$ and we call n the *dimension of* ρ.

A representation $\rho = (c; u, Q, v)$ will be called *pure*, if the condition $c = 0$ is satisfied. In this case, we will omit c and we will simply write $\rho = (u, Q, v)$.

As we mentioned above, each non-commutative rational function $\mathfrak{r}$ admits a representation $\rho = (c; u, Q, v)$. Moreover, by enlarging Q and u, v if necessary, we can always achieve $c = 0$. Indeed, if a representation $\rho = (c; u, Q, v)$ of dimension n is given, we can define the matrix

$$\tilde{Q} := \begin{pmatrix} Q & 0 \\ 0 & -1 \end{pmatrix} \in M_{n+1}(\mathbb{C}\langle x_1, \dots, x_g\rangle),$$

which is again full, and the following vectors of dimension $n+1$ over $\mathbb{C}\langle x_1, \dots, x_g\rangle$

$$\tilde{u} = \begin{pmatrix} u & c \end{pmatrix} \qquad \text{and} \qquad \tilde{v} = \begin{pmatrix} v \\ 1 \end{pmatrix}.$$

Obviously, $\tilde{\rho} := (\tilde{u}, \tilde{Q}, \tilde{v})$ yields a pure representation of the same non-commutative rational function.

Among all pure representations, we are mainly interested in those being linear in the following sense.

Definition III.2.28. We call a pure representation $\rho = (u, Q, v)$ *linear*, if

- Q is of the form
$$Q = Q^{(0)} + Q^{(1)}x_1 + \dots + Q^{(g)}x_g$$
for some matrices $Q^{(0)}, Q^{(1)}, \dots, Q^{(g)} \in M_n(\mathbb{C})$,
- and if u, v are row respectively column vectors over $\mathbb{C}$ of dimension n.

While passing from a general representation to a pure one is achieved by a simple algebraic manipulation, it is by no means clear whether we can always find a pure linear representation. That this is indeed the case relies on some "process of linearization by enlargement" (quoting [**CR99**]), which goes back to [**Hig40**] and became known among experts as *Higman's trick*. However, this knowledge did unfortunately not spread out widely, so that the authors of [**HT05, HST06**] reinvented similar methods for their purposes.

We point out that the framework of formal linear representations, which will be presented in Subsection III.2.3, provides an explicit algorithm for producing pure linear representations (by combining Algorithm III.2.45 with Corollary III.2.47). This gives then another proof of the following theorem, by which we summarize our previous observations.

Theorem III.2.29 (see Section 1, [**CR99**]). *Each non-commutative rational function admits a pure linear representation.*

If some non-commutative rational function $\mathfrak{r} \in \mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ can be written as $\mathfrak{r} = -u_0 Q_1^{-1} v_0$ with row and column vectors u_0 and v_0, respectively, over $\mathbb{C}$ and a full matrix of corresponding size over $\mathbb{C}\langle x_1, \dots, x_g \rangle$, which is more precisely of the form $Q_1 = Q_1^{(0)} + Q_1^{(1)} x_1 + \dots + Q_1^{(g)} x_g$ with scalar matrices $Q_1^{(0)}, Q_1^{(1)}, \dots, Q_1^{(g)}$, then $\rho = (u_0, Q_1, v_0)$ gives a pure linear representation of $\mathfrak{r}$. This is exactly how pure linear representations were defined in Definition III.2.27 and Definition III.2.28. If $\mathfrak{r}$ is given more generally by an expression of the form

$$\mathfrak{r} = (-1)^k u_0 Q_1^{-1} P_1 \cdots Q_{k-1}^{-1} P_{k-1} Q_k^{-1} v_0$$

with complex matrices u_0 of size $1 \times n_1$ and v_0 of size $n_k \times 1$, full matrices

$$Q_j = Q_j^{(0)} + Q_j^{(1)} x_1 + \dots + Q_j^{(g)} x_g \qquad \text{with} \qquad Q_j^{(0)}, Q_j^{(1)}, \dots, Q_j^{(g)} \in M_{n_j}(\mathbb{C}),$$

and rectangular matrices

$$P_j = P_j^{(0)} + P_j^{(1)} x_1 + \dots + P_j^{(g)} x_g \qquad \text{with} \qquad P_j^{(0)}, P_j^{(1)}, \dots, P_j^{(g)} \in M_{n_j \times n_{j+1}}(\mathbb{C}),$$

then a pure linear representation of $\mathfrak{r}$ can be obtained by

$$\rho = (u, Q, v) := \left(\begin{pmatrix} 0 & \dots & 0 & u_0 \end{pmatrix}, \begin{pmatrix} & & P_1 & Q_1 \\ & \iddots & Q_2 & \\ P_{k-1} & \iddots & & \\ Q_k & & & \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_0 \end{pmatrix} \right)$$

with a full matrix Q of size $n \times n$, where $n := n_1 + \dots + n_k$. This can be proven easily with the help of Corollary III.2.47 and Lemma IV.2.4, where the latter gets applied to $\mathcal{A} = \mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$. But since this will not be used in the sequel, we omit the details.

Let us come back now to the question how pure linear representations can be used to give an alternative construction of the free field $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$. So far, we have worked inside $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$, but the notion of pure linear representations makes perfectly sense without this underlying object. According to Theorem III.2.29, it is possible to encode each non-commutative rational functions by pure linear representations. However, if they should serve as a proper model for $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$, pure linear representations representing the same rational functions should be identified.

Definition III.2.30. Let $\rho = (u, Q, v)$ and $\rho' = (u', Q', v')$ be two pure linear representations. We call ρ and ρ' *equivalent*, written as $\rho \sim \rho'$, if they represent the same rational function $\mathfrak{r} \in \mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$, i.e., if

$$-uQ^{-1}v = \mathfrak{r} = -u'Q'^{-1}v'.$$

However, this definition of $\sim$ is far from being intrinsic for the setting of pure linear representations, since it involves the object $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$, for whose construction pure linear representations were introduced. We are thus looking for an alternative description of $\sim$, which does not require any previous knowledge about $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$. For this purpose, using the language of category theory turns out to be appropriate. Let us consider the (small) category whose objects are pure linear representations and whose morphisms are defined as follows.

Definition III.2.31. Let $\rho = (u, Q, v)$ and $\rho' = (u', Q', v')$ be two pure linear representations. A *morphism from* ρ *to* ρ' is a pair (S, T) of matrices of appropriate size over $\mathbb{C}$ (and, in fact, they are of the same size), such that

$$u' = uT, \qquad v = Sv', \qquad \text{and} \qquad SQ' = QT.$$

Indeed, one can easily verify that a category is obtained in this way. Note that if (S, T) is a morphism from ρ to ρ' and if (S', T') is a morphism from ρ' to another pure linear representation ρ'', then (SS', TT') yields a morphism from ρ tho ρ''.

The advantage of using this categorical language is that it naturally provides us with the notion of monomorphisms, epimorphisms, and hence isomorphisms. Note that, according to [**CR99**], an isomorphism in this categorical sense is nothing else than a morphism (S, T) with invertible matrices S, T.

However, by only identifying isomorphic pure linearizations, we would not recover the equivalence relation $\sim$ introduced in Definition III.2.30, since even non-isomorphic pure linear representation can represent the same rational functions. On closer inspection, it turns out that morphisms itself are already enough. More precisely, if $\rho = (u, Q, v)$ and $\rho' = (u', Q', v')$ are any two pure linear representations, representing rational functions $\mathfrak{r}$ and $\mathfrak{r}'$, respectively, and if we suppose that there is a morphism from ρ to ρ', then $\mathfrak{r} = \mathfrak{r}'$ follows. Indeed, given any morphism ρ to ρ', say (S, T), then its defining properties $u' = uT$, $v = Sv'$ and $SQ' = QT$, with the latter one reformulated as $TQ'^{-1} = Q^{-1}S$, give us that

$$\mathfrak{r} = -uQ^{-1}v = -uQ^{-1}Sv' = -uTQ'^{-1}v' = -u'Q'^{-1}v' = \mathfrak{r}'.$$

Thus, if two pure linear representations can be connected by some chain consisting of morphisms and inverse morphisms, then they must represent the same rational function. The following remarkable theorem tells us that even the converse is true, such that equivalence in the meaning of Definition III.2.30 is completely characterized within this categorical frame.

Theorem III.2.32 (Theorem 1.1 and Corollary 1.3, [**CR99**]). *Two pure linear representations are equivalent in the sense of Definition III.2.30 if and only if there is a chain of morphisms and inverse morphisms between them.*

Accordingly, the free skew field $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ can be identified with the set of equivalence classes of pure linear representations with respect to the equivalence relation induced by chains of morphisms. The arithmetic operations on the set of equivalence classes of pure linear representations are defined via their representatives, according to the rules collected in the next lemma.

Lemma III.2.33. *Within the frame of pure linear representations, the following rules hold true.*

(i) *For scalars $\lambda \in \mathbb{C}$ and the variables x_j, $j = 1, \dots, g$, pure linear representations are given by*

$$\text{(III.6)} \qquad \begin{aligned} \rho_{x_j} &:= \left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} x_j & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \qquad \text{and} \\ \rho_{\lambda} &:= \left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right). \end{aligned}$$

(ii) *If $\rho_1 = (u_1, Q_1, v_1)$ and $\rho_2 = (u_2, Q_2, v_2)$ are pure linear representations of $\mathfrak{r}_1$ and $\mathfrak{r}_2$, respectively, then*

$$\text{(III.7)} \qquad \rho_1 \oplus \rho_2 := \left(\begin{pmatrix} u_1 & u_2 \end{pmatrix}, \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)$$

gives a pure linear representation of $\mathfrak{r}_1 + \mathfrak{r}_2$.

(iii) *If $\rho_1 = (u_1, Q_1, v_1)$ and $\rho_2 = (u_2, Q_2, v_2)$ are pure linear representations of $\mathfrak{r}_1$ and $\mathfrak{r}_2$, respectively, then*

$$\text{(III.8)} \qquad \rho_1 \odot \rho_2 := \left(\begin{pmatrix} 0 & u_1 \end{pmatrix}, \begin{pmatrix} v_1 u_2 & Q_1 \\ Q_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \right)$$

gives a pure linear representation of $\mathfrak{r}_1 \cdot \mathfrak{r}_2$.

(iv) *If $\rho = (u, Q, v)$ is a pure linear representation of $\mathfrak{r} \neq 0$, then*

$$\text{(III.9)} \qquad \rho^{-1} := \left(\begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & u \\ v & -Q \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

gives a pure linear representation of $\mathfrak{r}^{-1}$.

These formulas, which are slight modifications of Equations (4), (5), and (6) in [**CR99**], can be proven readily by using the Schur complement formula (A.1) as presented in Lemma A.1, but they will also follow from the corresponding formulas (III.10), (III.11), (III.12), and (III.13) for formal linear representations. This will be discussed below. Furthermore, let us note that rule (III.8) is a special case of Lemma IV.2.4.

It would lead much to far to go into the details of the proof of Theorem III.2.32. Nevertheless, we want to highlight the following result, which constitutes not only an important step in [**CR99**] towards the proof of Theorem III.2.32, but which is also of independent interest.

Lemma III.2.34 (Lemma 1.2, [**CR99**]). *Let $\rho = (u, Q, v)$ be a pure linear representation of $\mathfrak{r} \in \mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$. Then $\mathfrak{r} = 0$ holds if and only if there are invertible matrices S and T of appropriate size over $\mathbb{C}$, such that*

$$uT = \begin{pmatrix} u_1 & 0 \end{pmatrix}, \quad SQT = \begin{pmatrix} Q_{1,1} & 0 \\ Q_{2,1} & Q_{2,2} \end{pmatrix}, \quad \text{and} \quad Sv = \begin{pmatrix} 0 \\ v_2 \end{pmatrix}$$

with square matrices $Q_{1,1}$ and $Q_{2,2}$ over $\mathbb{C}\langle x_1, \dots, x_g \rangle$.

For the seek of clarity, let us point out that the proof of Lemma III.2.34, as given in [**CR99**], heavily relies on Theorem III.2.26.

The construction of Cohn and Reutenauer is perfectly suited for defining an involution $*$ on the free field $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$, which extends the canonical involution on $\mathbb{C}\langle x_1, \dots, x_g \rangle$

as introduced in Definition I.1.16. This is certainly well-known to experts, but since we were not able to find some reference in the literature, where the same level of generality is discussed, we include here a self-contained proof; for the case of regular non-commutative rational functions, see [**HMV06**, Section A.3].

First, let us note that the involution of the $*$-algebra $\mathbb{C}\langle x_1,\dots,x_g\rangle$ immediately extends to matrices over $\mathbb{C}\langle x_1,\dots,x_g\rangle$, corresponding to which each $M_n(\mathbb{C}\langle x_1,\dots,x_g\rangle)$ forms a $*$-algebra. Then, in particular,

$$Q^* := (Q^{(0)})^* + (Q^{(1)})^* x_1 + \dots + (Q^{(g)})^* x_g$$

holds for any matrix of the form $Q = Q^{(0)} + Q^{(1)}x_1 + \dots + Q^{(g)}x_g$.

Lemma III.2.35. *Given any pure linear representation $\rho = (u, Q, v)$, then $\rho^* := (v^*, Q^*, u^*)$ is a pure linear representation as well. With respect to this notation, the following statements hold true:*

(i) *Let ρ_1 and ρ_2 be pure linear representations and let (S, T) a morphism from ρ_1 to ρ_2. Then (T^*, S^*) is a morphism from ρ_2^* to ρ_1^*. We will call (T^*, S^*) the* adjoint morphism.

(ii) *If ρ_1 and ρ_2 are equivalent pure linear representations, then also ρ_1^* and ρ_2^* are equivalent in the sense of Definition III.2.30.*

(iii) *If $\mathfrak{r}$ is a non-commutative rational function represented by a pure linear representation ρ, then we declare $\mathfrak{r}^*$ to be the non-commutative rational function induced by ρ^*. This gives rise to a well-defined involution $*$ on $\mathbb{C}\langle\!\langle x_1,\dots,x_g\rangle\!\rangle$, which satisfies the properties*
- $(\mathfrak{r}_1 + \mathfrak{r}_2)^* = \mathfrak{r}_1^* + \mathfrak{r}_2^*$ *and* $(\lambda\mathfrak{r})^* = \overline{\lambda}\mathfrak{r}^*$,
- $(\mathfrak{r}_1 \cdot \mathfrak{r}_2)^* = \mathfrak{r}_2^* \cdot \mathfrak{r}_1^*$,
- $(\mathfrak{r}^{-1})^* = (\mathfrak{r}^*)^{-1}$ *if* $\mathfrak{r} \neq 0$,
- *we have* $1^* = 1$ *and* $x_j^* = x_j$ *for* $j = 1,\dots,g$.

In particular, we have that $*$ agrees on $\mathbb{C}\langle x_1,\dots,x_g\rangle \subseteq \mathbb{C}\langle\!\langle x_1,\dots,x_g\rangle\!\rangle$ with the canonical involution introduced in Definition I.1.16.

Proof. Given a pure linear representation $\rho = (u, Q, v)$, the only issue that might prevent $\rho^* := (v^*, Q^*, u^*)$ from being a pure linear representation is that Q^* must be full over $\mathbb{C}\langle x_1,\dots,x_g\rangle$. For seeing that this is indeed the case, assume to the contrary that $Q^* \in M_n(\mathbb{C}\langle x_1,\dots,x_g\rangle)$ is not full. By Definition III.2.23, we then could find rectangular matrices

$$R_1 \in M_{n\times(n-1)}(\mathbb{C}\langle x_1,\dots,x_g\rangle) \qquad \text{and} \qquad R_2 \in M_{(n-1)\times n}(\mathbb{C}\langle x_1,\dots,x_g\rangle),$$

such that $Q^* = R_1R_2$ holds. Applying $*$ on both sides of this equation, we would obtain that $Q = R_2^*R_1^*$, where now

$$R_2^* \in M_{n\times(n-1)}(\mathbb{C}\langle x_1,\dots,x_g\rangle) \qquad \text{and} \qquad R_1^* \in M_{(n-1)\times n}(\mathbb{C}\langle x_1,\dots,x_g\rangle),$$

contradicting the assumption that Q is full. Hence, Q^* must be full as well.

(i) Let (S, T) be a morphism from $\rho_1 = (u_1, Q_1, v_1)$ to $\rho_2 = (u_2, Q_2, v_2)$. By Definition III.2.31, we have that

$$u_2 = u_1T, \qquad v_1 = Sv_2, \qquad \text{and} \qquad SQ_2 = Q_1T.$$

Taking adjoints of all these three relations results in

$$v_1^* = v_2^*S^*, \qquad u_2^* = T^*u_1^*, \qquad \text{and} \qquad T^*Q_1^* = Q_2^*S^*,$$

which simply means that (T^*, S^*) is a morphism from $\rho_2^* = (v_2^*, Q_2^*, v_2^*)$ to $\rho_1^* = (v_1^*, Q_1^*, u_1^*)$, as we wished to prove.

(ii) Given two pure linear representations ρ_1 and ρ_2 of the same non-commutative rational function, then Theorem III.2.32 tells us that there is a chain of morphisms connecting ρ_1 and ρ_2. Part (i) allows us to conclude that ρ_1^* and ρ_2^* are also connected by some chain of morphisms, namely the corresponding chain of adjoint morphisms, such that Theorem III.2.32 yields the equivalence of ρ_1^* and ρ_2^*.

(iii) If $\mathfrak{r}$ is a non-commutative rational function represented by a pure linear representation ρ, then $\mathfrak{r}^*$ is determined as the non-commutative rational function induced by ρ^*. This is well-defined, since choosing another representation will result in an equivalent representation for $\mathfrak{r}^*$ according to (ii).

It is therefore clear that $*$ yields an involution, because $(\rho^*)^* = \rho$ is satisfied for all pure linear representations ρ. That $*$ is moreover $\mathbb{C}$-antilinear, can be seen as follows: if $\rho = (u, Q, v)$ is a pure linear representation, then $\lambda\rho := (\lambda u, Q, v)$ yields obviously a pure linear representation of $\lambda\mathfrak{r}$. Thus, we deduce that $\overline{\lambda}\rho^* = (\overline{\lambda}v^*, Q^*, u^*)$ gives a pure linear representation of $\overline{\lambda}\mathfrak{r}^*$, whereas $(\lambda\rho)^* = (v^*, Q^*, \overline{\lambda}u^*)$ gives a pure linear representation of $(\lambda\mathfrak{r})^*$. In order to deduce the claimed formula $(\lambda\mathfrak{r})^* = \overline{\lambda}\mathfrak{r}^*$, we need to check that the pure linear representations $(\lambda\rho)^* = (v^*, Q^*, \overline{\lambda}u^*)$ and $\overline{\lambda}\rho^* = (\overline{\lambda}v^*, Q^*, u^*)$ are equivalent, for which it suffices to check, according to Theorem III.2.32, that there exists a chain of morphisms connecting them. Indeed, it is easy to see that $(\overline{\lambda}1, \overline{\lambda}1)$ gives a morphism from $(\lambda\rho)^* = (v^*, Q^*, \overline{\lambda}u^*)$ to $\overline{\lambda}\rho^* = (\overline{\lambda}v^*, Q^*, u^*)$.

For proving the other stated properties, we can use the rules collected in Lemma III.2.33. With their help, it only remains to note that $(\rho_1 \oplus \rho_2)^* = \rho_1^* \oplus \rho_2^*$, $(\rho_1 \odot \rho_2)^* = \rho_2^* \odot \rho_1^*$ and $(\rho^{-1})^* = (\rho^*)^{-1}$ holds. This concludes the proof. □

Instead of using the framework of pure linear representations, we could alternatively follow Amitsur's construction (see Paragraph III.2.2.2, especially Theorem III.2.19) in order to define an involution on the free field. This is carried out in the next lemma.

LEMMA III.2.36. *With respect to the natural involution on $\mathfrak{R}_{\mathbb{C}}(x_1, \ldots, x_g)$, which was considered in Remark III.2.5, the following statements hold true:*

(i) *If r_1 and r_2 are arbitrary non-commutative rational expressions, then $r_1 \sim r_2$ implies $r_1^* \sim r_2^*$, where $\sim$ now stands for $M(\mathbb{C})$-evaluation equivalence in the sense of Definition III.2.13.*

(ii) *If $\mathfrak{r}$ is a non-commutative rational function, written as $\mathfrak{r} = [r]$ for some non-commutative rational expression $r \in \mathfrak{R}_{\mathbb{C}}^0(x_1, \ldots, x_g)$, then we declare $\mathfrak{r}^\star$ to be the non-commutative rational function given by $\mathfrak{r}^\star := [r^*]$. This gives rise to a well-defined involution $\star$ on $\mathbb{C}\langle\!\langle x_1, \ldots, x_g \rangle\!\rangle$, which satisfies the properties*
- $(\mathfrak{r}_1 + \mathfrak{r}_2)^\star = \mathfrak{r}_1^\star + \mathfrak{r}_2^\star$ *and* $(\lambda\mathfrak{r})^\star = \overline{\lambda}\mathfrak{r}^\star$,
- $(\mathfrak{r}_1 \cdot \mathfrak{r}_2)^\star = \mathfrak{r}_2^\star \cdot \mathfrak{r}_1^\star$,
- $(\mathfrak{r}^{-1})^\star = (\mathfrak{r}^\star)^{-1}$ *if* $\mathfrak{r} \neq 0$,
- *we have* $1^\star = 1$ *and* $x_j^\star = x_j$ *for* $j = 1, \ldots, g$.

In particular, we have that $\star$ agrees on $\mathbb{C}\langle x_1, \ldots, x_g \rangle \subseteq \mathbb{C}\langle\!\langle x_1, \ldots, x_g \rangle\!\rangle$ with the canonical involution introduced in Definition I.1.16.

(iii) *The involution $\star$ coincides on all of $\mathbb{C}\langle\!\langle x_1, \ldots, x_g \rangle\!\rangle$ with the involution $*$, which was introduced in Lemma III.2.35.*

Proof. (i) Suppose that r_1 and r_2 are $M(\mathbb{C})$-evaluation equivalent non-commutative rational expressions in $\mathfrak{R}_{\mathbb{C}}(x_1,\dots,x_g)$. We choose $X \in \operatorname{dom}_{M_n(\mathbb{C})}(r_1^*) \cap \operatorname{dom}_{M_n(\mathbb{C})}(r_2^*)$ for any $n \in \mathbb{N}$. Lemma III.2.8 tells us that $X^* \in \operatorname{dom}_{M_n(\mathbb{C})}(r_1) \cap \operatorname{dom}_{M_n(\mathbb{C})}(r_2)$ holds with $r_1^*(X) = r_1(X^*)^*$ and $r_2^*(X) = r_2(X^*)^*$. Furthermore, since r_1 and r_2 are $M(\mathbb{C})$-evaluation equivalent, we have that $r_1(X^*) = r_2(X^*)$. By combining these facts, we derive that

$$r_1^*(X) = r_1(X^*)^* = r_2(X^*)^* = r_2^*(X).$$

Since $n \in \mathbb{N}$ and $X \in \operatorname{dom}_{M_n(\mathbb{C})}(r_1^*) \cap \operatorname{dom}_{M_n(\mathbb{C})}(r_2^*)$ were arbitrarily chosen, we infer $r_1^* \sim r_2^*$, as we wished to show.

(ii) In order to verify that $\star$ is well-defined, we must check that $\mathfrak{r}^\star$ does not depend on the concrete choice of a representative $r \in \mathfrak{R}^0_{\mathbb{C}}(x_1,\dots,x_g)$ of $\mathfrak{r}$. This is in fact the content of (i).

In order to establish the other properties of $\star$, we only need to recall that $\star$ and all of the involved arithmetic operations, namely $+$, $\cdot$, and $^{-1}$, are defined on representatives, thus

$$(\mathfrak{r}_1+\mathfrak{r}_2)^\star = ([r_1]+[r_2])^\star = [r_1+r_2]^\star = [(r_1+r_2)^*] = [r_1^*+r_2^*] = [r_1^*]+[r_2^*] = [r_1]^\star+[r_2]^\star = \mathfrak{r}_1^\star+\mathfrak{r}_2^\star,$$

and analogously

$$(\mathfrak{r}_1\cdot\mathfrak{r}_2)^\star = ([r_1]\cdot[r_2])^\star = [r_1\cdot r_2]^\star = [(r_1\cdot r_2)^*] = [r_2^*\cdot r_1^*] = [r_2^*]\cdot[r_1^*] = [r_2]^\star\cdot[r_1]^\star = \mathfrak{r}_2^\star\cdot\mathfrak{r}_1^\star,$$

and

$$(\lambda\mathfrak{r})^\star = (\lambda[r])^\star = [\lambda\cdot r]^\star = [(r\cdot\lambda)^*] = [r^*\cdot\overline{\lambda}] = [\overline{\lambda}\cdot r^*] = \overline{\lambda}[r^*] = \overline{\lambda}[r]^\star = \overline{\lambda}\mathfrak{r}^\star,$$

and finally

$$(\mathfrak{r}^{-1})^\star = ([r]^{-1})^\star = [r^{-1}]^\star = [(r^{-1})^*] = [(r^*)^{-1}] = [r^*]^{-1} = ([r]^\star)^{-1} = (\mathfrak{r}^\star)^{-1}.$$

In general, each non-commutative rational expression r, which is self-adjoint in the sense of Definition III.2.7, satisfies $r \sim r^*$ and hence $[r] = [r^*]$. Accordingly, the non-commutative rational function given by $\mathfrak{r} := [r]$ satisfies $\mathfrak{r}^\star = \mathfrak{r}$. From this observation, it follows that $1^\star = 1$ and $x_j^\star = x_j$ for $j = 1,\dots,g$.

(iii) For seeing that $\star$ coincides with $*$ on all of $\mathbb{C}\langle\!\langle x_1,\dots,x_g\rangle\!\rangle$, we consider the subset $K \subseteq \mathbb{C}\langle\!\langle x_1,\dots,x_g\rangle\!\rangle$ given by

$$K := \{\mathfrak{r} \in \mathbb{C}\langle\!\langle x_1,\dots,x_g\rangle\!\rangle \mid \mathfrak{r}^\star = \mathfrak{r}^*\}.$$

By using the properties of $\star$ appearing in part (ii) and the corresponding properties of $*$, which were collected in part (iii) of Lemma III.2.35, it is easily seen that K forms a skew sub-field of $\mathbb{C}\langle\!\langle x_1,\dots,x_g\rangle\!\rangle$, which contains $\mathbb{C}\langle x_1,\dots,x_g\rangle$. Since $\mathbb{C}\langle\!\langle x_1,\dots,x_g\rangle\!\rangle$ is known to be epic, this enforces $K = \mathbb{C}\langle\!\langle x_1,\dots,x_g\rangle\!\rangle$, so that $\mathfrak{r}^\star = \mathfrak{r}^*$ for all $\mathfrak{r} \in \mathbb{C}\langle\!\langle x_1,\dots,x_g\rangle\!\rangle$. □

Among all pure linear representations representing the same non-commutative rational function, it is natural to consider those having minimal dimension. In fact, as we will see in Chapter IV, working with pure linear representation of minimal (or at least small) dimension is of great advantage if they are used for numerical computations.

Definition III.2.37. A pure linear representation $\rho = (u, Q, v)$ of a non-commutative rational expression $\mathfrak{r}$ is called *minimal*, if the dimension of ρ is minimal among all pure linear representations $\rho' = (u', Q', v')$ of $\mathfrak{r}$.

Surprisingly, if a minimal pure linear representation ρ is given, then any other pure linear representation ρ' of the same non-commutative rational function can be transformed by some base change in such a way, that ρ' appears as a block in ρ. This is the content of the following theorem. In the regular case, Algorithm III.4.15 will make this construction more transparent.

Theorem III.2.38 (Theorem 1.4, [**CR99**]). *If $\rho = (u, Q, v)$ and $\rho' = (u', Q', v')$ are pure linear representations of the same non-commutative rational function, where ρ is minimal, then ρ' is isomorphic (in the sense of morphisms) to a representation $\rho'' = (u'', Q'', v'')$, which has a block decomposition of the form*

$$u'' = \begin{pmatrix} u_1 & u & 0 \end{pmatrix}, \quad Q'' = \begin{pmatrix} Q_{1,1} & 0 & 0 \\ Q_{2,1} & Q & 0 \\ Q_{3,1} & Q_{3,2} & Q_{3,3} \end{pmatrix}, \quad v'' = \begin{pmatrix} 0 \\ v \\ v_3 \end{pmatrix}.$$

Although Theorem III.2.38 tells us that we can reduce in principle any pure linear realization to a minimal one, it remains unclear how this cutting down works concretely. For the important case of non-commutative rational expressions, which are *regular (at* 0*)* (see Definition III.4.7 below), one can give an alternative and more intrinsic characterization of minimality – albeit its slightly different meaning within the theory of descriptor realizations – that results in an explicit algorithm, by which any descriptor realization can be reduced to a minimal one; see Definition III.4.11 and Algorithm III.4.15, which are taken from [**HMS15**]. Without going into details, we note that the authors of [**CR99**] provide a similar characterization of minimality within their framework of pure linear representations. The conditions given there (namely being *prime* and *monic*) seem to be closely related to those, which are formulated in Definition III.4.11 for the case of regular non-commutative functions, but the situation for general non-commutative rational expressions turns out to be much more complicated. Some striking progress was made by J. Volcic in [**Vol15**] and we hope to address possible applications of this approach in future work. In Paragraph III.2.3.3, where we work instead in the context of formal linear representations (see Subsection III.2.3 below), we will come back to these questions about reducing the size.

We record the following interesting corollary of Theorem III.2.38.

Corollary III.2.39 (Corollary 1.6 in [**CR99**]). *Any two minimal pure linear representations of the same non-commutative rational function in $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ are isomorphic.*

This has the following interesting consequence: let $\mathfrak{r}$ be a non-commutative rational function in $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ and let $\rho = (u, Q, v)$ be a pure linear representation in the sense of Definition III.2.27 and Definition III.2.28, which is minimal (see Definition III.2.37). Suppose now that $\mathfrak{r}$ is self-adjoint with respect to the involution $*$ introduced in Lemma III.2.35. Lemma III.2.35 yields then that $\rho^* = (v^*, Q^*, u^*)$ is a pure linear representation of $\mathfrak{r}^*$ and hence of $\mathfrak{r}$. Since ρ^* has the same size as the minimal pure linear representation ρ, it must itself be minimal. Therefore, Corollary III.2.39 tells us that ρ and ρ^* are isomorphic, which means in the language of Definition III.2.31 that there exists a morphism (S, T) from ρ to ρ^* consisting of invertible matrices S and T over $\mathbb{C}$ with the same size, such that

$$v^* = uT, \qquad v = Su^*, \qquad \text{and} \qquad SQ^* = QT$$

holds. Taking adjoints in all three equalities gives us

$$v^* = uS^*, \qquad v = T^*u^*, \qquad \text{and} \qquad T^*Q^* = QS^*,$$

which means that not only (S, T) but also (T^*, S^*) forms a morphism from ρ to ρ^*. But why should this be an interesting fact? For seeing this, let us take a look at Remark III.4.14, where similar arguments will be used to prove that among all descriptor realizations of a given self-adjoint non-commutative rational function $\mathfrak{r}$, which is *regular (at 0)* in the sense of Definition III.4.7, one can always find a self-adjoint one. It seems to be a reasonable guess that the same conclusion also holds in the setting of pure linear representations, but it is not clear how one should adapt the proof from the regular case as given in Remark III.4.14. The construction there relies on the state space similarity theorem as stated in Item (i) of Lemma III.4.13, and on the uniqueness of the similarity transforms between two monic minimal descriptor realizations in particular. While Corollary III.2.39 gives a suitable replacement of the state space similarity theorem within the framework of pure linear representations, we do not know here anything about the uniqueness of the corresponding morphisms (for which we possibly need some additional conditions) and we also do not see how to modify the construction of Remark III.4.14 in order to get along without having such uniqueness results. Thus, unfortunately, we cannot finalize our arguments here, but we hope to settle this interesting problem in future work.

Let us conclude our excursion with the following remark.

Remark III.2.40. The rich theory of pure linear representations, as invented in [**CR99**], provides undeniably some handy approach to the skew field $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ of non-commutative rational functions. However, for the intended applications in Chapter IV, one would like to have some extension of that theory, which applies even to (rectangular) matrices of non-commutative rational functions. While generalizations of Definition III.2.27 and Definition III.2.28 are pretty much straightforward, such that even Theorem III.2.29 remains true (as the discussion in Subsection III.3.2 will show), it is not immediately clear if other important results, such as Theorem III.2.32 or the characterization of minimality, stay valid. This could possibly lead to an extension of [**Vol15**]. We leave these interesting questions to future work.

III.2.3. Formal linear representations. If one is interested in evaluations of non-commutative rational expressions, the typically nested inversions, which they involve, might cause some high computational effort. We note that the appearance of nested inversions leads – both for non-commutative rational expressions and functions – to the notion of *inversion height*; this quantity is treated in [**Reu96**]. From this point of view, the concept of pure linear representations, which was presented in Paragraph III.2.2.4, is very appealing, since it means that non-commutative rational functions can be brought into matrix form, where only one inversion is needed and where, even better, the expression that has to be inverted is linear in the considered variables. The drawback, however, is that pure linear representations provide formulas for non-commutative rational functions, which are equivalence classes of non-commutative rational expressions, and not for non-commutative rational expressions itself, so that it remains questionable whether these representations behave well under evaluations – in fact, as we will see in Section III.5, it is necessary to impose an additional condition on the algebra, such that this works.

Here, we will present the concept of formal linear representations of non-commutative rational expressions, which is very much inspired by pure linear representations as discussed in Paragraph III.2.2.4, but which has the additional advantage to behave well under evaluations on every unital complex algebra without imposing further conditions. Although Amitsur's construction of the free field as well as several variants thereof rely on evaluations of non-commutative rational expressions, especially on matrix algebras, such questions were seemingly not explored in detail before. Our exposition relies on [**HMS15**].

III.2.3.1. *Affine linear pencils.* Nested inversions in non-commutative rational functions were resolved by pure linear representations, but the price that we have to pay for this great simplification is that we need to pass over to matrix-valued expressions. In particular, the linear expression

$$Q = Q^{(0)} + Q^{(1)}x_1 + \cdots + Q^{(g)}x_g$$

that comes along with the pure linear representation $\rho = (u, Q, v)$ has matricial coefficients $Q^{(0)}, Q^{(1)}, \dots, Q^{(g)}$; see Definition III.2.28. Such objects are more generally called linear pencils and they are defined as follows.

DEFINITION III.2.41. Let $x = (x_1, \dots, x_g)$ be a g-tuple of formal variables.

(i) An expression of the form

$$Q = Q^{(0)} + Q^{(1)}x_1 + \cdots + Q^{(g)}x_g$$

with matrices $Q^{(0)}, Q^{(1)}, \dots, Q^{(g)} \in M_{n\times m}(\mathbb{C})$ is called *affine linear pencil (of size $n \times m$) in x*. If moreover $Q^{(0)} = 0$ holds, it would be appropriate to call Q a *linear pencil (of size $n \times m$) in x*, whenever we want to emphasize this fact. However, since this case will never occur in the following, this distinction becomes irrelevant for us and so we take the freedom to use both terms as synonyms.

(ii) If affine linear pencils $Q_{k,l} = Q^{(0)}_{k,l} + Q^{(1)}_{k,l}x_1 + \cdots + Q^{(g)}_{k,l}x_g$ of size $n_k \times m_l$ are given for $1 \leq k \leq K$ and $l \leq l \leq L$, we write

$$Q = \begin{pmatrix} Q_{1,1} & \dots & Q_{1,L} \\ \vdots & \ddots & \vdots \\ Q_{K,1} & \dots & Q_{K,L} \end{pmatrix}$$

for the linear pencil

$$Q = Q^{(0)} + Q^{(1)}x_1 + \cdots + Q^{(g)}x_g$$

of size $(n_1 + \cdots + n_K) \times (m_1 + \cdots + m_L)$, which is represented by the matrices

$$Q^{(j)} := \begin{pmatrix} Q^{(j)}_{1,1} & \dots & Q^{(j)}_{1,L} \\ \vdots & \ddots & \vdots \\ Q^{(j)}_{K,1} & \dots & Q^{(j)}_{K,L} \end{pmatrix}, \qquad \text{for } j = 0, 1, \dots, g.$$

(iii) If an affine linear pencil $Q = Q^{(0)} + Q^{(1)}x_1 + \cdots + Q^{(g)}x_g$ of size $n \times m$ and matrices $S \in M_n(\mathbb{C})$ and $T \in M_m(\mathbb{C})$ are given, we denote by SQT the affine linear pencil that is defined by

$$SQT := (SQ^{(0)}T) + (SQ^{(1)}T)x_1 + \cdots + (SQ^{(g)}T)x_g.$$

(iv) If an affine linear pencil $Q = Q^{(0)} + Q^{(1)}x_1 + \cdots + Q^{(g)}x_g$ of size $n \times m$ is given, then Q^* denotes the affine linear pencil of size $m \times n$, which is defined by

$$Q^* = (Q^{(0)})^* + (Q^{(1)})^*x_1 + \cdots + (Q^{(g)})^*x_g.$$

Formal linear representations will like pure linear representations involve inverses of affine linear pencils. Accordingly, we will mostly work with affine linear pencils whose coefficients are square matrices (i.e. $n = m$). However, on this formal level, taking inverses of affine linear pencils does not make any sense.

The most obvious solution would be to view affine linear pencils simply as elements in $M_n(\mathbb{C}\langle x_1,\dots,x_g\rangle)$. Inverses of affine linear pencils would then be defined as inverses in the unital complex algebra $M_n(\mathbb{C}\langle x_1,\dots,x_g\rangle)$. This definition, however, turns out to be much too restrictive as we are interested in the general case of non-commutative rational functions. In fact, we would only cover non-commutative polynomials in this way, like linearizations did in [**BMS13**].

In the spirit of [**Coh85, Coh06**], it would be more appropriate to check instead whether an affine linear pencil of size $n\times n$ is full in $M_n(\mathbb{C}\langle x_1,\dots,x_g\rangle)$. The general theory [**Coh85, Coh06, Mal82**] tells us that such matrices become invertible as matrices over the universal skew-field of non-commutative rational functions in the variables x; see Subsection III.2.2. But since we are interested in evaluations, we prefer at the moment to work with non-commutative rational expressions instead of non-commutative rational functions.

Our expedient here is to impose some universality condition, which guarantees invertibility of the given affine linear pencil under evaluations on every unital complex algebra. Evaluations of affine linear pencils are in fact straightforward and the question of invertibility leads us directly to the notion of domains for the formal object Q^{-1}. Notably, this will be generalized in Section III.3, where Q^{-1} is considered as a particular example of matrix-valued rational expressions.

Definition III.2.42. Let $x=(x_1,\dots,x_g)$ be formal variables and let $\mathcal{A}$ be some unital complex algebra.

(i) Let $Q=Q^{(0)}+Q^{(1)}x_1+\cdots+Q^{(g)}x_g$ be an affine linear pencil of size $n\times m$ in the variables x. We define the evaluation $Q(X)\in M_{n\times m}(\mathcal{A})$ of Q at some point $X=(X_1,\dots,X_g)\in\mathcal{A}^g$ by

$$Q(X):=Q^{(0)}1_{\mathcal{A}}+Q^{(1)}X_1+\cdots+Q^{(g)}X_g.$$

(ii) Let Q be an affine linear pencil of size $n\times n$. We put

$$\operatorname{dom}_{\mathcal{A}}(Q^{-1}):=\{X\in\mathcal{A}^g \mid Q(X)\text{ is invertible in }M_n(\mathcal{A})\}.$$

III.2.3.2. *Definition and construction of formal linear representations.* We are prepared now to introduce the main actor of this Subsection, namely formal linear representations. By formal linear representations, we encode non-commutative rational expressions in such a way that evaluations on their domains can be computed by carrying out only inversion, even if the rational expression itself involves several nested inversions.

Our approach, which was presented for the first time in [**HMS15**], is very much inspired by the work of Cohn [**Coh85, Coh06**] and Malcolmson [**Mal78, Mal80, Mal82**] and is furthermore closely related to constructions appearing in [**CR99**]. It also builds on the well-established theory of descriptor realizations for the case of regular non-commutative rational expressions, which was of particular interest in [**HMS15**]; we will come back to this in Subsection III.4.3. The main difference between all previously developed concepts and the approach of formal linear representations is that non-commutative rational expressions are addressed instead of non-commutative rational functions and that domains and evaluations on arbitrary unital complex algebras are taken into account.

Definition III.2.43. Let r be any rational expression in the formal variables $x=(x_1,\dots,x_g)$. A *formal linear representation* $\rho=(u,Q,v)$ *of* r consists of

- an affine linear pencil Q of size $n \times n$,
- a $1 \times n$-matrix u (i.e. a row vector) over $\mathbb{C}$,
- and a $n \times 1$-matrix v (i.e. a column vector) over $\mathbb{C}$,

and it satisfies the following property:

For any unital complex algebra $\mathcal{A}$, we have that

$$\operatorname{dom}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(Q^{-1})$$

and it holds true for any $(X_1, \dots, X_g) \in \operatorname{dom}_{\mathcal{A}}(r)$ that

$$r(X_1, \dots, X_g) = -uQ(X_1, \dots, X_g)^{-1} v.$$

The main result of this paragraph is the following theorem, by which we ensure that formal linear representations exist for each non-commutative rational expression.

Theorem III.2.44. *Each non-commutative rational expression $r \in \mathfrak{R}_{\mathbb{C}}(x_1, \dots, x_g)$ possesses a formal linear representation in the sense of Definition III.2.43.*

Later, in Theorem III.3.11, we will present some variant of Theorem III.2.44 for the case of self-adjoint non-commutative rational expressions. Operator-valued generalizations of both of these results will be given in Theorems III.2.58 and III.3.20.

In a sense more important than the actual statement of Theorem III.2.44 is its proof, since it fully relies on constructive arguments. It is provided by the following algorithm, whose validity will be checked in Paragraph III.2.3.4.

Algorithm III.2.45. *Let r be a non-commutative rational expression in the formal variables $x = (x_1, \dots, x_g)$. A formal linear representation $\rho = (u, Q, v)$ of r can be constructed by using successively (see Remark III.2.2) the following rules:*

(i) *For scalars $\lambda \in \mathbb{C}$ and the variables x_j, $j = 1, \dots, g$, formal linear representations are given by*

$$\rho_{x_j} := \left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} x_j & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \quad \text{and}$$
$$\rho_{\lambda} := \left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right), \tag{III.10}$$

respectively.

(ii) *If $\rho_1 = (u_1, Q_1, v_1)$ and $\rho_2 = (u_2, Q_2, v_2)$ are formal linear representations for the rational expressions r_1 and r_2, respectively, then*

$$\rho_1 \oplus \rho_2 := \left(\begin{pmatrix} u_1 & u_2 \end{pmatrix}, \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right) \tag{III.11}$$

gives a formal linear representation of $r_1 + r_2$.

(iii) *If $\rho_1 = (u_1, Q_1, v_1)$ and $\rho_2 = (u_2, Q_2, v_2)$ are formal linear representations for the rational expressions r_1 and r_2, respectively, then*

$$\rho_1 \odot \rho_2 := \left(\begin{pmatrix} 0 & u_1 \end{pmatrix}, \begin{pmatrix} v_1 u_2 & Q_1 \\ Q_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \right) \tag{III.12}$$

gives a formal linear representation of $r_1 \cdot r_2$.

(iv) *If $\rho = (u, Q, v)$ is a formal linear representation of r, then*

$$\rho^{-1} := \left(\begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & u \\ v & -Q \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \tag{III.13}$$

gives a formal linear representation of r^{-1}.

Note that the operations (III.10), (III.11), (III.12), and (III.13), which are described in Algorithm III.2.45, have to be understood on the level of linear pencils as explained in Definition III.2.41.

In Theorem III.2.29, we claimed existence of pure linear representations in the sense of Cohn and Reutenauer [**CR99**]; see Definition III.2.27 and Definition III.2.28. Theorem III.2.44, in combination with Algorithm III.2.45, yields an alternative and even constructive proof of this important fact, as soon as we are able to relate formal linear representations with pure linear representations. This will be the content of Corollary III.2.47. But before we can formulate the precise statement, we need some preparation.

First of all, it is appropriate to distinguish between the variables $x = (x_1, \dots, x_g)$ of the universal skew field of fractions $\mathbb{C}\langle\!\!\langle x_1, \dots, x_g \rangle\!\!\rangle$ and the formal variables, out of which non-commutative rational expressions are built, say $z = (z_1, \dots, z_g)$. Let us furthermore agree on the following terminology.

Definition III.2.46. Let $\mathfrak{r} \in \mathbb{C}\langle\!\!\langle x_1, \dots, x_g \rangle\!\!\rangle$ and some rational expression $r \in \mathfrak{R}_{\mathbb{C}}(z)$ be given. Then we say that $\mathfrak{r}$ *is represented by* r (or that r *represents* $\mathfrak{r}$), if the conditions

$$x = (x_1, \dots, x_g) \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x_1,\dots,x_g \rangle\!\rangle}(r) \qquad \text{and} \qquad r(x) = \mathfrak{r}$$

are satisfied with respect to the complex unital algebra $\mathbb{C}\langle\!\!\langle x_1, \dots, x_g \rangle\!\!\rangle$.

The previous definition is actually nothing but a fancy wrapping of some trivial observation, which however allows us a formally clean treatment of the relation between non-commutative rational expressions and rational functions. Indeed, due to all the relations that are valid in $\mathbb{C}\langle\!\!\langle x_1, \dots, x_g \rangle\!\!\rangle$, we are used to write down a non-commutative rational functions $\mathfrak{r}$ by picking one concrete non-commutative rational expression r, for which $\mathfrak{r} = r(x_1, \dots, x_g)$ holds, but keeping in mind that the chosen r is far from being unique and can be changed by applying the arithmetic rules of the free field. For instance, one easily sees that in $\mathbb{C}\langle\!\!\langle x_1, x_2 \rangle\!\!\rangle$

$$\mathfrak{r} := x_1 x_2 (x_1 x_2 - x_2 x_1)^{-1} = 1 + x_2 x_1 (x_1 x_2 - x_2 x_1)^{-1}$$

holds true (this example is taken from [**KV12**]), such that the two non-commutative rational expressions $r_1, r_2 \in \mathfrak{R}_{\mathbb{C}}(z_1, z_2)$, which are given by

$$r_1 := (z_1 \cdot z_2) \cdot (z_1 \cdot z_2 + (-1) \cdot z_2 \cdot z_1)^{-1} \qquad \text{and} \qquad r_2 := 1 + (z_2 \cdot z_1) \cdot (z_1 \cdot z_2 + (-1) \cdot z_2 \cdot z_1)^{-1},$$

both represent $\mathfrak{r}$ in the sense of Definition III.2.46.

Now, we can formulate the promised result relating formal linear representations and pure linear representations, by which we furthermore recover Theorem III.2.29. A self-adjoint version will be presented in Corollary III.2.59.

Corollary III.2.47. *Let $\mathfrak{r} \in \mathbb{C}\langle\!\!\langle x_1, \dots, x_g \rangle\!\!\rangle$ be a non-commutative rational function in the variables $x = (x_1, \dots, x_g)$. Then the following statements hold true:*

(i) *There exists a non-commutative rational expression $r \in \mathfrak{R}_{\mathbb{C}}(z)$ in formal variables $z = (z_1, \dots, z_g)$, such that r represents $\mathfrak{r}$ in the sense of Definition III.2.46.*

(ii) *If $\mathfrak{r}$ is represented by some non-commutative rational expression $r \in \mathfrak{R}_{\mathbb{C}}(z)$, then each formal linear representation $\rho = (u, Q, v)$ of r induces by $(u, Q(x), v)$ a pure linear representation of $\mathfrak{r}$ in the sense of Definition III.2.27 and Definition III.2.28.*

Consequently, each non-commutative rational function has a pure linear representation.

Proof. (i) Let us denote by K the subset of $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ consisting of all non-commutative rational functions $\mathfrak{r}$, which are represented by some non-commutative rational expression $r \in \mathfrak{R}_{\mathbb{C}}(z)$. It is clearly the case that K contains all scalars $\lambda \in \mathbb{C}$ and the variables $x_1, \dots, x_g$. Furthermore, one easily sees that

- if $\mathfrak{r}_1$ and $\mathfrak{r}_2$ are elements in K, which are represented by r_1 and r_2, respectively, then $\mathfrak{r}_1 + \mathfrak{r}_2$ is represented by $r_1 + r_2$ and $\mathfrak{r}_1 \cdot \mathfrak{r}_2$ is represented by $r_1 \cdot r_2$, so that $\mathfrak{r}_1 + \mathfrak{r}_2 \in K$ and $\mathfrak{r}_1 \cdot \mathfrak{r}_2 \in K$;
- if $\mathfrak{r} \neq 0$ belongs to K, which is represented by r, then $\mathfrak{r}^{-1}$ is represented by r^{-1}, so that $\mathfrak{r}^{-1} \in K$ holds.

It follows that K forms a skew sub-field of $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$, which contains $\mathbb{C}\langle x_1, \dots, x_g \rangle$. Since $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ is epic, it follows $K = \mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ and hence the validity of (i).

(ii) Let us consider $r \in \mathfrak{R}_{\mathbb{C}}(z)$ and any formal linear representation $\rho = (u, Q, v)$ of r. If now the non-commutative rational function $\mathfrak{r} \in \mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ is represented by r, we have that $x \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x_1,\dots,x_g \rangle\!\rangle}(r)$ and $r(x) = \mathfrak{r}$, such that

$$x \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x_1,\dots,x_g \rangle\!\rangle}(r) \subseteq \operatorname{dom}_{\mathbb{C}\langle\!\langle x_1,\dots,x_g \rangle\!\rangle}(Q^{-1})$$

and $\mathfrak{r} = r(x) = -uQ(x)^{-1}v$ holds according to the defining properties of ρ. For the desired conclusion that $\rho(x) = (u, Q(x), v)$ is indeed a pure linear representation of $\mathfrak{r}$, it is only left to note that the invertibility of $Q(x)$ in $M_n(\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle)$ means equivalently that $Q(x)$ is a full matrix over the ring $\mathbb{C}\langle x_1, \dots, x_g \rangle$.

The additional assertion follows now from (i), (ii), and Theorem III.2.29. □

Let us mention that there is still another canonical relation between non-commutative rational expressions and elements in the free field. It comes from the construction of $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ as the set of equivalence classes $\mathfrak{R}^0_{\mathbb{C}}(z_1, \dots, z_g)/\!\sim$, which we presented in Paragraph III.2.2.2. Correspondingly, each non-commutative rational function $\mathfrak{r}$ can be written as $\mathfrak{r} = [r]$ for some non-commutative rational expression $r \in \mathfrak{R}^0_{\mathbb{C}}(z_1, \dots, z_g)$. We are tempted to believe that these two notions are in fact equivalent, meaning that $r \in \mathfrak{R}_{\mathbb{C}}(z_1, \dots, z_g)$ represents $\mathfrak{r} \in \mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ in the sense of Definition III.2.46 if and only if the conditions $r \in \mathfrak{R}^0_{\mathbb{C}}(z_1, \dots, z_g)$ (or, equivalently, $\operatorname{dom}_{M(\mathbb{C})}(r) \neq \emptyset$) and $[r] = \mathfrak{r}$ are both satisfied; the following proposition confirms that this guess is indeed correct.

Proposition III.2.48. *If we identify $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle = \mathfrak{R}^0_{\mathbb{C}}(z_1, \dots, z_g)/\!\sim$, meaning that $x_i = [z_i]$ for $i = 1, \dots, g$, then the following statements hold true:*

(a) *A non-commutative rational expression $r \in \mathfrak{R}_{\mathbb{C}}(z_1, \dots, z_g)$ is non-degenerate in the sense of Definition III.2.13 if and only if $(x_1, \dots, x_g) \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x_1,\dots,x_g \rangle\!\rangle}(r)$ holds. Whenever these equivalent conditions are satisfied, we have that*

$$r(x_1, \dots, x_g) = [r].$$

(b) *If $r \in \mathfrak{R}_{\mathbb{C}}(z_1, \dots, z_g)$ and $\mathfrak{r} \in \mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ are given, then the following statements are equivalent:*

(i) *r represents $\mathfrak{r}$ in the sense of Definition III.2.46, i.e., we have that*

$$(x_1,\dots,x_g) \in \mathrm{dom}_{\mathbb{C}\langle\!\!\langle x_1,\dots,x_g\rangle\!\!\rangle}(r) \qquad \text{and} \qquad r(x_1,\dots,x_g) = \mathfrak{r}.$$

(ii) *We have that*

$$\mathrm{dom}_{M(\mathbb{C})}(r) \neq \emptyset \qquad \text{and} \qquad [r] = \mathfrak{r}.$$

The proof of Proposition III.2.48 requires some preparation.

Let us first introduce some notation. If $\mathfrak{X}$ is any subset of $\mathfrak{R}_{\mathbb{C}}(z_1,\dots,z_g)$, we denote by $\langle\mathfrak{X}\rangle$ the closure of $\mathfrak{X}$ with respect to $+$ and $\cdot$, i.e., the smallest subset $\langle\mathfrak{X}\rangle \subseteq \mathfrak{R}_{\mathbb{C}}(z_1,\dots,z_g)$, which satisfies $\mathfrak{X} \subseteq \langle\mathfrak{X}\rangle$ and which is closed under both operations $+$ and $\cdot$ in the sense that $r_1, r_2 \in \langle\mathfrak{X}\rangle$ implies $r_1 + r_2 \in \langle\mathfrak{X}\rangle$ and $r_1 \cdot r_2 \in \langle\mathfrak{X}\rangle$.

Lemma III.2.49. *Let $\mathfrak{D} \subseteq \mathfrak{R}_{\mathbb{C}}(z_1,\dots,z_g)$ be any subset with the following properties:*

(i) *$\mathfrak{D}$ is closed under the arithmetic operations $+$ and $\cdot$, i.e., $r_1, r_2 \in \mathfrak{D}$ implies that $r_1 + r_2 \in \mathfrak{D}$ and $r_1 \cdot r_2 \in \mathfrak{D}$.*
(ii) *If $r \in \mathfrak{D}$ decomposes as $r = r_1 + r_2$ or $r = r_1 \cdot r_2$ for certain $r_1, r_2 \in \mathfrak{R}_{\mathbb{C}}(z_1,\dots,z_g)$, then necessarily $r_1, r_2 \in \mathfrak{D}$.*

Then, for each subset $\mathfrak{X}$ of $\mathfrak{R}_{\mathbb{C}}(z_1,\dots,z_g)$, we have that $\langle\mathfrak{X}\rangle \cap \mathfrak{D} = \langle\mathfrak{X} \cap \mathfrak{D}\rangle$.

Proof. Since $\mathfrak{X} \subseteq \langle\mathfrak{X}\rangle$ holds by definition of the closure $\langle\mathfrak{X}\rangle$, we have that $\mathfrak{X} \cap \mathfrak{D} \subseteq \langle\mathfrak{X}\rangle \cap \mathfrak{D}$, and since the properties of $\langle\mathfrak{X}\rangle$ and $\mathfrak{D}$ guarantee that the set $\langle\mathfrak{X}\rangle \cap \mathfrak{D}$ is closed under the arithmetic operations $+$ and $\cdot$, it follows $\langle\mathfrak{X} \cap \mathfrak{D}\rangle \subseteq \langle\mathfrak{X}\rangle \cap \mathfrak{D}$. For the opposite inclusion, let us consider the set $\mathfrak{R} \subseteq \langle\mathfrak{X}\rangle$, which is given by

$$\mathfrak{R} := \big\{ r \in \langle\mathfrak{X}\rangle \big|\ r \in \mathfrak{D} \implies r \in \langle\mathfrak{X} \cap \mathfrak{D}\rangle \big\}.$$

We clearly have $\mathfrak{X} \subseteq \mathfrak{R}$ and one can also show that $\mathfrak{R}$ is closed under $+$ and $\cdot$. We consider only $+$, the argument for $\cdot$ is completely analogous. Let $r_1, r_2 \in \mathfrak{R}$ be given. Since we have $r_1, r_2 \in \langle\mathfrak{X}\rangle$, the properties of $\langle\mathfrak{X}\rangle$ imply that $r_1 + r_2 \in \langle\mathfrak{X}\rangle$. Furthermore, whenever $r_1 + r_2 \in \mathfrak{D}$ is satisfied, our assumptions about $\mathfrak{D}$ enforce that $r_1, r_2 \in \mathfrak{D}$, from which $r_1, r_2 \in \langle\mathfrak{X} \cap \mathfrak{D}\rangle$ follows due to $r_1, r_2 \in \mathfrak{R}$ and finally $r_1 + r_2 \in \langle\mathfrak{X} \cap \mathfrak{D}\rangle$. This verifies $r_1 + r_2 \in \mathfrak{R}$. From these properties of $\mathfrak{R}$, we infer that $\langle\mathfrak{X}\rangle \subseteq \mathfrak{R}$ must hold, so that in total $\mathfrak{R} = \langle\mathfrak{X}\rangle$. The latter means that $\langle\mathfrak{X}\rangle \cap \mathfrak{D} \subseteq \langle\mathfrak{X} \cap \mathfrak{D}\rangle$, which concludes the proof. $\square$

In Remark III.2.2, we provided some useful characterization of $\mathfrak{R}_{\mathbb{C}}(z_1,\dots,z_g)$, by which we made precise how $\mathfrak{R}_{\mathbb{C}}(z_1,\dots,z_g)$ can be seen as being "generated" by $z_1,\dots,z_g$ with respect to the arithmetic operations $+$, $\cdot$, and $^{-1}$. For the proof of Proposition III.2.48, we will need a similar characterization for the subsets $\mathfrak{R}^0_{\mathbb{C}}(z_1,\dots,z_g)$ and $\widetilde{\mathfrak{R}}^0_{\mathbb{C}}(z_1,\dots,z_g)$ of $\mathfrak{R}_{\mathbb{C}}(z_1,\dots,z_g)$, which are given by

$$\mathfrak{R}^0_{\mathbb{C}}(z_1,\dots,z_g) := \{ r \in \mathfrak{R}_{\mathbb{C}}(z_1,\dots,z_g) |\ \mathrm{dom}_{M(\mathbb{C})}(r) \neq \emptyset \},$$
$$\widetilde{\mathfrak{R}}^0_{\mathbb{C}}(z_1,\dots,z_g) := \{ r \in \mathfrak{R}_{\mathbb{C}}(z_1,\dots,z_g) |\ (x_1,\dots,x_g) \in \mathrm{dom}_{\mathbb{C}\langle\!\!\langle x_1,\dots,x_g\rangle\!\!\rangle}(r) \}.$$

Note that $\mathfrak{R}^0_{\mathbb{C}}(z_1,\dots,z_g)$ already appeared in Theorem III.2.19, where we summarized the construction of the free field $\mathbb{C}\langle\!\!\langle x_1,\dots,x_g\rangle\!\!\rangle$ following the approach of Amitsur. Since these subsets are determined by certain constraints on the domains of their elements, they have in either case some rigid structure. With the following definition, we extract now the essential properties, which $\mathfrak{R}^0_{\mathbb{C}}(z_1,\dots,z_g)$ and $\widetilde{\mathfrak{R}}^0_{\mathbb{C}}(z_1,\dots,z_g)$ have in common.

Definition III.2.50. Consider a subset $\mathfrak{D} \subseteq \mathfrak{R}_{\mathbb{C}}(z_1, \ldots, z_g)$. We call $\mathfrak{D}$ a *natural domain*, if it satisfies the following three properties:

(i) $\mathfrak{D}$ is closed under the arithmetic operations $+$ and $\cdot$, i.e., $r_1, r_2 \in \mathfrak{D}$ implies that $r_1 + r_2 \in \mathfrak{D}$ and $r_1 \cdot r_2 \in \mathfrak{D}$.
(ii) If $r \in \mathfrak{D}$ decomposes as $r = r_1 + r_2$ or $r = r_1 \cdot r_2$ for certain $r_1, r_2 \in \mathfrak{R}_{\mathbb{C}}(z_1, \ldots, z_g)$, then necessarily $r_1, r_2 \in \mathfrak{D}$.
(iii) If $r \in \mathfrak{D}$ can be written as $r = r_0^{-1}$ for some $r_0 \in \mathfrak{R}_{\mathbb{C}}(z_1, \ldots, z_g)$, then $r_0 \in \mathfrak{D}$.

With the help of Definition III.2.3, we can easily check that both $\mathfrak{R}^0_{\mathbb{C}}(z_1, \ldots, z_g)$ and $\widetilde{\mathfrak{R}}^0_{\mathbb{C}}(z_1, \ldots, z_g)$ are natural domains in the sense of the previous definition. Next, we want to prove that natural domains enjoy a characterization similar to Remark III.2.2.

Corollary III.2.51. *Let $\mathfrak{D} \subseteq \mathfrak{R}_{\mathbb{C}}(z_1, \ldots, z_g)$ be a natural domain. Suppose that $\mathfrak{R}$ is a subset of $\mathfrak{D}$, which has the following properties:*

(i) *We have that $\mathfrak{P}_{\mathbb{C}}(z_1, \ldots, z_g) \cap \mathfrak{D} \subseteq \mathfrak{R}$.*
(ii) *The non-commutative rational expression r^{-1} belongs to $\mathfrak{R}$ for each $r \in \mathfrak{R}$, which satisfies $r^{-1} \in \mathfrak{D}$.*
(iii) *$\mathfrak{R}$ is closed under the arithmetic operations $+$ and $\cdot$, i.e., $r_1, r_2 \in \mathfrak{R}$ implies that $r_1 + r_2 \in \mathfrak{R}$ and $r_1 \cdot r_2 \in \mathfrak{R}$.*

In this case, we necessarily have that $\mathfrak{R} = \mathfrak{D}$ holds true.

Proof. Since $\mathfrak{R}$ is supposed to be a subset of $\mathfrak{D}$, it suffices to prove $\mathfrak{D} \subseteq \mathfrak{R}$. We consider the exhaustion $(\mathfrak{R}_n)_{n\geq 0}$ of $\mathfrak{R}_{\mathbb{C}}(z_1, \ldots, z_g)$, i.e.

$$\mathfrak{R}_{\mathbb{C}}(z_1, \ldots, z_g) = \bigcup_{m\geq 0} \mathfrak{R}_m \supset \cdots \supset \mathfrak{R}_{n+1} \supset \mathfrak{R}_n \supset \cdots \supset \mathfrak{R}_0,$$

which is defined inductively by $\mathfrak{R}_0 := \mathfrak{P}_{\mathbb{C}}(z_1, \ldots, z_g)$ and $\mathfrak{R}_{n+1} := \langle \mathfrak{R}_n \cup \{r^{-1} |\ r \in \mathfrak{R}_n\}\rangle$ for all $n \geq 0$. We clearly have $\mathfrak{D} = \bigcup_{m\geq 0} \mathfrak{D}_m$, where we put $\mathfrak{D}_m := \mathfrak{R}_m \cap \mathfrak{D}$ for $m \geq 0$, and we claim that

$$\mathfrak{D}_{n+1} = \langle \mathfrak{D}_n \cup \{r^{-1} |\ r \in \mathfrak{D}_n : r^{-1} \in \mathfrak{D}\}\rangle$$

for each $n \geq 0$. This follows easily with the help of Lemma III.2.49. Indeed,

$$\begin{aligned} \mathfrak{D}_{n+1} &= \mathfrak{R}_{n+1} \cap \mathfrak{D} \\ &= \langle \mathfrak{R}_n \cup \{r^{-1} |\ r \in \mathfrak{R}_n\}\rangle \cap \mathfrak{D} \\ &= \langle (\mathfrak{R}_n \cup \{r^{-1} |\ r \in \mathfrak{R}_n\}) \cap \mathfrak{D}\rangle \\ &= \langle \mathfrak{D}_n \cup (\{r^{-1} |\ r \in \mathfrak{R}_n\} \cap \mathfrak{D})\rangle \\ &= \langle \mathfrak{D}_n \cup \{r^{-1} |\ r \in \mathfrak{D}_n : r^{-1} \in \mathfrak{D}\}\rangle, \end{aligned}$$

where we used in the last step that $\mathfrak{D}$ enjoys the property stated in Item (iii). Now, by induction on m, we can prove that $\mathfrak{D}_m \subseteq \mathfrak{R}$ holds for all $m \geq 0$. Indeed, the initial case $m = 0$ is obtained by

$$\mathfrak{D}_0 = \mathfrak{R}_0 \cap \mathfrak{D} = \mathfrak{P}_{\mathbb{C}}(z_1, \ldots, z_g) \cap \mathfrak{D} \subseteq \mathfrak{R},$$

and if $\mathfrak{D}_m \subseteq \mathfrak{R}$ is already established, then the other assumptions on $\mathfrak{R}$ guarantee that

$$\mathfrak{D}_{n+1} = \langle \mathfrak{D}_n \cup \{r^{-1} |\ r \in \mathfrak{D}_n : r^{-1} \in \mathfrak{D}\}\rangle \subseteq \mathfrak{R}.$$

In summary, we get $\mathfrak{D} = \bigcup_{m\geq 0} \mathfrak{D}_m \subseteq \mathfrak{R}$, as we wished to show. □

Note that Corollary III.2.51 is really a generalization of Remark III.2.2, since $\mathfrak{R}_{\mathbb{C}}(z_1, \ldots, z_g)$ forms a natural domain itself. Having Corollary III.2.51 at hand, we are prepared to return to our actual goal.

Proof of Proposition III.2.48. The proof is carried out in three steps. In order to simplify the notation, we will abbreviate $x = (x_1, \ldots, x_g)$ and $z = (z_1, \ldots, z_g)$ in the sequel.

(1) Let us look at the set of all non-commutative rational expressions $r \in \mathfrak{R}^0_{\mathbb{C}}(z) \cap \widetilde{\mathfrak{R}}^0_{\mathbb{C}}(z)$, which satisfy the condition $r(x) = [r]$. It is a trivial but nonetheless helpful observation that this set can be viewed in two different ways, namely as a subset of $\mathfrak{R} \subseteq \mathfrak{R}^0_{\mathbb{C}}(z)$ and as a subset $\widetilde{\mathfrak{R}} \subseteq \widetilde{\mathfrak{R}}^0_{\mathbb{C}}(z)$. More precisely, we put

$$\begin{aligned} \mathfrak{R} &:= \{r \in \mathfrak{R}^0_{\mathbb{C}}(z) |\ x \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x \rangle\!\rangle}(r),\ r(x) = [r]\}, \\ \widetilde{\mathfrak{R}} &:= \{r \in \widetilde{\mathfrak{R}}^0_{\mathbb{C}}(z) |\ \operatorname{dom}_{M(\mathbb{C})}(r) \neq \emptyset,\ r(x) = [r]\}, \end{aligned}$$

where clearly $\mathfrak{R} = \widetilde{\mathfrak{R}}$ holds true. Our goal is to show that $\mathfrak{R} = \mathfrak{R}^0_{\mathbb{C}}(z)$ and $\widetilde{\mathfrak{R}} = \widetilde{\mathfrak{R}}^0_{\mathbb{C}}(z)$. As soon as we have established these results, we may infer that

$$\mathfrak{R}^0_{\mathbb{C}}(z) = \{r \in \mathfrak{R}^0_{\mathbb{C}}(z) \cap \widetilde{\mathfrak{R}}^0_{\mathbb{C}}(z) |\ r(x) = [r]\} = \widetilde{\mathfrak{R}}^0_{\mathbb{C}}(z),$$

so that the statements made in (a) and (b) become obvious.

(2) Let us first address the asserted equality $\mathfrak{R} = \mathfrak{R}^0_{\mathbb{C}}(z)$. One easily confirms with the help of Definition III.2.3 and Remark III.2.16 that the set $\mathfrak{R}$, which obviously contains all scalars $\lambda \in \mathbb{C}$ as well as the variables $x_1, \ldots, x_g$, is furthermore closed under the arithmetic operations $+$ and $\cdot$. Correspondingly, taking inverses is the only issue that remains. What we need to prove is according to Corollary III.2.51 that r^{-1} belongs to $\mathfrak{R}$ for each $r \in \mathfrak{R}$, which satisfies the condition $r^{-1} \in \mathfrak{R}^0_{\mathbb{C}}(z)$. Take any $r \in \mathfrak{R}$ with $r^{-1} \in \mathfrak{R}^0_{\mathbb{C}}(z)$. Let us check first that $x \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x \rangle\!\rangle}(r)$ holds. Since r and r^{-1} belong both to $\mathfrak{R}^0_{\mathbb{C}}(z)$, Remark III.2.16 allows us to compute $[r] \cdot [r^{-1}] = [r \cdot r^{-1}] = [1] \in \mathbb{C}\langle\!\langle x \rangle\!\rangle$, from which $[r] \neq 0$ follows with $[r]^{-1} = [r^{-1}]$. Furthermore, due to $r \in \mathfrak{R}$, we know that $x \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x \rangle\!\rangle}(r)$ with $r(x) = [r]$, so that the invertibility of $[r]$ yields $x \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x \rangle\!\rangle}(r^{-1})$ and finally $r^{-1}(x) = r(x)^{-1} = [r]^{-1} = [r^{-1}]$. Altogether, this implies that $r^{-1} \in \mathfrak{R}$, which is exactly what we had to show.

(3) Now, let us address $\widetilde{\mathfrak{R}} = \widetilde{\mathfrak{R}}^0_{\mathbb{C}}(z)$. With the help of Definition III.2.3 and Remark III.2.16, it is not hard to see that the set $\widetilde{\mathfrak{R}}$, which clearly contains all scalars $\lambda \in \mathbb{C}$ as well as the variables $x_1, \ldots, x_g$, is also closed under the arithmetic operations $+$ and $\cdot$. What remains to check is according to Corollary III.2.51 that for each $r \in \widetilde{\mathfrak{R}}$, which satisfies $r^{-1} \in \widetilde{\mathfrak{R}}^0_{\mathbb{C}}(z)$, the non-commutative rational expression r^{-1} necessarily belongs to $\widetilde{\mathfrak{R}}$. Let us confirm first that $\operatorname{dom}_{M(\mathbb{C})}(r^{-1}) \neq \emptyset$ holds true. For seeing this, assume to the contrary that $\operatorname{dom}_{M(\mathbb{C})}(r^{-1}) = \emptyset$. Since $r \in \mathfrak{R}^0_{\mathbb{C}}(z)$ is satisfied by the assumption $r \in \widetilde{\mathfrak{R}}$, we may conclude with the help of Remark III.2.18 that r must be $M(\mathbb{C})$-evaluation equivalent to 0, i.e. $[r] = 0$. Moreover, $r \in \widetilde{\mathfrak{R}}$ guarantees $r(x) = [r]$, so that finally $r(x) = [r] = 0$ follows. Let us take now any formal linear representation $\rho = (u, Q, v)$ of r, whose existence is guaranteed by Theorem III.2.44, and use formula (III.13) provided in Algorithm III.2.45 in order to construct an explicit formal linear representation of r^{-1}, namely

$$\rho^{-1} = (\hat{u}, \hat{Q}, \hat{v}) = \left(\begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & u \\ v & -Q \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right).$$

Since $r^{-1} \in \widetilde{\mathfrak{R}}^0_{\mathbb{C}}(z)$ means that $x \in \mathrm{dom}_{\mathbb{C}\langle\!\langle x\rangle\!\rangle}(r^{-1})$, we conclude according to Definition III.2.43 that in particular $x \in \mathrm{dom}_{\mathbb{C}\langle\!\langle x\rangle\!\rangle}(r^{-1}) \subseteq \mathrm{dom}_{\mathbb{C}\langle\!\langle x\rangle\!\rangle}(\hat{Q}^{-1})$, which enforces the matrix $\hat{Q}(x)$ to be full over $\mathbb{C}\langle x\rangle$. On the other hand, we have established in Corollary III.2.47 that the formal linear representation $\rho = (u, Q, v)$ of r induces by $(u, Q(x), v)$ a pure linear representation of the non-commutative rational function $r(x)$, which is 0 as we have seen above. Obviously, $(u, -Q(x), v)$ yields a pure linear representation of 0 as well, Theorem III.2.26 tells us then that its display

$$\begin{pmatrix} 0 & u \\ v & -Q(x) \end{pmatrix} = \hat{Q}(x)$$

cannot be full. This contradiction tells us that $\mathrm{dom}_{M(\mathbb{C})}(r^{-1}) \neq \emptyset$, as we wished to show. Finally, in order to finish the proof of $r^{-1} \in \widetilde{\mathfrak{R}}$, we must show that $r^{-1}(x) = [r^{-1}]$ holds true. We already noticed $x \in \mathrm{dom}_{\mathbb{C}\langle\!\langle x\rangle\!\rangle}(r^{-1})$, so that $r^{-1}(x) = r(x)^{-1}$, and since we have $r \in \widetilde{\mathfrak{R}}$, we know that $r(x) = [r]$. With these facts, we may derive that indeed $r^{-1}(x) = r(x)^{-1} = [r]^{-1} = [r^{-1}]$, which concludes the proof. □

III.2.3.3. *Reducing the size of formal linear representations.* Before we proceed to the proof of Algorithm III.2.45 in the next Paragraph III.2.3.4, let us first illustrate its manner of functioning by the following instructive example.

Example III.2.52. Consider the non-commutative rational expressions

$$r_1 = (x_1 \cdot x_2)^{-1} \qquad \text{and} \qquad r_2 = x_2^{-1} \cdot x_1^{-1}.$$

By applying Algorithm III.2.45, we obtain for r_1 the formal linear representation

$$\begin{aligned} \rho_1 &= (\rho_{x_1} \odot \rho_{x_2})^{-1} \\ &= \left(\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & x_1 & -1 \\ 0 & 1 & -1 & 0 \\ x_2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right)^{-1} \\ &= \left(\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -x_1 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -x_2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) \end{aligned}$$

and similarly for r_2 the formal linear representation

$$\begin{aligned} \rho_2 &= \rho_{x_2}^{-1} \odot \rho_{x_1}^{-1} \\ &= \left(\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & -x_2 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \odot \left(\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & -x_1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -x_2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -x_1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right). \end{aligned}$$

Concerning the corresponding $\mathcal{A}$-domains with respect to any unital complex algebra $\mathcal{A}$, we can make the following statements:

- The $\mathcal{A}$-domain $\operatorname{dom}_{\mathcal{A}}(r_1)$ of r_1 is the set of pairs $(X_1, X_2) \in \mathcal{A}^2$, whose product X_1X_2 is invertible in $\mathcal{A}$, whereas the $\mathcal{A}$-domain of r_2 consists of all pairs (X_1, X_2) of invertible elements $X_1, X_2 \in \mathcal{A}$. Thus, clearly, $\operatorname{dom}_{\mathcal{A}}(r_2) \subseteq \operatorname{dom}_{\mathcal{A}}(r_1)$.
- The defining properties of ρ_1 and ρ_2 guarantee the inclusions $\operatorname{dom}_{\mathcal{A}}(r_1) \subseteq \operatorname{dom}_{\mathcal{A}}(Q_1^{-1})$ and $\operatorname{dom}_{\mathcal{A}}(r_2) \subseteq \operatorname{dom}_{\mathcal{A}}(Q_2^{-1})$, but apart from this, nothing can be said about $\operatorname{dom}_{\mathcal{A}}(Q_1^{-1})$ and $\operatorname{dom}_{\mathcal{A}}(Q_2^{-1})$ without additional calculations.

This example highlights the computational disadvantage of Algorithm III.2.45, that roughly speaking the dimension of the linear pencil Q of a formal linear representation $\rho = (u, Q, v)$ increases rapidly with the complexity of the corresponding rational expression r. Clearly, since the rational expressions r_1 and r_2 given in Example III.2.52 are rather simple, we guess that neither ρ_1 nor ρ_2 are minimal.

In analogy to Definition III.2.37, we call a formal linear representation $\rho = (u, Q, v)$ of some fixed rational expression r *minimal*, if the corresponding linear pencil Q has minimal size among all linear pencils that come from other formal linear representations $\rho' = (u', Q', v')$ of r.

Accordingly, we expect that there are other formal linear representations of smaller dimensions, but how should we find them?

Unfortunately, since r_1 and r_2 do not fall into the setting of Subsection III.4.3, we cannot use the machinery of descriptor realizations (see Algorithm III.4.15) to cut down these realizations to minimal ones. One expedient could be to use the analogous but more general machinery, which was invented recently in [**Vol15**]. However, due to some technical subtleties, we did not yet succeed in formulating an implementable algorithm.

Fortunately, we can give some (albeit less sophisticated) ad hoc construction: if we arrange any formal linear representation $\rho = (u, Q, v)$ of a given rational expression r as

$$\begin{array}{c|c} & u \\ \hline v & Q \end{array},$$

we can try to bring this array into the form

$$\begin{array}{c|cc} & \widetilde{u} & u' \\ \hline \widetilde{v} & \widetilde{Q} & 0 \\ v' & 0 & Q' \end{array}$$

by acting by elementary row and column operations on Q, while bookkeeping their effect in the first row and column, respectively. Note that, more generally, acting with invertible matrices $S, T \in M_n(\mathbb{C})$ on ρ according to the rule

$$S \cdot \rho \cdot T := (uT, SQT, Sv)$$

always produces another formal linear representation of the same non-commutative rational expression r. If it happens that (u', Q', v') is a *formal linear representation of* 0 *relatively to* r, meaning that for each unital complex algebra $\mathcal{A}$ the conditions $\operatorname{dom}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}((Q')^{-1})$ and $-u'Q'(X)^{-1}v' = 0$ for each $X \in \operatorname{dom}_{\mathcal{A}}(r)$ are satisfied, then we can remove this additional part, obtaining $\widetilde{\rho} = (\widetilde{u}, \widetilde{Q}, \widetilde{v})$. This $\widetilde{\rho}$ gives accordingly another formal linear representation of r.

Example III.2.53. Consider the non-commutative rational expressions from Example III.2.52 above. We can show by using the previously discussed method that

$$\widetilde{\rho}_1 = (\widetilde{u}_1, \widetilde{Q}_1, \widetilde{v}_1) = \left(\begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -x_1 \\ -x_2 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

gives another formal linear representation of r_1 and that

$$\widetilde{\rho}_2 = (\widetilde{u}_2, \widetilde{Q}_2, \widetilde{v}_2) = \left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -x_2 \\ -x_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

gives another formal linear representation of r_2. Regarding their domains with respect to any unital complex algebra $\mathcal{A}$, we collect the following observations:

- It is easy to see that the formal linear representations $\widetilde{\rho}_1$ and $\widetilde{\rho}_2$ satisfy the relation $\tilde{\rho}_2 = U^{-1} \cdot \tilde{\rho}_1 \cdot U$, i.e.

  $$\widetilde{u}_2 = \widetilde{u}_1 U, \qquad \widetilde{v}_1 = U\widetilde{v}_2, \qquad \text{and} \qquad U\widetilde{Q}_2 = \widetilde{Q}_1 U,$$

  with respect to the matrix

  $$U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

  These formulas are analogous to those that capture the equivalence of pure linear representations in Definition III.2.31. In particular, we have $\operatorname{dom}_{\mathcal{A}}(\widetilde{Q}_1^{-1}) = \operatorname{dom}_{\mathcal{A}}(\widetilde{Q}_2^{-1})$.
- By using the Schur complement formula (A.1) given in Lemma A.1, we may check that $Q_1(X_1, X_2)$ and hence $Q_2(X_1, X_2)$ are invertible in $M_2(\mathcal{A})$ for some pair $(X_1, X_2) \in \mathcal{A}^2$, if and only if their product $X_1 X_2$ is invertible in $\mathcal{A}$.

In summary, we obtain

$$\operatorname{dom}_{\mathcal{A}}(r_2) \subsetneq \operatorname{dom}_{\mathcal{A}}(r_1) = \operatorname{dom}_{\mathcal{A}}(Q_1^{-1}) = \operatorname{dom}_{\mathcal{A}}(Q_2^{-1}).$$

III.2.3.4. *Proof of Rules in Algorithm III.2.45.* First of all, we examine the validity of rule (i). This is the content of the following lemma, which gives a slightly more general statement and allows a uniform proof for formal linear representations both of scalars $\lambda \in \mathbb{C}$ and of the variables $x_1, \dots, x_g$.

Lemma III.2.54. *Consider a rational expression r in the variables $x = (x_1, \dots, x_g)$, which is of the form*

$$r = \lambda_0 + \lambda_1 \cdot x_1 + \cdots + \lambda_g \cdot x_g$$

for some $\lambda_0, \lambda_1, \dots, \lambda_g \in \mathbb{C}$ with any fixed order of summation. Then a formal linear representation of r is given by

$$\rho := \left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} \lambda_0 & -1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix} x_1 + \cdots + \begin{pmatrix} \lambda_g & 0 \\ 0 & 0 \end{pmatrix} x_g, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

Proof. Write $\rho = (u, Q, v)$. Given any unital complex algebra $\mathcal{A}$, we may observe that the matrix $Q(X)$ is invertible for all $X \in \operatorname{dom}_{\mathcal{A}}(r) = \mathcal{A}^g$ with

$$Q(X)^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & -r(X) \end{pmatrix}.$$

Thus, $X \in \operatorname{dom}_{\mathcal{A}}(Q^{-1})$ holds and furthermore $-uQ(X)^{-1}v = r(X)$. According to Definition III.2.43, this means that ρ is a formal linear representation of r, as we wished to show. □

Next, we justify the rules (ii) and (iii) of Algorithm III.2.45.

LEMMA III.2.55. *Let $\rho_1 = (u_1, Q_1, v_1)$ and $\rho_2 = (u_2, Q_2, v_2)$ be formal linear representations of rational expressions r_1 and r_2, respectively. Then the following statements hold true:*

- *A formal linear representation of $r_1 + r_2$ is given by*
$$\rho_1 \oplus \rho_2 := \left(\begin{pmatrix} u_1 & u_2 \end{pmatrix}, \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right).$$
- *A formal linear representation of $r_1 \cdot r_2$ is given by*
$$\rho_1 \odot \rho_2 := \left(\begin{pmatrix} 0 & u_1 \end{pmatrix}, \begin{pmatrix} v_1 u_2 & Q_1 \\ Q_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \right).$$

PROOF. Let $\mathcal{A}$ be any unital complex algebra. Recall from Definition III.2.3 that
$$\operatorname{dom}_{\mathcal{A}}(r_1 + r_2) = \operatorname{dom}_{\mathcal{A}}(r_1) \cap \operatorname{dom}_{\mathcal{A}}(r_2) = \operatorname{dom}_{\mathcal{A}}(r_1 \cdot r_2).$$
We take any $X \in \operatorname{dom}_{\mathcal{A}}(r_1) \cap \operatorname{dom}_{\mathcal{A}}(r_2)$. Since ρ_1 and ρ_2 are both formal linear representations, we know that $X \in \operatorname{dom}_{\mathcal{A}}(Q_1^{-1}) \cap \operatorname{dom}_{\mathcal{A}}(Q_2^{-1})$ holds and furthermore
$$r_1(X) = -u_1 Q_1(X)^{-1} v_1 \qquad \text{and} \qquad r_2(X) = -u_2 Q_2(X)^{-1} v_1.$$
Let $\rho = (u, Q, v)$ denote either $\rho_1 \oplus \rho_2$ or $\rho_1 \odot \rho_2$. The invertibility of $Q_1(X)$ and $Q_2(X)$ guarantees the invertibility of
$$Q(X) = \begin{pmatrix} Q_1(X) & 0 \\ 0 & Q_2(X) \end{pmatrix} \qquad \text{respectively} \qquad Q(X) = \begin{pmatrix} v_1 u_2 & Q_1(X) \\ Q_2(X) & 0 \end{pmatrix},$$
and by some straightforward computation we may convince ourselves that more precisely
$$Q(X)^{-1} = \begin{pmatrix} Q_1(X)^{-1} & 0 \\ 0 & Q_2(X)^{-1} \end{pmatrix}$$
respectively
$$Q(X)^{-1} = \begin{pmatrix} 0 & Q_2(X)^{-1} \\ Q_1(X)^{-1} & -Q_1(X)^{-1} v_1 u_2 Q_2(X)^{-1} \end{pmatrix}.$$
This shows in either case that $X \in \operatorname{dom}_{\mathcal{A}}(Q^{-1})$ holds. Furthermore, we can check that
$$\begin{aligned} -uQ(X)^{-1}v &= -\begin{pmatrix} u_1 & u_2 \end{pmatrix} \begin{pmatrix} Q_1(X)^{-1} & 0 \\ 0 & Q_2(X)^{-1} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ &= -u_1 Q_1(X)^{-1} v_1 - u_2 Q_2(X)^{-1} v_2 \\ &= r_1(X) + r_2(X) \end{aligned}$$
respectively
$$\begin{aligned} -uQ(X)^{-1}v &= -\begin{pmatrix} 0 & u_1 \end{pmatrix} \begin{pmatrix} 0 & Q_2(X)^{-1} \\ Q_1(X)^{-1} & -Q_1(X)^{-1} v_1 u_2 Q_2(X)^{-1} \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} \\ &= u_1 Q_1(X)^{-1} v_1 u_2 Q_2(X)^{-1} v_2 \\ &= r_1(X) r_2(X). \end{aligned}$$
Thus, we have confirmed that $\rho_1 \oplus \rho_2$ and $\rho_1 \odot \rho_2$ are formal linear representations of $r_1 + r_2$ and $r_1 \cdot r_2$, respectively. □

Finally, rule (iv) of Algorithm III.2.45 is concerned in the following lemma.

Lemma III.2.56. *Let $\rho = (u, Q, v)$ be a formal linear representation of some rational expression r. Then*

$$\rho^{-1} := \left(\begin{pmatrix} 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & u \\ v & -Q \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$$

gives a formal linear representation of r^{-1}.

Proof. Take any $X \in \mathrm{dom}_{\mathcal{A}}(r^{-1})$, which means by Definition III.2.3 that $X \in \mathrm{dom}_{\mathcal{A}}(r)$ holds and that $r(X)$ is invertible in $\mathcal{A}$. Since ρ is assumed to be a formal linear representation of r, this ensures the invertibility of $Q(X)$ and $-uQ(X)^{-1}v = r(X)$. Hence, the Schur complement formula, Lemma (A.1), tells us that the matrix

$$\begin{pmatrix} 0 & u \\ v & -Q(X) \end{pmatrix}$$

must be invertible since both its Schur complement, which is given by $uQ(X)^{-1}v = -r(X)$, and the block $Q(X)$ are invertible. Hence, we infer

$$X \in \mathrm{dom}_{\mathcal{A}} \left(\begin{pmatrix} 0 & u \\ v & -Q \end{pmatrix}^{-1} \right)$$

and the Schur complement formula (A.1) tells us that in this case

$$- \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & u \\ v & -Q(X) \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -(uQ(X)^{-1}v)^{-1} = r(X)^{-1}.$$

Thus, we see that ρ^{-1} is indeed a formal linear representation of r^{-1}. □

III.2.4. Self-adjoint formal linear representations. The previous Subsection III.2.3 provides by the concept of formal linear representations some very effective tool to treat evaluations of non-commutative rational expressions in a universal way. Their effectiveness relies more precisely on the fact that a formal linear representation $\rho = (u, Q, v)$ does not only resolve all nested inversions in the corresponding rational expression r, but it also "linearizes" r in terms of the affine linear pencil $Q = Q^{(0)} + Q^{(1)}x_1 + \cdots + Q^{(g)}x_g$. Accordingly, one might hope that these techniques can be applied to a wide range of problems, especially in free probability theory, where non-commutative rational expressions are intended to provide a new class of "non-commutative test functions" for studying non-commutative distributions.

For instance, regularity questions in the spirit of Chapter VI but for other classes than non-commutative polynomials (see also Chapter VII) constitute an active topic of current research. First partial results in this direction wing the hope that "linearization" could provide the right techniques to settle such problems for non-commutative rational expressions.

Another kind of problem will be addressed in Chapter IV. Following [**HMS15**], we will explain there in particular how formal linear representations can be used to compute the analytic distribution of any self-adjoint non-commutative rational expressions in freely independent self-adjoint variables under the assumption that their individual distributions are known. For this purpose, it is appropriate to rewrite the given ρ as an affine linear pencil $L = L^{(0)} + L^{(1)}x_1 + \cdots + L^{(g)}x_g$ according to

$$L := \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix} = \begin{pmatrix} 0 & u \\ v & Q^{(0)} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & Q^{(1)} \end{pmatrix} x_1 + \cdots + \begin{pmatrix} 0 & 0 \\ 0 & Q^{(g)} \end{pmatrix} x_g.$$

The algorithmic solution to this problem, which we are going to provide in Chapter IV, will be based besides these algebraic techniques on more analytic results from operator-valued free probability theory. However, in order to guarantee that the algebraic and the analytic component fit together, we need to find for self-adjoint r some special kind of formal linear representation ρ – in the following called "self-adjoint formal linear representations" – for which the affine linear pencil L consists only of self-adjoint matrices. In the free probability community, such an improvement of the "linearization trick" from [**HT05, HST06**] was found by Anderson [**And12, And13, And15**] and was extensively used in [**BMS13**]. His ingenious approach was, however, limited to the case of non-commutative polynomial expressions. As we became aware of recently, similar arguments were known in non-commutative control theory long before; see, for instance, [**Kal63, Kal76, HMV06, KV09, KV12**] and [**HMS15**]. This has led to the insight, which was worked out in [**HMS15**], that formal linear representations not only provide some straightforward extension of the "linearization trick" from non-commutative polynomials to rational expressions, but also constitute some very general framework, into which many other techniques merge.

In this subsection, we want to explain how arguments similar to those of [**And12, And13, And15**] (see also [**BMS13**]) can be used in order to construct the desired self-adjoint formal linear representations. Let us first present their precise definition.

Definition III.2.57. Let r be any non-commutative rational expression in the formal variables $x = (x_1, \dots, x_g)$, which is self-adjoint in the sense of Definition III.2.7. A *self-adjoint formal linear representation* $\rho = (Q, v)$ consists of

- an affine linear pencil
$$Q = Q^{(0)} + Q^{(1)}x_1 + \dots + Q^{(g)}x_g$$
with self-adjoint matrices $Q^{(0)}, Q^{(1)}, \dots, Q^{(g)} \in M_n(\mathbb{C})$ of some dimension n
- and a $n \times 1$-matrix v over $\mathbb{C}$

and it satisfies the following property:

For any unital complex $*$-algebra $\mathcal{A}$, we have that
$$\mathrm{dom}^{\mathrm{sa}}_{\mathcal{A}}(r) \subseteq \mathrm{dom}_{\mathcal{A}}(Q^{-1})$$
and for any $(X_1, \dots, X_g) \in \mathrm{dom}^{\mathrm{sa}}_{\mathcal{A}}(r)$ it holds true that
$$r(X_1, \dots, X_g) = -v^* Q(X_1, \dots, X_g)^{-1} v.$$

Following ideas of [**And12, And13, And15**] (see also [**BMS13**]), we can prove now the announced self-adjoint version of Theorem III.2.44.

Theorem III.2.58. *Each non-commutative rational expression $r \in \mathfrak{R}_{\mathbb{C}}(x_1, \dots, x_g)$, which is self-adjoint in the sense of Definition III.2.7, admits a self-adjoint formal linear representation $\rho = (Q, v)$ in the meaning of Definition III.2.57.*

Proof. Given any self-adjoint non-commutative rational expression r, then Theorem III.2.44 guarantees the existence of a formal linear representation $\rho_0 = (u_0, Q_0, v_0)$ of r, say of size $n \times n$. Inspired by [**And12, And13, And15**], we put

$$\rho = (Q, v) := \left(\begin{pmatrix} 0 & Q_0^* \\ Q_0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}u_0^* \\ v_0 \end{pmatrix} \right) \tag{III.14}$$

and we claim that ρ forms a self-adjoint formal linear representation of r. First of all, we notice that the linear pencil Q consists according to Definition III.2.41 of self-adjoint matrices. Thus, it only remains to check that $\operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(Q^{-1})$ and $r(X) = -v^* Q(X)^{-1} v$ for all $X = (X_1, \dots, X_g) \in \operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r)$, whenever $\mathcal{A}$ is a unital complex $*$-algebra. Given a unital complex $*$-algebra $\mathcal{A}$, a straightforward computation confirms $Q_0^*(X) = Q_0(X^*)^*$ for arbitrary $X = (X_1, \dots, X_g) \in \mathcal{A}^g$ and furthermore that $Q(X) \in M_{2n}(\mathcal{A})$ is invertible if and only if $Q_0(X) \in M_n(\mathcal{A})$ is so. Thus, in summary, we have $\operatorname{dom}_{\mathcal{A}}(Q_0^{-1}) = \operatorname{dom}_{\mathcal{A}}(Q^{-1})$. Now, since ρ_0 is a formal linear representation, we have

$$\operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(Q_0^{-1}) = \operatorname{dom}_{\mathcal{A}}(Q^{-1})$$

and for each point $X \in \operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r)$ we get that

$$\begin{aligned} -v^* Q(X)^{-1} v &= -\begin{pmatrix} \frac{1}{2}u_0 & v_0^* \end{pmatrix} \begin{pmatrix} 0 & Q_0(X)^{-1} \\ (Q_0(X)^*)^{-1} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2}u_0^* \\ v_0 \end{pmatrix} \\ &= -\frac{1}{2} u_0 Q_0(X)^{-1} v_0 - \frac{1}{2} v_0^* (Q_0(X)^*)^{-1} u_0^* \\ &= -\frac{1}{2} u_0 Q_0(X)^{-1} v_0 - \frac{1}{2} \big(u_0 Q_0(X)^{-1} v_0\big)^* \\ &= \frac{1}{2} r(X) + \frac{1}{2} r(X)^* \\ &= r(X). \end{aligned}$$

In the last step, we have used that $r(X)^* = r(X)$ holds according to Definition III.2.7, since we have $X = X^*$ and r is supposed to be self-adjoint. This completes the proof. $\square$

In Corollary III.2.47, we have shown that a pure linear representation of a given non-commutative rational function $\mathfrak{r}$ can be obtained from a formal linear representation of any rational expression r that represents $\mathfrak{r}$. In combination with Algorithm III.2.45, this gave in particular an alternative proof of Theorem III.2.29, where the existence of a pure linear representation in the sense of Definition III.2.27 and Definition III.2.28 was asserted. Now, since we have established in Theorem III.2.58 (where we supplemented Algorithm III.2.45 with the rule given in (III.14)) the existence of self-adjoint formal linear representations for self-adjoint non-commutative rational expressions and since $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ forms a $*$-algebra (with respect to the $*$-structure introduced in Lemma III.2.35), it is natural to ask for self-adjoint counterparts of Corollary III.2.47 and Theorem III.2.29. This is answered to the affirmative by the next corollary.

Corollary III.2.59. *Let $\mathfrak{r} \in \mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ be a non-commutative rational function in the variables $x = (x_1, \dots, x_g)$. Then the following statements hold true:*

(i) *The non-commutative rational function $\mathfrak{r}$ is self-adjoint (i.e., satisfies $\mathfrak{r}^* = \mathfrak{r}$) with respect to the involution $*$ introduced in Lemma III.2.35 if and only if there exists a non-commutative rational expression $r \in \mathfrak{R}_{\mathbb{C}}(z)$ in formal variables $z = (z_1, \dots, z_g)$, which is self-adjoint in the sense of Definition III.2.7, such that $\mathfrak{r}$ is represented by r in the sense of Definition III.2.46.*

(ii) *If $\mathfrak{r}$ is represented by some self-adjoint non-commutative rational expression $r \in \mathfrak{R}_{\mathbb{C}}(z)$, then each self-adjoint formal linear representation $\rho = (Q, v)$ of r induces then by $(v^*, Q(x), v)$ a pure linear representation of $\mathfrak{r}$ in the sense of Definition III.2.27 and Definition III.2.28.*

In particular, each self-adjoint non-commutative rational function $\mathfrak{r}$ admits a pure linear representation $\rho = (u, Q, v)$, which is self-adjoint *in the sense that $u = v^*$ and $Q = Q^*$ are satisfied.*

Proof. (i) Assume first that $\mathfrak{r} \in \mathbb{C}\langle\!\langle x \rangle\!\rangle$ is represented by some self-adjoint non-commutative rational expression $r \in \mathfrak{R}_{\mathbb{C}}(z)$. According to Lemma III.2.35, $\mathbb{C}\langle\!\langle x \rangle\!\rangle$ forms a unital complex $*$-algebra, in which the variables $x_1, \dots, x_g$ are self-adjoint. Since r is supposed to represent $\mathfrak{r}$, Definition III.2.46 tells us that $x \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x \rangle\!\rangle}(r)$ is satisfied with $r(x) = \mathfrak{r}$, and we infer from $x = x^*$ that even $x \in \operatorname{dom}^{\mathrm{sa}}_{\mathbb{C}\langle\!\langle x \rangle\!\rangle}(r)$ holds true. Furthermore, since r is self-adjoint, Definition III.2.7 enforces $r(x)^* = r(x)$, so that in summary $\mathfrak{r} = \mathfrak{r}^*$ follows.

Conversely, let us assume that $\mathfrak{r}$ is self-adjoint. In Corollary III.2.47, we have seen that there exists $r \in \mathfrak{R}_{\mathbb{C}}(z)$, such that $\mathfrak{r}$ is represented by r. Let us consider the non-commutative rational expression given by $\widetilde{r} := \frac{1}{2} \cdot (r + r^*)$. Lemma III.2.8 and the rules of Definition III.2.3 tell us that for any unital complex $*$-algebra $\mathcal{A}$

$$\operatorname{dom}_{\mathcal{A}}(\widetilde{r}) = \operatorname{dom}_{\mathcal{A}}(r) \cap \operatorname{dom}_{\mathcal{A}}(r^*) = \operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r)$$

is satisfied with $\widetilde{r}(X) = \frac{1}{2}(r(X) + r(X)^*)$ for all $X \in \operatorname{dom}_{\mathcal{A}}(\widetilde{r})$, so that $\widetilde{r}$ is self-adjoint according to Definition III.2.7. Furthermore, by applying the previous observation to $\mathcal{A} = \mathbb{C}\langle\!\langle x \rangle\!\rangle$, we obtain that $\widetilde{r}$ represents $\mathfrak{r}$.

(ii) Suppose that $\mathfrak{r}$ is represented by some self-adjoint non-commutative rational expression $r \in \mathfrak{R}_{\mathbb{C}}(z)$ and let $\rho = (Q, v)$ be any self-adjoint formal linear representation of r. We have then $x \in \operatorname{dom}^{\mathrm{sa}}_{\mathbb{C}\langle\!\langle x \rangle\!\rangle}(r)$ with $\mathfrak{r} = r(x)$ and so we see by Definition III.2.57 that $x \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x \rangle\!\rangle}(Q^{-1})$ holds, meaning that $Q(x)$ is full over $\mathbb{C}\langle x \rangle$, with $\mathfrak{r} = r(x) = -v^* Q(x)^{-1} v$. The latter means that $(v^*, Q(x), v)$ is a pure linear representation of $\mathfrak{r}$, as desired.

The additional assertion is an immediate consequence of (i), (ii), and Theorem III.2.58. ☐

III.3. Operator-valued non-commutative rational expressions

Anderson's self-adjoint version of the linearization trick – after it was successfully applied in [**BMS13**] for computing distributions of self-adjoint non-commutative polynomials in freely independent variables – was generalized further in [**BSS15**] for studying even Brown measures of arbitrary non-commutative polynomials in freely independent variables. For that purpose, the authors provided an extension of Anderson's approach, which can be combined with the hermitization method as explained in Subsection I.3.2.

With an eye towards such applications, it is natural to ask, to which extend the theory presented in the previous Section III.2 can be generalized to matrices of non-commutative rational expressions and functions. Since most of these arguments even work in the more general operator-valued setting, which arises roughly speaking when the role of the complex numbers $\mathbb{C}$ is taken over by any other complex unital algebra $\mathcal{B}$, we find it appropriate to present this framework first; Section III.4 is then devoted to the matricial case.

III.3.1. Operator-valued rational expressions and their evaluations. Conceptually, an operator-valued non-commutative rational expression is obtained in the same way as scalar-valued non-commutative rational expressions were obtained above, but with the scalars $\mathbb{C}$ replaced by some unital complex algebra $\mathcal{B}$. The formal definition, which generalizes Definition III.2.1, reads as follows.

Definition III.3.1. Let $\mathcal{B}$ be a unital complex algebra and let $x = (x_1, \dots, x_g)$ be a g-tuple of formal variables. A *$\mathcal{B}$-valued (non-commutative) rational expression in x* is a syntactically valid combination of

- elements $b \in \mathcal{B}$ and the variables $x_1, \dots, x_g$,
- the arithmetic operations $+, \cdot, ^{-1}$, and
- parentheses $(,)$.

In the following, the set of all $\mathcal{B}$-valued non-commutative rational expressions in x will be denoted by $\mathfrak{R}_\mathcal{B}(x)$.

Like in the scalar-valued case, we suppose for $\mathcal{B}$-valued rational expressions that brackets are placed in such a way that they avoid any ambiguity concerning the order of sums and products. Accordingly, also Remark III.2.2 translates naturally to the $\mathcal{B}$-valued setting.

In Definition III.2.3, we introduced domains and evaluations of scalar-valued rational expressions. Their generalization to the operator-valued level is the content of the next definition.

Definition III.3.2. Let $\mathcal{A}$ be any unital complex algebra, in which $\mathcal{B}$ is unitally embedded. For any $\mathcal{B}$-valued non-commutative rational expression r in the formal variables $x = (x_1, \dots, x_g)$, we define its *$\mathcal{A}$-domain* $\operatorname{dom}_{\mathcal{A}/\mathcal{B}}(r)$ together with its *evaluation* $\operatorname{ev}_X(r)$ *for any* $X = (X_1, \dots, X_g) \in \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(r)$ by the following rules:

(i) For any b, we put $\operatorname{dom}_{\mathcal{A}/\mathcal{B}}(b) = \mathcal{A}^g$ and $\operatorname{ev}_X(b) = b$.
(ii) For $i = 1, \dots, g$, we put $\operatorname{dom}_{\mathcal{A}/\mathcal{B}}(x_i) = \mathcal{A}^g$ and $\operatorname{ev}_X(x_i) = X_i$.
(iii) If r_1, r_2 are $\mathcal{B}$-valued rational expressions in x, we have
$$\operatorname{dom}_{\mathcal{A}/\mathcal{B}}(r_1 \cdot r_2) = \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(r_1) \cap \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(r_2)$$
and
$$\operatorname{ev}_X(r_1 \cdot r_2) = \operatorname{ev}_X(r_1) \cdot \operatorname{ev}_X(r_2).$$
(iv) If r_1, r_2 are $\mathcal{B}$-valued rational expressions in x, we have
$$\operatorname{dom}_{\mathcal{A}/\mathcal{B}}(r_1 + r_2) = \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(r_1) \cap \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(r_2)$$
and
$$\operatorname{ev}_X(r_1 + r_2) = \operatorname{ev}_X(r_1) + \operatorname{ev}_X(r_2).$$
(v) If r is a $\mathcal{B}$-valued rational expression in x, we have
$$\operatorname{dom}_{\mathcal{A}/\mathcal{B}}(r^{-1}) = \{X \in \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(r) |\ \operatorname{ev}_X(r) \text{ is invertible in } \mathcal{A}\}$$
and
$$\operatorname{ev}_X(r^{-1}) = \operatorname{ev}_X(r)^{-1}.$$

In the following, for any given $\mathcal{B}$-valued rational expression r and each $X \in \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(r)$, we will abbreviate $r(X) := \operatorname{ev}_X(r)$.

We already introduced in Definition I.2.8 the unital complex algebra $\mathcal{B}\langle x_1, \dots, x_g\rangle$ of $\mathcal{B}$-valued non-commutative polynomials. Like in the scalar-valued case, $\mathcal{B}$-valued non-commutative polynomials should not be confused with $\mathcal{B}$-valued non-commutative polynomial expressions, which constitute for their part a subset $\mathfrak{P}_\mathcal{B}(x)$ of $\mathfrak{R}_\mathcal{B}(x)$. Let us give now the precise definition.

Definition III.3.3. Let $\mathcal{B}$ be a unital complex algebra and let $x = (x_1, \dots, x_g)$ be a g-tuple of formal variables. A *$\mathcal{B}$-valued (non-commutative) polynomial expression in x* is a syntactically valid combination of

- elements $b \in \mathcal{B}$ and the variables $x_1, \dots, x_g$,
- the arithmetic operations $+, \cdot$, and
- parentheses $(,)$.

In the following, the set of all $\mathcal{B}$-valued non-commutative rational expressions in x will be denoted by $\mathfrak{P}_{\mathcal{B}}(x)$.

Note that we have $\mathfrak{P}_{\mathcal{B}}(x) \subset \mathfrak{R}_{\mathcal{B}}(x)$ and $\operatorname{dom}_{\mathcal{A}/\mathcal{B}}(p) = \mathcal{A}^g$ for each $p \in \mathfrak{P}_{\mathcal{B}}(x)$ and any unital complex algebra $\mathcal{A}$, into which $\mathcal{B}$ unitally embeds.

Again, different elements p in $\mathfrak{P}_{\mathcal{B}}(x)$ may represent the same $\mathcal{B}$-valued non-commutative polynomial $\mathfrak{p} \in \mathcal{B}\langle x_1, \dots, x_g\rangle$, where "represent" is understood in complete analogy to the scalar-valued case. Since this is of no importance for our subsequent considerations, we omit the details.

In Section III.4, we will discuss matrices of non-commutative rational expressions. Loosely speaking, the main difference between matrix-valued non-commutative rational expressions and matrices of non-commutative rational expressions is that for the latter the formal variables should "commute" with scalar matrices. This peculiarity makes it necessary to develop a more refined terminology for domains and evaluations. In a first step, we generalize the notion of $\mathcal{A}$-domains, which was introduced previously in Definition III.3.2.

Definition III.3.4. Let r be a $\mathcal{B}$-valued non-commutative rational expression. Whenever $\mathcal{C} \subseteq \mathcal{A} \supseteq \mathcal{B}$ are a unital inclusion of unital complex algebras $\mathcal{C}$ and $\mathcal{A}$, then we define the *$\mathcal{C}\backslash\mathcal{A}$-domain* $\operatorname{dom}_{\mathcal{C}\backslash\mathcal{A}/\mathcal{B}}(r)$ *of r* by

$$\operatorname{dom}_{\mathcal{C}\backslash\mathcal{A}/\mathcal{B}}(r) := \mathcal{C}^g \cap \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(r).$$

Remark III.3.5. When choosing $\mathcal{B} = M_n(\mathbb{C})$, we recover the important case of matrix-valued rational expressions. In view of this circumstance, it is worthwhile to check some particular matricial expression, which appeared before: though not mentioned there explicitly, the definition of a linear pencil Q of size $n \times n$ identifies Q and consequently Q^{-1} as $M_n(\mathbb{C})$-valued rational expressions – of course, a formally correct treatment requires to insert brackets in order to avoid ambiguities, but since the relevant properties of Q do not change with any placement of brackets, we omit them for reasons of simplicity. Thus, we can obtain the domain $\operatorname{dom}_{\mathcal{A}}(Q^{-1})$, in which we were interested before, by

$$\operatorname{dom}_{\mathcal{A}}(Q^{-1}) = \mathcal{A}^g \cap \operatorname{dom}_{M_n(\mathcal{A})/M_n(\mathbb{C})}(Q^{-1}) = \operatorname{dom}_{\mathcal{A}\backslash M_n(\mathcal{A})/M_n(\mathbb{C})}(Q^{-1}),$$

where $\mathcal{A}$ is seen as a subalgebra of $M_n(\mathcal{A})$ by $X \mapsto X1_n = \begin{pmatrix} X & & 0 \\ & \ddots & \\ 0 & & X \end{pmatrix}$.

III.3.2. Operator-valued formal linear representations. In this subsection, we want to generalize the concept of formal linear representations, which was developed in the previous Subsection III.2.3 for scalar-valued non-commutative rational expressions, to the case of $\mathcal{B}$-valued non-commutative rational expressions. Such operator-valued formal linear representations are defined pretty much in the same way as scalar-valued formal linear representations, but we need to pay attention to some hidden particularities.

To begin with, let us introduce $\mathcal{B}$-valued (affine) linear pencils. Keeping in mind that most $\mathcal{B}$-valued concepts arise from their scalar-valued ancestors simply by replacing the complex numbers by $\mathcal{B}$, one is tempted to define a $\mathcal{B}$-valued linear pencil (of size $n \times m$) in the variables $x = (x_1, \dots, x_g)$ as an expression of the form

$$Q = Q^{(0)} + Q^{(1)} x_1 + \cdots + Q^{(g)} x_g$$

with matrices $Q^{(0)}, Q^{(1)}, \dots, Q^{(g)} \in M_{n\times m}(\mathcal{B})$. While most of the arguments would indeed go through, it is not clear how an operator-valued version of Theorem III.2.58 could be obtained. The problem is that the harmless-looking observation

$$Q^*(X_1, \dots, X_g) = Q(X_1^*, \dots, X_g^*)^* \tag{III.15}$$

relating Q and its formal adjoint

$$Q^* = (Q^{(0)})^* + (Q^{(1)})^* x_1 + \cdots + (Q^{(g)})^* x_g,$$

which was used in the proof of Theorem III.2.58 for a scalar-valued affine linear pencil Q, fails for $\mathcal{B}$-valued affine linear pencils like above, since the non-commutative random variables $X_1, \dots, X_g$ do not necessarily commute with elements from $\mathcal{B}$.

One expedient could be to work instead with $\mathcal{B}$-valued (affine) linear bi-pencils, which are defined as

$$Q = Q^{(0)} + A^{(1)} x_1 B^{(1)} + \cdots + A^{(g)} x_g B^{(g)}$$

with a matrix $Q^{(0)} \in M_{n\times m}(\mathcal{B})$ and matrices $A^{(j)} \in M_{n\times k_j}(\mathcal{B})$ and $B^{(j)} \in M_{k_j\times m}(\mathcal{B})$ for $j = 1, \dots, g$, where $k_1, \dots, k_g$ are arbitrary integers. It is easy to see that property (III.15) stays valid for $\mathcal{B}$-valued linear bi-pencils over any unital complex $*$-algebra $\mathcal{B}$, if the formal adjoint of Q is defined by

$$Q^* := (Q^{(0)})^* + (B^{(1)})^* x_1 (A^{(1)})^* + \cdots + (B^{(g)})^* x_g (A^{(g)})^*.$$

It is beyond all question that this would provide some very natural operator-valued extension of the notion of affine linear pencils. However, working with operator-valued affine linear bi-pencils in a way similar to Definition III.2.41 turns out to be rather cumbersome, especially with an eye towards the intended applications.

Therefore, we take up here a more pragmatic point of view: we regard $\mathcal{B}$ as a collection of additional variables, which we treat on par with $x_1, \dots, x_g$. This idea is captured by the following definition.

Definition III.3.6. Let $\mathcal{B}$ be a unital complex algebra and let $x = (x_1, \dots, x_g)$ be a g-tuple of formal variables.

(i) An expression of the form

$$Q = Q^{(0)} + Q^{(1)} x_1 + \cdots + Q^{(g)} x_g$$

with matrices $Q^{(0)} \in M_{n\times m}(\mathcal{B})$ and $Q^{(1)}, \dots, Q^{(g)} \in M_{n\times m}(\mathbb{C})$, with $M_{n\times m}(\mathbb{C})$ viewed as a subset of $M_{n\times m}(\mathcal{B})$, is called *$\mathcal{B}$-valued affine linear pencil (of size $n \times m$) in x*. Like in the scalar-valued case, we will mostly omit the term "affine", although for the seek of a formally clean terminology, an $\mathcal{B}$-valued affine linear pencil Q should be called *linear pencil (of size $n \times m$) in x* only when the condition $Q^{(0)} = 0$ is satisfied; however, since $\mathcal{B}$ contributes only to $Q^{(0)}$, this will anyway happen very rarely.

(ii) If $\mathcal{B}$-valued affine linear pencils $Q_{k,l} = Q_{k,l}^{(0)} + Q_{k,l}^{(1)} x_1 + \cdots + Q_{k,l}^{(g)} x_g$ of size $n_k \times m_l$ are given for $1 \leq k \leq K$ and $l \leq l \leq L$, we write

$$Q = \begin{pmatrix} Q_{1,1} & \dots & Q_{1,L} \\ \vdots & \ddots & \vdots \\ Q_{K,1} & \dots & Q_{K,L} \end{pmatrix}$$

for the $\mathcal{B}$-valued affine linear pencil

$$Q = Q^{(0)} + Q^{(1)} x_1 + \cdots + Q^{(g)} x_g$$

of size $(n_1 + \cdots + n_K) \times (m_1 + \cdots + m_L)$, which is represented by the matrices

$$Q^{(j)} := \begin{pmatrix} Q_{1,1}^{(j)} & \dots & Q_{1,L}^{(j)} \\ \vdots & \ddots & \vdots \\ Q_{K,1}^{(j)} & \dots & Q_{K,L}^{(j)} \end{pmatrix}, \qquad \text{for } j = 1, \dots, g.$$

(iii) If a $\mathcal{B}$-valued affine linear pencil $Q = Q^{(0)} + Q^{(1)} x_1 + \cdots + Q^{(g)} x_g$ of size $n \times m$ and matrices $S \in M_n(\mathbb{C})$ and $T \in M_m(\mathbb{C})$ are given, we denote by SQT the $\mathcal{B}$-valued affine linear pencil that is defined by

$$SQT := (SQ^{(0)}T) + (SQ^{(1)}T) x_1 + \cdots + (SQ^{(g)}T) x_g.$$

(iv) If $\mathcal{B}$ is even a $*$-algebra and $Q = Q^{(0)} + Q^{(1)} x_1 + \cdots + Q^{(g)} x_g$ a $\mathcal{B}$-valued affine linear pencil of size $n \times m$, then Q^* denotes the affine linear pencil of size $m \times n$, which is given by

$$Q^* = (Q^{(0)})^* + (Q^{(1)})^* x_1 + \cdots + (Q^{(g)})^* x_g.$$

Definition III.3.7. Let $\mathcal{B}$ be a unital complex algebra and let $x = (x_1, \dots, x_g)$ be a g-tuple of formal variables. Furthermore, we consider a unital complex algebra $\mathcal{A}$, into which $\mathcal{B}$ unitally embeds.

(i) Let $Q = Q^{(0)} + Q^{(1)} x_1 + \cdots + Q^{(g)} x_g$ be an $\mathcal{B}$-valued affine linear pencil of size $n \times m$ in the variables x. We define the evaluation $Q(X) \in M_{n \times m}(\mathcal{A})$ of Q at some point $X = (X_1, \dots, X_g) \in \mathcal{A}^g$ by

$$Q(X) := Q^{(0)} + Q^{(1)} X_1 + \cdots + Q^{(g)} X_g.$$

(ii) Let Q be a linear pencil of size $n \times n$. We put

$$\operatorname{dom}_{\mathcal{A}/\mathcal{B}}(Q^{-1}) := \{X \in \mathcal{A}^g |\ Q(X) \text{ is invertible in } M_n(\mathcal{A})\}.$$

Remark III.3.8. Note that $\operatorname{dom}_{\mathcal{A}/\mathcal{B}}(Q^{-1}) = \operatorname{dom}_{\mathcal{A}\backslash M_n(\mathcal{A})/M_n(\mathcal{B})}(Q^{-1})$, if we view Q^{-1} as an $M_n(\mathcal{B})$-valued non-commutative rational expression (for an arbitrary but admissible placement of brackets in Q).

Definition III.3.9. Let r be a $\mathcal{B}$-valued rational expression in the formal variables $x = (x_1, \dots, x_g)$. A *$\mathcal{B}$-valued formal linear representation $\rho = (u, Q, v)$ of r* consists of

- an affine linear pencil

$$Q = Q^{(0)} + Q^{(1)} x_1 + \cdots + Q^{(g)} x_g$$

with matrices $Q^{(0)} \in M_n(\mathcal{B})$ and $Q^{(1)}, \dots, Q^{(g)} \in M_n(\mathbb{C}) \subseteq M_n(\mathcal{B})$ of some dimension n,
- a $1 \times n$-matrix u over $\mathbb{C} \subseteq \mathcal{B}$,
- and a $n \times 1$-matrix v over $\mathbb{C} \subseteq \mathcal{B}$,

and it satisfies the following property:

> For any unital complex algebra $\mathcal{A}$ in which $\mathcal{B}$ is unitally embedded, we have that
> $$\mathrm{dom}_{\mathcal{A}/\mathcal{B}}(r) \subseteq \mathrm{dom}_{\mathcal{A}/\mathcal{B}}(Q^{-1})$$
> and it holds true for any $(X_1, \dots, X_g) \in \mathrm{dom}_{\mathcal{A}/\mathcal{B}}(r)$ that
> $$r(X_1, \dots, X_g) = -uQ(X_1, \dots, X_g)^{-1}v.$$

Algorithm III.2.45 extends immediately to the framework of $\mathcal{B}$-valued rational expressions. Its validity can be checked without any problems by repeating the arguments given in Paragraph III.2.3.4.

Algorithm III.3.10. *Let r be an $\mathcal{B}$-valued rational expression in the formal variables $x = (x_1, \dots, x_g)$. A formal linear representation $\rho = (u, Q, v)$ of r can be constructed by using successively the following rules:*

(i) *For scalars $b \in \mathcal{B}$ and the variables x_j, $j = 1, \dots, g$, $\mathcal{B}$-valued formal linear representations are given by*
$$\rho_{x_j} := \left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} x_j & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \qquad \textit{and}$$
$$\rho_b := \left(\begin{pmatrix} 0 & 1 \end{pmatrix}, \begin{pmatrix} b & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right),$$
respectively.

(ii) *If $\rho_1 = (u_1, Q_1, v_1)$ and $\rho_2 = (u_2, Q_2, v_2)$ are $\mathcal{B}$-valued formal linear representations for the $\mathcal{B}$-valued rational expressions r_1 and r_2, respectively, then $\rho_1 \oplus \rho_2$ defined verbatim like in* (III.11) *gives a $\mathcal{B}$-valued formal linear representation of the non-commutative rational expression $r_1 + r_2$.*

(iii) *If $\rho_1 = (u_1, Q_1, v_1)$ and $\rho_2 = (u_2, Q_2, v_2)$ are $\mathcal{B}$-valued formal linear representations for the $\mathcal{B}$-valued rational expressions r_1 and r_2, respectively, then $\rho_1 \odot \rho_2$ defined verbatim like in* (III.12) *gives a $\mathcal{B}$-valued formal linear representation of the non-commutative rational expression $r_1 \cdot r_2$.*

(iv) *If $\rho = (u, Q, v)$ is a $\mathcal{B}$-valued formal linear representation of r, then ρ^{-1} defined verbatim like in* (III.13) *gives a $\mathcal{B}$-valued formal linear representation of r^{-1}.*

Thus, we obtain the following operator-valued analogue of Theorem III.2.44.

Theorem III.3.11. *Let $\mathcal{B}$ be a unital complex algebra. Each $\mathcal{B}$-valued rational expression has an $\mathcal{B}$-valued formal linear representation in the sense of Definition III.3.9.*

We conclude this subsection by the following definition, which slightly generalizes the concept of $\mathcal{B}$-valued formal linear representations as introduced in Definition III.3.9. Like Definition III.3.4, by which we have extended the notion of domains, the refined terminology of Definition III.3.12 will become important in Section III.4, where we talk about matrices of non-commutative rational expressions.

Definition III.3.12. Let r be a $\mathcal{B}$-valued non-commutative rational expression in the formal variables $x = (x_1, \dots, x_g)$ and suppose that $\mathcal{C} \subseteq \mathcal{A} \supseteq \mathcal{B}$ is any unital embedding of unital complex algebra. A *$\mathcal{B}$-valued formal linear $\mathcal{C}\backslash\mathcal{A}$-representation $\rho = (u, Q, v)$ of* r consists of

- an affine linear pencil
$$Q = Q^{(0)} + Q^{(1)}x_1 + \cdots + Q^{(g)}x_g$$
with matrices $Q^{(0)} \in M_n(\mathcal{B})$ and $Q^{(1)},\dots,Q^{(g)} \in M_n(\mathbb{C}) \subseteq M_n(\mathcal{B})$ of some dimension n,
- a $1 \times n$-matrix u over $\mathbb{C} \subseteq \mathcal{B}$,
- and a $n \times 1$-matrix v over $\mathbb{C} \subseteq \mathcal{B}$

and it satisfies the following property:

> We have $\operatorname{dom}_{\mathcal{C}\backslash\mathcal{A}/\mathcal{B}}(r) \subseteq \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(Q^{-1})$ and at each given point $(X_1,\dots,X_g) \in \operatorname{dom}_{\mathcal{C}\backslash\mathcal{A}/\mathcal{B}}(r)$ it holds true that
> $$r(X_1,\dots,X_g) = -uQ(X_1,\dots,X_g)^{-1}v.$$

Remark III.3.13. Let r be a $\mathcal{B}$-valued non-commutative rational expression in the formal variables $x = (x_1,\dots,x_g)$. Clearly, whenever we have a chain of unital embeddings $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{A} \supseteq \mathcal{B}$ of unital complex algebras, then each $\mathcal{B}$-valued formal linear $\mathcal{C}_2\backslash\mathcal{A}$-representation of r is automatically a $\mathcal{B}$-valued formal linear $\mathcal{C}_1\backslash\mathcal{A}$-representation of r. Furthermore, we note that a $\mathcal{B}$-valued formal linear $\mathcal{A}\backslash\mathcal{A}$-representation of r is nothing else than a $\mathcal{B}$-valued formal linear representation in the sense of Definition III.3.9. Thus, in summary, Theorem III.3.11 guarantees the existence of a $\mathcal{B}$-valued formal linear $\mathcal{C}\backslash\mathcal{A}$-representation of r for each unital embedding $\mathcal{C} \subseteq \mathcal{A} \supseteq \mathcal{B}$ of unital complex algebras.

III.3.3. Self-adjoint operator-valued formal linear representations. Let us assume now that the unital complex algebra $\mathcal{B}$ is even a $*$-algebra. In this case, we may introduce – like it was done in Definition III.2.7 for the case of scalar-valued non-commutative rational expressions – the notion of self-adjoint $\mathcal{B}$-valued non-commutative rational expressions. This is the aim of the following definition.

Definition III.3.14. Let r be a $\mathcal{B}$-valued non-commutative rational expression in the formal variables $x = (x_1,\dots,x_g)$.

(i) If $\mathcal{A}$ is a unital complex $*$-algebra, into which $\mathcal{B}$ unitally embeds, such that also the $*$-structure is preserved, then we denote by $\operatorname{dom}^{\text{sa}}_{\mathcal{A}/\mathcal{B}}(r)$ the subset of $\operatorname{dom}_{\mathcal{A}/\mathcal{B}}(r)$, which consists of all points $X = (X_1,\dots,X_g) \in \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(r)$ satisfying $X = X^*$, where we put $X^* := (X_1^*,\dots,X_g^*)$ as before.

(ii) We call the $\mathcal{B}$-valued rational expression r *self-adjoint*, if it satisfies the condition $r(X)^* = r(X)$ for each $X \in \operatorname{dom}^{\text{sa}}_{\mathcal{A}/\mathcal{B}}(r)$, whenever $\mathcal{A}$ is a unital complex $*$-algebra, into which $\mathcal{B}$ unitally embeds, such that the $*$-structure is preserved.

Also the notion of self-adjoint formal linear representations, which was introduced in Definition III.2.57, can be carried over from the scalar-valued to the $\mathcal{B}$-valued setting. This is the content of the next definition.

Definition III.3.15. Let r be a self-adjoint $\mathcal{B}$-valued rational expression. A *self-adjoint $\mathcal{B}$-valued formal linear representation* $\rho = (Q,v)$ consists of

- an affine linear pencil
$$Q = Q^{(0)} + Q^{(1)}x_1 + \cdots + Q^{(g)}x_g$$
with self-adjoint matrices $Q^{(0)} \in M_n(\mathcal{B})$ and $Q^{(1)},\dots,Q^{(g)} \in M_n(\mathbb{C}) \subseteq M_n(\mathcal{B})$ of some dimension n

- and a $n \times 1$-matrix v over $\mathbb{C} \subseteq \mathcal{B}$

and it satisfies the following property:

> For any unital complex $*$-algebra $\mathcal{A}$, into which $\mathcal{B}$ unitally embeds, such that the $*$-structure is preserved, we have that
> $$\operatorname{dom}^{\text{sa}}_{\mathcal{A}/\mathcal{B}}(r) \subseteq \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(Q^{-1})$$
> and it holds true for any $(X_1, \dots, X_g) \in \operatorname{dom}^{\text{sa}}_{\mathcal{A}/\mathcal{B}}(r)$ that
> $$r(X_1, \dots, X_g) = -v^* Q(X_1, \dots, X_g)^{-1} v.$$

In order to complete the analogy to the scalar-valued case, we should provide now some counterpart of Theorem III.2.58. It is indeed the case that each self-adjoint $\mathcal{B}$-valued non-commutative rational expression possesses a self-adjoint $\mathcal{B}$-valued formal linear representation, but the proof of this statement can readily be formulated in a more refined framework, which we prefer to present first.

Let us begin with the following variant of Definition III.3.14.

Definition III.3.16. Let r be a $\mathcal{B}$-valued non-commutative rational expression in the formal variables $x = (x_1, \dots, x_g)$.

(i) Suppose that $\mathcal{C} \subseteq \mathcal{A} \supseteq \mathcal{B}$ is a unital embedding of unital complex $*$-algebras, both preserving the respective $*$-structures. We denote by $\operatorname{dom}^{\text{sa}}_{\mathcal{C}\backslash\mathcal{A}/\mathcal{B}}(r)$ the subset of $\operatorname{dom}_{\mathcal{C}\backslash\mathcal{A}/\mathcal{B}}(r)$, which consists of all points $X = (X_1, \dots, X_g) \in \operatorname{dom}_{\mathcal{C}\backslash\mathcal{A}/\mathcal{B}}(r)$ satisfying the condition $X = X^*$, where we put $X^* := (X_1^*, \dots, X_g^*)$ as before.

(ii) The $\mathcal{B}$-valued rational expression r is called *$\mathcal{C}\backslash\mathcal{A}$-self-adjoint*, if it satisfies the condition $r(X)^* = r(X)$ for any $X \in \operatorname{dom}^{\text{sa}}_{\mathcal{C}\backslash\mathcal{A}/\mathcal{B}}(r)$, whenever $\mathcal{C} \subseteq \mathcal{A} \supseteq \mathcal{B}$ is a unital embedding of unital complex $*$-algebras, both preserving the respective $*$-structures.

Remark III.3.17. Given a $\mathcal{B}$-valued non-commutative rational expression r, then the following statements are obviously equivalent:

(i) r is self-adjoint in the sense of Definition III.3.14.

(ii) r is $\mathcal{A}\backslash\mathcal{A}$-self-adjoint in the sense of Definition III.3.16 for each unital complex $*$-algebra $\mathcal{A}$, into which $\mathcal{B}$ unitally embeds, such that also the $*$-structure is preserved.

(iii) r is $\mathcal{C}\backslash\mathcal{A}$-self-adjoint in the sense of Definition III.3.16 for each unital embedding $\mathcal{C} \subseteq \mathcal{A} \supseteq \mathcal{B}$ of unital complex $*$-algebras, both preserving the respective $*$-structures.

Like Definition III.3.9 was generalized by Definition III.3.12, the next definition extends Definition III.3.15.

Definition III.3.18. Let r be a self-adjoint $\mathcal{B}$-valued rational expression and suppose that $\mathcal{C} \subseteq \mathcal{A} \supseteq \mathcal{B}$ is any unital embedding of unital complex $*$-algebra, where both embeddings also preserve the respective $*$-structures. A *self-adjoint $\mathcal{B}$-valued formal linear $\mathcal{C}\backslash\mathcal{A}$-representation* $\rho = (Q, v)$ consists of

- an affine linear pencil
$$Q = Q^{(0)} + Q^{(1)} x_1 + \dots + Q^{(g)} x_g$$

with self-adjoint matrices $Q^{(0)} \in M_n(\mathcal{B})$ and $Q^{(1)}, \dots, Q^{(g)} \in M_n(\mathbb{C}) \subseteq M_n(\mathcal{B})$ of some dimension n
- and a $n \times 1$-matrix v over $\mathbb{C} \subseteq \mathcal{B}$

and it satisfies the following property:

We have $\operatorname{dom}^{\text{sa}}_{\mathcal{C}\backslash\mathcal{A}/\mathcal{B}}(r) \subseteq \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(Q^{-1})$ and it holds true for any given point $(X_1, \dots, X_g) \in \operatorname{dom}^{\text{sa}}_{\mathcal{C}\backslash\mathcal{A}/\mathcal{B}}(r)$ that

$$r(X_1, \dots, X_g) = -v^* Q(X_1, \dots, X_g)^{-1} v.$$

With the following lemma, we adapt now the crucial argument in the proof of Theorem III.2.58 to the setting described in Definition III.3.12 and the previous Definition III.3.18.

Lemma III.3.19. *Let r be a $\mathcal{B}$-valued non-commutative rational expression in the formal variables $x = (x_1, \dots, x_g)$ and suppose that $\mathcal{C} \subseteq \mathcal{A} \supseteq \mathcal{B}$ is any unital embedding of unital complex $*$-algebra, where both of these embeddings preserve the respective $*$-structures. If r is $\mathcal{C}\backslash\mathcal{A}$-self-adjoint in the sense of Definition III.3.16, then any $\mathcal{B}$-valued formal linear $\mathcal{C}\backslash\mathcal{A}$-representation $\rho_0 = (u_0, Q_0, v_0)$ of r in the sense of Definition III.3.12 induces by*

$$\rho = (Q, v) := \left(\begin{pmatrix} 0 & Q_0^* \\ Q_0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} u_0^* \\ v_0 \end{pmatrix} \right) \tag{III.16}$$

a self-adjoint $\mathcal{B}$-valued formal linear $\mathcal{C}\backslash\mathcal{A}$-representation ρ of the $\mathcal{B}$-valued non-commutative rational expression r in the meaning of Definition III.3.18.

Proof. Apart from the sightly different terminology, the proof proceeds mainly along the same lines as the proof of Theorem III.2.58. Given any $\mathcal{B}$-valued formal linear $\mathcal{C}\backslash\mathcal{A}$-representation $\rho_0 = (u_0, Q_0, v_0)$ of r, say of size $n \times n$, we must prove that $\rho = (Q, v)$ as defined in (III.16) yields indeed a self-adjoint $\mathcal{B}$-valued formal linear $\mathcal{C}\backslash\mathcal{A}$-representation of r. First of all, in view of Definition III.3.6, it is obviously the case that the linear pencil Q consists only of self-adjoint matrices. Thus, it only remains to check that $\operatorname{dom}^{\text{sa}}_{\mathcal{C}\backslash\mathcal{A}/\mathcal{B}}(r) \subseteq \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(Q^{-1})$ and $r(X) = -v^* Q(X)^{-1} v$ for any given point $X = (X_1, \dots, X_g) \in \operatorname{dom}^{\text{sa}}_{\mathcal{C}\backslash\mathcal{A}/\mathcal{B}}(r)$. With our terminology of $\mathcal{B}$-valued affine linear pencils, it is easy to check that like in the scalar-valued case $Q_0^*(X) = Q_0(X^*)^*$ holds for arbitrary $X = (X_1, \dots, X_g) \in \mathcal{A}^g$. By a straightforward computation we can furthermore confirm that $Q(X) \in M_{2n}(\mathcal{A})$ is invertible if and only if $Q_0(X) \in M_n(\mathcal{A})$ is so. Thus, in summary, we have $\operatorname{dom}_{\mathcal{A}/\mathcal{B}}(Q^{-1}) = \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(Q_0^{-1})$. Now, since ρ_0 is a $\mathcal{B}$-valued formal linear representation, we have

$$\operatorname{dom}^{\text{sa}}_{\mathcal{C}\backslash\mathcal{A}/\mathcal{B}}(r) \subseteq \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(Q_0^{-1}) = \operatorname{dom}_{\mathcal{A}/\mathcal{B}}(Q^{-1})$$

and for each point $X \in \operatorname{dom}^{\text{sa}}_{\mathcal{C}\backslash\mathcal{A}/\mathcal{B}}(r)$ we get that

$$\begin{aligned} -v^* Q(X)^{-1} v &= -\begin{pmatrix} \frac{1}{2} u_0 & v_0^* \end{pmatrix} \begin{pmatrix} 0 & Q_0(X)^{-1} \\ (Q_0(X)^*)^{-1} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} u_0^* \\ v_0 \end{pmatrix} \\ &= -\frac{1}{2} u_0 Q_0(X)^{-1} v_0 - \frac{1}{2} v_0^* (Q_0(X)^*)^{-1} u_0^* \\ &= -\frac{1}{2} u_0 Q_0(X)^{-1} v_0 - \frac{1}{2} \big(u_0 Q_0(X)^{-1} v_0\big)^* \\ &= \frac{1}{2} r(X) + \frac{1}{2} r(X)^* \\ &= r(X). \end{aligned}$$

In the last step, we have used that $r(X) = r(X)^*$ holds according to Definition III.3.16, since we have $X = X^*$ and since r was assumed to be $\mathcal{C}\backslash\mathcal{A}$-self-adjoint. □

The following theorem states the announced $\mathcal{B}$-valued analogue of Theorem III.2.58 and provides some refined observations, which will become important in Section III.4.

THEOREM III.3.20.

(i) *Let $\mathcal{C} \subseteq \mathcal{A} \supseteq \mathcal{B}$ be any unital embedding of unital complex $*$-algebras, where both of these embeddings are supposed to preserve the respective $*$-structures. Then any $\mathcal{C}\backslash\mathcal{A}$-self-adjoint $\mathcal{B}$-valued non-commutative rational expression in the formal variables $x = (x_1, \dots, x_g)$ admits a self-adjoint $\mathcal{B}$-valued formal linear $\mathcal{C}\backslash\mathcal{A}$-representation.*

(ii) *Let r be any $\mathcal{B}$-valued non-commutative rational expression in the formal variables $x = (x_1, \dots, x_g)$. Then there exists $\rho = (Q, v)$, such that ρ is a self-adjoint $\mathcal{B}$-valued formal linear $\mathcal{C}\backslash\mathcal{A}$-representation, whenever $\mathcal{C} \subseteq \mathcal{A} \supseteq \mathcal{B}$ are unital embeddings of unital complex $*$-algebras, where both of these embeddings are supposed to preserve the respective $*$-structures, and r happens to be $\mathcal{C}\backslash\mathcal{A}$-self-adjoint.*

(iii) *In particular, any self-adjoint $\mathcal{B}$-valued rational expression admits a self-adjoint $\mathcal{B}$-valued formal linear representation in the sense of Definition III.3.15.*

PROOF. (i) Let r be an arbitrary $\mathcal{B}$-valued non-commutative rational expression in x. In Theorem III.3.11, we have proven the existence of a $\mathcal{B}$-valued formal linear representation $\rho_0 = (u_0, Q_0, v_0)$ for r. Suppose now that r is $\mathcal{C}\backslash\mathcal{A}$-self-adjoint. Since the matrices in the affine linear pencil Q_0 are not necessarily self-adjoint and since also the condition $u_0 = v_0^*$ cannot be taken for granted, the only issue that remains is how to find the desired self-adjoint formal linear $\mathcal{C}\backslash\mathcal{A}$-representation. By Definition III.3.15, the given $\mathcal{B}$-valued formal linear representation ρ_0 is in particular a formal $\mathcal{B}$-valued linear $\mathcal{C}\backslash\mathcal{A}$-representation in the sense of Definition III.3.12, for which due to Lemma III.3.19 a self-adjoint $\mathcal{B}$-valued formal linear $\mathcal{C}\backslash\mathcal{A}$-representation ρ exists; this $\rho = (Q, v)$ was defined explicitly in (III.16). This concludes the proof of the first assertion.

(ii) The second statement follows immediately from the proof of (i), because the universal construction of ρ in (III.16) was based on a $\mathcal{B}$-valued formal linear representation ρ_0. Thus, it follows by the same arguments as above that this fixed ρ forms a self-adjoint $\mathcal{B}$-valued formal linear $\mathcal{C}\backslash\mathcal{A}$-representation of r, whenever r happens to be $\mathcal{C}\backslash\mathcal{A}$-self-adjoint.

(iii) The statement of Item (iii) is contained in that of Item (ii). We only must note

- that r being self-adjoint means according to Remark III.3.17 simply that r is $\mathcal{A}\backslash\mathcal{A}$-self-adjoint for any unital complex $*$-algebra $\mathcal{A}$.
- that ρ being a self-adjoint $\mathcal{B}$-valued formal linear $\mathcal{A}\backslash\mathcal{A}$-representation for any unital complex $*$-algebra $\mathcal{A}$ means that ρ is a self-adjoint $\mathcal{B}$-valued formal linear representation in the sense of Definition III.3.15.

This completes the proof. □

III.4. Matrices of non-commutative rational expressions

How does all this relate now to matrices of non-commutative rational expressions? As we already pointed out in Remark III.3.5, the case of matrix-valued non-commutative rational expressions is contained in the considerations of the previous Section III.3 for the special choice $\mathcal{B} = M_n(\mathbb{C})$. However, matrix-valued non-commutative rational expressions should not be confused with the closely related but still different objects, which are matrices of non-commutative rational expressions. The main difference is that matrices of non-commutative rational expressions are built on some inherent commutativity between the formal variables and the matricial coefficients, while any possible relation is suppressed when working with non-commutative matrix-valued rational expressions.

III.4.1. Matrices of rational expressions and their evaluations. This crucial difference already shows up if one compares matrix-valued non-commutative polynomials and matrices of non-commutative polynomials. Here, it can simply be explained by the fact that $M_n(\mathbb{C})\langle x_1, \dots, x_g\rangle$ and $M_n(\mathbb{C}\langle x_1, \dots, x_g\rangle) \cong M_n(\mathbb{C}) \otimes \mathbb{C}\langle x_1, \dots, x_g\rangle$ are by definition non-isomorphic algebras. However, if we distinguish for a moment the formal variables in both algebras, we can easily find a canonical epimorphism

$$M_n(\mathbb{C})\langle \underline{x}_1, \dots, \underline{x}_g\rangle \to M_n(\mathbb{C}\langle x_1, \dots, x_g\rangle),$$

which is uniquely determined by the conditions that

$$1_n \mapsto 1_n \qquad \text{and} \qquad \underline{x}_i \mapsto x_i 1_n = \begin{pmatrix} x_i & & 0 \\ & \ddots & \\ 0 & & x_i \end{pmatrix} \quad \text{for } i = 1, \dots, g.$$

This means that for each $n \times n$ matrix $(p_{i,j}(x))_{i,j=1}^n$ of non-commutative polynomials belonging to $\mathbb{C}\langle x_1, \dots, x_g\rangle$, there exists a canonical lifting to a matrix-valued non-commutative polynomial $P \in M_n(\mathbb{C})\langle \underline{x}_1, \dots, \underline{x}_g\rangle$, such that $P(x_1 1_n, \dots, x_g 1_n) = (p_{i,j}(x))_{i,j=1}^n$. This observations will guide us when dealing with *matrices of non-commutative rational expressions*, i.e. square[1] matrices over $\mathfrak{R}_{\mathbb{C}}(x_1, \dots, x_g)$, although $\mathfrak{R}_{\mathbb{C}}(x_1, \dots, x_g)$ does not carry an algebra structure.

Definition III.4.1. Let $\underline{r} = (r_{i,j})_{i,j=1}^n$ be a matrix of non-commutative rational expressions belonging to $\mathfrak{R}_{\mathbb{C}}(x_1, \dots, x_g)$.

(i) If $\mathcal{C}$ is a unital complex algebra, we put

$$\operatorname{dom}_{\mathcal{C}}(\underline{r}) := \bigcap_{i,j=1}^n \operatorname{dom}_{\mathcal{C}}(r_{i,j})$$

and $\underline{r}(X) := (r_{i,j}(X))_{i,j=1}^n$ for any given $X \in \operatorname{dom}_{\mathcal{C}}(\underline{r})$.

(ii) If $\mathcal{C}$ is a unital complex $*$-algebra, then we put

$$\operatorname{dom}_{\mathcal{C}}^{\mathrm{sa}}(\underline{r}) := \bigcap_{i,j=1}^n \operatorname{dom}_{\mathcal{C}}^{\mathrm{sa}}(r_{i,j}).$$

(iii) We call $\underline{r}$ *self-adjoint*, if $\underline{r}(X)^* = \underline{r}(X)$ holds in $M_n(\mathcal{C})$ for any unital complex algebra $\mathcal{C}$ and any $X \in \operatorname{dom}_{\mathcal{C}}^{\mathrm{sa}}(\underline{r})$.

[1]Some of the arguments given below would even work for rectangular matrices, but for the seek of simplicity, we restrict to the for us relevant case of square matrices.

Lemma III.4.2. *Let* $\underline{r} = (r_{i,j})_{i,j=1}^n$ *be a matrix of non-commutative rational expressions belonging to* $\mathfrak{R}_{\mathbb{C}}(x_1,\dots,x_g)$. *We denote by* $(e_{i,j})_{i,j=1}^n$ *the canonical matrix-units in* $M_n(\mathbb{C})$. *Then (for any fixed order of summation)*

$$r = \sum_{i,j=1}^n e_{i,j} r_{i,j}(\underline{x})$$

yields some element in $\mathfrak{R}_{M_n(\mathbb{C})}(\underline{x}_1,\dots,\underline{x}_g)$, *which satisfies the following two properties:*

(i) *For any unital complex algebra* $\mathcal{C}$, *we have*

$$\mathrm{dom}_{\mathcal{C}}(\underline{r}) = \mathrm{dom}_{\mathcal{C}\backslash M_n(\mathcal{C})/M_n(\mathbb{C})}(r)$$

and $\underline{r}(X) = r(X)$ *holds for any* $X \in \mathrm{dom}_{\mathcal{C}}(\underline{r})$.

(ii) *The matrix* $\underline{r}$ *is self-adjoint in the sense of Definition III.4.1 if and only if* r *is* $\mathcal{C}\backslash M_n(\mathcal{C})$*-self-adjoint for any unital complex* $*$*-algebra* $\mathcal{C}$.

Proof. Let us point out that each $r \in \mathfrak{R}_{\mathbb{C}}(x_1,\dots,x_g)$ induces naturally $r(\underline{x}) \in \mathfrak{R}_{M_n(\mathbb{C})}(\underline{x}_1,\dots,\underline{x}_g)$, after identifying $\mathbb{C}$ with $\mathbb{C}1_n \subset M_n(\mathbb{C})$. The distinction between the formal variables $x = (x_1,\dots,x_g)$ and $\underline{x} = (\underline{x}_1,\dots,\underline{x}_g)$ helps us to separate the two meanings of r without using different letters for them. Keeping this in mind, on easily sees that $\mathrm{dom}_{\mathcal{C}\backslash M_n(\mathcal{C})/M_n(\mathbb{C})}(r(\underline{x})) = \mathrm{dom}_{\mathcal{C}}(r)$ holds for any $r \in \mathfrak{R}_{\mathbb{C}}(x_1,\dots,x_g)$. Thus, we may derive by the defining properties of $\mathrm{dom}_{M_n(\mathcal{C})/M_n(\mathbb{C})}(\cdot)$ that

$$\mathrm{dom}_{M_n(\mathcal{C})/M_n(\mathbb{C})}(r(\underline{x})) = \bigcap_{i,j=1}^n \mathrm{dom}_{M_n(\mathcal{C})/M_n(\mathbb{C})}(r_{i,j}(\underline{x}))$$

and hence, after intersecting both sides with $\mathcal{C}^g$, that

$$\mathrm{dom}_{\mathcal{C}\backslash M_n(\mathcal{C})/M_n(\mathbb{C})}(r(\underline{x})) = \bigcap_{i,j=1}^n \mathrm{dom}_{\mathcal{C}\backslash M_n(\mathcal{C})/M_n(\mathbb{C})}(r_{i,j}(\underline{x})) = \bigcap_{i,j=1}^n \mathrm{dom}_{\mathcal{C}}(r_{i,j}) = \mathrm{dom}_{\mathcal{C}}(\underline{r}),$$

as we claimed in (i). Clearly, if $X \in \mathrm{dom}_{\mathcal{C}}(\underline{r})$ is given, we also have

$$\underline{r}(X) = (r_{i,j}(X))_{i,j=1}^n = \sum_{i,j=1}^n e_{i,j} r_{i,j}(X) = r(X).$$

Furthermore, according to Definition III.4.1, the matrix $\underline{r}$ is self-adjoint if and only if $\underline{r}(X) = \underline{r}(X)^*$ holds for any unital complex $*$-algebra $\mathcal{C}$ and any $X \in \mathrm{dom}^{\mathrm{sa}}_{\mathcal{C}}(\underline{r})$. Because $\mathrm{dom}^{\mathrm{sa}}_{\mathcal{C}}(\underline{r}) = \mathrm{dom}^{\mathrm{sa}}_{\mathcal{C}\backslash M_n(\mathcal{C})/M_n(\mathbb{C})}(r)$ and

$$\underline{r}(X) = \sum_{i,j=1}^n e_{i,j} r_{i,j}(X) = r(X),$$

the latter condition is equivalent to r being $\mathcal{C}\backslash M_n(\mathcal{C})$-self-adjoint for any unital complex $*$-algebra $\mathcal{C}$. This proves (ii). □

III.4.2. Formal linear representations of matrices of rational expressions. In this subsection, we want to generalize the notion of formal linear representations from Subsection III.2.3 to the present setting of matrices of non-commutative rational expressions.

DEFINITION III.4.3. Let $\underline{r}$ be a $k \times k$-matrix of non-commutative rational expressions in the formal variables $x = (x_1, \dots, x_g)$. A *formal linear representation* $\rho = (u, Q, v)$ *of* $\underline{r}$ consists of

- an affine linear pencil Q of size $n \times n$,
- a $k \times n$-matrix u over $\mathbb{C}$,
- and a $n \times k$-matrix v over $\mathbb{C}$,

and it satisfies the following property:

For any unital complex algebra $\mathcal{C}$, we have that
$$\operatorname{dom}_{\mathcal{C}}(\underline{r}) \subseteq \operatorname{dom}_{\mathcal{C}}(Q^{-1})$$
and it holds true for any $(X_1, \dots, X_g) \in \operatorname{dom}_{\mathcal{C}}(r)$ that
$$\underline{r}(X_1, \dots, X_g) = -uQ(X_1, \dots, X_g)^{-1}v.$$

In analogy to Theorem III.2.44, we claim the following.

THEOREM III.4.4. *For each $k \times k$ matrix $\underline{r}$ of non-commutative rational expressions in the formal variables $x = (x_1, \dots, x_g)$ there exists a formal linear representation $\rho = (u, Q, v)$ in the sense of Definition III.4.3.*

Before proceeding to the proof, let us treat the self-adjoint case first.

DEFINITION III.4.5. Let $\underline{r}$ be a self-adjoint $k \times k$ matrix of non-commutative rational expressions in the formal variables $x = (x_1, \dots, x_g)$. A *self-adjoint formal linear representation* $\rho = (Q, v)$ *of* $\underline{r}$ consists of

- an affine linear pencil
$$Q = Q^{(0)} + Q^{(1)}x_1 + \dots + Q^{(g)}x_g$$
for self-adjoint matrices $Q^{(0)}, Q^{(1)}, \dots, Q^{(g)} \in M_n(\mathbb{C})$ of some dimension n
- and a $n \times k$-matrix v over $\mathbb{C}$

and it satisfies the following property:

For any unital complex $*$-algebra $\mathcal{C}$, we have that
$$\operatorname{dom}^{\mathrm{sa}}_{\mathcal{C}}(\underline{r}) \subseteq \operatorname{dom}_{\mathcal{A}}(Q^{-1})$$
and it holds true for any $(X_1, \dots, X_g) \in \operatorname{dom}^{\mathrm{sa}}_{\mathcal{C}}(r)$ that
$$\underline{r}(X_1, \dots, X_g) = -v^*Q(X_1, \dots, X_g)^{-1}v.$$

We obtain the following analogue of Theorem III.2.58.

THEOREM III.4.6. *Any $k \times k$ matrix $\underline{r}$ of non-commutative rational expressions in the formal variables $x = (x_1, \dots, x_g)$, which is self-adjoint in the sense of Definition III.4.1, admits a self-adjoint formal linear representation $\rho = (Q, v)$ in the sense of Definition III.4.5.*

PROOF OF THEOREM III.4.4 AND OF THEOREM III.4.6. Given any $k \times k$ matrix $\underline{r}$ of non-commutative rational expressions in the formal variables $x = (x_1, \dots, x_g)$, then Lemma III.4.2 tells us that we can find some $M_k(\mathbb{C})$-valued non-commutative rational expression r in x, such that for any unital complex algebra $\mathcal{C}$ the conditions $\operatorname{dom}_{\mathcal{C}}(\underline{r}) =$

$\mathrm{dom}_{\mathcal{C}\backslash M_k(\mathcal{C})/M_k(\mathbb{C})}(r)$ and $\underline{r}(X) = r(X)$ at any point $X \in \mathrm{dom}_{\mathcal{C}}(\underline{r}) = \mathrm{dom}_{\mathcal{C}\backslash M_k(\mathcal{C})/M_k(\mathbb{C})}(r)$ are satisfied.

(1) Let us first prove Theorem III.4.4. For the $M_k(\mathbb{C})$-valued non-commutative rational expression r, we can construct according to Theorem III.3.11 some $M_k(\mathbb{C})$-valued formal linear representation $\rho = (u, Q, v)$ of r. By Definition III.3.9, we know (if we chose $\mathcal{A} = M_k(\mathcal{C})$) that $\mathrm{dom}_{M_k(\mathcal{C})/M_k(\mathbb{C})}(r) \subseteq \mathrm{dom}_{M_k(\mathcal{C})}(Q^{-1})$ and $r(X) = -uQ(X)^{-1}v$ for any $X \in \mathrm{dom}_{M_k(\mathcal{C})/M_k(\mathbb{C})}(r)$ holds. Combining this with the properties of $\underline{r}$, we see that

$$\mathrm{dom}_{\mathcal{C}}(\underline{r}) = \mathrm{dom}_{\mathcal{C}\backslash M_k(\mathcal{C})/M_k(\mathbb{C})}(r) \subseteq \mathrm{dom}_{M_k(\mathcal{C})}(Q^{-1})$$

and for any $X \in \mathrm{dom}_{\mathcal{C}}(\underline{r})$ that

$$\underline{r}(X) = r(X) = -uQ(X)^{-1}v.$$

Since $\mathcal{C}$ was arbitrarily chosen, this means in the terminology of Definition III.4.3, that ρ forms a formal linear representation of $\underline{r}$. Note that Q is some $M_k(\mathbb{C})$-valued affine linear pencil of size $n \times n$ and can thus be naturally identified with a scalar-valued affine linear pencil of size $(nk) \times (nk)$. Similarly, u and v can be seen as row and column vectors over $M_k(\mathbb{C})$ and can thus be identified with matrices in $M_{k\times(nk)}(\mathbb{C})$ and $M_{(nk)\times k}(\mathbb{C})$, respectively.

(2) For proving Theorem III.4.6, we use instead Item (ii) of Theorem III.3.20. This tells us that there exists a universal $\rho = (Q, v)$, such that ρ is a self-adjoint $M_k(\mathbb{C})$-valued formal linear $\mathcal{C}\backslash\mathcal{A}$-representation, whenever $\mathcal{C} \subseteq \mathcal{A} \supseteq M_k(\mathbb{C})$ are unital embeddings of unital complex $*$-algebras, where both of these embeddings are supposed to preserve the respective $*$-structures, and r happens to be $\mathcal{C}\backslash\mathcal{A}$-self-adjoint. Since the statement in Item (ii) of Lemma III.4.2 rephrases our assumption that $\underline{r}$ is self-adjoint into the statement that r is $\mathcal{C}\backslash M_k(\mathcal{C})$-self-adjoint for any unital complex $*$-algebra $\mathcal{C}$, we may conclude that ρ is a self-adjoint $M_k(\mathbb{C})$-valued formal linear $\mathcal{C}\backslash M_k(\mathcal{C})$-representation for any unital complex $*$-algebra $\mathcal{C}$. In such cases, Definition III.3.18 tells us that $\mathrm{dom}^{\mathrm{sa}}_{\mathcal{C}\backslash M_k(\mathcal{C})/M_k(\mathbb{C})}(r) \subseteq \mathrm{dom}_{M_k(\mathcal{C})/M_k(\mathbb{C})}(Q^{-1})$ and $r(X) = -v^*Q(X)^{-1}v$ holds true for any $X \in \mathrm{dom}^{\mathrm{sa}}_{\mathcal{C}\backslash M_k(\mathcal{C})/M_k(\mathbb{C})}(r)$. Combining this with the properties of $\underline{r}$, yields

$$\mathrm{dom}^{\mathrm{sa}}_{\mathcal{C}}(\underline{r}) = \mathrm{dom}^{\mathrm{sa}}_{\mathcal{C}\backslash M_k(\mathcal{C})/M_k(\mathbb{C})}(r) \subseteq \mathrm{dom}_{M_k(\mathcal{C})/M_k(\mathbb{C})}(Q^{-1})$$

and, for any $X \in \mathrm{dom}^{\mathrm{sa}}_{\mathcal{C}}(\underline{r})$, that

$$\underline{r}(X) = r(X) = -v^*Q(X)^{-1}v.$$

Since $\mathcal{C}$ was arbitrarily chosen, this means in the terminology of Definition III.4.5, that ρ forms a self-adjoint formal linear representation of $\underline{r}$. We note that Q and v (regarded as a linear pencil and a vector over $M_k(\mathbb{C})$, respectively), naturally induce a self-adjoint formal linear representation of $\underline{r}$ in the sense of Definition III.4.5. □

III.4.3. Regular rational expressions and functions. In this section, we will specialize our discussion to the case of regular non-commutative rational expressions and functions, i.e. on non-commutative rational expressions and functions, whose evaluation at zero is well-defined.

Definition III.4.7.

(i) A non-commutative rational expressions r in non-commuting variables $x_1, \dots, x_g$ is called *regular (at 0)*, if $(0, \dots, 0) \in \mathrm{dom}_{M(\mathbb{C})}(r)$.

(ii) A non-commutative rational function $\mathfrak{r} \in \mathbb{C}\langle\!\langle x_1, \ldots, x_g \rangle\!\rangle$ is called *regular (at 0)*, if there exists a regular rational expression r in the variables $x_1, \ldots, x_g$, which represents $\mathfrak{r}$ in the sense of Definition III.2.46. According to Proposition III.2.48, this is equivalent to saying that the equivalence class of r with respect to $M(\mathbb{C})$-evaluation equivalence is $\mathfrak{r}$.

For regular non-commutative rational functions, there exists besides the concept of pure linear representations by Cohn and Reutenauer [**CR99**], which we presented in Paragraph III.2.2.4, another concept in the same spirit, which goes under the name non-commutative descriptor realizations and originates in the work of Schützenberger [**Sch61**] on recognizable rational series; see [**Kal63, Kal76, HMV06, KV09, KV12**]. The following definitions are taken from [**HMV06**]; see also [**HMS15**].

Definition III.4.8. By a *$d_1 \times d_2$-descriptor system (of size $d \times d$)*, we mean a collection $\mathbb{r} = (D; C, J, A, B)$, where

- $A = (A_1, \ldots, A_g)$ is a g-tuple of matrices $A_1, \ldots, A_g \in M_d(\mathbb{C})$,
- $B \in M_{d\times d_2}(\mathbb{C})$ and $C \in M_{d_1\times d}(\mathbb{C})$,
- $D \in M_{d_1\times d_2}(\mathbb{C})$ a matrix, sometimes called *the feed through term*,
- and J is a $d \times d$ *signature matrix*, i.e. a matrix $J \in M_d(\mathbb{C})$, which satisfies both $J = J^*$ and $J^2 = 1$.

For any unital complex algebra $\mathcal{A}$, we put

$$\mathrm{dom}_{\mathcal{A}}(\mathbb{r}) := \{X \in \mathcal{A}^g \mid J - L_A(X) \text{ is invertible in } M_d(\mathcal{A})\},$$

where L_A stands for the linear pencil given by $L_A := A_1 z_1 + \cdots + A_g z_g$ in formal variables $z = (z_1, \ldots, z_g)$.

Remark III.4.9. Given any $d_1 \times d_2$-descriptor system $\mathbb{r} = (D; C, J, A, B)$, say of size $d \times d$, then the matrix $Q := J - L_A(x) \in M_d(\mathbb{C}\langle x_1, \ldots, x_g\rangle)$ must be full. Indeed, if we assume to the contrary that Q is not full, we can find according to Definition III.2.23 rectangular matrices $R_1 \in M_{d\times(d-1)}(\mathbb{C}\langle x_1, \ldots, x_g\rangle)$ and $R_2 \in M_{(d-1)\times d}(\mathbb{C}\langle x_1, \ldots, x_g\rangle)$, such that $Q = R_1 R_2$ holds. Consider now the homomorphism $\mathrm{ev}_0 : \mathbb{C}\langle x_1, \ldots, x_g\rangle \to \mathbb{C}$ given by evaluation at 0, i.e. ev_0 is as a homomorphism determined by the conditions $\mathrm{ev}_0(1) = 1$ and $\mathrm{ev}_0(x_j) = 0$ for $j = 1, \ldots, g$. Since this homomorphism ev_0 extends naturally to a family $(\mathrm{ev}_0^{(d_1\times d_2)})_{d_1,d_2\geq 0}$ of linear mappings

$$\mathrm{ev}_0^{(d_1\times d_2)} :\ M_{d_1\times d_2}(\mathbb{C}\langle x_1, \ldots, x_g\rangle) \to M_{d_1\times d_2}(\mathbb{C}),\ (P_{i,j})_{\substack{i=1,\ldots,d_1\\ j=1,\ldots,d_2}} \mapsto (\mathrm{ev}_0(P_{i,j}))_{\substack{i=1,\ldots,d_1\\ j=1,\ldots,d_2}},$$

which are clearly compatible with matrix multiplication, we see that

$$J = \mathrm{ev}_0^{(d\times d)}(Q) = \mathrm{ev}_0^{(d\times d)}(R_1 R_2) = \mathrm{ev}_0^{(d\times(d-1))}(R_1)\,\mathrm{ev}_0^{((d-1)\times d)}(R_2) = R_1' R_2'$$

with scalar matrices $R_1' := \mathrm{ev}_0^{(d\times(d-1))}(R_1) \in M_{d\times(d-1)}(\mathbb{C})$ and $R_2' := \mathrm{ev}_0^{((d-1)\times d)}(R_2) \in M_{(d-1)\times d}(\mathbb{C})$. This, of course, contradicts the invertibility of J in $M_d(\mathbb{C})$, so that our assumption turns out to be wrong, meaning that Q must be full. Theorem III.2.24 and in particular the comments collected in Remark III.2.25 show that

$$\mathbb{r}(x) := D + C(J - L_A(x))^{-1} B$$

forms a $d_1 \times d_2$-matrix of regular non-commutative rational functions in the sense of Definition III.4.7.

This justifies the following definition.

Definition III.4.10. Let $\mathfrak{r} \in M_{d_1 \times d_2}(\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle)$ be a $d_1 \times d_2$-matrix of regular non-commutative rational functions in the variables $x = (x_1, \dots, x_g)$. A *(non-commutative) descriptor realization* means a $d_1 \times d_2$-descriptor system $\mathbb{r} = (D; C, J, A, B)$, such that

$$\mathfrak{r} = \mathbb{r}(x) = D + C(J - L_A(x))^{-1}B$$

holds over the free field $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$; note that, in order to simplify the terminology, we will often refer to

$$\mathbb{r}(x) = D + C(J - L_A(x))^{-1}B \qquad \text{with} \qquad L_A(x) = A_1 x_1 + \dots + A_g x_g$$

as the descriptor realization of $\mathfrak{r}$. A non-commutative descriptor realization is called *monic*, if we have $J = 1$. In the square case $d_1 = d_2$, a non-commutative descriptor realization is called *self-adjoint*, if $B = C^*$ holds and if the matrices $A_1, \dots, A_g$ and D are all self-adjoint.

Like before, the notion of minimality singles out some special descriptor realizations among all descriptor realizations of a given matrix of non-commutative rational functions.

Definition III.4.11. A descriptor realization $\mathbb{r}$ of $\mathfrak{r}$ like in Definition III.4.10 is called

(i) *controllable*, if the *controllable space*

$$\mathcal{S} = \operatorname{span}\left\{(JA_{i_1}) \cdots (JA_{i_k})JBv \,\middle|\, k \geq 0,\ 1 \leq i_1, \dots, i_k \leq g,\ v \in \mathbb{C}^{d_2}\right\}$$

is all of $\mathbb{C}^d$;

(ii) *observable*, if the *observable space*

$$\mathcal{Q} = \left\{v \in \mathbb{C}^d \,\middle|\, \forall k \geq 0,\ 1 \leq i_1, \dots, i_k \leq g:\ C(JA_{i_1}) \cdots (JA_{i_k})v = 0\right\}$$

is $\{0\}$;

(iii) *minimal*, if $\mathbb{r}$ is both controllable and observable.

Note that minimality is defined here in terms of two abstract properties, namely controllability and observability. Algorithm III.4.15 below shows that any descriptor realization, which is not both controllable and observable, can be cut down to some descriptor realization of smaller size. Hence, any descriptor realization of minimal size must be controllable and observable, and hence minimal in the sense of the previous Definition III.4.10. One would actually expect that the converse – even though not obvious – is also true, but as it is explained in [**HMV06**, Section 4.1.1.], there are controllable and observable descriptor realizations, which do not have minimal size. The problem is that descriptor realizations may have non-zero feed through term D: varying D gives an additional degree of freedom to minimize the size, while minimality in the sense of Definition III.4.11 concerns the class of descriptor realizations with fixed feed through term D. However, given two descriptor realizations of the same scalar-valued non-commutative rational function, which are minimal in the sense of Definition III.4.11, then their dimensions differ at most by one. Furthermore, if we stay inside the class of descriptor realizations with feed through term $D = 0$, then the two notions of minimality coincide; see also [**Vol15**], where a more general setting is discussed.

Remark III.4.12. Observability can be expressed as controllability for the adjoint system. Indeed, given a descriptor realization

$$\mathbb{r}(x) = D + C(J - L_A(x))^{-1}B \qquad \text{with} \qquad L_A(x) = A_1 x_1 + \dots + A_g x_g,$$

with the conventions laid down in Definition III.4.10, the adjoint descriptor realization looks like

$$\mathbb{r}^*(x) = D^* + B^*(J - L_{A^*}(x))^{-1}C^* \qquad \text{with} \qquad L_{A^*}(x) = A_1^* x_1 + \cdots + A_g^* x_g.$$

The controllability space $\mathcal{S}^*$, by definition, is given as

$$\mathcal{S}^* = \operatorname{span}\left\{(JA_{i_1}^*)\cdots(JA_{i_k}^*)JC^*v \,\middle|\, k \geq 0,\ 1 \leq i_1, \ldots, i_k \leq g,\ v \in \mathbb{C}^{d_1}\right\}$$

and we may observe that $(\mathcal{S}^*)^\perp = \mathcal{Q}$, where $\mathcal{Q}$ denotes the observable space of $\mathbb{r}$, i.e.

$$\mathcal{Q} = \left\{v \in \mathbb{C}^d \,\middle|\, \forall k \geq 0,\ 1 \leq i_1, \ldots, i_k \leq g:\ C(JA_{i_1})\cdots(JA_{i_k})v = 0\right\}.$$

For seeing this, we introduce the index set $I = \{(i_1, \ldots, i_k) |\ k \geq 0,\ 1 \leq i_1, \ldots, i_k \leq g\}$ and for each $i = (i_1, \ldots, i_k) \in I$ the matrix $T_{(i_1,\ldots,i_k)} := C(JA_{i_1})\cdots(JA_{i_k})$ (where the value for $i = \emptyset$ in the case $k = 0$ is understood as $T_\emptyset = C$). Since J is invertible and since for any $i \in I$

$$JT^*_{(i_1,\ldots,i_k)} = J(JA_{i_k})^*\cdots(JA_{i_1})^*C^* = J(A_{i_k}^*J)\cdots(A_{i_1}^*J)C^* = (JA_{i_k}^*)\cdots(JA_{i_1}^*)JC^*$$

holds, we may write $\mathcal{S}^* = \sum_{i\in I}\operatorname{ran}(T_i^*)$ and $\mathcal{Q} = \bigcap_{i\in I}\ker(T_i)$, which makes the assertion obvious. Likewise, controllability is the same as observability for the adjoint system.

The next lemma collects from [**HMV06**] some basic facts about descriptor realizations. Note that the statements below are formulated in the complex case, whereas the version appearing in [**HMV06**] covers the real case.

LEMMA III.4.13 (Lemma 4.1 in [**HMV06**]).

(i) *Any descriptor realization determines a matrix of regular non-commutative rational functions.*

Conversely, each matrix $\mathfrak{r}$ of regular non-commutative rational functions, has a minimal descriptor realization (which could be taken to be monic) with feed through term $D = 0$.

Moreover, any two monic minimal descriptor realizations

$$\begin{aligned} \mathbb{r} &= D + C(1 - L_A(x))^{-1}B \\ \widetilde{\mathbb{r}} &= D + \widetilde{C}(1 - L_{\widetilde{A}}(x))^{-1}\widetilde{B}, \end{aligned}$$

for $\mathfrak{r}$ with the same feed through term are similar via a unique similarity transform, where a similarity transform *means an invertible matrix S satisfying*

(III.17) $$SA_j = \widetilde{A}_jS \quad \text{for } j = 1, \ldots, g, \qquad SB = \widetilde{B}, \qquad \text{and} \qquad C = \widetilde{C}S.$$

(ii) *Any matrix of regular non-commutative rational functions, which admits a self-adjoint descriptor realization, is self-adjoint*[2].

(iii) *If $\mathfrak{r}$ is a self-adjoint matrix of regular non-commutative rational functions, then $\mathfrak{r}$ has a minimal descriptor realization, which is self-adjoint as well.*

These statements are translations of their real counterparts given in [**HMV06**, Lemma 4.1]. Since it is straightforward, how the proof for the real case goes over to the complex case, we omit the details here. However, a comment on the statement in Item (iii) of Lemma III.4.13 is in order.

[2]Recall that Lemma III.2.35 (see also Lemma III.2.36) established the existence of a canonical involution $*$ on the free field $\mathbb{C}\langle\!\langle x_1, \ldots, x_g \rangle\!\rangle$.

REMARK III.4.14. The statement of Item (iii) is quite remarkable, as it significantly improves the construction (III.14), which was given in the proof of Theorem III.2.58 and which doubled the size of the given formal linear representation by passing to a self-adjoint one. The proof of Item (iii) in the real version presented in [**HMV06**] relies on [**HMV06**, Lemma 4.2]. In order to convince the reader that the complex counterpart is straightforward, we sketch their construction: suppose that

$$\mathbb{r} = D + C(1 - L_A(x))^{-1}B$$

is a monic descriptor realization of some self-adjoint non-commutative rational function $\mathfrak{r}$, which satisfies $D = D^*$ and which is minimal. Then there exists a unique complex matrix S, such that S is both self-adjoint and invertible and satisfies

$$SA_jS^{-1} = A_j^* \quad \text{and} \quad SB = C^* \quad \text{for } j = 1, \dots, g.$$

If we write $S = RJR^*$ with a signature matrix J and some invertible matrix R, then

$$\widetilde{\mathbb{r}} = D + \widetilde{C}(J - L_{\widetilde{A}}(x))^{-1}\widetilde{C}^*$$

with

$$\widetilde{C} := C(R^{-1})^* \quad \text{and} \quad \widetilde{A}_j := JR^*A_j(R^{-1})^* \quad \text{for } j = 1, \dots, g$$

forms a self-adjoint descriptor realization of $\mathfrak{r}$. For proving these assertions, one considers

$$\mathbb{r}^* = D + B^*(1 - L_{A^*}(x))^{-1}C^*,$$

which gives another monic descriptor realization of $\mathfrak{r}$ as $\mathfrak{r}$ was assumed to be self-adjoint. According to Item (i), the desired matrix S is then given as the unique similarity transform between $\mathbb{r}$ and $\mathbb{r}^*$. The fact, that this matrix S is self-adjoint, follows by uniqueness of S, since also S^* gives a similarity transform between $\mathbb{r}$ and $\mathbb{r}^*$. Finally, since $\widetilde{\mathbb{r}}$ has the same size as the initial minimal descriptor realization $\mathbb{r}$, it must be minimal as well.

A construction from classical one variable system theory, which dates back at least to Kalman [**Kal63**], also works well in this much more general context, cf. [**BGM05**]. It is that of cutting down the descriptor realization of a rational expression r to controllability and observability spaces thereby obtaining a minimal realization (whose existence was claimed in Item (i) of Lemma III.4.13 above). This results in a decomposition like in Theorem III.2.38. Accordingly, the following construction can be seen as an algorithmic version of Theorem III.2.38 for the regular case.

ALGORITHM III.4.15. *Given any descriptor realization*

$$\mathbb{r} = D + C(J - L_A(x))^{-1}B \quad \text{with} \quad L_A(x) = A_1x_1 + \dots + A_gx_g,$$

of some matrix $\mathfrak{r}$ of non-commutative rational functions in $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$. Cutting down $\mathbb{r}$ to a minimal descriptor realization proceeds as follows.

STEP 0 *First of all, we rewrite our given descriptor realization $\mathbb{r}$ in monic form as*

$$\tilde{\mathbb{r}} = D + C(1 - L_{JA}(x))^{-1}(JB) \quad \text{with} \quad L_{JA}(x) = (JA_1)x_1 + \dots + (JA_g)x_g.$$

Note that, on the level of matrix-valued rational expressions, these are different descriptor realizations, but over the free field, they represent the same matrix of non-commutative rational functions $\mathfrak{r}$. Moreover, the controllability and observability spaces for $\mathbb{r}$ and $\tilde{\mathbb{r}}$ are the same.

STEP 1 *Next, we cut down* $\tilde{\mathfrak{r}}$ *to its controllability space* $\mathcal{S}$*: with respect to the subspace decomposition* $\mathbb{C}^d = \mathcal{S} + \mathcal{S}^\perp$*, the system of matrices in the monic descriptor realization* $\tilde{\mathfrak{r}}$ *has the block decomposition*

$$C = \begin{pmatrix} \hat{C} & C_2 \end{pmatrix}, \qquad JA = \begin{pmatrix} \hat{A} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad \text{and} \qquad JB = \begin{pmatrix} \hat{B} \\ 0 \end{pmatrix}$$

Note that by definition of $\mathcal{S}$*, the range of* JB *is contained in* $\mathcal{S}$ *and that each* JA_i *leaves* $\mathcal{S}$ *invariant. Thus, we get a new descriptor realization*

$$\hat{\mathfrak{r}} = D + \hat{C}(1 - L_{\hat{A}}(x))^{-1}\hat{B} \qquad \text{with} \qquad L_{\hat{A}}(x) = \hat{A}_1 x_1 + \cdots + \hat{A}_g x_g$$

of $\mathfrak{r}$*, which is controllable by construction.*

STEP 2 *Finally, since the controllable descriptor realization* $\hat{\mathfrak{r}}$ *obtained in Step 1 may not be observable, we repeat the dual of the construction of Step 1 on* $\hat{\mathfrak{r}}$*. More precisely, we consider the adjoint system*

$$\hat{\mathfrak{r}}^* = D^* + \hat{B}^*(1 - L_{\hat{A}^*}(x))^{-1}\hat{C}^* \qquad \text{with} \qquad L_{\hat{A}^*}(x) = \hat{A}_1^* x_1 + \cdots + \hat{A}_g^* x_g$$

and make it controllable by cutting down to its controllability space $\hat{\mathcal{S}}^*$*, which agrees with* $\hat{\mathcal{Q}}^\perp$*. Thus, with respect to the decomposition* $\mathcal{S} = \hat{\mathcal{Q}} + \hat{\mathcal{Q}}^\perp$*, the system of matrices in* $\hat{\mathfrak{r}}$ *decomposes as*

$$\hat{B}^* = \begin{pmatrix} \hat{B}_1^* & \check{B}^* \end{pmatrix}, \qquad \hat{A}^* = \begin{pmatrix} \hat{A}_{11}^* & 0 \\ \hat{A}_{12}^* & \check{A}^* \end{pmatrix}, \qquad \text{and} \qquad \hat{C}^* = \begin{pmatrix} 0 \\ \check{C}^* \end{pmatrix}.$$

This yields a controllable descriptor realization

$$\check{\mathfrak{r}}^* = D^* + \check{B}^*(1 - L_{\check{A}^*}(x))^{-1}\check{C}^* \qquad \text{with} \qquad L_{\check{A}^*}(x) = \check{A}_1^* x_1 + \cdots + \check{A}_g^* x_g,$$

which represents the same non-commutative rational function as $\hat{\mathfrak{r}}^*$*. Thus, the adjoint system*

$$\check{\mathfrak{r}} = D + \check{C}(1 - L_{\check{A}}(x))^{-1}\check{B} \qquad \text{with} \qquad L_{\check{A}}(x) = \check{A}_1 x_1 + \cdots + \check{A}_g x_g,$$

with

$$\hat{C} = \begin{pmatrix} 0 & \check{C} \end{pmatrix}, \qquad \hat{A} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \check{A} \end{pmatrix}, \qquad \text{and} \qquad \hat{B} = \begin{pmatrix} \hat{B}_1 \\ \check{B} \end{pmatrix},$$

which represents $\mathfrak{r}$ *as* $\hat{\mathfrak{r}}$ *does, is observable. Since controllability was not affected by this procedure, it results in a minimal (monic) descriptor realization* $\check{\mathfrak{r}}$ *of* $\mathfrak{r}$*.*

Combining Step 1 and Step 2, the matrices appearing in the descriptor realization $\tilde{\mathfrak{r}}$ *have a block decomposition*

$$C = \begin{pmatrix} 0 & \check{C} & C_2 \end{pmatrix}, \qquad JA = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} & A_{12}^1 \\ 0 & \check{A} & A_{12}^2 \\ 0 & 0 & A_{22} \end{pmatrix}, \qquad \text{and} \qquad JB = \begin{pmatrix} \hat{B}_1 \\ \check{B} \\ 0 \end{pmatrix},$$

with respect to the decomposition $\mathbb{C}^d = \hat{\mathcal{Q}} + \hat{\mathcal{Q}}^\perp + \mathcal{S}^\perp$*.*

III.5. Evaluations of non-commutative rational functions

Although we will mostly work with non-commutative rational expressions instead of non-commutative rational functions, the construction of the free field raises an interesting question, which is also of great importance for our subsequent considerations:

> Given any non-commutative rational expression r, we know that we can compute its evaluation $r(X)$ at any point X belonging to its $\mathcal{A}$-domain. Is there also a well-defined evaluation for non-commutative rational functions?

First of all, we observe that (like in the commutative world) passing from one representative of a rational function to another one typically changes the domain. Correspondingly, "evaluation of a rational function $\mathfrak{r}$" can only mean evaluation of some of its representatives r. Recall from Definition III.2.46 that a non-commutative rational function $\mathfrak{r}$ in $\mathbb{C}\langle\!\!\langle x_1, \dots, x_g \rangle\!\!\rangle$ is said to be represented by a non-commutative rational expression $r \in \mathfrak{R}_{\mathbb{C}}(z_1, \dots, z_g)$, if the conditions $(x_1, \dots, x_g) \in \operatorname{dom}_{\mathbb{C}\langle\!\langle x_1,\dots,x_g \rangle\!\rangle}(r)$ and $r(x_1, \dots, x_g) = \mathfrak{r}$ are both satisfied. Alternatively, in the spirit of Paragraph III.2.2.2, we can view $\mathfrak{r}$ as an equivalence class $[r]$ of non-degenerate non-commutative rational expressions with respect to $M(\mathbb{C})$-evaluation equivalence, but we have seen in Proposition III.2.48 that r represents $\mathfrak{r}$ if and only if r is non-degenerate and $\mathfrak{r} = [r]$ holds. Since In this sense, it is natural to put

$$\operatorname{dom}_{\mathcal{A}}(\mathfrak{r}) := \bigcup_{r \text{ represents } \mathfrak{r}} \operatorname{dom}_{\mathcal{A}}(r).$$

Nevertheless, it is not clear that we can introduce a well-defined evaluation of $\mathfrak{r}$ on $\operatorname{dom}_{\mathcal{A}}(\mathfrak{r})$. In fact, this boils down to the question, whether two non-commutative rational expressions r_1 and r_2, which both represent the same non-commutative rational function, are $\mathcal{A}$-evaluation equivalent.

III.5.1. Stably finite algebras. It is certainly not surprising – although maybe not obvious at first sight – that this fails for general algebras $\mathcal{A}$. Let us look at some prototypical example.

Example III.5.1. Consider the two rational expressions $r_1, r_2 \in \mathfrak{R}_{\mathbb{C}}(z_1, z_2)$ given by

$$r_1 = z_1 \cdot ((z_2 \cdot z_1)^{-1} \cdot z_2) \qquad \text{and} \qquad r_2 = 1.$$

Clearly, for any unital complex algebra $\mathcal{A}$, we have

$$\operatorname{dom}_{\mathcal{A}}(r_1) = \{(X_1, X_2) \in \mathcal{A}^2 |\ X_2 X_1 \text{ is invertible in } \mathcal{A}\}$$

and $\operatorname{dom}_{\mathcal{A}}(r_2) = \mathcal{A}^2$. It is clear, that r_1 and r_2 both represent the same non-commutative rational function $\mathfrak{r}$ in $\mathbb{C}\langle\!\!\langle x_1, x_2 \rangle\!\!\rangle$. If we chose now $\mathcal{A} = B(\ell^2(\mathbb{N}))$ and for $\ell^2(\mathbb{N})$ the standard orthonormal basis $(e_n)_{n\in\mathbb{N}}$ for $\ell^2(\mathbb{N})$, then the corresponding right shift operator l and its adjoint l^* (i.e. the left shift operator) satisfy

$$l^* l = 1_{\mathcal{A}} \qquad \text{and} \qquad l l^* = p,$$

where p denotes the orthogonal projection onto the orthogonal complement of e_1. Accordingly, we have

$$(l, l^*) \in \operatorname{dom}_{\mathcal{A}}(r_1) \cap \operatorname{dom}_{\mathcal{A}}(r_2)$$

but $r_1(l, l^*) = p \neq 1_{\mathcal{A}} = r_2(l, l^*)$. Thus, r_1 and r_2 are not $\mathcal{A}$-evaluation equivalent.

In summary, the actual reason why the algebra $\mathcal{A} = B(\ell^2(\mathbb{N}))$ considered in the previous example does not admit well-defined evaluations for non-commutative rational functions is that it contains by l an operator, which has a left but no right inverse. In the terminology of the following definition, this means that $\mathcal{A}$ is not *stably finite*.

Definition III.5.2. A unital complex algebra $\mathcal{A}$ is called *stably finite algebra* $\mathcal{A}$ (sometimes also addressed as *weakly finite*) if it has the following property: for each $n \in \mathbb{N}$, every $A \in M_n(\mathcal{A})$ with either a left inverse or a right inverse has an inverse. More precisely, if we have $A, B \in M_n(\mathcal{A})$, then $AB = 1_{\mathcal{A}}$ implies $BA = 1_{\mathcal{A}}$.

Surprisingly, Example III.5.1 highlights in some sense the only obstacle in getting well-defined evaluations for non-commutative rational functions. Indeed, Theorem III.5.4 below will confirm that $\mathcal{A}$ being stably finite guarantees that all non-commutative rational expressions representing the same non-commutative rational expression must yield the same value under evaluation at $X = (X_1, \dots, X_g) \in \mathcal{A}^g$, as long as they are both defined at the considered point X. However, it is at first sight not clear how restrictive this condition is and how much work it requires to check its validity in concrete situations. Fortunately, for the purposes of free probability, stably finite is not a real issue. In fact, we often work in the setting of non-commutative C^*-probability spaces, which are equipped with faithful tracial states, and the following lemma tells us that the condition of being stably finite is automatically satisfied in such cases.

Lemma III.5.3. *A unital C^*-algebra, on which a faithful tracial state exists, is stably finite.*

The lemma is a standard fact in operator algebras and shows up for example in [**RLL00**] as an exercise. The interested reader is also referred to [**HMS15**] for a sketch of the proof.

III.5.2. Evaluation equivalence of rational expressions. The following theorem, which is a special case of [**Coh06**, Theorem 7.8.3], states that being stably finite is exactly the right condition on $\mathcal{A}$ that we need in order to be sure that $\mathcal{A}$-evaluation equivalence holds for non-commutative rational expressions, which represent the same non-commutative rational function.

Theorem III.5.4. *Let $\mathcal{A}$ be a unital complex algebra.*

(i) *If two non-commutative rational expressions r_1 and r_2 represent the same non-commutative rational function $\mathfrak{r}$, then they are $\mathcal{A}$-evaluation equivalent, provided $\mathcal{A}$ is a stably finite algebra.*
(ii) *If $\mathcal{A}$ is not stably finite then there exist rational expressions r_1 and r_2 in finitely many variables $z_1, \dots, z_g$, which represent the same non-commutative rational function in $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$, and $X \in \mathcal{A}^g$, such that $X \in \operatorname{dom}_{\mathcal{A}}(r_1) \cap \operatorname{dom}_{\mathcal{A}}(r_2)$ but $r_1(X) \neq r_2(X)$.*

For the readers convenience we include a proof of Theorem III.5.4

Proof of Theorem III.5.4. (i) Let two non-commutative rational expressions r_1 and r_2 be given. According to Theorem III.2.44, we can find formal linear representations $\rho_1 = (u_1, Q_1, v_1)$ and $\rho_2 = (u_2, Q_2, v_2)$ of r_1 and r_2, respectively. One can easily check that

$$\rho = (u, Q, v) := \left(\begin{pmatrix} u_1 & u_2 \end{pmatrix}, \begin{pmatrix} Q_1 & 0 \\ 0 & -Q_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right)$$

gives a formal linear representation of the non-commutative rational expression $r := r_1 + (-1) \cdot r_2$. Note that $\operatorname{dom}_{\mathcal{A}}(r) = \operatorname{dom}_{\mathcal{A}}(r_1) \cap \operatorname{dom}_{\mathcal{A}}(r_2)$ for any unital complex algebra $\mathcal{A}$. If we assume now that r_1 and r_2 represent the same non-commutative rational function $\mathfrak{r}$. Then r represents 0 in $\mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$, and Corollary III.2.47 tells us that the formal linear representation $\rho = (u, Q, v)$ of r yields a pure linear representation $(u, Q(x), v)$ of

0 in the sense of Cohn and Reutenauer [**CR99**]; see Definition III.2.27 and Definition III.2.28. Thus, Theorem III.2.26 tells us that the display

$$\begin{pmatrix} 0 & u \\ v & Q(x) \end{pmatrix}$$

cannot be full in $M_{n+1}(\mathbb{C}\langle x_1,\dots,x_g\rangle)$, which means by Definition III.2.23 that there are rectangular matrices

$$R_1 \in M_{(n+1)\times n}(\mathbb{C}\langle x_1,\dots,x_g\rangle) \qquad \text{and} \qquad R_2 \in M_{n\times(n+1)}(\mathbb{C}\langle x_1,\dots,x_g\rangle),$$

such that

$$\begin{pmatrix} 0 & u \\ v & Q(x) \end{pmatrix} = R_1 R_2.$$

We write

$$R_1 = \begin{pmatrix} p_1 \\ P_1 \end{pmatrix} \qquad \text{and} \qquad R_2 = \begin{pmatrix} p_2 & P_2 \end{pmatrix}$$

with $P_1, P_2 \in M_n(\mathbb{C}\langle x_1,\dots,x_g\rangle)$ and matrices p_1 and p_2, which are of size $1\times n$ respectively $n\times 1$, with entries in $\mathbb{C}\langle x_1,\dots,x_g\rangle$. This matrix-valued but polynomial identity can be evaluated at any given point in $\mathcal{A}^g$, especially at $X \in \operatorname{dom}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(Q^{-1})$, yielding

$$\begin{aligned} \begin{pmatrix} 0 & u \\ v & Q(X) \end{pmatrix} &= R_1(X)R_2(X) \\ &= \begin{pmatrix} p_1(X) \\ P_1(X) \end{pmatrix} \begin{pmatrix} p_2(X) & P_2(X) \end{pmatrix} \\ &= \begin{pmatrix} 0 & p_1(X) \\ 0 & P_1(X) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ p_2(X) & P_2(X) \end{pmatrix}. \end{aligned} \tag{III.18}$$

Applying formula (A.2), which appears in the proof of the Schur complement formula in Lemma A.1, to the evaluation of the display gives

$$\begin{pmatrix} 0 & u \\ v & Q(X) \end{pmatrix} = \begin{pmatrix} 1 & uQ(X)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -uQ(X)^{-1}v & 0 \\ 0 & Q(X) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Q(X)^{-1}v & 1 \end{pmatrix},$$

which can be reformulated as

$$\begin{pmatrix} 1 & -uQ(X)^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & u \\ v & Q(X) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Q(X)^{-1}v & 1 \end{pmatrix} = \begin{pmatrix} r(X) & 0 \\ 0 & Q(X) \end{pmatrix},$$

since $r(X) = -uQ(X)^{-1}v$ holds due to the defining properties of ρ. Finally, we get that

$$\begin{pmatrix} 1 & -uQ(X)^{-1} \\ 0 & Q(X)^{-1} \end{pmatrix} \begin{pmatrix} 0 & u \\ v & Q(X) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Q(X)^{-1}v & 1 \end{pmatrix} = \begin{pmatrix} r(X) & 0 \\ 0 & 1 \end{pmatrix}.$$

Combined with the decomposition (III.18), this yields

$$\begin{aligned} \begin{pmatrix} r(X) & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & -uQ(X)^{-1} \\ 0 & Q(X)^{-1} \end{pmatrix} \begin{pmatrix} 0 & u \\ v & Q(X) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Q(X)^{-1}v & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -uQ(X)^{-1} \\ 0 & Q(X)^{-1} \end{pmatrix} \begin{pmatrix} 0 & p_1(X) \\ 0 & P_1(X) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ p_2(X) & P_2(X) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Q(X)^{-1}v & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & p_1(X) - uQ(X)^{-1}P_1(X) \\ 0 & Q(X)^{-1}P_1(X) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ p_2(X) - P_2(X)Q(X)^{-1}v & P_2(X) \end{pmatrix}, \end{aligned}$$

which can be written in the form

$$\begin{pmatrix} r(X) & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b & B \end{pmatrix}$$

for some square matrices $A, B \in M_n(\mathcal{A})$ and with a and b of appropriate size. After multiplying the matrices on the right hand side of this equality and comparing the result with its left hand side we get that

$$r(X) = ab, \quad aB = 0, \quad Ab = 0, \quad \text{and} \quad AB = 1.$$

Since $\mathcal{A}$ is assumed to be stably finite, the equation $AB = 1$ implies that A and B are both invertible. Thus, from $aB = 0$ and $Ab = 0$, it follows that $a = 0$ and $b = 0$, which gives $r(X) = ab = 0$. Finally, by definition of r, we deduce $r_1(X) = r_2(X)$. Since $X \in \operatorname{dom}_{\mathcal{A}}(r) = \operatorname{dom}_{\mathcal{A}}(r_1) \cap \operatorname{dom}_{\mathcal{A}}(r_2)$ was arbitrarily chosen, we conclude that r_1 and r_2 are $\mathcal{A}$-evaluation equivalent, as stated.

(ii) If $\mathcal{A}$ is not stably finite then there exists square matrices Q, P over $\mathcal{A}$, say of size $n \times n$, such that $PQ = 1$, but $QP \neq 1$. Let T and S be $n \times n$ matrices over the free field with indeterminate entries. We have then $T(ST)^{-1}S - 1 = 0$. This gives n^2 equations in the entries of S, T, and $(ST)^{-1}$, all of which are 0 in the free field. However, not all of them are true in our algebra $\mathcal{A}$, though all expressions make sense there. ☐

III.5.3. Evaluation of pure linear representations. The proof of Item (i) of Theorem III.5.4 suggests the following generalization to the case where non-commutative rational expressions and their realizations are compared after evaluation.

COROLLARY III.5.5. *Let $\mathfrak{r}$ be a non-commutative rational function in $\mathbb{C}\langle\!\langle x_1, \ldots, x_g \rangle\!\rangle$, which is represented by some non-commutative rational expression r in the formal variables $z_1, \ldots, z_g$, and let $(u, Q(x), v)$, with some linear pencil Q in $z_1, \ldots, z_g$, be any pure linear representation of $\mathfrak{r}$. If $\mathcal{A}$ is stably finite, then*

$$r(X) = -uQ(X)^{-1}v \qquad \text{for all } X \in \operatorname{dom}_{\mathcal{A}}(r) \cap \operatorname{dom}_{\mathcal{A}}(Q^{-1}).$$

PROOF. For the given rational expression r, we may find according to Theorem III.2.44 a formal linear representation $\rho_0 = (u_0, Q_0, v_0)$. By Definition III.2.43, it satisfies

$$\operatorname{dom}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(Q_0^{-1})$$

and

$$r(X) = -u_0 Q_0(X)^{-1} v_0 \qquad \text{for all } X = (X_1, \ldots, X_g) \in \operatorname{dom}_{\mathcal{A}}(r).$$

Moreover, as it was observed in Corollary III.2.47, this formal linear representation ρ_0 induces a pure linear representation $(u_0, Q_0(x), v_0)$ of $\mathfrak{r}$ in the sense of Definition III.2.28. Now, since $(u, Q(x), v)$ and $(u_0, Q_0(x), v_0)$ are both pure linear representations of the same non-commutative rational rational $\mathfrak{r}$, we may deduce (for the most part with the help of Rule III.7 of Lemma III.2.33) that

$$(\tilde{u}, \tilde{Q}(x), \tilde{v}) := \left(\begin{pmatrix} u_0 & u \end{pmatrix}, \begin{pmatrix} Q_0(x) & 0 \\ 0 & -Q(x) \end{pmatrix}, \begin{pmatrix} v_0 \\ v \end{pmatrix} \right)$$

gives a pure linear representation of 0 in $\mathbb{C}\langle\!\langle x_1, \ldots, x_g \rangle\!\rangle$, where $\tilde{Q}$ denotes the linear pencil in the variables $z_1, \ldots, z_g$, which is given by

$$\tilde{Q} := \begin{pmatrix} Q_0 & 0 \\ 0 & -Q \end{pmatrix}.$$

Thus, we may continue now like in the proof of Theorem III.5.4. Indeed, Theorem III.2.26 tells us that the corresponding display

$$\begin{pmatrix} 0 & \tilde{u} \\ \tilde{v} & \tilde{Q}(x) \end{pmatrix}$$

cannot be full, which allows us to deduce that $-\tilde{u}\tilde{Q}(X)^{-1}\tilde{v} = 0$ holds, whenever X belongs to $\operatorname{dom}_{\mathcal{A}}(\tilde{Q}^{-1})$ for any stably finite algebra $\mathcal{A}$. Since

$$\operatorname{dom}_{\mathcal{A}}(\tilde{Q}^{-1}) = \operatorname{dom}_{\mathcal{A}}(Q_0^{-1}) \cap \operatorname{dom}_{\mathcal{A}}(Q^{-1}) \supseteq \operatorname{dom}_{\mathcal{A}}(r) \cap \operatorname{dom}_{\mathcal{A}}(Q^{-1}),$$

we obtain by using that ρ_0 is a formal linear representation of r that

$$0 = -\tilde{u}\tilde{Q}(X)^{-1}\tilde{v} = -u_0 Q_0(X)^{-1} v_0 + uQ(X)^{-1} v = r(X) + uQ(X)^{-1} v$$

and hence $r(X) = -uQ(X)^{-1}v$ for all $X \in \operatorname{dom}_{\mathcal{A}}(r) \cap \operatorname{dom}_{\mathcal{A}}(Q^{-1})$, as claimed. □

The following lemma is a slight modification and generalization of [**HMS15**, Lemma 3.6]. Nevertheless, since it can be obtained basically in the same way, we go without giving a proof.

LEMMA III.5.6. *Suppose $\mathcal{A}$ is a unital complex algebra. Then the following two statements are equivalent:*

(i) *If a $k \times k$ block triangular matrix*

$$Q = \begin{pmatrix} Q_{1,1} & & 0 \\ \vdots & \ddots & \\ Q_{k,1} & \dots & Q_{k,k} \end{pmatrix}$$

with entries $Q_{i,j} \in M_{m_i \times m_j}(\mathcal{A})$ of dimensions $m_1, \dots, m_k \in \mathbb{N}$ is invertible, then all its diagonal entries $Q_{i,i} \in M_{m_i}(\mathcal{A})$ are invertible.

(ii) *$\mathcal{A}$ is stably finite.*

The following proposition records an interesting consequence of the previous lemma.

PROPOSITION III.5.7. *Let $\mathfrak{r} \in \mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ be any non-commutative rational function in the variables $x = (x_1, \dots, x_g)$.*

(i) *Two minimal pure linear representations of $\mathfrak{r}$ have the same $\mathcal{A}$-domain for any unital complex algebra $\mathcal{A}$. Furthermore, they take the same values on the intersection of their $\mathcal{A}$-domains.*

(ii) *Consider two pure linear representations $\rho = (u, Q(x), v)$ and $\rho_0 = (u_0, Q_0(x), v_0)$ of $\mathfrak{r}$, where we assume that ρ_0 is minimal. Then, for any unital complex algebra $\mathcal{A}$, which is stably finite, it holds true that*

$$\operatorname{dom}_{\mathcal{A}}(Q^{-1}) \subseteq \operatorname{dom}_{\mathcal{A}}(Q_0^{-1}) \tag{III.19}$$

and

$$-uQ(X)^{-1}v = -u_0 Q_0(X)^{-1} v_0 \qquad \textit{for all } X = (X_1, \dots, X_g) \in \operatorname{dom}_{\mathcal{A}}(Q^{-1}).$$

PROOF. (i) Given two minimal pure linear representations $\rho = (u, Q(x), v)$ and $\rho' = (u', Q'(x), v')$ of the same non-commutative rational function $\mathfrak{r}$, then Corollary III.2.39 tells us that ρ and ρ' are isomorphic in the sense that there exists a morphism (S, T) from ρ to ρ' with invertible matrices S and T. Recall that (S, T) being a morphism from ρ to ρ' means that

$$u' = uT, \qquad v = Sv', \qquad \text{and} \qquad SQ'(x) = Q(x)T$$

holds. Since S and T are both invertible, it follows from $SQ'(x) = Q(x)T$ that $SQ' = QT$ and hence that $\operatorname{dom}_{\mathcal{A}}(Q^{-1}) = \operatorname{dom}_{\mathcal{A}}((Q')^{-1})$ holds for any unital complex algebra $\mathcal{A}$.

Furthermore, we see for any point X in the joint $\mathcal{A}$-domain that $SQ'(X) = Q(X)T$ and hence $Q(X)^{-1}S = TQ'(X)^{-1}$ holds, such that finally

$$-uQ(X)^{-1}v = -uQ(X)^{-1}Sv' = -uTQ'(X)^{-1}v' = -u'Q'(X)^{-1}v'$$

follows as claimed.

(ii) Theorem III.2.38 tells us that we can find invertible matrices S and T, such that

$$uT = \begin{pmatrix} u_1 & u_0 & 0 \end{pmatrix}, \quad SQ(x)T = \begin{pmatrix} Q_{1,1}(x) & 0 & 0 \\ Q_{2,1}(x) & Q_0(x) & 0 \\ Q_{3,1}(x) & Q_{3,2}(x) & Q_{3,3}(x) \end{pmatrix}, \quad Sv = \begin{pmatrix} 0 \\ v_0 \\ v_3 \end{pmatrix}.$$

If we take now any unital complex algebra $\mathcal{A}$, which is stably finite, and if we assume that the matrix $Q(X)$ is invertible, then $SQ(X)T$ is also invertible and Lemma III.5.6 finally gives the invertibility of $Q_0(X)$. Thus, the stated inclusion $\operatorname{dom}_\mathcal{A}(Q^{-1}) \subseteq \operatorname{dom}_\mathcal{A}(Q_0^{-1})$ for stably finite $\mathcal{A}$ follows. In this case, given any $X \in \operatorname{dom}_\mathcal{A}(Q^{-1})$, it is easy to see that

$$-uQ(X)^{-1}v = -(uT)(SQ(X)T)^{-1}(Sv) = -u_0Q_0(X)^{-1}v_0,$$

as we wished to show. □

We may finalize now our observations by the following theorem.

THEOREM III.5.8. *Let r be a rational expression in the formal variables $x = (x_1, \dots, x_g)$ and let $\mathfrak{r} \in \mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$ denote the non-commutative rational function induced by r.*

(i) *The non-commutative rational function $\mathfrak{r}$ admits a pure linear representation $\rho = (u, Q(x), v)$, which enjoys the following property:*
If $\mathcal{A}$ is a unital complex algebra (not necessarily stably finite), then

$$\operatorname{dom}_\mathcal{A}(r) \subseteq \operatorname{dom}_\mathcal{A}(Q^{-1})$$

and

$$r(X) = -uQ(X)^{-1}v \qquad \text{if } X = (X_1, \dots, X_g) \in \operatorname{dom}_\mathcal{A}(r).$$

(ii) *Any minimal pure linear representation $\rho_0 = (u_0, Q_0(x), v_0)$ of $\mathfrak{r}$ satisfies the following property:*
If $\mathcal{A}$ is a unital complex algebra, which is stably finite, then

$$\operatorname{dom}_\mathcal{A}(r) \subseteq \operatorname{dom}_\mathcal{A}(Q_0^{-1}) \tag{III.20}$$

and

$$r(X) = -u_0Q_0(X)^{-1}v_0 \qquad \text{for all } X = (X_1, \dots, X_g) \in \operatorname{dom}_\mathcal{A}(r). \tag{III.21}$$

PROOF. (i) The assertion in (i) is a direct consequence of Theorem III.2.44, which shows that a formal linear representation (u, Q, v) of r exists (and hence satisfies the requested property according to Definition III.2.43), and of Corollary III.2.47, which says that $\rho = (u, Q(x), v)$ is a pure linear representation.

(ii) In order to check the validity of (ii), we chose a pure linear representation $\rho = (u, Q, v)$ like in (i) and for any unital complex algebra $\mathcal{A}$, which is stably finite, we observe

... by part (i) that $\operatorname{dom}_\mathcal{A}(r) \subseteq \operatorname{dom}_\mathcal{A}(Q^{-1})$ holds and that we have

$$r(X) = -uQ(X)^{-1}v \qquad \text{for any } X \in \operatorname{dom}_\mathcal{A}(r).$$

... by Item (ii) of Proposition III.5.7 that $\operatorname{dom}_{\mathcal{A}}(Q^{-1}) \subseteq \operatorname{dom}_{\mathcal{A}}(Q_0^{-1})$ holds and that

$$-uQ(X)^{-1}v = -u_0 Q_0(X)^{-1} v_0 \qquad \text{for all } X \in \operatorname{dom}_{\mathcal{A}}(Q^{-1}).$$

Thus, it follows that for any stably finite $\mathcal{A}$

$$\operatorname{dom}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(Q^{-1}) \subseteq \operatorname{dom}_{\mathcal{A}}(Q_0^{-1})$$

holds and moreover

$$r(X) = -uQ(X)^{-1}v = -u_0 Q_0(X)^{-1} v_0 \qquad \text{for all } X \in \operatorname{dom}_{\mathcal{A}}(r).$$

This concludes the proof. □

Proposition III.5.7 and Theorem III.5.8 are prototypical for very similar investigations in the setting of descriptor realizations. However, the results that we will obtain in the sequel are not just reformulations of the previous ones. In fact, Proposition III.5.9 and Theorem III.5.10 apply even to (rectangular) matrices of non-commutative rational functions, and the additional Theorem III.5.11 will provide a self-adjoint version of Theorem III.5.10. As already outlined in Remark III.2.40, we expect that the theory of pure linear representations can be extended to the case of (rectangular) matrices of non-commutative rational functions, and that also (despite the slightly different notions of minimality) an analogue of [**HMV06**, Lemma 4.2] exists. Recall that the construction of [**HMV06**, Lemma 4.2], which was presented in Remark III.4.14, explained how the minimal descriptor realizations can be used to construct a self-adjoint descriptor realization of the same size, which is therefore minimal as well.

III.5.4. Evaluations of descriptor realizations. In the previous subsection, we have explored some applications of formal linear representations within the theory of pure linear representations. Here, we turn our attention to the case of (matrices of) non-commutative rational expressions, which are regular at zero, and we aim at proving similar statements about non-commutative descriptor realizations. As it will turn out, the excellent evaluation properties of formal linear representations with respect to stably finite algebras pass on descriptor realizations under the assumption of minimality in the sense of Definition III.4.11.

Proposition III.5.9.

(i) *Any two minimal descriptor realizations of the same matrix-valued non-commutative rational function, which both have the same feed through term, have the same $\mathcal{A}$-domain for any unital complex algebra $\mathcal{A}$. Furthermore, they take the same values on the intersection of their $\mathcal{A}$-domains.*

(ii) *Suppose that*

$$\mathbb{r}(x) = D + C(J - L_A(x))^{-1}B$$
$$\hat{\mathbb{r}}(x) = D + \hat{C}(\hat{J} - L_{\hat{A}}(x))^{-1}\hat{B},$$

are both (self-adjoint) descriptor realizations for the same (self-adjoint) matrix-valued non-commutative rational expression with $\hat{\mathbb{r}}$ being minimal. If $\mathcal{A}$ is a unital ($*$-)algebra, which is stably finite, then

$$\operatorname{dom}_{\mathcal{A}}\left((J - L_A)^{-1}\right) \subseteq \operatorname{dom}_{\mathcal{A}}\left((\hat{J} - L_{\hat{A}})^{-1}\right) \tag{III.22}$$

and

$$\mathbb{r}(X) = \hat{\mathbb{r}}(X) \qquad \textit{for all } X = (X_1, \dots, X_g) \in \operatorname{dom}_{\mathcal{A}}(\mathbb{r}).$$

Proof. The proof proceeds along the same lines as the proof of Proposition III.5.7. The only differences are

- that Corollary III.2.39 gets replaced by the state space similarity theorem stated in Item (i) of Lemma III.4.13, where both minimal descriptor realization must first be brought into monic form (like in Step 0 of Algorithm III.4.15), which does not affect their minimality, and
- that Theorem III.2.38 gets replaced by Algorithm III.4.15.

We conclude by noting that the self-adjoint case of Item (ii) is clearly covered by the more general statement that was proven above, since any minimal self-adjoint realization is in particular a minimal realization; see Remark III.4.14. ☐

Now, we may proceed to the following counterpart of Theorem III.5.8.

Theorem III.5.10. *Let r be a matrix of rational expressions in variables $x = (x_1, \dots, x_g)$, which are regular at zero, and denote by $\mathfrak{r}$ the induced matrix of non-commutative rational functions. Then the following statements hold true:*

(i) *The matrix $\mathfrak{r}$ admits a monic realization of the form*
$$\mathbb{r}(x) = D + C(1 - L_A(x))^{-1}B,$$
where the feed through term $D \in M_k(\mathbb{C})$ can be prescribed arbitrarily, which enjoys the following property:
If $\mathcal{A}$ is a unital complex algebra (not necessarily stably finite), then
$$\operatorname{dom}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(\mathbb{r})$$
and
$$r(X) = \mathbb{r}(X) \qquad \text{if } X \in \operatorname{dom}_{\mathcal{A}}(r).$$
(ii) *Any minimal realization*
$$\hat{\mathbb{r}}(x) = D + \hat{C}(\hat{J} - L_{\hat{A}}(x))^{-1}\hat{B},$$
of $\mathfrak{r}$ satisfies the following property:
If $\mathcal{A}$ is a unital complex algebra, which is stably finite, then
$$\operatorname{dom}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(\hat{\mathbb{r}}) \tag{III.23}$$
and
$$r(X) = \hat{\mathbb{r}}(X) \qquad \text{if } X \in \operatorname{dom}_{\mathcal{A}}(r). \tag{III.24}$$

Proof. (i) For proving (i), we proceed as follows: by Theorem III.2.58 we may find some matrix-valued formal linear representation $\rho = (u, Q, v)$ of $r - D$. Since $0 \in \operatorname{dom}_{\mathcal{A}}(r) = \operatorname{dom}_{\mathcal{A}}(r - D)$ holds by the regularity assumption and since we have $\operatorname{dom}_{\mathcal{A}}(r - D) \subseteq \operatorname{dom}_{\mathcal{A}}(Q^{-1})$ due to Definition III.3.9, we see that the linear pencil Q entails an invertible matrix $Q^{(0)}$. Thus, we may introduce
$$\mathbb{r}_0(x) := -u\big(1 + (Q^{(0)})^{-1}Q^{(1)}x_1 + \dots + (Q^{(0)})^{-1}Q^{(g)}x_g\big)^{-1}(Q^{(0)})^{-1}v,$$
which is of the form $C(1 - L_A(x))^{-1}B$ with $C = -u$, $B = (Q^{(0)})^{-1}v$ and $A_j = -(Q^{(0)})^{-1}Q^{(j)}$ for $j = 1, \dots, n$. Again by Definition III.3.9, we know that $\operatorname{dom}_{\mathcal{A}}(r - D) \subseteq \operatorname{dom}_{\mathcal{A}}(\mathbb{r})$ holds for any unital complex algebra $\mathcal{A}$ and in addition
$$r(X) - D = \mathbb{r}_0(X) \qquad \text{for all } X \in \operatorname{dom}_{\mathcal{A}}(r),$$

i.e.

$$r(X) = D + C(1 - L_A(x))^{-1}B \qquad \text{for all } X \in \operatorname{dom}_{\mathcal{A}}(r).$$

Since this applies in particular in the case $\mathcal{A} = \mathbb{C}\langle x_1, \dots, x_g \rangle$, we see that $\mathbb{r}(x) = D + C(1 - L_A(x))^{-1}B$ is the desired monic descriptor realization of r.

(ii) For seeing (ii), we start with any descriptor realization $\mathbb{r}$ of r as in part (i). For the given minimal realization $\hat{\mathbb{r}}$, we know

... by part (i) that $\operatorname{dom}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(\mathbb{r})$ holds and that we have

$$r(X) = \mathbb{r}(X) \qquad \text{for all } X \in \operatorname{dom}_{\mathcal{A}}(r).$$

... by Item (ii) of Proposition III.5.9 that $\operatorname{dom}_{\mathcal{A}}(\mathbb{r}) \subseteq \operatorname{dom}_{\mathcal{A}}(\hat{\mathbb{r}})$ holds and that

$$\mathbb{r}(X) = \hat{\mathbb{r}}(X) \qquad \text{for all } X \in \operatorname{dom}_{\mathcal{A}}(\mathbb{r}).$$

This yields the chain of inclusions $\operatorname{dom}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(\mathbb{r}) \subseteq \operatorname{dom}_{\mathcal{A}}(\hat{\mathbb{r}})$, which proves (III.23), and furthermore $r(X) = \mathbb{r}(X) = \hat{\mathbb{r}}(X)$ for any point $X \in \operatorname{dom}_{\mathcal{A}}(r)$, which shows the validity of (III.24). □

Similarly, self-adjoint representations allow us to construct self-adjoint realizations. This will be the content of the following theorem, which can be seen as a self-adjoint counterpart of Theorem III.5.10.

THEOREM III.5.11. *Let r be a self-adjoint matrix of rational expressions in formal variables $x_1, \dots, x_g$, which is regular at zero, and denote by $\mathfrak{r}$ the induced matrix of noncommutative rational functions. Then the following statements hold true:*

(i) *The matrix $\mathfrak{r}$ admits a self-adjoint realization of the form*

$$\mathbb{r}(x) = D + C(J - L_A(x))^{-1}C^*,$$

where the feed through term $D \in M_k(\mathbb{C})$ can be prescribed arbitrarily, which enjoys the following property:

If $\mathcal{A}$ is a unital complex $*$-algebra (not necessarily stably finite), then

$$\operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(\mathbb{r})$$

and

$$r(X) = \mathbb{r}(X) \qquad \text{if } X \in \operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r).$$

(ii) *Any minimal self-adjoint realization*

$$\hat{\mathbb{r}}(x) = D + \hat{C}(\hat{J}_0 - L_{\hat{A}}(x))^{-1}\hat{C}^*,$$

of $\mathfrak{r}$ satisfies the following property:

If $\mathcal{A}$ is a unital complex $*$-algebra, which is stably finite, then

$$\operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(\hat{\mathbb{r}}) \tag{III.25}$$

and

$$r(X) = \hat{\mathbb{r}}(X) \qquad \text{if } X \in \operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r). \tag{III.26}$$

PROOF. (i) For proving (i), we need a refinement of the argument that was used in the proof of Item (i) in Theorem III.5.10: Since r is assumed to be regular at zero, we know that for any formal linear representation $\rho_0 = (u_0, Q_0, v_0)$ of $r - D$, the matrix $Q_0^{(0)}$ appearing in the linear pencil

$$Q_0 = Q_0^{(0)} + Q_0^{(1)}x_1 + \dots + Q_0^{(g)}x_g$$

has to be invertible. Thus, we may form with $\widetilde{Q}_0^{(j)} := (Q_0^{(0)})^{-1}Q_0^{(j)}$ for $j = 0,\dots,g$ the linear pencil

$$\widetilde{Q}_0 = \widetilde{Q}_0^{(0)} + \widetilde{Q}_0^{(1)}x_1 + \cdots + \widetilde{Q}_0^{(g)}x_g \qquad \text{where} \qquad \widetilde{Q}_0^{(0)} = 1.$$

We define in addition $\widetilde{u}_0 := u_0$ and $\widetilde{v}_0 := (Q_0^{(0)})^{-1}v_0$. By this construction, we clearly obtain another formal linear representation $\widetilde{\rho}_0 = (\widetilde{u}_0, \widetilde{Q}_0, \widetilde{v}_0)$ of $r - D$. If we proceed now with the construction that was presented in (III.14), this yields a self-adjoint formal linear representation

$$\rho = (Q, v) := \left(\begin{pmatrix} 0 & \widetilde{Q}_0^* \\ \widetilde{Q}_0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2}\widetilde{u}_0^* \\ \widetilde{v}_0 \end{pmatrix} \right).$$

Now, we continue like in the proof of Item (i) in Theorem III.5.10. Starting with the self-adjoint formal linear representation $\rho = (Q, v)$, we introduce

$$\mathbb{r}_0(x) := -v^*\big(Q^{(0)} + Q^{(1)}x_1 + \cdots + Q^{(g)}x_g\big)^{-1}v,$$

which is of the form $\mathbb{r}_0(x) = C(J - L_A(x))^{-1}C^*$ with $C = v$, $J = -Q^{(0)}$, and $A_j = Q^{(j)}$ for $j = 1,\dots,g$. Note that indeed $J^* = J$ and $J^2 = 1$. Finally, we put

$$\mathbb{r}(x) := D + C(J - L_A(x))^{-1}C^*.$$

Thus, by construction, we have for any unital complex $*$-algebra $\mathcal{A}$ that

$$\mathrm{dom}_{\mathcal{A}}^{\mathrm{sa}}(r) \subseteq \mathrm{dom}_{\mathcal{A}}(\mathbb{r})$$

and

$$r(X) = \mathbb{r}(X) \qquad \text{if } X \in \mathrm{dom}_{\mathcal{A}}^{\mathrm{sa}}(r).$$

It remains to prove that $\mathbb{r}$ is indeed a realization of $\mathfrak{r}$. For that purpose, we apply the previous observation in the case $\mathcal{A} = \mathbb{C}\langle\!\!\langle x_1,\dots,x_g \rangle\!\!\rangle$, which is known to be a unital complex $*$-algebra according to Lemma III.2.35 (see also Lemma III.2.36). This yields then $\mathfrak{r} = r(x) = \mathbb{r}(x)$, as desired.

(ii) The assertion in (ii) can be proven as follows. Given $\mathfrak{r}$, which is represented by r, we may consider besides its minimal self-adjoint realization

$$\hat{\mathbb{r}}(x) = D + \hat{C}(\hat{J}_0 - L_{\hat{A}}(x))^{-1}\hat{C}^*,$$

any other self-adjoint descriptor realization

$$\mathbb{r}(x) = D + C(J - L_A(x))^{-1}C^*.$$

with the prescribed feed through term D, as constructed in (i). Thus, if $\mathcal{A}$ is any unital complex $*$-algebra, which is stably finite, we know

... by part (i) that $\mathrm{dom}_{\mathcal{A}}^{\mathrm{sa}}(r) \subseteq \mathrm{dom}_{\mathcal{A}}(\mathbb{r})$ holds and that we have

$$r(X) = \mathbb{r}(X) \qquad \text{for any } X \in \mathrm{dom}_{\mathcal{A}}^{\mathrm{sa}}(r).$$

... by Proposition III.5.9 that $\mathrm{dom}_{\mathcal{A}}(\mathbb{r}) \subseteq \mathrm{dom}_{\mathcal{A}}(\hat{\mathbb{r}})$ holds and that moreover

$$\mathbb{r}(X) = \hat{\mathbb{r}}(X) \qquad \text{for any } X \in \mathrm{dom}_{\mathcal{A}}^{\mathrm{sa}}(\mathbb{r}).$$

Combining both observation proves the stated inclusion (III.23) and also the representation given in (III.24). □

CHAPTER IV

Distributions and Brown measures of non-commutative polynomial and rational expression in freely independent variables

In Chapter I, we have learned that non-commutative distributions $\mu_{X_1,\dots,X_g}$ of freely independent elements $X_1,\dots,X_g$ are fully determined by their single-variable distributions $\mu_{X_1},\dots,\mu_{X_g}$. This has the important consequence that for any given non-commutative polynomial $P \in \mathbb{C}\langle x_1,\dots,x_g\rangle$ the distribution $\mu_{P(X_1,\dots,X_g)}$ of its evaluation $P(X_1,\dots,X_g)$ depends only on P and $\mu_{X_1},\dots,\mu_{X_g}$. Following [**BV93**], we denote this dependency by

$$\mu_{P(X_1,\dots,X_g)} = P^{\square}(\mu_{X_1},\dots,\mu_{X_g}).$$

However, making this relation more explicit remained a challenging problem for quite a while. It was tackled only in a few special cases and a general machinery, which could be applied uniformly to all non-commutative polynomials, seemed to be out of reach. For instance, if we choose $P(x_1,x_2) = x_1 + x_2$ and $P(x_1,x_2) = x_1 \cdot x_2$, we recover the free additive convolution $\mu_{X_1} \boxplus \mu_{X_2}$ and the free multiplicative convolution $\mu_{X_1} \boxtimes \mu_{X_2}$, respectively, which can effectively be treated with the help of the R- and S-transform. Apart from these examples, only the commutator $P(x_1,x_2) = i(x_1x_2 - x_2x_1)$ and the anti-commutator $P(x_1,x_2) = x_1x_2 + x_2x_1$ were discussed in detail; see [**NS98, Vas03**].

In this chapter, we finally want to close this gap. Following [**BMS13, BSS15, HMS15**], we develop some systematic approach to this question, which will be based on the method of linearization as developed in Chapter III and on tools from operator-valued free probability theory as presented in Section I.2 of Chapter I, especially on the operator-valued subordination result given in Theorem I.2.18. Our work will merge into explicit algorithms, which are easily accessible to numerical computations. Of course, we can only hope for some deeper insights, which go beyond the combinatorial description of non-commutative distributions, if the underlying non-commutative probability space carries some analytic structure. We will work here in the general setting of C^*-probability spaces and we will suppose in addition that the freely independent variables $X_1,\dots,X_n$ are self-adjoint. In this case, the initial distributions $\mu_{X_1},\dots,\mu_{X_g}$ can be identified with Borel probability measures on $\mathbb{R}$ and they are thus conveniently encoded via their Cauchy transforms as analytic functions on the complex upper half-plane. The type of the output $\mu_{P(X_1,\dots,X_g)}$, however, depends on whether the given non-commutative polynomial P is self-adjoint or not. Correspondingly, we are faced with two different types of questions:

(i) If the non-commutative polynomial P is self-adjoint, how can we compute the analytic distribution of $P(X_1,\dots,X_g)$?
(ii) If the non-commutative polynomial P is not self-adjoint, how can we compute the Brown measure of $P(X_1,\dots,X_g)$?

These are addressed in Algorithm IV.4.1 and Algorithm IV.4.2, respectively.

Since the linearization trick is by no means limited to the case of non-commutative polynomials, but applies equally well to rational expressions, it is a very natural question to ask whether the algorithms also extend to this case. Indeed, we will see that for any non-commutative rational expression r in g formal variables, whose domain contains the given non-commutative random variables $(X_1, \dots, X_g)$, the distribution $\mu_{r(X_1,\dots,X_g)}$ of its evaluation $r(X_1, \dots, X_g)$ is fully determined by r and $\mu_{X_1}, \dots, \mu_{X_g}$ and our Algorithms IV.4.1 and IV.4.2 allow to make this relationship explicit.

IV.1. The $*$-distribution of evaluated rational expressions

Consider a C^*-probability space $(\mathcal{A}, \phi)$ and self-adjoint non-commutative random variables $X_1, \dots, X_g \in \mathcal{A}$, which are freely independent. We have observed in Remark I.1.37 that freeness results in certain universal rules for computing mixed moments, such that $\mu_{P(X_1,\dots,X_g)}$ for any non-commutative polynomial $P \in \mathbb{C}\langle x_1, \dots, x_g\rangle$ depends on nothing more than P and the individual distributions $\mu_{X_1}, \dots, \mu_{X_g}$. However, if we take now any non-commutative rational expression in formal non-commuting variables $x_1, \dots, x_g$, say r, whose $\mathcal{A}$-domain contains $(X_1, \dots, X_g)$, it is not readily clear, why the same should hold true for $\mu_{r(X_1,\dots,X_g)}$.

It is instructive to consider the case of faithful expectations first: if $(\mathcal{A}, \phi)$ and $(\mathcal{B}, \psi)$ are C^*-probability spaces, such that ϕ and ψ are both faithful, and if $X_1, \dots, X_g$ and $Y_1, \dots, Y_g$ are freely independent self-adjoint elements in $\mathcal{A}$ and $\mathcal{B}$, respectively, such that, as unital C^*-algebras, $\mathcal{A}$ is generated by $X_1, \dots, X_g$ and $\mathcal{B}$ by $Y_1, \dots, Y_g$, then

$$\mu_{X_i} = \mu_{Y_i} \qquad \text{for } i = 1, \dots, g$$

implies $\mu_{X,X^*} = \mu_{Y,Y^*}$ and hence by Theorem I.1.23 the existence of a unique isometric $*$-isomorphism $\Phi : \mathcal{A} \to \mathcal{B}$, such that $\Phi(X_i) = Y_i$ for all $i = 1, \dots, g$ and $\psi \circ \Phi = \phi$ holds. Then, for any given non-commutative rational expression r in g variables, we have $(X_1, \dots, X_g) \in \operatorname{dom}_{\mathcal{A}}(r)$ if and only if $(Y_1, \dots, Y_g) \in \operatorname{dom}_{\mathcal{B}}(r)$ and in this case $\Phi(r(X_1, \dots, X_g)) = r(Y_1, \dots, Y_g)$. This statement is confirmed by the following easy lemma.

Lemma IV.1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and let $\Phi : \mathcal{A} \to \mathcal{B}$ be an isometric and unital $*$-isomorphism. Then, for each non-commutative rational expression r in the formal variables $x = (x_1, \dots, x_g)$, we have that*

$$\big\{(\Phi(Z_1), \dots, \Phi(Z_g)) \big| \, (Z_1, \dots, Z_g) \in \operatorname{dom}_{\mathcal{A}}(r)\big\} = \operatorname{dom}_{\mathcal{B}}(r) \tag{IV.1}$$

and $\Phi(r(Z_1, \dots, Z_g)) = r(\Phi(Z_1), \dots, \Phi(Z_g))$ for each $(Z_1, \dots, Z_g) \in \operatorname{dom}_{\mathcal{A}}(r)$.

Proof. Consider the subset $\mathfrak{R}$ of $\mathfrak{R}_{\mathbb{C}}(x)$, which consists of all scalar-valued non-commutative rational expressions r satisfying

$$\big\{(\Phi(Z_1), \dots, \Phi(Z_g)) \big| \, (Z_1, \dots, Z_g) \in \operatorname{dom}_{\mathcal{A}}(r)\big\} \subseteq \operatorname{dom}_{\mathcal{B}}(r)$$

and $\Phi(r(Z_1, \dots, Z_g)) = r(\Phi(Z_1), \dots, \Phi(Z_g))$ for each $(Z_1, \dots, Z_g) \in \operatorname{dom}_{\mathcal{A}}(r)$. It is sufficient to prove that $\mathfrak{R} = \mathfrak{R}_{\mathbb{C}}(x)$ holds; the equality claimed in (IV.1) follows then by switching the roles of $\mathcal{A}$ and $\mathcal{B}$ and by replacing Φ by its inverse Φ^{-1}.

One easily sees that $\mathfrak{P}_{\mathbb{C}}(x) \subseteq \mathfrak{R}$ holds. Thus, according to Remark III.2.2, it remains to check that $\mathfrak{R}$ is closed under the arithmetic operations $+$, $\cdot$, and $^{-1}$. We leave out the trivial cases $+$ and $\cdot$ and discuss here only $^{-1}$. If $(Z_1, \dots, Z_g) \in \operatorname{dom}_{\mathcal{A}}(r^{-1})$ is given, then we know by Definition III.2.3 that $(Z_1, \dots, Z_g) \in \operatorname{dom}_{\mathcal{A}}(r)$ holds and that

$r(Z_1, \ldots, Z_g)$ is invertible in $\mathcal{A}$. Since $r \in \mathfrak{R}$, it follows from $(Z_1, \ldots, Z_g) \in \mathrm{dom}_{\mathcal{A}}(r)$ that $(\Phi(Z_1), \ldots, \Phi(Z_g)) \in \mathrm{dom}_{\mathcal{B}}(r)$ and $\Phi(r(Z_1, \ldots, Z_g)) = r(\Phi(Z_1), \ldots, \Phi(Z_g))$ holds. Since Φ is a unital homomorphism, the latter allows to deduce the invertibility of $r(\Phi(Z_1), \ldots, \Phi(Z_g))$ from the invertibility of $r(Z_1, \ldots, Z_g)$, where moreover

$$r(\Phi(Z_1), \ldots, \Phi(Z_g))^{-1} = \Phi(r(Z_1, \ldots, Z_g))^{-1} = \Phi(r(Z_1, \ldots, Z_g)^{-1}) = \Phi(r^{-1}(Z_1, \ldots, Z_g)).$$

In summary, we have according to Definition III.2.3 that $(\Phi(Z_1), \ldots, \Phi(Z_g)) \in \mathrm{dom}_{\mathcal{B}}(r^{-1})$ holds with $r^{-1}(\Phi(Z_1), \ldots, \Phi(Z_g)) = \Phi(r^{-1}(Z_1, \ldots, Z_g))$. Since $(Z_1, \ldots, Z_g) \in \mathrm{dom}_{\mathcal{A}}(r^{-1})$ was arbitrarily chosen, it follows $r^{-1} \in \mathfrak{R}$, as we wished to show. □

Hence, the $*$-moments of $r(X_1, \ldots, X_g)$ with respect to ϕ coincide with the $*$-moments of $r(Y_1, \ldots, Y_g)$ with respect to ψ, such that finally $\mu_{r(X),r(X)^*} = \mu_{r(Y),r(Y)^*}$ follows. We conclude that the distribution of $r(X_1, \ldots, X_g)$ does not depend on the concrete choice of non-commutative random variables $X_1, \ldots, X_g$, but only on their individual distributions $\mu_{X_1}, \ldots, \mu_{X_g}$. This is what was claimed above.

In this chapter, we want to present another, more explicit approach to this question: if r is a self-adjoint non-commutative rational expression, which guarantees that the non-commutative random variable $r(X_1, \ldots, X_g)$ is self-adjoint, then we know that the (analytic) distribution of $r(X_1, \ldots, X_g)$ is determined by its Cauchy transform; otherwise, if r fails to be self-adjoint, then we know that the Brown measures of $r(X_1, \ldots, X_g)$ can be obtained from the $M_2(\mathbb{C})$-valued Cauchy-transform of

$$\begin{pmatrix} 0 & r(X_1, \ldots, X_g) \\ r(X_1, \ldots, X_g)^* & 0 \end{pmatrix},$$

as we explained in Section I.3. This leads to the following two problems:

Problem IV.1.2. *Given a self-adjoint rational expression r in formal variables $x = (x_1, \ldots, x_g)$. Let $X_1, \ldots, X_g$ be freely independent self-adjoint elements in some non-commutative C^*-probability space $(\mathcal{A}, \phi)$ for which the evaluation $r(X_1, \ldots, X_g)$ is well-defined. If the distributions of all of the X_j's are known, how can we compute the distribution of $r(X_1, \ldots, X_g)$?*

Problem IV.1.3. *Given an arbitrary rational expression r in formal variables $x = (x_1, \ldots, x_g)$. Let $X_1, \ldots, X_g$ be freely independent self-adjoint elements in some tracial W^*-probability space $(\mathcal{A}, \phi)$ for which the evaluation $r(X_1, \ldots, X_g)$ is well-defined. If the distributions of all of the X_j's are known, how can we compute the Brown-measure of $r(X_1, \ldots, X_g)$?*

In either case, we are faced with the problem to compute – at least numerically – the (matrix-valued) Cauchy transform of certain matrices over the unital C^*-algebra, which is generated by the variables $X_1, \ldots, X_g$, if their distributions $\mu_{X_1}, \ldots, \mu_{X_g}$ are given. The two main ingredients of Algorithm IV.4.1 and Algorithm IV.4.2, by which we will solve Problem IV.1.2 and Problem IV.1.3, respectively, are the operator-valued subordination result given in Theorem I.2.18 and the method of linearization. The next section will supply some technical framework, which opens the rich toolbox of Chapter III and makes these powerful tools accessible to us.

IV.2. Linearization of rational expressions

In Chapter III, we have outlined several algebraic techniques, summarized under the name "linearization", which allow an effective treatment of non-commutative rational expressions. These are:

- *Formal linear representations*: Each formal linear representation $\rho = (u, Q, v)$ for a given non-commutative rational expression r enjoys by definition the very important feature, that it provides some universal formula for the evaluation of r on its corresponding $\mathcal{A}$-domain for each unital complex algebra $\mathcal{A}$. If r is self-adjoint, it is appropriate to work instead with self-adjoint formal linear representations $\rho = (Q, v)$, which enjoy on their part very similar properties. In addition, we have seen that our arguments even extend to operator-valued non-commutative rational expressions and to matrices of non-commutative rational expressions, in particular. However, a not quite concealable disadvantage is that the size of formal linear representations grows very fast with the complexity of the considered non-commutative rational expression, while there is no effective way to reduce the size.
- *Pure linear representations*: In contrast to formal linear representations, pure linear representations concern non-commutative rational functions instead of non-commutative rational expressions. This becomes an issue as soon as evaluations are considered. Fortunately, if we restrict ourselves to stably finite algebras $\mathcal{A}$, which is actually quite natural from a free probability point of view, then rational identities are preserved under evaluations on their $\mathcal{A}$-domains, so that we can still hope to use them for our purposes. A great advantage is that pure linear representations of minimal size can be characterized, so that it becomes possible to formulate algorithms by which the size of arbitrary pure linear representations can be reduced, albeit sometimes in a less explicit way.
- *Non-commutative descriptor realizations*: Descriptor realizations deal like pure linear representations with non-commutative rational functions, but under the additional condition that the rational function is regular at 0. If one is willing to accept this restriction, one is recompensed by some very effective algorithm that allows to cut down any descriptor realization to a minimal one.

In this section, we will develop some unifying framework, which bridges between the algebraic setting of Chapter III and the analytic setting of Theorem I.2.18, so that all these tools from Chapter III can readily be used to attack Problem IV.1.2 and IV.1.3.

The following definition captures the crucial idea of linearization.

Definition IV.2.1. Suppose that $(\mathcal{A}, \phi)$ is a non-commutative C^*-probability space and fix a point $X = (X_1, \dots, X_g) \in \mathcal{A}_{\text{sa}}^g$. With respect to these initial data, we introduce the following terminology: a self-adjoint element $Y \in M_k(\mathcal{A})$ is said to be *linearized at X by* $(\Delta; \Lambda, \Xi)$, if

- Λ is a affine linear pencil of the form
$$\Lambda = \Lambda^{(0)} + \Lambda^{(1)} x_1 + \dots + \Lambda^{(g)} x_g$$
with self-adjoint matrices $\Lambda^{(0)}, \Lambda^{(1)}, \dots, \Lambda^{(g)} \in M_n(\mathbb{C})$ for some $n \in \mathbb{N}$,
- $\Xi \in M_{n \times k}(\mathbb{C})$ a rectangular matrix,
- and $\Delta \in M_k(\mathbb{C})$ a self-adjoint matrix,

such that $\Lambda(X)$ is invertible in $M_n(\mathcal{A})$ and

$$Y = \Delta - \Xi^* \Lambda(X)^{-1} \Xi.$$

For any such $(\Delta; \Lambda, \Xi)$, we define an affine linear pencil $L_{(\Delta;\Lambda,\Xi)}$ by

$$L_{(\Delta;\Lambda,\Xi)} := \begin{pmatrix} \Delta & \Xi^* \\ \Xi & \Lambda \end{pmatrix} = L^{(0)}_{(\Delta;\Lambda,\Xi)} + L^{(1)}_{(\Delta;\Lambda,\Xi)} x_1 + \cdots + L^{(g)}_{(\Delta;\Lambda,\Xi)} x_g.$$

If Y is linearized at X by $(\Delta; \Lambda, \Xi)$, we call

$$\hat{Y}_{(\Delta;\Lambda,\Xi)} := L_{(\Delta;\Lambda,\Xi)}(X) = \begin{pmatrix} \Delta & \Xi^* \\ \Xi & \Lambda(X) \end{pmatrix}$$

the *linearization of* Y *at* X *associated to* $(\Delta; \Lambda, \Xi)$. If it is clear from the context, to which $(\Delta; \Lambda, \Xi)$ the linearization $\hat{Y}_{(\Delta;\Lambda,\Xi)}$ and the accompanying linear pencil $L_{(\Delta;\Lambda,\Xi)}$ are associated, we will often abbreviate $\hat{Y}_{(\Delta;\Lambda,\Xi)}$ by $\hat{Y}$ and $L_{(\Delta;\Lambda,\Xi)}$ by L.

This definition is very close to the concept of self-adjoint formal linear representations, which was introduced

- in Definition III.2.57 for the case of a scalar-valued non-commutative rational expression,
- in Definition III.3.15 for operator-valued rational expressions, and
- in Definition III.4.5 for matrices of non-commutative rational expressions.

These notions, however, should not be confused, since Definition IV.2.1 starts with some concrete operator $Y \in M_k(\mathcal{A})$, whereas formal linear representations in general apply to non-commutative rational expressions as abstract objects. We note that the terminology of Definition IV.2.1 is in accordance with [**BMS13**], where a prestage of this theory, based on [**And12, And13, And15**], for the case of non-commutative polynomials was presented.

Two further remarks are in order.

Remark IV.2.2.

- Although the above definition applies to arbitrary operators $Y \in M_k(\mathcal{A})$, it is clear from the definition that the only elements $Y \in M_k(\mathcal{A})$, which could in principle be linearized at a given point $X \in \mathcal{A}_{\text{sa}}^g$ by some triple $(\Delta; \Lambda, \Xi)$, are those matrices Y, whose entries belong to the rational closure of $\mathbb{C}\langle X_1, \ldots, X_g\rangle$ in $\mathcal{A}$ with respect to its natural embedding; see Definition III.2.21.
- In contrast to the concept of formal linear representations, we allow $(\Delta; \Lambda, \Xi)$ to depend on the concrete choice of variables $X_1, \ldots, X_g$ in $(\mathcal{A}, \phi)$.

While the setting of Chapter III differs from that of Definition IV.2.1 above, these concepts are nonetheless related in the sense that the methods provided by Chapter III fit into the general picture of Definition IV.2.1. This important fact, which is one of the reasons, why we have introduced the terminology of Definition IV.2.1, is the content of the next lemma.

Lemma IV.2.3. *Let $x = (x_1, \ldots, x_g)$ be a g-tuple of formal variables.*

(a) *Let r be a self-adjoint $k \times k$ matrix of rational expressions in x. Consider any self-adjoint formal linear representation $\rho = (Q, v)$ in the sense of Definition III.4.5 (whose existence is guaranteed by Theorem III.4.6). Furthermore, let $X_1, \ldots, X_g$*

be self-adjoint elements in some C^*-probability space $(\mathcal{A},\phi)$, such that the condition $X=(X_1,\dots,X_g)\in \mathrm{dom}_{\mathcal{A}}(r)$ is satisfied. Then $r(X)$ is linearized at X by $(0;Q,v)$.

(b) *Let r be a non-degenerate self-adjoint scalar-valued rational expression in x. According to Proposition III.2.48 and Corollary III.2.59, r induces a self-adjoint non-commutative rational function $\mathfrak{r}$. Consider any self-adjoint pure linear representation $\rho=(v^*,Q(x),v)$ of $\mathfrak{r}$ (whose existence is guaranteed by Corollary III.2.59). Furthermore, let $X_1,\dots,X_g$ be self-adjoint elements in some C^*-probability space $(\mathcal{A},\phi)$ with a faithful tracial state ϕ. Assume that $X=(X_1,\dots,X_g)\in \mathrm{dom}_{\mathcal{A}}(r)\cap\mathrm{dom}_{\mathcal{A}}(Q^{-1})$ is satisfied. Then $r(X)$ is linearized at X by $(0;Q,v)$.*

(c) *Let r be a self-adjoint $k\times k$ matrix of regular rational expressions in formal variables $x=(x_1,\dots,x_g)$. Take any self-adjoint realization*

$$\mathbb{r}(x)=D+C(J-L_A(x))^{-1}B \qquad \text{with} \qquad L_A(x):=A_1x_1+\dots+A_gx_g$$

of the matrix $\mathfrak{r}$ of non-commutative rational functions, which is represented by r. Furthermore, let $X_1,\dots,X_g$ be self-adjoint elements in some C^*-probability space $(\mathcal{A},\phi)$ with a faithful tracial state ϕ. Assume
(i) *either that $X=(X_1,\dots,X_g)\in\mathrm{dom}_{\mathcal{A}}(r)\cap\mathrm{dom}_{\mathcal{A}}(\mathbb{r})$ is satisfied,*
(ii) *or that r is minimal and that $X=(X_1,\dots,X_g)\in\mathrm{dom}_{\mathcal{A}}(r)$ is satisfied.*
Then $r(X)$ is linearized at X by $(D;J-L_A,B)$.

Proof. (a) Given any self-adjoint formal linear representation $\rho=(Q,v)$ of r in the sense of Definition III.4.5 and any $X=X^*$ in $\mathrm{dom}_{\mathcal{A}}(r)$ (i.e. $X\in\mathrm{dom}^{\mathrm{sa}}_{\mathcal{A}}(r)$), we know from Definition III.4.5 that X also belongs to $\mathrm{dom}_{\mathcal{A}}(Q^{-1})$ and moreover that $r(X)=-uQ(X)^{-1}v$ holds. This means that $r(X)$ is linearized at X by $(0;Q,v)$.

(b) Consider any self-adjoint pure linear representation $\rho=(v^*,Q(x),v)$ of $\mathfrak{r}$ and take any $X\in\mathrm{dom}_{\mathcal{A}}(r)\cap\mathrm{dom}_{\mathcal{A}}(Q^{-1})$. Since ρ is in particular a pure linear representation of $\mathfrak{r}$, Corollary III.5.5 tells us that $r(X)=-v^*Q(X)^{-1}v$ holds, which means that $r(X)$ is linearized at X by $(0;Q,v)$, as we wished to show.

(c) First of all, we note that the additional assumption on ϕ being a faithful trace guarantees according to Lemma III.5.3 that $\mathcal{A}$ is stably finite.

(i) Since $X\in\mathrm{dom}_{\mathcal{A}}(r)\cap\mathrm{dom}_{\mathcal{A}}(\mathbb{r})$, Corollary III.5.5 tells us that the values $r(X)$ and $\mathbb{r}(X)$ coincide, i.e.

$$r(X)=D+C(J-L_A(X))^{-1}B.$$

(ii) Since $X=X^*$ belongs to $\mathrm{dom}^{\mathrm{sa}}_{\mathcal{A}}(r)$, Theorem III.5.11 tells us that X belongs to the domain $\mathrm{dom}_{\mathcal{A}}(\mathbb{r})$ of our given minimal realization $\mathbb{r}$ and that evaluating r and $\mathbb{r}$ at X yields the same result, i.e.

$$r(X)=D+C(J-L_A(x))^{-1}B.$$

In both cases (i) and (ii), we conclude that $r(X)$ is linearized at X by $(D;J-L_A,B)$. □

When working with concrete examples like in Section IV.5, it is very helpful to have certain shortcuts at hand, which allow a more clever construction of linearizations than just using the general algorithms. One such observation is presented in the next lemma.

Lemma IV.2.4. *Let $r \in \mathfrak{R}_{\mathbb{C}}(z_1, \dots, z_g)$ be a non-commutative rational expression. Furthermore, let k and $n_1, \dots, n_k$ be positive integers, u_0 and v_0 scalar matrices, and $P_1, \dots, P_{k-1}$ and $Q_1, \dots, Q_k$ affine linear pencils, such that*

- *for each $j = 1, \dots, k$, the affine linear pencil Q_j is of size $n_j \times n_j$,*
- *for each $j = 1, \dots, k-1$, the affine linear pencil P_j is of size $n_j \times n_{j+1}$,*
- *u_0 is of size $1 \times n_1$ and v_0 of size $n_k \times 1$.*

Consider the triple $\rho = (u, Q, v)$, which is obtained by

$$\rho = (u, Q, v) := \left(\begin{pmatrix} 0 & \dots & 0 & u_0 \end{pmatrix}, \begin{pmatrix} & & P_1 & Q_1 \\ & \cdot^{\cdot^{\cdot}} & Q_2 & \\ P_{k-1} & \cdot^{\cdot^{\cdot}} & & \\ Q_k & & & \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_0 \end{pmatrix} \right)$$

with an affine linear pencil Q of size $n \times n$, where $n := n_1 + \dots + n_k$. If $\mathcal{A}$ is some unital complex algebra $\mathcal{A}$, for which the conditions

$$\operatorname{dom}_{\mathcal{A}}(r) \subseteq \bigcap_{j=1}^{k} \operatorname{dom}_{\mathcal{A}}(Q_j^{-1}) \tag{IV.2}$$

and

$$r(X) = (-1)^k u_0 Q_1(X)^{-1} P_1(X) \cdots Q_{k-1}(X)^{-1} P_{k-1}(X) Q_k(X)^{-1} v_0 \quad \text{for all } X \in \operatorname{dom}_{\mathcal{A}}(r) \tag{IV.3}$$

hold true, then $\operatorname{dom}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(Q^{-1})$ is satisfied and we have

$$r(X) = -uQ(X)^{-1}v \quad \text{for all } X \in \operatorname{dom}_{\mathcal{A}}(r).$$

In particular, if the conditions (IV.2) *and* (IV.3) *are both satisfied for all unital complex algebras $\mathcal{A}$, then ρ is a formal linear representation in the sense of Definition III.2.43.*

Proof. Suppose that the conditions (IV.2) and (IV.3) are satisfied for some unital complex algebras $\mathcal{A}$ and take any $X \in \operatorname{dom}_{\mathcal{A}}(r)$ (note that there is nothing to prove, if $\operatorname{dom}_{\mathcal{A}}(r)$ happens to be empty). We want to prove first that $X \in \operatorname{dom}_{\mathcal{A}}(Q^{-1})$ holds, i.e., that $Q(X)$ is invertible in $M_n(\mathcal{A})$. For doing this, we proceed by mathematical induction on k. In the case $k = 1$, there is clearly nothing to prove. Assume now that we have verified the statement under question for any affine linear pencil that is build out of $k-1$ affine linear pencils. If we consider for $Q(X)$ the block decomposition

$$Q(X) = \begin{pmatrix} \tilde{P}(X) & \tilde{Q}(X) \\ Q_k(X) & 0 \end{pmatrix} \quad \text{with} \quad \tilde{Q} := \begin{pmatrix} & & P_1 & Q_1 \\ & \cdot^{\cdot^{\cdot}} & Q_2 & \\ P_{k-2} & \cdot^{\cdot^{\cdot}} & & \\ Q_{k-1} & & & \end{pmatrix}, \quad \tilde{P} := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ P_{k-1} \end{pmatrix},$$

then we know by induction hypothesis that $\tilde{Q}(X)$ must be invertible and we can easily verify by a straightforward computation that the matrices

$$Q(X) = \begin{pmatrix} \tilde{P}(X) & \tilde{Q}(X) \\ Q_k(X) & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & Q_k(X)^{-1} \\ \tilde{Q}(X)^{-1} & -\tilde{Q}(X)^{-1}\tilde{P}(X)Q_k(X)^{-1} \end{pmatrix}$$

are inverses of each other. In consequence, $Q(X)$ must be invertible, as we wished to show. This recursive construction moreover allows us to check that

$$-uQ(X)^{-1}v = (-1)^k u_0 Q_1(X)^{-1} P_1(X) \cdots Q_{k-1}(X)^{-1} P_{k-1}(X) Q_k(X)^{-1} v_0 = r(X)$$

holds. Thus, we see that $\rho = (u, Q, v)$ satisfies the claimed properties for the given $\mathcal{A}$, and if the conditions (IV.2) and (IV.3) are both satisfied for all unital complex algebras $\mathcal{A}$, then ρ forms obviously a formal linear representation. □

IV.3. Representation of Cauchy transforms

In Lemma IV.2.3, we have seen that the concepts of Chapter III merge into the unifying frame of Definition IV.2.1. What we want to show next, is that Definition IV.2.1 can be connected with operator-valued free probability theory, especially with Theorem I.2.18. Indeed, the terminology of Definition IV.2.1 is modeled according to the Schur complement formula A.1, such that matrix-valued Cauchy transforms for elements $Y \in M_k(\mathcal{A})$, which are linearized by some $(\Delta; \Lambda, \Xi)$, can be computed explicitly via the matrix-valued Cauchy transform of the linearization of Y associated to $(\Delta; \Lambda, \Xi)$. The following theorem makes this relation explicit; see [**HMS15**].

Theorem IV.3.1. *Let $X_1, \dots, X_g$ be self-adjoint elements in a non-commutative C^*-probability space $(\mathcal{A}, \phi)$. Furthermore, let $Y = Y^* \in M_k(\mathcal{A})$ be given and suppose that Y is linearized at $X = (X_1, \dots, X_g)$ by $(\Delta; \Lambda, \Xi)$, where the linear pencil Λ is of size $n \times n$ and Δ, Ξ are of appropriate size. We put*

$$\hat{Y} := \hat{Y}_{(\Delta;\Lambda,\Xi)} = \begin{pmatrix} \Delta & \Xi^* \\ \Xi & \Lambda(X) \end{pmatrix}.$$

Then the following statements hold true:

(i) *For all $b \in \mathbb{H}^+(M_k(\mathbb{C}))$, the point*

$$\hat{b} := \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \in M_{n+k}(\mathbb{C})$$

belongs to the $M_{n+k}(\mathbb{C})$-valued resolvent set $\rho_{M_{n+k}(\mathcal{A})/M_{n+k}(\mathbb{C})}(\hat{Y})$ of $\hat{Y}$ in $M_{n+k}(\mathcal{A})$ and we have that

$$(b - Y)^{-1} = \begin{pmatrix} 1_k & 0 \end{pmatrix} \left(\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} - \hat{Y} \right)^{-1} \begin{pmatrix} 1_k \\ 0 \end{pmatrix}. \tag{IV.4}$$

(ii) *If we consider the operator-valued C^*-probability spaces*

$$(M_{n+k}(\mathcal{A}), E_{n+k}, M_{n+k}(\mathbb{C})) \qquad \text{and} \qquad (M_k(\mathcal{A}), E_k, M_k(\mathbb{C})),$$

which were introduced in Example I.2.10, then the $M_k(\mathbb{C})$-valued Cauchy transform G_Y is determined by the $M_{n+k}(\mathbb{C})$-valued Cauchy transform $G_{\hat{Y}}$ via

$$G_Y(b) = \lim_{\varepsilon \searrow 0} \begin{pmatrix} 1_k & 0 \end{pmatrix} G_{\hat{Y}} \left(\begin{pmatrix} b & 0 \\ 0 & i\varepsilon 1_n \end{pmatrix} \right) \begin{pmatrix} 1_k \\ 0 \end{pmatrix} \tag{IV.5}$$

for all $b \in \mathbb{H}^+(M_k(\mathbb{C}))$.

Proof. (i) Let $Y = Y^* \in C^*(X_1, \dots, X_g)$ be given and assume that Y is linearized at $X = (X_1, \dots, X_g)$ by $(\Delta; \Lambda, \Xi)$ with Λ of size $n \times n$. According to Definition IV.2.1, we know that $\Lambda(X)$ is invertible in $M_n(\mathcal{A})$. Thus, for any given $b \in M_k(\mathbb{C})$, the Schur complement formula (A.1) stated in Lemma A.1 tells us that the matrix

$$\hat{b} - \hat{Y} = \begin{pmatrix} b - \Delta & -\Xi^* \\ -\Xi & -\Lambda(X) \end{pmatrix}$$

is invertible in $M_{n+k}(\mathcal{A})$ if and only if its Schur complement

$$b - (\Delta - \Xi^* \Lambda(X)^{-1} \Xi) = b - Y$$

is invertible in $M_k(\mathcal{A})$. Now, if we take any $b \in \mathbb{H}^+(M_k(\mathbb{C}))$, then $b - Y$ must be invertible, because Y is self-adjoint. Thus, we may conclude that $\hat{b} - \hat{Y}$ is invertible, i.e., that $\hat{b}$ belongs to the $M_{n+k}(\mathbb{C})$-valued resolvent set $\rho_{M_{n+k}(\mathcal{A})/M_{n+k}(\mathbb{C})}(\hat{Y})$ of $\hat{Y}$ in $M_{n+k}(\mathcal{A})$, and the Schur complement formula (A.1) yields in addition that

$$\begin{pmatrix} 1_k & 0 \end{pmatrix} \big(\hat{b} - \hat{Y}\big)^{-1} \begin{pmatrix} 1_k \\ 0 \end{pmatrix} = \begin{pmatrix} 1_k & 0 \end{pmatrix} \begin{pmatrix} b - \Delta & -\Xi^* \\ -\Xi & -\Lambda(X) \end{pmatrix}^{-1} \begin{pmatrix} 1_k \\ 0 \end{pmatrix} = (b - Y)^{-1},$$

which is the stated formula (IV.4). This proves (i).

(ii) For seeing (IV.5), we first note that by definition

$$E_k\bigg[\begin{pmatrix} 1_k & 0 \end{pmatrix} W \begin{pmatrix} 1_k \\ 0 \end{pmatrix} \bigg] = \begin{pmatrix} 1_k & 0 \end{pmatrix} E_{n+k}[W] \begin{pmatrix} 1_k \\ 0 \end{pmatrix} \qquad \text{for all } W \in M_{n+k}(\mathcal{A}).$$

Thus, we get by applying E_k to both sides of (IV.4) that

$$E_k\big[(b - Y)^{-1}\big] = \begin{pmatrix} 1_k & 0 \end{pmatrix} E_{n+k}\big[\big(\hat{b} - \hat{Y}\big)^{-1}\big] \begin{pmatrix} 1_k \\ 0 \end{pmatrix}$$

and hence, by definition of G_Y, that

$$G_Y(b) = \begin{pmatrix} 1_k & 0 \end{pmatrix} E_{n+k}\big[\big(\hat{b} - \hat{Y}\big)^{-1}\big] \begin{pmatrix} 1_k \\ 0 \end{pmatrix}. \tag{IV.6}$$

Now, we want to relate the expression $E_{n+k}[(\hat{b} - \hat{Y})^{-1}]$ appearing on the right hand side of equation (IV.6) with the $M_{n+k}(\mathbb{C})$-valued Cauchy transform $G_{\hat{Y}}$ of $\hat{Y}$. Unfortunately, we cannot do this directly, since this expression is not precisely an evaluation of $G_{\hat{Y}}$. In fact, it is rather a boundary value of it, since $\hat{b}$ does not belong itself to the upper half-plane $\mathbb{H}^+(M_{n+k}(\mathbb{C}))$, but can be approximated in the operator norm on $M_{n+k}(\mathbb{C})$ by

$$\hat{b}_\varepsilon := \begin{pmatrix} b & 0 \\ 0 & i\varepsilon 1_n \end{pmatrix} \in \mathbb{H}^+(M_{n+k}(\mathbb{C}))$$

as $\varepsilon \searrow 0$. Nevertheless, we have established in Item (i) that $\hat{b}$ belongs to the $M_{n+k}(\mathbb{C})$-valued resolvent set $\rho_{M_{n+k}(\mathcal{A})/M_{n+k}(\mathbb{C})}(\hat{Y})$, onto which the $M_{n+k}(\mathbb{C})$-valued Cauchy transform $G_{\hat{Y}}$ can be extended analytically according to Subsection I.2.3 by

$$G_{\hat{Y}}:\ \rho_{M_{n+k}(\mathcal{A})/M_{n+k}(\mathbb{C})}(\hat{Y}) \to M_{n+k}(\mathbb{C}),\ b \mapsto E_{n+k}\big[(b - \hat{Y})^{-1}\big].$$

Of course, we could be content with this observation, since it allows us to write

$$E_{n+k}\big[\big(\hat{b} - \hat{Y}\big)^{-1}\big] = G_{\hat{Y}}(b),$$

but we prefer to work on the natural domain $\mathbb{H}^+(M_{n+k}(\mathbb{C}))$ of $G_{\hat{Y}}$, because only here the properties of $G_{\hat{Y}}$ can be controlled in such a way that all our analytic tools apply. This is the reason why the representation given in (IV.5) involves a limit procedure, which replaces the point $\hat{b}$ of $\rho_{M_{n+k}(\mathcal{A})/M_{n+k}(\mathbb{C})}(\hat{Y})$ by $\hat{b}_\varepsilon \in \mathbb{H}^+(M_{n+k}(\mathbb{C}))$. In order to convince ourselves that the representation given in (IV.5) is indeed correct, we need to recall first that the analytic extension of $G_{\hat{Y}}$ is in particular continuous (see Theorem B.4), such that

$$E_{n+k}\big[\big(\hat{b}-\hat{Y}\big)^{-1}\big] = G_{\hat{Y}}(b) = \lim_{\varepsilon\searrow 0} G_{\hat{Y}}(\hat{b}_\varepsilon). \tag{IV.7}$$

Due to the continuity of the map compressing $M_{n+k}(\mathbb{C})$ to $M_k(\mathbb{C})$, a combination of (IV.6) and (IV.7) yields finally the stated formula (IV.5). This proves (ii). □

IV.4. How to calculate distributions and Brown measures of rational expressions

According to Lemma IV.2.3 and Theorem IV.3.1, Definition IV.2.1 establishes some direct connection between Chapter III and operator-valued free probability theory. In this section, we want to explain how this connection can be used to solve Problem IV.1.2 and Problem IV.1.3.

In either case, we will arrive at the problem how Cauchy transforms $G_{\hat{Y}}$ of operators of the form

$$\hat{Y} = L(X_1,\dots,X_g) = L^{(0)} + L^{(1)}X_1 + \dots + L^{(g)}X_g$$

with square scalar matrices $L^{(0)}, L^{(1)}, L^{(g)}$ and freely independent non-commutative random variables $X_1,\dots,X_g$ can be computed. Lemma I.2.17 tells us that this is essentially nothing but the operator-valued free additive convolution of the operators $L^{(1)}X_1,\dots,L^{(g)}X_g$ and so Theorem I.2.18 and Algorithm I.2.11 will allow us to conclude.

IV.4.1. An algorithmic solution of Problem IV.1.2. Let us first discuss the solution to Problem IV.1.2. This is the content of the following algorithm.

Algorithm IV.4.1. *Let r be a self-adjoint non-commutative rational expression in formal variables $x=(x_1,\dots,x_g)$ and let $X_1,\dots,X_g$ be freely independent self-adjoint elements in some C^*-probability space $(\mathcal{A},\phi)$, such that $X=(X_1,\dots,X_g)\in \operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r)$ holds. If the scalar-valued Cauchy transforms $G_{X_1},\dots,G_{X_g}$ are given, then the distribution $\mu_{r(X_1,\dots,X_g)}$ of $r(X_1,\dots,X_g)$ can be obtained as follows:*

(i) *By means of Lemma IV.2.3 find $(\Delta;\Lambda,\Xi)$, such that $Y := r(X_1,\dots,X_g)$ is linearized by $(\Delta;\Lambda,\Xi)$ in the sense of Definition IV.2.1, where the affine linear pencil Λ is of size $n\times n$ for some n.*

(ii) *Consider the affine linear pencil*

$$L = L_{(\Delta;\Lambda,\Xi)} = L^{(0)} + L^{(1)}x_1 + \dots + L^{(g)}x_g,$$

associated to $(\Delta;\Lambda,\Xi)$, which consists by construction of self-adjoint matrices $L^{(0)}, L^{(1)},\dots,L^{(g)}\in M_{n+1}(\mathbb{C})$; see Definition IV.2.1.

(iii) *Apply Theorem IV.3.1 in the case $k=1$ and deduce from (IV.5) that the scalar-valued Cauchy transform of $r(X_1,\dots,X_g)$ is determined by the $M_{n+1}(\mathbb{C})$-valued Cauchy transform the linearization*

$$\hat{Y} := \hat{Y}_{(\Delta;\Lambda,\Xi)} = L(X_1,\dots,X_g) = L^{(0)} + L^{(1)}X_1 + \dots + L^{(g)}X_g.$$

In fact, we have

$$G_{r(X_1,\dots,X_g)}(z) = \lim_{\varepsilon\searrow 0} \begin{pmatrix}1 & 0\end{pmatrix} G_{\hat{Y}}\left(\begin{pmatrix} z & 0\\ 0 & i\varepsilon 1_n\end{pmatrix}\right)\begin{pmatrix}1\\0\end{pmatrix}$$

for each $z\in\mathbb{C}^+$.

(iv) *According to Lemma I.2.17, the operators $L^{(1)}X_1, \dots, L^{(g)}X_g$ are freely independent with amalgamation over $M_{n+1}(\mathbb{C})$. Hence, the $M_{n+1}(\mathbb{C})$-valued Cauchy transform of $\hat{Y} - L^{(0)}$ can be computed by means of Theorem I.2.18; note that the matrix-valued Cauchy-transforms of $L^{(1)}X_1, \dots, L^{(g)}X_g$ can be computed by Algorithm I.2.11. The desired $M_{n+1}(\mathbb{C})$-valued Cauchy transform of $\hat{Y}$ is then obtained by the following shift*

$$G_{\hat{Y}}(b) = G_{\hat{Y}-L^{(0)}}(b - L^{(0)}) \qquad \text{for all } b \in \mathbb{H}^+(M_{n+1}(\mathbb{C})).$$

(v) *With $G_{r(X_1,\dots,X_g)}$, the desired distribution of $r(X_1, \dots, X_g)$ is then obtained by Stieltjes inversion formula; see Theorem I.1.29.*

IV.4.2. An algorithmic solution of Problem IV.1.3. In this subsection, we want to discuss the algorithmic solution of Problem IV.1.3.

ALGORITHM IV.4.2. *Let r be any non-commutative rational expression in formal variables $x = (x_1, \dots, x_g)$ and let $X_1, \dots, X_g$ be freely independent self-adjoint elements in some C^*-probability space $(\mathcal{A}, \phi)$, for which $X = (X_1, \dots, X_g) \in \operatorname{dom}_{\mathcal{A}}(r)$ holds. If the scalar-valued Cauchy transforms $G_{X_1}, \dots, G_{X_g}$ are given, then the Brown measure $\nu_{r(X_1,\dots,X_g)}$ of $r(X_1, \dots, X_g)$ can be obtained then as follows:*

(i) *Consider the following matrix of non-commutative rational expressions*

$$\underline{r} := \begin{pmatrix} 0 & r \\ r^* & 0 \end{pmatrix}, \tag{IV.8}$$

where r^* denotes the adjoint of r in the sense of Remark III.2.5. Due to Lemma III.2.8, the matrix $\underline{r}$ is self-adjoint in the sense of Definition III.4.1.

(ii) *By means of Lemma IV.2.3 find $(\Delta; \Lambda, \Xi)$, such that $\underline{Y} := \underline{r}(X_1, \dots, X_g)$ is linearized by $(\Delta; \Lambda, \Xi)$ in the sense of Definition IV.2.1, where the affine linear pencil Λ is of size $n \times n$ for some n.*

(iii) *Consider the affine linear pencil*

$$L = L_{(\Delta;\Lambda,\Xi)} = L^{(0)} + L^{(1)}x_1 + \dots + L^{(g)}x_g,$$

associated to $(\Delta; \Lambda, \Xi)$, which consists by construction of self-adjoint matrices $L^{(0)}, L^{(1)}, \dots, L^{(g)} \in M_{n+2}(\mathbb{C})$; see Definition IV.2.1.

(iv) *Apply Theorem IV.3.1 in the case $k = 2$ and deduce from* (IV.5) *that the scalar-valued Cauchy transform of $r(X_1, \dots, X_g)$ is determined by the $M_{n+1}(\mathbb{C})$-valued Cauchy transform the linearization*

$$\hat{\underline{Y}} := \hat{\underline{Y}}_{(\Delta;\Lambda,\Xi)} = L(X_1, \dots, X_g) = L^{(0)} + L^{(1)}X_1 + \dots + L^{(g)}X_g.$$

In fact, we have

$$G_{\underline{r}(X_1,\dots,X_g)}(b) = \lim_{\varepsilon \searrow 0} \begin{pmatrix} 1_2 & 0 \end{pmatrix} G_{\hat{\underline{Y}}}\left(\begin{pmatrix} b & 0 \\ 0 & i\varepsilon 1_n \end{pmatrix} \right) \begin{pmatrix} 1_2 \\ 0 \end{pmatrix}$$

for each $b \in \mathbb{H}^+(M_2(\mathbb{C}))$.

(v) *According to Lemma I.2.17, the operators $L^{(1)}X_1, \dots, L^{(g)}X_g$ are freely independent with amalgamation over $M_{n+2}(\mathbb{C})$. Hence, the $M_{n+2}(\mathbb{C})$-valued Cauchy transform of $\hat{\underline{Y}} - L^{(0)}$ can be computed by means of Theorem I.2.18; note that the matrix-valued Cauchy-transforms of $L^{(1)}X_1, \dots, L^{(g)}X_g$ can be computed by*

Algorithm I.2.11. The desired $M_{n+2}(\mathbb{C})$-valued Cauchy transform of $\underline{\hat{Y}}$ is then obtained by the following shift

$$G_{\underline{\hat{Y}}}(b) = G_{\underline{\hat{Y}}-L^{(0)}}(b - L^{(0)}) \qquad \text{for all } b \in \mathbb{H}^+(M_{n+2}(\mathbb{C})).$$

(vi) *The regularized Cauchy transform $G_{r(X_1,\dots,X_g),\varepsilon}$ is then determined by* (I.16), *i.e. we have*

$$G_{r(X_1,\dots,X_g),\varepsilon}(z) = \big[G_{\underline{r}(X_1,\dots,X_g)}(\Lambda_\varepsilon(z))\big]_{2,1} \qquad \text{with} \qquad \Lambda_\varepsilon(z) = \begin{pmatrix} i\varepsilon & z \\ \overline{z} & i\varepsilon \end{pmatrix}$$

for all $z \in \mathbb{C}$.

(vii) *The regularized Brown measure $\nu_{r(X_1,\dots,X_g),\varepsilon}$ can be obtained, according to* (I.15), *from the regularized Cauchy transform by*

$$d\nu_{r(X_1,\dots,X_g),\varepsilon}(z) = \frac{1}{\pi}\frac{\partial}{\partial \overline{z}} G_{r(X_1,\dots,X_g),\varepsilon}(z)\, d\lambda^2(z).$$

(viii) *As $\varepsilon \searrow 0$, the regularized Brown measure $\nu_{r(X_1,\dots,X_g),\varepsilon}$ converges weakly to the Brown measure $\nu_{r(X_1,\dots,X_g)}$.*

We conclude by the useful observation that a self-adjoint realization of the matrix $\underline{\mathfrak{r}}$, which we introduced above in (IV.8), can be constructed from any realization of the involved rational expression r. The precise statement, which in addition covers the case of rational expressions, which are not necessarily regular at 0, reads as follows.

LEMMA IV.4.3. *Let r be a scalar-valued non-commutative rational expression in the formal variables $z = (z_1, \dots, z_g)$ and denote by $\mathfrak{r}$ the non-commutative rational function, which is represented by r. Consider the matrix $\underline{r}$ of non-commutative rational expressions, which was introduced in* (IV.8), *and denote by $\underline{\mathfrak{r}}$ the matrix of non-commutative rational functions, which is induced by $\underline{r}$, i.e.*

$$\underline{r} = \begin{pmatrix} 0 & r \\ r^* & 0 \end{pmatrix} \qquad \text{and} \qquad \underline{\mathfrak{r}} := \begin{pmatrix} 0 & \mathfrak{r} \\ \mathfrak{r}^* & 0 \end{pmatrix}.$$

(i) *If $\rho = (u, Q, v)$ is any formal linear representation of r, then*

$$\underline{\rho} = (\underline{Q}, \underline{v}) := \left(\begin{pmatrix} 0 & Q \\ Q^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & v \\ u^* & 0 \end{pmatrix} \right)$$

gives a self-adjoint formal linear representation of $\underline{r}$.

(ii) *Suppose that $\mathfrak{r}$ is regular at 0. If*

$$\mathbb{r}(x) = D + C(J - L_A(x))^{-1}B \qquad \text{with} \qquad L_A(x) = A_1x_1 + \dots + A_gx_g,$$

is any descriptor realization of $\mathfrak{r}$, then we may obtain a self-adjoint descriptor realization of $\underline{\mathfrak{r}}$ by

$$\underline{\mathbb{r}} = \underline{D} + \underline{C}(\underline{J} - L_{\underline{A}}(x))^{-1}\underline{B} \qquad \text{with} \qquad L_{\underline{A}}(x) = \underline{A}_1x_1 + \dots + \underline{A}_gx_g,$$

where we have $\underline{D} = \begin{pmatrix} 0 & D \\ D^* & 0 \end{pmatrix}$, $\underline{C}^* = \underline{B} = \begin{pmatrix} 0 & B \\ C^* & 0 \end{pmatrix}$, $\underline{J} = \begin{pmatrix} 0 & J \\ J^* & 0 \end{pmatrix}$ *and*

$$\underline{A}_j = \begin{pmatrix} 0 & A_j \\ A_j^* & 0 \end{pmatrix} \qquad \text{for } j = 1, \dots, g.$$

Proof. (i) Let $\mathcal{A}$ be any $*$-algebra. We clearly have that $\operatorname{dom}_{\mathcal{A}}(Q^{-1}) = \operatorname{dom}_{\mathcal{A}}(\underline{Q}^{-1})$, and since ρ is a formal linear representation of r, we have by definition $\operatorname{dom}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(Q^{-1})$. In combination, this gives $\operatorname{dom}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(\underline{Q}^{-1})$ and in particular $\operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r) \subseteq \operatorname{dom}_{\mathcal{A}}(\underline{Q}^{-1})$. Furthermore, ρ enjoys the property that $r(X) = -uQ(X)^{-1}v$ and hence by Lemma III.2.8

$$r^*(X^*) = r(X)^* = (-uQ(X)^{-1}v)^* = -v^*Q^*(X^*)^{-1}u^* \qquad \text{for any } X \in \operatorname{dom}_{\mathcal{A}}(r).$$

Thus, if we take $X \in \operatorname{dom}^{\mathrm{sa}}_{\mathcal{A}}(r)$, we may deduce that $r(X) = -uQ(X)^{-1}v$ and $r^*(X) = -v^*Q^*(X)^{-1}u^*$ holds, so that

$$\begin{aligned}
-\underline{v}^*\underline{Q}(X)^{-1}\underline{v} &= -\begin{pmatrix} 0 & u \\ v^* & 0 \end{pmatrix}\begin{pmatrix} 0 & Q(X) \\ Q^*(X) & 0 \end{pmatrix}^{-1}\begin{pmatrix} 0 & v \\ u^* & 0 \end{pmatrix}\\
&= -\begin{pmatrix} 0 & u \\ v^* & 0 \end{pmatrix}\begin{pmatrix} 0 & Q^*(X)^{-1} \\ Q(X)^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & v \\ u^* & 0 \end{pmatrix}\\
&= -\begin{pmatrix} 0 & u \\ v^* & 0 \end{pmatrix}\begin{pmatrix} Q^*(X)^{-1}u^* & 0 \\ 0 & Q(X)^{-1}v \end{pmatrix}\\
&= -\begin{pmatrix} 0 & uQ(X)^{-1}v \\ v^*Q^*(X)^{-1}u^* & 0 \end{pmatrix}\\
&= \begin{pmatrix} 0 & r(X) \\ r^*(X) & 0 \end{pmatrix}\\
&= \underline{r}(X).
\end{aligned}$$

This shows that $\underline{\rho}$ is indeed a self-adjoint formal linear representation.

(ii) First of all, we note that $\underline{J}^* = \underline{J}$ and $\underline{J}^2 = 1_{2k}$, since by assumption $J^* = J$ and $J^2 = 1_k$ holds. Thus, $\mathbb{r}(x)$ is indeed a self-adjoint descriptor realization of some matrix of non-commutative rational functions. It remains to prove that it forms in fact a descriptor realization of $\mathfrak{r}$. Let us first check that we have, given any unital complex $*$-algebra $\mathcal{A}$,

$$\underline{r}(X) = \mathbb{r}(X) \qquad \text{for all } X = X^* \in \operatorname{dom}_{\mathcal{A}}(\underline{r}) \cap \operatorname{dom}_{\mathcal{A}}(\mathbb{r}).$$

For doing so, let us take any $X = X^* \in \operatorname{dom}_{\mathcal{A}}(\underline{r}) \cap \operatorname{dom}_{\mathcal{A}}(\mathbb{r})$. Since

$$\underline{J} - L_{\underline{A}}(X) = \begin{pmatrix} 0 & J - L_A(X) \\ J^* - L_{A^*}(X) & 0 \end{pmatrix},$$

we have that X also belongs to the domain of $\mathbb{r}$ and furthermore

$$\begin{aligned}
&\underline{C}(\underline{J} - L_{\underline{A}}(X))^{-1}\underline{B}\\
&\quad= \begin{pmatrix} 0 & C \\ B^* & 0 \end{pmatrix}\begin{pmatrix} 0 & (J^* - L_{A^*}(X))^{-1} \\ (J - L_A(X))^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & B \\ C^* & 0 \end{pmatrix}\\
&\quad= \begin{pmatrix} 0 & C \\ B^* & 0 \end{pmatrix}\begin{pmatrix} (J^* - L_{A^*}(X))^{-1}C^* & 0 \\ 0 & (J - L_A(X))^{-1}B \end{pmatrix}\\
&\quad= \begin{pmatrix} 0 & C(J - L_A(X))^{-1}B \\ B^*(J^* - L_{A^*}(X))^{-1}C^* & 0 \end{pmatrix}
\end{aligned}$$

so that

$$\mathbb{r}(X) = \underline{D} + \underline{C}(\underline{J} - L_{\underline{A}}(X))^{-1}\underline{B} = \begin{pmatrix} 0 & \mathbb{r}(X) \\ \mathbb{r}(X)^* & 0 \end{pmatrix}$$

Moreover, since $\mathbb{r}$ is a realization of r and therefore $\mathbb{r}(X) = r(X)$ holds, we may continue

$$\underline{\mathbb{r}}(X) = \begin{pmatrix} 0 & \mathbb{r}(X) \\ \mathbb{r}(X)^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & r(X) \\ r(X)^* & 0 \end{pmatrix} = \underline{r}(X).$$

Finally, we apply this observation to the complex unital $*$-algebra $\mathcal{A} = \mathbb{C}\langle\!\langle x_1, \dots, x_g \rangle\!\rangle$; see Lemma III.2.35 or Lemma III.2.36. This gives $\underline{\mathfrak{r}} = \underline{r}(x) = \underline{\mathbb{r}}(x)$ and concludes the proof. □

A more complicated construction underlies the minimal symmetric realization asserted in Item (iii) of Lemma III.4.13.

IV.5. Examples

In this section, we conclude the work that was done in the last chapters by some concrete examples. This should convince the readers that the Algorithms IV.4.1 and IV.4.2, by which we solved in a uniform and systematic way the fundamental Problems IV.1.2 and IV.1.3, respectively, are accessible for numerical computations.

We must confess, however, that numerical simulations are by now the only known applications of these algorithms, mainly because finding explicit solutions for the matrix-valued equations, by which the subordination functions are determined, is typically a heavy task, even for very basic examples. Sometimes, the special shape of these equations arouses the guess that their solution should show a certain kind of "symmetry", which would then allow us to reduce the number of involved indeterminates, but a theoretical justification for such shortcuts is still missing. Phenomena of this type are currently under investigation, as well as the question whether one can go beyond the case of freely independent variables, for instance to the setting of Boolean independence.

The numerical computations based on Algorithm IV.4.1 and Algorithm IV.4.2 will be performed in the case of freely independent elements $X_1, \dots, X_g$, for which random matrix models $(X_1^{(N)}, \dots, X_g^{(N)})$ are available. Therefore, we can compare the obtained analytic distributions and Brown measures of $r(X_1, \dots, X_g)$ with the empirical eigenvalue distributions of the corresponding random matrices $r(X_1^{(N)}, \dots, X_g^{(N)})$. Fascinatingly, this will show an nice conformity in all considered cases, though this is not always theoretically justified by now; for more details, see Chapter II. But in those cases, where the observed conformity is already known to be true, it demonstrates in a very impressive way (namely by only one single picture!), how different theories, such as random matrix theory, free probability theory and Banach space-valued complex analysis, fit together. Compared to other areas of pure mathematics, this situation is arguably quite unique.

IV.5.1. How to use Algorithm IV.4.1. Let us begin with the polynomial case. In both Example IV.5.1 and Example IV.5.2, the impressive conformity between the considered random matrix simulations and the outcome of Algorithm IV.4.1 as shown in Figure IV.1 and Figure IV.2, respectively, is explained and justified by Lemma II.4.1 and Theorem II.3.9.

Example IV.5.1 (Anti-commutator, see Figure IV.1). We consider the self-adjoint noncommutative polynomial expression

$$p := x_1 \cdot x_2 + x_2 \cdot x_1 \in \mathfrak{R}_{\mathbb{C}}(x_1, x_2).$$

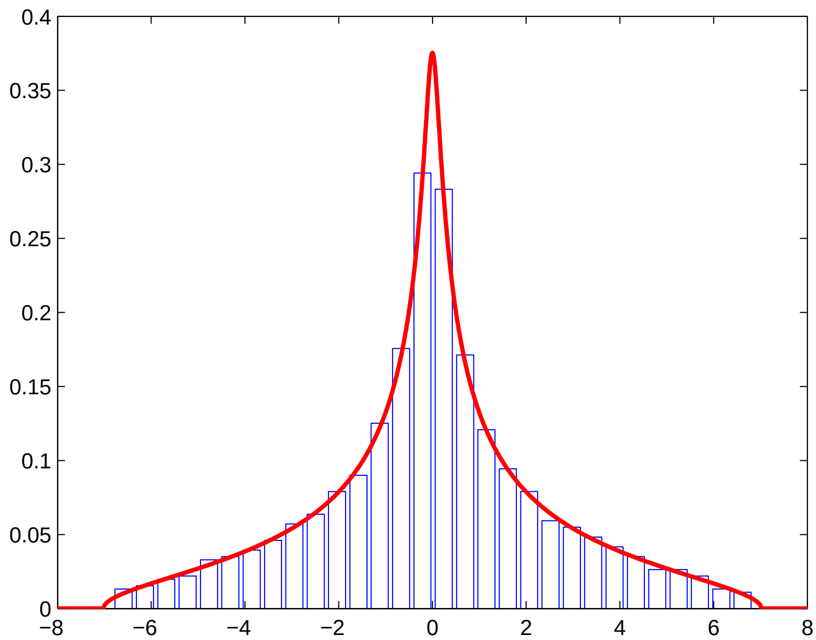


FIGURE IV.1. Histogram of eigenvalues of $p(X_1^{(N)}, X_2^{(N)})$, where p was defined in Example IV.5.1, for one realization of independent random matrices $X_1^{(N)}, X_2^{(N)}$, where $X_1^{(N)}$ is a Wishart random matrix and $X_2^{(N)}$ a Gaussian random matrix, both of size $N = 1000$, compared with the distribution of $p(X_1, X_2)$ for freely independent elements X_1, X_2, where X_1 is a free Poisson element and X_2 a semicircular element.

For given freely independent elements $X_1 = X_1^*$ and $X_2 = X_2^*$ living in some non-commutative C^*-probability space $(\mathcal{A}, \phi)$, the evaluation of p at (X_1, X_2) yields $p(X_1, X_2) = X_1X_2 + X_2X_1$, the so-called *anti-commutator* of X_1 and X_2. We want to use Algorithm IV.4.1 in order to compute $\mu_{p(X_1,X_2)}$ if the individual distributions μ_{X_1} and μ_{X_2} are prescribed.

Fist, we must find some $(\Delta; \Lambda, \Xi)$ by which $p(X_1, X_2)$ is linearized at the point (X_1, X_2). According to Lemma IV.2.3, it is sufficient to construct a formal linear representation $\rho = (Q, v)$ of p, because we can chose then $(\Delta; \Lambda, \Xi) = (0; Q, v)$. For finding ρ, we could of course just apply the algorithm that we presented in detail in Section IV.4. Alternatively, since we can write (actually for each unital complex algebra $\mathcal{A}$ and all points $(X_1, X_2) \in \mathcal{A}^2$)

$$p(X_1, X_2) = \begin{pmatrix} X_1 & X_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

we can use Lemma IV.2.4, which leads us to the self-adjoint pure linear representation

$$\rho = (Q, v) := \left(\begin{pmatrix} 0 & x_1 & x_2 & -1 \\ x_1 & 0 & -1 & 0 \\ x_2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

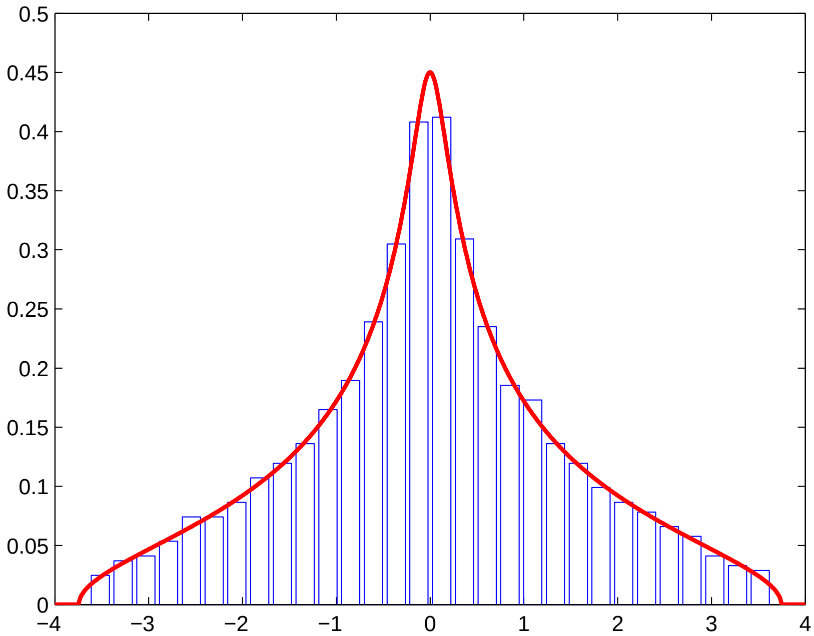


FIGURE IV.2. Histogram of eigenvalues of $q(X_1^{(N)}, X_2^{(N)})$, where the polynomial q was defined in Example IV.5.2, for one realization of independent random matrices $X_1^{(N)}, X_2^{(N)}$, where $X_1^{(N)}$ is a Wishart random matrix and $X_2^{(N)}$ a Gaussian random matrix, both of size $N = 1000$, compared with the distribution of $q(X_1, X_2)$ for freely independent elements X_1, X_2, where X_1 is a free Poisson element and X_2 a semicircular element.

Note that, strictly speaking, Lemma IV.2.4 only produces a pure linear representation, but it happens incidentally to be self-adjoint without further modification. The associated linear pencil $L = L_{(\Delta;\Lambda,\Xi)}$ looks then as follows

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & x_1 & x_2 & -1 \\ 0 & x_1 & 0 & -1 & 0 \\ 0 & x_2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix},$$

which decomposes as $L = L^{(0)} + L^{(1)}x_1 + L^{(2)}x_2$, where

$$L^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad L^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad L^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, $\hat{Y} := L(X_1, X_2)$ yields a linearization of the anti-commutator $Y = p(X_1, X_2) = X_1X_2 + X_2X_1$. According to Theorem IV.3.1, the $M_5(\mathbb{C})$-valued Cauchy transform of $\hat{Y}$ determines the scalar-valued Cauchy transform $G_{p(X_1,X_2)}$ of $p(X_1, X_2)$ and in consequence the distribution of $p(X_1, X_2)$.

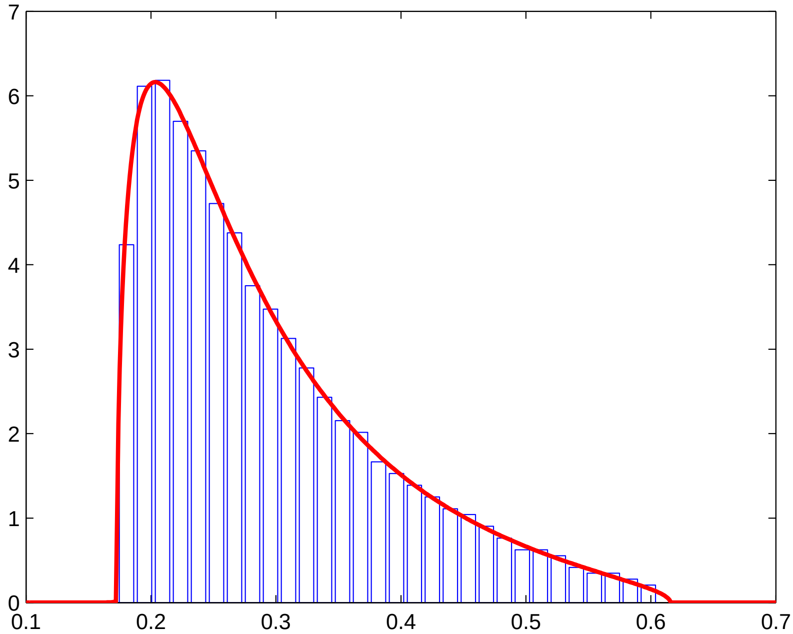


FIGURE IV.3. Histogram of eigenvalues of $r(X_1^{(N)}, X_2^{(N)})$ for one realization of independent Gaussian random matrices $X_1^{(N)}, X_2^{(N)}$ of size $N = 1000$, compared with the distribution of $r(X_1, X_2)$ for freely independent semicircular elements X_1, X_2. See Example IV.5.3.

EXAMPLE IV.5.2 (Commutator, see Figure IV.2). We consider the self-adjoint noncommutative polynomial expression

$$q := i(x_1x_2 - x_2x_1) \in \mathfrak{P}_{\mathbb{C}}(x_1, x_2).$$

For given freely independent elements $X_1 = X_1^*$ and $X_2 = X_2^*$ living in some noncommutative C^*-probability space $(\mathcal{A}, \phi)$, the evaluation of q at (X_1, X_2) yields the so-called *commutator* of X_1 and X_2 (multiplied by i in order to stay inside the class of self-adjoint operators). We want to use Algorithm IV.4.1 in order to compute $\mu_{p(X_1,X_2)}$ if the individual distributions μ_{X_1} and μ_{X_2} are prescribed. Like it was done for the anticommutator in Example IV.5.1, we will produce the triple $(\Delta; \Lambda, \Xi)$ linearizing $q(X_1, X_2)$ out of a self-adjoint formal linearization $\rho = (Q, v)$ of q, following the suggestion of Lemma IV.2.3. The latter can again be constructed by Lemma IV.2.4 and the decomposition

$$q(X_1, X_2) = \begin{pmatrix} X_1 & X_2 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^{-1} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

which yields the formal linear representation

$$\rho = (Q, v) = \left(\begin{pmatrix} 0 & x_1 & x_2 & -1 \\ x_1 & 0 & i & 0 \\ x_2 & -i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right).$$

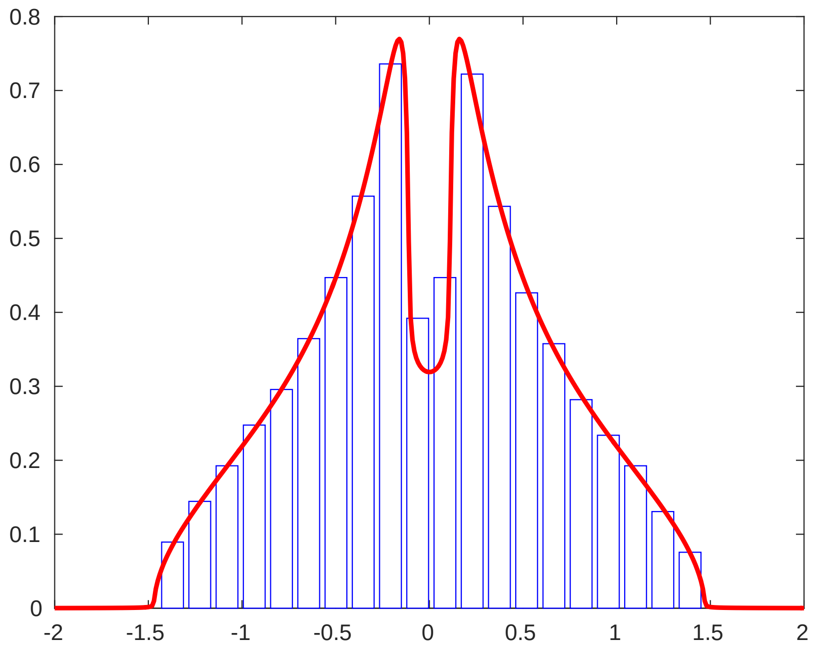


FIGURE IV.4. Histogram of eigenvalues of $r(X_1^{(N)}, X_2^{(N)})$ for one realization of independent random matrices $X_1^{(N)}, X_2^{(N)}$ of size $N = 1000$, $X_1^{(N)}$ being a standard self-adjoint complex Gaussian and $X_2^{(N)}$ being a standard self-adjoint complex Wishart random matrix, compared with the distribution of $r(X_1, X_2)$ for freely independent elements X_1, X_2, where X_1 is a semicircular and X_2 a free Poisson element. See Example IV.5.4.

The associated linear pencil $L = L_{(\Delta;\Lambda,\Xi)}$ looks then as follows:

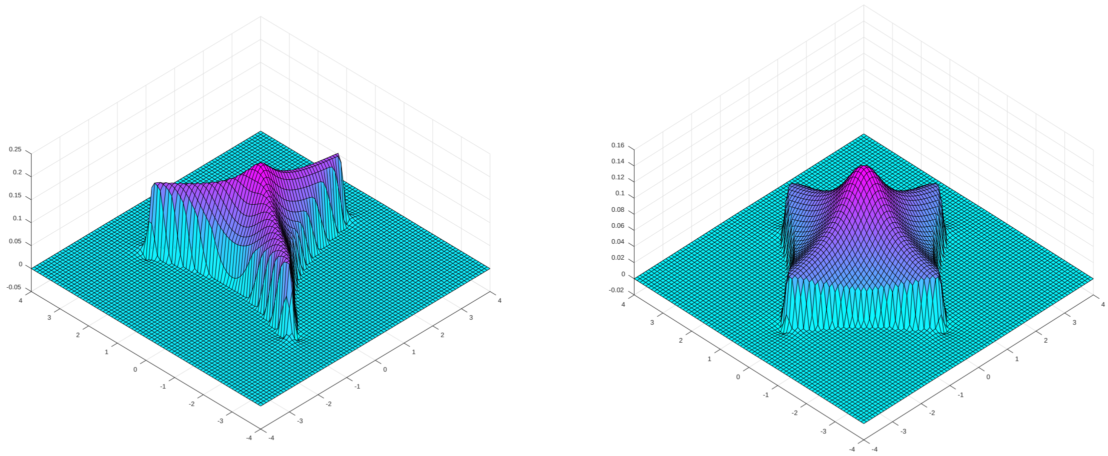
$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & x_1 & x_2 & -1 \\ 0 & x_1 & 0 & i & 0 \\ 0 & x_2 & -i & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

which decomposes as $L = L^{(0)} + L^{(1)}x_1 + L^{(2)}x_2$, where now

$$L^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad L^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad L^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, $\hat{Y} := L(X_1, X_2)$ gives a linearization of the commutator $Y = q(X_1, X_2) = i(X_1X_2 - X_2X_1)$. According to Theorem IV.3.1, the $M_5(\mathbb{C})$-valued Cauchy transform of $\hat{Y}$ determines the scalar-valued Cauchy transform $G_{q(X_1,X_2)}$ of $q(X_1, X_2)$ and so the distribution of $q(X_1, X_2)$.

Next, we turn our attention to the case of rational expressions. In Example IV.5.3 and Example IV.5.4, we will again see an astounding conformity between the random matrix



(a) $g = 3$

(b) $g = 4$

FIGURE IV.5. Brown measure of $p_g(X_1, \dots, X_g)$ for the polynomial p_g introduced in Example IV.5.5 with freely independent semicircular elements $X_1, \dots, X_g$ for different values of g.

simulations and the result of our Algorithm IV.4.1. Here, it is explained by Lemma II.4.4 and the strong convergence (see Definition II.4.2) of the considered random matrix models.

EXAMPLE IV.5.3. Consider the regular non-commutative rational function

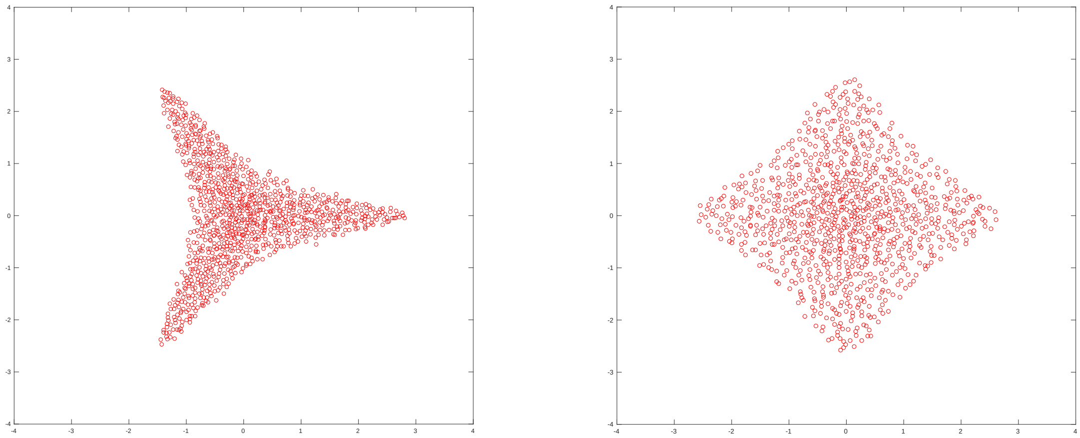
$$\mathfrak{r} = (4 - x_1)^{-1} + (4 - x_1)^{-1} x_2 \left((4 - x_1) - x_2 (4 - x_1)^{-1} x_2\right)^{-1} x_2 (4 - x_1)^{-1} \in \mathbb{C}\langle\!\langle x_1, x_2 \rangle\!\rangle.$$

One can check (either by some direct computation based on the Schur complement formula, Lemma A.1, or by following the construction proposed in the proof of Theorem III.5.11, namely by producing first a self-adjoint descriptor realization out of any formal linear representations and then cutting down to a minimal one; see Algorithm III.4.15) that $\mathfrak{r}$ admits the self-adjoint monic descriptor realization $\mathbb{r}(x) = D + C^*(1 - L_A(x))^{-1} C$ with feed through term $D = 0$ defined by

$$\begin{aligned} \mathbb{r}(x) &:= \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 1 - \frac{1}{4} x_1 & -\frac{1}{4} x_2 \\ -\frac{1}{4} x_2 & 1 - \frac{1}{4} x_1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix} \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} x_1 - \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix} x_2 \right)^{-1} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}. \end{aligned}$$

Now, chose any non-commutative rational expression r, which represents $\mathfrak{r}$, and consider some C^*-probability space $(\mathcal{A}, \phi)$, which comes endowed with a tracial state ϕ, and furthermore freely independent elements $X_1 = X_1^*$ and $X_2 = X_2^*$ in $\mathcal{A}$, such that the condition $(X_1, X_2) \in \mathrm{dom}^{\mathrm{sa}}_{\mathcal{A}}(r)$ is satisfied. Lemma IV.2.3 then tells us that $Y = r(X_1, X_2)$ is linearized at (X_1, X_2) by $(\Delta; \Lambda, \Xi) = (0; 1 - L_A, C)$. The associated linear pencil $L = L_{(\Delta;\Lambda,\Xi)}$ decomposes as $L = L^{(0)} + L^{(1)} x_1 + L^{(2)} x_2$, where

$$L^{(0)} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad L^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}, \quad \text{and} \quad L^{(2)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{pmatrix}.$$



(a) $g = 3$ and $N = 4000$ (b) $g = 4$ and $N = 4000$

FIGURE IV.6. Eigenvalues of $p_g(X_1^{(N)}, \dots, X_g^{(N)})$ for the polynomial p_g introduced in Example IV.5.5 with independent Gaussian random matrices $X_1^{(N)}, \dots, X_g^{(N)}$ of size N.

Finally, $\hat{Y} := L(X_1, X_2)$ gives a linearization of $Y = r(X_1, X_2)$ and according to Theorem IV.3.1, the $M_3(\mathbb{C})$-valued Cauchy transform of $\hat{Y}$ determines the scalar-valued Cauchy transform $G_{r(X_1,X_2)}$ of $r(X_1, X_2)$ and so the distribution of $r(X_1, X_2)$. In Figure IV.3, we compare the histogram of eigenvalues of $r(X_1^{(N)}, X_2^{(N)})$ for one realization of independent Gaussian random matrices $X_1^{(N)}, X_2^{(N)}$ of size $N = 1000$ with the distribution of $r(X_1, X_2)$ for freely independent semicircular elements X_1, X_2, calculated according to our algorithm.

EXAMPLE IV.5.4. Let us consider the non-commutative rational expression

$$r(x_1, x_2) := \big((x_1 \cdot x_2 + (-i))^{-1} \cdot x_1\big) \cdot \big(x_2 \cdot x_1 + i\big)^{-1}.$$

It is not hard to see that r is self-adjoint in the sense of Definition III.2.7. Assume that X_1 and X_2 are freely independent self-adjoint elements in some C^*-probability space $(\mathcal{A}, \phi)$ with tracial state ϕ, where the distribution of X_1 is the standard semicircular distribution and the distribution of X_2 the Marchenko-Pastur distribution with rate $\lambda = 1$ and jump size $\alpha = 1$. In Figure IV.4, the distribution of $r(X_1, X_2)$ computed with the help of Algorithm IV.4.1 is shown, together with the normalized histogram of all eigenvalues of $r(X_1^{(N)}, X_2^{(N)})$ for independent random matrices $X_1^{(N)}$ and $X_2^{(N)}$ of size $N = 1000$, where $X_1^{(N)}$ is a self-adjoint complex Gaussian random matrix and $X_2^{(N)}$ is a self-adjoint complex Wishart random matrix.

IV.5.2. How to use Algorithm IV.4.2. We present now tow applications of Algorithm IV.4.2. Again, the chosen random matrix models will produce eigenvalue distributions being in perfect accordance with the computed Brown measures. However, for non-commutative polynomials like in Example IV.5.5 – not to mention the case of non-commutative rational expressions in Example IV.5.6 – we do not have a theoretical justification of this, but the produced figures support the conjecture that this is indeed correct.

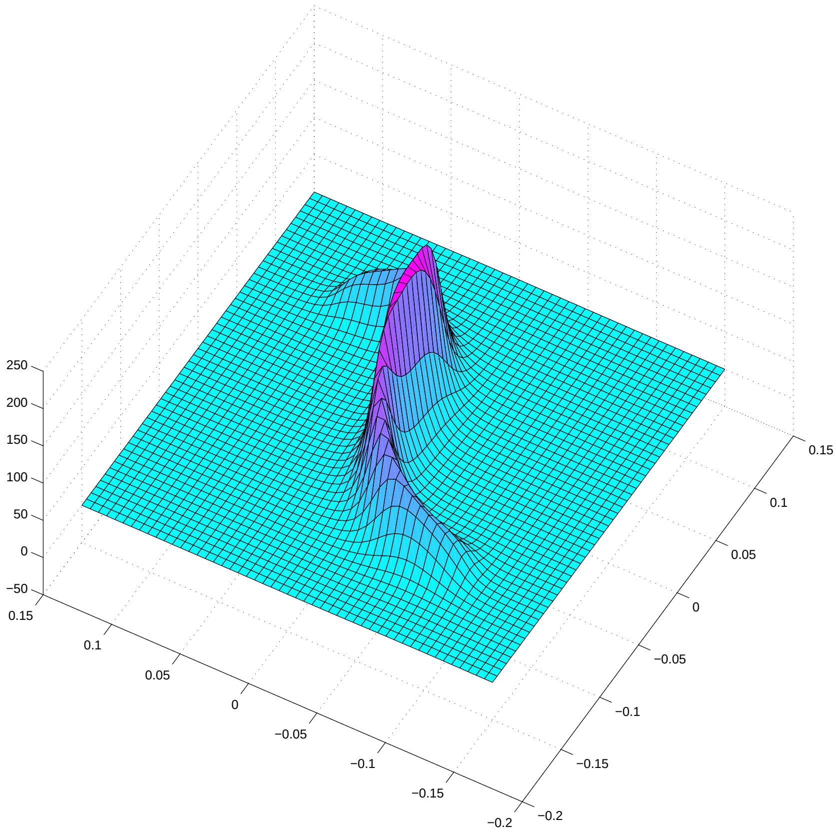


FIGURE IV.7. Brown measure of $r(X_1, X_2)$ for the rational expression $r(x_1, x_2)$ defined in Example IV.5.6, evaluated in freely independent semicircular elements X_1, X_2.

EXAMPLE IV.5.5. For any given $g \in \mathbb{N}$, $g \geq 2$, we denote by $\mathfrak{p}_g$ the non-commutative polynomial in $\mathbb{C}\langle x_1, \dots, x_g\rangle$ that is defined by

$$\mathfrak{p}_g = x_1x_2 + x_2x_3 + \cdots + x_{g-1}x_g + x_gx_1.$$

Choosing any non-commutative polynomial expression $p_g \in \mathfrak{R}_\mathbb{C}(z_1, \dots, z_g)$, which represents $\mathfrak{p}_g$, we can use Algorithm IV.4.2 in order to compute the Brown measure of $p_g(X_1, \dots, X_g)$ for freely independent semicircular variables $X_1, \dots, X_g$. For $g = 3$ and $g = 4$, the outcome is shown in Figure IV.5. This should be compared with the eigenvalues of $p_g(X_1^{(N)}, \dots, X_g^{(N)})$ for one realization of independent Gaussian random matrices $X_1^{(N)}, \dots, X_g^{(N)}$ as shown in Figure IV.6.

EXAMPLE IV.5.6. Let us consider the non-commutative descriptor realization

$$\mathbb{r}(x_1, x_2) := \begin{pmatrix} 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 - \frac{1}{4}x_1 & -ix_2 \\ -\frac{1}{4}x_2 & 1 - \frac{1}{4}x_1 \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$$

and denote by $\mathfrak{r}$ the non-commutative rational function in $\mathbb{C}\langle\!\langle x_1, x_2\rangle\!\rangle$, which is represented by $\mathbb{r}$. If $r \in \mathfrak{R}_\mathbb{C}(z_1, z_g)$ represents $\mathfrak{r}$, then the matrix $\underline{r}$ of non-commutative rational expressions introduced in (IV.8) represents the matrix $\underline{\mathfrak{r}}$ of non-commutative rational functions,

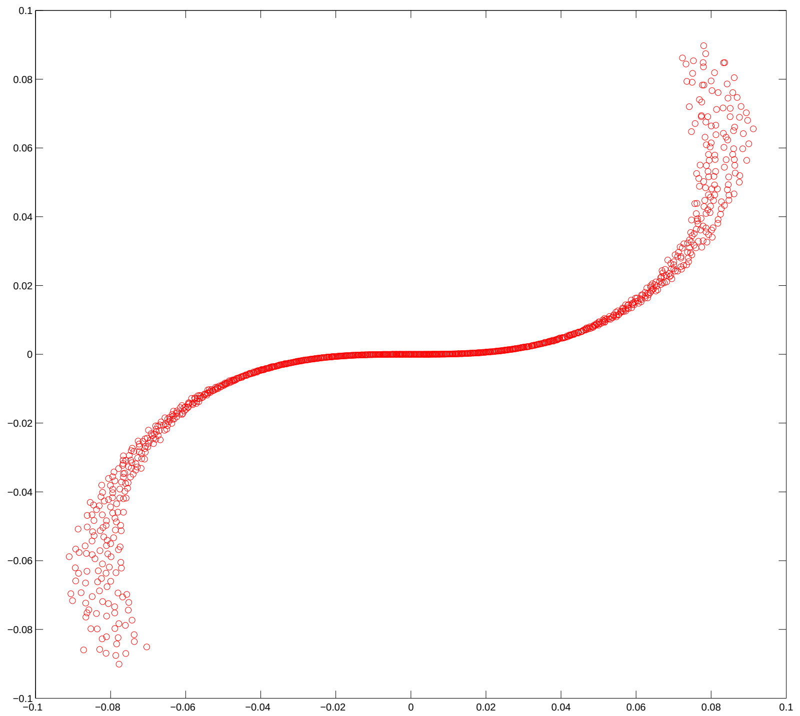


FIGURE IV.8. Eigenvalues of $r(X_1^{(N)}, X_2^{(N)})$ for the rational expression $r(x_1, x_2)$ defined in Example IV.5.6 with independent Gaussian random matrices $X_1^{(N)}, X_2^{(N)}$ of size $N = 1000$.

for which Lemma IV.4.3, yields the descriptor realization

$$\underline{\mathbb{r}}(x_1,x_2) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1-\frac{1}{4}x_1 & -ix_2 \\ 0 & 0 & -\frac{1}{4}x_2 & 1-\frac{1}{4}x_1 \\ 1-\frac{1}{4}x_1 & -\frac{1}{4}x_2 & 0 & 0 \\ ix_2 & 1-\frac{1}{4}x_1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix}.$$

According to Theorem IV.3.1, we introduce now

$$L(x_1,x_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & -1+\frac{1}{4}x_1 & ix_2 \\ 0 & 0 & 0 & 0 & \frac{1}{4}x_2 & -1+\frac{1}{4}x_1 \\ 0 & 0 & -1+\frac{1}{4}x_1 & \frac{1}{4}x_2 & 0 & 0 \\ \frac{1}{2} & 0 & -ix_2 & -1+\frac{1}{4}x_1 & 0 & 0 \end{pmatrix}.$$

Again, $L(x_1, x_2)$ decomposes as $L(x_1, x_2) = L^{(0)} + L^{(1)}x_1 + L^{(2)}x_2$, which provides the initial data for our algorithm: if (X_1, X_2) is a tuple of freely independent semicircular elements, then in $(X_1, X_2) \in \operatorname{dom}_{\mathcal{A}}(r)$ and the obtained density of the Brown measure of $r(X_1, X_2)$ is shown in Figure IV.7; Figure IV.8 shows the eigenvalues of $r(X_1^{(N)}, X_2^{(N)})$ for one realization of independent Gaussian random matrices $X_1^{(N)}, X_2^{(N)}$ of size $N = 1000$.

CHAPTER V

Non-commutative derivatives and derivations

The advances of free probability and the vast improvement in our understanding of non-commutative distributions in particular goes from its beginning hand in hand with the development of a non-commutative analysis – often called free analysis – which provides the right frame for analytic questions similar to the classical case and hence completes the combinatorial picture of non-commutative distributions. It is certainly not surprising that the backbone of free analysis is a suitable notion of derivatives. But since free analysis is built at the highest degree of non-commutativity, these derivatives are of completely different nature than their classical ancestors. Accordingly, studying the class of all "differentiable non-commutative functions" is a rather intricate endeavor, so that one typically restricts oneself to certain subclasses.

The most important class of functions, for which these derivations can be introduced in purely algebraic terms, are non-commutative polynomials. On the $*$-algebra $\mathbb{C}\langle x_1, \dots, x_n\rangle$ consisting of non-commutative polynomials in n formal non-commuting variables $x_1, \dots, x_n$ (see part (i) of Definition I.1.12), the so-called *non-commutative derivatives* are given as linear mappings

$$\partial_i : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle \otimes \mathbb{C}\langle x_1, \dots, x_n\rangle \qquad \text{for } i = 1, \dots, n.$$

Notably, in contrast to the classical derivatives, these non-commutative derivatives take their values in the tensor product $\mathbb{C}\langle x_1, \dots, x_n\rangle \otimes \mathbb{C}\langle x_1, \dots, x_n\rangle$. Roughly speaking, ∂_i removes the variable x_i, one after another, at each position where it appears, like the classical derivatives do, but it memorizes in addition the former position of x_i by putting a tensor sign there instead. We will give the precise definition of ∂_i in Section V.1.

Non-commutative polynomials in $\mathbb{C}\langle x_1, \dots, x_n\rangle$ have compared to more general classes of non-commutative functions the advantage that they provide some kind of "universal rule", which allows us to evaluate them easily on arbitrary tuples of elements living in some unital complex algebra. More precisely, as we have seen in part (ii) of Definition I.1.12, there is a natural homomorphism

$$\mathrm{ev}_X : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathcal{A}, \ P \mapsto P(X_1, \dots, X_n)$$

for each n-tuple $X = (X_1, \dots, X_n)$ consisting of elements in a unital complex algebra $\mathcal{A}$. In the case where the considered elements $X_1, \dots, X_n$ do not satisfy any non-trivial algebraic relation, this functional calculus yields an isomorphism between $\mathbb{C}\langle x_1, \dots, x_n\rangle$ and the unital subalgebra $\mathbb{C}\langle X_1, \dots, X_n\rangle$ of $\mathcal{A}$, which is generated by the elements $X_1, \dots, X_n$. This allows us in particular to reinterpret the non-commutative derivatives ∂_i as $\mathbb{C}\langle X_1, \dots, X_n\rangle \otimes \mathbb{C}\langle X_1, \dots, X_n\rangle$-valued derivations on $\mathbb{C}\langle X_1, \dots, X_n\rangle$. With this point of view, non-commutative derivatives fall into the more general class of *non-commutative derivations*. We will study these objects in detail in Section V.3. This will provide some very important tools for our considerations in the subsequent chapters.

This chapter is organized as follows. In Section V.1, we first discuss the purely algebraic theory of non-commutative derivatives

$$\partial_i : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle \otimes \mathbb{C}\langle x_1, \dots, x_n\rangle \qquad \text{for } i = 1, \dots, n.$$

on the algebra $\mathbb{C}\langle x_1, \dots, x_n\rangle$ of non-commutative polynomials in n non-commuting variables $x_1, \dots, x_n$. The question, under which conditions $\partial_1, \dots, \partial_n$ can be lifted for given elements $X_1, \dots, X_n$ in a unital complex algebra $\mathcal{A}$ to derivations

$$\hat{\partial}_i : \mathbb{C}\langle X_1, \dots, X_n\rangle \to \mathbb{C}\langle X_1, \dots, X_n\rangle \otimes \mathbb{C}\langle X_1, \dots, X_n\rangle \qquad \text{for } i = 1, \dots, n,$$

which are compatible with the evaluation map ev_X, will be addresses in Section V.2. Finally, in Section V.3, we will put several results from [**Voi98**] and [**Dab10**] (see also [**Dab14**]) in a uniform framework. Based on this, we will obtain Proposition V.6.1, which is taken from [**Mai15**] and which provides a significant generalization of the previous result that was obtained in [**MSW17**].

V.1. Non-commutative derivatives

This section is devoted to non-commutative derivatives. We introduce them here in purely algebraic terms as linear mappings on the unital $*$-algebra $\mathbb{C}\langle x_1, \dots, x_n\rangle$ of non-commutative polynomials in n formal non-commuting variables $x_1, \dots, x_n$ with values in the twofold tensor product $\mathbb{C}\langle x_1, \dots, x_n\rangle \otimes \mathbb{C}\langle x_1, \dots, x_n\rangle$.

Definition V.1.1. For $i = 1, \dots, n$, the *non-commutative derivative* ∂_i is the unique linear map

$$\partial_i : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle \otimes \mathbb{C}\langle x_1, \dots, x_n\rangle,$$

which satisfies the *Leibniz rule*

$$\partial_i(P_1P_2) = (\partial_i P_1)(1 \otimes P_2) + (P_1 \otimes 1)(\partial_i P_2) \qquad \text{for all } P_1, P_2 \in \mathbb{C}\langle x_1, \dots, x_n\rangle \tag{V.1}$$

and the condition

$$\partial_i x_j = \delta_{i,j} 1 \otimes 1 \qquad \text{for } j = 1, \dots, n. \tag{V.2}$$

Indeed, the claimed uniqueness can be checked as follows: fix $i \in \{1, \dots, n\}$ and assume that there would be another linear map

$$\tilde{\partial}_i : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle \otimes \mathbb{C}\langle x_1, \dots, x_n\rangle,$$

which satisfies both the Leibniz rule (V.1) and condition (V.2). Consider the set

$$D := \{P \in \mathbb{C}\langle x_1, \dots, x_n\rangle \mid \partial_i P = \tilde{\partial}_i P\}.$$

Because ∂_i and $\tilde{\partial}_i$ are both linear and satisfy the Leibniz rule, one can easily deduce that D is in fact a subalgebra of $\mathbb{C}\langle x_1, \dots, x_n\rangle$. Now, since the Leibniz rule forces both ∂_i and $\tilde{\partial}_i$ to vanish on scalars, we clearly have $\mathbb{C} \subseteq D$. Moreover, due to (V.2), we also have that $x_1, \dots, x_n \in D$. Hence, in summary, it follows $D = \mathbb{C}\langle x_1, \dots, x_n\rangle$, which yields the stated uniqueness.

More naturally, non-commutative derivatives should be considered within the frame of derivations. Recall the following definition.

Definition V.1.2. A linear mapping $\delta : \mathcal{A} \to \mathcal{M}$, defined on a unital complex algebra $\mathcal{A}$ and taking values in an $\mathcal{A}$-bimodule $\mathcal{M}$, is called a *$\mathcal{M}$-valued derivation on $\mathcal{A}$*, if it satisfies the *generalized Leibniz rule*

$$\delta(a_1a_2) = \delta(a_1) \cdot a_2 + a_1 \cdot \delta(a_2) \qquad \text{for all } a_1, a_2 \in \mathcal{A}.$$

Note that $\mathbb{C}\langle x_1,\dots,x_n\rangle \otimes \mathbb{C}\langle x_1,\dots,x_n\rangle$ carries naturally the structure of a $\mathbb{C}\langle x_1,\dots,x_n\rangle$-bimodule by

$$P_1 \cdot (Q_1 \otimes Q_2) \cdot P_2 := (P_1 \otimes 1)(Q_1 \otimes Q_2)(1 \otimes P_2) = (P_1Q_1) \otimes (Q_2P_2).$$

Accordingly, we can rewrite the Leibniz rule (V.1) as

$$\partial_j(P_1P_2) = (\partial_j P_1) \cdot P_2 + P_1 \cdot (\partial_j P_2) \qquad \text{for all } P_1, P_2 \in \mathbb{C}\langle x_1,\dots,x_n\rangle \tag{V.3}$$

and we see that Definition V.1.1 identifies ∂_i as a $\mathbb{C}\langle x_1,\dots,x_n\rangle \otimes \mathbb{C}\langle x_1,\dots,x_n\rangle$-valued derivation on $\mathbb{C}\langle x_1,\dots,x_n\rangle$.

Remark V.1.3. The non-commutative derivatives $\partial_1,\dots,\partial_n$ are as linear mappings uniquely determined by their values on all *monomials*, i.e., on any polynomial of the form $x_{i_1}\cdots x_{i_m}$ for $m \in \mathbb{N}_0$ and $1 \leq i_1,\dots,i_m \leq n$ (where the monomial is understood as the constant monomial 1 in the case $m = 0$). If P is any such monomial, we have for any fixed $i = 1,\dots,n$

$$\partial_i P := \sum_{P = P_1 x_i P_2} P_1 \otimes P_2, \tag{V.4}$$

where the sum runs over all decompositions of P in the form $P = P_1 x_i P_2$ with some monomials P_1, P_2. The validity of this formula can be checked easily by means of the Leibniz rule, (V.1) and (V.3), respectively.

The next remark clarifies some structure, which appears repeatedly when working in the non-commutative setting of bimodules.

Remark V.1.4. Let $\mathcal{A}$ and $\mathcal{B}$ be complex algebras. If $\mathcal{M}$ is an $\mathcal{A}$-$\mathcal{B}$-bimodule, then we denote by $\sharp$ the operation $(\mathcal{A} \otimes \mathcal{B}) \times \mathcal{M} \to \mathcal{M}$ that is determined by linear extension of $(a \otimes b)\sharp m := a \cdot m \cdot b$. Furthermore, if we would replace here $\mathcal{B}$ by its *opposite algebra* $\mathcal{B}^{\mathrm{op}}$, then $\sharp$ would give rise to a left action of the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}$ on $\mathcal{M}$. But since the multiplicative structure of $\mathcal{A} \otimes \mathcal{B}$ will play a minor role in our considerations, we will not care about this subtlety in the following.

The following remarks record some crucial properties of non-commutative derivatives.

Remark V.1.5. The collection $(\partial_1,\dots,\partial_n)$ of non-commutative derivatives is "universal" in the following sense:

(i) Consider any $\mathbb{C}\langle x_1,\dots,x_n\rangle$-bimodule $\mathcal{M}$ and let $\delta : \mathbb{C}\langle x_1,\dots,x_n\rangle \to \mathcal{M}$ be a derivation. Then

$$\delta(P) = \sum_{j=1}^{n} (\partial_j P)\sharp\delta(x_j) \qquad \text{for all } P \in \mathbb{C}\langle x_1,\dots,x_n\rangle. \tag{V.5}$$

This formula is an immediate consequence of the generalized Leibniz rule for δ and it describes in an explicit way how each such derivation $\delta : \mathbb{C}\langle x_1,\dots,x_n\rangle \to \mathcal{M}$ is fully determined by its values $\delta(x_1),\dots,\delta(x_n)$ on the generators $x_1,\dots,x_n$.

(ii) More generally, consider a unital complex algebra $\mathcal{A}$ and some $\mathcal{A}$-bimodule $\mathcal{M}$. Then, for each derivation $\delta : \mathcal{A} \to \mathcal{M}$ and for any n-tuple $X = (X_1,\dots,X_n)$ of elements in $\mathcal{A}$, we have that

$$\delta(P(X_1,\dots,X_n)) = \sum_{j=1}^{n} (\partial_j P)(X_1,\dots,X_n)\sharp\delta(X_j) \qquad \text{for all } P \in \mathbb{C}\langle x_1,\dots,x_n\rangle, \tag{V.6}$$

where we put $Q(X_1,\dots,X_n) := (\mathrm{ev}_X \otimes \mathrm{ev}_X)(Q)$ for any $Q \in \mathbb{C}\langle x_1,\dots,x_n\rangle^{\otimes 2}$.

Remark V.1.6. A very important property of non-commutative derivatives $\partial_1, \dots, \partial_n$ on $\mathbb{C}\langle x_1, \dots, x_n\rangle$ is that each of them satisfies the so-called *coassociativity relation*, i.e., we have that

$$(\mathrm{id} \otimes \partial_i) \circ \partial_i = (\partial_i \otimes \mathrm{id}) \circ \partial_i \qquad \text{for all } i = 1, \dots, n.$$

We point out, that more generally

$$(\mathrm{id} \otimes \partial_i) \circ \partial_j = (\partial_j \otimes \mathrm{id}) \circ \partial_i \qquad \text{for all } i, j = 1, \dots, n$$

holds, which can be checked easily by a direct computation on monomials.

We admit that their definition might seem quite artificial at first sight, but these non-commutative derivatives show up naturally in Voiculescu's non-microstates approach to free entropy in [**Voi98**] and also at several other places in mathematics, loosely speaking, whenever one tries to differentiate functions in highly non-commuting variables.

Example V.1.7. Let us consider a non-commutative polynomial P in formal non-commuting variables $x_1, \dots, x_n$. By evaluation, P induces naturally a function

$$P: \ M_N(\mathbb{C})^n \to M_N(\mathbb{C}), \ (X_1, \dots, X_n) \mapsto \mathrm{ev}_{(X_1,\dots,X_n)}(P) = P(X_1, \dots, X_n)$$

on the space $M_N(\mathbb{C})^n$ of n-tuples of complex matrices of any fixed dimension N. Similarly, evaluation of $\partial_i P$ yields

$$(\mathrm{ev}_{(X_1,\dots,X_n)} \otimes \mathrm{ev}_{(X_1,\dots,X_n)})(\partial_i P) = (\partial_i P)(X_1, \dots, X_n),$$

which is an element in $M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$. Note that any element $Y \in M_N(\mathbb{C}) \otimes M_N(\mathbb{C})$ induces naturally a linear map $M_N(\mathbb{C}) \to M_N(\mathbb{C})$ by $X \mapsto Y \sharp X$, where $\sharp$ is defined as above by considering $M_N(\mathbb{C})$ as a bimodule over itself, i.e. the operation

$$\sharp: \ (M_N(\mathbb{C}) \otimes M_N(\mathbb{C})) \times M_N(\mathbb{C}) \to M_N(\mathbb{C})$$

is given by bilinear extension of $(Y_1 \otimes Y_2)\sharp X := Y_1 X Y_2$. If we endow now $M_N(\mathbb{C})$ with the usual operator-norm (by identifying matrices in $M_N(\mathbb{C})$ with bounded linear operators on the Hilbert space $\mathbb{C}^N$), we can ask whether P admits directional derivatives. Indeed, it turns out that

$$\lim_{t\to 0} \frac{1}{t}\big(P(X_1 + tY_1, \dots, X_n + tY_n) - P(X_1, \dots, X_n)\big) = \sum_{j=1}^{n} (\partial_j P)(X_1, \dots, X_n)\sharp Y_j$$

for any point $(X_1, \dots, X_n) \in M_N(\mathbb{C})^n$ and for any direction $(Y_1, \dots, Y_n) \in M_N(\mathbb{C})^n$.

It is a rather surprising feature of free analysis, that these derivatives arise also from purely algebraic operations as soon as we let our functions act on matrices of different sizes. In fact, we have that

$$P\Big(\begin{pmatrix} X_1 & Y_1 \\ 0 & X_1 \end{pmatrix}, \dots, \begin{pmatrix} X_n & Y_n \\ 0 & X_n \end{pmatrix}\Big) = \begin{pmatrix} P(X_1, \dots, X_n) & \displaystyle\sum_{j=1}^{n} (\partial_j P)(X_1, \dots, X_n)\sharp Y_j \\ 0 & P(X_1, \dots, X_n) \end{pmatrix}.$$

This observation is at the base of free non-commutative function theory as developed in [**KV14**], but goes back in this particular situation to [**Tay72, Tay73**].

V.2. Non-commutative derivatives and algebraic relations

In the previous section, non-commutative derivatives were treated as purely algebraic objects. Following [**Voi98**], we will put them in a much more analytic setting: if $X_1, \dots, X_n$ are certain (self-adjoint) elements in some W^*-probability space (M, τ), we can consider the unital $(*\text{-})$subalgebra $\mathbb{C}\langle X_1, \dots, X_n \rangle$ of M, which is generated by $X_1, \dots, X_n$. Our goal is to define

$$\hat{\partial}_i : \mathbb{C}\langle X_1, \dots, X_n \rangle \to \mathbb{C}\langle X_1, \dots, X_n \rangle \otimes \mathbb{C}\langle X_1, \dots, X_n \rangle \qquad \text{for } i = 1, \dots, n$$

as $\mathbb{C}\langle X_1, \dots, X_n \rangle \otimes \mathbb{C}\langle X_1, \dots, X_n \rangle$-valued derivations on $\mathbb{C}\langle X_1, \dots, X_n \rangle$, which satisfy the property $\hat{\partial}_i X_j = \delta_{i,j} 1 \otimes 1$ for all $i, j = 1, \dots, n$.

In [**Voi98**], this was achieved by assuming that the considered variables $X_1, \dots, X_n$ do not satisfy any algebraic relation. Indeed, this assumption guarantees that the evaluation homomorphism

$$\mathrm{ev}_X : \mathbb{C}\langle x_1, \dots, x_n \rangle \to \mathbb{C}\langle X_1, \dots, X_n \rangle$$

induced by $X = (X_1, \dots, X_n)$ is an isomorphism, so that $\hat{\partial}_i$ defined by $\hat{\partial}_i := \mathrm{ev}_X \circ \partial_i \circ \mathrm{ev}_X^{-1}$ clearly does the job.

The following proposition shows that the absence of algebraic relations is, among other conditions, equivalent to the existence of such derivations $\hat{\partial}_1, \dots, \hat{\partial}_n$.

Proposition V.2.1. *Let (M, τ) be a tracial W^*-probability space and let self-adjoint $X_1, \dots, X_n \in M$ be given. Then the following statements are equivalent:*

(i) *The variables $X_1, \dots, X_n$ do not satisfy any algebraic relation.*
(ii) *For any non-commutative polynomial $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$, the following implication holds true:*

(V.7) $$P(X_1, \dots, X_n) = 0 \quad \Longrightarrow \quad \forall j = 1, \dots, n : \ (\partial_j P)(X_1, \dots, X_n) = 0$$

(iii) *For each $j = 1, \dots, n$, there is a derivation*

$$\hat{\partial}_j : \mathbb{C}\langle X_1, \dots, X_n \rangle \to \mathbb{C}\langle X_1, \dots, X_n \rangle \otimes \mathbb{C}\langle X_1, \dots, X_n \rangle$$

such that the following diagram commutes.

(V.8) $$\begin{array}{ccc} \mathbb{C}\langle x_1, \dots, x_n \rangle & \xrightarrow{\ \partial_j\ } & \mathbb{C}\langle x_1, \dots, x_n \rangle^{\otimes 2} \\ \downarrow{\scriptstyle \mathrm{ev}_X} & & \downarrow{\scriptstyle \mathrm{ev}_X \otimes \mathrm{ev}_X} \\ \mathbb{C}\langle X_1, \dots, X_n \rangle & \xrightarrow{\ \hat{\partial}_j\ } & \mathbb{C}\langle X_1, \dots, X_n \rangle^{\otimes 2} \end{array}$$

(iv) *For each $j = 1, \dots, n$, there is a derivation*

$$\hat{\partial}_j : \mathbb{C}\langle X_1, \dots, X_n \rangle \to \mathbb{C}\langle X_1, \dots, X_n \rangle \otimes \mathbb{C}\langle X_1, \dots, X_n \rangle$$

such that $\hat{\partial}_j(X_i) = \delta_{j,i} 1 \otimes 1$ for $i = 1, \dots, n$.

In particular, if the equivalent conditions (i) – (iv) are satisfied, then each derivation $\hat{\partial}_j$ in (iii) as well as in (iv) is uniquely determined, and they both coincide.

Proof. First of all, let us note that the implication "(i) $\Longrightarrow$ (ii)" is trivial, since $P = 0$ is under the assumption (i) the only non-commutative polynomial satisfying $P(X_1, \dots, X_n) = 0$.

(ii) $\Longrightarrow$ (i): For $j = 1, \dots, n$, we may define $\Delta_j := ((\tau \circ \mathrm{ev}_X) \otimes \mathrm{id}) \circ \partial_j$, which gives a linear mapping

$$\Delta_j : \mathbb{C}\langle x_1, \dots, x_n \rangle \to \mathbb{C}\langle x_1, \dots, x_n \rangle.$$

The assumption (V.7) made in (ii) yields for any $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ the implication

$$P(X) = 0 \quad \Longrightarrow \quad \forall j = 1, \dots, n : \ (\Delta_j P)(X) = 0. \tag{V.9}$$

Take now any polynomial $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$, for which $P(X_1, \dots, X_n) = 0$ holds, and assume that P is non-zero. Thus, we can write P as

$$P = a_0 + \sum_{k=1}^{d} \sum_{i_1, \dots, i_k = 1}^{n} a_{i_1, \dots, i_k} x_{i_1} \dots x_{i_k},$$

where $d \geq 1$ denotes the total degree of P. We choose any summand of highest degree

$$a_{i_1, \dots, i_d} x_{i_1} \dots x_{i_d}$$

of P. Since $\Delta_{i_d} \dots \Delta_{i_1}$ is clearly zero on constants, any monomial of degree strictly less than d, and furthermore on any monomial of degree d, where the variables do not appear in the prescribed order, we see that $\Delta_{i_d} \dots \Delta_{i_1} P = a_{i_1, \dots, i_d}$. Hence, we deduce by iterating (V.9)

$$a_{i_1, \dots, i_d} = (\Delta_{i_d} \dots \Delta_{i_1} P)(X) = 0,$$

which finally leads to a contradiction. Therefore, we must have $P = 0$, which shows (i).

(ii) $\Longrightarrow$ (iii): For any given element $Y \in \mathbb{C}\langle X_1, \dots, X_n \rangle$, we choose $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ with $Y = P(X_1, \dots, X_n)$ and we put

$$\hat{\partial}_j Y := (\partial_j P)(X_1, \dots, X_n) \qquad \text{for } j = 1, \dots, n.$$

By assumption (ii), this gives a well-defined mapping

$$\hat{\partial}_j : \mathbb{C}\langle X_1, \dots, X_n \rangle \to \mathbb{C}\langle X_1, \dots, X_n \rangle \otimes \mathbb{C}\langle X_1, \dots, X_n \rangle,$$

which is in fact a derivation, as a straightforward calculation shows. Moreover, by definition of $\hat{\partial}_j$, it is clear that the diagram in (V.8) commutes.

(iii) $\Longrightarrow$ (ii): Let $P \in \mathbb{C}\langle x_1, \dots, x_n \rangle$ with $P(X_1, \dots, X_n) = 0$ be given. By assumption, we get

$$\begin{aligned}
(\partial_j P)(X_1, \dots, X_n) &= ((\mathrm{ev}_X \otimes \mathrm{ev}_X) \circ \partial_j)(P) \\
&= (\hat{\partial}_j \circ \mathrm{ev}_X)(P) \\
&= \hat{\partial}_j(P(X_1, \dots, X_n)) \\
&= \hat{\partial}_j(0) \\
&= 0,
\end{aligned}$$

which shows (ii).

(iii) $\Longrightarrow$ (iv): Let any $j \in \{1, \ldots, n\}$ be given. If there is a derivation $\hat{\partial}_j$, for which the diagram in (V.8) commutes, then we can check that

$$\begin{aligned}\hat{\partial}_j(X_i) &= \hat{\partial}_j(\mathrm{ev}_X(x_i))\\ &= (\mathrm{ev}_X \otimes \mathrm{ev}_X)(\partial_j x_i)\\ &= (\mathrm{ev}_X \otimes \mathrm{ev}_X)(\delta_{j,i} 1 \otimes 1)\\ &= \delta_{j,i} 1 \otimes 1.\end{aligned}$$

holds for $i = 1, \ldots, n$, i.e. $\hat{\partial}_j$ satisfies the condition of (iv).

(iv) $\Longrightarrow$ (iii): Note that $\mathbb{C}\langle X_1, \ldots, X_n\rangle^{\otimes 2}$ becomes a $\mathbb{C}\langle x_1, \ldots, x_n\rangle$-bimodule via the evaluation map ev_X, i.e., we define $P_1 \cdot Q \cdot P_2 := P_1(X) Q P_2(X)$ for all $P_1, P_2 \in \mathbb{C}\langle x_1, \ldots, x_n\rangle$ and each $Q \in \mathbb{C}\langle X_1, \ldots, X_n\rangle^{\otimes 2}$. Now, fix $j \in \{1, \ldots, n\}$ and assume that a derivation

$$\hat{\partial}_j : \ \mathbb{C}\langle X_1, \ldots, X_n\rangle \to \mathbb{C}\langle X_1, \ldots, X_n\rangle \otimes \mathbb{C}\langle X_1, \ldots, X_n\rangle$$

exists, which satisfies the condition $\hat{\partial}_j(X_i) = \delta_{j,i} 1 \otimes 1$ for $i = 1, \ldots, n$. With respect to this bimodule structure, the linear mapping

$$d_j := \hat{\partial}_j \circ \mathrm{ev}_X : \ \mathbb{C}\langle x_1, \ldots, x_n\rangle \to \mathbb{C}\langle X_1, \ldots, X_n\rangle \otimes \mathbb{C}\langle X_1, \ldots, X_n\rangle$$

turns out to be a $\mathbb{C}\langle X_1, \ldots, X_n\rangle^{\otimes 2}$-valued derivation on $\mathbb{C}\langle x_1, \ldots, x_n\rangle$, which moreover enjoys the property that $d_j(x_i) = \hat{\partial}_j(X_i) = \delta_{j,i} 1 \otimes 1$ holds for $i = 1, \ldots, n$. Thus, using (V.5), we obtain for each $P \in \mathbb{C}\langle x_1, \ldots, x_n\rangle$ that

$$d_j(P) = \sum_{i=1}^{n} (\partial_i P) \sharp d_j(x_i) = (\partial_j P) \sharp (1 \otimes 1) = (\mathrm{ev}_X \otimes \mathrm{ev}_X)(\partial_j P)$$

and finally, according to the definition of d_j, that

$$(\hat{\partial}_j \circ \mathrm{ev}_X)(P) = d_j(P) = ((\mathrm{ev}_X \otimes \mathrm{ev}_X) \circ \partial_j)(P).$$

This means precisely that the diagram in (V.8) commutes.

If the equivalent conditions (i) – (iv) hold true, then the derivations in (iii) and (iv) are uniquely determined. Indeed, if $\hat{\partial}_j$ satisfies (iii) or (iv), then its value on each element in $\mathbb{C}\langle X_1, \ldots, X_n\rangle$, represented according to (i) as $P(X_1, \ldots, X_n)$ for some unique $P \in \mathbb{C}\langle x_1, \ldots, x_n\rangle$, must be given by

$$\hat{\partial}_j(P(X_1, \ldots, X_n)) = (\hat{\partial}_j \circ \mathrm{ev}_X)(P) = (\mathrm{ev}_X \circ \partial_j)(P) = (\partial_j P)(X_1, \ldots, X_n)$$

(according to the commutativity of the diagram in (V.8)) or

$$\hat{\partial}_j(P(X_1, \ldots, X_n)) = (\hat{\partial}_j \circ \mathrm{ev}_X)(P) = d_j(P) = (\partial_j P)(X_1, \ldots, X_n)$$

(according to the proof of "(iv) $\Longrightarrow$ (iii)"), respectively. Furthermore, we see that both derivations must coincide. □

These derivations $\hat{\partial}_1, \ldots, \hat{\partial}_n$ were finally used in [**Voi98**] for defining the so-called *free Fisher information* $\Phi^*(X_1, \ldots, X_n)$. Accordingly, talking about the Fisher information $\Phi^*(X_1, \ldots, X_n)$ always required to impose the a priori condition of absence of algebraic relations. This is actually not an issue, since the condition $\Phi^*(X_1, \ldots, X_n) < \infty$ is expected to imply some strong kind of regularity for the tuple $(X_1, \ldots, X_n)$, which should be incongruous with algebraic relations anyway. Nevertheless, one would prefer another approach to free Fisher information that circumvents the initial assumption of absence of algebraic relations and rather establishes this as a consequence of the condition $\Phi^*(X_1, \ldots, X_n) < \infty$. In fact, such a slight modification of the definition of $\Phi^*(X_1, \ldots, X_n)$ will be presented in Section VI.1 of Chapter VI, culminating in Theorem VI.1.5, where Proposition V.2.1 is used to prove that $\Phi^*(X_1, \ldots, X_n) < \infty$ indeed excludes algebraic relations.

V.3. Non-commutative derivations

In the previous section, we have discussed the non-commutative derivatives

$$\hat{\partial}_i : \mathbb{C}\langle X_1, \ldots, X_n\rangle \to \mathbb{C}\langle X_1, \ldots, X_n\rangle \otimes \mathbb{C}\langle X_1, \ldots, X_n\rangle \qquad \text{for } i = 1, \ldots, n,$$

which were introduced in [**Voi98**]. In fact, they fit nicely in the much more general frame of *non-commutative derivations*. Their theory arises from the work of Voiculescu [**Voi98, Voi99**] and of Dabrowski [**Dab10, Dab14**], and the corresponding generalization of methods originating from [**MSW17**].

At the beginning, a few words on tensor products are in order. Throughout this section, the purely algebraic tensor product of complex vector spaces or complex algebras will be denoted by $\odot$, whereas the more familiar symbol $\otimes$ is reserved for its "natural" closure in the corresponding analytic setting, as for instance for Hilbert spaces or von Neumann algebras. Since the tensor sign will appear mostly in its "closed version", this convention saves us from decorating the tensor signs repeatedly with fancy tags and hence keeps the notation as simple as possible.

Derivations are mainly characterized by the Leibniz rule, which is a straightforward generalization of the Leibniz rule for usual derivatives. Hence, these objects can be introduced and studied in a purely algebraic setting. But since we are interested more in the analytic rather than the purely algebraic properties of derivations, we will impose here some additional conditions on the algebra $\mathcal{A}$ and the $\mathcal{A}$-bimodule $\mathcal{M}$. For doing this, we clearly have a lot of flexibility. The most general notion of such analytic derivations is probably the one that is presented in [**CS03**, Definition 4.1]. However, the feasibility of our arguments here depends strongly on more restrictive assumptions, due to which those derivations will behave pretty much like the usual non-commutative derivatives as discussed in Section V.1. Accordingly, we shall call them non-commutative derivations.

Throughout this section, let (M, τ) be a tracial W^*-probability space.

Definition V.3.1. A linear map

$$\delta : M \supseteq D(\delta) \to L^2(M, \tau) \otimes L^2(M, \tau)$$

is called a *non-commutative derivation on* M if the following two conditions are satisfied:

- The domain $D(\delta)$ of δ is a unital $*$-subalgebra of M, which is moreover weakly dense in M.
- The linear map δ satisfies the *Leibniz rule* (or *product rule*)

$$\delta(X_1X_2) = \delta(X_1) \cdot X_2 + X_1 \cdot \delta(X_2)$$

for all $X_1, X_2 \in D(\delta)$, where $\cdot$ denotes the natural bimodule operation of M on the Hilbert space

$$L^2(M, \tau) \otimes L^2(M, \tau) \cong L^2(M \otimes M, \tau \otimes \tau).$$

Remark V.3.2. Assume that $\delta : M \supseteq D(\delta) \to L^2(M,\tau) \otimes L^2(M,\tau)$ is any non-commutative derivation in the sense of Definition V.3.1. If $X_1, \dots, X_n$ are self-adjoint elements in $D(\delta)$, then the formula (V.6) given in Remark V.1.5 applies and yields

$$\delta(P(X_1,\dots,X_n)) = \sum_{i=1}^{n} (\partial_i P)(X_1,\dots,X_n)\sharp\delta(X_i) \tag{V.10}$$

for any $P \in \mathbb{C}\langle x_1,\dots,x_n\rangle$. In other words, the non-commutative derivatives $\partial_1,\dots,\partial_n$ are universal in the sense that they provide an explicit expression for the restriction of any non-commutative derivation δ to a subalgebra $\mathbb{C}\langle X_1,\dots,X_n\rangle$ of its domain $D(\delta)$ in terms of its values on the generators $X_1,\dots,X_n$.

Following [**Voi98, Voi99**], we change now our point of view by considering any non-commutative derivation $\delta : M \supseteq D(\delta) \to L^2(M,\tau) \otimes L^2(M,\tau)$ in the sense of Definition V.3.1 as an unbounded linear operator

$$\delta :\ L^2(M,\tau) \supset D(\delta) \to L^2(M,\tau) \otimes L^2(M,\tau).$$

Since $D(\delta)$ is clearly dense in $L^2(M,\tau)$ with respect to the L^2-norm $\|\cdot\|_2$ induced by τ, we can also consider its adjoint operator

$$\delta^* :\ L^2(M,\tau) \otimes L^2(M,\tau) \supseteq D(\delta^*) \to L^2(M,\tau).$$

The theory that we are going to presented in the next sections concerns properties of δ and its adjoint δ^*. More precisely, we will discuss the question of closability for δ and we will show that δ and δ^*, which are unbounded operators by definition, can nevertheless be controlled in appropriate norms. For most of these results, the condition $1 \otimes 1 \in D(\delta^*)$ turns out to be essential.

V.4. Voiculescu's formulas for δ^*

In [**Voi98**], Voiculescu deduced formulas for the adjoint operator δ^* of a non-commutative derivation δ under the assumption that $1 \otimes 1 \in D(\delta^*)$. This was shown in [**Voi98**] only in the case of the non-commutative derivatives that are defined on the algebra of finitely many generators, but it was noted and worked out in [**Voi99**] that the same arguments apply in more general situations. Although this is commonly accepted as a well-known fact, we give here for reader's convenience a complete introduction to this circle of ideas, since these beautiful results are of great importance for our considerations.

For the rest of this subsection, let $\delta : M \supseteq D(\delta) \to L^2(M,\tau) \otimes L^2(M,\tau)$ be some fixed non-commutative derivation in the sense of Definition V.3.1, viewed as an unbounded linear operator

$$\delta :\ L^2(M,\tau) \supset D(\delta) \to L^2(M,\tau) \otimes L^2(M,\tau).$$

Following Voiculescu's strategy, we begin by deducing some very useful product rules for its adjoint operator δ^*.

Clearly, we may extend the involution $*$ on M from M uniquely to an involution on $L^2(M,\tau)$, and the canonical involution $*$ on $M \otimes M$ from $M \otimes M$ uniquely to an involution $L^2(M,\tau) \otimes L^2(M,\tau)$. Consequently,

$$\langle X, Y\rangle = \langle Y^*, X^*\rangle \tag{V.11}$$

holds for all $X, Y \in L^2(M,\tau)$.

LEMMA V.4.1. *Let $U \in D(\delta^*) \cap (M \odot M)$ and $X \in D(\delta)$ be given. Then*

$$\text{(V.12)} \qquad \begin{aligned} \delta^*(X \cdot U) &= X\delta^*(U) - (\tau \otimes \mathrm{id})(U\sharp\delta(X^*)^*), \\ \delta^*(U \cdot X) &= \delta^*(U)X - (\mathrm{id} \otimes \tau)(U\sharp\delta(X^*)^*), \end{aligned}$$

where $\sharp$ is defined according to Remark V.1.4 with respect to the M-M-bimodule $L^2(M,\tau) \otimes L^2(M,\tau)$. In particular, for any $U \in D(\delta^*) \cap (M \odot M)$, we have

$$\{X_1 \cdot U \cdot X_2 |\ X_1, X_2 \in D(\delta)\} \subseteq D(\delta^*).$$

PROOF. Let $U \in D(\delta^*) \cap (M \odot M)$ and $X \in D(\delta)$ be given. For any $Y \in D(\delta)$, we observe that

$$\begin{aligned} \langle \delta(Y), X \cdot U \rangle &= \langle X^* \cdot \delta(Y), U \rangle \\ &= \langle \delta(X^*Y), U \rangle - \langle \delta(X^*) \cdot Y, U \rangle \\ &= \langle X^*Y, \delta^*(U) \rangle - \langle 1 \otimes Y, U\sharp\delta(X^*)^* \rangle \\ &= \langle Y, X\delta^*(U) \rangle - \langle Y, (\tau \otimes \mathrm{id})(U\sharp\delta(X^*)^*) \rangle \\ &= \langle Y, X\delta^*(U) - (\tau \otimes \mathrm{id})(U\sharp\delta(X^*)^*) \rangle, \end{aligned}$$

from which $X \cdot U \in D(\delta^*)$ and the first formula in (V.12) follows. Analogously, we obtain by

$$\begin{aligned} \langle \delta(Y), U \cdot X \rangle &= \langle \delta(Y) \cdot X^*, U \rangle \\ &= \langle \delta(YX^*), U \rangle - \langle Y \cdot \delta(X^*), U \rangle \\ &= \langle YX^*, \delta^*(U) \rangle - \langle Y \otimes 1, U\sharp\delta(X^*)^* \rangle \\ &= \langle Y, \delta^*(U)X \rangle - \langle Y, (\mathrm{id} \otimes \tau)(U\sharp\delta(X^*)^*) \rangle \\ &= \langle Y, \delta^*(U)X - (\mathrm{id} \otimes \tau)(U\sharp\delta(X^*)^*) \rangle \end{aligned}$$

that $U \cdot X \in D(\delta^*)$ and the second formula in (V.12). A combination of both observations immediately yields the stated inclusion

$$\{X_1 \cdot U \cdot X_2 |\ X_1, X_2 \in D(\delta)\} \subseteq D(\delta^*)$$

for any $U \in D(\delta^*) \cap (M \odot M)$. □

In the case $1 \otimes 1 \in D(\delta^*)$, Lemma V.4.1 yields an explicit formula for δ^* on $D(\delta) \odot D(\delta)$ in terms of $\delta^*(1 \otimes 1)$ and δ. It takes its nicest form if we require an additional property of δ. In fact, we will assume a certain compatibility between the involution $*$ on M and some involution $\dagger$ on $M \otimes M$, where the latter is defined as follows.

DEFINITION V.4.2. On $M \otimes M$, the involution $\dagger$ is determined by anti-linear extension of

$$(X_1 \otimes X_2)^\dagger := X_2^* \otimes X_1^* \qquad \text{for all } X_1, X_2 \in M.$$

Note that $\dagger$ differs from the canonical involution $*$ on $M \otimes M$ only by the flip mapping $\sigma : M \otimes M \to M \otimes M$, i.e., we have $U^\dagger = \sigma(U^*)$.

Clearly, we may extend the involution $\dagger$ from $M \otimes M$ uniquely to an involution $L^2(M,\tau) \otimes L^2(M,\tau)$. Accordingly, for all $U, V \in L^2(M,\tau) \otimes L^2(M,\tau)$, it holds true that

$$\text{(V.13)} \qquad \langle U, V \rangle = \langle V^\dagger, U^\dagger \rangle.$$

DEFINITION V.4.3. A non-commutative derivation

$$\delta : M \supseteq D(\delta) \to L^2(M,\tau) \otimes L^2(M,\tau)$$

on (M,τ) is called *real*, if it satisfies

$$\delta(X)^\dagger = \delta(X^*) \qquad \text{for all } X \in D(\delta). \tag{V.14}$$

Often, condition (V.14) can be weakened. We record this here as a remark.

REMARK V.4.4. We point out that condition (V.14) is automatically satisfied if the unital $*$-algebra $D(\delta)$ is generated by self-adjoint elements X_i, $i \in I$, for some index set $I \neq \emptyset$, such that $\delta(X_i)^\dagger = \delta(X_i)$ holds for all $i \in I$.

Indeed, if we define $\tilde{\delta}$ with $D(\tilde{\delta}) := D(\delta)$ by

$$\tilde{\delta}:\ M \supseteq D(\tilde{\delta}) \to L^2(M,\tau) \otimes L^2(M,\tau),\ X \mapsto \delta(X^*)^\dagger,$$

we can easily check that $\tilde{\delta}$ is a non-commutative derivation as well. Thus, the set

$$D := \{X \in D(\delta) |\ \delta(X) = \tilde{\delta}(X)\}$$

is closed under multiplication, i.e. $X_1, X_2 \in D$ implies $X_1X_2 \in D$. Since it contains the generators $\{X_i |\ i \in I\}$ by assumption, we must have that $D = D(\delta)$, from which it follows by construction that $\delta(X)^\dagger = \delta(X^*)$ holds for all $X \in D(\delta)$.

The following lemma collects some useful formulas for real non-commutative derivations.

LEMMA V.4.5. *Let $\delta : M \supseteq D(\delta) \to L^2(M,\tau) \otimes L^2(M,\tau)$ be a real non-commutative derivation on (M,τ), Then, for all $X \in D(\delta)$, it holds true that*

$$\begin{aligned}(\mathrm{id} \otimes \tau)(\delta(X))^* &= (\tau \otimes \mathrm{id})(\delta(X^*)),\\ (\tau \otimes \mathrm{id})(\delta(X))^* &= (\mathrm{id} \otimes \tau)(\delta(X^*)).\end{aligned}$$

Furthermore, for any $U \in D(\delta^*)$, we have also $U^\dagger \in D(\delta^*)$ and it holds true that

$$\delta^*(U^\dagger) = \delta^*(U)^*.$$

In particular, if $1 \otimes 1 \in D(\delta^*)$, we have $\delta^*(1 \otimes 1) = \delta^*(1 \otimes 1)^*$.

PROOF. The first statement is an immediate consequence of the defining property of real derivations, since in general

$$\begin{aligned}(\mathrm{id} \otimes \tau)(U)^* &= (\tau \otimes \mathrm{id})(U^\dagger),\\ (\tau \otimes \mathrm{id})(U)^* &= (\mathrm{id} \otimes \tau)(U^\dagger)\end{aligned} \tag{V.15}$$

holds for each $U \in L^2(M,\tau) \otimes L^2(M,\tau)$. For seeing the second statement, we take any $Y \in D(\delta)$ and we observe by using (V.13) that

$$\langle U^\dagger, \delta(Y)\rangle = \langle \delta(Y)^\dagger, U\rangle = \langle \delta(Y^*), U\rangle = \langle Y^*, \delta^*(U)\rangle = \langle \delta^*(U)^*, Y\rangle.$$

This yields $U^\dagger \in D(\delta^*)$ with $\delta^*(U^\dagger) = \delta^*(U)^*$, as desired. □

Now, we can combine formulas (V.12) of Lemma V.4.1.

LEMMA V.4.6. *If the condition* $1 \otimes 1 \in D(\delta^*)$ *is satisfied, then*

$$D(\delta) \odot D(\delta) \subseteq D(\delta^*).$$

If δ *is a real derivation in the sense of Definition V.4.3, then we have more explicitly for all* $U \in D(\delta) \odot D(\delta)$ *that*

$$\delta^*(U) = U \sharp \delta^*(1 \otimes 1) - m_1(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\delta \otimes \mathrm{id} + \mathrm{id} \otimes \delta)(U), \tag{V.16}$$

where, in general, we denote by m_η *for any* $\eta \in L^2(M,\tau)$ *the linear mapping* $m_\eta : M \odot M \to L^2(M,\tau)$ *that is determined by* $m_\eta(v) = v\sharp\eta$*, so that* m_1 *is nothing else than the multiplication map* $m_1(X_1 \otimes X_2) = X_1X_2$.

The formula (V.16) given in Lemma V.4.6 immediately implies that in particular

$$\begin{aligned} \delta^*(X \otimes 1) &= X\delta^*(1 \otimes 1) - (\mathrm{id} \otimes \tau)(\delta(X)), \\ \delta^*(1 \otimes X) &= \delta^*(1 \otimes 1)X - (\tau \otimes \mathrm{id})(\delta(X)), \end{aligned} \tag{V.17}$$

which we record here for later reference.

PROOF OF LEMMA V.4.6. The first assertion, namely that $D(\delta) \odot D(\delta) \subseteq D(\delta^*)$ holds under the condition $1 \otimes 1 \in D(\delta^*)$, is an immediate consequence of Lemma V.4.1. Note that we did not use for this conclusion the assumption that δ is real.

For seeing (V.16), we proceed as follows. First of all, we note that the validity of (V.14) guarantees according to Lemma V.4.5 that

$$\begin{aligned} (\mathrm{id} \otimes \tau)(\delta(X^*)^*) &= (\tau \otimes \mathrm{id})(\delta(X)), \\ (\tau \otimes \mathrm{id})(\delta(X^*)^*) &= (\mathrm{id} \otimes \tau)(\delta(X)) \end{aligned}$$

for each $X \in D(\delta)$. Next, for any $U = X_1 \otimes X_2$ with $X_1, X_2 \in D(\delta)$, we check by using consecutively both formulas of (V.12) and Lemma V.4.5 that

$$\begin{aligned} \delta^*(U) &= \delta^*(X_1 \cdot (1 \otimes X_2)) \\ &= X_1\delta^*((1 \otimes 1) \cdot X_2) - (\tau \otimes \mathrm{id})((1 \otimes X_2)\sharp\delta(X_1^*)^*) \\ &= X_1\delta^*(1 \otimes 1)X_2 - X_1(\mathrm{id} \otimes \tau)(\delta(X_2^*)^*) - (\tau \otimes \mathrm{id})((1 \otimes X_2)\sharp\delta(X_1^*)^*) \\ &= U\sharp\delta^*(1 \otimes 1) - X_1(\tau \otimes \mathrm{id})(\delta(X_2)) - (\mathrm{id} \otimes \tau)(\delta(X_1))X_2 \\ &= U\sharp\delta^*(1 \otimes 1) - m_1(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\delta \otimes \mathrm{id} + \mathrm{id} \otimes \delta)(U). \end{aligned}$$

By linearity, this shows (V.16) for all $U \in D(\delta) \odot D(\delta)$. This concludes the proof. □

V.5. Dabrowski's inequalities

Based on Voiculescu's formulas, Dabrowski deduced in [**Dab10**] a collection of interesting inequalities concerning the boundedness of the non-commutative derivatives, which are very surprising from a classical point of view. In [**Dab14**], he noted that the same arguments also apply in a more general setting. More precisely, he observed (without carrying out the proof) that his result remain valid for any real derivation, which satisfies in addition the so-called coassociativity relation.

DEFINITION V.5.1. Let $\delta : M \supseteq D(\delta) \to L^2(M,\tau) \otimes L^2(M,\tau)$ be a non-commutative derivation on (M,τ). We say that δ satisfies the *coassociativity relation*,

- if δ takes its values in $D(\delta) \odot D(\delta)$,

- and if δ has the property that

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta. \tag{V.18}$$

For reader's convenience, we state here those of Dabrowski's formulas, which we need for our purposes. Since it is instructive, we also include a slightly simplified proof thereof.

Theorem V.5.2. *Let $\delta : M \supseteq D(\delta) \to L^2(M,\tau) \otimes L^2(M,\tau)$ be a non-commutative derivation on a tracial W^*-probability space (M,τ), which*

- *is real in the sense of Definition V.4.3*
- *and satisfies the coassociativity relation as formulated in Definition V.5.1.*

If the condition $1 \otimes 1 \in D(\delta^*)$ is satisfied, we have for all $X \in D(\delta)$ that

$$\begin{aligned} \|\delta^*(X \otimes 1)\|_2 &\leq \|\delta^*(1 \otimes 1)\|_2 \|X\| \\ \|\delta^*(1 \otimes X)\|_2 &\leq \|\delta^*(1 \otimes 1)\|_2 \|X\| \end{aligned} \tag{V.19}$$

and

$$\begin{aligned} \|(\mathrm{id} \otimes \tau)(\delta(X))\|_2 &\leq 2\|\delta^*(1 \otimes 1)\|_2 \|X\| \\ \|(\tau \otimes \mathrm{id})(\delta(X))\|_2 &\leq 2\|\delta^*(1 \otimes 1)\|_2 \|X\| \end{aligned} \tag{V.20}$$

Before proceeding with to the proof of Theorem V.5.2, we record here the following formula for its later use therein.

Lemma V.5.3. *In the situation of Theorem V.5.2, let $X \in D(\delta)$ be given and put*

$$Y := (\mathrm{id} \otimes \tau)(\delta(X)).$$

Then $Y \in D(\delta)$ holds and we have that

$$(\mathrm{id} \otimes \tau)(\delta(Y)) = (\mathrm{id} \otimes \langle \cdot, \delta^*(1 \otimes 1)\rangle)(\delta(X)).$$

Proof. Since δ is assumed to satisfy the coassociativity relation, we know by Definition V.5.1 that in particular $D(\delta) \odot D(\delta)$ holds, which gives $Y \in D(\delta)$. Furthermore, according to the coassociativity relation formulated in (V.18), we see that

$$\begin{aligned} \delta(Y) &= (\mathrm{id} \otimes \mathrm{id} \otimes \tau)\big((\delta \otimes \mathrm{id})(\delta(X))\big) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes \tau)\big((\mathrm{id} \otimes \delta)(\delta(X))\big) \end{aligned}$$

holds. Since we have on $D(\delta)$ the identity $(\tau \otimes \tau) \circ \delta = \langle \cdot, \delta^*(1 \otimes 1)\rangle$, we get

$$\begin{aligned} (\mathrm{id} \otimes \tau)(\delta(Y)) &= (\mathrm{id} \otimes \tau \otimes \tau)\big((\mathrm{id} \otimes \delta)(\delta(X))\big) \\ &= \big(\mathrm{id} \otimes ((\tau \otimes \tau) \circ \delta)\big)(\delta(X)) \\ &= \big(\mathrm{id} \otimes \langle \cdot, \delta^*(1 \otimes 1)\rangle\big)(\delta(X)), \end{aligned}$$

which is the desired formula. □

Additionally, the proof of Theorem V.5.2 will be based on the following observation.

Lemma V.5.4. *Let (M,τ) be a W^*-probability space and let $T : D(T) \to M$ be a linear operator on a unital $*$-subalgebra $D(T)$ of M. Assume that the following conditions are satisfied:*

(i) *There exists a constant $C > 0$ such that*

$$\|T(X)\|_2^2 \leq C\|T(X^*X)\|_2 \qquad \text{for all } X \in D(T).$$

(ii) *For each* $X \in D(T)$, *we have that*

$$\limsup_{m\to\infty} \|T(X^m)\|_2^{\frac{1}{m}} \leq \|X\|.$$

Then T *satisfies* $\|T(X)\|_2 \leq C\|X\|$ *for all* $X \in D(T)$.

Proof. Let $X \in D(T)$ be given. For each $n \in \mathbb{N}_0$, we define $Z_n := (X^*X)^{2^n} \in D(T)$. By assumption (i), we see that

$$\|T(Z_n)\|_2^2 \leq C\|T(Z_{n+1})\|_2 \qquad \text{for all } n \in \mathbb{N}_0,$$

which yields inductively

$$\|T(Z_0)\|_2 \leq C^{\frac{1}{2}+\dots+\frac{1}{2^n}} \|T(Z_n)\|_2^{\frac{1}{2^n}} \qquad \text{for all } n \in \mathbb{N}_0.$$

Since

$$\limsup_{n\to\infty} \|T(Z_n)\|_2^{\frac{1}{2^n}} = \limsup_{n\to\infty} \|T((X^*X)^{2^n})\|_2^{\frac{1}{2^n}} \leq \|X^*X\| = \|X\|^2$$

due to (ii), it follows that

$$\|T(Z_0)\|_2 \leq C\|X\|^2.$$

By using (ii) once again, we obtain

$$\|T(X)\|_2^2 \leq C\|T(Z_0)\|_2 \leq C^2\|X\|^2$$

and hence $\|T(X)\|_2 \leq C\|X\|$, as stated. □

Proof of Theorem V.5.2. First of all, we note that it suffices to prove (V.19), since (V.20) follows from (V.19) and Voiculescu's formula (V.17) by an application of the triangle inequality.

For proving (V.19), we want to use Lemma V.5.4. We consider the linear mapping $T : D(T) \to M$ on $D(T) := D(\delta)$ given by

$$T(X) := \delta^*(X \otimes 1) \qquad \text{for all } X \in D(\delta).$$

Since Lemma V.4.6 guarantees $D(\delta) \odot D(\delta) \subseteq D(\delta^*)$, the mapping T is indeed well-defined.

Now, we just have to follow the receipt given in Lemma V.5.4.

(i) For any given $X \in D(\delta)$, we have to compare $\|T(X^*X)\|_2$ and $\|T(X)\|_2$. In fact, we will show that

$$\|T(X)\|_2^2 = \langle T(X^*X), \delta^*(1 \otimes 1)\rangle \tag{V.21}$$

from which

$$\|T(X)\|_2^2 \leq \|\delta^*(1 \otimes 1)\|_2 \|T(X^*X)\|_2$$

immediately follows by an application of the Cauchy-Schwarz inequality.

Formula (V.21) can be shown as follows. Let $X \in D(\delta)$ be given and put $Y := (\mathrm{id} \otimes \tau)(\delta(X))$. Since

$$Y^* = (\mathrm{id} \otimes \tau)(\delta(X))^* = (\tau \otimes \mathrm{id})(\delta(X^*))$$

according to Lemma V.4.5, we may observe by using in turn Lemma V.5.3 and Lemma V.4.5 in the version (V.17) that

$$\begin{aligned}\|Y\|_2^2 &= \langle Y, (\mathrm{id}\otimes\tau)(\delta(X))\rangle\\ &= \langle Y\otimes 1, \delta(X)\rangle\\ &= \langle \delta^*(Y\otimes 1), X\rangle\\ &= \langle Y\delta^*(1\otimes 1), X\rangle - \langle(\mathrm{id}\otimes\tau)(\delta(Y)), X\rangle\\ &= \langle \delta^*(1\otimes 1)X^*, Y^*\rangle - \langle(\mathrm{id}\otimes\langle\cdot,\delta^*(1\otimes 1)\rangle)(\delta(X)), X\rangle\\ &= \langle 1\otimes\delta^*(1\otimes 1)X^*, \delta(X^*)\rangle - \langle\delta(X), X\otimes\delta^*(1\otimes 1)\rangle.\end{aligned}$$

Because moreover

$$\begin{aligned}\langle\delta(X),&X\otimes\delta^*(1\otimes 1)\rangle\\ &= \langle X^*\cdot\delta(X), 1\otimes\delta^*(1\otimes 1)\rangle\\ &= \langle\delta(X^*X), 1\otimes\delta^*(1\otimes 1)\rangle - \langle\delta(X^*)\cdot X, 1\otimes\delta^*(1\otimes 1)\rangle\\ &= \langle\delta(X^*X), 1\otimes\delta^*(1\otimes 1)\rangle - \langle\delta(X^*), 1\otimes\delta^*(1\otimes 1)X^*\rangle,\end{aligned}$$

we may conclude

$$\|Y\|_2^2 = 2\Re\big(\langle 1\otimes\delta^*(1\otimes 1)X^*, \delta(X^*)\rangle\big) - \langle\delta(X^*X), 1\otimes\delta^*(1\otimes 1)\rangle.$$

Furthermore, since $T(X) = X\delta^*(1\otimes 1) - Y$ due to (V.17), we get that

$$\begin{aligned}\|T(X)\|_2^2 &= \langle X\delta^*(1\otimes 1) - Y, X\delta^*(1\otimes 1) - Y\rangle\\ &= \|X\delta^*(1\otimes 1)\|_2^2 + \|Y\|_2^2 - 2\Re\big(\langle X\delta^*(1\otimes 1), Y\rangle\big)\\ &= \|X\delta^*(1\otimes 1)\|_2^2 + \|Y\|_2^2 - 2\Re\big(\langle X\delta^*(1\otimes 1)\otimes 1, \delta(X)\rangle\big).\end{aligned}$$

We check now

$$\begin{aligned}\langle X\delta^*&(1\otimes 1)\otimes 1, \delta(X)\rangle\\ &= \langle\delta^*(1\otimes 1), (\mathrm{id}\otimes\tau)(X^*\cdot\delta(X))\rangle\\ &= \langle(\mathrm{id}\otimes\tau)(X^*\cdot\delta(X))^*, \delta^*(1\otimes 1)\rangle && \text{(by (V.11))}\\ &= \langle(\mathrm{id}\otimes\tau)(\delta(X))^*X, \delta^*(1\otimes 1)\rangle\\ &= \langle(\tau\otimes\mathrm{id})(\delta(X^*))X, \delta^*(1\otimes 1)\rangle && \text{(by Lemma V.4.5)}\\ &= \langle(\tau\otimes\mathrm{id})(\delta(X^*)), \delta^*(1\otimes 1)X^*\rangle\\ &= \langle\delta(X^*), 1\otimes\delta^*(1\otimes 1)X^*\rangle,\end{aligned}$$

so that

$$\Re\big(\langle X\delta^*(1\otimes 1)\otimes 1, \delta(X)\rangle\big) = \Re\big(\langle 1\otimes\delta^*(1\otimes 1)X^*, \delta(X^*)\rangle\big).$$

A combination of our previous computations leads us to

$$\|T(X)\|_2^2 = \|X\delta^*(1\otimes 1)\|_2^2 - \langle\delta(X^*X), 1\otimes\delta^*(1\otimes 1)\rangle \tag{V.22}$$

Furthermore, due to (V.17), we have

$$T(X^*X) = X^*X\delta^*(1\otimes 1) - (\mathrm{id}\otimes\tau)(\delta(X^*X)),$$

and hence

$$\langle T(X^*X), \delta^*(1\otimes 1)\rangle = \|X\delta^*(1\otimes 1)\|_2^2 - \langle\delta(X^*X), \delta^*(1\otimes 1)\otimes 1\rangle. \tag{V.23}$$

Since (V.22) implies that $\langle\delta(X^*X), 1\otimes\delta^*(1\otimes 1)\rangle$ must be real, we get by using (V.13), Lemma V.4.5, and (V.14) that

$$\langle\delta(X^*X), \delta^*(1\otimes 1)\otimes 1\rangle = \langle 1\otimes\delta^*(1\otimes 1), \delta(X^*X)\rangle = \langle\delta(X^*X), 1\otimes\delta^*(1\otimes 1)\rangle.$$

Thus, comparing (V.22) and (V.23) gives

$$\|T(X)\|_2^2 = \langle T(X^*X), \delta^*(1\otimes 1)\rangle,$$

which is the stated formula (V.21).

(ii) To begin with, we observe that for any polynomial P and any $X\in D(\delta)$

$$\|T(P(X))\|_2 \leq \|P(X)\|\|\delta^*(1\otimes 1)\|_2 + \|(\partial P)(X)\|_\pi\|\delta(X)\|_2, \tag{V.24}$$

where $\|\cdot\|_\pi$ denotes the projective norm on $D(\delta)\odot D(\delta)$, which is given by

$$\|U\|_\pi := \inf\Big\{\sum_{j=1}^N \|A_j\|\|B_j\| \;\Big|\; N\in\mathbb{N},\, A_1,\dots,A_N,B_1,\dots,B_N\in D(\delta):\; U=\sum_{j=1}^N A_j\otimes B_j\Big\}$$

for any $U\in D(\delta)\odot D(\delta)$.

Indeed, according to (V.17), we have for each polynomial P and $X\in D(\delta)$

$$\begin{aligned} T(P(X)) &= \delta^*(P(X)\otimes 1)\\ &= P(X)\delta^*(1\otimes 1) - (\mathrm{id}\otimes\tau)(\delta(P(X)))\\ &= P(X)\delta^*(1\otimes 1) - (\mathrm{id}\otimes\tau)((\partial P)(X)\sharp\delta(X)), \end{aligned}$$

where we used that $\delta(P(X)) = (\partial P)(X)\sharp\delta(X)$ according to formula (V.10), which was given in Remark V.3.2. This yields as desired

$$\begin{aligned} \|T(P(X))\|_2 &\leq \|P(X)\delta^*(1\otimes 1)\|_2 + \|(\mathrm{id}\otimes\tau)((\partial P)(X)\sharp\delta(X))\|_2\\ &\leq \|P(X)\delta^*(1\otimes 1)\|_2 + \|(\partial P)(X)\sharp\delta(X)\|_2\\ &\leq \|P(X)\|\|\delta^*(1\otimes 1)\|_2 + \|(\partial P)(X)\|_\pi\|\delta(X)\|_2. \end{aligned}$$

If we apply (V.24) to the polynomial $P(X) = X^m$ for any $m\in\mathbb{N}$, we may deduce that

$$\|T(X^m)\|_2 \leq \|X\|^m\|\delta^*(1\otimes 1)\|_2 + m\|X\|^{m-1}\|\delta(X)\|_2$$

since $\|(\partial P)(X)\|_\pi \leq m\|X\|^{m-1}$ holds. From this, we immediately get that

$$\limsup_{m\to\infty}\|T(X^m)\|_2^{\frac{1}{m}} \leq \|X\|.$$

Thus, condition (ii) of Lemma V.5.4 is satisfied.

Lemma V.5.4 tells us now that $\|T(X)\|_2 \leq \|\delta^*(1\otimes 1)\|_2\|X\|$, which is by definition of T exactly the first inequality in (V.19). The second one can simply be deduced from the first one by using that $\delta^*(U^\dagger) = \delta^*(U)^*$ holds for any $U\in D(\delta^*)$ according to Lemma V.4.5, since δ was assumed to be real. □

Combining Theorem V.5.2 with Lemma V.4.6 yields the following corollary.

Corollary V.5.5. *Let $\delta: M\supseteq D(\delta)\to L^2(M,\tau)\otimes L^2(M,\tau)$ a non-commutative derivation on a tracial W^*-probability space (M,τ). We assume that δ is a real derivation in the sense of V.4.3 and that is satisfies the coassociativity relation formulated in V.5.1. Then, for all $X_1, X_2\in D(\delta)$, it holds true that*

$$\|\delta^*(X_1\otimes X_2)\|_2 \leq 3\|\delta^*(1\otimes 1)\|_2\|X_1\|\|X_2\| \tag{V.25}$$

and

$$\text{(V.26)}\qquad \begin{aligned} \|(\mathrm{id}\otimes\tau)(\delta(X_1)\cdot X_2)\|_2 &\leq 4\|\delta^*(1\otimes 1)\|_2\|X_1\|\|X_2\|,\\ \|(\tau\otimes\mathrm{id})(X_1\cdot\delta(X_2))\|_2 &\leq 4\|\delta^*(1\otimes 1)\|_2\|X_1\|\|X_2\|. \end{aligned}$$

Proof. According to Lemma V.4.6, we have for all $X_1, X_2 \in D(\delta)$ that

$$\begin{aligned} \delta^*(X_1\otimes X_2) &= X_1\delta^*(1\otimes 1)X_2 - m_1(\mathrm{id}\otimes\tau\otimes\mathrm{id})(\delta\otimes\mathrm{id}+\mathrm{id}\otimes\delta)(X_1\otimes X_2)\\ &= X_1\delta^*(1\otimes 1)X_2 - (\mathrm{id}\otimes\tau)(\delta(X_1))X_2 - X_1(\tau\otimes\mathrm{id})(\delta(X_2))\\ &= \delta^*(X_1\otimes 1)X_2 - X_1(\tau\otimes\mathrm{id})(\delta(X_2)) \end{aligned}$$

and thus, by applying the estimates (V.20) and (V.19), that

$$\begin{aligned} \|\delta^*(X_1\otimes X_2)\|_2 &\leq \|\delta^*(X_1\otimes 1)\|_2\|X_2\| + \|X_1\|\|(\tau\otimes\mathrm{id})(\delta(X_2))\|_2\\ &\leq 3\|\delta^*(1\otimes 1)\|_2\|X_1\|\|X_2\|. \end{aligned}$$

This shows the validity of (V.25). For proving (V.26), we first use integration by parts in order to obtain

$$\begin{aligned} (\mathrm{id}\otimes\tau)(\delta(X_1)\cdot X_2) &= (\mathrm{id}\otimes\tau)(\delta(X_1X_2)) - (\mathrm{id}\otimes\tau)(X_1\cdot\delta(X_2))\\ &= (\mathrm{id}\otimes\tau)(\delta(X_1X_2)) - X_1(\mathrm{id}\otimes\tau)(\delta(X_2)) \end{aligned}$$

for arbitrary $X_1, X_2 \in D(\delta)$. From this, we can easily deduce by using (V.20) that

$$\begin{aligned} &\|(\mathrm{id}\otimes\tau)(\delta(X_1)\cdot X_2)\|_2\\ &\qquad\leq \|(\mathrm{id}\otimes\tau)(\delta(X_1X_2))\|_2 + \|X_1\|\|(\mathrm{id}\otimes\tau)(\delta(X_2))\|_2\\ &\qquad\leq 4\|\delta^*(1\otimes 1)\|_2\|X_1\|\|X_2\| \end{aligned}$$

which is the first inequality of (V.26). The second inequality can either be proven similarly or can be deduced from the first one by using that δ is real. □

We conclude this subsection by highlighting Formula (V.21), which was obtained in the proof of Theorem V.5.2. Since we think that this observation might be of independent interest and could be helpful for future investigations, we record (V.21) here by the following corollary.

Corollary V.5.6. *Let $\delta : M \supseteq D(\delta) \to L^2(M,\tau)\otimes L^2(M,\tau)$ be a non-commutative derivation on a tracial W^*-probability space (M,τ), which is real and satisfies the coassociativity relation. Assume additionally that $1\otimes 1 \in D(\delta^*)$. Then, for each $X \in D(\delta)$, it holds true that*

$$\|\delta^*(X\otimes 1)\|_2^2 = \langle \delta^*((X^*X)\otimes 1), \delta^*(1\otimes 1)\rangle.$$

Assume, for instance, that in the situation of Corollary V.5.6 the conditions

$$\delta^*(1\otimes 1) \in D(\overline{\delta})\cap M \qquad \text{and} \qquad \overline{\delta}(\delta^*(1\otimes 1)) \in M\otimes M$$

are satisfied in addition. Corollary V.5.6 allows us then to conclude that for any $X \in D(\delta)$

$$\|\delta^*(X\otimes 1)\|_2^2 = \langle \delta^*((X^*X)\otimes 1), \delta^*(1\otimes 1)\rangle = \langle (X^*X)\otimes 1, \overline{\delta}(\delta^*(1\otimes 1))\rangle = \langle X\otimes 1, X\cdot\overline{\delta}(\delta^*(1\otimes 1))\rangle$$

and hence $\|\delta^*(X\otimes 1)\|_2 \leq \|\overline{\delta}(\delta^*(1\otimes 1))\|^{1/2}\|X\|_2$ holds. Like in Theorem V.5.2, we can use this in combination with (V.17) in order to deduce that

$$\|(\mathrm{id}\otimes\tau)(\delta(X))\|_2 \leq \big(\|\delta^*(1\otimes 1)\| + \|\overline{\delta}(\delta^*(1\otimes 1))\|^{1/2}\big)\|X\|_2$$

holds for each $X \in D(\delta)$. Analogous inequalities can of course be proven for $\delta^*(1\otimes X)$ and $(\tau\otimes\mathrm{id})(\delta(X))$. In other words, we can strengthen the bounds that were obtained in

Theorem V.5.2 by imposing some stronger "regularity conditions" on $\delta^*(1\otimes 1)$. Note that this in fact slightly improves similar estimates that were deduced in [**Dab14**].

V.6. Survival of zero divisors

We are mainly interested here in applications of the theory of non-commutative derivations to regularity questions for certain distributions; see Chapter VI and VII. The basic idea that originates in [**MSW14, MSW17**] is that, in order to exclude atoms, one should reformulate this question in more algebraic terms as a question about the existence of zero-divisors, where the latter can be excluded by a successive reduction of the degree by applying non-commutative derivations.

Note that a *zero-divisor* means here in fact a left zero divisor, and we are typically interested in the setting of von Neumann algebras. Thus, a zero-divisor is understood as an element X in some von Neumann algebra M, which is non-zero and for which another non-zero element $u \in M$ can be found, such that $Xu = 0$ holds.

The key for excluding zero divisors by some kind of iterative reduction argument is a certain inequality which allows the conclusion that zero-divisors $Xu = 0$ survive under applying operators of the form

$$\Delta_p(X) := (\tau \otimes \mathrm{id})(p \cdot \delta(X))$$

for any non-commutative derivation δ satisfying certain conditions and some non-trivial projection p. This inequality will be given below in Proposition V.6.1. As we will see, it will more generally relate products Xu and X^*v for elements X in the domain of the given non-commutative derivation δ and arbitrary elements u, v in the corresponding von Neumann algebra with an expression of the form $v^* \cdot \delta(X) \cdot u$.

We point out that although the inequality itself holds in a considerably large generality, the feasibility of the whole strategy for excluding zero-divisors relies heavily on the structure of the given non-commutative derivation. Roughly speaking, applying δ has to "reduce the degree" of the given element X. More formally, one should think of a grading on the space of distributions under consideration that is compatible with δ. We do not want to give a definition in full generality, but we want to mention that the grading that was used in [**MSW14, MSW17**] was given by the monomials of fixed degree and, as we will see in Section VII.3, that there is a closely related grading on the finite Wigner chaos.

The crucial inequality will now be formulated in the following proposition.

PROPOSITION V.6.1. *Let $\delta : L^2(M,\tau) \supseteq D(\delta) \to L^2(M,\tau) \otimes L^2(M,\tau)$ be a non-commutative derivation. We assume that δ is real and satisfies the coassociativity relation.*

Then, if in addition $1 \otimes 1 \in D(\delta^*)$ holds, we have for all $X \in D(\overline{\delta})$, where $\overline{\delta}$ denotes the closure of δ, and $u, v \in M$ the inequality

$$|\langle v^* \cdot \overline{\delta}(X) \cdot u, Y_1 \otimes Y_2\rangle| \leq 4\|\delta^*(1\otimes 1)\|_2\big(\|v\|\|Xu\|_2 + \|u\|\|X^*v\|_2\big)\|Y_1\|\|Y_2\| \tag{V.27}$$

for all $Y_1, Y_2 \in D(\delta)$.

In particular, if we have $Xu = 0$ and $X^*v = 0$ for any $X \in D(\overline{\delta})$ and some $u, v \in M$, then also $v^* \cdot \overline{\delta}(X) \cdot u = 0$ holds.

Before giving the proof of Proposition V.6.1, we first mention an easy but useful application of Kaplansky's density theorem.

LEMMA V.6.2. *In the given setting of a tracial W^*-probability space (M, τ), let D be a $*$-subalgebra of M, which is weakly dense in M. Then, for each $w \in M$, there eXists a sequence $(w_k)_{k\in\mathbb{N}}$ of elements in D such that*

(i) $\sup_{k\in\mathbb{N}} \|w_k\| \leq \|w\|$,
(ii) $\|w_k - w\|_2 \to 0$ *as* $k \to \infty$.

If $w = w^*$, then we may assume in addition that $w_k = w_k^*$ for all $k \in \mathbb{N}$.

PROOF. First of all, we note that for proving the existence of a sequence $(w_k)_{k\in\mathbb{N}}$ of elements in D, which satisfies conditions (i) and (ii), it suffices to find a net $(w_\lambda)_{\lambda\in\Lambda}$ of elements in D, which satisfies

(i)' $\sup_{\lambda\in\Lambda} \|w_\lambda\| \leq \|w\|$,
(ii)' $\|w_\lambda - w\|_2 \overset{\lambda\in\Lambda}{\longrightarrow} 0$.

Indeed, given such a net $(w_\lambda)_{\lambda\in\Lambda}$, we may choose a sequence $(\lambda_k)_{k\in\mathbb{N}}$ in Λ, such that $\|w_{\lambda_k} - w\|_2 < \frac{1}{k}$ holds for all $k \in \mathbb{N}$. Hence, the sequence $(w_{\lambda_k})_{k\in\mathbb{N}}$ satisfies (i) and (ii), as desired.

Now, for finding a net of elements in D, which satisfies (i)' and (ii)', we apply Kaplansky's density theorem. Indeed, this theorem guarantees the existence of a net $(w_\lambda)_{\lambda\in\Lambda}$ of elements in D, such that $\|w_\lambda\| \leq \|w\|$ holds for all $\lambda \in \Lambda$, and which converges to w in the strong operator topology. Thus, the net $(w_\lambda)_{\lambda\in\Lambda}$ already satisfies condition (i)' and it remains to show the validity of (ii)'.

For seeing (ii)', we note that with respect to the weak operator topology,

$$w_\lambda^* w \overset{\lambda\in\Lambda}{\longrightarrow} w^*w, \qquad w^* w_\lambda \overset{\lambda\in\Lambda}{\longrightarrow} w^*w, \qquad \text{and} \qquad w_\lambda^* w_\lambda \overset{\lambda\in\Lambda}{\longrightarrow} w^*w,$$

such that according to the continuity of τ

$$\begin{aligned} \|w_\lambda - w\|_2^2 &= \tau((w_\lambda - w)^*(w_\lambda - w)) \\ &= \tau(w_\lambda^* w_\lambda) - \tau(w_\lambda^* w) - \tau(w^* w_\lambda) + \tau(w^* w) \\ &\overset{\lambda\in\Lambda}{\longrightarrow} 0, \end{aligned}$$

as claimed in (ii)'. This concludes the proof of the first part of the lemma.

For proving the additional statement, we just have to observe that in the case $w = w^*$, we can take any sequence $(w_k)_{k\in\mathbb{N}}$ that satisfies (i) and (ii), and replace each w_k by its real part $\Re(w_k) = \frac{1}{2}(w_k + w_k^*)$. Indeed, for the sequence $(w_k)_{k\in\mathbb{N}}$ obtained in this way, conditions (i) and (ii) are still valid, but we have achieved $w_k = w_k^*$ for all $k \in \mathbb{N}$ in addition. □

Now, we may proceed by

Proof of Proposition V.6.1. Firstly, we assume that $X \in D(\delta)$ as well as $u, v \in D(\delta)$. In this particular case, we may compute

$$\begin{aligned}\langle Xu, \delta^*(vY_1 \otimes Y_2)\rangle &= \langle \delta(Xu), vY_1 \otimes Y_2\rangle \\ &= \langle \delta(X)\cdot u, vY_1 \otimes Y_2\rangle + \langle X \cdot \delta(u), vY_1 \otimes Y_2\rangle \\ &= \langle v^* \cdot \delta(X) \cdot u, Y_1 \otimes Y_2\rangle + \langle \delta(u) \cdot Y_2^*, X^* v Y_1 \otimes 1\rangle \\ &= \langle v^* \cdot \delta(X) \cdot u, Y_1 \otimes Y_2\rangle + \langle (\mathrm{id} \otimes \tau)(\delta(u)\cdot X_2^*), X^* v Y_1\rangle.\end{aligned}$$

Rearranging the terms yields

$$\langle v^* \cdot \delta(X) \cdot u, Y_1 \otimes Y_2\rangle = \langle Xu, \delta^*(vY_1 \otimes Y_2)\rangle - \langle (\mathrm{id} \otimes \tau)(\delta(u)\cdot Y_2^*), X^* v Y_1\rangle,$$

from which we deduce by the inequalities in Corollary V.5.5 that

$$\begin{aligned}&|\langle v^* \cdot \delta(X) \cdot u, Y_1 \otimes Y_2\rangle| \\ &\quad\le |\langle Xu, \delta^*(vY_1 \otimes Y_2)\rangle| + |\langle (\mathrm{id} \otimes \tau)(\delta(u)\cdot Y_2^*), X^* v Y_1\rangle| \\ &\quad\le \|Xu\|_2 \|\delta^*(vY_1 \otimes Y_2)\|_2 + \|(\mathrm{id} \otimes \tau)(\delta(u)\cdot Y_2^*)\|_2 \|X^* v Y_1\|_2 \\ &\quad\le 4\|\delta^*(1 \otimes 1)\|_2 \big(\|v\|\|Xu\|_2 + \|u\|\|X^*v\|_2\big)\|Y_1\|\|Y_2\|,\end{aligned}$$

as desired. Due to Lemma V.6.2, this inequality extends to arbitrary $u, v \in M$.

Thus, we have proven (V.27) for $X \in D(\delta)$ and $u, v \in M$. It remains to show that we may extend it from $X \in D(\delta)$ to $X \in D(\overline{\delta})$.

Since $D(\overline{\delta})$ turns out to be the closure of $D(\delta)$ with respect to the norm $\|\cdot\|_{2,1}$ defined by

$$\|X\|_{2,1} := \big(\|X\|_2^2 + \|\delta(X)\|_2^2\big)^{\frac{1}{2}} \qquad \text{for any } X \in D(\delta),$$

we can find for any $X \in D(\overline{\delta})$ a sequence $(X_k)_{k\in\mathbb{N}}$ in $D(\delta)$ such that both conditions $\|X_k - X\|_2 \to 0$ and $\|\delta(X_k) - \overline{\delta}(X)\|_2 \to 0$ as $k \to \infty$ are satisfied. Hence, for given $u, v \in M$, we observe

$$\lim_{k\to\infty} \langle v^* \cdot \delta(X_k) \cdot u, y_1 \otimes y_2\rangle = \langle v^* \cdot \delta(X_k) \cdot u, Y_1 \otimes Y_2\rangle$$

and

$$\lim_{k\to\infty} \big(\|v\|\|X_k u\|_2 + \|u\|\|X_k^* v\|_2\big) = \|v\|\|Xu\|_2 + \|u\|\|X^*v\|_2,$$

from which (V.27) immediately follows in full generality.

Finally, if we have $Xu = 0$ and $X^*v = 0$, then (V.27) implies that

$$\langle v^* \cdot \overline{\delta}(X) \cdot u, Y_1 \otimes Y_2\rangle = 0 \qquad \text{for all } Y_1, Y_2 \in D(\delta)$$

and hence by linearity

$$\langle v^* \cdot \overline{\delta}(X) \cdot u, U\rangle = 0 \qquad \text{for all } U \in D(\delta) \odot D(\delta).$$

Since $D(\delta) \odot D(\delta)$ is dense in $L^2(\mathcal{S}, \tau) \otimes L^2(\mathcal{S}, \tau)$, we obtain $v^* \cdot \overline{\delta}(X) \cdot u = 0$, as stated. □

Proposition V.6.1 will be used essentially twice in the subsequent chapters. The first application will be in Chapter VI, where we prove that conditions like finiteness of the Fisher information or maximality of the non-microstates free entropy dimension exclude zero divisors and hence atoms in the distribution of any non-constant self-adjoint polynomial in the considered variables. This follows the exposition in [**MSW14, MSW17**]. The second application will be presented in Chapter VII, where we extend these methods from the "discrete case" of non-commutative polynomials to the "continuous case" of the Wigner chaos. This will be based on [**Mai15**].

CHAPTER VI

Absence of algebraic relations and of zero divisors under the assumption of full non-microstates free entropy dimension

In a groundbreaking series of papers [**Voi93, Voi94, Voi96, Voi97, Voi98, Voi99**] (see also the survey [**Voi02b**]), Voiculescu transferred the notion of entropy and Fisher information to the world of non-commutative probability theory. Free entropy and free Fisher information are some of the core quantities in free probability theory, with fundamental importance both for operator algebraic and random matrix questions. One of the most striking results which came out of this program is arguably the proof of the fact [**Voi96**] that the free group factors do not possess Cartan subalgebras. This gave the solution of the longstanding open question of whether every separable II_1-factor contains Cartan subalgebras. But despite such deep results and applications, still many of the basic analytic properties of free entropy and Fisher information are poorly understood.

Voiculescu gave actually two different approaches to entropy and Fisher information in the non-commutative setting. The first one is based on the notion of matricial microstates and defines free entropy χ first and then, based on this, the free Fisher information Φ; the second approach is based on the notion of conjugate variables with respect to certain non-commutative derivatives and defines free Fisher information Φ^* first and then, based on this, free entropy χ^*. Both constructions lead independently to objects χ and χ^* (as well as Φ and Φ^*) which are, in analogy with the classical theory, justifiably called entropy (and Fisher information). But it is still not known whether they coincide in general, while equality in the case of a single variable was already shown by Voiculescu [**Voi98**]. Among the various attempts to settle this problem, we want to mention here the ingenious work [**BCG03**], where the inequality $\chi \leq \chi^*$ was established, and also the more recent and very impressive work [**Dab16**], where the author seeks to prove equality $\chi = \chi^*$ for a large class of tuples of non-commutative random variables. For many questions the actual value of these quantities is not important, essential is whether they are finite or infinite. There exist also more refined quantities, so-called free entropy dimensions (again in various variations), which allow a further distinction of the case of infinite entropy. In particular, finiteness of free entropy or of free Fisher information implies that the microstates free entropy dimension δ^* takes on its maximal value.

In the classical case, finiteness of entropy or of Fisher information imply some regularity of the corresponding distribution of the variables; in particular, they have a density (with respect to Lebesgue measure). In the non-commutative situation, the notion of a density does not make any direct sense, but still it is believed that the existence of finite free entropy or finite free Fisher information (in any of the two approaches) should correspond to some regularity properties of the considered non-commutative distributions. Thus one expects many "smooth" properties for random variables $X_1, \dots, X_n$ for which either one of the quantities $\chi(X_1, \dots, X_n)$, $\chi^*(X_1, \dots, X_n)$, $\Phi(X_1, \dots, X_n)$, or $\Phi^*(X_1, \dots, X_n)$ is finite. In particular, it is commonly expected that such a finiteness implies that

- there cannot exist non-trivial algebraic relations between the considered random variables;
- and that such algebraic relations can also not hold locally on non-trivial Hilbert subspaces; more formally this means that there are no zero divisors in the affiliated von Neumann algebra.

Up to now there has been no proof of such general statements. We will show here such results. In [**MSW14**], this was done under the assumption of finite non-microstates free Fisher information $\Phi^*(X_1 \dots, X_n)$. Inspired by this preprint, Shlyakhtenko could prove in [**Shl14**] (see [**CS16**] for an extended version joint with Charlesworth), by combining our ideas with his earlier work in [**CS05**], our results under the weaker assumption of maximal non-microstates free entropy dimension $\delta^*(X_1, \dots, X_n) = n$. In the light of this, we reexamined our original arguments and noticed that they can also be extended without much extra effort to this most general case; this has finally led to [**MSW17**].

Our work was originally inspired by the realization that in the usual approaches to conjugate variables one usually assumes that there exist no algebraic relations between the considered variables. Though this is not necessary for the definition of conjugate variables themselves, more advanced arguments (which rely on the existence of non-commutative derivative operators) only work in the absence of such algebraic relations. As alluded to above one actually expected that the existence of conjugate variables (and thus the finiteness of Φ^*) implies the absence of such relations. But since this has not been shown up to now, there was a kind of an annoying gap in the theory. This gap will be closed in Section VI.1.

It turned out that our ideas for this could also be extended to the much deeper question whether relations could hold locally; instead of asking whether for a non-trivial polynomial P we can have algebraically $P(X_1, \dots, X_n) = 0$, we weaken this to the question whether $P(X_1, \dots, X_n)$ could be zero on an affiliated Hilbert subspace; if u denotes the projection onto this subspace, then this is the question whether $P(X_1, \dots, X_n)$ can be a zero divisor, i.e., whether it is possible to find some non-zero element u in the von Neumann algebra generated by $X_1, \dots, X_n$, such that $P(X_1, \dots, X_n)u = 0$. We will show that already the condition $\delta^*(X_1, \dots, X_n) = n$ for the non-microstates free entropy dimension excludes such zero divisors.

In particular, this result allows us to conclude that the distribution of any non-trivial self-adjoint non-commutative polynomial in the generators does not have atoms, if the generators have full non-microstates free entropy dimension. Note that in a random matrix language this allows the conclusion that the asymptotic eigenvalue distribution of polynomials in random matrices has, under the above assumption, no atomic parts. Questions on the absence of atoms for polynomials in non-commuting random variables (or for polynomials in random matrices) have been an open problem for quite a while. Only recently there was some progress on this in such generality; in [**SS15**], Shlyakhtenko and Skoufranis showed that polynomials in free variables exhibit (under the assumption of no atoms for each of the variables) no atoms; here we give a vast generalization of this, by showing that the crucial issue is not freeness but the maximality of the free entropy dimension.

VI.1. Existence of conjugate variables and absence of algebraic relations

Our setting will be as presented in Section V.1: we consider $\mathbb{C}\langle x_1, \dots, x_n\rangle$, which is the $*$-algebra of non-commutative polynomials in n self-adjoint (formal) variables $x_1, \dots, x_n$.

For $j = 1, \dots, n$, we denote by ∂_j the non-commutative derivative with respect to x_j, i.e. ∂_j is the unique derivation

$$\partial_j : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle \otimes \mathbb{C}\langle x_1, \dots, x_n\rangle$$

that satisfies $\partial_j x_i = \delta_{i,j} 1 \otimes 1$ for $i = 1, \dots, n$.

Throughout the following, let (M, τ) be a tracial W^*-probability space, which means that M is a von Neumann algebra and τ is a faithful normal tracial state on M. For self-adjoint $X_1, \dots, X_n \in M$ we denote by $\mathrm{vN}(X_1, \dots, X_n) \subset M$ the von Neumann subalgebra of M which is generated by $X_1, \dots, X_n$ and by $L^2(X_1, \dots, X_n, \tau) \subset L^2(M, \tau)$ the L^2-space which is generated by $X_1, \dots, X_n$ with respect to the inner product given by $\langle P, Q\rangle := \tau(PQ^*)$.

As before, we will denote by ev_X the evaluation $*$-homomorphism

$$\mathrm{ev}_X : \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle X_1, \dots, X_n\rangle \subset M$$

for a given n-tuple $X = (X_1, \dots, X_n)$ of self-adjoint elements of M and we put $P(X) := \mathrm{ev}_X(P)$ for any $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ and $Q(X) := (\mathrm{ev}_X \otimes \mathrm{ev}_X)(Q)$ for any $Q \in \mathbb{C}\langle x_1, \dots, x_n\rangle^{\otimes 2}$.

Definition VI.1.1. Let $X_1, \dots, X_n \in M$ be self-adjoint elements. If there are elements $\xi_1, \dots, \xi_n \in L^2(M, \tau)$, such that

$$(\tau \otimes \tau)((\partial_j P)(X_1, \dots, X_n)) = \tau(\xi_j P(X_1, \dots, X_n)) \tag{VI.1}$$

is satisfied for each non-commutative polynomial $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ and for $j = 1, \dots, n$, then we say that $(\xi_1, \dots, \xi_n)$ *satisfies the conjugate relations for* $(X_1, \dots, X_n)$.

If, in addition, $\xi_1, \dots, \xi_n$ belong to $L^2(X_1, \dots, X_n, \tau)$, we say that $(\xi_1, \dots, \xi_n)$ is the *conjugate system for* $(X_1, \dots, X_n)$.

Like in [**Voi98**], we note the following.

Remark VI.1.2. Let π be the orthogonal projection from $L^2(M, \tau)$ to $L^2(X_1, \dots, X_n, \tau)$. It is easy to see that if $(\xi_1, \dots, \xi_n)$ satisfies the conjugate relations for $(X_1, \dots, X_n)$, then $(\pi(\xi_1), \dots, \pi(\xi_n))$ satisfies the conjugate relations for $(X_1, \dots, X_n)$ as well, and is therefore a conjugate system for $(X_1, \dots, X_n)$.

It is an easy consequence of its defining property (VI.1) that a conjugate system $(\xi_1, \dots, \xi_n)$ for $(X_1, \dots, X_n)$ is unique if it exists.

In [**Voi98**, Proposition 3.5] we find that in the case $n = 1$ a conjugate variable $\xi \in L^2(X, \tau) \cong L^2(\mathbb{R}, \mu_X)$ for X exists, if the analytic distribution μ_X is absolutely continuous with respect to the Lebesgue measure λ^1 on $\mathbb{R}$ and has a density ρ_X, which belongs to $L^3(\mathbb{R}, d\lambda^1)$; in this case, the conjugate variable is (up to some constant factor) given by the *Hilbert transform of* ρ_X. A recent result by S. Belinschi and H. Bercovici states that even the converse is true; the proof appeared in [**MS16**]. In this sense, the existence of a conjugate variable enforces strong regularity properties for X.

Note that our notion of conjugate relations and conjugate variables differs from the usual definition which was given by Voiculescu in [**Voi98**], roughly speaking, just by the placement of brackets. To be more precise, in (VI.1), we first apply the derivative ∂_j to the given non-commutative polynomial P before we apply the evaluation at $X = (X_1, \dots, X_n)$, instead of applying the evaluation first, which consequently makes it necessary to have in the second step a well-defined derivation on $\mathbb{C}\langle X_1, \dots, X_n\rangle$ corresponding to ∂_j.

From a more abstract point of view, this idea is in the same spirit as [**Shl00**, Lemma 3.2] but only on a purely algebraic level. In fact, we used the surjective homomorphism $\mathrm{ev}_X : \mathbb{C}\langle x_1,\dots,x_n\rangle \to \mathbb{C}\langle X_1,\dots,X_n\rangle$ in order to pass from $(\mathbb{C}\langle X_1,\dots,X_n\rangle,\tau)$ to the non-commutative probability space $(\mathbb{C}\langle x_1,\dots,x_n\rangle,\tau_X)$, where $\tau_X := \tau\circ\mathrm{ev}_X$. Due to this lifting, the algebraic relations between the generators disappear whereas the relevant information about their joint distribution remains unchanged.

In this section, our aim is to show that the existence of a conjugate system guarantees that $X_1,\dots,X_n$ do not satisfy any algebraic relations. This will be the content of Theorem VI.1.5 below. Its proof proceeds in two steps, which are performed in the following two propositions.

PROPOSITION VI.1.3. *Let (M,τ) be a tracial W^*-probability space and let self-adjoint $X_1,\dots,X_n\in M$ be given. Assume that there are elements $\xi_1,\dots,\xi_n\in L^2(M,\tau)$, such that $(\xi_1,\dots,\xi_n)$ satisfies the conjugate relations* (VI.1) *for $X=(X_1,\dots,X_n)$. Then the following implication holds true for any non-commutative polynomial $P\in\mathbb{C}\langle x_1,\dots,x_n\rangle$:*

$$P(X)=0 \quad\Longrightarrow\quad \forall j=1,\dots,n:\ (\partial_j P)(X)=0 \tag{VI.2}$$

Before beginning with the proof, let us introduce a binary operation $\sharp$ on the algebraic tensor product $M\otimes M$ by bilinear extension of

$$(a_1\otimes a_2)\sharp(b_1\otimes b_2) := (a_1b_1)\otimes(b_2a_2).$$

Note that, since $M\otimes M$ is naturally a M-bimodule, this corresponds exactly to Remark V.1.4.

PROOF OF PROPOSITION VI.1.3. Assume that $P\in\mathbb{C}\langle x_1,\dots,x_n\rangle$ satisfies $P(X)=0$ for $X=(X_1,\dots,X_n)$ and choose any $j=1,\dots,n$. If we take arbitrary $P_1,P_2\in\mathbb{C}\langle x_1,\dots,x_n\rangle$, we have by iterating the product rule (V.3) twice that

$$\partial_j(P_1PP_2) = (\partial_jP_1)PP_2 + P_1P(\partial_jP_2) + P_1(\partial_jP)P_2$$

and therefore, by evaluating this identity at X and applying $\tau\otimes\tau$ subsequently,

$$(\tau\otimes\tau)\big((\partial_j(P_1PP_2))(X)\big) = (\tau\otimes\tau)(P_1(X)(\partial_jP)(X)P_2(X)).$$

Furthermore, according to (VI.1), we may deduce that

$$(\tau\otimes\tau)\big((\partial_j(P_1PP_2))(X)\big) = \tau(\xi_j(P_1PP_2)(X)) = 0.$$

Thus, we observe that

$$(\tau\otimes\tau)((P_1\otimes P_2)(X)\sharp(\partial_jP)(X)) = (\tau\otimes\tau)(P_1(X)(\partial_jP)(X)P_2(X)) = 0$$

for all $P_1,P_2\in\mathbb{C}\langle x_1,\dots,x_n\rangle$ and hence by linearity

$$(\tau\otimes\tau)(Q(X)\sharp(\partial_jP)(X)) = 0$$

for all $Q\in\mathbb{C}\langle x_1,\dots,x_n\rangle^{\otimes 2}$. If we apply this observation to $Q=(\partial_jP)^*$, the faithfulness of $\tau\otimes\tau$ (recall that τ was assumed to be faithful) implies $(\partial_jP)(X)=0$, as claimed. □

The second proposition shows now that the validity of (VI.2) is already sufficient to exclude algebraic relations.

PROPOSITION VI.1.4. *Let (M,τ) be a tracial W^*-probability space and let self-adjoint $X_1,\dots,X_n\in M$ be given, such that* (VI.2) *holds. Then the non-commutative random variables $X_1,\dots,X_n$ do not satisfy any non-trivial algebraic relation, i.e., if $P(X_1,\dots,X_n)=0$ holds for any non-commutative polynomial $P\in\mathbb{C}\langle x_1,\dots,x_n\rangle$, then we must have $P=0$.*

PROOF. This follows immediately from Proposition V.2.1. Indeed, the implication "(ii) → (i)" thereof tells us that the validity of condition (VI.2) excludes any non-trivial algebraic relation among the variables $X_1, \dots, X_n$. □

Combining now the above Proposition VI.1.3 with Proposition VI.1.4 leads us directly to the following theorem.

THEOREM VI.1.5. *As before, let (M, τ) be a tracial W^*-probability space. Let $X_1, \dots, X_n \in M$ be self-adjoint and assume that there are elements $\xi_1, \dots, \xi_n \in L^2(M, \tau)$, such that $(\xi_1, \dots, \xi_n)$ satisfies the conjugate relations for $(X_1, \dots, X_n)$, i.e.* (VI.1) *holds for $j = 1, \dots, n$. Then we have the following statements:*

(a) *$X_1, \dots, X_n$ do not satisfy any non-trivial algebraic relation, i.e. there exists no non-zero polynomial $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ such that $P(X_1, \dots, X_n) = 0$.*
(b) *For $j = 1, \dots, n$, there is a unique derivation*

$$\hat{\partial}_j : \mathbb{C}\langle X_1, \dots, X_n\rangle \to \mathbb{C}\langle X_1, \dots, X_n\rangle \otimes \mathbb{C}\langle X_1, \dots, X_n\rangle$$

which satisfies $\hat{\partial}_j(X_i) = \delta_{j,i} 1 \otimes 1$ for $i = 1, \dots, n$.

Note that part (b) is an immediate consequence of part (a): Since (a) tells us that the evaluation homomorphism ev_X is in fact an isomorphism, we can immediately define a non-commutative derivation

$$\hat{\partial}_j : \mathbb{C}\langle X_1, \dots, X_n\rangle \to \mathbb{C}\langle X_1, \dots, X_n\rangle \otimes \mathbb{C}\langle X_1, \dots, X_n\rangle,$$

where the terminology derivation has to be understood with respect to the $\mathbb{C}\langle X_1, \dots, X_n\rangle$-bimodule structure of $\mathbb{C}\langle X_1, \dots, X_n\rangle^{\otimes 2}$. The uniqueness can be deduced from (V.5); see also Proposition V.2.1.

Following [**Voi98**], we may proceed now by defining (non-microstates) free Fisher information.

DEFINITION VI.1.6. Let (M, τ) be a tracial W^*-probability space and let self-adjoint elements $X_1, \dots, X_n \in M$ be given. We define their *(non-microstates) free Fisher information* $\Phi^*(X_1, \dots, X_n)$ by

$$\Phi^*(X_1, \dots, X_n) := \sum_{j=1}^{n} \|\xi_j\|_2^2$$

if a conjugate system $(\xi_1, \dots, \xi_n)$ for $(X_1, \dots, X_n)$ in the sense of Definition VI.1.1 exists, and we put $\Phi^*(X_1, \dots, X_n) := \infty$ if no such conjugate system for $(X_1, \dots, X_n)$ exists.

This $\Phi^*(X_1, \dots, X_n)$ is just the usual non-microstates free Fisher information as defined in [**Voi98**]. However, we have now the advantage that it can be defined even without assuming the algebraic freeness of $X_1, \dots, X_n$ right from the beginning. Actually, our result can now be stated as follows: $\Phi^*(X_1, \dots, X_n) < \infty$ implies the absence of algebraic relations between $X_1, \dots, X_n$.

Let (M, τ) be a W^*-probability space and let self-adjoint elements $X_1, \dots, X_n \in M$ be given such that the condition $\Phi^*(X_1, \dots, X_n) < \infty$ is fulfilled. Part (a) of Theorem VI.1.5 tells us then that $X_1, \dots, X_n$ do not satisfy any algebraic relation, which in other words means that the evaluation homomorphism ev_X induces an isomorphism between the abstract polynomial algebra $\mathbb{C}\langle x_1, \dots, x_n\rangle$ and the subalgebra $\mathbb{C}\langle X_1, \dots, X_n\rangle$ of M. Thus, each of the non-commutative derivatives

$$\hat{\partial}_j : \mathbb{C}\langle X_1, \dots, X_n \rangle \to \mathbb{C}\langle X_1, \dots, X_n \rangle \otimes \mathbb{C}\langle X_1, \dots, X_n \rangle,$$

whose existence is claimed in part (b) of Theorem VI.1.5, is naturally induced under this identification by the corresponding non-commutative derivative ∂_j on $\mathbb{C}\langle x_1, \dots, x_n \rangle$. Due to this strong relationship, we do not have to distinguish anymore between ∂_j and $\hat{\partial}_j$.

We finish this section by noting that $\Phi^*(X_1, \dots, X_n) < \infty$ moreover excludes analytic relations. More precisely, this means that there is no non-zero non-commutative power series P, which is convergent on a polydisc

$$D_R := \{(Y_1, \dots, Y_n) \in M^n | \ \forall j = 1, \dots, n : \ \|Y_j\| < R\}$$

for some $R > 0$, such that $(X_1, \dots, X_n) \in D_R$ and $P(X_1, \dots, X_n) = 0$. Based on Voiculescu's original definition of the non-microstates free Fisher information and hence under the additional assumption that $X_1, \dots, X_n$ are algebraically free, this was shown by Dabrowski in [**Dab14**, Lemma 37].

VI.2. Non-microstates free entropy dimension and zero divisors

Inspired by the methods used in the proof of Theorem VI.1.5, we address now the more general question of existence of zero divisors under the assumption of full non-microstates free entropy dimension.

First of all, we shall make more precise what we mean by this. We postpone the definition of the non-microstates free entropy dimension $\delta^*(X_1, \dots, X_n)$ and related quantities to Subsection VI.2.1, but we state here the result that we aim to prove.

THEOREM VI.2.1. *Let (M, τ) be a tracial W^*-probability space. Furthermore, let $X_1, \dots, X_n \in M$ be self-adjoint elements and assume that $\delta^*(X_1, \dots, X_n) = n$ holds. Then, for any non-zero non-commutative polynomial P, there exists no non-zero element $w \in \mathrm{vN}(X_1, \dots, X_n)$ such that*

$$P(X_1, \dots, X_n)w = 0.$$

Recall (see Definition I.1.18) that to each element $X = X^* \in M$ corresponds a unique probability measure μ_X on the real line $\mathbb{R}$, which has the same moments as X, i.e. it satisfies

$$\tau(X^k) = \int_{\mathbb{R}} t^k \, d\mu_X(t) \qquad \text{for } k = 0, 1, 2, \dots.$$

It is an immediate consequence of Theorem VI.2.1 that the distribution $\mu_{P(X_1,\dots,X_n)}$ of $P(X_1, \dots, X_n)$ for any non-constant self-adjoint polynomial P cannot have atoms, if $\delta^*(X_1, \dots, X_n) = n$. Note that a Borel probability measure μ on $\mathbb{R}$ is said to have an *atom* if there exists some $\alpha \in \mathbb{R}$, such that $\mu(\{\alpha\}) \neq 0$. In this case, we call α an *atom of* μ. The precise statement reads as follows.

COROLLARY VI.2.2. *Let (M, τ) be a tracial W^*-probability space and let $X_1, \dots, X_n \in M$ be self-adjoint with $\delta^*(X_1, \dots, X_n) = n$. Then, for any non-constant self-adjoint non-commutative polynomial P, the distribution $\mu_{P(X_1,\dots,X_n)}$ of $P(X_1, \dots, X_n)$ does not have atoms.*

Indeed, any atom α of the distribution $\mu_{P(X_1,\dots,X_n)}$, i.e. any $\alpha \in \mathbb{R}$ satisfying $\mu_{P(X_1,\dots,X_n)}(\{\alpha\}) \neq 0$, leads by the spectral theorem for bounded self-adjoint operators on

Hilbert spaces to a non-zero projection u satisfying $(P(X_1,\dots,X_n)-\alpha 1)u=0$; see Lemma I.1.20 and the comment thereafter. Thus, applying Theorem VI.2.1 yields immediately the statement of Corollary VI.2.2.

We point out that the conclusions of both Theorem VI.2.1 and Corollary VI.2.2 were shown in [**MSW14**] under the stronger assumption of finite non-microstates free Fisher information. In the above stated generality they appeared first in [**Shl14**], where the proof is based on results from [**CS05**]. We are going to prove those statements here in a more direct way by refining the initial methods of [**MSW14**]; our exposition follows [**MSW17**].

More precisely, in Subsection VI.2.2, we will give a quantitative version of our key idea that under the assumption of finite non-microstates Fisher information there is a strong relation between kernels of polynomials and the kernels of their derivatives.

Since the semicircular perturbation is inherent in the definition of the non-microstates free entropy as well as the corresponding entropy dimension, we find ourselves in the setting of finite free Fisher information. Therefore, we will study in Subsection VI.2.3 the behavior of the results found in Subsection VI.2.2 under semicircular perturbations – roughly speaking, we will be interested in the case where the perturbation tends to zero.

Finally, in Subsection VI.2.4, which is dedicated to the proof of Theorem VI.2.1, we will deduce from this observation a certain reduction argument that allows us to reduce successively the degree of the polynomial P satisfying the conditions of Theorem VI.2.1.

VI.2.1. Non-microstates free entropy and free entropy dimension. We want to catch up now on the definition of the non-microstates free entropy dimension. Let (M,τ) be a tracial W^*-probability space and let self-adjoint elements $X_1,\dots,X_n\in M$ be given. By enlarging (M,τ), if necessary, we may always assume that (M,τ) contains additionally semicircular elements $S_1,\dots,S_n$ such that

$$\{X_1,\dots,X_n\},\{S_1\},\dots,\{S_n\}$$

are freely independent. Indeed, this can be done by replacing (M,τ) by the free product $(M,\tau)*_{\mathbb{C}}(L(\mathbb{F}_n),\tau_n)$ of (M,τ) with the free group factor $(L(\mathbb{F}_n),\tau_n)$. Following Voiculescu [**Voi98**], we define the *non-microstates free entropy* $\chi^*(X_1,\dots,X_n)$ of $X_1,\dots,X_n$ by

$$\chi^*(X_1,\dots,X_n):=\frac{1}{2}\int_0^\infty\Big(\frac{n}{1+t}-\Phi^*(X_1+\sqrt{t}S_1,\dots,X_n+\sqrt{t}S_n)\Big)\,dt+\frac{n}{2}\log(2\pi e).$$

We need to note that the function

$$t\mapsto\Phi^*(X_1+\sqrt{t}S_1,\dots,X_n+\sqrt{t}S_n)$$

is well-defined, since [**Voi98**, Corollary 3.9] tells us that there exists a conjugate system of $(X_1+\sqrt{t}S_n,\dots,X_n+\sqrt{t}S_n)$ for all $t>0$. Moreover, we have the inequalities (cf. [**Voi98**, Corollary 6.14])

$$\text{(VI.3)}\qquad \frac{n^2}{C^2+nt}\le\Phi^*(X_1+\sqrt{t}S_1,\dots,X_n+\sqrt{t}S_n)\le\frac{n}{t}\quad\text{for all } t>0,$$

where $C^2:=\tau(X_1^2+\dots+X_n^2)$. The left inequality in (VI.3) particularly implies that (cf. [**Voi98**, Proposition 7.2])

$$\chi^*(X_1,\dots,X_n)\le\frac{n}{2}\log(2\pi e n^{-1}C^2).$$

This allows to define the *non-microstates free entropy dimension* $\delta^*(X_1,\dots,X_n)$ by

$$\delta^*(X_1,\dots,X_n) := n - \liminf_{\varepsilon\searrow 0} \frac{\chi^*(X_1+\sqrt{\varepsilon}S_1,\dots,X_n+\sqrt{\varepsilon}S_n)}{\log(\sqrt{\varepsilon})}.$$

We note that there is actually a variant of $\delta^*(X_1,\dots,X_n)$, given by

$$\hat{\delta}^\star(X_1,\dots,X_n) := n - \liminf_{t\searrow 0} t\Phi^*(X_1+\sqrt{t}S_1,\dots,X_n+\sqrt{t}S_n),$$

which is formally obtained by applying L'Hospital's rule to the lim inf appearing in the definition of $\delta^*(X_1,\dots,X_n)$. In [**CS05**], where $\hat{\delta}^\star$ was introduced, it was denoted by $\delta^\star$; we have slightly changed the notation for better legibility. Due to (VI.3), we have that $0 \leq \hat{\delta}^\star(X_1,\dots,X_n) \leq n$. It was already mentioned in [**Shl14**] that $\delta^*(X_1,\dots,X_n) = n$ or even $\hat{\delta}^\star(X_1,\dots,X_n) = n$ are the weakest possible assumptions where we can expect a version of Theorem VI.2.1 to hold true. Accordingly, it sits at the end of a longer chain of general implications, namely

$$\begin{aligned} \Phi^*(X_1,\dots,X_n) < \infty \quad &\Longrightarrow \quad \chi^*(X_1,\dots,X_n) > -\infty \\ &\Longrightarrow \quad \delta^*(X_1,\dots,X_n) = n \\ &\Longrightarrow \quad \hat{\delta}^\star(X_1,\dots,X_n) = n. \end{aligned}$$

The first implication follows by definition of $\chi^*(X_1,\dots,X_n)$, the second implication is a direct consequence of the definition of $\delta^*(X_1,\dots,X_n)$, and the last implication is justified by $\hat{\delta}^\star(X_1,\dots,X_n) \geq \delta^*(X_1,\dots,X_n)$, which was shown in [**CS05**, Lemma 4.1] by a straightforward computation.

VI.2.2. The case of finite non-microstates free Fisher information revisited. Let (M,τ) be a tracial W^*-probability space and let $X_1,\dots,X_n \in M$ be self-adjoint with $\Phi^*(X_1,\dots,X_n) < \infty$. As we have seen in Section VI.1, those conditions guarantee for each $i = 1,\dots,n$ the existence of a unique derivation

$$\partial_i :\ \mathbb{C}\langle X_1,\dots,X_n\rangle \to \mathbb{C}\langle X_1,\dots,X_n\rangle \otimes \mathbb{C}\langle X_1,\dots,X_n\rangle,$$

which is determined by the condition $\partial_i X_j = \delta_{i,j} 1 \otimes 1$ for $j = 1,\dots,n$. We may observe that the non-commutative derivatives $\partial_1,\dots,\partial_n$ fit perfectly into the setting of Chapter V. We collect below the relevant facts:

- Let us abbreviate $N := \mathrm{vN}(X_1,\dots,X_n)$. For each $i = 1,\dots,n$, we have that
$$\partial_i :\ N \supset D(\partial_i) \to L^2(N,\tau|_N) \otimes L^2(N,\tau|_N)$$
is a non-commutative derivation in the sense of Definition V.3.1, as its canonical domain $D(\partial_i) = \mathbb{C}\langle X_1,\dots,X_n\rangle$ is a unital $*$-subalgebra of N, which is weakly dense in N, and it also satisfies the Leibniz rule.
- For each $i = 1,\dots,n$, the non-commutative derivation ∂_i is real in the sense of Definition V.4.3, i.e., it satisfies $\partial_i(P^*) = (\partial_i P)^\dagger$ for all $P \in D(\partial_i)$. This can be checked easily by a direct computation.
- Each non-commutative derivation ∂_i takes its values in $D(\partial_i)\odot D(\partial_i)$ and satisfies the coassociativity relation
$$(\partial_i \otimes \mathrm{id}) \circ \partial_i = (\mathrm{id} \otimes \partial_i) \circ \partial_i$$
in the sense of Definition V.5.1. This was already observed in Remark V.1.6;

- For each $i = 1, \dots, n$ may consider ∂_j as a densely defined unbounded linear operator
$$\partial_i : \ L^2(N, \tau|_N) \supseteq D(\partial_i) \to L^2(N, \tau|_N) \otimes L^2(N, \tau|_N).$$
Since $\Phi^*(X_1, \dots, X_n) < \infty$ implies that a conjugate system $(\xi_1, \dots, \xi_n)$ for $(X_1, \dots, X_n)$ exists, we see by (VI.1) that $1 \otimes 1$ belongs to the domain of definition of each of the adjoints $\partial_1^*, \dots, \partial_n^*$. Accordingly, for any $i = 1, \dots, n$, the initial condition $1 \otimes 1 \in D(\partial_i^*)$ is satisfied and we have that $\xi_i = \partial_i^*(1 \otimes 1)$.

The following Proposition VI.2.3 is the main result of this subsection. In [**MSW14, MSW17**], where it showed up for the first time, it was proven by using results of [**Voi98, Dab10**]. But here, we can shorten the exposition, since we have already discussed results of this type in Chapter V, even in a slightly more general setting. Proposition VI.2.3 is thus largely a direct consequence of Proposition V.6.1. However, the concrete situation allows us to include some additional statements. These will involve the projective norm $\|\cdot\|_\pi$ on $\mathbb{C}\langle X_1, \dots, X_n\rangle^{\otimes 2}$, which is given by
$$\|Q\|_\pi := \inf \left\{ \sum_{k=1}^N \|Q_{k,1}\| \|Q_{k,2}\| \,\middle|\, Q = \sum_{k=1}^N Q_{k,1} \otimes Q_{k,2} \right\}$$
for any $Q \in \mathbb{C}\langle X_1, \dots, X_n\rangle^{\otimes 2}$, where the infimum is taken over all possible decompositions of Q with $Q_{k,1}, Q_{k,2} \in \mathbb{C}\langle X_1, \dots, X_n\rangle$ for $k = 1, \dots, N$ and some $N \in \mathbb{N}$. Whenever it becomes necessary in the following to mention explicitly the dependence of $\|\cdot\|_\pi$ on the underlying set of generators $X = (X_1, \dots, X_n)$, we will also write $\|\cdot\|_{\pi,X}$ instead of $\|\cdot\|_\pi$.

Proposition VI.2.3. *Let (M, τ) be a tracial W^*-probability space and let self-adjoint elements $X_1, \dots, X_n \in M$ be given such that the condition $\Phi^*(X_1, \dots, X_n) < \infty$ is satisfied. Let $(\xi_1, \dots, \xi_n)$ be the conjugate system for $(X_1, \dots, X_n)$. Then, for all $P \in \mathbb{C}\langle X_1, \dots, X_n\rangle$ and all $u, v \in \mathrm{vN}(X_1, \dots, X_n)$, we have*
$$|\langle v^*(\partial_i P)u, Q\rangle| \leq 4\|\xi_i\|_2 \big(\|Pu\|_2 \|v\| + \|u\| \|P^* v\|_2\big) \|Q\|_\pi \tag{VI.4}$$
for all $Q \in \mathbb{C}\langle X_1, \dots, X_n\rangle^{\otimes 2}$ and $i = 1, \dots, n$. In particular, we have
$$\begin{aligned} &\sum_{i=1}^n |\langle v^*(\partial_i P)u, Q\rangle|^2 \\ &\qquad \leq 16\big(\|Pu\|_2 \|v\| + \|u\| \|P^* v\|_2\big)^2 \Phi^*(X_1, \dots, X_n) \|Q\|_\pi^2 \end{aligned} \tag{VI.5}$$
for all $Q \in \mathbb{C}\langle X_1, \dots, X_n\rangle^{\otimes 2}$.

Proof. As we already mentioned above, the first part of the statement concerning (VI.4) is an immediate consequence of Proposition V.6.1. Indeed, since
$$\partial_i : \ N \supset D(\partial_i) \to L^2(N, \tau|_N) \otimes L^2(N, \tau|_N)$$
forms a non-commutative derivation, which is both real and satisfies the coassociativity relation, and since also $1 \otimes 1 \in D(\partial_i^*)$ holds, Proposition V.6.1 can be applied. This yields
$$|\langle v^*(\partial_i P)u, Q_1 \otimes Q_2\rangle| \leq 4\|\xi_i\|_2 \big(\|Pu\|_2 \|v\| + \|u\| \|P^* v\|_2\big) \|Q_1\| \|Q_2\|$$
for all $Q_1, Q_2 \in \mathbb{C}\langle X_1, \dots, X_n\rangle$ and hence, by the triangle inequality
$$|\langle v^*(\partial_i P)u, Q\rangle| \leq 4\|\xi_i\|_2 \big(\|Pu\|_2 \|v\| + \|u\| \|P^* v\|_2\big) \sum_{k=1}^N \|Q_{k,1}\| \|Q_{k,2}\|$$

for any $Q \in \mathbb{C}\langle X_1, \dots, X_n\rangle^{\otimes 2}$ that is decomposed as

$$Q = \sum_{k=1}^{N} Q_{k,1} \otimes Q_{k,2} \qquad \text{with } Q_{k,1}, Q_{k,2} \in \mathbb{C}\langle X_1, \dots, X_n\rangle \text{ for } k = 1, \dots, N.$$

Since this decomposition of Q can be chosen arbitrarily, the inequality above goes over to

$$|\langle v^*(\partial_i P)u, Q\rangle| \leq 4\|\xi_i\|_2\big(\|Pu\|_2\|v\| + \|u\|\|P^*v\|_2\big)\|Q\|_\pi,$$

which is just the claimed inequality (VI.4). The second inequality follows by taking squares on both sides of (VI.4) for each $i = 1, \dots, n$ and summing over $i = 1, \dots, n$. Since $\Phi^*(X_1, \dots, X_n) = \sum_{i=1}^n \|\xi_i\|_2^2$, this concludes the proof. □

We want to stress that Proposition VI.2.3 is in fact a quantitative version of a previous result of ours that allowed us in [**MSW14**] (which is an earlier version of [**MSW17**], on which this chapter is based) to give a proof of Theorem VI.2.1 under the stronger assumption of finite non-microstates Fisher information. This was based on the following corollary.

Corollary VI.2.4. *Let (M, τ) be a tracial W^*-probability space and let self-adjoint elements $X_1, \dots, X_n \in M$ be given, such that*

$$\Phi^*(X_1, \dots, X_n) < \infty$$

holds. We consider $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$. Then, for arbitrary $u, v \in \mathrm{vN}(X_1, \dots, X_n)$, the following implication holds true:

$$P(X)u = 0 \quad \textit{and} \quad P(X)^*v = 0 \implies \forall i = 1, \dots, n: \ v^*(\partial_i P)(X)u = 0,$$

where we abbreviate $X = (X_1, \dots, X_n)$.

Proof. The inequality (VI.4), which was stated in Proposition VI.2.3, immediately implies that $\langle v^*(\partial_i P)u, Q\rangle = 0$ for all $Q \in \mathbb{C}\langle X_1, \dots, X_n\rangle^{\otimes 2}$. Hence, since $\mathbb{C}\langle X_1, \dots, X_n\rangle^{\otimes 2}$ is dense in $L^2(M, \tau)$ with respect to $\|\cdot\|_2$, this yields $v^*(\partial_i P)(X)u = 0$ as claimed. □

Thus, readers interested in the proof of Theorem VI.2.1 only under the stronger assumption $\Phi^*(X_1, \dots, X_n) < \infty$ may skip Subsection VI.2.3 and proceed directly to Subsection VI.2.4, since the final step in the proof of Theorem VI.2.1 will only need the above reduction argument.

VI.2.3. Treating the case of full entropy dimension via semicircular perturbations. Since the non-microstates free entropy dimension $\delta^*(X_1, \dots, X_n)$ and its variant $\hat{\delta}^\star(X_1, \dots, X_n)$ are both determined in a more or less explicit way by the behavior of the function

$$t \mapsto \Phi^*(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n)$$

as $t \searrow 0$, one is tempted to apply the results obtained in Proposition VI.2.3 to the semicircular perturbation $(X_1+\sqrt{t}S_1, \dots, X_n+\sqrt{t}S_n)$. In fact, as we will see in Proposition VI.2.5 below, in this way the quantity

$$\begin{aligned}\alpha(X_1, \dots, X_n) &:= n - \hat{\delta}^\star(X_1, \dots, X_n)\\ &= \liminf_{t\searrow 0} t\Phi^*(X_1 + \sqrt{t}S_1, \dots, X_n + \sqrt{t}S_n),\end{aligned}$$

which also appeared in [**CS05**, Section 4], emerges naturally from the inequality given in Proposition VI.2.3 and allows us to study its influence.

It is therefore not surprising that some of the technical arguments that we will use below for dealing with semicircular perturbations are similar to [**CS05**, Section 4]. However, the proof itself is conceptually independent and follows a different strategy, since it is a straightforward continuation of Proposition VI.2.3 and hence relies on the inequalities due to Dabrowski [**Dab10**]; a general version of these inequalities was presented in Theorem V.5.2.

More precisely, we will show the following.

PROPOSITION VI.2.5. *Let (M,τ) be a tracial W^*-probability space and let self-adjoint elements $X_1,\dots,X_n \in M$ be given. Moreover, let $P \in \mathbb{C}\langle x_1,\dots,x_n\rangle$ be a non-commutative polynomial for which there are elements $u,v \in \mathrm{vN}(X_1,\dots,X_n)$ such that*

$$P(X_1,\dots,X_n)u = 0 \qquad \text{and} \qquad P(X_1,\dots,X_n)^*v = 0.$$

Then, for all $Q \in \mathbb{C}\langle X_1,\dots,X_n\rangle^{\otimes 2}$,

$$\begin{aligned}\sum_{i=1}^n &|\langle v^*(\partial_i P)(X)u, Q\rangle|^2\\ &\leq 16\big(\|(\partial P)(X)u\|_2\|v\| + \|u\|\|(\partial P)(X)^*v\|_2\big)^2\alpha(X)\|Q\|_\pi^2,\end{aligned}$$

where we abbreviate $X = (X_1,\dots,X_n)$.

Here, we use the notation ∂P for the gradient $(\partial_1 P,\dots,\partial_n P)$ of a polynomial $P \in \mathbb{C}\langle x_1,\dots,x_n\rangle$. Evaluation $(\partial P)(X)$, taking adjoints $(\partial P)(X)^*$, and multiplication by elements from M like in $(\partial P)(X)u$ and $(\partial P)(X)^*v$ are then defined component-wise.

Furthermore, we point out that the space $L^2(M,\tau)^n$ becomes a Hilbert space in the obvious way. We denote its induced norm also by $\|\cdot\|_2$.

PROOF OF PROPOSITION VI.2.5. (i) Without any restriction, we may assume that our underlying W^*-probability space (M,τ) contains n normalized semicircular elements $S_1,\dots,S_n$ such that

$$\{X_1,\dots,X_n\}, \{S_1\},\dots,\{S_n\}$$

are freely independent. We define variables

$$X_j^t := X_j + \sqrt{t}S_j \qquad \text{for } t \geq 0 \text{ and } j = 1,\dots,n$$

and denote by $N_t := \mathrm{vN}(X_1^t,\dots,X_n^t)$ the von Neumann algebras they generate. In particular, N_0 is the von Neumann algebra generated by $X_1,\dots,X_n$. We abbreviate $X^t = (X_1^t,\dots,X_n^t)$ for $t \geq 0$, so that in particular $X^0 = X = (X_1,\dots,X_n)$.

Since N_t, for each $t \geq 0$, is a von Neumann subalgebra of M, we may consider the unique trace-preserving conditional expectation $\mathbb{E}_t$ from M onto N_t; see Remark I.2.7. Finally, we introduce $u_t := \mathbb{E}_t[u] \in N_t$ and $v_t := \mathbb{E}_t[v] \in N_t$.

It follows then that $P(X^t)u_t = P(X^t)\mathbb{E}_t[u] = \mathbb{E}_t[P(X^t)u]$ and hence

$$\|P(X^t)u_t\|_2 = \|\mathbb{E}_t[P(X^t)u]\|_2 \leq \|P(X^t)u\|_2.$$

Now, since $t \mapsto P(X^t)u$ is a polynomial in $\sqrt{t}$, which vanishes at $t=0$, we may observe that

$$\lim_{t\searrow 0}\frac{1}{\sqrt{t}}P(X^t)u = \sum_{i=1}^n (\partial_i P)(X)u\sharp S_i.$$

Since the linear subspaces

$$\operatorname{span}\{a_1 S_j a_2 |\ a_1, a_2 \in N_0\}, \qquad j = 1, \dots, n,$$

of $L^2(M,\tau)$ are pairwise orthogonal and since the mapping

$$L^2(N_0,\tau) \otimes L^2(N_0,\tau) \to L^2(M,\tau),\ U \mapsto U\sharp S_j$$

is in fact an isometry, which are both consequences of the assumed freeness of $\{X_1,\dots,X_n\},\{S_1\},\dots,\{S_n\}$ (more precisely, we can use the formula provided in Example I.1.36), it follows that

$$\lim_{t\searrow 0} \frac{1}{\sqrt{t}} \|P(X^t)u\|_2 = \Big(\sum_{i=1}^n \|(\partial_i P)(X)u\|_2^2\Big)^{1/2} = \|(\partial P)(X)u\|_2$$

Similarly, we may deduce that

$$\lim_{t\searrow 0} \frac{1}{\sqrt{t}} \|P(X^t)^* v\|_2 = \Big(\sum_{i=1}^n \|(\partial_i P)(X)^* v\|_2^2\Big)^{1/2} = \|(\partial P)(X)^* v\|_2.$$

(ii) We note that for each $Q \in \mathbb{C}\langle x_1,\dots,x_n\rangle^{\otimes 2}$

$$\limsup_{t\searrow 0} \|Q(X^t)\|_{\pi,X^t} \leq \|Q(X)\|_{\pi,X}.$$

Indeed, for any given $Q \in \mathbb{C}\langle x_1,\dots,x_n\rangle^{\otimes 2}$, we may consider an arbitrary decomposition

$$Q = \sum_{k=1}^N Q_{k,1} \otimes Q_{k,2}.$$

For $k = 1,\dots,N$, we may write

$$Q_{k,1}(X^t) \otimes Q_{k,2}(X^t) = Q_{k,1}(X) \otimes Q_{k,2}(X) + \sum_{l=1}^{d_k} t^{l/2} R_{k,l}$$

for some $d_k \geq 1$ with certain elements

$$R_{k,1},\dots,R_{k,d_k} \in \mathbb{C}\langle X_1,\dots,X_n,S_1,\dots,S_n\rangle^{\otimes 2},$$

which are independent of t. Since the norm $\|\cdot\|$ on $M\overline{\otimes}M$ is a cross norm, we get

$$\begin{aligned}\|Q_{k,1}(X^t)\|\|Q_{k,2}(X^t)\| &= \|Q_{k,1}(X^t) \otimes Q_{k,2}(X^t)\| \\ &\leq \|Q_{k,1}(X) \otimes Q_{k,2}(X)\| + \sum_{l=1}^{d_k} t^{l/2}\|R_{k,l}\| \\ &= \|Q_{k,1}(X)\|\|Q_{k,2}(X)\| + \sum_{l=1}^{d_k} t^{l/2}\|R_{k,l}\|\end{aligned}$$

for all $k = 1,\dots,N$ and thus

$$\begin{aligned}\|Q(X^t)\|_{\pi,X^t} &\leq \sum_{k=1}^N \|Q_{k,1}(X^t)\|\|Q_{k,2}(X^t)\| \\ &\leq \sum_{k=1}^N \|Q_{k,1}(X)\|\|Q_{k,2}(X)\| + \sum_{k=1}^N \sum_{l=1}^{d_k} t^{l/2}\|R_{k,l}\|,\end{aligned}$$

so that

$$\limsup_{t\searrow 0}\|Q(X^t)\|_{\pi,X^t}\leq\sum_{k=1}^N\|Q_{k,1}(X)\|\|Q_{k,2}(X)\|.$$

Since the decomposition of Q was arbitrarily chosen, we get as desired

$$\limsup_{t\searrow 0}\|Q(X^t)\|_{\pi,X^t}\leq\|Q(X)\|_{\pi,X}.$$

(iii) Let $Q\in\mathbb{C}\langle x_1,\dots,x_n\rangle^{\otimes 2}$ be given. Since $\Phi^*(X^t)<\infty$, we obtain by Proposition VI.2.3 that

$$\begin{aligned}
&\sum_{i=1}^n|\langle v_t^*(\partial_i P)(X^t)u_t,Q(X^t)\rangle|^2\\
&\quad\leq 16\big(\|P(X^t)u_t\|_2\|v_t\|+\|u_t\|\|P(X^t)^*v_t\|_2\big)^2\Phi^*(X^t)\|Q(X^t)\|^2_{\pi,X^t}\\
&\quad\leq 16\big(\|P(X^t)u\|_2\|v\|+\|u\|\|P(X^t)^*v\|_2\big)^2\Phi^*(X^t)\|Q(X^t)\|^2_{\pi,X^t}\\
&\quad= 16\bigg(\frac{1}{\sqrt{t}}\|P(X^t)u\|_2\|v\|+\|u\|\frac{1}{\sqrt{t}}\|P(X^t)^*v\|_2\bigg)^2 t\Phi^*(X^t)\|Q(X^t)\|^2_{\pi,X^t},
\end{aligned}$$

so that, since $\limsup_{t\searrow 0}\|Q(X^t)\|_{\pi,X^t}\leq\|Q(X)\|_{\pi,X}$ according to (ii),

$$\begin{aligned}
&\liminf_{t\searrow 0}\sum_{i=1}^n|\langle v_t^*(\partial_i P)(X^t)u_t,Q(X^t)\rangle|^2\\
&\qquad\leq 16\big(\|(\partial P)(X)u\|_2\|v\|+\|u\|\|(\partial P)(X)^*v\|_2\big)^2\alpha(X)\|Q(X)\|^2_{\pi}.
\end{aligned}$$

(iv) It remains to show that in fact

$$\liminf_{t\searrow 0}\sum_{i=1}^n|\langle v_t^*(\partial_i P)(X^t)u_t,Q(X^t)\rangle|^2=\sum_{i=1}^n|\langle v^*(\partial_i P)(X)u,Q(X)\rangle|^2.$$

We first check that

$$\begin{aligned}
\langle v_t^*(\partial_i P)(X^t)u_t,Q(X^t)\rangle &= \langle\mathbb{E}_t[v^*](\partial_i P)(X^t)\mathbb{E}_t[u],Q(X^t)\rangle\\
&= \langle(\mathbb{E}_t\otimes\mathbb{E}_t)[v^*(\partial_i P)(X^t)u],Q(X^t)\rangle\\
&= \langle v^*(\partial_i P)(X^t)u,Q(X^t)\rangle,
\end{aligned}$$

which means in particular that this expression is actually a complex polynomial in $\sqrt{t}$. This guarantees that

$$\lim_{t\searrow 0}\langle v_t^*(\partial_i P)(X^t)u_t,Q(X^t)\rangle=\langle v^*(\partial_i P)(X)u,Q(X)\rangle$$

and finally completes the proof. □

VI.2.4. Proof of Theorem VI.2.1. Now, we are prepared to give a proof of Theorem VI.2.1. Even more, we will do this under the (possibly) weaker assumption that $\hat{\delta}^\star(X_1,\dots,X_n)=n$.

The main tool is the following corollary of Proposition VI.2.5.

Corollary VI.2.6. *Let* (M, τ) *be a tracial* W^**-probability space and let self-adjoint elements* $X_1, \dots, X_n \in M$ *be given, such that*

$$\hat{\delta}^\star(X_1, \dots, X_n) = n$$

holds. We consider $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$*. Then, for arbitrary* $u, v \in \mathrm{vN}(X_1, \dots, X_n)$*, the following implication holds true:*

$$P(X)u = 0 \quad \text{and} \quad P(X)^* v = 0 \implies \forall i = 1, \dots, n: \ v^*(\partial_i P)(X)u = 0.$$

Proof. Note that our assumption $\hat{\delta}^\star(X_1, \dots, X_n) = n$ is equivalent to the fact that $\alpha(X_1, \dots, X_n) = 0$. If $P(X)u = 0$ and $P(X)^* v = 0$, then Proposition VI.2.5 yields

$$\sum_{i=1}^n |\langle v^*(\partial_i P)(X)u, Q\rangle|^2 = 0$$

for all $Q \in \mathbb{C}\langle X_1, \dots, X_n\rangle^{\otimes 2}$. Since $\mathbb{C}\langle X_1, \dots, X_n\rangle^{\otimes 2}$ is by definition dense in $L^2(M, \tau)$ with respect to $\|\cdot\|_2$, we conclude that $v^*(\partial_i P)(X)u = 0$ for $i = 1, \dots, n$. □

Remark VI.2.7. Let $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ be given and assume that there are $u = u^*, v = v^* \in \mathrm{vN}(X_1, \dots, X_n)$ such that $P(X)u = P(X)^* v = 0$ holds. Then, according to Corollary VI.2.6, we know that $v(\partial_j P)(X)u = 0$ for any $j = 1, \dots, n$. If we replace now P by P^*, the statement of Corollary VI.2.6 also gives $u(\partial_j P^*)(X)v = 0$ for $j = 1, \dots, n$. But we want to point out that this does not lead to any new information.

For seeing this, let us consider the involution † on $M \otimes M$, which was introduced in Definition V.4.2. If we apply † to the initial statement

$$v(\partial_j P)(X)u = 0,$$

we get

$$u(\partial_j P)(X)^\dagger v = 0.$$

An easy calculation on monomials shows that $(\partial_j P)(X)^\dagger = (\partial_j P^*)(X)$, such that the above result reduces exactly to the statement obtained by replacing P with P^*.

Before doing the final step, we first want to test in two examples how strong the result in Corollary VI.2.6 is. In both of these examples, we therefore suppose that (M, τ) is a tracial W^*-probability space and that $X_1, \dots, X_n \in M$ for $n \geq 2$ are self-adjoint elements satisfying the condition $\hat{\delta}^\star(X_1, \dots, X_n) = n$.

Example VI.2.8. For the self-adjoint polynomial $P = x_1 x_2 x_1$, we calculate $\partial_2 P = x_1 \otimes x_1$, such that $P(X)w = 0$ for some $w \in \mathrm{vN}(X_1, \dots, X_n)$ implies according to Corollary VI.2.6 that $wX_1 \otimes X_1 w = 0$ and therefore $\tilde{P}(X)w = 0$ holds with $\tilde{P} = x_1$.

Applying Corollary VI.2.6 once again, but now to the non-commutative polynomial $\tilde{P}$ and with respect to ∂_1, we end up with $w \otimes w = 0$, such that $w = 0$ follows.

Example VI.2.9. Take $P = x_1 x_2 + x_2 x_1$. We have

$$\partial_1 P = 1 \otimes x_2 + x_2 \otimes 1,$$
$$\partial_2 P = x_1 \otimes 1 + 1 \otimes x_1$$

and thus according to Corollary VI.2.6

$$(X_2 w)^*(X_2 w) = m_{X_2}(w(\partial_1 P)(X)w) = 0,$$
$$(X_1 w)^*(X_1 w) = m_{X_1}(w(\partial_2 P)(X)w) = 0.$$

We conclude $X_1 w = X_2 w = 0$, from which we may deduce like above by a second application of Corollary VI.2.6 that $w = 0$.

Although the above examples might give the feeling that Corollary VI.2.6 is strong enough to allow directly a successive reduction of any polynomial, the needed algebraic manipulations turn out to be obscure in general; a skeptical reader might convince himself by having a try at the polynomial $P = x_1x_2x_3 + x_3x_2x_1$, for instance.

Moreover, in contrast to Theorem VI.2.1, any such reduction argument that is based only on Corollary VI.2.6 would need a symmetric starting condition. Fortunately, in our situation, we can go around these complications, since we can use the following well-known general lemma, which is an easy consequence of the polar decomposition and encodes the additional information that our statement is formulated in a tracial setting.

LEMMA VI.2.10. *Let X be an element of any tracial W^*-probability space (M, τ) over some complex Hilbert space H. Let $p_{\ker(X)}$ and $p_{\ker(X^*)}$ denote the orthogonal projections onto* $\ker(X)$ *and* $\ker(X^*)$*, respectively. The projections $p_{\ker(X)}$ and $p_{\ker(X^*)}$ belong both to M and satisfy*

$$\tau(p_{\ker(X)}) = \tau(p_{\ker(X^*)}).$$

Thus, in particular, if $\ker(X)$ *is non-zero, then also* $\ker(X^*)$ *is a non-zero subspace of H.*

PROOF. We consider the polar decomposition $X = V(X^*X)^{1/2} = (XX^*)^{1/2}V$ of X, where $V \in M$ is a partial isometry mapping $\overline{\operatorname{ran}(X^*)}$ to $\overline{\operatorname{ran}(X)}$, such that

$$V^*V = p_{\overline{\operatorname{ran}(X^*)}} \qquad \text{and} \qquad VV^* = p_{\overline{\operatorname{ran}(X)}}.$$

Hence, it follows that

$$1 - V^*V = p_{\operatorname{ran}(X^*)^\perp} = p_{\ker(X)} \qquad \text{and} \qquad 1 - VV^* = p_{\operatorname{ran}(X)^\perp} = p_{\ker(X^*)},$$

from which we may deduce by traciality of τ that indeed

$$\tau(p_{\ker(X)}) = \tau(1 - V^*V) = \tau(1 - VV^*) = \tau(p_{\ker(X^*)}).$$

This concludes the proof. □

Combining Lemma VI.2.10 with Corollary VI.2.6 will provide us with the desired reduction argument. Before giving the precise statement, let us introduce some notation. If $p \in M$ is any projection, we define a linear mapping

$$\Delta_{p,j} : \ \mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}\langle x_1, \dots, x_n\rangle$$

for $j = 1, \dots, n$ by

$$\Delta_{p,j} P := (\tau \otimes \mathrm{id})\big(p(\mathrm{ev}_X \otimes \mathrm{id})(\partial_j P)\big)$$

for any $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$.

COROLLARY VI.2.11. *Let (M, τ) be a tracial W^*-probability space and let $X_1, \dots, X_n \in M$ be self-adjoint elements, which satisfy*

$$\hat{\delta}^\star(X_1, \dots, X_n) = n.$$

Moreover, let $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ and $w = w^* \in \mathrm{vN}(X_1, \dots, X_n)$ be given, such that $P(X)w = 0$ holds true. If $w \neq 0$, then there exists a projection $0 \neq p \in \mathrm{vN}(X_1, \dots, X_n)$ such that $(\Delta_{p,j}P)(X)w = 0$.

Proof. Since $P(X)w = 0$ and $w \neq 0$, we see that $\{0\} \neq \operatorname{ran}(w) \subseteq \ker(P(X))$, such that we also must have $\ker(P(X)^*) \neq \{0\}$ according to Lemma VI.2.10. The projection $p := p_{\ker(P(X)^*)} \in \mathrm{vN}(X_1, \dots, X_n)$ thus satisfies $p \neq 0$ and $P(X)^* p = 0$. Corollary VI.2.6 tells us that $p(\partial_j P)(X)w = 0$ for $j = 1, \dots, n$ holds true. Hence, we get that

$$(\Delta_{p,j} P)(X)w = (\tau \otimes \mathrm{id})\big(p(\partial_j P)(X)\big)w = (\tau \otimes \mathrm{id})\big(p(\partial_j P)(X)w\big) = 0,$$

which concludes the proof. □

Now, we are prepared to finish the proof of Theorem VI.2.1.

Proof of Theorem VI.2.1. Obviously, it suffices to show that, if any non-commutative polynomial $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ and $w \in \mathrm{vN}(X_1, \dots, X_n)$ with $w \neq 0$ are given such that $P(X)w = 0$, then $P = 0$ follows. By possibly replacing w by ww^*, we may assume in addition that $w = w^*$.

For proving $P = 0$, we proceed as follows. First, we write

$$P = a_0 + \sum_{k=1}^{d} \sum_{i_1,\dots,i_k=1}^{n} a_{i_1,\dots,i_k} x_{i_1} \dots x_{i_k}$$

and we assume that the total degree d of P satisfies $d \geq 1$. We choose then any summand of P of highest degree, which is non-zero, say $a_{i_1,\dots,i_d} x_{i_1} \dots x_{i_d}$. Iterating Corollary VI.2.11, we see that there are non-zero projections $p_1, \dots, p_d \in \mathrm{vN}(X_1, \dots, X_n)$ such that

$$(\Delta_{p_d,i_d} \dots \Delta_{p_1,i_1} P)(X)w = 0.$$

But since we can easily check that

$$(\Delta_{p_d,i_d} \dots \Delta_{p_1,i_1} P)(X) = \tau(p_d) \dots \tau(p_1) a_{i_1,\dots,i_d},$$

this leads us to $a_{i_1,\dots,i_d} = 0$, which contradicts our assumption. Thus, P must be constant, and since $w \neq 0$, we end up with $P = 0$. This concludes the proof of Theorem VI.2.1. □

We finish by noting that Theorem VI.2.1 yields now, with Proposition V.2.1 in mind, the following generalization of Theorem VI.1.5.

Corollary VI.2.12. *Let (M, τ) be a tracial W^*-probability space. Furthermore, let $X_1, \dots, X_n \in M$ be self-adjoint elements and assume that $\delta^*(X_1, \dots, X_n) = n$ holds. Then the following statements hold true:*

(a) *$X_1, \dots, X_n$ do not satisfy any non-trivial algebraic relation, i.e. there exists no non-zero polynomial $P \in \mathbb{C}\langle x_1, \dots, x_n\rangle$ such that $P(X_1, \dots, X_n) = 0$.*
(b) *For $j = 1, \dots, n$, there is a unique derivation*

$$\hat{\partial}_j : \mathbb{C}\langle X_1, \dots, X_n\rangle \to \mathbb{C}\langle X_1, \dots, X_n\rangle \otimes \mathbb{C}\langle X_1, \dots, X_n\rangle$$

which satisfies $\hat{\partial}_j(X_i) = \delta_{j,i} 1 \otimes 1$ for $i = 1, \dots, n$.

CHAPTER VII

Regularity of distributions of Wigner integrals

In 1998, P. Biane and R. Speicher established with their seminal work [**BS98**] a non-commutative counterpart of classical stochastic calculus and Malliavin calculus in the realm of free probability. In particular, they introduced there the so-called (multiple) Wigner integrals

$$I_n^S(f) = \int_{\mathbb{R}_+^n} f(t_1, \dots, t_n)\, dS_{t_1} \cdots dS_{t_n}$$

for $f \in L^2(\mathbb{R}_+^n)$ on $\mathbb{R}_+ = [0, \infty)$ as the free counterpart of the classical (multiple) Wiener-Itô integrals [**Wie38, Itô51, Itô52**]. Despite some clear peculiarities of these free objects, their construction proceeds to a great extend parallel to the classical case, roughly speaking by replacing the classical Brownian motion by its free relative $(S_t)_{t \geq 0}$. In analogy to the classical Wiener-Itô chaos, these Wigner integrals form the so-called Wigner chaos, which likewise enjoys many properties similar to the classical Wiener-Itô chaos; e.g. [**KNPS12**].

We point out that the increments of the free Brownian motion $(S_t)_{t \geq 0}$ carry the semicircular distribution as the free equivalent of the normal distribution from classical probability theory. It might seem strange at first sight that the nomenclature of Wigner integrals refers explicitly to Wigner, although his work clearly predates the birth of free stochastic calculus. However, this simply highlights the very important fact that the semicircular distribution already appeared in Wigner's famous semicircle law and that this rather surprising connection to random matrix theory, which was later clarified by Voiculescu, marks the starting point of an extremely fruitful interaction between random matrix theory and the theory of operator algebras.

Classical Malliavin calculus has many important applications (cf. [**Nua06, Nua09**]). In particular, it became prominent for its use in treating regularity questions in different situations, as e.g. for distributions of random variables in the Wiener-Itô chaos. For instance, it was used by Shigekawa [**Shi80**] for proving that any non-trivial element in the finite Wiener-Itô chaos, i.e. any non-constant finite sum of Wiener-Itô integrals, has a distribution which is absolutely continuous with respect to the Lebesgue measure.

In contrast, in the world of free probability, distributions of non-commutative random variables that appear in the Wigner chaos are poorly understood. The aim of this chapter, which is based on the paper [**Mai15**], is a first step towards a better understanding of these distributions by answering one of the fundamental questions formulated by Nourdin and Peccati in [**NP13**, Remark 1.6], namely: can the distribution of any non-constant self-adjoint Wigner integral have atoms or not? We will see that the answer to this question is no in full generality. Even more, we will show that the distribution of self-adjoint elements in the finite Wigner chaos, i.e. non-commutative random variables of the form

$$I_1^S(f_1) + I_2^S(f_2) + \cdots + I_N^S(f_N)$$

with mirror-symmetric $f_n \in L^2(\mathbb{R}_+^n)$ for $n = 1, \dots, N$ and $f_N \neq 0$, cannot have atoms. This is the content of of our main Theorem VII.1.4.

Although this result is clearly in accord with the classical result of Shigekawa [**Shi80**], the proof of Theorem VII.1.4 uses completely different methods. Shigekawa's approach is based on arguments which are specially adapted to the commutative setting. In fact, he uses Malliavin's Lemma, which is a powerful result that provides a sufficient condition for a measure on $\mathbb{R}^d$ to be absolutely continuous with respect to Lebesgue measure. The non-commutativity in our situation forces us therefore to follow a totally different strategy, which is inspired by recently developed methods [**MSW17, Shl14**].

In free probability, regularity questions of this type were successfully addressed only quite recently [**SS15, MSW14, Shl14, MSW17, CS16**]. Our considerations here are very much based on the paper [**MSW17**], where it was shown that in a von Neumann algebra M, which is endowed with a faithful normal tracial state τ, the distribution of any non-constant self-adjoint polynomial expression $P(X_1, \ldots, X_n)$ in finitely many self-adjoint variables $X_1, \ldots, X_n \in M$ does not have atoms if the so-called non-microstates free entropy dimension $\delta^*(X_1, \ldots, X_n)$ is maximal, i.e., if it satisfies $\delta^*(X_1, \ldots, X_n) = n$.

We note that the quantity $\delta^*(X_1, \ldots, X_n)$ has its origin among other important quantities in the work of Voiculescu. He transferred in a groundbreaking series of papers [**Voi93, Voi94, Voi96, Voi97, Voi98, Voi99**] (see also the survey article [**Voi02b**]) the classical notions of entropy and Fisher information to the non-commutative world. At the base of our work are techniques from the so-called non-microstates approach presented in [**Voi98, Voi99**].

Formulated in general terms, so that it can be applied in our situation, the method of [**MSW17**] works as follows:

(i) Rephrase the question of absence of atoms in more algebraic terms as a question about the absence of (certain) zero-divisors.
(ii) Prove that zero-divisors survive under special operations that are built on non-commutative derivations. This means that zero-divisors for some particular non-commutative random variable induce zero-divisors for some other non-commutative random variables of "lower degree", where the term "degree" refers to the degree of the considered polynomial, or in general to some natural grading on the space of non-commutative random variables under consideration.
(iii) Iterate the procedure of (ii) until reaching a non-commutative random variable of degree zero and check that the obtained element cannot be zero under the imposed conditions on the initial non-commutative random variable. This will lead to a contradiction and hence excludes zero-divisors.

It might be of independent interest that Step (i) establishes a very interesting relationship to the work of Linnell [**Lin91, Lin92, Lin93, Lin98**] on analytic versions of the zero divisor conjecture, particularly in the case of the free group. In fact, we will prove the more general statement that the product of any non-commutative random variable in the finite Wigner chaos, which is non-zero, with any non-zero element from the von Neumann algebra generated by the underlying free Brownian motion cannot be zero as well.

The crucial part is Step (ii), which relies in [**MSW17**] as well as in our considerations heavily on results of Dabrowski [**Dab10, Dab14**], concerning bounds for the non-commutative derivatives that underlie the non-microstates approach to free Fisher information and free entropy of [**Voi97**] and also for more general derivations.

In contrast to the preceding studies, which especially concern the case of finitely many variables, the underlying von Neumann algebra in the setting of Wigner integrals is generated by a free Brownian motion $(S_t)_{t\geq 0}$ and therefore by an uncountable family of semicircular elements, indexed by the continuous parameter $t \geq 0$. Accordingly, the role of non-commutative derivatives in [**MSW14, MSW17**] is taken over here by the directional gradient operators of free Malliavin calculus. Thus, the subsequent investigations can be seen as a continuous extension of the previous work [**SS15, MSW14, Shl14, MSW17, CS16**].

In [**MSW14**], which is an earlier version of [**MSW17**], the absence of atoms in the distribution of $P(X_1,\dots,X_n)$ for non-constant self-adjoint polynomials P was first shown under the stronger assumption of finite non-microstates free Fisher information $\Phi^*(X_1,\dots,X_n)$. Based on these ideas, Shlyakhtenko [**Shl14**] was able to prove a significant extension, namely to the most general case of full non-microstates entropy dimension $\delta^*(X_1,\dots,X_n)$, by involving different techniques from [**CS05**]. However, shortly after [**Shl14**], the authors of [**MSW14**] were also able to upgrade their own methods to this generality, which led to the final version [**MSW17**].

Deep results of Shlyakhtenko and Skoufranis [**SS15**] characterize the possible sizes of atoms that can appear in distributions of polynomial expressions $P(X_1,\dots,X_n)$ in non-commutative random variables $X_1,\dots,X_n$, which have not necessarily non-atomic distributions, (and even more matrices $(P_{ij}(X_1,\dots,X_n))_{i,j=1}^d$ thereof) under the assumption that $X_1,\dots,X_n$ are freely independent. Since the non-microstates free entropy is additive for freely independent variables and since in the case of a single self-adjoint variable X the maximality condition $\delta^*(X)=1$ holds if and only if the distribution of X has no atomic part, the results from [**MSW17, Shl14**] clearly generalize some parts of the statements given in [**SS15**]. However, the full range of regularity results presented in [**SS15**] is still out of reach in this generality, but nevertheless, one expects that indeed for most of these properties rather the maximality of the non-microstates free entropy dimension matters than the free independence of the involved variables.

We point out that certain questions concerning the non-singularity and absolute continuity of distributions were addressed recently by Charlesworth and Shlyakhtenko [**CS16**], in continuation of [**Shl14**].

This chapter is organized as follows. In Section VII.1, we state our main result Theorem VII.1.4 on the regularity of distributions of Wigner integrals. For reader's convenience, we recall there also the fundamental definition of a free Brownian motion and the construction of Wigner integrals, as it can be found in the seminal work [**BS98**]. This exposition of the foundations of free stochastic calculus will then be continued in Section VII.2. In particular, we will define there the main operators of free Malliavin calculus and collect some results from [**BS98**], which will be used later on.

Finally, in Section VII.3, we will piece together these ingredients for the actual proof of Theorem VII.1.4. For this purpose, we will introduce the notion of directional gradients. The proof itself relies then on the fact that directional gradients, which belong by definition to free Malliavin calculus as presented Section VII.2, fit also nicely into the general framework of non-commutative derivations as considered in Chapter V. Indeed, this will allow us to follow the aforementioned strategy in the spirit of [**MSW14, MSW17**].

VII.1. Wigner integrals and regularity of their distributions

In this section, we provide all basic terminology and background knowledge as far as it is needed for stating our main result, Theorem VII.1.4.

We will give the definition of a free Brownian motion and present the construction of free Wigner integrals as they were introduced by Biane and Speicher in [**BS98**]; see also [**Spe03**] and [**KNPS12**].

The introduction to free stochastic calculus will be continued later in Section VII.2.

Note that, as in Section V.3 of Chapter V, we will denote here the algebraic tensor product (over $\mathbb{C}$) by $\odot$, whereas the symbol $\otimes$ is reserved for all different kinds of closures of the algebraic tensor product.

VII.1.1. Free Brownian motion. Like the classical Brownian motion in the case of Wiener-Itô integrals, the free Brownian motion is the fundamental object in free stochastic analysis and underlies in particular the construction of Wigner integrals. Thus, we want to recall now its definition.

Note that the definition itself will reflect the important fact that the role of the normal distribution in classical probability is taken over in free probability by the semicircular distribution as its free counterpart. As introduced in Definition I.1.43, the *semicircular distribution with mean* 0 *and variance* $t > 0$ will be denoted by σ_t. Recall from Remark I.1.45 that $(\sigma_t)_{t\geq 0}$ forms a semi-group with respect to the free additive convolution, i.e. we have that $\sigma_s \boxplus \sigma_t = \sigma_{s+t}$ holds for all $s, t \geq 0$.

Definition VII.1.1. Let (M, τ) be a tracial W^*-probability space. A family $(S_t)_{t\geq 0}$ of operators in (M, τ) is called *free Brownian motion*, if there exists a *filtration* $(M_t)_{t\geq 0}$ *of* M, i.e. a family $(M_t)_{t\geq 0}$ of von Neumann subalgebras M_t of M with

$$M_s \subseteq M_t \qquad \text{whenever } s \leq t,$$

such that the following conditions are satisfied:

- We have $S_0 = 0$ and $S_t = S_t^* \in M_t$ for all $t \geq 0$.
- For each $t > 0$, the distribution of S_t is the semicircular distribution σ_t.
- For all $0 \leq s < t$, the distribution of $S_t - S_s$ is the semicircular distribution σ_{t-s}.
- For all $0 \leq s < t$, the *increment* $S_t - S_s$ is free from M_s, which means more precisely that the unital subalgebra generated by $S_t - S_s$ is free from M_s.

A free Brownian motion can be constructed in several ways. For instance, one construction gives the free Brownian motion as the limit of matrix-valued classical Brownian motions as the dimension tends to infinity. In contrast to this certainly appealing but rather indirect approach, we will present in Subsection VII.2.2 a construction of the free Brownian motion on the full Fock space over the Hilbert space $L^2(\mathbb{R}_+)$ of all square-integrable functions on the positive real half-line $\mathbb{R}_+ := [0, \infty)$. This has the advantage that it will not only prove the existence of the free Brownian motion but it will also give an additional structure to this important object, which is in fact the starting point of free Malliavin calculus. However, for the moment, we take the existence of a free Brownian motion for granted.

VII.1.2. Wigner integrals. Presuming the existence of a free Brownian motion $(S_t)_{t\geq 0}$ in a W^*-probability space (M, τ) with respect to a filtration $(M_t)_{t\geq 0}$ of M, we may introduce now (multiple) Wigner integrals integrals with respect to $(S_t)_{t\geq 0}$.

DEFINITION VII.1.2. Let $n \in \mathbb{N}$ be given. We denote by $D^n \subset \mathbb{R}_+^n$ the collection of all diagonals in $\mathbb{R}_+^n$, i.e.

$$D^n := \{(t_1, \dots, t_n) \in \mathbb{R}_+^n |\ t_i = t_j \text{ for some } 1 \leq i, j \leq n \text{ with } i \neq j\}.$$

The construction of the *(multiple) Wigner integral* $I_n^S(f)$ for any function $f \in L^2(\mathbb{R}_+^n)$ proceeds as follows.

- For any indicator function $f = 1_E$ of some set
$$E = [s_1, t_1] \times \cdots \times [s_n, t_n] \subset \mathbb{R}_+^n$$
that satisfies $E \cap D^n = \emptyset$, we define $I_n^S(f)$ by
$$I_n^S(f) = (S_{t_1} - S_{s_1}) \cdots (S_{t_n} - S_{s_n}).$$
- By linearity, we extend I_n^S to all *off-diagonal step functions*, i.e. to all step functions
$$f = \sum_{j=1}^m a_j 1_{E_j}$$
on $\mathbb{R}_+^n$, where each set $E_j \subset \mathbb{R}_+^n$ is of the form
$$E_j = [s_{j,1}, t_{j,1}] \times \cdots \times [s_{j,n}, t_{j,n}]$$
and satisfies $E_j \cap D^n = \emptyset$.
- Since off-diagonal step functions are dense in $L^2(\mathbb{R}_+^n)$ (an important fact, which is actually not hard to prove, but which is definitely worth to think about for a moment) and since the *Itô isometry*
$$\tau(I_n^S(f)^* I_n^S(g)) = \langle g, f \rangle_{L^2(\mathbb{R}_+^n)}$$
holds for all off-diagonal step functions f and g, we may finally extend I_n^S isometrically to $L^2(\mathbb{R}_+^n)$.

For given $f \in L^2(\mathbb{R}_+^n)$, we will write

$$I_n^S(f) = \int_{\mathbb{R}_+^n} f(t_1, \dots, t_n)\, dS_{t_1} \cdots dS_{t_n}.$$

Note that multiple Wigner integrals $I_n^S(f)$ are for general $f \in L^2(\mathbb{R}_+^n)$ by definition elements of $L^2(M, \tau)$. But in fact, it turns out that $I_n^S(f)$ belongs to M for each $f \in L^2(\mathbb{R}_+^n)$ (and actually, to be more precise, it belongs to the C^*-subalgebra of M that is generated by the free Brownian motion $(S_t)_{t\geq 0}$). This is an immediate consequence of the fact that off-diagonal step functions are dense in $L^2(\mathbb{R}_+^n)$ and of [**BS98**, Theorem 5.3.4], which tells us that the operator norm can be bounded by a kind of Haagerup inequality, namely

$$\text{(VII.1)} \qquad \Big\| \int_{\mathbb{R}_+^n} f(t_1, \dots, t_n)\, dS_{t_1} \cdots dS_{t_n} \Big\| \leq (n+1) \|f\|_{L^2(\mathbb{R}_+^n)} \qquad \text{for all } f \in L^2(\mathbb{R}_+^n).$$

Since Wigner integrals are bounded linear operators, we are of course allowed to multiply them, and it is therefore natural to ask, whether one can describe this operation also on the level of the corresponding functions. Indeed, this turns out to be possible and it leads to a free counterpart of *Itô's formula* (see, for example, [**Spe03**, Theorem 2.11]). Although this result appears in many different formulations, it always reflects the same inherent structure that shows up, roughly speaking, under multiplication. We mention here the following version, which allows us to decompose products of Wigner integrals explicitly as linear combinations of Wigner integrals.

Theorem VII.1.3 (Biane and Speicher, 1998, [**BS98**]). *Let* $f \in L^2(\mathbb{R}_+^n)$ *and* $g \in L^2(\mathbb{R}_+^m)$. *For any* $0 \leq p \leq \min\{n, m\}$, *we define the* p-th contraction of f and g *by*

$$f \overset{p}{\frown} g(t_1, \dots, t_{n+m-2p}) = \int_{\mathbb{R}_+^p} f(t_1, \dots, t_{n-p}, s_1, \dots, s_p)$$
$$g(s_p, \dots, s_1, t_{n-p+1}, \dots, t_{n+m-2p})\, ds_1 \dots ds_p.$$

Then the Itô formula

$$I_n^S(f) I_m^S(g) = \sum_{p=0}^{\min\{n,m\}} I_{n+m-2p}^S(f \overset{p}{\frown} g)$$

holds.

In principle, all previously collected facts about Wigner integrals put them in the most convenient setting of non-commutative probability, such that we can already talk about their (joint) distributions in a purely combinatorial sense. However, since we work here in the regular setting of W^*-probability spaces, we also want to study distributions of Wigner integrals in a stronger analytic sense, namely as (compactly supported) probability measures. Thus, we should have a criterion on the level of integrands that allows us to guarantee that the corresponding Wigner integral is self-adjoint. This criterion is provided by mirror symmetry.

It follows immediately from the definition of Wigner integrals that

$$I_n^S(f)^* = I_n^S(f^*) \qquad \text{for all } f \in L^2(\mathbb{R}_+^n)$$

holds, where the function $f^* \in L^2(\mathbb{R}_+^n)$ is determined for any $f \in L^2(\mathbb{R}_+^n)$ by

$$f^*(t_1, t_2, \dots, t_n) = \overline{f(t_n, \dots, t_2, t_1)}$$

for Lebesgue almost all $(t_1, \dots, t_n) \in \mathbb{R}_+^n$. As a consequence, any $f \in L^2(\mathbb{R}_+^n)$ satisfying $f = f^*$ gives a self-adjoint Wigner integral $I_n^S(f)$. We will call such $f \in L^2(\mathbb{R}_+^n)$ *mirror symmetric*.

VII.1.3. Main Theorem. Here, we are interested in properties of the distributions of Wigner integrals

$$I_n^S(f) = \int_{\mathbb{R}_+^n} f(t_1, \dots, t_n)\, dS_{t_1} \cdots dS_{t_n}$$

for mirror symmetric functions $f \in L^2(\mathbb{R}_+^n)$, and, more generally, in distributions of finite sums of such Wigner integrals like

$$Y = I_1^S(f_1) + I_2^S(f_2) + \cdots + I_N^S(f_N)$$

for some $N \in \mathbb{N}$ and mirror symmetric functions $f_n \in L^2(\mathbb{R}_+^n)$ for $n = 1, \dots, N$ with $f_N \neq 0$.

Surely one of the most basic questions one can ask about distributions in general is whether their support is connected or not. Basic functional analysis yields that this question can be reformulated in more operator algebraic terms to a question about the existence of non-trivial projections in the C^*-algebra that is generated by the considered operator. Fortunately, this translation is also helpful in our situation: As we have mentioned above, Wigner integrals are in fact elements of the C^*-algebra that is generated by the free Brownian motion $(S_t)_{t\geq 0}$. Hence, by quoting a results obtained by Guionnet and Shlyakhtenko in [**GS09**], which excludes non-trivial projections in $C^*(\{S_t|\ t \geq 0\})$, we may conclude without further effort that the distribution μ_Y of any operator Y as above must have connected support.

However, apart from this observation, almost nothing was known until now about regularity properties of these distributions. In particular, as it was formulated by Nourdin and Peccati in [**NP13**, Remark 1.6], it remained an open questions whether the distribution of Wigner integrals of mirror symmetric functions being non-zero (except, of course, in the chaos of order zero) may have atoms or not. We are going to answer this question here by showing that the distribution of any such Wigner integral of a non-zero mirror symmetric function (and even of any non-constant finite sum of such Wigner integrals) does not have atoms.

Recall that an *atom* of a Borel probability measure μ on $\mathbb{R}$ means some $\alpha \in \mathbb{R}$ satisfying the condition $\mu(\{\alpha\}) \neq 0$.

The statement of the main theorem of this chapter reads as follows.

Theorem VII.1.4. *For given $N \in \mathbb{N}$, we consider mirror symmetric functions $f_n \in L^2(\mathbb{R}_+^n)$ for $n = 1, \dots, N$, where we assume that $f_N \neq 0$. Then, the distribution μ_Y of*

$$Y := I_1^S(f_1) + I_2^S(f_2) + \cdots + I_N^S(f_N),$$

regarded as an element in (M, τ), has no atoms.

The proof of Theorem VII.1.4 will be given in Section VII.3. We stress that the above statement clearly stays valid if we add to Y a constant multiple of the identity. In fact, this will be a direct outcome of the proof of Theorem VII.1.4, since we will use the chaos decomposition to deal with such shifts in a uniform way. More precisely, we can just encode constant multiples of the identity by the chaos of order zero.

Furthermore, we point out that Theorem VII.1.4 corresponds nicely to a classical result of Shigekawa [**Shi78, Shi80**] (although its proof uses completely different methods for which there are by now no free analogues), which states that any non-trivial finite sum of Wiener-Itô integrals has an absolutely continuous distribution, and hence cannot have atoms. Thus, confident of the far reaching parallelism between classical and free probability, we are tempted to conjecture in accordance with [**Spe13**] that the analogy between Wiener-Itô integrals and Wigner integrals goes even further, namely that any Y like in Theorem VII.1.4 has in fact an absolutely continuous distribution. We leave this question to further investigations.

VII.2. Free stochastic calculus

One of the main pillars on which the proof of Theorem VII.1.4 rests is free stochastic calculus as it was introduced by Biane and Speicher in [**BS98**]. For readers convenience, we recall in this section the basic definitions and some results of this theory as far as necessary.

First of all, we will introduce the notion of biprocesses. Secondly, we will describe the concrete realization of the free Brownian motion on the full Fock space over $L^2(\mathbb{R}_+)$. This additional structure will finally allow us to introduce the basic operators of Malliavin calculus.

VII.2.1. Biprocesses. We broach now the theory of biprocesses. Our exposition here heavily relies on [**BS98**], [**Spe03**], and [**KNPS12**].

Let us first introduce a few general notions. We denote by $\mathcal{E}(\mathbb{R}_+)$ the space of all complex valued functions f on $\mathbb{R}_+$, which can be written as a finite sum

$$f = \sum_{j=1}^{n} a_j \, 1_{E_j}$$

for some intervals $E_1, \ldots, E_n \subseteq \mathbb{R}_+$ of the form $E_j = [s_j, t_j)$ with $0 \leq s_j < t_j < \infty$ for $j = 1, \ldots, n$ and complex numbers $a_1, \ldots, a_n \in \mathbb{C}$. As usually, 1_E denotes the indicator function of a subset $E \subseteq \mathbb{R}_+$. It is easy to see that $\mathcal{E}(\mathbb{R}_+)$ is in fact a complex algebra.

For any unital complex algebra $\mathcal{A}$, the algebraic tensor product $\mathcal{E}(\mathbb{R}_+, \mathcal{A}) := \mathcal{E}(\mathbb{R}_+) \odot \mathcal{A}$ consists of all functions f defined on $\mathbb{R}_+$ and taking values in $\mathcal{A}$, which can be written as

$$f = \sum_{j=1}^{n} A_j \, 1_{E_j}$$

for some intervals $E_1, \ldots, E_n \subseteq \mathbb{R}_+$ of the form $E_j = [s_j, t_j)$ with $0 \leq s_j < t_j < \infty$ for $j = 1, \ldots, n$ and elements $A_1, \ldots, A_n \in \mathcal{A}$.

VII.2.1.1. *Definition of biprocesses.* We are prepared now to define biprocesses. For the remaining part of this subsection, we fix a tracial W^*-probability space (M, τ) for which a filtration $(M_t)_{t \geq 0}$ exists.

Definition VII.2.1. We distinguish several types of biprocesses, which are built on each other. Their definition proceeds as follows:

(i) The elements
$$U : \ \mathbb{R}_+ \to M \odot M, \ t \mapsto U_t$$
of $\mathcal{E}(\mathbb{R}_+, M \odot M)$ are called *simple biprocesses*.

(ii) A simple biprocess $U : \mathbb{R}_+ \to M \odot M$ is called *adapted*, if the condition $U_t \in M_t \odot M_t$ is satisfied for all $t \geq 0$. The set of all adapted simple biprocesses will be denoted by $\mathcal{E}^a(\mathbb{R}_+, M \odot M)$.

(iii) We denote by $\mathcal{B}_p$ for $1 \leq p \leq \infty$ the completion of $\mathcal{E}(\mathbb{R}_+, M \odot M)$, with respect to the norm $\| \cdot \|_{\mathcal{B}_p}$, which is given by
$$\|U\|_{\mathcal{B}_p} := \left(\int_{\mathbb{R}_+} \|U_t\|^2_{L^p(M \otimes M, \tau \otimes \tau)} \, dt \right)^{\frac{1}{2}}.$$
An element of $\mathcal{B}_p$ is called an *L^p-biprocess*.

(iv) For $1 \leq p \leq \infty$, the closure of $\mathcal{E}^a(\mathbb{R}_+, M \odot M)$ with respect to $\| \cdot \|_{\mathcal{B}_p}$ will be denoted by $\mathcal{B}^a_p$. Elements of $\mathcal{B}^a_p$ are called *adapted L^p-biprocesses*.

VII.2.1.2. *Integration of biprocesses.* For our purposes, the integration theory of biprocesses is of great importance. We focus here first on the integration of L^p-biprocesses with respect to functions in $L^2(\mathbb{R}_+)$.

On the basic level of simple biprocesses, such integrals can be introduced quite easily: if U is any simple biprocess, we may write

$$U = \sum_{j=1}^{n} U^{(j)} \, 1_{E_j} \tag{VII.2}$$

for some intervals $E_1, \dots, E_n \subseteq \mathbb{R}_+$ of the form $E_j = [s_j, t_j)$ with $0 \leq s_j < t_j < \infty$ for $j = 1, \dots, n$ and certain elements $U^{(1)}, \dots, U^{(n)} \in M \odot M$. Then, we put

$$\int_{\mathbb{R}_+} U_t \, \overline{h(t)} \, dt := \sum_{j=1}^{n} \langle 1_{E_j}, h \rangle_{L^2(\mathbb{R}_+)} \, U^{(j)},$$

and it is easy to see that the value of this integral does not depend on the concrete choice of the representation (VII.2).

Sometimes, it is more appropriate to write a given simple biprocess U in *standard form*, i.e. in the form of (VII.2), where the intervals $E_1, \dots, E_n \subseteq \mathbb{R}_+$ are assumed to be pairwise disjoint.

By the construction presented above, we obtain a sesqui-linear pairing

$$\langle \cdot, \cdot \rangle : \ \mathcal{E}(\mathbb{R}_+, M \odot M) \times L^2(\mathbb{R}_+) \to M \odot M, \qquad \langle U, h \rangle := \int_{\mathbb{R}_+} U_t \, \overline{h(t)} \, dt.$$

Since we want to extend $\langle \cdot, \cdot \rangle$ to a sesqui-linear paring between $\mathcal{B}_p$ and $L^2(\mathbb{R}_+)$, we need to study its continuity with respect to $\| \cdot \|_{\mathcal{B}_p}$. This will be done in the following lemma. In the case $p = \infty$, this property of $\langle \cdot, \cdot \rangle$ was already mentioned in [**BS98**]. The general case is probably also well-known to experts, but for the seek of completeness, we include here the straightforward proof.

Lemma VII.2.2. *Let $1 \leq p \leq \infty$ be given. For any $U \in \mathcal{E}(\mathbb{R}_+, M \odot M)$ and $h \in L^2(\mathbb{R}_+)$, it holds true that*

$$\|\langle U, h \rangle\|_{L^p(M \otimes M, \tau \otimes \tau)} \leq \|U\|_{\mathcal{B}_p} \|h\|_{L^2(\mathbb{R}_+)}.$$

Proof. Take $U \in \mathcal{E}(\mathbb{R}_+, M \odot M)$ and $h \in L^2(\mathbb{R}_+)$ and write U in standard form $U = \sum_{j=1}^{n} U^{(j)} \, 1_{E_j}$. For any fixed $1 \leq p \leq \infty$, we may check that

$$\|U\|_{\mathcal{B}_p} = \Big(\int_{\mathbb{R}_+} \|U_t\|^2_{L^p(M \otimes M, \tau \otimes \tau)}, dt \Big)^{\frac{1}{2}} = \Big(\sum_{j=1}^{n} \lambda^1(E_j) \|U^{(j)}\|^2_{L^p(M \otimes M, \tau \otimes \tau)} \Big)^{\frac{1}{2}},$$

where λ^1 denotes the Lebesgue measure on $\mathbb{R}$. Thus, applying twice the Cauchy-Schwarz inequality yields as desired

$$\begin{aligned}
&\|\langle U, h \rangle\|_{L^p(M \otimes M, \tau \otimes \tau)} \\
&\quad \leq \sum_{j=1}^{n} |\langle 1_{E_j}, h \rangle_{L^2(\mathbb{R}_+)}| \, \|U^{(j)}\|_{L^p(M \otimes M, \tau \otimes \tau)} \\
&\quad = \sum_{j=1}^{n} |\langle 1_{E_j}, 1_{E_j} h \rangle_{L^2(\mathbb{R}_+)}| \, \|U^{(j)}\|_{L^p(M \otimes M, \tau \otimes \tau)} \\
&\quad \leq \sum_{j=1}^{n} \|1_{E_j} h\|_{L^2(\mathbb{R}_+)} \|1_{E_j}\|_{L^2(\mathbb{R}_+)} \|U^{(j)}\|_{L^p(M \otimes M, \tau \otimes \tau)} \\
&\quad \leq \Big(\sum_{j=1}^{n} \|1_{E_j} h\|^2_{L^2(\mathbb{R}_+)} \Big)^{\frac{1}{2}} \Big(\sum_{j=1}^{n} \|1_{E_j}\|^2_{L^2(\mathbb{R}_+)} \|U^{(j)}\|^2_{L^p(M \otimes M, \tau \otimes \tau)} \Big)^{\frac{1}{2}} \\
&\quad \leq \|h\|_{L^2(\mathbb{R}_+)} \|U\|_{\mathcal{B}_p},
\end{aligned}$$

where we used in addition that due to the orthogonality of $\{1_{E_j}h|\ j=1,\dots,n\}$

$$\Big(\sum_{j=1}^{n}\|1_{E_j}h\|^2_{L^2(\mathbb{R}_+)}\Big)^{\frac{1}{2}} \leq \|h\|_{L^2(\mathbb{R}_+)}$$

holds and that we have $\|1_E\|^2_{L^2(\mathbb{R}_+)} = \lambda^1(E)$ for any Borel set $E \subseteq \mathbb{R}_+$ with finite Lebesgue measure. □

Due to the inequality that we have established in Lemma VII.2.2, the definition of $\langle\cdot,\cdot\rangle$ extends now naturally to $\mathcal{B}_p$.

Definition VII.2.3. For any $1 \leq p \leq \infty$, the sesqui-linear pairing

$$\langle\cdot,\cdot\rangle:\ \mathcal{E}(\mathbb{R}_+, M\odot M)\times L^2(\mathbb{R}_+)\to M\odot M,\qquad \langle U,h\rangle = \int_{\mathbb{R}_+} U_t\overline{h(t)}\,dt,$$

extends continuously according to

$$\|\langle U,h\rangle\|_{L^p(M\otimes M,\tau\otimes\tau)} \leq \|U\|_{\mathcal{B}_p}\|h\|_{L^2(\mathbb{R}_+)}.$$

to a sesqui-linear pairing

$$\langle\cdot,\cdot\rangle:\ \mathcal{B}_p\times L^2(\mathbb{R}_+)\to L^p(M\otimes M,\tau\otimes\tau).$$

VII.2.1.3. *Stochastic integrals of biprocesses.* Next, we are going to define stochastic integrals $\int_{\mathbb{R}_+} U_t\sharp dS_t$ of biprocesses U with respect to the free Brownian motion $(S_t)_{t\geq 0}$. For this purpose, we will use again the $\sharp$-notation, which was introduced in Remark V.1.4.

Definition VII.2.4. Let $(S_t)_{t\geq0}$ be a free Brownian motion in M with respect to its given filtration $(M_t)_{t\geq 0}$.

- For any simple biprocess $U\in\mathcal{E}(\mathbb{R}_+, M\odot M)$, we define

$$\int_{\mathbb{R}_+} U_t\sharp dS_t := \sum_{j=1}^{n} U^{(j)}\sharp(S_{t_j}-S_{s_j}) = \sum_{j=1}^{n}\sum_{i=1}^{m_j} A_i^{(j)}(S_{t_j}-S_{s_j})B_i^{(j)},$$

  where U is written in the form (VII.2) for intervals $E_j=[s_j,t_j)$ with $0\leq s_j < t_j < \infty$ and elements $U^{(j)}\in M\odot M$ of the form $U^{(j)}=\sum_{i=1}^{m_j} A_i^{(j)}\otimes B_i^{(j)}$ for $j=1,\dots,n$.
- If $U,V\in\mathcal{E}^a(\mathbb{R}_+, M\odot M)$ are simple adapted biprocesses, then the *general Wigner-Itô isometry* (cf. [**Spe03**, Proposition 2.7]) tells us that

$$\langle\int_{\mathbb{R}_+} U_t\sharp dS_t, \int_{\mathbb{R}_+} V_t\sharp dS_t\rangle = \int_{\mathbb{R}_+}\langle U_t,V_t\rangle\,dt =: \langle U,V\rangle_{\mathcal{B}_2}$$

  holds. Thus, we have in particular that

$$\Big\|\int_{\mathbb{R}_+} U_t\sharp dS_t\Big\|_2 = \|U\|_{\mathcal{B}_2}$$

  for all $U\in\mathcal{E}^a(\mathbb{R}_+, M\odot M)$. Therefore, the integral $\int_{\mathbb{R}_+} U_t\sharp dS_t$ extends from simple adapted biprocesses to any adapted L^2-biprocess $U\in\mathcal{B}_2^a$ in such a way that the induced mapping

$$U\mapsto\int_{\mathbb{R}_+} U_t\sharp dS_t$$

  is isometric from $\mathcal{B}_2^a$ to $L^2(M,\tau)$.

VII.2.2. The free Brownian motion on the full Fock space. We come back now to the construction of the free Brownian motion. As we announced earlier, we will do this here in an explicit way on the full Fock space over $L^2(\mathbb{R}_+)$. These techniques will be used to build up free Malliavin calculus, in the same way as classical Malliavin calculus is built on the symmetric Fock space.

VII.2.2.1. *The full Fock space and field operators.* We first recall the construction of the full Fock space over an arbitrary complex Hilbert space.

Recall that in the context of complex Hilbert spaces, the symbol $\odot$ stands for the algebraic tensor product (over the complex numbers $\mathbb{C}$), whereas its completion with respect to the canonical inner product will be denoted by $\otimes$.

Definition VII.2.5. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a complex Hilbert space. We define the *full Fock space* $\mathcal{F}(H)$ associated to H as the complex Hilbert space that is given by

$$\mathcal{F}(H) := \bigoplus_{n=0}^{\infty} H^{\otimes n},$$

where $\bigoplus$ is understood as Hilbert space operation. Therein, we declare that $H^{\otimes 0} := \mathbb{C}\Omega$ for some fixed vector Ω of norm 1, which we call the *vacuum vector* of $\mathcal{F}(H)$.

More explicitly, the inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{F}(H)$ is determined by the following rules: We have

$$\langle g_1 \otimes \cdots \otimes g_m, h_1 \otimes \cdots \otimes h_n\rangle = 0 \qquad \text{if } m \neq n$$

and in the case $m = n$

$$\langle g_1 \otimes \cdots \otimes g_m, h_1 \otimes \cdots \otimes h_m\rangle = \langle g_1, h_1\rangle_H \cdots \langle g_m, h_m\rangle_H.$$

Later on, we will also work with some special (non-closed) subspaces of the full Fock space $\mathcal{F}(H)$, involving an infinite but algebraic direct sum, namely

- $\mathcal{F}_{\text{alg}}(H) := \bigoplus_{n=0}^{\infty}{}_{\text{alg}} H^{\odot n}$, i.e. the subspace of $\mathcal{F}(H)$ that consists of finite sums of tensor products of vectors in H, and
- $\mathcal{F}_{\text{fin}}(H) := \bigoplus_{n=0}^{\infty}{}_{\text{alg}} H^{\otimes n}$, i.e. the subspace of $\mathcal{F}(H)$ that consists of finite sums of elements in the Hilbert spaces $H^{\otimes n}$.

It is clear by definition that we have the inclusions $\mathcal{F}_{\text{alg}}(H) \subseteq \mathcal{F}_{\text{fin}}(H) \subseteq \mathcal{F}(H)$ and that both subspaces $\mathcal{F}_{\text{alg}}(H)$ and $\mathcal{F}_{\text{fin}}(H)$ are dense in $\mathcal{F}(H)$.

On the full Fock space $\mathcal{F}(H)$, we may introduce the so-called field operators. In the case $H = L^2(\mathbb{R}_+)$, these operators will provide the desired realization of the free Brownian motion.

Definition VII.2.6. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a complex Hilbert space. For each $h \in H$ we introduce the following operators on the full Fock space $\mathcal{F}(H)$ over H:

(i) The *creation operator* $l(h) \in B(\mathcal{F}(H))$ is determined by

$$\begin{aligned} l(h)\, h_1 \otimes \cdots \otimes h_n &= h \otimes h_1 \otimes \cdots \otimes h_n, \\ l(h)\,\Omega &= h. \end{aligned}$$

(ii) The *annihilation operator* $l^*(h) \in B(\mathcal{F}(H))$ is given by

$$\begin{aligned} l^*(h)\, h_1 \otimes \cdots \otimes h_n &= \langle h, h_1 \rangle_H h_2 \otimes \cdots \otimes h_n, \qquad n \geq 2, \\ l^*(h)\, h_1 &= \langle h, h_1 \rangle_H \Omega, \\ l^*(h)\, \Omega &= 0. \end{aligned}$$

(iii) The *field operator* $X(h) \in B(\mathcal{F}(H))$ is defined by

$$X(h) := l(h) + l^*(h).$$

An easy calculation shows that we have $l^*(h) = l(h)^*$ for all $h \in H$, as the notation suggests. As an immediate consequence, $X(h) = X(h)^*$ holds for each $h \in H$.

In order to obtain a W^*-probability space, in which the free Brownian lives, it is natural to consider the von Neumann algebra generated by field operators $X(h)$ for a sufficiently large family of vectors h. As it turns out, the right choice for this purpose are the "real" vectors h. More formally, we will consider the full Fock space over the *complexification* $H_{\mathbb{C}} = H \oplus iH$ of any real Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$. The "real" vectors are then naturally those, which are coming from H. We shall make this more precise with the following definition.

Definition VII.2.7. Let H be a real Hilbert space and denote by $H_{\mathbb{C}} = H \oplus iH$ its complexification. We define the von Neumann algebra $\mathcal{S}(H) \subseteq B(\mathcal{F}(H_{\mathbb{C}}))$ by

$$\mathcal{S}(H) = \mathrm{vN}\big(\{X(h) |\ h \in H\}\big).$$

We may endow $\mathcal{S}(H)$ with the *vacuum expectation* $\tau : \mathcal{S}(H) \to \mathbb{C}$ given by

$$\tau(X) = \langle X\Omega, \Omega \rangle.$$

Due to the fact that H is a real Hilbert space, we are in the nice situation that τ gives a faithful normal tracial state on $\mathcal{S}(H)$. Thus, we have obtained a W^*-probability space $(\mathcal{S}(H), \tau)$.

Later on, we will also use the unital $*$-algebra $\mathcal{S}_{\mathrm{alg}}(H)$ that is given by

$$\mathcal{S}_{\mathrm{alg}}(H) := \mathrm{alg}\big(\{X(h) |\ h \in H\}\big).$$

Clearly, $\mathcal{S}_{\mathrm{alg}}(H) \subseteq \mathcal{S}(H) \subseteq B(\mathcal{F}(H_{\mathbb{C}}))$.

It is a very nice feature of $(\mathcal{S}(H), \tau)$ that its L^2-space $L^2(\mathcal{S}(H), \tau)$ can be identified in a natural way with the corresponding full Fock space $\mathcal{F}(H_{\mathbb{C}})$. This important observation is at the base of free Malliavin calculus.

Since we have for all $X_1, X_2 \in \mathcal{S}(H)$ that

$$\langle X_1, X_2 \rangle_{L^2(\mathcal{S}(H),\tau)} = \tau(X_2^* X_1) = \langle (X_2^* X_1)\Omega, \Omega \rangle_{\mathcal{F}(H)} = \langle X_1\Omega, X_2\Omega \rangle_{\mathcal{F}(H)},$$

we see that the map

$$\Phi_0 :\ \mathcal{S}(H) \to \mathcal{F}(H_{\mathbb{C}}),\ X \mapsto X\Omega$$

admits an isometric extension

$$\Phi :\ L^2(\mathcal{S}(H), \tau) \to \mathcal{F}(H_{\mathbb{C}}).$$

The following lemma allows us to conclude that Φ is even more surjective and hence gives the desired isometric isomorphism between $L^2(\mathcal{S}(H), \tau)$ and $\mathcal{F}(H_{\mathbb{C}})$. A proof can be found in [**BS98**, Section 5.1].

LEMMA VII.2.8. *Given $h_1, \dots, h_n \in H_{\mathbb{C}}$, then there exists a unique operator*

$$W(h_1 \otimes \dots \otimes h_n) \in \mathcal{S}(H),$$

called the Wick product of $h_1 \otimes \dots \otimes h_n$, *such that*

$$W(h_1 \otimes \dots \otimes h_n)\Omega = h_1 \otimes \dots \otimes h_n.$$

More precisely, if $(e_j)_{j \in J}$ is an orthonormal basis of H then

$$W(e_{j_1}^{\otimes k_1} \otimes \dots \otimes e_{j_n}^{\otimes k_n}) = U_{k_1}(X(e_{j_1})) \cdots U_{k_n}(X(e_{j_n})),$$

where $j_1 \neq j_2 \neq \dots \neq j_n$ and U_k denotes the k'th (normalized) Chebyshev polynomial of the second kind. *These polynomials are determined by $U_0(X) = 1$, $U_1(X) = X$ and the recursion $U_{k+1}(X) = XU_k(X) - U_{k-1}(X)$ for $k \geq 1$.*

Note that the lemma implies in particular that $\Phi_0(\mathcal{S}_{\text{alg}}(H)) = \mathcal{F}_{\text{alg}}(H)$.

VII.2.2.2. *$\mathcal{F}(L^2(\mathbb{R}_+))$ and the free Brownian motion.* We return now to the actual goal of this subsection, namely the construction of the free Brownian motion. This is achieved by applying the foregoing constructions to the real Hilbert $H = L^2(\mathbb{R}_+, \mathbb{R})$, whose complexification is clearly given by $H_{\mathbb{C}} \cong L^2(\mathbb{R}_+)$.

In the W^*-probability space $(\mathcal{S}, \tau)$ where we abbreviate $\mathcal{S} := \mathcal{S}(L^2(\mathbb{R}_+, \mathbb{R}))$, the free Brownian motion $(S_t)_{t \geq 0}$ is obtained by putting

$$S_t := X(1_{[0,t]}) \qquad \text{for all } t \geq 0.$$

The corresponding filtration $(\mathcal{S}_t)_{t \geq 0}$ of $\mathcal{S}$ is given by

$$\mathcal{S}_t := \text{vN}\left(\{X(h) |\ h \in L^2([0,t], \mathbb{R})\}\right),$$

where we regard $L^2([0,t], \mathbb{R})$ as a subspace of $L^2(\mathbb{R}_+, \mathbb{R})$ via extension by zero. In fact, $\mathcal{S}_t$ is generated as a von Neumann algebra by $\{S_s |\ 0 \leq s \leq t\}$, while $\mathcal{S}$ is generated by $\{S_s |\ s \geq 0\}$.

The very concrete realization of the free Brownian motion in the W^*-probability space $(\mathcal{S}, \tau)$ has the advantage that it carries the rich structure provided by the underlying Fock space $\mathcal{F} := \mathcal{F}(L^2(\mathbb{R}_+))$ by the isometric isomorphism

$$\Phi : \ L^2(\mathcal{S}, \tau) \to \mathcal{F},$$

which was obtained by isometric extension of the map $\Phi_0 : \mathcal{S} \to \mathcal{F}$ given by $\Phi_0(X) = X\Omega$. This will be used in the next subsection on free Malliavin calculus.

But before continuing in this direction, we first discuss the chaos decomposition for arbitrary elements in $L^2(\mathcal{S}, \tau)$, which emerges from the isomorphism Φ. In the simplest case, it boils down to a nice relation between Wigner integrals and the Wick products as introduced in Lemma VII.2.8. More precisely, we have for all $h_1, \dots, h_n \in L^2(\mathbb{R}_+)$ that

$$W(h_1 \otimes \dots \otimes h_n) = I_n^S(h_1 \otimes \dots \otimes h_n) = \int_{\mathbb{R}_+^n} h_1(t_1) \cdots h_n(t_n) \, dS_{t_1} \cdots dS_{t_n}.$$

This observation is generalized by the following result.

PROPOSITION VII.2.9 (Proposition 5.3.2. in [**BS98**]). *The inverse of the isomorphism $\Phi : L^2(\mathcal{S}, \tau) \to \mathcal{F}$ is given by*

$$I^S : \ \mathcal{F} \to L^2(\mathcal{S}, \tau), \ f \mapsto I^S(f),$$

where

$$I^S(f) := \sum_{n=0}^{\infty} I_n^S(f_n)$$

for any

$$f = (f_n)_{n=0}^{\infty} \in \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+^n) \cong \mathcal{F}.$$

This means that each element of $L^2(\mathcal{S},\tau)$ has a unique representation in the form $I^S(f)$ for some $f \in \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}_+^n)$, to which we refer as its *chaos decomposition*.

There is a similar decomposition for L^2-biprocesses. Since the mapping $I^S : \mathcal{F} \to L^2(\mathcal{S},\tau)$ gives rise to an isometric isomorphism

$$I^S \otimes I^S : \ \mathcal{F} \otimes \mathcal{F} \to L^2(\mathcal{S},\tau) \otimes L^2(\mathcal{S},\tau),$$

we see, by using the natural isometric identifications

$$L^2(\mathbb{R}_+, \mathcal{F} \otimes \mathcal{F}) \cong \mathcal{F} \otimes L^2(\mathbb{R}_+) \otimes \mathcal{F}$$

and

$$L^2(\mathbb{R}_+, L^2(\mathcal{S},\tau) \otimes L^2(\mathcal{S},\tau)) \cong L^2(\mathcal{S},\tau) \otimes L^2(\mathbb{R}_+) \otimes L^2(\mathcal{S},\tau) \cong \mathcal{B}_2,$$

that $I^S \otimes I^S$ induces an isometric isomorphism

$$I^S \otimes I^S : \ L^2(\mathbb{R}_+, \mathcal{F} \otimes \mathcal{F}) \to \mathcal{B}_2,$$

which is again denoted by $I^S \otimes I^S$. More explicitly, this induced isomorphism sends each $f : \mathbb{R}_+ \to \mathcal{F} \otimes \mathcal{F}, t \mapsto f_t$ that belongs to $L^2(\mathbb{R}_+, \mathcal{F} \otimes \mathcal{F})$ to the L^2-biprocess that is given by $t \mapsto (I^S \otimes I^S)(f_t)$.

The following diagram offers a clear view on the situation described above.

$$\begin{array}{ccc} \mathcal{F} \otimes L^2(\mathbb{R}_+) \otimes \mathcal{F} & \xrightarrow{\ \cong\ } & L^2(\mathbb{R}_+, \mathcal{F} \otimes \mathcal{F}) \\ \downarrow {\scriptstyle I^S \otimes \mathrm{id} \otimes I^S} & & \downarrow {\scriptstyle I^S \otimes I^S} \\ L^2(\mathcal{S},\tau) \otimes L^2(\mathbb{R}_+) \otimes L^2(\mathcal{S},\tau) & \xrightarrow{\ \cong\ } & \mathcal{B}_2 \end{array}$$

We call $U = (I^S \otimes I^S)(f)$ for $f \in L^2(\mathbb{R}_+, \mathcal{F} \otimes \mathcal{F})$ the *Wigner chaos expansion of the L^2-biprocess U.*

VII.2.3. Free Malliavin calculus. Like in the classical case, the basic operators of free Malliavin calculus are constructed first on the side of the full Fock space and are then transferred to the algebra of field operators via the identification that is provided by the map $X \mapsto X\Omega$.

VII.2.3.1. *Free Malliavin calculus on $\mathcal{F}(H)$.* As above in the construction of the free Brownian motion, we begin with the general case of an arbitrary complex Hilbert space H. On the full Fock space $\mathcal{F}(H)$ over H, we consider

- an unbounded linear operator
$$\tilde{\nabla}:\ \mathcal{F}(H) \supseteq D(\tilde{\nabla}) \to \mathcal{F}(H) \otimes H \otimes \mathcal{F}(H)$$
with domain $D(\tilde{\nabla}) = \mathcal{F}_{\mathrm{alg}}(H)$, which is determined by the conditions $\tilde{\nabla}\Omega = 0$ and
$$\tilde{\nabla}(h_1 \otimes \cdots \otimes h_n) := \sum_{j=1}^{n} (h_1 \otimes \cdots \otimes h_{j-1}) \otimes h_j \otimes (h_{j+1} \otimes \cdots \otimes h_n),$$
where the tensor products appearing in the brackets are understood as Ω if the corresponding set of indices happens to be empty.
- an unbounded linear operator
$$\tilde{\delta}:\ \mathcal{F}(H) \otimes H \otimes \mathcal{F}(H) \supseteq D(\delta) \to \mathcal{F}(H)$$
with domain $D(\tilde{\delta}) = \mathcal{F}_{\mathrm{alg}}(H) \odot H \odot \mathcal{F}_{\mathrm{alg}}(H)$ by linear extension of
$$\begin{aligned} \tilde{\delta}((h_1 \otimes \cdots \otimes h_n) \otimes h \otimes (g_1 \otimes \cdots \otimes g_m)) &:= h_1 \otimes \cdots \otimes h_n \otimes h \otimes g_1 \otimes \cdots \otimes g_m, \\ \tilde{\delta}(\Omega \otimes h \otimes (g_1 \otimes \cdots \otimes g_m)) &:= h \otimes g_1 \otimes \cdots \otimes g_m, \\ \tilde{\delta}((h_1 \otimes \cdots \otimes h_n) \otimes h \otimes \Omega) &:= h_1 \otimes \cdots \otimes h_n \otimes h, \\ \tilde{\delta}(\Omega \otimes h \otimes \Omega) &:= h \end{aligned}$$
- an unbounded linear operator
$$\tilde{N}:\ \mathcal{F}(\mathcal{H}) \supseteq D(N) \to \mathcal{F}(\mathcal{H})$$
with domain $D(\tilde{N}) = \mathcal{F}_{\mathrm{alg}}(H)$, which is defined by $\tilde{N}\Omega = 0$ and
$$\tilde{N}(h_1 \otimes \cdots \otimes h_n) := n\, h_1 \otimes \cdots \otimes h_n.$$

We collect now a few observations related to the operators $\tilde{\nabla}$ and $\tilde{\delta}$. We grant that some of these statements might appear quite artificial at the first sight, but there actual meaning will become clear after passing from the Fock space to operators defined on it.

Remark VII.2.10. Consider the setting that was described above.

(a) A straightforward calculation shows that

(VII.3)
$$\langle \tilde{\nabla} y, u\rangle_{\mathcal{F}(H)\otimes H\otimes\mathcal{F}(H)} = \langle y, \tilde{\delta}(u)\rangle_{\mathcal{F}(H)}$$

holds for all $y \in D(\tilde{\nabla})$ and $u \in D(\tilde{\delta})$.

(b) If we endow $\mathcal{F}_{\mathrm{alg}}(H)$ with the multiplication induced by the tensor product $\otimes$ (in fact, we obtain in this way the tensor algebra over H), we may easily check that $\tilde{\nabla}$ satisfies a kind of product rule, namely

(VII.4)
$$\tilde{\nabla}(y_1 \otimes y_2) = (\tilde{\nabla} y_1) \cdot y_2 + y_1 \cdot (\tilde{\nabla} y_2)$$

for all $y_1, y_2 \in D(\tilde{\nabla})$, where $\cdot$ denotes the canonical left and right action, respectively, of $\mathcal{F}_{\mathrm{alg}}(H)$ on $\mathcal{F}_{\mathrm{alg}}(H) \otimes H \otimes \mathcal{F}_{\mathrm{alg}}(H)$ that is induced by $\otimes$, i.e.
$$y_1 \cdot (x_1 \otimes h \otimes x_2) \cdot y_2 = (y_1 \otimes x_1) \otimes h \otimes (x_2 \otimes y_2).$$

Instead of the tensor product operation, we can also consider another binary operation $\frown$ on $\mathcal{F}_{\text{alg}}(H)$, which is defined by bilinear extension of

$$g_1 \otimes \cdots \otimes g_m \frown h_1 \otimes \cdots \otimes h_n := \sum_{p=1}^{\min\{m,n\}} \langle g_m, h_1 \rangle \cdots \langle g_{m+1-p}, h_p \rangle g_1 \otimes \cdots \otimes g_{m-p} \otimes h_{p+1} \otimes \cdots \otimes h_n.$$

By a careful bookkeeping of all expressions appearing on both sides of the stated formula, one can convince oneself that

$$\tilde{\nabla}(y_1 \frown y_2) = (\tilde{\nabla} y_1) \cdot y_2 + y_1 \cdot (\tilde{\nabla} y_2) \tag{VII.5}$$

for all $y_1, y_2 \in D(\tilde{\nabla})$, where $\cdot$ denotes now the left and right action, respectively, of $\mathcal{F}_{\text{alg}}(H)$ on $\mathcal{F}_{\text{alg}}(H) \otimes H \otimes \mathcal{F}_{\text{alg}}(H)$ that is induced by $\frown$, i.e.

$$y_1 \cdot (x_1 \otimes h \otimes x_2) \cdot y_2 = (y_1 \frown x_1) \otimes h \otimes (x_2 \frown y_2).$$

(c) Since the range of $\tilde{\nabla}$ is by definition contained in the domain of $\tilde{\delta}$, the composition $\tilde{\delta} \circ \tilde{\nabla}$ is well-defined. In fact, one has $\tilde{N} = \tilde{\delta} \circ \tilde{\nabla}$.

VII.2.3.2. *Free Malliavin calculus on $\mathcal{F}(L^2(\mathbb{R}_+))$.* We apply now the preceding construction in the special case, where the Hilbert space H is given by $L^2(\mathbb{R}_+)$. Thus, we may use the isomorphisms

$$I^S : \mathcal{F} \to L^2(\mathcal{S}, \tau) \qquad \text{and} \qquad I^S \otimes I^S : L^2(\mathbb{R}_+, \mathcal{F} \otimes \mathcal{F}) \to \mathcal{B}_2$$

to pull over

- the operator
$$\tilde{\nabla} : \mathcal{F} \supseteq D(\tilde{\nabla}) \to L^2(\mathbb{R}_+, \mathcal{F} \otimes \mathcal{F})$$
to the so-called *gradient operator*
$$\nabla : L^2(\mathcal{S}, \tau) \supseteq D(\nabla) \to \mathcal{B}_2$$
with domain $D(\nabla) = I^S(D(\tilde{\nabla}))$,
- and the operator
$$\tilde{\delta} : L^2(\mathbb{R}_+, \mathcal{F} \otimes \mathcal{F}) \supseteq D(\tilde{\delta}) \to \mathcal{F}$$
to the so-called *divergence operator*
$$\delta : \mathcal{B}_2 \supseteq D(\delta) \to L^2(\mathcal{S}, \tau)$$
with domain $D(\delta) = (I^S \otimes I^S)(D(\tilde{\delta}))$,

in the obvious way as shown in the following two commutative diagrams.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{I^S} & L^2(\mathcal{S}, \tau) \\ \uparrow & & \uparrow \\ D(\tilde{\nabla}) & \xrightarrow{I^S} & D(\nabla) \\ \downarrow \tilde{\nabla} & & \downarrow \nabla \\ L^2(\mathbb{R}_+, \mathcal{F} \otimes \mathcal{F}) & \xrightarrow{I^S \otimes I^S} & \mathcal{B}_2 \end{array} \qquad \begin{array}{ccc} L^2(\mathbb{R}_+, \mathcal{F} \otimes \mathcal{F}) & \xrightarrow{I^S \otimes I^S} & \mathcal{B}_2 \\ \uparrow & & \uparrow \\ D(\tilde{\delta}) & \xrightarrow{I^S \otimes I^S} & D(\delta) \\ \downarrow \tilde{\delta} & & \downarrow \delta \\ \mathcal{F} & \xrightarrow{I^S} & L^2(\mathcal{S}, \tau) \end{array}$$

In fact, the above definitions amount to

$$D(\nabla) = \mathcal{S}_{\text{alg}} \qquad \text{and} \qquad D(\delta) = \mathcal{S}_{\text{alg}} \odot L^2(\mathbb{R}_+) \odot \mathcal{S}_{\text{alg}},$$

where we abbreviate $\mathcal{S}_{\text{alg}} := \mathcal{S}_{\text{alg}}(L^2(\mathbb{R}_+, \mathbb{R}))$.

REMARK VII.2.11. We may observe that the properties of the operators $\tilde{\nabla}$ and $\tilde{\delta}$, which were formulated in (a) and (b) of Remark VII.2.10 take now a much more natural form. Indeed,

- formula (VII.3) reduces to

$$\langle \nabla Y, U \rangle_{\mathcal{B}_2} = \langle Y, \delta(U) \rangle_{L^2(\mathcal{S},\tau)} \tag{VII.6}$$

  for all $Y \in D(\nabla)$ and $U \in D(\delta)$,
- and formula (VII.5) implies that ∇ is a derivation in the sense that a kind of Leibniz rule

$$\nabla(Y_1 Y_2) = (\nabla Y_1) \cdot Y_2 + Y_1 \cdot (\nabla Y_2) \tag{VII.7}$$

  holds for all $Y_1, Y_2 \in D(\nabla)$, where $\cdot$ denotes the left and right action, respectively, of $\mathcal{S}$ on $\mathcal{B}_2$. For seeing this, note that the Itô formula given in Theorem VII.1.3 reduces to $I^S(g)I^S(h) = I^S(g \frown h)$ for all $g, h \in \mathcal{F}_{\text{alg}} := \mathcal{F}_{\text{alg}}(L^2(\mathbb{R}_+))$, which means that $I^S : \mathcal{F}_{\text{alg}} \to \mathcal{S}_{\text{alg}}$ becomes multiplicative with respect to $\frown$. It follows then for all $Y_1 = I^S(g)$ and $Y_2 = I^S(h)$ in $\mathcal{S}_{\text{alg}} = I^S(\mathcal{F}_{\text{alg}})$ that

$$\begin{aligned}
\nabla(Y_1 Y_2) &= \nabla(I^S(g) I^S(h)) \\
&= \nabla(I^S(g \frown h)) \\
&= (I^S \otimes I^S)(\tilde{\nabla}(g \frown h)) \\
&= (I^S \otimes I^S)\big((\tilde{\nabla} g) \cdot h + g \cdot (\tilde{\nabla} h)\big) \\
&= (I^S \otimes I^S)\big(\tilde{\nabla} g) \cdot I^S(h) + I^S(g) \cdot (I^S \otimes I^S)\big(\tilde{\nabla} h\big) \\
&= \nabla(I^S(g)) \cdot I^S(h) + I^S(g) \cdot \nabla(I^S(h)) \\
&= (\nabla Y_1) \cdot Y_2 + Y_1 \cdot (\nabla Y_2).
\end{aligned}$$

We recall [**KNPS12**, Proposition 3.23], which is itself a combination of Propositions 5.3.9 and 5.3.10 in [**BS98**].

PROPOSITION VII.2.12. *The gradient operator*

$$\nabla : \; L^2(\mathcal{S}, \tau) \supseteq D(\nabla) \to \mathcal{B}_2$$

is densely defined and closable. The domain $D(\overline{\nabla})$ *of the closure*

$$\overline{\nabla} : \; L^2(\mathcal{S}, \tau) \supseteq D(\overline{\nabla}) \to \mathcal{B}_2$$

can be characterized by the chaos expansion in the following way

$$D(\overline{\nabla}) = \Big\{ I^S(f) \Big| \; f = (f_n)_{n=0}^\infty \in \mathcal{F} : \; \sum_{n=0}^\infty n \|f_n\|^2_{L^2(\mathbb{R}_+^n)} < \infty \Big\}.$$

In fact, if we write $Y \in D(\overline{\nabla})$ *in the form* $Y = I^S(f)$ *with* $f \in \mathcal{F}$*, we have that*

$$\|\overline{\nabla} Y\|^2_{\mathcal{B}_2} = \sum_{n=0}^\infty n \|f_n\|^2_{L^2(\mathbb{R}_+^n)}.$$

Moreover, the action of $\overline{\nabla}$ *on its domain* $D(\overline{\nabla})$ *is determined by*

$$\begin{aligned}&\overline{\nabla}_t\Big(\int f(t_1,\dots,t_n)\,dS_{t_1}\cdots dS_{t_n}\Big)\\&=\sum_{j=1}^n\int f(t_1,\dots,t_{j-1},t,t_{j+1},\dots,t_n)\,dS_{t_1}\cdots dS_{t_{j-1}}\otimes dS_{t_{j+1}}\cdots dS_{t_n}\end{aligned}$$

for $f\in L^2(\mathbb{R}_+^n)$.

Remark VII.2.13. We point out that Proposition 5.2.3 in [**BS98**] shows beyond this that ∇ is also closable as an unbounded linear operator from $L^p(\mathcal{S},\tau)$ to $\mathcal{B}_p$ for each $1\le p<\infty$. The domain of its closure, which will be denoted by $\mathbb{D}^p$, is given as the closure of $\mathcal{S}_{\mathrm{alg}}$ with respect to the norm $\|\cdot\|_{1,p}$ defined by

$$\|Y\|_{1,p}:=\big(\|Y\|^p_{L^p(\mathcal{S},\tau)}+\|\nabla Y\|^p_{\mathcal{B}_p}\big)^{\frac{1}{p}}.$$

We will use this observation only in the case $p=2$, where $D(\overline{\nabla})=\mathbb{D}^2$ gives an alternative description of the domain $D(\overline{\nabla})$ of the closure of the gradient operator ∇, which was characterized in Proposition VII.2.12 in terms of the chaos decomposition.

Concerning now the divergence operator, we record here [**KNPS12**, Proposition 3.25], which combines Propositions 5.3.9 and 5.3.11 of [**BS98**].

Proposition VII.2.14. *The divergence operator*

$$\delta:\ \mathcal{B}_2\supseteq D(\delta)\to L^2(\mathcal{S},\tau)$$

is densely defined and closable. The domain $D(\overline{\delta})$ *of its closure*

$$\overline{\delta}:\ \mathcal{B}_2\supseteq D(\overline{\delta})\to L^2(\mathcal{S},\tau)$$

contains all adapted L^2*-biprocesses* $\mathcal{B}_2^a$ *and for each* $U\in\mathcal{B}_2^a$*, we have*

$$\overline{\delta}(U)=\int_{\mathbb{R}_+}U_t\sharp dS_t.$$

In general, the action of $\overline{\delta}$ *on its domain* $D(\overline{\delta})$ *is determined by*

$$\begin{aligned}&\overline{\delta}\Big(\int f_t(t_1,\dots,t_n;s_1,\dots,s_m)\,dS_{t_1}\cdots dS_{t_n}\otimes dS_{s_1}\cdots dS_{s_m}\Big)\\&=\int f_t(t_1,\dots,t_n;s_1,\dots,s_m)\,dS_{t_1}\cdots dS_{t_n}dS_tdS_{s_1}\cdots dS_{s_m}\end{aligned}$$

for any $f\in L^2(\mathbb{R}_+,L^2(\mathbb{R}_+^n)\otimes L^2(\mathbb{R}_+^m))$.

Finally, we also take the operator $\tilde{N}$ into account. This operator induces the so-called *number operator*

$$N:\ L^2(\mathcal{S},\tau)\supseteq D(N)\to L^2(\mathcal{S},\tau)$$

with domain $D(N) := I^S(\mathcal{F}_{\text{alg}}) = \mathcal{S}_{\text{alg}}$ as shown in the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\quad I^S \quad} & L^2(\mathcal{S},\tau) \\
\uparrow & & \uparrow \\
D(\tilde{N}) & \xrightarrow{\quad I^S \quad} & D(N) \\
\Big\downarrow{\scriptstyle \tilde{N}} & & \Big\downarrow{\scriptstyle N} \\
\mathcal{F} & \xrightarrow{\quad I^S \quad} & L^2(\mathcal{S},\tau)
\end{array}$$

Remark VII.2.15. The relation $\tilde{N} = \tilde{\delta} \circ \tilde{\nabla}$ on $\mathcal{F}_{\text{alg}}$, which was recorded in part (c) of Remark VII.2.10, translates by definition immediately to the relation $N = \delta \circ \nabla$ on $\mathcal{S}_{\text{alg}}$.

We recall now [**KNPS12**, Remark 3.24].

Proposition VII.2.16. *The number operator*

$$N: \ L^2(\mathcal{S},\tau) \supseteq D(N) \to L^2(\mathcal{S},\tau)$$

is densely defined and closable. The domain $D(\overline{N})$ of its closure can be characterized by using the chaos expansion in the following way

$$D(\overline{N}) = \Big\{ I^S(f) \Big| \ f = (f_n)_{n=0}^\infty \in \mathcal{F}: \ \sum_{n=0}^\infty n^2 \|f_n\|^2_{L^2(\mathbb{R}_+^n)} < \infty \Big\}.$$

In particular, the closure of the gradient $\overline{\nabla}$ maps $D(\overline{N})$ into $D(\overline{\delta})$, and on $D(\overline{N})$, it holds true that $\overline{N} = \overline{\delta} \circ \overline{\nabla}|_{D(\overline{N})}$.

VII.3. Proof of Theorem VII.1.4

We are prepared now to build the proof of Theorem VII.1.4 on its two pillars that we raised in the previous sections, namely free Malliavin calculus as presented in Subection VII.2.3 and the theory of non-commutative derivatives as developped in Chapter V, in particular in Section V.6.

In the light of free Malliavin calculus, it seems natural that methods from Chapter V could be used for a proof of Theorem VII.1.4 based on the same reduction method as in [**MSW14, MSW17**]. Nevertheless, there is the fundamental obstacle that in the world of free stochastic calculus, the role of non-commutative derivatives which were used in the "discrete setting" of [**MSW14, MSW17**], is taken over by the Malliavin operators as their "continuous counterparts". These operators are seemingly of completely different nature.

But on closer inspection, it turns out that the right object for this purpose, which bridges – somehow as an architrave, if one wants to strain the architecture language again – between free stochastic calculus and the theory of non-commutative derivatives are directional gradients. We will introduce this concept in the following subsection.

VII.3.1. Directional gradients. Roughly speaking, directional gradients are obtained from the gradient operator by integrating out the (for us obstructive) time dependence against any function in $L^2(\mathbb{R}_+)$. More formally, we shall introduce these objects as follows.

Definition VII.3.1. For each $h \in L^2(\mathbb{R}_+)$, we define an unbounded linear operator

$$\nabla^h : L^2(\mathcal{S}, \tau) \supseteq D(\nabla^h) \to L^2(\mathcal{S}, \tau) \otimes L^2(\mathcal{S}, \tau)$$

with domain $D(\nabla^h) := D(\nabla) = \mathcal{S}_{\mathrm{alg}}$ by

$$\nabla^h Y := \langle \nabla Y, h \rangle = \int_{\mathbb{R}_+} \nabla_t Y \, \overline{h(t)} \, dt,$$

where we refer to the pairing $\langle \cdot, \cdot \rangle$ that was introduced in Definition VII.2.3. We call ∇^h the *directional gradient (in the direction h)*.

This terminology goes in fact parallel to classical Malliavin calculus, where corresponding expressions are also interpreted as directional derivatives.

We collect some basic but very important properties of directional gradients in the following lemma.

Lemma VII.3.2. *Let $h \in L^2(\mathbb{R}_+)$ be given.*

(a) *If $\cdot$ denotes the left and right action of $\mathcal{S}$ on $L^2(\mathcal{S}, \tau) \otimes L^2(\mathcal{S}, \tau)$, respectively, then the Leibniz rule*

$$\nabla^h (Y_1 Y_2) = (\nabla^h Y_1) \cdot Y_2 + Y_1 \cdot (\nabla^h Y_2)$$

holds for all $Y_1, Y_2 \in D(\nabla^h) = \mathcal{S}_{\mathrm{alg}}$.

(b) *For all $Y \in D(\nabla^h)$, it holds true that*

$$\nabla^h (Y^*) = (\nabla^{\overline{h}} Y)^\dagger.$$

Thus, if $h \in L^2(\mathbb{R}_+, \mathbb{R})$, we have in particular that

$$\nabla^h (Y^*) = (\nabla^h Y)^\dagger$$

holds for all $Y \in D(\nabla^h)$.

(c) *The directional gradient ∇^h takes its values in $\mathcal{S}_{\mathrm{alg}} \odot \mathcal{S}_{\mathrm{alg}}$ and we have that*

$$(\nabla^h \otimes \mathrm{id}) \nabla^h = (\mathrm{id} \otimes \nabla^h) \nabla^h.$$

More generally, it holds true for all $h_1, h_2 \in L^2(\mathbb{R}_+)$ that

$$(\nabla^{h_1} \otimes \mathrm{id}) \nabla^{h_2} = (\mathrm{id} \otimes \nabla^{h_2}) \nabla^{h_1}.$$

Proof. The fact that ∇^h satisfies the Leibniz rule stated in (a) follows immediately from the Leibniz rule (VII.7) for ∇ on $D(\nabla)$, since the domains $D(\nabla)$ and $D(\nabla^h)$ agree.

For seeing (b), we consider $Y = X(h_1) \dots X(h_n) \in \mathcal{S}_{\text{alg}}$ for $h_1, \dots, h_n \in L^2(\mathbb{R}_+, \mathbb{R})$. A straightforward calculation confirms that

$$\begin{aligned}\nabla^h(Y^*) &= \sum_{j=1}^{n} \langle h_j, h \rangle X(h_n) \cdots X(h_{j+1}) \otimes X(h_{j-1}) \cdots X(h_1)\\ &= \Big(\sum_{j=1}^{n} \langle h_j, \overline{h} \rangle X(h_1) \cdots X(h_{j-1}) \otimes X(h_{j+1}) \cdots X(h_n) \Big)^\dagger\\ &= (\nabla^{\overline{h}} Y)^\dagger.\end{aligned}$$

Because $h = \overline{h}$ holds for any $h \in L^2(\mathbb{R}_+, \mathbb{R})$, the additional statement in (b) is an immediate consequence of the formula $\nabla^h(Y^*) = (\nabla^{\overline{h}} Y)^\dagger$. Alternatively, by referring to Remark V.4.4, it suffices to check $\nabla^h(Y^*) = (\nabla^{\overline{h}} Y)^\dagger$ on the algebraic generators $(X(g))_{g \in L^2(\mathbb{R}_+, \mathbb{R})}$ of $\mathcal{S}_{\text{alg}}$. But in this case, the statement is obvious since $X(g)$ is self-adjoint and since we have $\nabla^h X(g) = \langle g, h \rangle_{L^2(\mathbb{R}_+)} 1 \otimes 1$ for any $g \in L^2(\mathbb{R}_+, \mathbb{R})$.

For proving (c), since ∇^h clearly takes its values in $\mathcal{S}_{\text{alg}} \odot \mathcal{S}_{\text{alg}}$, it only remains to show the stated formula. For doing this, it suffices by linearity to prove

$$(\nabla^{h_1} \otimes \text{id}) \nabla^{h_2} Y = (\text{id} \otimes \nabla^{h_2}) \nabla^{h_1} Y$$

for all $h_1, h_2 \in L^2(\mathbb{R}_+)$ and any element $Y \in \mathcal{S}_{\text{alg}}$ of the form

$$Y = X(g_1) X(g_2) \cdots X(g_n).$$

If $1 \leq j_1 < j_2 \leq n$ are given, we will abbreviate in the following

$$\check{X}_{j_1, j_2} := X(g_1) \cdots X(g_{j_1 - 1}) \otimes X(g_{j_1 + 1}) \cdots X(g_{j_2 - 1}) \otimes X(g_{j_2 + 1}) \cdots X(g_n),$$

where as usually empty products are understood as 1. Firstly, we compute

$$\nabla^{h_2} Y = \sum_{1 \leq j_2 \leq n} \langle g_{j_2}, h_2 \rangle X(g_1) \cdots X(g_{j_2 - 1}) \otimes X(g_{j_2 + 1}) \cdots X(g_n),$$

which yields

$$(\nabla^{h_1} \otimes \text{id}) \nabla^{h_2} Y = \sum_{1 \leq j_1 < j_2 \leq n} \langle g_{j_1}, h_1 \rangle \langle g_{j_2}, h_2 \rangle \check{X}_{j_1, j_2}$$

Similarly, we compute

$$\nabla^{h_1} Y = \sum_{1 \leq j_1 \leq n} \langle g_{j_1}, h_1 \rangle X(g_1) \cdots X(g_{j_1 - 1}) \otimes X(g_{j_1 + 1}) \cdots X(g_n),$$

which yields

$$(\text{id} \otimes \nabla^{h_2}) \nabla^{h_1} Y = \sum_{1 \leq j_1 < j_2 \leq n} \langle g_{j_1}, h_1 \rangle \langle g_{j_2}, h_2 \rangle \check{X}_{j_1, j_2}$$

Because the right hand sides of both results agree, we finally obtain the desired equality. This concludes the proof. □

Combining the properties of directional gradients that we have established in the previous Lemma VII.3.2 leads us immediately to the following crucial observation.

Corollary VII.3.3. *For any $h \in L^2(\mathbb{R}_+)$, the directional gradient*

$$\nabla^h : \ L^2(\mathcal{S}, \tau) \supseteq D(\nabla^h) \to L^2(\mathcal{S}, \tau) \otimes L^2(\mathcal{S}, \tau),$$

induces a non-commutative derivation on $\mathcal{S}$ in the sense of Definition V.3.1, which satisfies additionally the coassociativity relation that was formulated in Definition V.5.1. If *we choose particularly any $h \in L^2(\mathbb{R}_+, \mathbb{R})$, then ∇^h is also a real derivation in the sense of Definition V.4.3.*

The importance of this observations is perfectly clear now, since it puts directional gradients in the setting non-commutative derivations and gives therefore access to the general theory that was presented in Chapter V.

However, there is still one key property missing that is needed to fully open this powerful toolbox, namely the condition $1 \otimes 1 \in D(\delta^h)$, where δ^h denotes the adjoint operator of ∇^h, i.e.

$$\delta^h := (\nabla^h)^* : \ L^2(\mathcal{S}, \tau) \otimes L^2(\mathcal{S}, \tau) \supseteq D(\delta^h) \to L^2(\mathcal{S}, \tau),$$

We shall call δ^h the *directional divergence operator (in the direction h)* in the following.

The condition $1 \otimes 1 \in D(\delta^h)$ would in particular guarantee according to Proposition V.4.6 that δ^h is densely defined and hence that ∇^h is closable. But there is actually a shortcut in our situation. We insert here the following lemma which expresses the directional divergence operator δ^h in terms of the divergence operator δ and which will allow us to conclude directly that the domain of δ^h is sufficiently large.

Lemma VII.3.4. *For any $h \in L^2(\mathbb{R}_+)$, the domain $D(\delta^h)$ of the directional divergence operator δ^h contains $\mathcal{S}_{\text{alg}} \odot \mathcal{S}_{\text{alg}}$ and we have explicitly*

$$\delta^h(U) = \delta(U \sharp 1 \otimes h \otimes 1) \qquad \textit{for all } U \in \mathcal{S}_{\text{alg}} \odot \mathcal{S}_{\text{alg}}.$$

In particular, δ^h is densely defined and we have that $1 \otimes 1 \in D(\delta^h)$ with $\delta^h(1 \otimes 1) = X(h)$.

Proof. We just have to note that by definition $U \sharp 1 \otimes h \otimes 1 \in D(\delta)$ for any $U \in \mathcal{S}_{\text{alg}} \odot \mathcal{S}_{\text{alg}}$ and that the corresponding element $\delta(U \sharp 1 \otimes h \otimes 1) \in L^2(\mathcal{S}, \tau)$ satisfies

$$\langle Y, \delta(U \sharp 1 \otimes h \otimes 1) \rangle = \langle \nabla Y, U \sharp 1 \otimes h \otimes 1 \rangle_{\mathcal{B}_2} = \langle \nabla^h Y, U \rangle.$$

This means that $\mathcal{S}_{\text{alg}} \odot \mathcal{S}_{\text{alg}} \subseteq D(\delta^h)$ and even more explicit

$$\delta^h(U) = \delta(U \sharp 1 \otimes h \otimes 1) \qquad \text{for all } U \in \mathcal{S}_{\text{alg}} \odot \mathcal{S}_{\text{alg}}.$$

In particular, we may deduce that δ^h is densely defined and that $1 \otimes 1 \in D(\delta^h)$ holds true with $\delta^h(1 \otimes 1) = \delta(1 \otimes h \otimes 1) = X(h)$. ☐

The closability of ∇^h, which is implied by the lemma above, will be recorded in the following proposition. But we discuss there in addition that the domain of the closure of ∇^h contains the domain of the closure of ∇.

Proposition VII.3.5. *Given $h \in L^2(\mathbb{R}_+)$. The directional gradient*

$$\nabla^h : \ L^2(\mathcal{S}, \tau) \supseteq D(\nabla^h) \to L^2(\mathcal{S}, \tau) \otimes L^2(\mathcal{S}, \tau)$$

is densely defined and closable. The domain $D(\overline{\nabla}^h)$ of its closure

$$\overline{\nabla}^h : \ L^2(\mathcal{S}, \tau) \supseteq D(\overline{\nabla}^h) \to L^2(\mathcal{S}, \tau) \otimes L^2(\mathcal{S}, \tau)$$

contains the domain $D(\overline{\nabla})$ of $\overline{\nabla}$.

Proof. Basic functional analysis tells us that in this case $D(\overline{\nabla}^h)$ is obtained as the closure of $\mathcal{S}_{\text{alg}}$ with respect to the norm $\|\cdot\|_{2,1}^h$ that is given by

$$\|Y\|_{2,1}^h := \big(\|Y\|_2^2 + \|\nabla^h Y\|_2^2\big)^{\frac{1}{2}} \qquad \text{for all } Y \in \mathcal{S}_{\text{alg}},$$

whereas the domain $D(\overline{\nabla})$ of $\overline{\nabla}$ is obtained as the closure of $\mathcal{S}_{\text{alg}}$ with respect to the norm

$$\|Y\|_{2,1} = \big(\|Y\|_2^2 + \|\nabla Y\|_{\mathcal{B}_2}^2\big)^{\frac{1}{2}} \qquad \text{for all } Y \in \mathcal{S}_{\text{alg}},$$

as we pointed out in Remark VII.2.13. Therefore, the desired inclusion $D(\overline{\nabla}) \subseteq D(\overline{\nabla}^h)$ follows as soon as we have established that

$$\text{(VII.8)} \qquad \|Y\|_{2,1}^h \leq \max\{1, \|h\|_{L^2(\mathbb{R}_+)}\} \, \|Y\|_{2,1} \qquad \text{for all } Y \in \mathcal{S}_{\text{alg}}.$$

For that purpose, we make use of Lemma VII.2.2. This yields

$$\|\nabla^h Y\|_2 = \|\langle \nabla Y, h\rangle\|_2 \leq \|h\|_{L^2(\mathbb{R}_+)} \|\nabla Y\|_{\mathcal{B}_2}.$$

Now, the desired inequality (VII.8) immediately follows. □

VII.3.2. Reduction by directional gradients. In the previous subsection, we have seen that directional gradients fit nicely into the general frame of non-commutative derivations. The following proposition, which will be a the core of our reduction method, is therefore an immediate consequence of Proposition V.6.1.

PROPOSITION VII.3.6. *Take any $Y \in \mathcal{S}_{\text{fin}}$. If there are $u, v \in \mathcal{S}$ such that the conditions $Yu = 0$ and $Y^*v = 0$ are satisfied, then it holds true that*

$$v^* \cdot (\overline{\nabla}^h Y) \cdot u = 0 \qquad \textit{for all } h \in L^2(\mathbb{R}_+, \mathbb{R}).$$

PROOF. Let $h \in L^2(\mathbb{R}_+, \mathbb{R})$ be given. Firstly, we recall that the directional gradient

$$\nabla: \ L^2(\mathcal{S}, \tau) \supseteq D(\nabla^h) \to L^2(\mathcal{S}, \tau) \otimes L^2(\mathcal{S}, \tau),$$

induces according to Corollary VII.3.3 a real non-commutative derivation, which satisfies in addition the coassociativity relation. Furthermore, its adjont operator, the directional divergence operator δ^h, satisfies due to Lemma VII.3.4 the condition $1 \otimes 1 \in D(\delta^h)$. Thus, we can apply Proposition V.6.1, which yields the desired statement. □

REMARK VII.3.7. In the proof of Proposition VII.3.6 above, we used crucially the properties of directional gradients, which put them nicely in the setting of non-commutative derivations and which therefore allowed us to turn on by Proposition V.6.1 the powerful machinery that was built up in Chapter V.

But recall that one of the crucial ingredients in the proof of Proposition V.6.1 were Dabrowski's inequalities V.5.2. Thus, concealed in the larger apparatus, we deduced particularly for any $Y \in D(\nabla^h)$ according to the inequalities (V.19) that

$$\text{(VII.9)} \qquad \begin{aligned} \|\delta^h(Y \otimes 1)\|_2 &\leq \|h\|_{L^2(\mathbb{R}_+)} \|Y\|, \\ \|\delta^h(1 \otimes Y)\|_2 &\leq \|h\|_{L^2(\mathbb{R}_+)} \|Y\|, \end{aligned}$$

and according to the inequalities (V.20) that

$$\text{(VII.10)} \qquad \begin{aligned} \|(\mathrm{id} \otimes \tau)(\nabla^h Y)\|_2 &\leq 2\|h\|_{L^2(\mathbb{R}_+)} \|Y\|, \\ \|(\tau \otimes \mathrm{id})(\nabla^h Y)\|_2 &\leq 2\|h\|_{L^2(\mathbb{R}_+)} \|Y\|, \end{aligned}$$

since we have $\|\delta^h(1 \otimes 1)\|_2 = \|h\|_{L^2(\mathbb{R}_+)}$.

However, the semicircular generators that underlie our situation force in fact a much stronger result than the inequalities above. In fact, for any $Y \in \mathcal{S}_{\text{fin}}$, we have that

$$\text{(VII.11)} \qquad \begin{aligned} \|\delta^h(Y \otimes 1)\|_2 &= \|h\|_{L^2(\mathbb{R}_+)} \|Y\|_2, \\ \|\delta^h(1 \otimes Y)\|_2 &= \|h\|_{L^2(\mathbb{R}_+)} \|Y\|_2, \end{aligned}$$

and

$$\begin{aligned}\|(\mathrm{id}\otimes\tau)(\nabla^h Y)\|_2 &\le \|h\|_{L^2(\mathbb{R}_+)}\|Y\|_2,\\ \|(\tau\otimes\mathrm{id})(\nabla^h Y)\|_2 &\le \|h\|_{L^2(\mathbb{R}_+)}\|Y\|_2.\end{aligned}\tag{VII.12}$$

This can be seen by considering the chaos decomposition of Y and by using the formulas

$$\begin{aligned}\delta^h(I_n^S(f)\otimes 1) &= I_{n+1}^S(f\otimes h),\\ \delta^h(1\otimes I_n^S(f)) &= I_{n+1}^S(h\otimes f)\end{aligned}\tag{VII.13}$$

and

$$\begin{aligned}(\mathrm{id}\otimes\tau)(\nabla^h I_n^S(f)) &= I_{n-1}^S(f \overset{1}{\frown} h),\\ (\tau\otimes\mathrm{id})(\nabla^h I_n^S(f)) &= I_{n-1}^S(h \overset{1}{\frown} f).\end{aligned}\tag{VII.14}$$

The author is grateful to Yoann Dabrowski for pointing out that this fact should be included for reasons of clarity.

Of course, one could argue now that in view of this observation, the discussion around Theorem V.5.2 becomes superfluous in the context of this chapter. But since there is absolutely no chance to avoid completely a detour through the realm of non-commutative derivations – even by taking this shortcut – we decided to present the theory of non-commutative derivations (and in particular the result of Proposition V.6.1) in full generality, in order to show the complete picture and to make it ready for its possible use in future investigations.

VII.3.3. How to control the reduction. Because Theorem VII.1.4 is a statement about elements $f\in\mathcal{F}$, which break off after finitely many non-zero terms, namely about elements in $\mathcal{F}_{\text{fin}} := \mathcal{F}_{\text{fin}}(L^2(\mathbb{R}))$, we shall take now a closer look on

$$\mathcal{S}_{\text{fin}} := \Big\{ I^S(f) \Big|\ f = (f_n)_{n=0}^\infty \in \mathcal{F}_{\text{fin}} \Big\}$$

as the corresponding space of Wigner integrals, called the *finite Wigner chaos*. By definition, $\mathcal{S}_{\text{fin}}$ is only a subset of $L^2(\mathcal{S},\tau)$, but due to (VII.1) and Proposition VII.1.3 it turns out to be in fact a $*$-subalgebra of $\mathcal{S}$. Combining this with the easy observation that $\mathcal{S}_{\text{alg}}$ is contained in $\mathcal{S}_{\text{fin}}$, we may localize $\mathcal{S}_{\text{fin}}$ as intermediate $*$-algebra $\mathcal{S}_{\text{alg}} \subseteq \mathcal{S}_{\text{fin}} \subseteq \mathcal{S}$.

Following the lines of the proof in [**MSW14, MSW17**], we shall introduce now certain operators, which will later allow us to reduce any zero-divisor in $\mathcal{S}_{\text{fin}}$ in a controllable way to a zero-divisor in the chaos of order zero by means of Proposition VII.3.6.

DEFINITION VII.3.8. For any $h\in L^2(\mathbb{R}_+)$ and any projection $p\in\mathcal{S}$, we consider the linear operator $\Delta_{p,h}:\mathcal{S}_{\text{fin}}\to\mathcal{S}_{\text{fin}}$ that is defined by

$$\Delta_{p,h}Y := (\tau\otimes\mathrm{id})\big(p\otimes 1\,(\overline{\nabla}^h Y)\big) \qquad \text{for all } Y\in\mathcal{S}_{\text{fin}}.$$

Note that these operators are indeed well-defined since $\mathcal{S}_{\text{fin}} \subseteq D(\overline{\nabla}) \subseteq D(\overline{\nabla}^h)$ holds by Proposition VII.3.5. The fact that $\Delta_{p,h}$ takes its values in $\mathcal{S}_{\text{fin}}$ and is made more precise in the following lemma, which moreover shows that $\Delta_{p,h}$ "reduces the degree" with respect to the natural grading on $\mathcal{S}_{\text{fin}}$, which is induced by $\mathcal{F}_{\text{fin}}$.

LEMMA VII.3.9. *Let $h\in L^2(\mathbb{R}_+)$ and any projection $p\in\mathcal{S}$ be given. Let τ_p be the bounded linear functional on $\mathcal{F}$ that is given by*

$$\tau_p:\ \mathcal{F}\to\mathbb{C},\ f\mapsto \tau(pI^S(f)).$$

In fact, if we make use of the chaos decomposition of p*, we can write* $p = I^S(g)$ *for some* $g = (g_n)_{n=0}^\infty \in \mathcal{F}$*, so that* $\tau_p(f) = \langle f, g\rangle_{\mathcal{F}}$ *holds for all* $f \in \mathcal{F}$*.*

Now, let $f \in L^2(\mathbb{R}_+^n)$ *be given. For* $1 \leq k \leq n$*, we may regard* f *as an element* $f^{(k-1,n-k)}$ *in* $L^2(\mathbb{R}_+, L^2(\mathbb{R}_+^{k-1}) \otimes L^2(\mathbb{R}_+^{n-k})) \subset L^2(\mathbb{R}_+, \mathcal{F} \otimes \mathcal{F})$*. Using this notation, it holds true that*

$$\Delta_{p,h} I_n^S(f) = \sum_{k=1}^n I_{n-k}^S\Big((\tau_p \otimes \mathrm{id}_{\mathcal{F}})\Big(\int_{\mathbb{R}_+} f_t^{(k-1,n-k)}\, \overline{h(t)}\, dt\Big)\Big). \tag{VII.15}$$

Proof. It is very easy to check the validity of the formula under question in the case $f = f_1 \otimes \cdots \otimes f_n$. Indeed, we have

$$\overline{\nabla}^h I_n^S(f) = \sum_{k=1}^n \langle f_k, h\rangle I_{k-1}^S(f_1 \otimes \cdots \otimes f_{k-1}) \otimes I_{n-k}^S(f_{k+1} \otimes \cdots \otimes f_n)$$

and hence

$$\begin{aligned}\Delta_{p,h} I_n^S(f) &= \sum_{k=1}^n \langle f_k, h\rangle \tau(p I_{k-1}^S(f_1 \otimes \cdots \otimes f_{k-1})) I_{n-k}^S(f_{k+1} \otimes \cdots \otimes f_n)\\ &= \sum_{k=1}^n I_{n-k}^S\Big(\langle f_k, h\rangle \tau_p(f_1 \otimes \cdots \otimes f_{k-1})\, f_{k+1} \otimes \cdots \otimes f_n\Big)\\ &= \sum_{k=1}^n I_{n-k}^S\Big((\tau_p \otimes \mathrm{id}_{\mathcal{F}})\Big(\int_{\mathbb{R}_+} f_t^{(k-1,n-k)}\, \overline{h(t)}\, dt\Big)\Big),\end{aligned}$$

which confirms the desired formula (VII.15) in the case $f = f_1 \otimes \cdots \otimes f_n$. By linearity of both of its sides, we conclude that formula (VII.15) also holds for any function in the linear span of

$$\{f_1 \otimes \cdots \otimes f_n |\ f_1, \dots, f_n \in L^2(\mathbb{R}_+)\},$$

i.e. for any function in $L^2(\mathbb{R}_+)^{\odot n}$. Since this linear space is dense in $L^2(\mathbb{R}_+^n)$ with respect to $\|\cdot\|_{L^2(\mathbb{R}_+^n)}$, it remains to note that (VII.15) stays valid under taking limits with respect to $\|\cdot\|_{L^2(\mathbb{R}_+^n)}$, which means that we prove the continuity of the left and the right hand side of the formula under question with respect to $\|\cdot\|_{L^2(\mathbb{R}_+^n)}$.

Concerning first the left hand side, we note that

$$\|\overline{\nabla} I_n^S(f)\|_2 = \sqrt{n}\, \|f\|_{L^2(\mathbb{R}_+^n)}.$$

Indeed, we have according to Proposition VII.2.16 and the Itô isometry that

$$\|\overline{\nabla} I_n^S(f)\|_2^2 = \langle \overline{\nabla} I_n^S(f), \overline{\nabla} I_n^S(f)\rangle = \langle (\overline{\delta}\, \overline{\nabla}) I_n^S(f), I_n^S(f)\rangle = n\, \|I_n^S(f)\|_2^2 = n\, \|f\|_{L^2(\mathbb{R}_+^n)}^2.$$

Thus, we obtain the desired bound

$$\|\Delta_{p,h} I_n^S(f)\|_2 \leq \|p\|\, \|\overline{\nabla} I_n^S(f)\|_2 = \sqrt{n}\, \|p\|\, \|f\|_{L^2(\mathbb{R}_+^n)}.$$

Concerning now the right hand side of the formula under question, we note that

$$\Big\| \int_{\mathbb{R}_+} f_t^{(k-1,n-k)}\, \overline{h(t)}\, dt \Big\|_{L^2(\mathbb{R}_+^{k-1}) \otimes L^2(\mathbb{R}_+^{n-k})} \leq \|h\|_{L^2(\mathbb{R}_+)} \|f\|_{L^2(\mathbb{R}_+^n)}$$

holds for $1 \leq k \leq n$, which yields by the Itô isometry

$$\begin{aligned}
\Big\| I^S_{n-k}\Big((\tau_p \otimes \mathrm{id}_{\mathcal{F}})\Big(\int_{\mathbb{R}_+} f_t^{(k-1,n-k)}\,\overline{h(t)}\,dt\Big)\Big)\Big\|_2
&\leq \Big\|(\tau_p \otimes \mathrm{id}_{\mathcal{F}})\Big(\int_{\mathbb{R}_+} f_t^{(k-1,n-k)}\,\overline{h(t)}\,dt\Big)\Big\|_{L^2(\mathbb{R}_+^{n-k})}\\
&\leq \|p\|_2 \Big\|\int_{\mathbb{R}_+} f_t^{(k-1,n-k)}\,\overline{h(t)}\,dt\Big\|_{L^2(\mathbb{R}_+^{k-1})\otimes L^2(\mathbb{R}_+^{n-k})}\\
&\leq \|p\|_2 \|h\|_{L^2(\mathbb{R}_+)} \|f\|_{L^2(\mathbb{R}_+^n)}.
\end{aligned}$$

This concludes the proof. □

By applying iteratively operators of the form $\Delta_{p,h}$ to a fixed element in the finite Wigner chaos $\mathcal{S}_{\mathrm{fin}}$, we will therefore reach the chaos of order zero after finitely many steps. The following proposition provides an explicit formula for the output of this procedure.

Proposition VII.3.10. *Let $f = (f_n)_{n=0}^\infty \in \mathcal{F}_{\mathrm{fin}}$ be given and let $N \in \mathbb{N}$ be chosen such that $f_n = 0$ for all $n \geq N+1$. Then, for any choice of functions $h_1, \dots, h_N \in L^2(\mathbb{R}_+)$ and projections $p_1, \dots, p_N$, it holds true that*

$$\Delta_{p_N,h_N} \cdots \Delta_{p_1,h_1} I^S(f) = \tau(p_1)\cdots\tau(p_N)\,\langle f_N, h_1 \otimes \cdots \otimes h_N\rangle\, 1.$$

Before continuing with the proof of the general statement, we first focus on the special case of simple functions.

Remark VII.3.11. We note that for any $Y \in \mathcal{S}_{\mathrm{alg}}$

$$\Delta_{p_N,h_N} \cdots \Delta_{p_1,h_1} Y = (\tau^{\otimes N} \otimes \mathrm{id})\big(p_1 \otimes \cdots \otimes p_N \otimes 1\,(\overline{\nabla}^{h_1,\dots,h_N} Y)\big)$$

holds, where the *iterated gradient* $\overline{\nabla}^{h_1,\dots,h_N} : \mathcal{S}_{\mathrm{alg}} \to \mathcal{S}_{\mathrm{alg}}^{\odot(N+1)}$ is defined by

$$\overline{\nabla}^{h_1,\dots,h_N} := (\mathrm{id}^{\otimes(N-1)} \otimes \overline{\nabla}^{h_N}) \dots (\mathrm{id} \otimes \overline{\nabla}^{h_2})\overline{\nabla}^{h_1}.$$

Thus, the statement of Proposition VII.3.10 becomes apparent in the case where $f = (f_n)_{n=0}^\infty \in \mathcal{F}_{\mathrm{fin}}$ consists of simple functions $f_n \in \mathcal{E}(\mathbb{R}_+^n)$. Indeed, since $I^S(f)$ decomposes by the conditions that are imposed on f as

$$I^S(f) = I_0^S(f_0) + I_1^S(f_1) + \cdots + I_N^S(f_N)$$

and because obviously

$$\overline{\nabla}^{h_1,\dots,h_N} I_k^S(f_k) = 0 \qquad \text{for } 0 \leq k \leq N-1,$$

we see that

$$\overline{\nabla}^{h_1,\dots,h_N} I^S(f) = \overline{\nabla}^{h_1,\dots,h_N} I_N^S(f_N).$$

By using Proposition VII.2.12, we get

$$\overline{\nabla}^{h_1,\dots,h_N} I_N^S(f_N) = \langle f, h_1 \otimes \cdots \otimes h_N\rangle\, 1^{\otimes(N+1)}.$$

Combining these observations yields

$$\Delta_{p_N,h_N} \cdots \Delta_{p_1,h_1} I^S(f) = \tau(p_1)\cdots\tau(p_N)\,\langle f_N, h_1 \otimes \cdots \otimes h_N\rangle\, 1,$$

which is the stated formula.

Proof of Proposition VII.3.10. For the general case, we make use of Lemma VII.3.9. Applying formula (VII.15) iteratively, yields that for $1 \leq m \leq N$

$$\Delta_{p_m,h_m} \dots \Delta_{p_1,h_1} I^S(f) = I^S(f^{(m)}),$$

for some $f^{(m)} \in \mathcal{F}_{\text{fin}}$, where $f_n^{(m)} = 0$ for all $n \geq N - m + 1$. Moreover, if we put $f^{(0)} := f$, we see that

$$f_{N-m}^{(m)}(t_{m+1}, \dots, t_N) = \tau(p_m) \int_{\mathbb{R}_+} f_{N-m+1}^{(m-1)}(t_m, t_{m+1}, \dots, t_N)\, \overline{h_m(t_m)}\, dt_m$$

for all $1 \leq m \leq N - 1$ and

$$f_0^{(N)} = \tau(p_N) \int_{\mathbb{R}_+} f_1^{(N-1)}(t_N)\, \overline{h_N(t_N)}\, dt_N.$$

Hence, the only term that survives in $\Delta_{p_N,h_N} \dots \Delta_{p_1,h_1} I^S(f)$ is induced by

$$f_0^{(N)} = \tau(p_1) \cdots \tau(p_N)\, \langle f_N, h_1 \otimes \cdots \otimes h_N \rangle,$$

which gives the stated formula. □

VII.3.4. Absence of zero divisors. Our discussion in the previous subsections has shown that directional gradients allow us to transfer tools from the theory of non-commutative derivations as presented in Chapter V to the setting of free stochastic calculus. Moreover, we have convinced ourselves that directional gradients ∇^h induce operators $\Delta_{p,h}$, which satisfy the general conditions for performing our reduction method.

Putting things together, we obtain the following theorem, of which the desired Theorem VII.1.4 will be a corollary.

Theorem VII.3.12. *There are no zero divisors in $\mathcal{S}_{\text{fin}}$. More precisely, if $0 \neq Y \in \mathcal{S}_{\text{fin}}$ is given, then there is no $0 \neq u \in \mathcal{S}$ such that $Yu = 0$.*

Proof. Contrarily, assume that there are $0 \neq Y \in \mathcal{S}_{\text{fin}}$ and $0 \neq u \in \mathcal{S}$ such that $Yu = 0$. We may write $Y = I^S(f)$ for some $f \in \mathcal{F}_{\text{fin}}$ of the form $f = (f_n)_{n=0}^{\infty}$. Moreover, we may choose $N \in \mathbb{N}$ such that $f_N \neq 0$ but $f_n = 0$ for all $n \geq N + 1$.

Now, we fix arbitrary functions $h_1, \dots, h_N \in L^2(\mathbb{R}_+, \mathbb{R})$. Recall from Lemma VI.2.10 that whenever we have an element $X \in \mathcal{S}$ such that $Xu = 0$ holds, then there exists a non-zero projection $p \in \mathcal{S}$ such that $X^* p = 0$. Thus, by applying Proposition VII.3.6 iteratively, we may find non-zero projections $p_1, \dots, p_N \in \mathcal{S}$ such that

$$(\Delta_{p_N,h_N} \dots \Delta_{p_1,h_1} Y) u = 0.$$

According to Proposition VII.3.10, this means that

$$\tau(p_1) \cdots \tau(p_N)\, \langle f_N, h_1 \otimes \cdots \otimes h_N \rangle\, u = 0.$$

Since we have by assumption $u \neq 0$ and furthermore $\tau(p_1) \cdots \tau(p_N) \neq 0$, because $p_1, \dots, p_N$ are non-zero projections, it follows

$$\langle f_N, h_1 \otimes \cdots \otimes h_N \rangle = 0.$$

Inasmuch as the linear span of

$$\{h_1 \otimes \cdots \otimes h_N |\ h_1, \dots, h_N \in L^2(\mathbb{R}_+, \mathbb{R})\}$$

is dense in $L^2(\mathbb{R}_+^N)$ with respect to $\|\cdot\|_{L^2(\mathbb{R}_+^N)}$, the previous insight yields $f_N = 0$, which contradicts the condition according to which N was chosen. Thus, the assumption made above was wrong, so that the statement of the theorem must be true. $\square$

We finish by showing that Theorem VII.1.4 is indeed a consequence of Theorem VII.3.12 above. In fact, we will deduce Theorem VII.1.4 exactly in the same way as it was done for the analogous statement in [**MSW17**].

Proof of Theorem VII.1.4. More generally, by allowing right from the beginning a constant summand $I_0^S(f_0)$, we show the following: the distribution μ_Y of any self-adjoint element $Y \in \mathcal{S}_{\text{fin}}$, which does not belong to the chaos of order zero, cannot have atoms.

Let $Y \in \mathcal{S}_{\text{fin}}$ be given. If Y does not belong to the chaos of order zero, we can write it as

$$Y = I^S(f) = I_0^S(f_0) + I_1^S(f_1) + \cdots + I_N^S(f_N)$$

for some $f = (f_n)_{n=0}^\infty \in \mathcal{F}_{\text{fin}}$, which is stationary zero after $f_N \neq 0$ for some $N \in \mathbb{N}$. (Note that $N \neq 0$ means abstractly speaking that Y is not constant, as it was assumed in [**MSW17**].) Then, we observe that any atom α of the distribution μ_Y of Y, i.e. any $\alpha \in \mathbb{R}$ satisfying $\mu_Y(\{\alpha\}) \neq 0$, leads by the spectral theorem for bounded self-adjoint operators on Hilbert spaces to a non-zero projection u satisfying $(Y - \alpha 1)u = 0$; see Lemma I.1.20 and the comment thereafter. Now, Theorem VII.3.12 tells us that $Y = \alpha 1$, which contradicts $f_N \neq 0$. $\square$

APPENDIX A

The Schur complement formula

In this short chapter, we address one of the most important tools, which are used in this thesis. The so-called Schur complement formula belongs certainly to the general knowledge of many mathematicians, as it turns out to be extremely helpful in many respects, where it can significantly simplify life, but it seems not to be spread out in a uniform way over all different mathematical communities.

In this thesis, the Schur complement formula is mainly used in the context of the "linearization trick", as presented in Chapters III and IV. But since it also appears at some other places (see for instance Example I.2.10), we decided to take it out of the main flow and present it separately in the appendix. Its statement is formulated in the following lemma.

Lemma A.1 (Schur complement formula). *Let $\mathcal{A}$ be a complex and unital algebra. Let matrices $a \in M_k(\mathcal{A})$, $b \in M_{k\times l}(\mathcal{A})$, $c \in M_{l\times k}(\mathcal{A})$ and $d \in M_l(\mathcal{A})$ be given and assume that d is invertible in $M_l(\mathcal{A})$. Then the following statements are equivalent:*

(i) *The matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible in $M_{k+l}(\mathcal{A})$.*
(ii) *The* Schur complement *$a - bd^{-1}c$ is invertible in $M_k(\mathcal{A})$.*

If the equivalent conditions (i) and (ii) are satisfied, we have the relation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & d^{-1} \end{pmatrix} + \begin{pmatrix} 1 \\ -d^{-1}c \end{pmatrix} (a - bd^{-1}c)^{-1} \begin{pmatrix} 1 & -bd^{-1} \end{pmatrix}, \tag{A.1}$$

which is often called the Schur complement formula*.*

For convenience of those readers, who are not yet familiar with the Schur complement formula, and since this statement and some of the formulas appearing in its proof are crucial for our purposes, we include here the straightforward proof.

Proof of Lemma A.1. A direct calculation shows that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix} \tag{A.2}$$

holds. Since the matrices

$$\begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix}$$

are both invertible in $M_{k+l}(\mathcal{A})$, the stated equivalence of (i) and (ii) immediately follows from (A.2). Moreover, if (i) and (ii) are satisfied, (A.2) leads to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{pmatrix} \begin{pmatrix} (a - bd^{-1}c)^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix} \begin{pmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{pmatrix},$$

from which (A.1) directly follows. □

APPENDIX B

Some results from Banach space valued complex analysis

Many of our considerations, which we have presented in the previous chapters, rely on results from Banach space valued complex analysis. For readers convenience, we collect them here for references and we also recall the underlying definitions, as far as this is needed in order to understand their meaning. For a more detailed introduction to this rich field, we refer the interested reader to [**HP74**].

Throughout the following, let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be complex Banach spaces. There are basically two extensions of the classical concept of holomorphy for functions defined on open subsets of E and taking values in F.

Definition B.1. A function $f\colon U \to F$ defined on an open subset $\emptyset \neq U \subseteq E$ will be called

(a) *Gâteaux analytic on* U, if with respect to $\|\cdot\|_F$ the limit

$$\delta f(x;h) := \lim_{\substack{z\to 0\\ z\in U(x;h)\setminus\{0\}}} \frac{1}{z}(f(x+zh)-f(x))$$

exists for all $x \in U$ and all $h \in E$, where we put

$$U(x;h) := \{z \in \mathbb{C} |\ x + zh \in U\}.$$

(b) *Fréchet analytic on* U, if
  (i) it is Gâteaux analytic on U,
  (ii) its Gâteaux derivative $\delta f(x;\cdot)$ at any point $x \in U$ is a bounded linear operator, and
  (iii) it holds true that

$$\lim_{\|h\|_E\to 0} \frac{1}{\|h\|_E}\|f(x+h)-f(x)-\delta f(x;h)\|_F = 0.$$

In this case, we write $f'(x)$ for the linear operator given by the Gâteaux derivative $\delta f(x;\cdot)$ at any point $x \in U$.

Note that sometimes, we will use the term "holomorphic" instead of "analytic".

Remark B.2.

(i) It can be shown (see for instance [**Zor45**]) that the Gâteaux derivative $\delta f(x;\cdot)$ is always a linear mapping. Therefore, condition (ii) in part (b) of Definition B.1 reduces to the requirement of the boundedness of $\delta f(x;\cdot)$.
(ii) Moreover, a result of Zorn (see [**Zor46**]) says that, in part (b) of Definition B.1, condition (iii) is satisfied automatically if both (i) and (ii) hold true.

It is easy to see from the definition that any Fréchet analytic function is Gâteaux analytic and continuous. But the converse is also true. In fact, the continuity assumption can be replaced by local boundedness. This leads to the notion of analyticity.

Definition B.3. A function $f\colon D \to F$ defined on a domain $\emptyset \neq D \subseteq E$ will be called *analytic*, if the following conditions are satisfied:

(i) f is *locally bounded on* D, i.e. for every $x \in D$ there exists $r = r(x) > 0$ such that
$$\sup_{y \in U\colon \|y-x\|_E < r} \|f(y)\|_F < \infty.$$
(ii) f is Gâteaux analytic on D.

Indeed, the following remarkable result holds true.

Theorem B.4 (Theorem 3.17.1, [**HP74**]). *If the function $f\colon D \to F$ defined on a domain $\emptyset \neq D \subseteq E$ is analytic, then it is continuous and Fréchet analytic.*

Without giving the precise statement here, we want to point out that Theorem 3.17.1 in [**HP74**] guarantees in addition that f can be locally expanded in a uniformly convergent series, which is a natural analogue of the classical Taylor series.

Of particular interest for our considerations is the so called Earle-Hamilton Theorem (see [**Din89**, Theorem 11.1]), which can be seen as a holomorphic version of Banach's contracting mapping theorem.

Theorem B.5 (Earle-Hamilton Theorem). *Let $h\colon D \to E$ be a bounded Fréchet analytic function, defined on a domain $\emptyset \neq D \subseteq E$, such that $h(D)$* lies strictly inside *D, i.e.*
$$\inf\{\|u-v\| \mid u \in h(D), v \in E \setminus D\} > 0.$$
Then h has a unique fixed point x and the sequence of iterates $h^{\circ n}(x_0)$, converges for any $x_0 \in D$ with respect to $\|\cdot\|_E$ to x as $n \to \infty$.

Although we do not want to give a detailed proof of this statement here, we want to explain roughly its strategy. The reader interested in a complete proof of this theorem is referred to [**Din89**]. We will follow here mainly the exposition given in [**Har03**].

The most surprising fact about the proof is that it is indeed based on Banach's contracting mapping theorem. For this purpose, one constructs a special pseudometric ρ on D, called the *Carathéodory-Riffen-Finsler pseudometric*, in which h is a contraction and whose restriction to bounded subsets of D induces a complete metric space. This construction proceeds as follows. For any $x \in D$ and $v \in E$, we put
$$\alpha(x,v) := \sup\{|g'(x)v| \mid g : D \to \mathbb{C} \text{ Fréchet analytic with } g(D) \subseteq \mathbb{D}\},$$
where $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$, i.e. $\mathbb{D} := \{z \in \mathbb{C}\colon |z| < 1\}$. Denote by Γ the set of all curves $\gamma : [0,1] \to D$ with piecewise continuous derivative γ' and define
$$L(\gamma) := \int_0^1 \alpha(\gamma(t), \gamma'(t))\, dt$$
for each $\gamma \in \Gamma$. Finally, we define the desired pseudometric ρ by
$$\rho(x,y) := \inf\{L(\gamma) | \ \gamma \in \Gamma \text{ with } \gamma(0) = x \text{ and } \gamma(1) = y\}$$
for all $x, y \in D$. Since $(g \circ h)'(x)v = g'(h(x))(h'(x)v)$ by the chain rule for the Fréchet derivative, we may deduce that $\alpha(h(x), h'(x)v) \leq \alpha(x,v)$ for all $x \in D$ and $v \in E$. Since $(h \circ \gamma)'(t) = h'(\gamma(t))\gamma'(t)$, we get $L(h \circ \gamma) \leq L(\gamma)$ and thus the *Schwarz-Pick inequality*
$$\rho(h(x), h(y)) \leq \rho(x,y) \qquad \text{for all } x, y \in D.$$

Since h maps D strictly into D, there exists $\varepsilon > 0$ such that $\{y \in E | \, \|y-h(x)\|_E < \varepsilon\} \subseteq D$ for all $x \in E$. Thus, if we replace D by

$$\bigcup_{x\in D} \{y \in E| \, \|y - h(x)\|_E < \varepsilon\},$$

we may assume that D is bounded.

Under this strengthened condition, we can show that there exists $0 < c < 1$ such that $\rho(h(x), h(y)) \leq c\rho(x, y)$ holds for all $x, y \in D$. This allows us to apply Banach's contracting mapping theorem in order to conclude that h has a unique fixed point x, which is moreover attracting in the sense that, for any starting point $x_0 \in D$, the iterates $h^{\circ n}(x_0)$ converge with respect to ρ to x as $n \to \infty$.

It remains to observe that we can find a constant $m > 0$ (in fact, m determined by

$$\frac{1}{m} = \sup\{\|x - y\|_E | \; x, y \in D\}$$

works) such that

$$\rho(x, y) \geq m\|x - y\|_E \qquad \text{for all } x, y \in D. \tag{B.1}$$

This means in particular that the iterates converge to the desired fixed point x not only with respect to the rather abstract metric ρ but more naturally with respect to the norm $\| \cdot \|_E$ as well.

APPENDIX C

Measures on topological spaces

This chapter aims at collecting some basic facts about Radon measures and topologies defined on sets of Radon measures, since these concepts and the corresponding terminology are used repeatedly in this thesis. Although we are mainly interested in Borel probability measures on $\mathbb{R}$ or $\mathbb{C}$, we find it instructive to work here in a more general and unifying setting. Our exposition, which builds on [**Els11**, Chapter VIII], supposes that the reader is already familiar with the basic concepts of topology, functional analysis, and measure theory.

C.1. Radon measures

Let X be a Hausdorff topological space. We denote by $\mathcal{B}(X)$ the Borel σ-algebra on X, which is the σ-algebra on X generated by the open subsets of X. Let us focus first on the case of positive measures defined on $\mathcal{B}(X)$.

Definition C.1 (Definition VIII.1.1 in [**Els11**]). A positive measure $\mu : \mathcal{B}(X) \to [0, \infty]$ is called

- *inner regular*, if we have
$$\mu(B) = \sup_{K \subseteq B \text{ compact}} \mu(K) \qquad \text{for each } B \in \mathcal{B}(X);$$
- *outer regular*, if we have
$$\mu(B) = \inf_{U \supseteq B \text{ open}} \mu(U) \qquad \text{for each } B \in \mathcal{B}(X);$$
- *regular*, if it is both inner and outer regular;
- *locally finite* or *Borel measure*, if each point $x \in X$ has an open neighborhood U, which satisfies that $\mu(U) < \infty$;
- *finite*, if we have $\mu(X) < \infty$;
- *Radon measure*, if it is both inner regular and locally finite.

The set of all (positive) Radon measures on X will be denoted by $\mathcal{M}^+(\mathcal{B})$.

A few comments are in order.

Remark C.2.

- Clearly, each locally finite measure on $\mathcal{B}(X)$ is automatically finite on all compact subsets of X. If we suppose in addition that the underlying space X is locally compact, then the converse is also true, i.e., a measure $\mu : \mathcal{B}(X) \to [0, \infty]$ is locally finite if and only if it is finite on each compact subset of X; see [**Els11**, Corollary VIII.1.2 (c)].
- Each finite Radon measure is automatically regular; see [**Els11**, Corollary VIII.1.2 (f)].

- Recall that a Hausdorff topological space is called *σ-compact*, if it can be written as the union of countably many compact subsets. If each open subset of X is σ-compact, then each Borel measure $\mu : \mathcal{B}(X) \to [0,\infty]$ is automatically a Radon measure; see [**Els11**, Corollary VIII.1.6].

When working with Radon measures, the notion of support turns out to be important. The following definition is taken from [**Els11**, Section VIII.2.5].

DEFINITION C.3. Let $\mu : \mathcal{B}(X) \to [0,\infty]$ be a Radon measure. The *support* $\mathrm{supp}(\mu)$ *of* μ is defined by

$$\mathrm{supp}(\mu) := X \backslash V,$$

where V denotes the union of all open sets U in X satisfying $\mu(U) = 0$.

From this definition, it is immediately clear that $\mathrm{supp}(\mu)$ always forms a closed subset of X, since V, given as $V = \bigcup_{U \in \mathcal{U}} U$ with

$$\mathcal{U} := \{U |\ U \text{ open in } X :\ \mu(U) = 0\},$$

is open as a union of open sets. Less obvious, however, is that V itself satisfies $\mu(V) = 0$, because $\mathcal{U}$ is typically not a countable family. This is proven in [**Els11**, Lemma VIII.2.15] and the validity of this statement strongly depends on the assumption that μ is inner regular.

An important consequence is that, with respect to each Radon measure μ, all *μ-integrable* functions f on X, i.e., all measurable functions $f : X \to \mathbb{C}$ with the property that $\int_X |f(x)|\, d\mu(x) < \infty$, satisfy

$$\int_X f(x)\, d\mu(x) = \int_{\mathrm{supp}(\mu)} f(x)\, d\mu(x).$$

C.2. The Riesz representation theorems

Let X be a locally compact Hausdorff topological space. We will work with the following complex vector spaces of continuous functions on X:

- $C_b(X)$, the space of all bounded continuous functions $f : X \to \mathbb{C}$;
- $C_0(X)$, the space of all continuous functions $f : X \to \mathbb{C}$, which "vanish at infinity" in the sense that for each $\varepsilon > 0$ a compact subset $K \subseteq X$ exists, such that $|f(x)| < \varepsilon$ holds for all points $x \in X \backslash K$;
- $C_c(X)$, the space of all continuous functions $f : X \to \mathbb{C}$, which are "compactly supported" in the sense that a compact set K in X exists, such that $f(x) = 0$ for all $x \in X \backslash K$.

Note that we have the inclusions $C_c(X) \subseteq C_0(X) \subseteq C_b(X)$ and that both $C_0(X)$ and $C_b(X)$ become Banach spaces if they are endowed with the uniform norm $\|\cdot\|_\infty$.

THEOREM C.4 (Riesz representation theorem for $C_c(X)$). *Let X be a locally compact Hausdorff topological space. If $I : C_c(X) \to \mathbb{C}$ is a positive linear functional, then there exists a unique Radon measure $\mu : \mathcal{B}(X) \to [0,\infty]$, such that*

$$I(f) = \int_X f(x)\, d\mu(x) \qquad \text{for all } f \in C_c(X).$$

For a proof, we refer the reader to [**Els11**, Theorem VIII.2.5]. Note that $C(X) = C_c(X)$ holds for compact X, such that the previous theorem immediately yields the following characterization for compact Hausdorff topological spaces X.

Corollary C.5. *Let X be a compact Hausdorff topological space. If $I : C(X) \to \mathbb{C}$ is a positive linear functional, then there exists a unique Radon measure $\mu : \mathcal{B}(X) \to [0, \infty]$, such that*

$$I(f) = \int_X f(x)\,d\mu(x) \qquad \text{for all } f \in C(X).$$

In fact, this Radon measure is finite, since we have $\mu(X) = I(\mathbf{1}) < \infty$ as the constant function $\mathbf{1}$ belongs to $C(X)$.

In the next theorem, which is taken from [**Els11**, Theorem VIII.2.10], positive linear functions on $C_0(X)$ are discussed.

Theorem C.6 (Riesz representation theorem for $C_0(X)$). *Let X be a locally compact Hausdorff topological space. If $I : C_0(X) \to \mathbb{C}$ is a positive linear functional, then there exists a unique Radon measure $\mu : \mathcal{B}(X) \to [0, \infty]$, such that each $f \in C_0(X)$ is μ-integrable, and*

$$I(f) = \int_X f(x)\,d\mu(x) \qquad \text{for all } f \in C_0(X).$$

In fact, this Radon measure μ is even finite.

C.3. The weak and the vague topology

Let X be a locally compact Hausdorff topological space. We denote by

- $\mathcal{M}^+(X)$ the set of all (positive) Radon measures $\mu : \mathcal{B}(X) \to [0, \infty]$;
- $\mathcal{M}^+_{\text{fin}}(X)$ the set of all finite (positive) Radon measures $\mu : \mathcal{B}(X) \to [0, \infty)$.

Definition C.7.

(i) The *weak topology on $\mathcal{M}^+_{\text{fin}}(X)$* is defined as the weakest topology on $\mathcal{M}^+_{\text{fin}}(X)$, for which all mappings

$$\mathcal{M}^+_{\text{fin}}(X) \to \mathbb{C},\ \mu \mapsto \int_X f(x)\,d\mu(x)$$

with $f \in C_b(X)$ are continuous. In particular, if $(\mu_n)_{n\in\mathbb{N}}$ is any sequence in $\mathcal{M}^+_{\text{fin}}(X)$, we say that $(\mu_n)_{n\in\mathbb{N}}$ *converges weakly to* μ for some $\mu \in \mathcal{M}^+_{\text{fin}}(X)$, if

$$\lim_{n\to\infty} \int_X f(x)\,d\mu_n(x) = \int_X f(x)\,d\mu(x) \qquad \text{for all } f \in C_b(X).$$

(ii) The *vague topology on $\mathcal{M}^+(X)$* is given as the weakest topology, for which all mappings

$$\mathcal{M}^+(X) \to \mathbb{C},\ \mu \mapsto \int_X f(x)\,d\mu(x)$$

with $f \in C_c(X)$ are continuous. In particular, if $(\mu_n)_{n\in\mathbb{N}}$ is any sequence in $\mathcal{M}^+(X)$, we say that $(\mu_n)_{n\in\mathbb{N}}$ *converges vaguely to* μ for some $\mu \in \mathcal{M}^+(X)$, if

$$\lim_{n\to\infty} \int_X f(x)\,d\mu_n(x) = \int_X f(x)\,d\mu(x) \qquad \text{for all } f \in C_c(X).$$

Due to the inclusion $C_0(X) \subset C_b(X)$, weak convergence clearly implies vague convergence. The converse is wrong in general, but there is an additional condition that guarantees equivalence (see [**Els11**, Exercise VIII.4.4 (b)]): a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathcal{M}^+_{\text{fin}}(X)$ converges weakly to some $\mu \in \mathcal{M}^+_{\text{fin}}(\mathbb{R})$, if and only if $(\mu_n)_{n\in\mathbb{N}}$ converges vaguely to μ and

$$\lim_{n\to\infty} \mu_n(X) = \mu(X).$$

Theorem C.4 implies that the vague topology is Hausdorff. In particular, limits of vague convergent sequences are unique. One can show that also the limit of a weakly convergent sequence of finite Radon measures is unique; see [**Els11**, Exercise VIII.4.4 (a)].

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