Existence and regularity theorems for variants of the TV-image inpainting method in higher dimensions with vector-valued data

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Abstract

In this thesis we are mainly concerned with a modification of the classical total variation image inpainting model. This alteration, which leads to a variational problem with linear growth, has been suggested by M. Bildhauer and M. Fuchs and is of interest since it describes inpainting with simultaneous denoising, i.e., we jointly reconstruct the region of the image for which data are missing or inaccessible and denoise the generated image on the entire domain. First numerical experiments in collaboration with J. Weickert have revealed that the above modification is numerically comparable to the standard total variation image inpainting model with the advantage of a comprehensive existence and regularity theory of the corresponding solutions. The main focus of this thesis lies on establishing such a theory for any dimension together with arbitrary codimension, i.e., vector-valued images are included in our investigations.
More precisely we first show existence of generalized minimizers (in a suitable sense) and pass to the associated dual problem. In this context we prove new density results for functions of bounded variation and for Sobolev functions. Afterwards we investigate the regularity behavior of generalized minimizers.
As a slight advancement we moreover study a special non-autonomous variant of the above variational problem in the context of the denoising of images for which we establish existence and regularity results of generalized minimizers.

Zusammenfassung

Diese Arbeit beschäftigt sich hauptsächlich mit einer Abwandlung des klassischen TV-image inpainting Modells. Diese Modifikation, welche ein Variationsproblem mit linearem Wachstum beschreibt, wurde von M. Bildhauer und M. Fuchs vorgeschlagen und vereinigt das sogenannte inpainting mit simultanem Entrauschen. Numerische Experimente in Zusammenarbeit mit J. Weickert haben gezeigt, dass die obige Modifikation im Vergleich zu den bekannten TV-image inpainting Verfahren numerisch vergleichbare Ergebnisse erzielt. Ein klarer Vorteil des neuen Modells ist jedoch, dass eine ganzheitliche Existenz- und Regularitätstheorie für Lösungen existiert, wobei es ein Kernanliegen dieser Arbeit ist, eine solche Theorie für beliebige Dimensionen in Kombination mit beliebiger Kodimension zu entwickeln.
Zunächst wird dabei die Existenz verallgemeinerter Minimierer (in einem geeigneten Sinne) gezeigt, bevor wir anschließend das duale Problem untersuchen. Als Hilfsmittel werden wir neue Dichtheitssätze für Funktionen von beschränkter Variation und für Sobolevfunktionen beweisen. Im Anschluss diskutieren wir die Regularität verallgemeinerter Minimierer.
Ferner werden wir eine nicht-autonome Modifikation des obigen Variationsproblems im Kontext des reinen Entrauschens von Bildern untersuchen. Dabei werden wir Existenz und Regularität von verallgemeinerten Minimieren beweisen sowie die Existenz und Eindeutigkeit dualer Lösungen verifizieren.

Contents

Chapter 1

Introduction

The calculus of variations is an important and intensively studied field in mathematical analysis. One of the most basic problems occuring in this context is the minimization of (strictly) convex functionals subject to additional constraints (such as prescribed Dirichlet boundary data). Problems of such type often arise, e.g., in engineering, mathematical physics, geometry, economics or digital image processing (see, e.g., [58] for an overview of historical facts, examples and references).
The subject of this thesis is the study of (strictly) convex energy functionals occuring in digital image processing. Before going into analytical details, we give a short introduction in the field of image analysis.

Image analysis

Digital image processing is about transforming a digital image into another digital image that allows a better interpretation by humans or computers. In the mathematical sense we understand images as mappings $w : \Omega \to \mathbb{R}^M$, $M \geq 1$, where $\Omega \subset \mathbb{R}^n$, $n \geq 2$, usually denotes a bounded Lipschitz domain, e.g., a rectangular area in the case $n = 2$ or a cuboid if $n = 3$. In fact there are numerous types of images depending on the dimension of the domain and the codomain, respectively.

- For $n = 2$ and $M = 1$ we are concerned with a classical digital image (see Figure 1.1)) whose co-domain specifies the grey value. Normally, low grey levels are dark and high grey levels are bright in this context (see, e.g., [13, 66, 73]).

- The case $n = 3$ together with $M = 1$ covers three-dimensional images that are of fundamental meaning in medical imaging, e.g., computerized tomography or magnetic resonance imaging (MRI) (see, e.g., [72, 73, 97] and the references quoted therein).

- Considering the vectorial setting, i.e., $M > 1$, with arbitrary dimension $n \geq 2$, we are confronted with color images where each channel (or dimension) represents a corresponding color (see, e.g., [29, 48]). Another example of a vector-valued image is a multi-spectral image (e.g. satellite images) containing a variety of channels representing different frequency bands (see, e.g., [101]). Moreover we can consider tensor-valued images, e.g., matrix-valued images $w : \Omega \to \mathcal{M}_M(\mathbb{R})$ ($\mathcal{M}_M(\mathbb{R})$ denotes the set of $M \times M$-square matrices with real entries). Such tensor-valued data arise, e.g., in diffusion tensor magnetic resonance imaging (DT-MRI) and physical measurements of anisotropic behavior (see, e.g.,[101, 102] and the references quoted therein).

Figure 1.1: A classical digital image. Courtesy of J. Weickert

Image denoising

In what follows we are concerned with one of the oldest and most fundamental problems arising in image processing, the so-called image restoration. Before going into details we fix our basic setup and our underlying assumptions: we suppose that we are given an observed image represented by a $\mathcal{L}^n$-measurable ($\mathcal{L}^n$ denoting Lebesgue's measure on $\mathbb{R}^n$), at least locally integrable, function $f : \Omega \to \mathbb{R}^M$ ($M \geq 1$). We assume this image to be corrupted by a statistical phenomenon called noise (see Figure 1.2). In applications, this noise is usually caused by technical issues, such as faulty accquistion or errenous transmission. Usually, an additive noise model is assumed which means that the decomposition

$$f = u + n, \tag{1.0.1}$$

holds. Here, the function $n : \Omega \to \mathbb{R}^M$ models the noise and the function $u : \Omega \to \mathbb{R}^M$ stands for "meaningful" image data. In the discrete setting, the functions f, u and n are represented as a (higher-dimensional) array of real values and the noise n affects the values of u in a specific way, depending on the special nature of n, that is, the model for n. A famous and very popular noise model is the so-called Gaussian noise (we refer the reader to [13], p.61, where some examples of image degradation by noise are presented).

(a) Original image

(b) Original image with additive Gaussian noise

Figure 1.2: Example of image degradation. Courtesy of J. Weickert

The general goal of image denoising is to recover the unknown u from the given data f. There is a huge amount of methods and techniques to achieve this goal which roughly can be divided into three types: the probabilistic methods (see, e.g., [30, 56] and the references quoted therein), methods based on partial differential equations (PDEs), and variational methods relying on the minimization of certain energy functionals. We are strictly concerned with techniques from the latter type and here especially with variants of the so-called TV-regularization (TV stands for "total variation"). For detailed information on PDE-based methods and the minimization of the total variation or of related functionals having superlinear growth we refer to [1, 13, 14, 22, 29, 31, 34–37, 42, 70, 82, 85, 98, 100] and the references quoted therein where among theoretical aspects, numerical investigations are partially carried out as well.
In the variational approach to image denoising, the original image u is thought to be the (hopefully) unique minimizer of a functional of type (see, e.g., [1, 13, 31] and the references quoted therein)

$$J[w] := \int_{\Omega} |w - f|^2 dx + \lambda \int_{\Omega} \psi(|\nabla w|) dx. \tag{1.0.2}$$

The function $w : \Omega \to \mathbb{R}^M$ is an element of an adequate energy class $\mathbb{K}$, $f \in L^2(\Omega)^M$, ψ is a suitably pre-selected (strictly) convex and increasing function and finally λ is a positive parameter steering the amount of regularization. In this thesis we restrict ourselves to unconstrained problems, hence we do not take into account any boundary condition of Dirichlet-, Neumann- or any other type.
Interpreting the structure of the functional J, the first term in (1.0.2) can be regarded as a measure for the quality of "data fitting", i.e., the deviation of the image $w : \Omega \to \mathbb{R}^M$ from the original data f on Ω. The second term in (1.0.2) allows to incorporate apriori information of the sought minimizer into the minimization process and can be interpreted as a regularizer. In our case, this apriori information is a prescribed degree of smoothness of w, and then, the regularizing term is sometimes called "fidelity" term. At this point we like to point out that the name "fidelity term" is often used for the data term as well (in this case, the data term is sometimes called "data fidelity"-term). Nevertheless, in this thesis, the label fidelity term will be only used for the regularizing

term. Observe that the presence of noise in an image decreases its fidelity by the very nature of noise. As a consequence, a large (relative) weight on the data term favoures close-to-data-minimizers while a relatively large weight on the fidelity term increases the smoothness of the minimizer. Roughly speaking, it is the balance between the opposite effects of the data fitting- and fidelity-term that determines the minimization process and the characteristics of the minimizer.

It is natural that the choice of the function ψ in (1.0.2) crucially influences the minimization process and the properties of the minimizer, i.e., the result of the denoising process. One common and classical choice of the function ψ is

$$\psi(t) := t^2, \quad t \geq 0, \tag{1.0.3}$$

i.e., we choose the fidelity term to be Dirichlet's energy. This idea has been proposed in 1977 by Arsenin and Tikhonov (see [95]), where the original aim of adding a regularization term was to overcome the ill-posedness of the mere data fitting minimization problem. To be precise we seek minimizers of the functional

$$J_2[w] := \int_\Omega |w - f|^2 dx + \lambda \int_\Omega |\nabla w|^2 dx, \tag{1.0.4}$$

with functions $w : \Omega \to \mathbb{R}^M$ in the appropriate Sobolev space $W^{1,2}(\Omega)^M$ (for details concerning those spaces the reader is referred to [4]). As a slight generalization of (1.0.3) we can consider $\psi(t) := t^p$ with $p > 1$, i.e., we look for minimizers of the functional

$$J_p[w] := \int_\Omega |w - f|^2 dx + \lambda \int_\Omega |\nabla w|^p dx$$

among functions $w : \Omega \to \mathbb{R}^M$ in the space $W^{1,p}(\Omega)^M \cap L^2(\Omega)^M$. Here, the intersection of the Sobolev space $W^{1,p}(\Omega)^M$ and the Lebesgue space $L^2(\Omega)^M$ ensures well-definedness of the functional J_p. At this point we give a short comment on the space $W^{1,p}(\Omega)^M \cap L^2(\Omega)^M$: if Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, then there exists a continuous embedding $W^{1,p}(\Omega)^M \hookrightarrow L^q(\Omega)^M$ for all $1 \leq q < \frac{np}{n-p}$ (see, e.g., [4], Theorem 5.4, p. 97/98). Thus, under certain assumptions on the dimension n and the exponent p, the problem “$J_p \to \min$“ can be investigated on the entire space $W^{1,p}(\Omega)^M$. Otherwise, the requirement “$w \in L^2(\Omega)^M$“ is an additional constraint. Using the direct method in the calculus of variations we see, that the problem “$J_p \to \min$“ admits at least one minimizer $u \in W^{1,p}(\Omega)^M$. Uniqueness of the minimizer then is a consequence of the strict convexity of the data fitting term $\int_\Omega |w - f|^2 dx$ w.r.t. w.

Thus, by involving p-growth with some finite exponent $p > 1$ in the fidelity term we are actually dealing with superlinear problems. As a consequence, we are able to remove the noise and retrieve smooth solutions, where the procedure does not cause numerical problems. Since large gradients indicate edges in an image, a severe penalization of the gradient norm produces "over-smoothed" images, where essential features such as edges are blurred (see Figure 1.3 in the case $p = 2$). The penalizing effect and hence the blurring effect is as smaller as closer the exponent $p > 1$ is to one.

(a) noisy image

(b) filtered image

Figure 1.3: Regularization with Dirichlet's energy (The regularization parameter λ has been optimized such that the smallest possible mean square error w.r.t. the original image without noise is obtained).
Courtesy of J. Weickert

Now we are facing two technical obstacles. First, in view of the above observation, the natural question arises, how to find the optimal choice of the exponent p for preserving edges and other characteristic ingredients of the observed image f. If we consider the limit case $p = 1$ in the functional J_p, we are confronted with the problem

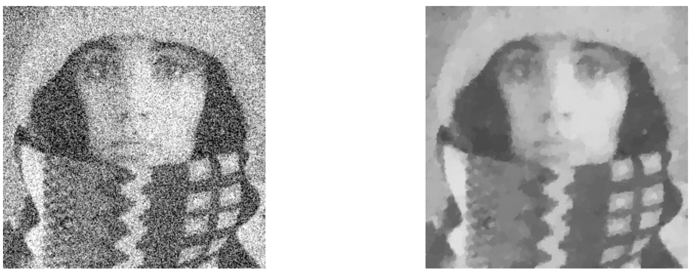
$$J_1[w] := \int\limits_\Omega |w - f|^2 dx + \lambda \int\limits_\Omega |\nabla w| dx \to \min \text{ in } W^{1,1}(\Omega)^M \cap L^2(\Omega)^M.$$

Recalling the continuous embedding $W^{1,1}(\Omega)^M \hookrightarrow L^{\frac{n}{n-1}}(\Omega)^M$ it becomes evident that the requirement "$w \in L^2(\Omega)^M$" is an additional constraint if $n \geq 3$. Furthermore, a functional analytical problem arises since the Sobolev space $W^{1,1}(\Omega)^M$ is not reflexive, i.e., apriori we can not assume existence of a weakly convergent subsequence of minimizing sequences being uniformly bounded in $W^{1,1}(\Omega)^M$. Hence, we cannot expect solvability of the above problem.
Secondly, the Sobolev spaces in general are not suitable for image processing since they simply do not contain "important" images such as cartoon-like images. To be more precise, even a characteristic function $\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$ with $A \subsetneq \Omega$ (A is assumed to be at least a set of finite perimeter in Ω, see, e.g., [63] for more details) does not belong to any Sobolev space. Already in the one-dimensional case with $A = (-\varepsilon, \varepsilon) \subset \Omega := (-1, 1)$, it is obvious that $\mathbb{1}_A \in L^1((-1, 1))$ whereas its distributional derivative is a difference $\delta_{-\varepsilon} - \delta_\varepsilon$ of Dirac δ-distributions representing a signed Radon measure with finite total variation. Now, simple calculations show that $\mathbb{1}_A$ does not belong to any Sobolev space $W^{1,p}(\Omega)$ with $p \geq 1$. However, the space of functions of bounded variation $BV(\Omega)^M$ covers all L^1-functions $w : \Omega \to \mathbb{R}^M$ whose distributional

gradient ∇w is represented by a tensor-valued Radon measure on Ω with finite total variation $\int\limits_\Omega |\nabla w|$ (for details, we refer to the monographies of Ambrosio, Fusco and Pallara [7] and Giusti [63]). This suggests that the space $BV(\Omega)^M$ has to be considered when dealing with the denoising of images and it is therefore natural to consider an extension of the functional J_1 to the larger space $BV(\Omega)^M \cap L^2(\Omega)^M$. One of the first contributions in this direction has been made by Rudin, Osher and Fatemi in 1992. In their work [85], they suggested to involve the total variation of w into the minimization process of J_1. This leads to the TV-regularization in its most elementary form which means that we are interested in solutions of the problem

$$\widetilde{J}[w] := \int\limits_\Omega |w - f|^2 dx + \lambda \int\limits_\Omega |\nabla w| \to \min \text{ in } BV(\Omega)^M \cap L^2(\Omega)^M. \tag{1.0.5}$$

Recalling the continuous embedding $BV(\Omega)^M \hookrightarrow L^{\frac{n}{n-1}}(\Omega)^M$ (see, e.g., [7], Corollary 3.49, p.152) we see that the requirement "$w \in L^2(\Omega)^M$" represents an additional constraint if $n \geq 3$. Using well-known properties of functions of bounded variation we state, that $\widetilde{J}$ admits at least one solution. Once more, uniqueness follows by strict convexity of the data fitting quantity w.r.t. w. Provided that our original function f is bounded we can even derive a maximum principle for u. At this point, the question arises if the $\widetilde{J}$-minimizer u can be linked with the original problem formulated in the space $W^{1,1}(\Omega)^M$. Clearly, u is not J_1-minimizing but it can be viewed as a generalized minimizer of J_1 in a suitable sense, as will be clarified in the second chapter of this thesis.
As a method for image denoising, TV-regularization has clearly its merits: no generation of new discontinuities, preservation of already existing edges, oversmoothing and blurring does not occur and in general, cartoon-like images are preserved to a large extent. Nevertheless, there are also drawbacks of this technique: regions, where the function values are changing in a smooth manner turn into regions, where the function values are piecewise constant. This phenomenon is called staircase effect (see, e.g., [39]).



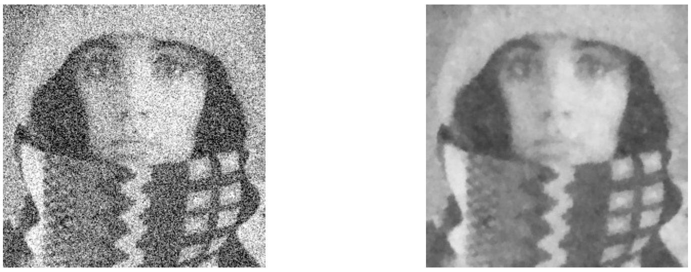
(a) noisy image (b) filtered image

Figure 1.4: TV-regularization (The regularization parameter λ has been optimized such that the smallest possible mean square error w.r.t. the original image without noise is obtained).
Courtesy of J. Weickert

An interesting compromise between the case $p > 1$ and the limit case $p = 1$ provides the choice $\psi(t) := h(t) := t \log(1+t), t \geq 0$. The function ψ satisfies a “nearly linear growth“ condition (“$L \log L$ case“). This leads to the so-called “logarithmic regularization“ which boils down to the minimization of the functional

$$\int_\Omega |f-w|^2 dx + \lambda \int_\Omega h(|\nabla w|) dx \tag{1.0.6}$$

with $w \in W^{1,h}(\Omega)^M \cap L^2(\Omega)^M$. Here, $W^{1,h}(\Omega)^M$ represents the Orlicz-Sobolev space (see, e.g., [4] for more details) generated by the N-function h. It is worth noting that we can choose any finite iteration of the logarithm for the function h as well. As outlined in [22] in the two-dimensional setting, the usage of such a N-function as density ψ indeed leads to a less strong smoothing effect if compared to the regularization results obtained with densities of p-growth for any exponent $p > 1$. These theoretical statements are at least on a visual level confirmed by numerical experiments in collaboration with J. Weickert (see below).



(a) noisy image

(b) filtered image

Figure 1.5: logarithmic regularization (The regularization parameter λ has been optimized such that the smallest possible mean square error w.r.t. the original image without noise is obtained).
Courtesy of J. Weickert

Denoising by means of TV-regularization, that is the usage of the functional $\widetilde{J}$, has its clear advantages. However, due to the lack of ellipticity we cannot expect regular solutions of the corresponding variational problem in general. In order to overcome this effect we exploit an idea from [22] and replace the rather unpleasant TV-density $|\nabla w|$ by a strictly convex integrand $F(\nabla w)$. This density F is assumed to be of linear growth w.r.t. the tensor-valued Radon measure ∇w but possesses better ellipticity properties. The corresponding functional now reads

$$J[w] := \lambda \int_\Omega F(\nabla w) dx + \int_\Omega |w-f|^2 dx, \quad w \in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M. \tag{1.0.7}$$

Nonetheless, the study of smoothness properties of corresponding solutions of (1.0.7) remains a difficult problem since the required linear growth of F admits only weak and anisotropic ellipticity conditions and it is not surprising that regularity crucially depends on the modulus of ellipticity that we propose. In accordance with [22] (see equation (1.13) therein) the following choice of F provides a natural class of examples w.r.t. approximating the TV-density $|P|$, $P \in \mathbb{R}^{nM}$: let us fix a real number $\mu > 1$ and define the following family of densities

$$\varphi_\mu(r) := \int_0^r \int_0^s (1+t)^{-\mu} dt ds, \quad r \in \mathbb{R}_0^+. \tag{1.0.8}$$

We observe for $\mu \neq 2$

$$\varphi_\mu(r) = \frac{r}{\mu-1} + \frac{1}{\mu-1}\frac{1}{\mu-2}(r+1)^{-\mu+2} - \frac{1}{\mu-1}\frac{1}{\mu-2}, \tag{1.0.9}$$

while in the case $\mu = 2$ we get

$$\varphi_2(r) = r - \log(1+r).$$

Since, in the TV-case, the density just depends on the modulus of ∇w, it is advantageous to introduce

$$\Phi_\mu(Z) := \varphi_\mu(|Z|), \quad Z \in \mathbb{R}^{nM}. \tag{1.0.10}$$

Standard calculations show that Φ_μ is strictly increasing, strictly convex, of class $C^2(\mathbb{R}^{nM})$, satisfies $\Phi_\mu(0) = 0$, $D\Phi_\mu(0) = 0$, is of linear growth and approximates the TV-density $|\nabla w|$ in the following sense

$$(\mu - 1)\Phi_\mu(Z) \to |Z| \quad \text{as } \mu \to \infty \tag{1.0.11}$$

for all $Z \in \mathbb{R}^{nM}$. Thus, the $\Phi_\mu(\nabla w)$ are integrands F of linear growth that approximate $|\nabla w|$. Furthermore, the Φ_μ satisfy much better ellipticity properties since they are even μ-elliptic with prescribed elliptic parameter $\mu > 1$. For the notion of μ-ellipticity we refer the reader to, e.g., [17], Assumption 4.1, p.97. Section 3.1 of this thesis is devoted to this concept as well.

Now, the properties of Φ_μ motivate to look at the following modification of the TV-regularization

$$\lambda \int_\Omega \Phi_\mu(\nabla w) + \int_\Omega |w - f|^2 dx \to \min \text{ in } BV(\Omega)^M \cap L^2(\Omega)^M \tag{1.0.12}$$

which admits a unique solution $u \in BV(\Omega)^M \cap L^2(\Omega)^M$. At first glance, the above problem seems somewhat artificial but numerical experiments carried out in collaboration with J. Weickert (see [28]) indicate that the above modification of the TV-regularization (1.0.12) is numerically comparable to the standard TV-model (see Figure 1.6). However, in contrast to the classical TV-model (1.0.5), we can provide a complete analysis of the regularity properties of the BV-minimizer of (1.0.12) which is one of the main goals of this thesis.

(a) noisy image

(b) filtered image, $\mu = 3/2$

(c) filtered image, $\mu = 20$

Figure 1.6: Φ_μ-regularization (The regularization parameter λ has been optimized such that the smallest possible mean square error w.r.t. the original image without noise is obtained).
Courtesy of J. Weickert

It is clear that we can admit arbitrary real values of the ellipticity parameter μ in the problem (1.0.12). If $\mu < 1$, then Φ_μ is a strictly increasing and strictly convex function having superlinear growth with power growth order $p := 2-\mu > 1$ which allows to investigate problem (1.0.12) in the classical setting of the Sobolev space $W^{1,p}(\Omega)^M$. Considering $\mu = 1$ we are located in the "nearly linear growth" situation which means that we can discuss problem (1.0.12) in the Orlicz-Sobolev space $W^{1,h}(\Omega)^M$, where $h(t) := t\log(1+t), t \geq 0$, denotes the corresponding N-function.

Image inpainting

Now we turn to another interesting and well-known problem in image restoration, the so-called image inpainting. We provide a short description of the image inpainting problem: suppose that we are given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ and a $\mathcal{L}^n$-measurable subset D of Ω having the property

$$0 \leq \mathcal{L}^n(D) < \mathcal{L}^n(\Omega).$$

Furthermore we assume that we are given an image described by a $\mathcal{L}^n$-measurable function $f : \Omega - D \to \mathbb{R}^M$ representing the partial and usually distorted observation of our image on $\Omega - D$. In this context, the subset D, that is called (at least in the scalar case $M = 1$ together with $n = 2$) "inpainting domain"(see, e.g., [33]), is a certain part of the image for which image data are not available. Thus, the aim is to develop methods and techniques to restore the image values for the missing part D from the known values of the part $\Omega - D$. In other words we want to generate an image $u : \Omega \to \mathbb{R}^M$ which interpolates the incomplete image $f : \Omega - D \to \mathbb{R}^M$. This kind of image interpolation fills in the missing image data on the set D and is called, at least in the scalar case $M = 1$ together with $n = 2$, "image inpainting" or just "inpainting" (see, e.g., [33, 84, 88]).
In accordance with [84] there are essentially four different types of approaches

which are concerned with the inpainting problem. They can be either variational or non-variational and local or non-local.
As outlined in [84], local inpainting techniques take the information needed for the filling-in process only from neighboring points of the boundary ∂D of D. This is the most common type of methods employed if the inpainting region D is rather small (see, e.g., [16, 38, 40, 41, 50, 84] and the references quoted therein). In contrast to this, non-local methods use the entire range of information available on the known part $\Omega - D$ of the image (see, e.g., [10–12]). As elucidated in [84], these techniques are desirable if one wants to fill in structures and textures into patch-like sets. However, these types of techniques often cause high computational costs.
Non-variational approaches for instance might be based on the direct use of a PDE not derived from a functional to fill in the missing information.

In this thesis we discuss a variational approach towards image inpainting being of non-local type. To be more precise we study a modification of the total variation (="TV") image inpainting problem, where the original TV image inpainting method is given by the problem (see, e.g., [84])

$$\int_\Omega |\nabla w| + \frac{\lambda}{2} \int_{\Omega - D} |w - f|^2 dx \to \min \quad \text{in} \quad BV(\Omega)^M \cap L^2(\Omega - D)^M.$$

Here, as usual, $\lambda > 0$ is a regularization parameter and we assume that the incomplete image f is of class $L^2(\Omega - D)^M$. At this point we like to point to the fact that the regularization parameter usually acts as a prefactor of the regularization term (see, e.g., [11]). Nonetheless, in the above problem, the prefactor λ still balances the data term and the fidelity term, where small values of λ increase the effect of smoothness of the minimizer. In contrast, large values of λ rather generate close-to-data minimizers. Using standard properties of BV-functions we observe that the above functional admits at least one BV-minimizer but in contrast to the TV-regularization used for the pure denoising of images we lose the uniqueness of BV-solutions in this context. Indeed, we merely have uniqueness on the set $\Omega - D$. Besides, as already mentioned above, we apriori cannot expect any smoothness properties of solutions of variational problems where a total variation component is involved.
In [24] extending ideas from [22], M. Bildhauer and M. Fuchs replaced the TV-density $|\nabla w|$ by a modified density $F(\nabla w)$ with linear growth that satisfies better ellipticity properties and approximates the TV-density in a suitable sense. Here, the density Φ_μ from (1.0.10) appears again as a natural example. Hence, we study the problem

$$\begin{aligned} I[w] := \int_\Omega F(\nabla w) + \frac{\lambda}{2} \int_{\Omega - D} |w - f|^2 dx \to \min \\ \text{in } BV(\Omega)^M \cap L^2(\Omega - D)^M. \end{aligned} \tag{1.0.13}$$

Now a short comment on the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$ is in order: in accordance with the continuous embedding $BV(\Omega)^M \hookrightarrow L^{\frac{n}{n-1}}(\Omega)^M$, the requirement

"$w \in L^2(\Omega - D)^M$" represents an additional constraint if $n \geq 3$.
Note, that in view of (1.0.12) the variational approach in (1.0.13) actually describes inpainting with simultaneous denoising. This means that if the incomplete image f on $\Omega - D$ is corrupted by noise we can jointly denoise f on $\Omega - D$ and generate a completion u defined on the whole domain Ω.
It is also worth mentioning that various modifications of the above technique of inpainting with simultaneous denoising can be constructed. To this end, the reader is referred to [23] where some of the modifications of the TV-image inpainting method have been studied in the scalar case together with $n = 2$ and possibly additional assumptions on D.

Following our considerations in the context of image denoising, it is reasonable to choose $F(Z) := \Phi_\mu(Z)$, $Z \in \mathbb{R}^{nM}$, in (1.0.13) with $\Phi_\mu(Z)$ as before from (1.0.10). Hence, we again exploit the approximation of the TV-density by means of Φ_μ (compare (1.0.11)). Although the investigation of problem (1.0.13) with the special choice $F(Z) = \Phi_\mu(Z)$ has been triggered by theoretical interest, numerical examinations performed in collaboration with J. Weickert have illustrated that this variational approach yields results that are numerically comparable to the standard TV-image inpainting model.

The investigation of the general functional (1.0.13) is a major subject of this thesis as well. We will provide a comprehensive analysis of the smoothness properties of its BV-minimizers for any dimension $n \geq 2$ and any codimension $M \geq 1$.

A short summary of already known results

Considering the scalar case $M = 1$ together with $n = 2$, problem (1.0.13) has been studied extensively by M. Bildhauer and M. Fuchs in [23–26] and for the case of pure denoising ("$D = \emptyset$") in the related work [22]. It is worth mentioning, that in [22], the case of vector-valued images together with additional (Dirichlet-) boundary data u_0 has been included as well. In [24], the existence of a unique minimizer of the functional

$$\int_\Omega F(\nabla w)dx + \frac{\lambda}{2} \int_{\Omega - D} |w - f|^2 dx \tag{1.0.14}$$

has been shown in the classical Sobolev space $W^{1,1}(\Omega)$ (compare [24], Theorem 1.3). Note that in accordance with Sobolev's embedding theorem, the functional in (1.0.14) is well-defined on the entire space $W^{1,1}(\Omega)$. This existence result was proven under rather strong ellipticity conditions on the density F. To be more precise, among other appropriate conditions, F is supposed to be μ-elliptic with prescribed ellipticity parameter $\mu \in (1, 2)$, i.e., the problem is located near the "nearly linear growth" situation. Furthermore, the incomplete image f is assumed to be of class $L^\infty(\Omega - D)$. Under these assumptions, it was possible to establish a maximum principle for the unique $W^{1,1}$-minimizer u. As an additional main result of [24], it was shown the continuity of $DF(\nabla u)$ on Ω which implies strong partial regularity of u on Ω in the sense that $\dim_{\mathcal{H}}(\mathrm{Sing}_{\Omega}(u)) = 0$, where $\mathrm{Sing}_{\Omega}(u)$ describes the set of interior singularities of u in Ω. By definition, $\dim_{\mathcal{H}}(\mathrm{Sing}_{\Omega}(u)) = 0$ means that ($\mathcal{H}^{\varepsilon}$ denoting the Hausdorff measure of dimension ε) $\mathcal{H}^{\varepsilon}(\mathrm{Sing}_{\Omega}(u)) = 0$ for any $\varepsilon > 0$, i.e., the set $\mathrm{Sing}_{\Omega}(u)$ is very small in a measuretheoretical sense.

In view of the results of [24], it is natural to ask how to deal with the functional in (1.0.14) for large values of the parameter μ, i.e., for $\mu > 2$. Here, an immediate problem arises from the lack of reflexivity of the Sobolev space $W^{1,1}(\Omega)$ implying that in general we cannot expect existence of $W^{1,1}$-minimizers of (1.0.14). Clearly, for $\mu \in (1, 2)$, the Sobolev space $W^{1,1}(\Omega)$ remains not reflexive and we apriori have the same problems as for large values of μ but in case $\mu \in (1, 2)$, Bildhauer and Fuchs were able to overcome this difficulty. Unfortunately, their strategy fails and cannot be extended to $\mu > 2$.
In [25], they used the concept of a convex function of a measure (see, e.g., [46] or [8, 60]) and a suitable relaxation of the functional in (1.0.14) formulated on the space $BV(\Omega)$. Then, it could be shown that the relaxed version of functional (1.0.14) under some suitable assumptions on F is solvable (see [25], Theorem 1.2). As a remarkable byproduct, the authors established a maximum principle for each BV-minimizer of the relaxed functional. Furthermore, they justified that each minimizer of the relaxation can be seen as generalized minimizer of the original functional in (1.0.14) and vice versa.
The dual variational problem associated to the original problem (1.0.13) has been considered in [25] as well. For the dual problem, the authors showed existence of a solution $\sigma \in L^{\infty}(\Omega)^2$ and the validity of the so-called inf-sup relation. Uniqueness of the dual solution σ has been derived under the assumption that the conjugate function F^* to F is strictly convex on the set $\{p \in \mathbb{R}^2, F^*(p) < \infty\}$ (see [25], Theorem 1.4). We refer to [49] or Section 2.1 for more details concerning the dual problem and facts from convex analysis and from duality theory.
Assuming that the set of interior points $\mathrm{Int}(D)$ of D is non-empty, Bildhauer and Fuchs stated some regularity results for the BV-minimizers and the dual solution σ on $\mathrm{Int}(D)$ under appropriate assumptions on the density F and the given data (compare [24], Theorem 1.2, 5., and Theorem 1.4, 3. and 4.).

Taking the partial regularity statement of Theorem 1.4 in [24] and therewith the existence of a non-empty, small set of singularities as a basis, the question arises if it is possible to exclude these interior singularities, i.e., to show full interior $C^{1,\alpha}$-regularity of the unique $W^{1,1}$-minimizer u of the functional in (1.0.14). In the joint work [27] we could give, at least in the scalar setting together with $n = 2$ and for fixed ellipticity parameter $\mu \in (1, 2)$, a satisfying answer to this question. We showed everywhere $C^{1,\alpha}$-regularity of u by performing a refined De Giorgi-type iteration (in contrast to the type of De Giorgi-type iteration that has been carried out in [17], Theorem 4.28, pp. 119) which has been modified to the situation at hand (see [27], Theorem 2).

Structure and main results of the thesis

Inspired by [24, 25] and from the joint work [27], one major effort in this thesis and more exactly of the second and third chapter, respectively, consists in generalizing the results of [24, 25, 27] to the case of any dimension $n \geq 2$ and any codimension M.
We start by giving some comments on the structure and the main results of the second chapter of the thesis: assuming from now on $n \geq 2$ and $M \geq 1$ we consider the functional (recall (1.0.13))

$$I[w] := \int\limits_{\Omega} F(\nabla w)dx + \frac{\lambda}{2} \int\limits_{\Omega - D} |w - f|^2 dx. \tag{1.0.15}$$

Here, F is supposed to be a strictly convex function being of linear growth w.r.t. the modulus of the gradient and satisfying some additional (rather weak) conditions which will be specified in appropriate places. Thus, I is well-defined for functions $w \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$ where in accordance with Sobolev's embedding theorem, the requirement "$w \in L^2(\Omega - D)^M$ " then acts as an additional constraint if $n \geq 3$.
Since as usual we cannot expect existence of I-minimizers in the suitable space $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$ we either have the possibility to weaken the notion of an I-minimizer, i.e., to introduce a suitable concept of a generalized minimizer or to pass to the dual variational problem associated to problem "$I \to \min$".
In the second chapter we investigate both aspects where we emphasize that we actually present the material of the joint article [55] with M. Fuchs. After introducing a suitable relaxation of the original functional I formulated on the space $BV(\Omega)^M$ and defining the set of generalized minimizers of the functional I we can show

Theorem 1.0.1
Let us assume $n \geq 2$ and $M \geq 1$ together with $f \in L^2(\Omega - D)^M$. Further, let F satisfy some appropriate (weak) conditions. Then there exists at least one generalized I-minimizer u (w.r.t. to a natural relaxation of the functional I) from the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$. Further, the set of all generalized minimizers coincides with the set of all minimizers of the corresponding relaxed variant of the functional I.

Concerning the dual problem we will prove

Theorem 1.0.2
Let us assume $n \geq 2$ and $M \geq 1$ together with $f \in L^2(\Omega - D)^M$. Further, the density F is required to satisfy some appropriate (weak) conditions. Then, there exists a unique dual solution $\sigma \in L^\infty(\Omega)^{nM}$ of the dual problem associated to the problem "$I \to min$" and the inf-sup relation holds true. Moreover we have the validity of the duality formula

$$\sigma = DF(\nabla^a u) \quad a.e. \text{ on } \Omega,$$

where u denotes any generalized minimizer from Theorem 1.0.1 and $\nabla^a u$ denotes the regular part of the tensor-valued Radon measure ∇u w.r.t. Lebesgue's measure being unique $\mathcal{L}^n$-a.e. on Ω.

Comparing the results from Theorem 1.0.1 with the results from Theorem 1.2 in [25] there are in fact two novelties in Theorem 1.0.1: on the one hand we generalize Theorem 1.2 in [25] to $n \geq 2$ with arbitrary codimension M and on the other hand these generalizations work under slightly weaker assumptions on F and f. Thus, we even have a generalization in the special case $n = 2$ at hand. For instance, we can drop the structure condition on F and avoid the usage of a maximum principle. It is worth mentioning that the latter point leads to severe problems and requires the derivation of a new density result for BV-functions (see Lemma 2.2.6).
However, by imposing a structure condition on F we can essentially use the same arguments as presented in [24], Theorem 1.2, for discussing the case $n \geq 3$ with arbitrary codimension M. In this context, an appropriate maximum principle which will be deduced for arbitrary minimizing sequences of the relaxed functional acts as an essential tool.
Comparing Theorem 1.0.2 and [24], Theorem 1.4 there are in fact three novelties: at first there is a generalization of the results from [24], Theorem 1.4, to the case $n \geq 2$ with arbitrary codimension M. In this context, it is to remark, that slightly weaker assumptions on the density F and the data f are needed. A second novelty is, that a new density result for Sobolev functions is used in order to prove solvability of the dual problem associated to the problem "$I \to \min$" (see Lemma 2.2.4). As the last novelty, uniqueness of the dual solution is proven without requiring, that the conjugate function F^* to F is strictly convex on the set $\{P \in \mathbb{R}^{nM}, F^*(P) < \infty\}$ and in addition the duality formula is valid for $\mathcal{L}^n$-a.e. $x \in \Omega$.

The second chapter of this thesis is organized as follows: in Section 2.1 we fix our assumptions and carry out the ideas in order to overcome the difficulty of non-solvability of the problem "$I \to \min$" in the non-reflexive Sobolev space $W^{1,1}(\Omega)^M$. To this purpose we use the concept of convex functions of a measure (see [46] or [60]) and define a suitable relaxed variant K of I which is formulated on the space $BV(\Omega)^M$. Afterwards we state Theorem 1.0.1 from above (see Theorem 2.1.1).
Subsequently we briefly recap some basic facts from convex analysis and from duality theory which leads to the definition of the so-called Lagrangian and therewith to the formulation of the dual problem associated to our original problem "$I \to \min$". Our results on the dual problem then will be summarized in Theorem 2.1.6 and Theorem 2.1.7.
In Section 2.2 we provide some important tools and auxiliary results which will be of fundamental meaning during the proof of Theorem 2.1.1 and Theorem 2.1.6 as well: starting with Section 2.2.1 we discuss the general chain rule for functions of BV-type and prove an important but known inequality which follows from the general chain rule for BV-functions. This inequality is

a crucial argument when proving new density results for Sobolev functions and BV-functions in Section 2.2.2. Here we again emphasize, that the proofs of our main results in this chapter crucially rely on the validity of these approximation arguments. In Section 2.2.3 we prove a slightly more general but already known variant of Poincaré's inequality which serves as an important tool during the proof of Theorem 2.1.1 and Theorem 2.1.6.
In Section 2.3 we give a proof of Theorem 2.1.1 while Section 2.4 is dedicated to the proof of Theorem 2.1.6. Finally, the goal of Section 2.5 is to derive uniqueness of the dual solution and to establish the duality formula which proves Theorem 2.1.7.
As the last point we think it is worth remarking that Theorem 1.0.1 and Theorem 1.0.2 actually extend to more general data fitting terms under consideration. For details, the reader is referred to the joint article [81] with J. Müller (see Theorem 1.1, Theorem 1.2 and Theorem 1.3 therein).

In the third chapter we discuss regularity properties of generalized minimizers of the functional I (w.r.t. to a suitable relaxation). As outlined in, e.g. [17], we may essentially expect three different types of regularity results for an arbitrary generalized minimizer u in our situation:

(i) $n \geq 2$, $M \geq 1$: suppose that $f \in L^\infty(\Omega - D)^M$ and that F satisfies some appropriate ellipticity conditions. Then there is an open set $\Omega_0 \subset \Omega$ such that $u \in C^{1,\alpha}(\Omega_0)^M$ for any $\alpha \in (0,1)$ with $\mathcal{L}^n(\Omega - \Omega_0) = 0$.

(ii) $n \geq 2$, $M = 1$: full interior $C^{1,\alpha}$-regularity of u for any $\alpha \in (0,1)$. Here, we additionally assume that $f \in L^\infty(\Omega - D)^M$ and that the density F satisfies some appropriate ellipticity conditions.

(iii) $n \geq 2$, $M > 1$: with the assumptions on f and F as stated in (ii) we further suppose, that F satisfies $F(Z) = g(|Z|^2)$ for some function $g \in C^2([0,\infty),[0,\infty))$ of class C^2 and impose some additional Hölder condition on the second derivatives of F. Then, u is of class $C^{1,\alpha}(\Omega)^M$ for any $\alpha \in (0,1)$.

The aim of Chapter 3 is to establish the claims (i)–(iii) in our setting where due to the presence of the data fitting term in the functional I from (1.0.13), it is not possible to refer to, e.g., [17], and adding some obvious modifications. As a consequence, our goal is to develop a suitable analysis including the penalty term $\int\limits_{\Omega - D} |w - f|^2 dx$ from (1.0.13).

<u>*ad (i).*</u> As already mentioned above, a strong partial $C^{1,\beta}$-regularity result for u together with full regularity of $DF(\nabla u)$ has been stated in [24], Theorem 1.4 for μ-elliptic energies under the assumption $\mu \in (1,2)$. Thus, the question arises if it is possible to establish partial regularity in the usual sense, i.e., $\mathcal{L}^n(\mathrm{Sing}_\Omega(u)) = 0$, of each generalized minimizer for arbitrary large values of μ. In Theorem 3.1.9 in Section 3.1 we can give a satisfying answer where we crucially benefit from a recent article of T. Schmidt [86]. In fact, our partial

regularity result extends to more general densities F satisfying among other conditions

$$0 < D^2F(P)(Q,Q) \le \nu_3(1+|P|)^{-1}|Q|^2 \tag{1.0.16}$$

for all $P, Q \in \mathbb{R}^{nM}$, $Q \neq 0$. We additionally have to assume a structure condition for F in the sense that there exists a function $\Phi \in C^2([0,\infty),[0,\infty))$ with $F(P) = \Phi(|P|)$ for all $P \in \mathbb{R}^{nM}$ (in the scalar case we can weaken this condition). This assumption allows to derive a maximum principle for each generalized minimizer (see Theorem 3.1.4 and Theorem 3.1.5 in this thesis) and this result ensures to get partial $C^{1,\beta}$-regularity, where the limit $\beta = \frac{1}{2}$ serves as an optimal choice in the setting of [86].

ad (ii). In the joint article [27] with M. Bildhauer and M. Fuchs we could establish the regularity statement (ii) in the scalar case $M = 1$ together with $n = 2$ (see [27], Theorem 2). Taken the arguments and the assumptions on F (in particular, F shall be μ-elliptic for some $\mu \in (1,2)$), that were given in [27], as a basis, Section 3.5.1 is devoted to the generalization of [27], Theorem 2, to the scalar case $M = 1$ together with arbitrary dimension $n \ge 3$ which implies (ii) from above in our setting (see Theorem 3.1.19 in the scalar case).
At first, a generalization of Theorem 1.3 in [24] to the case $n \ge 2$ with arbitrary codimension M can be established (see Theorem 3.1.15 in Section 3.1). To become more precise, we show, that under the above assumptions (in particular for $\mu \in (1,2)$), there exists a unique I-minimizer u from the space $W^{1,1}(\Omega)^M \cap L^\infty(\Omega)^M$. The boundedness of u will be derived by proving a maximum principle. A common approach towards the regularity of generalized minimizers (not only in the scalar case) relies on an approximation of our problem "$I \to \min$" by a sequence of more regular problems admitting smooth solutions with useful convergence properties. In our situation, for $\delta \in (0,1]$ being fixed, we look at the problem

$$I_\delta[w] := \int\limits_\Omega F_\delta(\nabla w)dx + \frac{\lambda}{2}\int\limits_{\Omega - D}|w-f|^2dx \to \min \text{ in } W^{1,2}(\Omega)^M \tag{1.0.17}$$

where

$$F_\delta(Z) := \frac{\delta}{2}|Z|^2 + F(Z), \quad Z \in \mathbb{R}^{nM}. \tag{1.0.18}$$

The problem (1.0.17) has a unique solution $u_\delta \in W^{1,2}(\Omega)^M$ which (under particular assumptions) is of class $W^{2,2}_{\mathrm{loc}}(\Omega)^M \cap C^{1,\alpha}(\Omega)^M \cap L^\infty(\Omega)^M$ for any $\alpha \in (0,1)$ (see Lemma 7.1.1 in the appendix of this thesis). Further we can show that $u_\delta \to u$ in $L^1(\Omega)^M$ and a.e. on Ω after passing to a suitable subsequence $\delta \to 0$ (see Theorem 3.1.15).
For proving (ii) we essentially use the same arguments and procedure as already carried out in [27]. However, there is a fundamental difference between the case $n = 2$ and $n \ge 3$: if $n = 2$, local apriori uniform (in δ) L^p-estimates of ∇u_δ

for any finite $p > 1$ have been derived as a byproduct during the proof that it holds $u_\delta \in W^{1,2}_{\mathrm{loc}}(\Omega)$ uniformly in δ (see the proof of Theorem 1.3 in [24]). This uniform higher integrability of ∇u_δ plays an important role in the proof of Theorem 2 in [27]. Adopting the arguments of the above proof in case $n \geq 3$, we merely obtain local apriori uniform (in δ) L^q-estimates for any $1 \leq q \leq \frac{2n}{n-2}$ (see Section 3.3 and Remark 3.5.9) and this initial starting integrability is not enough for carrying out a De Giorgi-type iteration in order to obtain local uniform (in δ) apriori gradient bounds for u_δ.
Thus, one major effort during the proof of (ii) is to verify local uniform (in δ) L^q-estimates of ∇u_δ for any finite $q > 1$. This is the main statement of Lemma 3.5.1. During the proof of this lemma, the derivation of an appropriate variant of Caccioppoli's inequality acts as an essential ingredient (see Lemma 3.5.2) where we follow the basic idea in [17], Lemma 4.19 (i), p. 108, and include the data fitting term in our calculations which causes severe difficulties. Note, that in order to obtain a reasonable modification of Caccioppoli's inequality from [17], Lemma 4.19 (i), p. 108, we strongly need the assumption $\mu \in (1, 2)$. Afterwards we adopt the refined iteration argument which has been given in [17], Theorem 4.25, p. 116. Here, it is important to assume $\mu \in (1, 2)$ and to use the modified variant of Caccioppoli's inequality (see Lemma 3.5.2 again). This yields the desired local uniform higher integrability result. Subsequently we may essentially follow the arguments of [27] for establishing $u \in C^{1,\alpha}(\Omega)$. In this context, we adopt the De Giorgi-type iteration as carried out in [27], proof of Theorem 2. This procedure actually represents a substantial refinement of the De Giorgi-iteration performed in [17], Theorem 4.28, pp.119, where we additionally include the data fitting term in our calculations. One major observation is that we actually do not need the entire range of local uniform L^p-estimates of ∇u_δ for getting local uniform gradient bounds of u_δ (see Lemma 3.5.6).

ad (iii). In addition to the requirement that F is μ-elliptic with $\mu \in (1, 2)$ we have to impose some stronger structure conditions on F. On one hand we require that there exists a function $g \in C^2([0, \infty), [0, \infty))$ such that $F(Z) = g(|Z|^2)$ and on the other hand we assume that D^2F satisfies a suitable Hölder-condition. As a consequence we can benefit from the arguments and techniques that have been used in the scalar case $M = 1$ for obtaining local Lipschitz continuity of the unique I-minimizer u from the space $W^{1,1}(\Omega)^M \cap L^\infty(\Omega)^M$. It is worth remarking that we actually do not need the Hölder condition on D^2F for deducing local Lipschitz continuity of u in Ω. An essential difference between the scalar case $M = 1$ and the vectorial case $M > 1$ is that in the latter situation, it is not possible to apply the well-known theory of De Giorgi, Moser and Nash in order to close the gap between local Lipschitz continuity of u in Ω and local Hölder continuity of ∇u in Ω. At this point we follow the basic idea of Mingione and Siepe in [78] and modify our original integrand F to an integrand $\widetilde{F}$ being of class C^2 satisfying some appropriate isotropic growth and ellipticity conditions. Afterwards we show that u is a local minimizer of the functional

$$\int_\Omega \widetilde{F}(\nabla w)dx + \frac{\lambda}{2}\int_{\Omega-D} |w-f|^2 dx,$$

being well-defined for functions from a suitable Sobolev space (here we crucially use local Lipschitz continuity of u). Subsequently we apply Theorem 4.1.7 which proves full interior $C^{1,\alpha}$-regularity for local minimizers of appropriate isotropic variational integrals, and finally guarantees the same regularity for u since the constructed auxiliary integrand allows to use this lemma.

Finally, the third chapter is organized as follows: in Section 3.1 we fix our notation and assumptions before we formulate our main results on the regularity behavior of generalized minimizers. In Section 3.2 we prove a maximum principle for generalized minimizers of the functional I from (1.0.13) (see Theorem 3.1.4 and Theorem 3.1.5 in the scalar case $M = 1$) whereas in Section 3.3 we give a proof of (i) from above (see Theorem 3.1.9). Section 3.4 is devoted to the proof of Theorem 3.1.15 and in Section 3.5 we discuss (ii) (see Section 3.5.1) as well as (iii) (see Section 3.5.2), i.e., we prove Theorem 3.1.19. We remark that in Section 3.4 and Section 3.5.1 we present the material of the forthcoming paper [94]. Furthermore, it is worth saying that Theorem 3.1.4, Theorem 3.1.5 and Theorem 3.1.9 extend to more general data terms under consideration (we refer the reader to the joint article [81] with J. Müller (see Theorem 1.4 therein)).

In the fourth chapter of this thesis we address to image denoising and study a non-autonomous modification of the well-known TV-regularization. In extension of the analysis started in [22] we discuss the minimization problem

$$\begin{aligned} I[w] :=& \lambda\int_\Omega F(x,\nabla w)dx + \int_\Omega |w-f|^2 dx \to \min \\ & \text{in } W^{1,1}(\Omega)^M \cap L^2(\Omega)^M. \end{aligned} \tag{1.0.19}$$

As usual, $\lambda > 0$ denotes a positive regularization parameter and we additionally require $f \in L^\infty(\Omega)^M$. In this context, the major novelty is that we admit a smooth x-dependence on our density F. This is motivated by the model density

$$F(x,P) := F_{\mu(x)}(P) := \int_0^{\sqrt{\varepsilon+|P|^2}} \int_0^s (1+r)^{-\mu(x)} dr ds, \quad \varepsilon > 0, \tag{1.0.20}$$

where $x \in \overline{\Omega}$ and $P \in \mathbb{R}^{nM}$. Here, μ denotes a function of class $C^2(\overline{\Omega})$ taking its values in the interval $(1,\infty)$. With this it follows existence of some suitable real numbers $1 < \mu_0 \leq \mu_1 < \infty$ such that $\mu(x) \in [\mu_0,\mu_1]$ for all $x \in \overline{\Omega}$. The idea to consider the functional I from (1.0.19) by making the choice $F(x,P) := F_{\mu(x)}(P)$ generating functionals of linear growth (uniformly in x) originates from [28]. In this forthcoming article, the problem (1.0.19) is investigated in the scalar setting together with $n = 2$ and it is to be mentionend that the authors include a slight modification of the density $F_{\mu(x)}(P)$

from above in their discussions. In general, it seems to be reasonable to involve an additional x-dependence in the corresponding variational model to image denoising. Considering the above problem (1.0.19) and according to the current subregion of Ω in which we are located, we may expect a different regularity behavior of our minimizer. To become more precise, on the subregion $\{x \in \Omega, 1 < \mu(x) < 2\}$, it stands to reason that we are confronted with slightly "oversmoothed" images where the edges probably appear to be blurred while on zones with large values of μ we obtain rather irregular solutions. In the latter case, essential characteristics of our generated image as edges will be preserved. Thus, as outlined in [28], the basic idea of involving an additional x-dependence in our context is, that the generated image shows a different degree of regularity on prescribed zones of Ω and provides more flexibility to denoise a given image by regarding the special structure of this image. From the analytical point of view, it is "convenient" to consider at least a continuous x-dependence of F. Among other conditions (the reader is referred to [7], p. 312), the continuity of $x \mapsto F(x, P)$ for all $P \in \mathbb{R}^{nM}$ is a basic requirement for defining a suitable relaxed variant of the functional I from (1.0.19) on $BV(\Omega)^M$.
In literature, it seems to be common that the regularization parameter λ is considered to depend on the independent variable x. As outlined in, e.g., [92], λ then plays the role of balance parameter and steers the amount of regularization w.r.t. the measured data.

The aim of Chapter 4 is to establish a comprehensive existence and regularity theory for minimizers of the problem (1.0.19) where for the sake of simplicity and of practical relevance we constantly consider the model integrand $F(x, P) = F_{\mu(x)}(P)$, $P \in \mathbb{R}^{nM}$. Summarizing we are going to prove the following theorem where part (a) and (c) of this theorem generalize Theorem 4 in [28] to any dimension $n \geq 2$ together with arbitrary codimension M. In addition to Theorem 4 in [28] we discuss the dual approach in the non-autonomous setting as well.

Theorem 1.0.3
Suppose that we are given a $\mathcal{L}^n$-measurable function $f : \Omega \to \mathbb{R}^M$ ($M \geq 1$) of class $L^\infty(\Omega)^M$. Moreover let us fix a parameter $\lambda > 0$ and let $\mu \in C^2(\overline{\Omega})$ taking its values in the interval $(1, \infty)$. Then it holds (with $F(x, P) := F_{\mu(x)}(P)$):

(a) There exists a unique generalized I-minimizer u (w.r.t. a suitable relaxation of I with I from (1.0.19)) from the space $BV(\Omega)^M$ which in addition satisfies
$$\sup_\Omega |u| \leq \sup_\Omega |f|.$$

(b) The dual problem associated to (1.0.19) admits a unique solution and the inf-sup relation holds. In addition we have the duality formula (with u from part (a))
$$\sigma = \nabla_P F_{\mu(\cdot)}(\nabla^a u)$$

being valid $\mathcal{L}^n$-a.e. on Ω. As usual, $\nabla^a u$ is the regular part of the tensor-valued Radon measure ∇u w.r.t. Lebesgue's measure.

(c) *We have $u \in C^{1,\beta}(\Omega_2)^M$ for any $\beta \in (0,1)$, where $\Omega_2 := \{x \in \Omega, 1 < \mu(x) < 2\}$.*

Now some comments on the above theorem are in order: in the course of the proof of part (a) we may essentially follow the lines of the proof of Theorem 2.1.1 given in Section 2.3. For proving part (b) we note that an additional (smooth) x-dependence does not affect the results from convex analysis as used in the proof of Theorem 2.1.6 for instance. However we again prefer to give a more direct proof relying on the analysis of solutions of a suitable regularization of our original problem (1.0.19). It turns out that we can take into account the same regularization as used in the proof of Theorem 2.1.6 where we replace F by our model density $F_{\mu(x)}(P)$. This does not cause any problems during the calculations. As an important byproduct, it turns out that the corresponding regularizing sequence (u_δ) is an I-minimizing sequence satisfying $u_\delta \to u$ in $L^1(\Omega)^M$ and a.e. in Ω by passing to appropriate subsequences $\delta \to 0$.
In part (c) we essentially apply the same procedure as already carried out in the third chapter making minor adjustments. In fact we construct appropriate test functions having compact support in the open set $\{x \in \Omega, 1 < \mu(x) < 2\}$. With this we derive helpful variants of Caccioppoli's inequality. Here, the strong uniform (in x) ellipticity properties of $F_{\mu(x)}(P)$ are of fundamental importance. After carrying out the same De Giorgi-type iteration as already done in Section 3.5.1 we get local Lipschitz continuity of u on Ω_2. For obtaining full interior $C^{1,\beta}$-regularity of u in the scalar case we can quote elliptic regularity theory while in the vectorial case we crucially benefit from Theorem 4.1.7.

In the final stage of the introduction of the fourth chapter (compare Section 4.1), we discuss the regularity properties of (local) minimizers of a properly defined class of non-autonomous isotropic variational problems which is the subject of Theorem 4.1.7. This theorem is of crucial meaning during the proof of Theorem 3.1.19 in the vectorial case $M > 1$ and in the proof Theorem 4.1.4 in case $M > 1$.

In the fifth chapter we return to the setting of image inpainting where we investigate a model for restoring images consisting only of completely black or completely white regions. This model has already been proposed in [26] where the scalar case together with $n = 2$ has been discussed. In this thesis we deal with the scalar case $M = 1$ together with arbitrary dimension $n \geq 2$. Among the common assumptions on Ω as well as D we suppose that our incomplete image is given by a $\mathcal{L}^n$-measurable, real-valued function $f : \Omega - D \to \mathbb{R}$ attaining its values in the interval $[0, 1]$. For points $x \in \Omega - D$, $f(x)$ can be interpreted as a measure of the intensity of the grey level. To become more precise, the model under our consideration has its origin in the TV-image inpainting method where one seeks minimizers of the functional

$$J[w] := \int_{\Omega} |\nabla w| + \frac{\lambda}{2} \int_{\Omega - D} (w - f)^2 dx.$$

in a suitable subspace of the space $BV(\Omega)$. Now we impose the requirement "$u(x) \in \{0, 1\}$" on our generated image $u : \Omega \to \mathbb{R}$ and it is therefore natural to look for J-minimizers among characteristic functions. We thus consider the energy

$$\mathcal{F}[E] := P(E, \Omega) + \frac{\lambda}{2} \int_{\Omega - D} (\chi_E - f)^2 dx, \tag{1.0.21}$$

where $P(E, \Omega) := \int_{\Omega} |\nabla \chi_E|$ denotes the perimeter of the Borel set E in Ω and χ_E its characteristic function. Further, E is assumed to have finite perimeter (i.e., E is a Caccioppoli set in Ω) and we refer to [7] or [63] for more details concerning sets of finite perimeter and the behavior of characteristic functions of such sets.

In Section 5.1 we prove that there exists at least one $\mathcal{F}$-minimizing set E of finite perimeter whose boundary part $\partial E \cap \Omega$ has some nice smoothness and geometric properties. In fact, the analytical behavior of $\partial E \cap \Omega$ crucially depends on the dimension n. Here we essentially use the same arguments as Bildhauer and Fuchs in [26] (see Theorem 1 therein). A slight novelty will be discussed in Section 5.2: we minimize (1.0.21) among all Caccioppoli sets on which we impose a volume constraint in the sense that we require $\mathcal{L}^n(E) = m$ where $m \in (0, \mathcal{L}^n(\Omega))$ denotes a fixed number. Vividly we are confronted with the task to restore the incomplete image using merely a given amount of black color. In Theorem 5.2.2 we show that the described problem has at least one solution E whose boundary ∂E is regular in some sense. Once again, this regularity result crucially depends on the dimension n.

The sixth chapter of this thesis is devoted to some final remarks about our achieved results and the comparison of these to already known results. Furthermore we will briefly discuss some extensions of the models that have been under our consideration. In particular we sketch the problem of higher order denoising which, from the mathematical point of view, is interesting to study. From the numerical point of view, the higher order denoising model could lead to difficulties. This motivates us to say a few words about coupled variants in this context as well. In the last part of the sixth chapter we briefly discuss the idea to study the classical TV-image inpainting model in some appropriate subclasses of the space $BV(\Omega)$ as, e.g., in the space of special functions of bounded variation $SBV(\Omega)$.

In the appendix of this thesis we collect and prove some auxiliary lemmas that are of important meaning in the course of this thesis. In Section 7.1 we intensively discuss the regularity properties of the regularizing sequence (u_δ) (see Lemma 7.1.1). Finally we show an algebraic proposition which serves as a helpful tool when performing a De Giorgi-type iteration in Section 3.5.1 (see Lemma 7.1.5). At last, Section 7.2 is devoted to an overview about the notation and the conventions that we use in this thesis.

Chapter 2

A modified TV-image inpainting method: existence results

2.1 The basic setup and statement of the main results

In this chapter we start our analysis of a modification of the total variation image inpainting method which has already been shortly presented in the introduction (see (1.0.14) therein). In fact, this modification boils down to the minimization of a functional that is initially formulated on the non-reflexive Sobolev space $W^{1,1}(\Omega)^M$. As outlined in [17], pp. 5, there are essentially two possibilities for overcoming this problem: we either consider a suitable relaxation of our original functional defined on the space $BV(\Omega)^M$ and introduce solutions of this relaxed variant as generalized minimizers (in a suitable sense) of the original problem or pass to the dual variational problem associated to our original problem. The aim of this chapter is to discuss both concepts applied to the modified TV-image inpainting method where we emphasize that we present the material from the joint article with M. Fuchs [55].
Finally we note that in connection with variational problems of linear growth, it is reasonable to give an interpretation of the underlying problem in the more adequate space (or in a suitable subspace of) $BV(\Omega)^M$ since in contrast to the Sobolev space $W^{1,1}(\Omega)^M$, we can make use of compactness properties of the space $BV(\Omega)^M$.

Before going into details we fix our setup and specify our assumptions: suppose that we are given a bounded Lipschitz domain Ω in $\mathbb{R}^n$ with $n \geq 2$ (e.g. a rectangle in the case $n = 2$ or a cuboid in the case $n = 3$) and a $\mathcal{L}^n$-measurable subset D of Ω satisfying

$$0 \leq \mathcal{L}^n(D) < \mathcal{L}^n(\Omega). \tag{2.1.1}$$

Note that the case $D = \emptyset$ corresponds to "pure denoising".
Moreover we assume that we are given an observed (possibly vector-valued) image described through a measurable function $f : \Omega - D \to \mathbb{R}^M$, where we require

$$f \in L^2(\Omega - D)^M. \tag{2.1.2}$$

Now our goal is to recover the missing part $D \to \mathbb{R}^M$ of the observed image by means of the given data. As already described in the introduction, there is a variety of different methods in order to handle the inpainting problem. In what follows we concentrate our studies on a TV-like variational approach being of non-local type. As proposed in [24] (see also the subsequent papers [23, 25–28]) we consider the functional ($\lambda > 0$ denotes a regularization parameter)

$$I[w] := \int\limits_{\Omega} F(\nabla w)dx + \frac{\lambda}{2} \int\limits_{\Omega - D} |w - f|^2 dx \tag{2.1.3}$$

for functions $w : \Omega \to \mathbb{R}^M$ from a suitable class. In formula (2.1.3), $F : \mathbb{R}^{nM} \to [0, \infty)$ is a given density function of class $C^1(\mathbb{R}^{nM})$ satisfying the following assumptions:

$$\text{F is strictly convex and (w.l.o.g.) } F(0) = 0, \tag{2.1.4}$$

$$|DF(P)| \leq \nu_1, \tag{2.1.5}$$

$$F(P) \geq \nu_2 |P| - \nu_3 \tag{2.1.6}$$

with constants $\nu_1, \nu_2 > 0, \nu_3 \in \mathbb{R}$, for all $P \in \mathbb{R}^{nM}$. From (2.1.5) and $F(0) = 0$ we immediately obtain

$$F(P) \leq \nu_1 |P|$$

for all $P \in \mathbb{R}^{nM}$ which shows that F is of linear growth in the following sense

$$\nu_2 |P| - \nu_3 \leq F(P) \leq \nu_1 |P|. \tag{2.1.7}$$

We then introduce the problem

$$I \to \min \quad \text{in } W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M. \tag{2.1.8}$$

Here we recall the continuous embedding $W^{1,1}(\Omega)^M \hookrightarrow L^{\frac{n}{n-1}}(\Omega)^M$ (see, e.g., [4], Theorem 5.4, p. 97/98), which means that the additional constraint $w \in L^2(\Omega - D)^M$ is automatically satisfied for functions $w \in W^{1,1}(\Omega)^M$, provided $n = 2$.

Without imposing stronger conditions on our density F (see Section 3 of this thesis) we cannot expect solvability of problem (2.1.8) in the non-reflexive

Sobolev space $W^{1,1}(\Omega)^M$. So the question arises how to give a reasonable extension and an interpretation of problem (2.1.8) in the setting of the more adequate function space $BV(\Omega)^M$. Exploiting that I-minimizing sequences (w_m) from the space $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$ are uniformly bounded in the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$ it follows $w_m \to: u$ in $L^1(\Omega)^M$ and a.e. up to a subsequence for a function $u \in BV(\Omega)^M$ by BV-compactness (see, e.g., [7], Theorem 3.23, p. 132). In addition we obtain $u \in L^2(\Omega - D)^M$ by using Fatou's lemma as well. For this reason it is natural to address the elements of the set

$$\begin{aligned}\mathcal{M} := \{u \in BV(\Omega)^M \cap L^2(\Omega - D)^M : u \text{ is } L^1 - \text{limit of an } I - \text{minimizing}\\ \text{sequence from the space}\\ W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M\}\end{aligned}$$

as generalized minimizers of problem (2.1.8). Another, formally different, point is to investigate a suitable relaxed variant K of our functional I from (2.1.3) on the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$ by applying the concept of convex functions of a measure (see, e.g., [9, 46] or [60]). Then, generalized minimizers of problem (2.1.8) are seen as minimizers of the relaxed version K (we refer, e.g., to [7], pp. 298 or [64]). To become more precise, in accordance with [46] or [9] we let for $w \in BV(\Omega)^M \cap L^2(\Omega - D)^M$

$$K[w] := \int_\Omega F(\nabla^a w)dx + \int_\Omega F^\infty\left(\frac{\nabla^s w}{|\nabla^s w|}\right)d|\nabla^s w| + \frac{\lambda}{2}\int_{\Omega - D} |w - f|^2 dx. \quad (2.1.9)$$

Here, for tensor-valued Radon measures ρ we denote by $\rho^a(\rho^s)$ the regular (singular) part of ρ w.r.t. to Lebesgue's measure $\mathcal{L}^n$. Quoting Radon-Nikodým's theorem (see, e.g., [7], Theorem 1.28, p.14) it follows, that the densities $\nabla^a w$ and $\frac{\nabla^s w}{|\nabla^s w|}$ are of class $L^1(\Omega)^{nM}$ being unique $\mathcal{L}^n$-a.e. on Ω and of class $L^1(\Omega, |\nabla^s w|)^{nM}$ being unique $|\nabla^s w|$-a.e. on Ω, respectively.

Moreover, F^∞ denotes the recession function of F and is defined by

$$F^\infty(P) := \lim_{t \to \infty} \frac{F(tP)}{t}, \quad P \in \mathbb{R}^{nM}. \quad (2.1.10)$$

From the definition of F^∞ we directly get that F^∞ is a 1-homogenous function. Based on the (strict) convexity and since F is of linear growth, it follows that F^∞ is well-defined (it even defines a norm on $\mathbb{R}^{nM}$ in this case).
Now, the idea is to seek minimizers of the relaxed variational problem

$$K \to \min \quad \text{in } BV(\Omega)^M \cap L^2(\Omega - D)^M \quad (2.1.11)$$

and to introduce them as generalized solutions of problem (2.1.8).

After the above preparations we will state a first theorem that is concerned with problem (2.1.11): first we will prove solvability of problem (2.1.11) in the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$. Moreover we will show uniqueness of the absolutely continuous part $\nabla^a u$ of the gradient of BV-solutions on the entire domain Ω and will additionally prove the uniqueness of BV-solutions outside of the inpainting region D. In part (c) we justify that each K-minimizer can be introduced as a generalized minimizer of the original functional I from (2.1.3) whereas in part (d) we verify that each K-minimizer is an element of the set $\mathcal{M}$ of generalized minimizers of the functional I and vice versa.

Theorem 2.1.1
Let us assume the validity of (2.1.1) *as well as* (2.1.2) *and suppose that* F *satisfies* (2.1.4)–(2.1.6). *It then holds:*

(a) Problem (2.1.11) *has at least one solution.*

(b) Suppose that u *and* $\widetilde{u}$ *are* K*-minimizing. We then have*

$$u = \widetilde{u} \text{ a.e. on } \Omega - D \quad \text{and} \quad \nabla^a u = \nabla^a \widetilde{u} \text{ a.e. on } \Omega.$$

(c)

$$\inf_{W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M} I = \inf_{BV(\Omega)^M \cap L^2(\Omega - D)^M} K.$$

(d) As defined above, we consider the set $\mathcal{M}$ *of generalized minimizers of the functional* I *from* (2.1.3). *Then* $\mathcal{M}$ *coincides with the set of all* K*-minimizers from the space* $BV(\Omega)^M \cap L^2(\Omega - D)^M$.

Remark 2.1.2

- *Part (b) of Theorem 2.1.1 shows uniqueness of solutions on* $\Omega - D$ *and the measures* ∇u *and* $\nabla \widetilde{u}$ *of minima* $u, \widetilde{u}$ *may only differ in their singular parts.*
- *The statements (c) and (d) in Theorem 2.1.1 reveal that the minimization of* K *in* $BV(\Omega)^M \cap L^2(\Omega - D)^M$ *represents a natural extension of the original variational problem* (2.1.8) *which in general fails to have solution in the non-reflexive Sobolev space* $W^{1,1}(\Omega)^M$. *Furthermore, it holds* $I = K$ *on* $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$ *and this fact also stresses that the functional* K *is a reasonable extension of the functional* I. *Further, it remains to be said that part (d) implies, that the (at least formally) different points of view from above actually describe the same set of functions.*
- *We emphasize that no additional topological assumptions on the inpainting region* D *are needed for establishing Theorem 2.1.1.*
- *The assumptions on our density* F *in Theorem 2.1.1 can be weakend in such a way that we just require* F *to be strictly convex and of linear growth in the sense of* (2.1.7). *In particular, we do not need (continuous) differentiability of* F.

Remark 2.1.3
Note that Theorem 2.1.1 also extends to more general (strictly) convex data terms. This and other issues have been discussed in the joint article with J.

Müller [81]. In this work we studied the following modification of the TV image inpainting method: for a given finite number $\zeta > 1$ we let

$$I_\zeta[w] := \int_\Omega F(\nabla w)dx + \frac{\lambda}{\zeta}\int_{\Omega - D} |w - f|^\zeta dx, \tag{2.1.12}$$

where we suppose the hypotheses from Theorem 2.1.1 and require $f \in L^\zeta(\Omega - D)^M$ in addition. Choosing $\zeta = 2$ in (2.1.12)*, I_2 coincides with the functional I from* (2.1.3)*. The corresponding relaxed version of the functional I_ζ formulated on the space $BV(\Omega)^M \cap L^\zeta(\Omega - D)^M$ consequently reads as:*

$$\begin{aligned} K_\zeta[w] := &\int_\Omega F(\nabla^a w)dx + \int_\Omega F^\infty\Big(\frac{\nabla^s w}{|\nabla^s w|}\Big)d|\nabla^s w| \\ &+ \frac{\lambda}{\zeta}\int_{\Omega - D} |w - f|^\zeta dx. \end{aligned} \tag{2.1.13}$$

As discussed in [81], Theorem 1.1, we can transfer the results from Theorem 2.1.1 to any value $\zeta > 1$ replacing K by K_ζ (note that by definition we have $K = K_2$ with K from (2.1.9)*). Furthermore, Theorem 2.1.1 partially extends to the case $\zeta = 1$ (in this case we lose uniqueness of BV-solutions in $\Omega - D$). It is worth remarking that the minimization of* (2.1.13) *among all functions of class $BV(\Omega)^M \cap L^\zeta(\Omega - D)^M$ in fact is a modification of the $TV - L^\zeta$-inpainting problem given by (see, e.g., [89], for the $TV - L^\zeta$-regularization in the context of pure denoising of images, i.e., for the choice $D = \emptyset$ below)*

$$\int_\Omega |\nabla w| + \frac{\lambda}{\zeta}\int_{\Omega - D} |w - f|^\zeta dx \to \mathit{min} \text{ in } BV(\Omega)^M \cap L^\zeta(\Omega - D)^M.$$

Here, at least in the context of pure denoising of images, the choice $\zeta = 1$ plays an important role in various applications (see, e.g., [65]), where the minimization of the particular functional

$$\lambda\int_\Omega |\nabla w| + \int_\Omega |w - f|dx \to \mathit{min} \text{ in } BV(\Omega)^M,$$

is also known as $TV - L^1$-regularization and seems to be reasonable for removing impulsive noise (see, e.g., [65] again).

Taking into account assertion (b) of Theorem 2.1.1 we may derive the uniqueness in case of $W^{1,1}$-solvability. Moreover, in the general case, the $L^{\frac{n}{n-1}}$-deviation $\|u - v\|_{L^{\frac{n}{n-1}}}$ of different solutions u, v on the inpainting region can be estimated in terms of $\nabla^s(u - v)$, i.e., it is governed by the total variation of the singular part $\nabla^s(u - v)$ of the tensor-valued Radon measure $\nabla(u - v)$.

Corollary 2.1.4 *(a) If there exists $u \in \mathcal{M}$ such that $u \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$, then it follows $\mathcal{M} = \{u\}$.*

(b) Suppose that $\overline{D} \subset \Omega$. Then there is a constant $c = c(n, M)$ such that for $u, v \in \mathcal{M}$ it holds

$$\|u - v\|_{L^{\frac{n}{n-1}}(\Omega)} = \|u - v\|_{L^{\frac{n}{n-1}}(D)} \leq c|\nabla^s(u - v)|(\overline{D}).$$

In particular, the constant c on the right-hand side does not depend on the free parameter λ.

Remark 2.1.5
For the proof of Corollary 2.1.4 we just note that Corollary 1.1 in [25] extends to any dimension $n \geq 2$. Furthermore, the statements remain valid for vector-valued functions, i.e., for the case $M \geq 2$. The corresponding references are given during the proof of [25], Corollary 1.1.

Motivated by the dual variational formulation of problems in the theory of minimal surfaces or of problems of plasticity (see [54] for a survey), the dual approach to problem (2.1.8) seems to be very natural. One essential motivation for studying dual variational problems is the uniqueness of solutions (for more detailed information we refer to Section 2.2 in [17]). On top of this, the dual solution σ usually admits a clear geometric or physical interpretation. For instance, we remark that in the theory of minimal surfaces, σ corresponds to the normal of the surface and in the theory of plasticity, σ represents the stress tensor. However it should be emphasized that we do not know an adequate interpretation of the dual solution σ in the context of image processing.

In order to formulate the dual problem associated to (2.1.8) we briefly recap some facts from convex analysis and duality theory, where we mainly follow the monograph of Ekeland and Temam (see [49]). Let us consider a function $G : V \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ defined on a Banach space V. Then the so-called conjugate function G^* to G is defined by

$$G^*(v^*) := \sup_{v \in V} [\langle v, v^* \rangle - G(v)], \quad v^* \in V^*,$$

where V^* denotes the dual space to V.
Further, the so-called biconjugate function G^{**} to G is given by

$$G^{**}(v) := \sup_{v^* \in V^*} [\langle v, v^* \rangle - G^*(v^*)], \quad v \in V.$$

Assuming that the function G is convex and lower semicontinuous, it is shown in [49], Proposition 4.1, p. 18 that it holds

$$G^{**}(v) = G(v) \quad \text{for all } v \in V. \tag{2.1.14}$$

Since convex functions $G : V \to \overline{\mathbb{R}}$ do not necessarily need to be differentiable in a given point $v_0 \in V$, we introduce a replacement for the differential of G in v_0, the so-called subdifferential $\partial G(v_0)$ which is defined by

$$\partial G(v_0) := \{v^* \in V^* : G(v) \geq G(v_0) + \langle v - v_0, v^* \rangle \quad \text{for all } v \in V\}, \tag{2.1.15}$$

if $G(v_0) < \infty$ and $\partial G(v_0) = \emptyset$ otherwise (see pp.20 in [49]). If G is differentiable in v_0 we obtain $v^* = DG(v_0)$ (see, e.g., [44], Theorem 2.6, p. 39).
With this notation and under the assumption that $\partial G(v_0) \neq \emptyset$, we can establish the duality relation (see [49], Proposition 5.1, p.21), i.e.,

$$v^* \in \partial G(v_0) \Longleftrightarrow G(v_0) + G^*(v^*) = \langle v_0, v^* \rangle. \tag{2.1.16}$$

Now we apply the abstract setting from above to our context: let F satisfy (2.1.4)–(2.1.6) and suppose that (2.1.1) as well as (2.1.2) hold. In what follows, our goal is to derive an alternative integral representation for our functional I from (2.1.3) which involves the conjugate function F^*. For that reason we consider the functional $G : L^1(\Omega)^{nM} \to \overline{\mathbb{R}}$,

$$G(P) := \int_\Omega F(P)dx, \quad P \in L^1(\Omega)^{nM}.$$

Applying [49], Proposition 2.1, p.271, together with the relation (2.1.14) from above we see (see, e.g., [17], p.15)

$$\int_\Omega F(P)dx = \sup_{\varkappa \in L^\infty(\Omega)^{nM}} \Big\{ \int_\Omega \varkappa : P dx - \int_\Omega F^*(\varkappa)dx \Big\}.$$

where the symbol $Q : Z$ denotes the standard scalar product in $\mathbb{R}^{nM}$. Since $P = \nabla w$, $w \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$, is an admissible choice in the above formula, we derive the following alternative representation formula for the functional I

$$I[w] = \sup_{\varkappa \in L^\infty(\Omega)^{nM}} \Big\{ \int_\Omega \varkappa : \nabla w - F^*(\varkappa)dx \Big\} + \frac{\lambda}{2} \int_{\Omega - D} |w - f|^2 dx. \tag{2.1.17}$$

Inspired by (2.1.17) we define the Lagrangian $l(w, \varkappa)$ (for more details, see [49], pp.51) for all $(w, \varkappa) \in (W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M) \times L^\infty(\Omega)^{nM}$ through the formula

$$l(w, \varkappa) := \int_\Omega [\varkappa : \nabla w - F^*(\varkappa)]dx + \frac{\lambda}{2} \int_{\Omega - D} |w - f|^2 dx. \tag{2.1.18}$$

and by virtue of (2.1.18) we can introduce the dual functional in terms of the Lagrangian, precisely

$$\begin{aligned} R : L^\infty(\Omega)^{nM} &\to [-\infty, \infty], \\ R[\varkappa] &:= \inf_{w \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M} l(w, \varkappa). \end{aligned}$$

Consequently, the dual problem reads as: to maximize R among all functions $\varkappa \in L^\infty(\Omega)^{nM}$.

In what follows we present the main results on the dual variational problem associated to (2.1.8). Among proving solvability of the dual problem and showing the validity of the inf-sup relation we prove uniqueness of the dual solution under the condition that F^* is strictly convex on the set $\{P \in \mathbb{R}^{nM} :$

$F^*(P) < \infty\}$. Another interesting result is part (c) from below which states a surprising compactness property of I-minimizing sequences from the space $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$.

Theorem 2.1.6
Suppose that (2.1.1) *and* (2.1.2) *hold. Further we let F satisfy* (2.1.4)–(2.1.6). *Then we have:*

(a) the dual problem

$$R \to \textit{max} \quad \textit{in} \quad L^\infty(\Omega)^{nM}$$

admits at least one solution. Moreover, the inf-sup *relation*

$$\inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M} I[v] = \sup_{\sigma \in L^\infty(\Omega)^{nM}} R[\sigma]$$

is valid;

(b) we have uniqueness of the dual solution if the conjugate function F^* is strictly convex on the set $\{P \in \mathbb{R}^{nM} : F^*(P) < \infty\}$;

(c) consider any I-minimizing sequence (u_m) from the space $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$. Then it holds

$$u_m \to u \quad \textit{in } L^2(\Omega - D)^M,$$

where u is the unique restriction of any generalized minimizer $\overline{u}$ from Theorem 2.1.1 to the set $\Omega - D$.

In fact, in order to verify uniqueness of the dual solution, the additional requirement on F^* in assertion (b) of Theorem 2.1.6 can be dropped. Further, the unique dual solution is related to the BV-solutions from Theorem 2.1.1 through an equation of stress-strain type.

Theorem 2.1.7
Let (2.1.1), (2.1.2) *hold and assume that we have* (2.1.4)–(2.1.6) *for the density F. Then the dual problem*

$$R \to \textit{max} \quad \textit{in } L^\infty(\Omega)^{nM}$$

admits a unique solution σ. We further have the duality formula

$$\sigma = DF(\nabla^a u) \quad \textit{a.e. on } \Omega,$$

where u denotes any K-minimizer from the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$.

Remark 2.1.8
Note that no additional topological assumptions on the inpainting region D are needed for proving Theorem 2.1.6 and Theorem 2.1.7.

Remark 2.1.9
Let us fix a finite number $\zeta > 1$ and consider the problem

$$I_\zeta \to min \quad in\; W^{1,1}(\Omega)^M \cap L^\zeta(\Omega - D)^M$$

with I_ζ from (2.1.12)*. Quoting [81], Theorem 1.2, we can state that part (a) and part (b) of Theorem 2.1.6 extend to the dual variational problem associated to "$I_\zeta \to min$". Moreover, we have uniqueness of the corresponding dual solution and the validity of the duality formula as well (see [81], Theorem 1.3).*

Remark 2.1.10
Let us compare our results with the recent results of M. Bildhauer and M. Fuchs stated in their joint article [25]: Theorem 2.1.1 and Theorem 2.1.6 have been proven in the scalar case together with $n = 2$ (see Theorem 1.2 and Theorem 1.4 therein) under stronger assumptions on the data and on the density F. Based on their assumptions, Bildhauer and Fuchs could verify that K-minimizing sequences can be chosen in such a way that they satisfy a maximum principle which gives compactness in $BV(\Omega)$ (see [25], proof of Theorem 1.2 (a)). If we impose the same requirements on the data and F as done in [25], we may use exactly the same arguments in order to generalize the results from [25] to the case of any dimension n together with arbitrary codimension M since we can show that K-minimizing sequences (u_m) may be chosen in such a way that

$$\sup_\Omega |u_m| \le \sup_{\Omega - D} |f|$$

yielding compactness in $BV(\Omega)^M$. As a direct consequence of the above inequality we do not need the density result for BV-functions as stated in Lemma 2.2.6, since continuity of the relaxed functional K follows after using the continuity theorem of Reshetnyak (see, e.g., [9], Proposition 2.2 or [61], Theorem 2, p.92) and dominated convergence. However, as it will be discussed in the following sections, our imposed assumptions on the data and on F are too weak for deducing a maximum principle for K-minimizing sequences which means that we strongly need an appropriate density result for BV-functions as given in Lemma 2.2.6. At least in the case $n \ge 3$ together with arbitrary codimension M, our results represent a substantial generalization of those in [25]. An analogous remark applies to Theorem 2.1.6: under the assumptions stated in [25], we can use (more or less) the same arguments in order to generalize the results for any dimension n together with arbitrary codimension M and we can avoid the usage of the density result for Sobolev functions (see Lemma 2.2.4).
Finally we remark that the uniqueness of the dual solution together with the validity of the duality formula (see Theorem 2.1.7), even under stronger assumptions, is a new result w.r.t. the above modification of the TV-image inpainting method.

The rest of the chapter is organized as follows: in Section 2.2 we provide some tools and auxiliary results that are important for proving Theorem 2.1.1 and

Theorem 2.1.6. In Section 2.3 we study generalized minimizers while in Section 2.4 we discuss the dual variational problem associated to problem (2.1.8). The uniqueness of the corresponding dual solution together with the validity of the duality formula will be established in Section 2.5.

2.2 Some tools and auxiliary results

The aim of this section is to provide some tools and auxiliary results that are of important meaning in the course of the proofs of Theorem 2.1.1 and Theorem 2.1.6. This section is splitted into three parts: in Section 2.2.1 we give some comments on the general chain rule formula for functions of BV-type and derive an inequality acting as an important tool for proving the density of smooth functions in spaces like $BV(\Omega)^M \cap L^2(\Omega - D)^M$ in Section 2.2.2. Section 2.2.3 is devoted to the discussion of an appropriate variant of Poincaré's inequality which will play a fundamental role in the proof of Theorem 2.1.1 and Theorem 2.1.6.

2.2.1 The chain rule for functions of BV-type

Suppose that we are given a function $u : \Omega \to \mathbb{R}^M$ of bounded variation and a Lipschitz function $\Phi : \mathbb{R}^M \to \mathbb{R}^L$. Considering the function $v := \Phi \circ u : \Omega \to \mathbb{R}^L$ we will see in Lemma 2.2.1 that v is still of class $BV(\Omega)^L$ and that the modulus of its distributional gradient represented by the tensor-valued Radon measure ∇v is absolutely continuous w.r.t. to the modulus of ∇u in the sense of measures. In this context, the problem arises how to derive a relation between the distributional derivatives ∇v, ∇u and the "derivative" of Φ, i.e., to prove a chain rule for functions of BV-type.
Before going into details we want to give a short overview about classical results in this context: assuming first that the Lipschitz function Φ is in addition continuously differentiable, a chain rule for functions in BV is proven in, e.g., [7], Theorem 3.96, p.189. If we assume that Φ is merely Lipschitz we are confronted with severe difficulties, e.g., that Φ might be nowhere differentiable in the range of u. However, Ambrosio and Dal Maso were the first who proved a general chain rule in BV under the condition that Φ is merely Lipschitz (see [6], Theorem 2.1), where one of the crucial tools in this context was the concept of tangential differentiability.
Before we briefly present the general chain rule for BV-functions we provide some useful notation. Recall the Lebesgue decomposition $\nabla u = \nabla^a u \llcorner \mathcal{L}^n + \nabla^s u$ valid for any function $u \in BV(\Omega)^M$ and motivated by the example of the so-called Cantor-Vitali function (see, e.g., [7], Example 3.34, p.142) it makes sense to study a decomposition of the singular part $\nabla^s u$ of ∇u. As outlined in [7], Section 3.9, p.184, we can split the singular part $\nabla^s u$ into two parts: the jump part

$$\nabla^j u := \nabla^s u \llcorner J_u$$

and the Cantor part

$$\nabla^c u := \nabla^s u \llcorner (\Omega - S_u).$$

Here, J_u denotes the set of approximate jump points (see [7], Definition 3.67, p.163) and using [7], Theorem 3.77, p.171, $\nabla^j u$ can be calculated through (see [7], equation (3.90), p.184)

$$\nabla^j u(B) = \int\limits_{B \cap J_u} (u^+(x) - u^-(x)) \otimes \nu_u(x) d\mathcal{H}^{n-1}(x), \quad B \in \mathcal{B}(\Omega).$$

Here, $\mathcal{B}(\Omega)$ is the Borel σ-algebra on Ω, ν_u represents the direction of jump and orientates the countably $\mathcal{H}^{n-1}$-rectifiable set J_u while the quantities $u^\pm$ denote the one-sided approximate limits. As stated in [7], Definition 3.63, p.160 and the comments after this definition, S_u is called the approximate discontinuity set where we remark that for any $x \in \Omega - S_u$ there is a (uniquely determined) vector $z \in \mathbb{R}^M$ being denoted by $\widetilde{u}$ and called the approximate limit of u at x, satisfying

$$\lim_{\rho \to 0} \fint\limits_{B_\rho(x)} |u(y) - z| dy = 0.$$

Altogether we have the following decomposition of ∇u (compare [7], relation (3.89), p.184):

$$\nabla u = \nabla^a u \llcorner \mathcal{L}^n + \nabla^j u + \nabla^c u = \widetilde{\nabla} u + \nabla^j u,$$

where the quantity $\widetilde{\nabla} u$ is called the diffuse part of the distributional derivative ∇u. In other words, $\widetilde{\nabla} u$ represents the sum $\nabla^a u \llcorner \mathcal{L}^n + \nabla^c u$.

With the above preparations and keeping the above notation in mind, the following relation between ∇v, ∇u and Φ is given and proven in [7], Theorem 3.96, p.189, where it is to emphasize that the Lipschitz function Φ is assumed to be of class C^1 at this stage:

$$\begin{aligned} \widetilde{\nabla} v &= \nabla \Phi(u) \nabla^a u \llcorner \mathcal{L}^n + \nabla \Phi(\widetilde{u}) \nabla^c u = \nabla \Phi(\widetilde{u}) \widetilde{\nabla} u, \\ \nabla^j v &= (\Phi(u^+) - \Phi(u^-)) \otimes \nu_u \mathcal{H}^{n-1} \llcorner J_u. \end{aligned} \tag{2.2.1}$$

Here, it is notable that from (2.2.1) we observe that in contrast to the jump part $\nabla^j v$ the diffuse part $\widetilde{\nabla} v$ actually resembles the classical chain rule formula of differentiable functions. Moreover it is possible to summarize (2.2.1) in a single formula (for details we refer to [7], Remark 3.98, p.191).
As already mentioned above, we can drop the requirement that the Lipschitz function Φ is of class C^1. Following the arguments of the proof of (2.2.1) (see the proof of Theorem 3.96 in [7]) we can see that Lipschitz continuity of Φ is sufficient for deriving the representation of the jump part $\nabla^j v$ in (2.2.1). Unfortunately due to the lack of differentiability, we cannot expect a corresponding representation of the diffuse part $\widetilde{\nabla} v$ of the distributional derivative ∇v in the spirit of (2.2.1) since the range of u might be contained in regions where Φ is not differentiable. Nevertheless, recalling Rademacher's theorem (see, e.g., [7],

Theorem 2.14, p.47) we know that Φ is differentiable at $\mathcal{L}^n$-a.e. point of $\mathbb{R}^M$ and quoting an extension of this theorem in the context of geometric measure theory (see, e.g., [7], Theorem 2.90, p.99) it makes sense to introduce the notion of tangential differentiabilty. This concept acts as basic idea in order to derive a general chain rule formula for BV-functions. Roughly speaking, in [6], Ambrosio and Dal Maso introduced a suitable tangent space and showed that the restriction of Φ to this tangent space is differentiable at the approximate limit $\widetilde{u}$ of u for $|\nabla u|$-a.e. point $x \in \Omega - S_u$. Taken this statement as a basis, they then established a formula for the diffuse part $\widetilde{\nabla} v$ of ∇v involving the "tangential differential" of Φ at $\widetilde{u}(x)$ for $|\nabla u|$-a.e. $x \in \Omega - S_u$.
We think it is worth to mention that in the case $M = L = 1$ (and under the assumption that Φ is merely Lipschitz) it is possible to give a more explicit representation of the general chain rule formula in BV (compare [7], Theorem 3.99, p.192):

$$\nabla v = \Phi'(u)\nabla^a u \llcorner \mathcal{L}^n + ((\Phi(u^+) - \Phi(u^-))\nu_u \mathcal{H}^{n-1} \llcorner J_u + \Phi'(\widetilde{u})\nabla^c u. \tag{2.2.2}$$

At this stage we want to discuss an interesting inequality which arises in the context of the proof of the chain rule for functions of BV-type.

Lemma 2.2.1
Let $u \in BV(\Omega)^M$ and consider a Lipschitz function $\Phi : \mathbb{R}^M \to \mathbb{R}^L$. Then $v := \Phi \circ u$ belongs to $BV(\Omega)^L$ and it holds

$$|\nabla v| \leq Lip(\Phi)|\nabla u|. \tag{2.2.3}$$

Proof of Lemma 2.2.1. First, we assume that Φ is in addition of class C^1. Then we reproduce the arguments that are given at the beginning of the proof of Theorem 3.96 in [7]: we fix an open set $U \subset \Omega$ and find a sequence $(u_n) \subset C^\infty(U)$ with $u_n \to u$ in $L^1(\Omega)^M$ and $|\nabla u_n|(U) \to |\nabla u|(U)$ by using the standard approximation procedure (see, e.g., [7], Theorem 3.9, p.122), where as usual $|\nabla u|$ denotes the total variation of u.
Exploiting that Φ is Lipschitz we immediately get that $v_n := \Phi(u_n)$ converges to $v = \Phi(u)$ strongly in $L^1(\Omega)^L$. Further it follows

$$\begin{aligned} |\nabla v_n|(U) &= \int_U |\nabla v_n| dx = \int_U |\nabla \Phi(u_n) \nabla u_n| dx \\ &\leq M \int_U |\nabla u_n| dx = M |\nabla u_n|(U), \end{aligned}$$

where $M := \sup\limits_z |\nabla \Phi(z)|_\infty$ is the Lipschitz constant of Φ. Next, we pass to the limit $n \to \infty$ and use the lower semicontinuity of the total variation w.r.t. strong L^1-convergence (see, e.g., [7], Proposition 3.6, p.120 or [63], Theorem 1.9, p.7) . This yields

$$|\nabla v|(U) \leq \liminf_{n\to\infty} |\nabla v_n|(U) \leq \lim_{n\to\infty} M|\nabla u_n|(U) = M|\nabla u|(U).$$

Thus, by the regularity of the measures $|\nabla v|$ and $|\nabla u|$ (see, e.g., [7], Proposition 1.43 (ii), p.19/20), we have

$$|\nabla v| \leq M|\nabla u|,$$

which means $|\nabla v|(E) \leq M|\nabla u|(E)$ for any Borel set $E \subset \mathcal{B}(\Omega)$.
In the next step we assume that Φ is merely Lipschitz. For a given and sufficiently small $\varepsilon > 0$, we denote by Φ_ε a mollification of Φ, for which we can state

$$\mathrm{Lip}(\Phi_\varepsilon) \leq \mathrm{Lip}(\Phi) \quad \text{and} \quad \Phi_\varepsilon \to \Phi \text{ uniformly as } \varepsilon \downarrow 0.$$

Consequently, $v_\varepsilon := \Phi_\varepsilon(u) \to \Phi(u) = v$ in $L^1(\Omega)^L$ and a.e. as $\varepsilon \downarrow 0$. Recalling the arguments from above it follows

$$|\nabla v_\varepsilon| \leq \mathrm{Lip}(\Phi_\varepsilon)|\nabla u|.$$

Using lower semicontinuity of the total variation once again we finally obtain

$$|\nabla v| \leq \liminf_{\varepsilon \downarrow 0} |\nabla v_\varepsilon| \leq \liminf_{\varepsilon \downarrow 0} \mathrm{Lip}(\Phi_\varepsilon)|\nabla u| \leq \mathrm{Lip}(\Phi)|\nabla u|,$$

which proves the desired inequality (2.2.3). □

Remark 2.2.2
We need the generalization of the inequality (2.2.3) *to the case that* Φ *is only Lipschitz continuous in the course of this chapter. As we will see in the next section, the inequality* (2.2.3) *plays an important role in order to establish the density result for BV-functions stated in Section 2.3 (see Lemma 2.2.6). Further, we strongly need the inequality for proving a maximum principle for generalized minimizers in the third chapter of this thesis (see Section 3.1, Theorem 3.1.4 and Theorem 3.1.5).*

We finish this section by adding some comments concerning the chain rule for Sobolev functions. For this reason we assume that $u : \Omega \to \mathbb{R}^M$ is of class $W^{1,t}(\Omega)^M$ for some $1 \leq t < \infty$ as well as that $\Phi : \mathbb{R}^M \to \mathbb{R}^L$ is Lipschitz continuous and of class C^1 in addition. Setting $v := \Phi \circ u : \Omega \to \mathbb{R}^L$ it is well known (see, e.g., [79], Theorem 3.1.9) that we have $v \in W^{1,t}(\Omega)^M$ (for this statement we merely need Lipschitz continuity of Φ) and that the corresponding chain rule reads as

$$\nabla v = \nabla\Phi(u)\nabla u.$$

Removing the hypothesis of continuous differentiability imposed on the Lipschitz function Φ before, a chain rule formula has been stated by Stampacchia in [91] (without a proof). Furthermore, an inequality being in the spirit of (2.2.3) has been proven by M. Bildhauer and M. Fuchs in [19], Lemma B.1, in the context of Sobolev functions. Precisely, with the above notation, it then holds

$$|\nabla v| \leq \mathrm{Lip}(\Phi)|\nabla u|. \tag{2.2.4}$$

Note that (2.2.4) in the slightly weaker form

$$|\nabla v| \leq \sqrt{L}\mathrm{Lip}(\Phi)|\nabla u|$$

has been shown by M. Meier in [75].

Remark 2.2.3
Note that the inequality (2.2.4) *with* Φ *just being Lipschitz continuous is of important meaning during the proof of a density result for Sobolev functions in Section 2.3 (see Lemma 2.2.4).*

2.2.2 Some density results

As outlined in [17] (see Section B.1 therein), the standard approximation procedure for functions of bounded variation can be found in [8] (we further refer to, e.g., [63], Theorem 1.17, p.14). In this context, a sequence of smooth functions, converging in $L^1(\Omega)^M$, is constructed in such a way that we also get convergence of the total variations, where this method represents a BV-version of the well-known Meyers-Serrin approximation which was given in the context of Sobolev spaces (see [76]). In order to obtain continuity of the functional

$$\widetilde{K}[w] := \int_\Omega F(\nabla^a w)dx + \int_\Omega F^\infty\left(\frac{\nabla^s w}{|\nabla^s w|}\right)d|\nabla^s w|, \quad w \in BV(\Omega)^M,$$

w.r.t. to a suitable notion of convergence $w_m \to w$, it turns out that the standard approximation of BV-functions as described above is too weak. For that reason, there exists a slight and well-known modification of the standard approximation procedure in the literature (see, e.g., [9], Proposition 2.3 or [17], Lemma B.1, pp.185, where a proof is given in addition) which provides the required stronger convergences in order to apply the well-known continuity theorem of Reshetnyak (see, e.g., [9], Proposition 2.2 or [61], Theorem 2, p.92) that implies the desired continuity of $\widetilde{K}$ w.r.t. the corresponding convergences.
Having the relaxed functional K from (2.1.9) of the inpainting model at hand, the data fitting term $\int_{\Omega - D} |w - f|^2 dx$ causes severe problems during this procedure since it is only defined on the $\mathcal{L}^n$-measurable subset $\Omega - D$ of Ω and as a consequence it seems to be a delicate problem to adopt the standard construction for approximating BV-functions by smooth functions such that the data fitting term is continuous w.r.t. to the corresponding convergences as well. We conjecture that without further topological assumptions on D we cannot expect to modify the standard approximation procedure to the situation of inpainting at hand. In this context we refer the reader to [80] where density results for Sobolev functions and functions of (higher order) bounded variation with additional integrability constraints are proven by adapting the standard Meyers-Serrin approximation procedure. In contrast to the investigations in [25] (see Lemma 2.1 therein) we also do not assume some structure condition on our density F, i.e., it is not possible to derive a maximum principle for

K-minimizing sequences (compare Theorem 3.1.4 in Section 3.1) such that we get continuity of the data fitting term after using dominated convergence (see [25], proof of Theorem 1.2). For these reasons we have to prove the density of smooth functions in spaces like $BV(\Omega)^M \cap L^2(\Omega - D)^M$, which directly gives the appropriate convergences that allow to apply Reshetnyak's continuity theorem and additionally provide continuity of the data fitting term. Again it should be emphasized that the procedure of construction that we present requires no further topological assumptions on the inpainting region D.

To get into the matter we start with a density result for Sobolev functions which will be useful in the proof of Theorem 2.1.6 in Section 2.4.

Lemma 2.2.4
Let $\Omega \subset \mathbb{R}^n$ denote a bounded Lipschitz domain and consider a measurable subset D of Ω such that $\mathcal{L}^n(D) < \mathcal{L}^n(\Omega)$. Consider $p \in [1, n)$ and let $q \in (\frac{np}{n-p}, \infty)$. Suppose further that $u \in W^{1,p}(\Omega)^M \cap L^q(\Omega - D)^M$ is given. Then there exists a sequence $(u_k) \subset C^\infty(\overline{\Omega})^M$ such that (as $k \to \infty$)

$$\|u_k - u\|_{W^{1,p}(\Omega)} + \|u_k - u\|_{L^q(\Omega - D)} \to 0. \tag{2.2.5}$$

Remark 2.2.5
- *By the continuity of Sobolev's embedding $W^{1,p}(\Omega)^M \hookrightarrow L^{\frac{np}{n-p}}(\Omega)^M$, our choice of q is reasonable, since otherwise we may directly apply [4], Theorem 3.18, p.54.*
- *During the proof of Lemma 2.2.4 we need at least Lipschitz regularity of $\partial\Omega$ in order apply extension theorems for Sobolev functions.*

Proof of Lemma 2.2.4. Let us choose a smooth bounded domain $\widetilde{\Omega}$ such that $\Omega \Subset \widetilde{\Omega}$. According to [74], Remark 1.60, p.34, we may extend $u \in W^{1,p}(\Omega)^M$ to a function $\widetilde{u} \in W^{1,p}(\widetilde{\Omega})^M$ (compare [5], Fortsetzungssatz A 5.12, p.174, as well). For $m \in \mathbb{N}$ we set $\Phi_m : \mathbb{R}^M \to \mathbb{R}^M$,

$$\Phi_m(y) := \begin{cases} y, & |y| \le m \\ m\frac{y}{|y|}, & |y| \ge m \end{cases}$$

and claim for the sequence $\widetilde{u}_m := \Phi_m \circ \widetilde{u}$ the validity of (as $m \to \infty$)

$$\|\widetilde{u}_m - u\|_{L^q(\Omega - D)} \to 0, \tag{2.2.6}$$

$$\|\widetilde{u}_m - \widetilde{u}\|_{W^{1,p}(\widetilde{\Omega})} \to 0. \tag{2.2.7}$$

In fact, from $|\widetilde{u}_m - u| \le 2|u|$ a.e. on $\Omega - D$ together with $\widetilde{u}_m \to \widetilde{u}$ a.e. on $\widetilde{\Omega}$ it follows by dominated convergence that (2.2.6) is true (recall our assumption $u \in L^q(\Omega - D)^M$). In the same way we obtain $\widetilde{u}_m \to \widetilde{u}$ in $L^p(\widetilde{\Omega})^M$. The chain rule in its general form (see, e.g., [7], Theorem 3.96, p. 189) shows $\widetilde{u}_m \in W^{1,p}(\widetilde{\Omega})^M$ together with $|\nabla\widetilde{u}_m| \le \mathrm{Lip}(\Phi_m)|\nabla\widetilde{u}| = |\nabla\widetilde{u}|$ (see the comments at the end of Section 2.2.1).
From $\widetilde{u}_m = \widetilde{u}$ a.e. on $\{x \in \widetilde{\Omega} : |\widetilde{u}(x)| \le m\} =: \widetilde{\Omega}_m$ it follows that $\nabla\widetilde{u}_m = \nabla\widetilde{u}$

on $\widetilde{\Omega}_m$ (see [62], Lemma 7.7, p.145), in particular we get $\nabla\widetilde{u}_m \to \nabla\widetilde{u}$ a.e. on $\widetilde{\Omega}$, and $\|\nabla\widetilde{u}_m - \nabla\widetilde{u}\|_{L^p(\widetilde{\Omega})} \to 0$ again is a consequence of dominated convergence. In view of (2.2.6) and (2.2.7) we find a subsequence $(\widetilde{u}_{m_k})$, $k \in \mathbb{N}$, such that

$$\|\widetilde{u}_{m_k} - \widetilde{u}\|_{W^{1,p}(\widetilde{\Omega})} + \|\widetilde{u}_{m_k} - u\|_{L^q(\Omega-D)} \le \frac{1}{k} \tag{2.2.8}$$

for any $k \in \mathbb{N}$. As the next step we consider a suitable sequence of radii $\rho_k \downarrow 0$ such that ($(\cdot)_{\rho_k}$ denoting the mollification operator)

$$\|\widetilde{u}_{m_k} - (\widetilde{u}_{m_k})_{\rho_k}\|_{W^{1,p}(\Omega)} + \|\widetilde{u}_{m_k} - (\widetilde{u}_{m_k})_{\rho_k}\|_{L^q(\Omega)} \le \frac{1}{k} \tag{2.2.9}$$

for each integer k. In order to justify (2.2.9) we note that from $\widetilde{u}_{m_k} \in W^{1,p}(\widetilde{\Omega})^M \cap L^q(\widetilde{\Omega})^M$ (it actually holds $\widetilde{u}_{m_k} \in L^\infty(\widetilde{\Omega})^M$) and by recalling $\Omega \Subset \widetilde{\Omega}$ we obtain $(\widetilde{u}_{m_k})_\rho \to \widetilde{u}_{m_k}$ as $\rho \downarrow 0$ in $W^{1,p}(\Omega)^M \cap L^q(\Omega)^M$. Obviously the functions $u_k := (\widetilde{u}_{m_k})_{\rho_k}$ belong to the class $C^\infty(\overline{\Omega})^M$, and (2.2.5) is a consequence of (2.2.8) and (2.2.9). □

Let us now come to the density result for functions of BV-type that has already been mentioned above. Having the convergences from below at hand it follows continuity of the relaxed functional K from (2.1.9) and this statement plays a vital role when proving Theorem 2.1.1.

Lemma 2.2.6
With Ω and D as in Lemma 2.2.4 consider $u \in BV(\Omega)^M \cap L^q(\Omega - D)^M$ for some $q \in (\frac{n}{n-1}, \infty)$. Then there exists a sequence $(u_m) \subset C^\infty(\overline{\Omega})^M$ such that (as $m \to \infty$)

(i) $u_m \to u$ *in* $L^1(\Omega)^M$,

(ii) $u_m \to u$ *in* $L^q(\Omega - D)^M$,

(iii) $\int_\Omega \sqrt{1 + |\nabla u_m|^2} dx \to \int_\Omega \sqrt{1 + |\nabla u|^2}$.

Remark 2.2.7
According to the continuous embedding $BV(\Omega)^M \hookrightarrow L^{\frac{n}{n-1}}(\Omega)^M$ valid for "bounded extension domains" Ω (see, e.g., [7], Corollary 3.49, p.152) it makes sense to consider exponents $q > \frac{n}{n-1}$ in Lemma 2.2.6.

Remark 2.2.8
Clearly, (iii) implies the convergence

$$\int_\Omega |\nabla u_m| dx \to \int_\Omega |\nabla u|$$

by applying the continuity theorem of Reshetnyak as stated in, e.g., [9], Proposition 2.2 and choosing $F(P) := |P|$, $P \in \mathbb{R}^{nM}$, in this reference. However, this kind of convergence is actually weaker than the notion from (iii).

Proof of Lemma 2.2.6. As already done in the proof of Lemma 2.2.4 we choose a smooth bounded domain $\widetilde{\Omega}$ such that $\Omega \Subset \widetilde{\Omega}$. Given a function $w \in BV(\widetilde{\Omega})^M$ we recall that in accordance with the general concept of applying a convex function to a measure (see, e.g., [9], Definition 2.1, or [46], p. 675) the quantity $\int_\Omega \sqrt{1+|\nabla w|^2}$ is defined as follows

$$\sqrt{1+|\nabla w|^2}(\Omega) = \int_\Omega \sqrt{1+|\nabla w|^2} := \int_\Omega \sqrt{1+|\nabla^a w|^2}dx + |\nabla^s w|(\Omega),$$

where we remark that the above definition also extends to Borel sets $B \in \mathcal{B}(\Omega)$. Now let $u_0 \in L^1(\partial\Omega)^M$ denote the trace on $\partial\Omega$ of the given function $u \in BV(\Omega)^M \cap L^q(\Omega - D)^M$ whose properties are summarized in e.g. [7], Theorem 3.87, p.180/181 and set $u_0 := 0$ on $\partial\widetilde{\Omega}$, thus $u_0 \in L^1(\partial G)^M$ where $G := \widetilde{\Omega} - \overline{\Omega}$. Referring to [63], Theorem 2.16, p.39, we can find $v \in W^{1,1}(G)^M$ having trace u_0 on ∂G and such that

$$\|v\|_{W^{1,1}(G)} \leq c\|u_0\|_{L^1(\partial G)} \tag{2.2.10}$$

with c depending on ∂G but independent of u_0 and v. We then let

$$\widetilde{u} := \begin{cases} u, & \text{on } \Omega \\ v, & \text{on } \widetilde{\Omega} - \overline{\Omega} \end{cases}$$

and observe $\widetilde{u} \in BV(\widetilde{\Omega})^M$, which follows from [7], Corollary 3.89, p.183, and the fact that (2.2.10) implies $v \in BV(G)^M$. Viewing ∇u (resp. ∇v) as measures on $\widetilde{\Omega}$ concentrated on Ω (resp. $\widetilde{\Omega} - \overline{\Omega}$) and recalling the definition of v we further deduce from the above reference the identity

$$\nabla\widetilde{u} = \nabla u + \nabla v \tag{2.2.11}$$

as measures on $\widetilde{\Omega}$. As in the proof of Lemma 2.2.4 we consider ($m \in \mathbb{N}$)

$$\widetilde{u}_m := \Phi_m \circ \widetilde{u}$$

and observe (compare Lemma 2.2.1 in Section 2.2.1)

$$\widetilde{u}_m \in BV(\widetilde{\Omega})^M, \quad |\nabla\widetilde{u}_m| \leq \mathrm{Lip}(\Phi_m)|\nabla\widetilde{u}| = |\nabla\widetilde{u}|. \tag{2.2.12}$$

In particular, from $|\nabla\widetilde{u}|(\partial\Omega) = 0$ (recall (2.2.11)) it follows that

$$|\nabla\widetilde{u}_m|(\partial\Omega) = 0, \quad m \in \mathbb{N}. \tag{2.2.13}$$

As a consequence from (2.2.12) we obtain (compare [9], Proposition 2.1)

$$\sqrt{1+|\nabla\widetilde{u}_m|^2}(E) \leq \sqrt{1+|\nabla\widetilde{u}|^2}(E) \tag{2.2.14}$$

for any Borel set $E \subset \widetilde{\Omega}$ while on the other hand we get by using (2.2.13)

$$\sqrt{1+|\nabla\widetilde{u}_m|^2}(\partial\Omega) = 0, \quad m \in \mathbb{N}. \tag{2.2.15}$$

As in the proof of Lemma 2.2.4, by dominated convergence, it holds (as $m \to \infty$)

$$\widetilde{u}_m \to \widetilde{u} \quad \text{in } L^1(\widetilde{\Omega})^M, \tag{2.2.16}$$
$$\widetilde{u}_m \to u \quad \text{in } L^q(\Omega - D)^M, \tag{2.2.17}$$

and (2.2.16) combined with lower semicontinuity (see Lemma 2.3.1 in Section 2.3 of this thesis) implies

$$\sqrt{1+|\nabla \widetilde{u}|^2}(\widetilde{\Omega}) \le \liminf_{m\to\infty} \sqrt{1+|\nabla \widetilde{u}_m|^2}(\widetilde{\Omega}).$$

From (2.2.14) we get

$$\sqrt{1+|\nabla \widetilde{u}_m|^2}(\widetilde{\Omega}) \le \sqrt{1+|\nabla \widetilde{u}|^2}(\widetilde{\Omega}),$$

thus

$$\sqrt{1+|\nabla \widetilde{u}_m|^2}(\widetilde{\Omega}) \to \sqrt{1+|\nabla \widetilde{u}|^2}(\widetilde{\Omega}), \quad m \to \infty. \tag{2.2.18}$$

Clearly we can replace $\widetilde{\Omega}$ in (2.2.18) by the domain Ω, so that in combination with (2.2.16) and (2.2.17) it holds for a subsequence (recall that by (2.2.11) $|\nabla \widetilde{u}|(\Omega) = |\nabla u|(\Omega)$)

$$\begin{aligned} &\|\widetilde{u}_{m_k} - u\|_{L^1(\Omega)} + \|\widetilde{u}_{m_k} - u\|_{L^q(\Omega - D)} \\ &+ \left|\sqrt{1+|\nabla \widetilde{u}_{m_k}|^2}(\Omega) - \sqrt{1+|\nabla u|^2}(\Omega)\right| \le \frac{1}{k}, \quad k \in \mathbb{N}. \end{aligned} \tag{2.2.19}$$

In a next step we consider a radius $\rho > 0$ (being sufficiently small) and introduce the mollification $(\widetilde{u}_{m_k})_\rho$. By quoting well-known convergence properties of the mollification (see, e.g., [4], Lemma 2.18, p.29/30) we may directly infer (note that we have the convergence $(\widetilde{u}_{m_k})_\rho \to \widetilde{u}_{m_k}$ in $L^p_{\text{loc}}(\widetilde{\Omega})^M$ as $\rho \downarrow 0$ for any $p \in [1, \infty)$)

$$\|(\widetilde{u}_{m_k})_\rho - \widetilde{u}_{m_k}\|_{L^1(\Omega)} + \|(\widetilde{u}_{m_k})_\rho - \widetilde{u}_{m_k}\|_{L^q(\Omega - D)} \to 0 \quad \text{as } \rho \downarrow 0. \tag{2.2.20}$$

In order to show the required convergence for the modification of the total variation we adopt ideas as applied in [63], Proposition 1.15, p.12 and follow the procedure carried out in [17], Proof of Lemma B.1, p.185-188, i.e. we consider the function

$$g(Z) = \sqrt{1+|Z|^2} - 1, \quad Z \in \mathbb{R}^{nM},$$

and notice that its conjugate function g^* is given by

$$g^*(Q) = \begin{cases} +\infty, & \text{if } |Q| > 1, \\ 1 - \sqrt{1-|Q|^2}, & \text{if } |Q| \le 1. \end{cases}$$

for any $Q \in \mathbb{R}^{nM}$. Further, we note that g^* is convex satisfying $g^*(0) = 0$ as well as $g^* \ge 0$.

Following [46], Definition 1.2, we can state the following representation formula for the measure $\int_{\widetilde{\Omega}} g(\nabla w)$ with $w \in BV(\widetilde{\Omega})^M$ (compare also (3) in [17] on p.186)

$$\int_{\widetilde{\Omega}} g(\nabla w) = \sup_{\varkappa \in C_0^\infty(\widetilde{\Omega})^{nM}, |\varkappa| \le 1} \Bigg[- \int_{\widetilde{\Omega}} w \operatorname{div} \varkappa \, dx - \int_{\widetilde{\Omega}} g^*(\varkappa) dx \Bigg].$$

Note that the above representation formula extends to any Borel set $E \subset \widetilde{\Omega}$, i.e., it holds (compare [46], Definition 1.2 again)

$$\int_E g(\nabla w) = \sup_{\varkappa \in C_0^\infty(\widetilde{\Omega})^{nM}, |\varkappa| \le 1} \Bigg[\int_E \varkappa : \nabla w - \int_{\widetilde{\Omega}} g^*(\varkappa) dx \Bigg].$$

By lower semicontinuity (recall (2.2.20)) we have

$$\int_{\Omega} g(\nabla \widetilde{u}_{m_k}) \le \liminf_{\rho \downarrow 0} \int_{\Omega} g(\nabla (\widetilde{u}_{m_k})_\rho). \tag{2.2.21}$$

For verifying the reverse inequality we fix $\varkappa \in C_0^\infty(\Omega)^{nM}$ satisfying $|\varkappa| \le 1$ and get (we identify $\varkappa$ with its zero-extension to $\mathbb{R}^n$)

$$\begin{aligned} & - \int_{\widetilde{\Omega}} (\widetilde{u}_{m_k})_\rho \operatorname{div} \varkappa \, dx - \int_{\widetilde{\Omega}} g^*(\varkappa) dx \\ & = - \int_{\widetilde{\Omega}} \widetilde{u}_{m_k} \operatorname{div}(\varkappa)_\rho \, dx - \int_{\widetilde{\Omega}} g^*(\varkappa) dx. \end{aligned} \tag{2.2.22}$$

Besides, we obtain (recall $\varkappa \equiv 0$ on $\mathbb{R}^n - \Omega$ and $g^*(0) = 0$)

$$\begin{aligned} - \int_{\widetilde{\Omega}} g^*(\varkappa) dx &= - \int_{\mathbb{R}^n} g^*(\varkappa) dx \\ &= - \int_{\mathbb{R}^n} g^*((\varkappa)_\rho) dx + \Bigg\{ \int_{\mathbb{R}^n} [g^*((\varkappa)_\rho) - g^*(\varkappa)] dx \Bigg\} \end{aligned} \tag{2.2.23}$$

In order to handle the second integral on the r.h.s. of (2.2.23) we use Jensen's inequality which gives

$$g^*((\varkappa)_\rho) \le g^*(\varkappa)_\rho. \tag{2.2.24}$$

By means of (2.2.24) and by performing standard calculations we may derive

$$\int_{\mathbb{R}^n} [g^*((\varkappa)_\rho) - g^*(\varkappa)] dx \le \int_{\mathbb{R}^n} [g^*(\varkappa)_\rho - g^*(\varkappa)] dx = 0. \tag{2.2.25}$$

As a result of (2.2.23)–(2.2.25), (2.2.22) turns into

$$\begin{aligned} & - \int_{\widetilde{\Omega}} (\widetilde{u}_{m_k})_\rho \operatorname{div} \varkappa \, dx - \int_{\widetilde{\Omega}} g^*(\varkappa) dx \\ & \le - \int_{\widetilde{\Omega}} (\widetilde{u}_{m_k}) \operatorname{div}(\varkappa)_\rho \, dx - \int_{\widetilde{\Omega}} g^*((\varkappa)_\rho) dx. \end{aligned} \tag{2.2.26}$$

Taking standard properties of the mollification into account (see, e.g.,[63], p.11) we get $|(\varkappa)_\rho| \le 1$ since $|\varkappa| \le 1$ as well as $\operatorname{spt}(\varkappa)_\rho \subset \Omega_\rho := \{x; \operatorname{dist}(x,\Omega) \le \rho\}$ since $\operatorname{spt}\varkappa \subset \Omega$. Consequently, (2.2.26) turns into

$$-\int\limits_{\widetilde{\Omega}} (\widetilde{u}_{m_k})_\rho \operatorname{div}\varkappa \, dx - \int\limits_{\widetilde{\Omega}} g^*(\varkappa) dx \le \int\limits_{\Omega_\rho} g(\nabla \widetilde{u}_{m_k}).$$

At this point we take the supremum over all such $\varkappa$ and get

$$\int\limits_{\Omega} g(\nabla(\widetilde{u}_{m_k})_\rho) \le \int\limits_{\Omega_\rho} g(\nabla \widetilde{u}_{m_k}).$$

Thus, we obtain

$$\limsup_{\rho\downarrow 0} \int\limits_{\Omega} g(\nabla(\widetilde{u}_{m_k})_\rho) \le \lim_{\rho\downarrow 0} \int\limits_{\Omega_\rho} g(\nabla \widetilde{u}_{m_k}) = \int\limits_{\overline{\Omega}} g(\nabla \widetilde{u}_{m_k}).$$

On account of (2.2.15) it holds

$$\int\limits_{\partial\Omega} g(\nabla \widetilde{u}_{m_k}) = 0,$$

thus we arrive at

$$\limsup_{\rho\downarrow 0} \int\limits_{\Omega} g(\nabla(\widetilde{u}_{m_k})_\rho) \le \int\limits_{\Omega} g(\nabla \widetilde{u}_{m_k}). \tag{2.2.27}$$

Combining (2.2.27) with (2.2.21) we have

$$\lim_{\rho\downarrow 0} \int\limits_{\Omega} g(\nabla(\widetilde{u}_{m_k})_\rho) = \int\limits_{\Omega} g(\nabla \widetilde{u}_{m_k}) \tag{2.2.28}$$

which clearly gives

$$\lim_{\rho\downarrow 0} \sqrt{1+|\nabla(\widetilde{u}_{m_k})_\rho|^2}(\Omega) = \sqrt{1+|\nabla \widetilde{u}_{m_k}|^2}(\Omega).$$

Passing now to a suitable subsequence of radii ρ_k ($k \in \mathbb{N}$) going to zero we can arrange

$$\begin{aligned} &\|\widetilde{u}_{m_k} - (\widetilde{u}_{m_k})_{\rho_k}\|_{L^1(\Omega)} + \|\widetilde{u}_{m_k} - (\widetilde{u}_{m_k})_{\rho_k}\|_{L^q(\Omega-D)} \\ &+ \left| \sqrt{1+|\nabla(\widetilde{u}_{m_k})_{\rho_k}|^2}(\Omega) - \sqrt{1+|\nabla \widetilde{u}_{m_k}|^2}(\Omega) \right| \\ &\le \frac{1}{k}. \end{aligned} \tag{2.2.29}$$

Putting together (2.2.19) and (2.2.29) we see that the sequence $u_k := (\widetilde{u}_{m_k})_{\rho_k}$ is of class $C^\infty(\overline{\Omega})^M$ and satisfies the desired convergences. □

Finally, in case that $\partial\Omega$ is Lipschitz, we like to give a short proof of the classical BV-approximation procedure as carried out in, e.g., [63], Theorem 1.17, pp.14.

Corollary 2.2.9
Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and let $u \in BV(\Omega)^M$. Then, there exists a sequence $(u_k) \subset C^\infty(\overline{\Omega})^M$ such that we have (as $k \to \infty$)

(i) $u_k \to u$ *in* $L^1(\Omega)^M$,

(ii) $\int_\Omega |\nabla u_k| dx \to \int_\Omega |\nabla u|$.

Proof of Corollary 2.2.9. With $\widetilde{\Omega}$, u_0, $\widetilde{u}$ and $\widetilde{u}_m$ as in the proof of Lemma 2.2.6 we recall the following facts (see (2.2.12) and (2.2.13))

$$\widetilde{u}_m \in BV(\widetilde{\Omega})^M, \quad |\nabla \widetilde{u}_m| \le |\nabla \widetilde{u}|, \quad |\nabla \widetilde{u}_m|(\partial\Omega) = 0. \tag{2.2.30}$$

An application of dominated convergence directly yields by using the definition of $\widetilde{u}_m$

$$\widetilde{u}_m \to \widetilde{u} \quad \text{in } L^1(\widetilde{\Omega})^M. \tag{2.2.31}$$

Combining lower semicontinuity (recall (2.2.31)) with the estimate $|\nabla \widetilde{u}_m| \le |\nabla \widetilde{u}|$ from above we obtain

$$\lim_{m\to\infty} |\nabla \widetilde{u}_m|(\widetilde{\Omega}) = |\nabla \widetilde{u}|(\widetilde{\Omega}). \tag{2.2.32}$$

Replacing $\widetilde{\Omega}$ through Ω and passing to a suitable subsequence we get (recall (2.2.31), (2.2.32) and $|\nabla \widetilde{u}|(\Omega) = |\nabla u|(\Omega)$ by (2.2.11))

$$\|\widetilde{u}_{m_k} - u\|_{L^1(\Omega)} + \Big| |\nabla \widetilde{u}_{m_k}|(\Omega) - |\nabla u|(\Omega) \Big| \le \frac{1}{k}, \quad k \in \mathbb{N}. \tag{2.2.33}$$

From [63], Proposition 1.15, p.12, and on account of $|\nabla \widetilde{u}_m|(\partial\Omega) = 0$ we can find a sequence of radii $\rho_k \downarrow 0$ such that ($(\cdot)_{\rho_k}$ denoting the mollification operator) the functions $u_k := (\widetilde{u}_{m_k})_{\rho_k}$ satisfy

$$\Big| |\nabla u_k|(\Omega) - |\nabla \widetilde{u}_{m_k}|(\Omega) \Big| \le \frac{1}{k}, \quad k \in \mathbb{N}. \tag{2.2.34}$$

Moreover due to the convergence $(\widetilde{u}_{m_k})_\rho \to \widetilde{u}_{m_k}$ in $L^p_{\mathrm{loc}}(\widetilde{\Omega})^M$ as $\rho \downarrow 0$ for any $p \in [1, \infty)$ we can arrange

$$\|u_k - \widetilde{u}_{m_k}\|_{L^1(\Omega)} \le \frac{1}{k} \tag{2.2.35}$$

for any $k \in \mathbb{N}$. Putting together (2.2.33)–(2.2.35) we see that the sequence $(u_k) \subset C^\infty(\overline{\Omega})^M$ has the desired properties. $\square$

Remark 2.2.10
Quoting [63], Theorem 2.11, p.37, it follows from Corollary 2.2.9 that we even get convergence of the particular traces. Precisely, assuming the hypotheses of Corollary 2.2.9 and denoting by $\varphi_k, \varphi \in L^1(\partial\Omega)^M$ the traces of the functions u_k, u from above we obtain

$$\lim_{k\to\infty} \int\limits_{\partial\Omega} |\varphi_k - \varphi| d\mathcal{H}^{n-1} = 0.$$

2.2.3 A variant of Poincaré's inequality

In this section we are going to show a variant of Poincaré's inequality that proves to be one of the main tools for verifying Theorem 2.1.1 and Theorem 2.1.6 as well. With Ω and D as above we consider a function $u \in W^{1,p}(\Omega)$ with $p \in [1,\infty)$. Then the following well-known version of Poincaré's inequality holds (see, e.g., [52], Theorem 1, p. 275): there exists a constant $c = c(p,\Omega)$ such that

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \le c\|\nabla u\|_{L^p(\Omega)} \tag{2.2.36}$$

where $(u)_\Omega := \fint_\Omega u\,dx := \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega u\,dx$ is the mean value of u on Ω. As it will be shortly discussed in Lemma 2.2.12, a standard approximation as stated in Corollary 2.2.9 shows that (2.2.36) extends to $u \in BV(\Omega)$.

Considering the case that Ω is convex it is even possible to calculate a constant c for which we get the above inequality. Precisely it holds (see, e.g., [62], (7.45), p.157)

$$\|u - (u)_\Omega\|_{L^p(\Omega)} \le \left(\frac{\omega_n}{\mathcal{L}^n(\Omega)}\right)^{1-\frac{1}{n}} d^n \|\nabla u\|_{L^p(\Omega)},$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$ and $d := \mathrm{diam}(\Omega)$.

Unfortunately, the inequality (2.2.36) does not precisely fit in the setting of image inpainting. We make use of the following version of Poincaré's inequality (see, e.g., [7], Exercise 7.7, p. 380).

Lemma 2.2.11
Let $\Omega \subset \mathbb{R}^n$ denote a bounded Lipschitz domain and consider a $\mathcal{L}^n$-measurable subset $E \subset \Omega$ with $\mathcal{L}^n(E) > 0$. Suppose further that $u \in W^{1,p}(\Omega)$ is given where $p \in [1,\infty)$ is a fixed number. Then there exists a constant $c = c(n,p,E,\Omega)$ such that

$$\|u - (u)_E\|_{L^p(\Omega)} \le c\|\nabla u\|_{L^p(\Omega)}. \tag{2.2.37}$$

Proof of Lemma 2.2.11. If the statement (2.2.37) is false, we can find a sequence $(u_k) \subset W^{1,p}(\Omega)$ such that

$$\|u_k - (u_k)_E\|_{L^p(\Omega)} > k\|\nabla u_k\|_{L^p(\Omega)} \tag{2.2.38}$$

for all $k \geq 1$. Now we let $v_k := u_k - (u_k)_E$ and observe

$$(v_k) \subset W^{1,p}(\Omega), \quad (v_k)_E = 0, \quad \text{as well as}$$
$$\|v_k\|_{L^p(\Omega)} > k\|\nabla v_k\|_{L^p(\Omega)}$$

for all $k \geq 1$, where the last inequality holds on account of (2.2.38).
Passing to the normalized sequence $w_k := v_k/\|v_k\|_{L^p(\Omega)}$ it follows

$$\|w_k\|_{L^p(\Omega)} = 1 \quad \text{and} \quad \|\nabla w_k\|_{L^p(\Omega)} < \frac{1}{k} \tag{2.2.39}$$

for all $k \geq 1$.
As a consequence, (2.2.39), in particular, yields $\sup\limits_k \|w_k\|_{W^{1,p}(\Omega)} < \infty$ and since the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact (see, e.g., [4] and note that Ω is a Lipschitz domain) we get

$$w_k \to: w \quad \text{in } L^p(\Omega) \tag{2.2.40}$$

at least for a subsequence. Besides, (2.2.39) and (2.2.40) further yield

$$(w_k, \nabla w_k) \to (w, 0) \quad \text{in } L^p(\Omega)^{1+n}.$$

Since $W^{1,p}(\Omega)$ is a closed subspace of $L^p(\Omega)^{1+n}$ we directly see

$$w \in W^{1,p}(\Omega), \quad \nabla w = 0 \text{ a.e. in } \Omega, \quad w_k \to w \quad \text{in } W^{1,p}(\Omega),$$

where $\nabla w = 0$ in Ω implies that w is constant in Ω since Ω, in particular, is connected.
Moreover, it holds

$$(w)_E = 0, \tag{2.2.41}$$

which implies $w = 0$ a.e. in E (recall that w is constant). From $\mathcal{L}^n(E) > 0$ we immediately derive $w = 0$ a.e. in Ω. By (2.2.40), this gives

$$0 = \|w\|_{L^p(\Omega)} = \lim_{k\to\infty} \|w_k\|_{L^p(\Omega)} = 1,$$

which is a contradiction. This proves Lemma 2.2.11. □

By using standard approximation arguments it is possible to extend Lemma 2.2.11 to all functions u of bounded variation.

Lemma 2.2.12
With Ω and E as in Lemma 2.2.11 consider $u \in BV(\Omega)$. Then there is a constant $c = c(n, E, \Omega)$ such that

$$\int_\Omega |u - (u)_E| dx \leq c|\nabla u|(\Omega) \tag{2.2.42}$$

Proof of Lemma 2.2.12. Let $u \in BV(\Omega)$ be given. From Lemma 2.2.11 we get

$$\int_\Omega |v - (v)_E| dx \le c \int_\Omega |\nabla v| dx$$

for all functions $v \in W^{1,1}(\Omega)$.
Quoting Corollary 2.2.9 from Section 2.2.2 we may choose a sequence $(v_k) \subset W^{1,1}(\Omega)$ such that (as $k \to \infty$)

$$v_k \to u \quad \text{in } L^1(\Omega),$$
$$\int_\Omega |\nabla v_k| dx \to |\nabla u|(\Omega),$$

and an application of these convergences finally yields the desired inequality (2.2.42). □

2.3 Weak minimizers. Proof of Theorem 2.1.1

Assuming the validity of the hypotheses from Theorem 2.1.1 we first recall (see [25], Lemma 2.2) the following auxiliary result concerning the continuity of (the relaxed variant of) the fidelity term occuring in the functional defined in formula (2.1.3).

Lemma 2.3.1
For $w \in BV(\Omega)^M$ let

$$\widetilde{K}[w] := \int_\Omega F(\nabla^a w) dx + \int_\Omega F^\infty\left(\frac{\nabla^s w}{|\nabla^s w|}\right) d|\nabla^s w|.$$

(a) Suppose that $w_m, w \in BV(\Omega)^M$ are such that $w_m \to w$ in $L^1(\Omega)^M$. Then it holds:

$$\widetilde{K}[w] \le \lim_{m\to\infty} \widetilde{K}[w_m]. \tag{2.3.1}$$

(b) If we know in addition

$$\int_\Omega \sqrt{1 + |\nabla w_m|^2} \to \int_\Omega \sqrt{1 + |\nabla w|^2},$$

then it follows

$$\lim_{m\to\infty} \widetilde{K}[w_m] = \widetilde{K}[w]. \tag{2.3.2}$$

Remark 2.3.2
The reader should note that Lemma 2.2 of [25] clearly extends to any any dimension $n \geq 2$, moreover, the statement remains valid for vector-valued functions, i.e., for the case $M \geq 2$. The corresponding references are given during the proof of [25], Lemma 2.2.

Now, let us start proving Theorem 2.1.1: first, we note that assertion (b) is immediate by using strict convexity of F and of the data fitting term w.r.t. w. For showing part (a) we denote by $(u_m) \subset BV(\Omega)^M \cap L^2(\Omega - D)^M$ a K-minimizing sequence for which we have

$$\sup_m \int_\Omega |\nabla u_m| < \infty, \tag{2.3.3}$$

$$\sup_m \int_{\Omega - D} |u_m|^2 dx < \infty. \tag{2.3.4}$$

Note that (2.3.3) is valid since F is of linear growth.
By virtue of (2.3.4) we apply the variant of Poincaré's inequality as stated in Lemma 2.2.12 in Section 2.2.3 (note that (2.1.1) trivially gives $\mathcal{L}^n(\Omega - D) > 0$) which together with (2.3.3) yields

$$\sup_m \int_\Omega |u_m| dx < \infty. \tag{2.3.5}$$

Combining (2.3.3) and (2.3.5), the BV-compactness theorem guarantees the existence of a function $\overline{u} \in BV(\Omega)^M$ such that $u_m \to: \overline{u}$ in $L^1(\Omega)^M$ and a.e. up to a subsequence. Further, (2.3.4) combined with Fatou's lemma implies $\overline{u} \in L^2(\Omega - D)^M$, i.e., $\overline{u} \in BV(\Omega)^M \cap L^2(\Omega - D)^M$ and $K[\overline{u}]$ is well-defined. From (2.3.1), we then get

$$\widetilde{K}[\overline{u}] \leq \liminf_{m \to \infty} \widetilde{K}[u_m],$$

whereas Fatou's lemma gives

$$\int_{\Omega - D} |\overline{u} - f|^2 dx \leq \liminf_{m \to \infty} \int_{\Omega - D} |u_m - f|^2 dx.$$

Hence,

$$\begin{aligned} K[\overline{u}] &\leq \liminf_{m \to \infty} \widetilde{K}[u_m] + \liminf_{m \to \infty} \frac{\lambda}{2} \int_{\Omega - D} |u_m - f|^2 dx \\ &\leq \liminf_{m \to \infty} K[u_m] = \inf_{BV(\Omega)^M \cap L^2(\Omega - D)^M} K. \end{aligned}$$

As a consequence, $\overline{u}$ is K-minimizing showing assertion (a) of Theorem 2.1.1.

For proving assertion (c) we set

$$\alpha := \inf_{BV(\Omega)^M \cap L^2(\Omega - D)^M} K, \quad \beta := \inf_{W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M} I$$

and observe that $\alpha \le \beta$ is obvious since $I = K$ on $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$.
Conversely we fix an arbitrary K-minimizer $u \in BV(\Omega)^M \cap L^2(\Omega - D)^M$ and choose a sequence (u_m) according to Lemma 2.2.6 applied with exponent $q = 2$. Quoting Lemma 2.3.1 we then have $\widetilde{K}[u_m] \to \widetilde{K}[u]$, and, by Lemma 2.2.6 (ii), we conclude $K[u_m] \to K[u]$. Thus,

$$\beta = \inf_{W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M} I \le I[u_m] = K[u_m] \longrightarrow K[u] = \alpha,$$

which shows (c).

To establish part (d) we first consider $u \in \mathcal{M}$, i.e. $u_m \to u$ in $L^1(\Omega)^M$ for an I-minimizing sequence (u_m) from $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$, where may assume in addition $u_m \to u$ a.e. on Ω. In view of Fatou's lemma we may conclude

$$\int\limits_{\Omega - D} |u - f|^2 dx \le \liminf_{m \to \infty} \int\limits_{\Omega - D} |u_m - f|^2 dx,$$

whereas by Lemma 2.3.1 (a)

$$\widetilde{K}[u] \le \liminf_{m \to \infty} \widetilde{K}[u_m].$$

Thus we arrive at

$$K[u] \le \liminf_{m \to \infty} K[u_m] = \liminf_{m \to \infty} I[u_m] = \inf_{BV(\Omega)^M \cap L^2(\Omega - D)^M} K,$$

where the last equality follows from assertion (c). This gives the K-minimality of u.
Conversely consider a K-minimizer $u \in BV(\Omega)^M \cap L^2(\Omega - D)^M$. If we choose u_m according to Lemma 2.2.6 and apply Lemma 2.3.1 (b), we obtain (as $m \to \infty$)

$$I[u_m] = K[u_m] \to K[u].$$

Using assertion (c) again, it follows that (u_m) is an I-minimizing sequence for which (see Lemma 2.2.6 (i)) $u_m \to u$ in $L^1(\Omega)^M$. This proves $u \in \mathcal{M}$ and completes the proof of Theorem 2.1.1. □

2.4 Dual solutions. Proof of Theorem 2.1.6

Let the assumptions of Theorem 2.1.6 hold. Primarily we note that a proof of assertion (a) probably can be deduced from [54], Theorem 1.2.1, p.15/16 or [49], Proposition 2.3, Chapter III, p.52. Inspired by [25], proof of Theorem 1.4, we decide to give a more constructive proof relying on an approximation of our original variational problem (2.1.8) by a sequence of more regular problems admitting smooth solutions with suitable convergence properties. Consequently, this sequence might be of interest for numerical computations. To become more precise we consider for fixed $\delta \in (0,1]$ the problem

$$I_\delta[w] := \int\limits_\Omega F_\delta(\nabla w)dx + \frac{\lambda}{2}\int\limits_{\Omega-D} |w-f|^2 dx \to \min \quad \text{in } W^{1,2}(\Omega)^M \tag{2.4.1}$$

where

$$F_\delta(P) := \frac{\delta}{2}|P|^2 + F(P), \quad P \in \mathbb{R}^{nM}. \tag{2.4.2}$$

In the following lemma we show that (2.4.1) is uniquely solvable in the appropriate Sobolev space $W^{1,2}(\Omega)^M$ and in addition we will state some useful uniform (in δ) features of the unique solution u_δ.

Lemma 2.4.1
The problem (2.4.1) *admits a unique solution* $u_\delta \in W^{1,2}(\Omega)^M$ *and we additionally have the following uniform bounds on* u_δ

(i) $\sup\limits_\delta \|\nabla u_\delta\|_{L^1(\Omega)} < \infty$,

(ii) $\sup\limits_\delta \|u_\delta - f\|_{L^2(\Omega-D)} < \infty$,

(iii) $\sup\limits_\delta \ \delta \int\limits_\Omega |\nabla u_\delta|^2 dx < \infty$,

(iv) $\sup\limits_\delta \|u_\delta\|_{W^{1,1}(\Omega)} < \infty$.

Proof of Lemma 2.4.1. Clearly, the problem (2.4.1) admits at most one solution $u_\delta \in W^{1,2}(\Omega)^M$. In fact, if u_1, u_2 are solutions of (2.4.1), we then have $\nabla u_1 = \nabla u_2$ on Ω together with $u_1 = u_2$ on $\Omega - D$. But then $u_1 = u_2$ on Ω on account of (2.1.1). Next, with δ being fixed, we consider a minimizing sequence (u_m) for (2.4.1). Using the linear growth of F it holds

$$\begin{aligned}
&\sup_m \|\nabla u_m\|_{L^2(\Omega)} \le c(\delta) < \infty,\\
&\sup_m \|\nabla u_m\|_{L^1(\Omega)} < \infty,\\
&\sup_m \|u_m - f\|_{L^2(\Omega-D)} < \infty.
\end{aligned}$$

The quadratic variant of the Poincaré inequality from Section 2.2.3 (choose $p = 2$ in Lemma 2.2.11) then yields

$$\sup_m \|u_m\||_{W^{1,2}(\Omega)} < \infty,$$

so that $u_m \rightharpoonup: u_\delta$ in $W^{1,2}(\Omega)^M$ at least for a subsequence of (u_m). Standard theorems on lower semicontinuity (see, e.g., [57], Theorem 2.3, p.18 or [2]) then show that u_δ solves (2.4.1).

Observing that we have the uniform estimate $I_\delta[u_\delta] \le I_\delta[0] = I[0]$ we can directly derive the assertions (i)-(iii) where the linear growth of F has to be exploited once again. Combining part (i) with Poincaré's inequality from Section 2.2.3 (choose $p = 1$ in Lemma 2.2.11 and recall (ii)) we finally obtain

$$\sup_\delta \|u_\delta\|_{L^1(\Omega)} < \infty. \tag{2.4.3}$$

Summarizing, part (i) and (2.4.3) directly imply the last claim (iv) and this completes the proof of Lemma 2.4.1. $\square$

Proceeding with the proof of Theorem 2.1.6 we first use Lemma 2.4.1 (iv) which yields by BV-compactness (at least for a suitable sequence $\delta \downarrow 0$)

$$u_\delta \to: \overline{u} \quad \text{in } L^1(\Omega)^M \text{ and a.e.}$$

for a function $\overline{u} \in BV(\Omega)^M$. Moreover, Lemma 2.4.1 (ii) gives

$$u_\delta \rightharpoonup \overline{u} \quad \text{in } L^2(\Omega - D)^M$$

after passing to an appropriate subsequence. Thus,

$$\int_{\Omega - D} |\overline{u} - f|^2 dx \le \liminf_{\delta \downarrow 0} \int_{\Omega - D} |u_\delta - f|^2 dx.$$

Altogether our limit function $\overline{u}$ belongs to the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$. Next we set

$$\tau_\delta := DF(\nabla u_\delta) \quad \text{and} \quad \sigma_\delta := DF_\delta(\nabla u_\delta) = \delta \nabla u_\delta + \tau_\delta \tag{2.4.4}$$

and observe that Lemma 2.4.1 (iii) implies

$$||\delta \nabla u_\delta||^2_{L^2(\Omega)} = \delta \Big(\delta \int_\Omega |\nabla u_\delta|^2 dx \Big) \to 0 \quad \text{as } \delta \to 0, \tag{2.4.5}$$

whereas (2.1.5) shows that τ_δ is uniformly bounded w.r.t. δ, i.e.,

$$\sup_\delta ||\tau_\delta||_{L^\infty(\Omega)} < \infty. \tag{2.4.6}$$

After passing to suitable sequences $\delta \to 0$ we get from (2.4.4)–(2.4.6)

$$\sigma_\delta \rightharpoonup: \sigma \text{ in } L^2(\Omega)^{nM} \quad \text{and} \quad \tau_\delta \overset{*}{\rightharpoonup}: \tau \text{ in } L^\infty(\Omega)^{nM} \tag{2.4.7}$$

and by combining (2.4.7) with (2.4.5), it follows $\sigma = \tau$.

Next, we claim that $\sigma \in L^\infty(\Omega)^{nM}$ is a solution of the dual variational problem. To justify this, we first observe that u_δ solves the Euler equation

$$\int_\Omega \tau_\delta : \nabla \varphi dx + \delta \int_\Omega \nabla u_\delta : \nabla \varphi dx + \lambda \int_{\Omega - D} (u_\delta - f) \cdot \varphi dx = 0 \tag{2.4.8}$$

for all $\varphi \in W^{1,2}(\Omega)^M$.
Applying the duality relation (2.1.16) to our smooth integrand F we obtain the identity $F(\nabla u_\delta) = \tau_\delta : \nabla u_\delta - F^*(\tau_\delta)$ which implies

$$I_\delta[u_\delta] = \frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 dx + \int_\Omega [\tau_\delta : \nabla u_\delta - F^*(\tau_\delta)]dx + \frac{\lambda}{2}\int_{\Omega - D} |u_\delta - f|^2 dx.$$

Since u_δ is an admissible choice in (2.4.8) it further holds

$$\begin{aligned} I_\delta[u_\delta] &= -\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 dx + \int_\Omega (-F^*(\tau_\delta))dx + \frac{\lambda}{2}\int_{\Omega-D} |u_\delta - f|^2 dx \\ &\quad - \lambda \int_{\Omega-D} (u_\delta - f)\cdot u_\delta dx \\ &= -\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 dx + \int_\Omega (-F^*(\tau_\delta))dx - \frac{\lambda}{2}\int_{\Omega-D} |u_\delta|^2 dx \\ &\quad + \frac{\lambda}{2}\int_{\Omega-D} |f|^2 dx, \end{aligned} \tag{2.4.9}$$

where the quadratic structure of the data fitting term is essential in order to establish (2.4.9).
Now we let $v \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$. Taking into account (2.1.17), (2.1.18) and the definition of the dual functional R it then follows for any $\rho \in L^\infty(\Omega)^{nM}$

$$I[v] = \sup_{\varkappa \in L^\infty(\Omega)^{nM}} l(v, \varkappa) \ge l(v, \rho) \ge \inf_{w \in W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M} l(w, \rho) = R[\rho],$$

hence, we deduce

$$\sup_{\rho \in L^\infty(\Omega)^{nM}} R[\rho] \le \inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M} I[v].$$

Obviously we have the validity of

$$\inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M} I[v] \le I[u_\delta] \le I_\delta[u_\delta]$$

and by virtue of (2.4.9) we may conclude

$$\begin{aligned} \sup_{\rho \in L^\infty(\Omega)^{nM}} R[\rho] &\le \inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M} I[v] \\ &\le -\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 dx + \int_\Omega (-F^*(\tau_\delta))dx \\ &\quad - \frac{\lambda}{2}\int_{\Omega-D} |u_\delta|^2 dx + \frac{\lambda}{2}\int_{\Omega-D} |f|^2 dx. \end{aligned} \tag{2.4.10}$$

Neglecting the quantity $-\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 dx$ in (2.4.10) for the moment, we pass to the limit $\delta \to 0$. This gives by using upper semicontinuity of $\int_\Omega (-F^*(\cdot))dx$

w.r.t. weak-$*$ convergence and by recalling $\int_{\Omega-D} |\overline{u}|^2 dx \leq \liminf_{\delta\to 0} \int_{\Omega-D} |u_\delta|^2 dx$ (note that we have the appropriate signs in (2.4.10))

$$\begin{aligned}\sup_{L^\infty(\Omega)^{nM}} R &\leq \inf_{W^{1,1}(\Omega)^M\cap L^2(\Omega-D)^M} I \\ &\leq \int_\Omega (-F^*(\tau))dx - \frac{\lambda}{2}\int_{\Omega-D} |\overline{u}|^2 dx + \frac{\lambda}{2}\int_{\Omega-D} |f|^2 dx. \end{aligned} \tag{2.4.11}$$

Passing to the limit $\delta \to 0$ in Euler's equation (2.4.8) we obtain (recall (2.4.5), (2.4.7) and $u_\delta \rightharpoondown \overline{u}$ in $L^2(\Omega - D)^M$)

$$\int_\Omega \tau : \nabla\varphi dx + \lambda \int_{\Omega-D} (\overline{u} - f)\cdot\varphi dx = 0 \tag{2.4.12}$$

for any $\varphi \in W^{1,2}(\Omega)^M$ and by approximation, equation (2.4.12) extends to $\varphi \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$ (we refer to Lemma 2.2.4).
At the same time, it holds

$$\begin{aligned} R[\tau] :&= \inf_{v\in W^{1,1}(\Omega)^M\cap L^2(\Omega-D)^M} l(v,\tau) \\ &= \int_\Omega (-F^*(\tau))dx \\ &+ \inf_{v\in W^{1,1}(\Omega)^M\cap L^2(\Omega-D)^M} \left[\int_\Omega \tau : \nabla v dx + \frac{\lambda}{2}\int_{\Omega-D} |v-f|^2 dx\right] \\ &= \int_\Omega (-F^*(\tau))dx \\ &+ \inf_{v\in W^{1,1}(\Omega)^M\cap L^2(\Omega-D)^M} \left[-\lambda\int_{\Omega-D} (\overline{u}-f)\cdot v dx + \frac{\lambda}{2}\int_{\Omega-D} |v-f|^2 dx\right] \\ &= \int_\Omega (-F^*(\tau))dx \\ &+ \inf_{v\in W^{1,1}(\Omega)^M\cap L^2(\Omega-D)^M} \left[\frac{\lambda}{2}\int_{\Omega-D} |\overline{u}-v|^2 dx + \frac{\lambda}{2}\int_{\Omega-D} |f|^2 dx - \frac{\lambda}{2}\int_{\Omega-D} |\overline{u}|^2 dx\right], \end{aligned}$$

where we have used (2.4.12) with the admissible choice $\varphi = v$ as well as the quadratic structure of the data fitting term. As a consequence we obviously get

$$R[\tau] \geq \int_\Omega (-F^*(\tau))dx + \frac{\lambda}{2}\int_{\Omega-D} |f|^2 dx - \frac{\lambda}{2}\int_{\Omega-D} |\overline{u}|^2 dx$$

which implies (recall (2.4.11))

$$\sup_{L^\infty(\Omega)^{nM}} R \leq \inf_{W^{1,1}(\Omega)^M\cap L^2(\Omega-D)^M} I \leq R[\tau].$$

Hence τ is R-maximizing and the inf-sup relation is valid which proves assertion (a) of Theorem 2.1.6. Additionally by means of the above chain of inequalities we have shown that

$$\delta \int_\Omega |\nabla u_\delta|^2 dx \to 0 \tag{2.4.13}$$

$$(u_\delta) \text{ is an } I - \text{minimizing sequence} \tag{2.4.14}$$

at least for a subsequence $\delta_m \to 0$. Thanks to Theorem 2.1.1, (d) and (2.4.14) it further follows that $\overline{u}$ is K-minimizing in $BV(\Omega)^M \cap L^2(\Omega - D)^M$.

For assertion (b) of Theorem 2.1.6 we may proceed exactly as in [25], proof of Theorem 1.4: we fix $v \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$ and consider the functional $H_v : L^\infty(\Omega)^{nM} \to \mathbb{R}$

$$H_v[\varkappa] := \int_\Omega \left[\varkappa : \nabla v + \frac{\lambda}{2}\mathbb{1}_{\Omega - D}|v - f|^2\right] dx$$

that gives the representation

$$R[\varkappa] = \int_\Omega (-F^*(\varkappa))dx + \inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M} H_v[\varkappa]$$

where obviously, $v \mapsto \inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M} H_v[\varkappa]$ is concave.
Now, we suppose that F^* is strictly convex and assume that τ_1, τ_2 are R-maximizing, but $\tau_1 \neq \tau_2$ on a set $S \subset \Omega$ with $\mathcal{L}^n(S) > 0$. Except for a set of points with zero measure we further must have $F^*(\tau_i(x)) < \infty$, $i \in \{1, 2\}$, since otherwise $R[\tau_i] = -\infty$.
Next, we set $\varkappa := \frac{1}{2}(\tau_1 + \tau_2)$ and as a consequence we get on the set S

$$F^*(\varkappa) < \frac{1}{2}F^*(\tau_1) + \frac{1}{2}F^*(\tau_2)$$

where on $\Omega - S$ we just have "$\leq$" by quoting the convexity of F^* on this set. Thus

$$\int_\Omega (-F^*(\varkappa))dx > \frac{1}{2}\int_\Omega (-F^*(\tau_1))dx + \frac{1}{2}\int_\Omega (-F^*(\tau_2))dx$$

and we may conclude

$$R[\varkappa] > \frac{1}{2}R[\tau_1] + \frac{1}{2}R[\tau_2] = \sup_{L^\infty(\Omega)^{nM}} R.$$

which contradicts the maximizing property. Hence, by requiring strict convexity of F^* on the set $\{P \in \mathbb{R}^{nM}, F^*(P) < \infty\}$, we get uniqueness of the dual solution and as a consequence of uniqueness the convergences (2.4.7) and (2.4.13) hold for any sequence $\delta \to 0$.

For proving Theorem 2.1.6 (c) we proceed similar to the proof of Theorem 1.7 in [22]: let (u_m) denote an I-minimizing sequence from the space $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$. Using the previous notation, we deduce from (2.4.11) and (2.4.12) (with admissible choice $\varphi = u_m$)

$$\inf_{W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M} I \le \int_\Omega [\tau : \nabla u_m - F^*(\tau)]dx - \frac{\lambda}{2}\int_{\Omega-D} |\overline{u}|^2 dx + \frac{\lambda}{2}\int_{\Omega-D} |f|^2 dx + \lambda \int_{\Omega-D} (\overline{u} - f)\cdot u_m \, dx$$

where $\overline{u}, \tau$ have the same meaning as before. Hence,

$$\begin{aligned}\inf_{W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M} I \le & \int_\Omega F(\nabla u_m)dx + \frac{\lambda}{2}\int_{\Omega-D} |u_m - f|^2 dx \\ & - \frac{\lambda}{2}\int_{\Omega-D} |u_m - \overline{u}|^2 dx \\ = & I[u_m] - \frac{\lambda}{2}\int_{\Omega-D} |u_m - \overline{u}|^2 dx,\end{aligned}$$

and we obtain our claim by recalling that $\overline{u}$ is K-minimizing and that by Theorem 2.1.1 (b) we have uniqueness of K-minimizers on $\Omega - D$. Altogether the proof of Theorem 2.1.6 is complete. □

2.5 Uniqueness of the dual solution and the duality formula. Proof of Theorem 2.1.7

Let the assumptions of Theorem 2.1.7 hold and consider a K-minimizing function u from the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$, whose existence is guaranteed by Theorem 2.1.1. Remembering the decomposition $\nabla u = \nabla^a u \llcorner \mathcal{L}^n + \nabla^s u$ (see, e.g., [53], Theorem 3, p.42) with density $\nabla^a u$ being independent of the particular minimizer (recall Theorem 2.1.1 (b)) we claim

Lemma 2.5.1
The tensor $\rho := DF(\nabla^a u)$ is a maximizer of the dual problem.

Proof of Lemma 2.5.1. On account of (2.1.5), ρ is in admissible choice in the dual functional R since $\rho \in L^\infty(\Omega)^{nM}$ where we recall that R is given by

$$R[\rho] = \inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M} l(v, \rho). \tag{2.5.1}$$

For $v \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$ it holds

$$
\begin{aligned}
l(v,\rho) &= \int_\Omega [DF(\nabla^a u) : \nabla v - F^*(DF(\nabla^a u))]dx \\
&\quad + \frac{\lambda}{2} \int_{\Omega - D} |v - f|^2 dx \\
&= \int_\Omega F(\nabla^a u)dx + \int_\Omega (\nabla v - \nabla^a u) : DF(\nabla^a u)dx \\
&\quad + \frac{\lambda}{2} \int_{\Omega - D} |v - f|^2 dx,
\end{aligned}
\tag{2.5.2}
$$

where we have made use of the formula (recall the duality formula (2.1.16))

$$
F(P) + F^*(DF(P)) = P : DF(P), \quad P \in \mathbb{R}^{nM}.
$$

Since u is K-minimizing, we get (note that $\nabla^s(u + tv) = \nabla^s u$ holds for the singular parts of the measures)

$$
0 = \frac{d}{dt_{|0}} K[u + tv] = \int_\Omega DF(\nabla^a u) : \nabla v dx + \lambda \int_{\Omega - D} v \cdot (u - f)dx. \tag{2.5.3}
$$

Using the K-minimality of u once again (notice that we make use of $\nabla(u+tu) = (1 + t)\nabla u$)

$$
\begin{aligned}
0 = \frac{d}{dt_{|0}} K[u + tu] &= \int_\Omega DF(\nabla^a u) : \nabla^a u dx + \int_\Omega F^\infty\left(\frac{\nabla^s u}{|\nabla^s u|}\right) d|\nabla^s u| \\
&\quad + \lambda \int_{\Omega - D} u \cdot (u - f)dx.
\end{aligned}
\tag{2.5.4}
$$

Inserting (2.5.3) and (2.5.4) into (2.5.2) we find

$$
\begin{aligned}
l(v,\rho) = &\int_\Omega F(\nabla^a u)dx + \int_\Omega F^\infty\left(\frac{\nabla^s u}{|\nabla^s u|}\right) d|\nabla^s u| \\
&- \lambda \int_{\Omega - D} v \cdot (u - f)dx + \lambda \int_{\Omega - D} u \cdot (u - f)dx \\
&+ \frac{\lambda}{2} \int_{\Omega - D} |v - f|^2 dx.
\end{aligned}
\tag{2.5.5}
$$

Observing that a.e. on $\Omega - D$ it holds

$$
-\lambda v \cdot (u - f) + \lambda u \cdot (u - f) + \frac{\lambda}{2}|v - f|^2 = \frac{\lambda}{2}|u - f|^2 + \frac{\lambda}{2}|u - v|^2,
$$

we deduce from (2.5.5)

$$
l(v,\rho) \geq K[u],
$$

and (2.5.1) implies $R[\rho] \geq K[u]$. But then the claim of Lemma 2.5.1 is a consequence of the inf-sup relation that has been stated in Theorem 1.2 (a). □

By definition the dual solution ρ from Lemma 2.5.1 attains its values in the open set $\mathrm{Im}(DF)$. If the dual problem would admit a second solution $\widetilde{\rho} \neq \rho$, then exactly the same arguments as used during the proof of Theorem 2.15 in [17] would lead to a contradiction. In fact, as demonstrated in this reference, the assumption $\rho \neq \widetilde{\rho}$ (on a set of positive measure) yields the strict inequality

$$\int_\Omega (-F^*)\left(\frac{\rho+\widetilde{\rho}}{2}\right)dx > \frac{1}{2}\int_\Omega (-F^*)(\rho)dx + \frac{1}{2}\int_\Omega (-F^*)(\widetilde{\rho})dx.$$

At the same time we observe that

$$L^\infty(\Omega)^{nM} \ni \kappa \mapsto \inf_{v\in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M} \int_\Omega [\kappa : \nabla v - \mathbb{1}_{\Omega-D}|v-f|^2]dx$$

is a concave function, hence

$$R\left[\frac{\rho+\widetilde{\rho}}{2}\right] > \frac{1}{2}R[\rho] + \frac{1}{2}R[\widetilde{\rho}],$$

which is not possible.
Thus, $DF(\nabla^a u)$ is the only dual solution and the validity of the duality formula is a direct conclusion. The proof of Theorem 2.1.7 is complete. □

Chapter 3

A modified TV-image inpainting method: regularity results

3.1 The basic setup and statement of the main results

In this chapter we discuss the regularity behavior of generalized minimizers of the functional

$$\begin{aligned} I[w] := \int_{\Omega} F(\nabla w)dx + \frac{\lambda}{2} \int_{\Omega - D} |w - f|^2 dx, \\ w \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M, \end{aligned} \tag{3.1.1}$$

i.e., we talk about minimizers of the relaxed variant

$$K[w] := \int_{\Omega} F(\nabla^a w)dx + \int_{\Omega} F^{\infty}\left(\frac{\nabla^s w}{|\nabla^s w|}\right) d|\nabla^s w| + \frac{\lambda}{2} \int_{\Omega - D} |w - f|^2 dx \tag{3.1.2}$$

of the functional I from above formulated on the adequate space $BV(\Omega)^M \cap L^2(\Omega - D)^M$. The existence of such minimizers is guaranteed by Theorem 2.1.1. Before starting our discussion we want to mention that most of the material which will be presented here originates from the forthcoming paper [94]. Let Ω and D as in the second chapter, i.e., in particular, we still assume

$$0 \le \mathcal{L}^n(D) < \mathcal{L}^n(\Omega), \tag{3.1.3}$$

and further require

$$f \in L^{\infty}(\Omega - D)^M, \quad M \ge 1, \tag{3.1.4}$$

throughout the entire chapter, where -as usual- f represents the vector-valued partial observation on $\Omega - D$.
Apriori, generalized I-minimizers can admit points of discontinuity but in most cases it is however possible to show that generalized minimizers are continuously differentiable up to a small, relatively closed, subset with Lebesgue measure zero. This property is called almost everywhere regularity and one challenging problem which arises in that context is to give estimates for the size of the singular set of minimizers.
Considering the setup from the second chapter, the assumptions (2.1.4)–(2.1.6) on F are probably too weak in order to establish any regularity results for generalized minimizers in Ω. Thus we require the following (much stronger) hypotheses on our integrand $F : \mathbb{R}^{nM} \to [0, \infty)$.

$$F \in C^2(\mathbb{R}^{nM}) \quad \text{and (w.l.o.g.) } F(0) = 0, DF(0) = 0, \tag{3.1.5}$$

$$|DF(P)| \le \nu_1, \tag{3.1.6}$$

$$\nu_2 \frac{1}{(1+|P|)^\mu}|Q|^2 \le D^2F(P)(Q,Q) \le \nu_3 \frac{1}{1+|P|}|Q|^2. \tag{3.1.7}$$

with some positive constants ν_1, ν_2, ν_3, for all $P, Q \in \mathbb{R}^{nM}$ and for a fixed exponent $\mu > 1$. Note that an integrand $F \in C^2(\mathbb{R}^{nM})$ satisfying (3.1.6) and (3.1.7) with the prescribed ellipticity parameter $\mu > 1$ is called μ-elliptic. Moreover we suppose that F satisfies the structure condition

$$F(P) = \Phi(|P|), \quad \Phi \in C^2([0,\infty),[0,\infty)). \tag{3.1.8}$$

In order to get (3.1.6) and (3.1.7) we then require

$$\nu_2 \frac{1}{(1+t)^\mu} \le \min\left\{\frac{\Phi'(t)}{t}, \Phi''(t)\right\}, \quad \max\left\{\frac{\Phi'(t)}{t}, \Phi''(t)\right\} \le \nu_3 \frac{1}{1+t} \tag{3.1.9}$$

for all $t \ge 0$ with suitable positive constants ν_2 and ν_3, where we point out that the hypothesis (3.1.9) just corresponds to (1.4^*) and (1.4^*_μ) from [25]. Note that w.l.o.g. we can assume $\Phi(0) = \Phi'(0) = 0$ as well.

It is worth mentioning that despite the above rather strong hypotheses on F, the study of smoothness properties of generalized solutions remains a delicate problem since the linear growth of F admits only weak and anisotropic ellipticity conditions (see (3.1.7) and recall $\mu > 1$) and we will see in our forthcoming discussions that full regularity crucially depends on the size of the value of the ellipticity parameter $\mu > 1$.

Remark 3.1.1
Let us give some comment on the condition (3.1.5)*: we trivially may assume* $F(0) = 0$ *for our density* F*. W.l.o.g. we moreover may suppose that we have* $DF(0) = 0$ *for* F*: let us fix a point* $z_0 \in \mathbb{R}^{nM}$ *such that* $DF(z_0) = 0$ *as well as* $F(z_0) = 0$*. Letting* $w \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$*, the idea is to seek minimizers of the modified functional*

$$\widetilde{I}[w] := \int_\Omega \widetilde{F}(\nabla w)dx + \frac{\lambda}{2}\int_{\Omega - D} |w - \widetilde{f}|^2 dx$$

in the space $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$. Here, we have set $\widetilde{F}(P) := F(P + z_0)$ and $\widetilde{f}(x) := f(x) - z_0 \cdot x$, where we directly see $\widetilde{f} \in L^\infty(\Omega - D)^M$ (recall that Ω is a bounded domain). Further, it holds $\widetilde{F}(0) = 0$, $D\widetilde{F}(0) = 0$ and $\widetilde{F}$ satisfies (3.1.6) *together with* (3.1.7) *from above. Finally, a generalized $\widetilde{I}$-minimizer $\widetilde{u}$ can be modified to a generalized I-minimizer u (with I from* (3.1.1)*) by setting $\widetilde{u} = u - z_0 \cdot x$.*
Another justification that w.l.o.g. we may assume $DF(0) = 0$ works by arguing with Euler's equation, where we then insert test functions that are compactly supported in Ω. To become more precise, we then define the auxiliary integrand $\overline{F}(P) := F(P) - DF(0) : P$ and consider $F(P) = \overline{F}(P) + DF(0) : P$. Passing to Euler's equation and inserting test functions φ having compact support in Ω it follows that the additional term

$$\int_\Omega DF(0) : \nabla\varphi dx$$

vanishes. Thus, we merely argue with the density $\overline{F}(P)$ for which we clearly have $D\overline{F}(0) = 0$ at hand (and the conditions (3.1.5)–(3.1.7) *as well).*

At this point we want to discuss an example of a non-standard class of integrands that satisfy the conditions (3.1.5)–(3.1.7) with a prescribed parameter $\mu > 1$.

Example 3.1.2 *(i) In the context of pure denoising of images and image inpainting, respectively, the following example can serve as a model w.r.t. approximating the TV-density $|P|$, $P \in \mathbb{R}^{nM}$: for a given number $\mu > 1$ we let*

$$\varphi_\mu(r) := \int_0^r \int_0^s (1+t)^{-\mu} dt ds, \quad r \in \mathbb{R}_0^+. \tag{3.1.10}$$

As already mentioned in the introduction, it is reasonable to consider densities depending on the modulus, i.e., we let

$$\Phi_\mu(Z) := \varphi_\mu(|Z|), \quad Z \in \mathbb{R}^{nM}. \tag{3.1.11}$$

Clearly, $\Phi_\mu : \mathbb{R}^{nM} \to [0, \infty)$ is of class C^2 satisfying (3.1.5)–(3.1.7) *with the prescribed ellipticity parameter $\mu > 1$.*
Moreover, we have an explicit representation of $\varphi_\mu(r)$,

$$\varphi_\mu(r) = \frac{r}{\mu - 1} + \frac{1}{\mu - 1}\frac{1}{\mu - 2}(r+1)^{-\mu+2} - \frac{1}{\mu - 1}\frac{1}{\mu - 2}, \ \mu \neq 2, \tag{3.1.12}$$

whereas for $\mu = 2$ it holds

$$\varphi_2(r) = r - \log(1 + r).$$

Observing next that

$$(\mu - 1)\Phi_\mu(Z) \to |Z| \quad \text{as } \mu \to \infty \tag{3.1.13}$$

for all $Z \in \mathbb{R}^{nM}$, it becomes evident that the density $\Phi_\mu(Z)$ serves as a very good candidate for the approximation of $|Z|$ by a regular class of integrands with linear growth (with some suitable choice of the ellipticity parameter $\mu > 1$).

(ii) As outlined in, e.g. [24], Remark 1.4 (iv), for a given number $\mu > 1$, a slight modification of the integrands Φ_μ from (3.1.11) *is given by*

$$\widetilde{\Phi}_\mu(Z) := \int_0^{|Z|}\int_0^s (1+t^2)^{-\frac{\mu}{2}}\,dtds, \quad Z \in \mathbb{R}^{nM}.$$

Here, we refer to [17], Example 3.9, pp. 48, for a short sketch of the proof that $\widetilde{\Phi}_\mu(Z)$ satisfies the conditions (3.1.5)–(3.1.7) *for a prescribed parameter $\mu > 1$.*
Note that for the special choice $\mu = 3$ we obtain the minimal surface integrand $F(Z) := \widetilde{\Phi}_3(Z) = \sqrt{1+|Z|^2}$ which probably is the most prominent example that fulfills the conditions of μ-ellipticity (3.1.6) *and* (3.1.7) *from above.*

For the class of μ-elliptic integrands we collect some useful properties that have already been established in [17], Remark 4.2, p.97/98.

Lemma 3.1.3
Suppose that F satisfies (3.1.5)–(3.1.7) *for some number $\mu > 1$. Then F is strictly convex on $\mathbb{R}^{nM}$ and it holds:*

(i) there are real constants $\nu_1 > 0$, $\nu_2 \in \mathbb{R}$ such that for all $Z \in \mathbb{R}^{nM}$ we have

$$DF(Z) : Z \geq \nu_1|Z| - \nu_2,$$

i.e., $DF(Z) : Z$ is at least of linear growth;

(ii) F is of linear growth in the sense that for real numbers ν_3, $\nu_4 > 0$, ν_5, $\nu_6 \in \mathbb{R}$ and for all $Z \in \mathbb{R}^{nM}$ it holds

$$\nu_3|Z| - \nu_5 \leq F(Z) \leq \nu_4|Z| + \nu_6;$$

(iii) the integrand satisfies a balancing condition: there exists a real constant $\nu_7 > 0$ such that

$$|D^2F(Z)||Z|^2 \leq \nu_7(1 + F(Z))$$

for all $Z \in \mathbb{R}^{nM}$.

After the above preparations we first are concerned with showing almost everywhere $C^{1,\alpha}$-regularity of generalized minimizers in the usual sense, where we first prove the validity of a “maximum principle“ for each generalized I-minimizer u from the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$ which implies global boundedness of u.

Theorem 3.1.4
Suppose that we have (3.1.3) *and* (3.1.4). *Further we assume that* F *satisfies the structure condition* (3.1.8) *with* $\Phi \in C^2([0,\infty),[0,\infty))$ *satisfying* (3.1.9) *with the prescribed ellipticity parameter* $\mu > 1$. *It then holds*

$$\sup_{\Omega} |u| \leq \sup_{\Omega - D} |f|$$

for each generalized I*-minimizer* $u \in BV(\Omega)^M \cap L^2(\Omega - D)^M$.

For the sake of completeness let us look at the scalar case $M = 1$: assuming w.l.o.g. that the observed image $f : \Omega - D \to \mathbb{R}$ takes almost all of its values in the closed interval $[0,1]$ we then can derive the following slightly different maximum principle for each generalized I-minimizer.

Theorem 3.1.5
Let $M = 1$, *suppose that we have* (3.1.3) *and require that* $0 \leq f \leq 1$ *a.e. on* $\Omega - D$. *Further we assume that* F *satisfies the structure condition* (3.1.8) *with* $\Phi \in C^2([0,\infty),[0,\infty))$ *fulfilling* (3.1.9) *with the prescribed ellipticity parameter* $\mu > 1$. *For each generalized* I*-minimizer* $u \in BV(\Omega)^M \cap L^2(\Omega - D)^M$, *we then have that the inequality*

$$0 \leq u(x) \leq 1$$

is satisfied for a.a. $x \in \Omega$.

Remark 3.1.6
The reader should note that the maximum principles stated in Theorem 3.1.4 and Theorem 3.1.5, respectively, remain valid under much weaker assumptions on our function Φ. *In fact we do not need differentiability of* Φ *in both theorems. Considering the vectorial case* $M > 1$ *we have to require at least that* Φ *is a strictly increasing and a convex function being of linear growth (see [21], Theorem 1). Concerning the scalar case we can give up the structure condition* (3.1.8) *imposed on* F. *To become more precise, we can replace* (3.1.8) *through the weaker condition* $F(P) = F(-P)$ *for all* $P \in \mathbb{R}^n$. *Moreover, in the scalar case,* F *needs to be strictly convex and of linear growth (compare [21], Theorem 2).*

Remark 3.1.7
Considering the scalar case $M = 1$ *we note that* $f(x)$ *can be seen as a measure for the intensity of the grey level of the observed image for points* $x \in \Omega - D$, *where we recall that usually, low grey levels are dark and high grey levels are bright. As a consequence, Theorem 3.1.5 can be interpreted in such a way that each generalized* I*-minimizer represents a measure for the intensity of the grey level since they automatically satisfy the inequality* $0 \leq u(x) \leq 1$ *for a.a.* $x \in \Omega$. *Thus, from the point of view of applications in image processing, Theorem 3.1.5 seems to be an interesting result.*

Remark 3.1.8
At least in the scalar case $M = 1$ together with $n = 2$, a proof of Theorem 3.1.5 has been sketched in [25] (see the proof of Theorem 1.2 (i) therein).

Now we state our result about partial $C^{1,\alpha}$-regularity regularity of generalized I-minimizers.

Theorem 3.1.9
Suppose that we have (3.1.3) *as well as* (3.1.4). *Further we assume that F satisfies* (3.1.8) *with a function Φ of class C^2 fulfilling* (3.1.9) *with the prescribed ellipticity parameter $\mu > 1$. Then for each K-minimizer u with K from* (3.1.2), *there exists an open subset Ω_0^u of Ω such that $u \in C^{1,\frac{1}{2}}(\Omega_0^u)^M$ together with $\mathcal{L}^n(\Omega - \Omega_0^u) = 0$.*

Remark 3.1.10

- *Note that Theorem 3.1.9 has already been established in the joint article with J. Müller [81] (see Theorem 1.4 and choose $\zeta = 2$ in this reference).*
- *Considering the case $\mathcal{L}^n(D) > 0$ and assuming that Int$(D) \neq \emptyset$ (Int(D) denoting the set of interior points of D) as well as that F satisfies the hypotheses from Theorem 3.1.9, then for each K-minimizer u there exists an open subset G^u of $G :=$ Int(D) such that we have $u \in C^{1,\alpha}(G^u)$ for any $\alpha \in (0,1)$ and $\mathcal{L}^n(G - G^u) = 0$. This statement is an immediate consequence of Theorem 1.1 in [9] and we remark that we may drop the structure condition* (3.1.8) *on F in this situation.*
- *Since we quote Corollary 3.3 in [86] in order to prove Theorem 3.1.9 we get the bound $\alpha = \frac{1}{2}$ for the Hölder exponent α in Theorem 3.1.9.*

Remark 3.1.11
We note that Theorem 3.1.9 remains valid under weaker assumptions on F. For more details we refer to [81], Theorem 1.4.

Remark 3.1.12
It is worth mentioning that the size of the ellipticity parameter $\mu > 1$ does not affect the statement of Theorem 3.1.9. Moreover, it is not necessary to impose any topological assumptions on D.

Remark 3.1.13
Let us recall the following modification of the total variation image inpainting method which has been touched in Remark 2.1.3 from the second chapter: for a given finite number $\zeta \geq 1$ and for functions $w \in W^{1,1}(\Omega)^M \cap L^\zeta(\Omega - D)^M$ we consider the functional

$$I_\zeta[w] := \int_\Omega F(\nabla w)dx + \frac{\lambda}{\zeta} \int_{\Omega - D} |w - f|^\zeta dx,$$

where we suppose the hypotheses from Theorem 3.1.9. Choosing $\zeta = 2$ *we are in the setting of Theorem 3.1.9. The corresponding relaxed version of the functional* I_ζ *formulated on the space* $BV(\Omega)^M$ *is given by*

$$\begin{aligned} K_\zeta[w] := &\int_\Omega F(\nabla^a w)dx + \int_\Omega F^\infty\left(\frac{\nabla^s w}{|\nabla^s w|}\right)d|\nabla^s w| \\ &+ \frac{\lambda}{\zeta}\int_{\Omega - D} |w - f|^\zeta dx, \end{aligned} \tag{3.1.14}$$

and (3.1.14) *is well-defined for functions* $w \in BV(\Omega)^M \cap L^\zeta(\Omega - D)^M$*. As proven in the joint paper with J. Müller [81] (see Theorem 1.4 in this reference), Theorem 3.1.9 extends to each* K_ζ*-minimizer, where we may even include the case* $\zeta = 1$ *(we refer to Remark 1.8 in [81]).*

Remark 3.1.14
Fixing a number $\zeta > 1$ *let us consider the situation as described in Remark 3.1.13: quoting Theorem 1.3 in [81] we know that the dual problem associated to "*$I_\zeta \to min$*" admits a unique solution* $\sigma \in L^\infty(\Omega)^{nM}$ *satisfying the duality formula* $\sigma(x) = DF(\nabla^a u(x))$ *for* $\mathcal{L}^n$*-a.e.* $x \in \Omega$ *where* u *denotes a* K_ζ*-minimizer. Taking into account the result* $u \in C^{1,\frac{1}{2}}(\Omega_0^u)^M$ *together with* $\mathcal{L}^n(\Omega - \Omega_0^u) = 0$ *we directly obtain* $\sigma \in C^{0,\frac{1}{2}}(\Omega_0^u)^{nM}$*.*

Let us now discuss full interior $C^{1,\alpha}$-regularity of generalized I-minimizers. In this context we will intensively use the notion of μ-elliptic integrands with a prescribed ellipticity parameter $\mu > 1$ and as already mentioned above, the size of the parameter $\mu > 1$ will play a fundamental role in our investigations.

This already concerns the integrability properties of the distributional derivatives of generalized I-minimizers. To become more precise, under rather strong assumptions on our density F, we show unique solvability of the problem $I \to \min$ with I from (3.1.1) in the natural Sobolev space $W^{1,1}(\Omega)^M$ without passing to a suitable relaxed variant of I formulated on the space $BV(\Omega)^M$ as seen in the second chapter. As a byproduct we further prove a "maximum principle" for the unique I-minimizer. Finally we remark that we crucially need the requirement $\mu \in (1, 2)$.

Theorem 3.1.15
Suppose that we have (3.1.3) *and* (3.1.4) *for the data. Further we assume that* F *satisfies* (3.1.5)–(3.1.7) *for some* $\mu \in (1, 2)$*. In the case* $M > 1$ *we additionally assume* (3.1.8) *with* $\Phi \in C^2([0,\infty), [0,\infty))$ *fulfilling* (3.1.9) *for some* $\mu \in (1, 2)$*. Then the problem* $I \to min$ *in* $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$ *with* I *from* (3.1.1) *admits a unique solution* u *and this solution satisfies* $\sup_\Omega |u| \le \sup_{\Omega - D} |f|$*.*

Looking at the proof of Theorem 3.1.15 we can state the following regularity result about the dual solution from Theorem 2.1.6 and Theorem 2.1.7, respectively.

Corollary 3.1.16
Denote by $\sigma \in L^\infty(\Omega)^{nM}$ the unique dual solution from Theorem 2.1.6 and Theorem 2.1.7, respectively. Under the assumptions of Theorem 3.1.15, in particular assuming $\mu \in (1,2)$, we have $\sigma \in W^{1,2}_{loc}(\Omega)^{nM}$.

Remark 3.1.17

- *The structure condition* (3.1.8) *for F is needed for establishing a maximum principle for the unique I-minimizer u. Considering the scalar case we can drop the structure condition* (3.1.8) *and consult the remarks stated after Theorem 2 in [21].*
- *Considering the scalar case together with $n = 2$, Theorem 3.1.15 has already been proven in [24] (see Theorem 1.3 therein).*

Remark 3.1.18
We conjecture that the solvability of the problem $I \to min$ in $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$ crucially depends on the size of the ellipticity parameter μ: if we require $\mu \in (1,2)$, we can adjust the arguments from [24], proof of Theorem 1.3, for overcoming the problem of non-reflexivity of the Sobolev space $W^{1,1}(\Omega)^M$. Here, the restriction $\mu \in (1,2)$ is considered for technical reasons. Unfortunately, this strategy fails for $\mu > 2$. Furthermore, as already outlined in [17], Theorem 4.39, p. 133, the choice $\mu > 3$ seems to be critical. Thus, for large values of μ, we then can consider suitable relaxed versions of the original problem, e.g., an appropriate BV-variant by using the notion of a convex function of a measure or passing to the associated dual variational problem (we refer the reader to the second chapter of this thesis).

At the final stage we proceed by discussing full interior $C^{1,\alpha}$-regularity of the unique I-minimizer u being of class $W^{1,1}(\Omega)^M \cap L^\infty(\Omega)^M$. Considering the scalar case $M = 1$ together with $n = 2$ and underlying the assumptions of Theorem 3.1.15 (in particular we assume that the density F is μ-elliptic with some prescribed parameter $\mu \in (1,2)$) it was first shown a strong partial $C^{1,\beta}$-regularity result for u on Ω for any $\beta < 1$ where the relatively closed set $\mathrm{Sing}_u(\Omega)$ of singular points of u in Ω is very small in a measuretheoretical sense (see [24], Theorem 1.4). To become more precise, it holds $\dim_{\mathcal{H}}(\mathrm{Sing}_u(\Omega)) = 0$, which, by definition, means that $\mathcal{H}^\varepsilon(\mathrm{Sing}_u(\Omega)) = 0$ for any $\varepsilon > 0$. For establishing this kind of almost everywhere regularity of u, the idea was to use the well-known technique of Frehse and Seregin (see [87]) for proving first the continuity of $DF(\nabla u) =: \sigma$ in the interior of Ω. In a second step, "almost everywhere inversion ", by applying $(DF)^{-1}$ to $\sigma(x)$ for points x from a suitable subset Ω then leads to the continuity of ∇u apart from a singular set with vanishing Hausdorff-dimension.
Afterwards, in the joint paper [27] with M. Bildhauer and M. Fuchs, a substantial improvement of the above partial regularity result for I-minimizers u has

been established in the sense that it is possible to rule out interior singularities, i.e., to prove $u \in C^{1,\alpha}(\Omega)$ for any $\alpha \in (0,1)$ (see [27], Theorem 2).
For proving full interior $C^{1,\alpha}$-regularity of u we used modified DeGiorgi-type arguments based on a variant of Caccioppoli's inequality.

Inspired by the result obtained in [27] we concentrate on the following natural questions that might arise:

- Is it possible to establish $C^{1,\alpha}$-interior differentiability for the unique I-minimizer u from Theorem 3.1.15 in the scalar case together with $n \geq 3$?
- Does an analogous result hold in the vectorial case and what kind of assumptions do we have to impose on the given data as well as on the density F?

It turns out that we can give a positive answer to the above questions under rather natural assumptions on the data and on F. However, if we are concerned with vector-valued problems, a structure condition on the density F being in the spirit of (3.1.11) is sufficient for proving Lipschitz regularity of minimizers but it seems to be too weak in order to derive everywhere C^1-regularity of generalized minimizers. Therefore we suppose that in addition to (3.1.15) there are constants K and $0 < \alpha < 1$ such that for all $Z, \widetilde{Z} \in \mathbb{R}^{nM}$

$$F(Z) = g(|Z|^2), \quad g \in C^2([0,\infty),[0,\infty)), \tag{3.1.15}$$

$$|D^2F(Z) - D^2F(\widetilde{Z})| \leq K|Z - \widetilde{Z}|^\alpha, \tag{3.1.16}$$

where the structure condition (3.1.15) is considered for technical reasons.
A non-standard example of class $C^2(\mathbb{R}^{nM})$ which satisfies (3.1.6) as well as (3.1.7) with the prescribed parameter $\mu > 1$ and fulfills the above conditions (3.1.15) as well as (3.1.16) is given by the density

$$\widetilde{\Phi}_\mu(Z) := \int\limits_0^{\sqrt{\varepsilon+|Z|^2}} \int\limits_0^s (1+r)^{-\mu} dr ds, \quad \varepsilon > 0. \tag{3.1.17}$$

Taking into account Lemma 3.1.3 (ii), $\widetilde{\Phi}_\mu(Z)$ is therefore of linear growth (note that we have $D\widetilde{\Phi}_\mu(0) = 0$).
Additionally, based on (3.1.13), we can state

$$\lim_{\mu\to\infty} (\mu - 1)\widetilde{\Phi}_\mu(Z) = \sqrt{\varepsilon + |Z|^2}, \quad \varepsilon > 0. \tag{3.1.18}$$

Note that the choice of $\varepsilon > 0$ gives some additional flexibility for approximating the TV-density.

At this point we formulate one of the main results of this thesis. Note that the validity of this result crucially relies on the smallness of the ellipticity parameter μ.

Theorem 3.1.19
Suppose that we have (3.1.3) *and* (3.1.4) *for the data. Further we assume that* F *satisfies* (3.1.5)–(3.1.7) *for some* $\mu \in (1, 2)$. *In addition we require that* (3.1.15) *and* (3.1.16) *are true in the vector-valued setting* $M > 1$. *Then it holds* $u \in C^{1,\alpha}(\Omega)^M$ *for any* $0 < \alpha < 1$ *where* u *is the solution from Theorem 3.1.15.*

Let us give some comments on Theorem 3.1.19.

Remark 3.1.20
Recalling Theorem 2.1.7 from the second chapter we know that the dual problem associated to "$I \to \min$" with I *from* (3.1.1) *admits a unique solution* $\sigma \in L^\infty(\Omega)^{nM}$ *satisfying the duality relation* $\sigma(x) = DF(\nabla^a u(x))$ *for* $\mathcal{L}^n$*-a.a.* $x \in \Omega$, *where* u *denotes any* K*-minimizer with* K *from* (3.1.2). *Assuming the assumptions of Theorem 3.1.19 from above, in particular we require* $\mu \in (1, 2)$, *we immediately get the relation* $\sigma = DF(\nabla u)$, *where* u *denotes the unique* I*-minimizer being of class* $C^{1,\alpha}(\Omega)^M$ *for any* $\alpha \in (0, 1)$. *Thus,* σ *is of class* $C^{0,\alpha}(\Omega)^{nM}$ *for any* $\alpha \in (0, 1)$.

Remark 3.1.21

- *One cannot overemphasize the importance of the smallness condition* $\mu \in (1, 2)$ *imposed on the parameter* μ. *Furthermore we remark that the bound on the ellipticity parameter* μ *in Theorem 3.1.19 is not depending on the dimension* n. *In addition, we conjecture that the limit* $\mu = 2$ *serves as an optimal choice in Theorem 3.1.19 (as well as Theorem 3.1.15) in the presence of the inpainting quantity* $\int\limits_{\Omega - D} |w - f|^2 dx$.
- *It is easy to check that the results stated in this section also hold if* $D = \emptyset$ *("pure denoising of* f*"). Applying minor adjustments, the results of Theorem 3.1.15 as well as Theorem 3.1.19 remain valid in the case that an additional boundary condition as* $u = u_0$ *on* $\partial\Omega$ *is involved as well. Here* u_0 *denotes a sufficiently regular function satisfying in addition* $u_0 \in L^\infty(\Omega)^M$.
- *Note that no topological assumptions are imposed on* D *in Theorem 3.1.15 and Theorem 3.1.19.*

Remark 3.1.22
Considering the vectorial situation we have to rely on the conditions (3.1.15) *and* (3.1.16) *for* F. *However, we merely need structure condition* (3.1.15) *for establishing local Lipschitz continuity of the unique* I*-minimizer* u *and therewith local apriori gradient bounds of* u. *In order to improve the local Lipschitz continuity of* u *to* $C^{1,\alpha}$*-interior differentiability of* u *we additionally make use of the Hölder condition* (3.1.16) *on* D^2F. *We refer the reader to Section 3.5.2 for more details.*

Remark 3.1.23
Let us take a short look at the essential difference in the arguments between the

cases $n = 2$ and $n \geq 3$: as usual, our strategy is to investigate a suitable approximation of our original problem by a sequence of more regular problems admitting smooth solutions with useful convergence properties towards our designated minimizer u (we refer to the regularization stated in Section 2.4). Denoting the unique solution of the regularized problem by u_δ being of class $W^{1,2}(\Omega)$ (for details we refer to Section 2.4) and following the lines of the proof of Theorem 3.1.15 in this thesis it is proven that the function $\varphi_\delta := (1 + |\nabla u_\delta|)^{1-\frac{\mu}{2}}$ ($\mu \in (1, 2)$) is of class $W^{1,2}_{loc}(\Omega)^M$ uniformly in δ. Setting $n = 2$, Sobolev's embedding theorem then immediately implies $\nabla u_\delta \in L^p_{loc}(\Omega)^{2M}$ uniformly in δ for any finite exponent p and this result serves as a main ingredient for carrying out a De Giorgi-type iteration that provides local uniform apriori gradient bounds of u_δ. Considering $n \geq 3$, Sobolev's embedding theorem merely yields local uniform L^p-estimates of ∇u_δ for all $1 \leq p \leq \frac{2n}{n-2}$ and it turns out that this initial integrability of ∇u_δ does not suffice for performing De Giorgi-type arguments (we refer to Remark 3.5.9). Thus, in the final analysis, our major effort is to establish higher local uniform (in δ) integrability of ∇u_δ. Having this result at hand we more or less can adopt the procedure from [27], Theorem 2, in order to prove local uniform (in δ) apriori gradient bounds of u_δ and therewith local Lipschitz continuity of the unique I-minimizer u by applying the theorem of Arzelà-Ascoli.

Remark 3.1.24
Applying minor adjustments it is worth remarking that Theorem 3.1.15 and Theorem 3.1.19 extend to generalized K_ζ-minimizers (with K_ζ from (3.1.14)) with $1 < \zeta < \infty$. We emphasize that we crucially need the restriction $\mu \in (1, 2)$ on the ellipticity parameter μ.

The rest of this chapter is organized as follows: in Section 3.2 we will discuss a maximum principle for generalized I-minimizers whereas in Section 3.3 we prove partial regularity of generalized I-minimizers. Section 3.4 is devoted to the proof of Theorem 3.1.15 while in Section 3.5 we show Theorem 3.1.19. Section 3.5 itself is divided into two subsections: the first part and therewith Section 3.5.1 is concerned with the proof of Theorem 3.1.19 in the scalar case $M = 1$ while the goal of the second part and therewith of Section 3.5.2 is to show Theorem 3.1.19 in the vectorial setting $M > 1$.

3.2 A maximum principle for generalized minimizers. Proofs of Theorem 3.1.4 and Theorem 3.1.5

Let us first consider the vectorial case $M > 1$ and assume the validity of the hypotheses of Theorem 3.1.4. For proving Theorem 3.1.4 we basically follow [21], proof of Theorem 1. However, we cannot directly quote Theorem 1 in this reference since this result was given in another setting (for instance, some boundary data u_0 were included there).

We denote by $u \in BV(\Omega)^M \cap L^2(\Omega - D)^M$ an arbitrary K-minimizer with K from (3.1.2). By minimality we have

$$K[u] \leq K[w] \tag{3.2.1}$$

for all $w \in BV(\Omega)^M \cap L^2(\Omega - D)^M$. Setting $L := \sup\limits_{\Omega - D} |f|$ we now consider the projection

$$H:\ \mathbb{R}^M \to \mathbb{R}^M$$
$$y \mapsto \begin{cases} y, & |y| \leq L \\ L\frac{y}{|y|}, & |y| > L \end{cases}$$

and define $v := H \circ u$. In particular, H is Lipschitz continuous with Lipschitz constant $\mathrm{Lip}(H) = 1$.
Quoting Lemma 2.2.1 from Section 2.2.1 it holds $v \in BV(\Omega)^M$ and we get the important inequality

$$|\nabla v| \leq \mathrm{Lip}(H)|\nabla u| = |\nabla u|, \tag{3.2.2}$$

i.e., $|\nabla v|(E) \leq |\nabla u|(E)$ for any Borel set $E \subset \Omega$.
Further,

$$|v - f|^2 \leq |u - f|^2 \quad \text{a.e. on } \Omega - D, \tag{3.2.3}$$

which is immediate on $\{x \in \Omega - D, |u(x)| \leq L\}$. On $\{x \in \Omega - D, |u(x)| > L\}$, (3.2.3) is a consequence of the inequality $(|u| - L)^2 \geq 0$.
As the next step we define the functional $\widetilde{K}$ through (note that $\bar{c} := \lim\limits_{t\to\infty} \frac{\Phi(t)}{t}$ exists in $(0, \infty)$ since Φ is of linear growth)

$$\widetilde{K}[w] := \int_\Omega \Phi(|\nabla^a w|)dx + \bar{c}|\nabla^s w|(\Omega)$$

being well-defined for functions $w \in BV(\Omega)^M$.
Using the theorem of Besicovitch (see, e.g., [7], Theorem 2.22, p.54) it follows by virtue of (3.2.2) for $\mathcal{L}^n$-a.a. $x \in \Omega$

$$|\nabla^a v(x)| = \lim_{\rho\downarrow 0} \frac{|\nabla v|(B_\rho(x))}{\mathcal{L}^n(B_\rho(x))} \leq \lim_{\rho\downarrow 0} \frac{|\nabla u|(B_\rho(x))}{\mathcal{L}^n(B_\rho(x))} = |\nabla^a u(x)|.$$

Since Φ is increasing we additionally obtain

$$\int_\Omega \Phi(|\nabla^a v|)dx \leq \int_\Omega \Phi(|\nabla^a u|)dx. \tag{3.2.4}$$

In accordance with [7], Proposition 3.92 (a), p.184, we may write for functions $w \in BV(\Omega)^M$

$$\nabla^s w = \nabla w \llcorner S_w, \quad S_w := \left\{x \in \Omega : \lim_{\rho\downarrow 0} \frac{|\nabla w|(B_\rho(x))}{\mathcal{L}^n(B_\rho(x))} = \infty\right\}, \tag{3.2.5}$$

where thanks to (3.2.2) it holds

$$S_v \subset S_u, \tag{3.2.6}$$

since $|\nabla v|(B_\rho(x)) \le |\nabla u|(B_\rho(x))$. Combining (3.2.2), (3.2.5) and (3.2.6) we obtain

$$|\nabla^s v|(\Omega) = |\nabla v|(S_v) \le |\nabla u|(S_v) \le |\nabla u|(S_u) = |\nabla^s u|(\Omega). \tag{3.2.7}$$

Hence, recalling the definition of $\widetilde{K}$ and using (3.2.3), (3.2.4) as well as (3.2.7) together with (3.2.1) we then get

$$K[u] = K[v]$$

being only possible if (recall (3.2.3), (3.2.4) and (3.2.7) again)

$$\int\limits_\Omega \Phi(|\nabla^a u|)dx = \int\limits_\Omega \Phi(|\nabla^a v|)dx, \tag{3.2.8}$$

$$|\nabla^s u|(\Omega) = |\nabla^s v|(\Omega), \tag{3.2.9}$$

$$\int\limits_{\Omega - D} |v-f|^2 dx = \int\limits_{\Omega - D} |u-f|^2 dx. \tag{3.2.10}$$

From (3.2.10) we infer $u = v$ a.e. on $\Omega - D$ by strict convexity. Incorporating (3.2.8), $|\nabla^a v| \le |\nabla^a u|$ and the properties of the function Φ we arrive at

$$|\nabla^a u| = |\nabla^a v| \quad \mathcal{L}^n - \text{a.e. on } \Omega. \tag{3.2.11}$$

Further, by exploiting (3.2.2) and (3.2.6) we get for any Borel set $E \subset \Omega$

$$|\nabla^s v|(E) = |\nabla v|(S_v \cap E) \le |\nabla u|(S_u \cap E) = |\nabla^s u|(E). \tag{3.2.12}$$

On account of (3.2.9) it follows by using (3.2.12) with E replaced by $\Omega - E$

$$\begin{aligned} |\nabla^s v|(E) &= |\nabla^s v|(\Omega) - |\nabla^s v|(\Omega - E) \ge |\nabla^s v|(\Omega) - |\nabla^s u|(\Omega - E) \\ &= |\nabla^s u|(\Omega) - |\nabla^s u|(\Omega - E) = |\nabla^s u|(E). \end{aligned}$$

Altogether we have shown

$$|\nabla^s u| = |\nabla^s v|. \tag{3.2.13}$$

Now we suppose that

$$\mathcal{L}^n\bigg(\{x \in \Omega : \nabla^a u(x) \neq \nabla^a v(x)\}\bigg) > 0. \tag{3.2.14}$$

From (3.2.14) we may conclude

$$\int\limits_{[\nabla^a u \neq \nabla^a v]} (|\nabla^a u| + |\nabla^a v| - |\nabla^a u + \nabla^a v|)dx > 0, \tag{3.2.15}$$

since otherwise

$$|\nabla^a u + \nabla^a v| = |\nabla^a u| + |\nabla^a v|$$

a.e. on $[\nabla^a u \neq \nabla^a v]$ and as a consequence

$$\nabla^a u = \lambda \nabla^a v$$

on this set where λ is a non-negative function. By virtue of (3.2.11) it must hold $\lambda = 1$ which is a contradiction.
Applying the properties of Φ we obtain from (3.2.15)

$$\begin{aligned}\int\limits_\Omega \Phi\left(\left|\nabla^a\left(\frac{u+v}{2}\right)\right|\right)dx &< \int\limits_\Omega \Phi\left(\frac{1}{2}|\nabla^a u| + \frac{1}{2}|\nabla^a v|\right)dx \\ &\leq \frac{1}{2}\int\limits_\Omega \Phi(|\nabla^a u|)dx + \frac{1}{2}\int\limits_\Omega \Phi(|\nabla^a v|)dx\end{aligned}$$

and since $|\nabla^s(u+v)| \leq |\nabla^s u| + |\nabla^s v|$ we get from (3.2.11), (3.2.13) and $u = v$ a.e. on $\Omega - D$

$$K\left[\frac{u+v}{2}\right] < K[u]. \tag{3.2.16}$$

Observing that $\frac{u+v}{2}$ is an element of the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$ it follows that the strict inequality contradicts the minimizing property of u and as a consequence, the assumption (3.2.14) is wrong which implies

$$\nabla^a u = \nabla^a v \quad \mathcal{L}^n - \text{a.e. on } \Omega. \tag{3.2.17}$$

As the next step we justify $\nabla^s u = \nabla^s v$ $|\nabla^s u|$-a.e. To this purpose we consider the measure $\mu := |\nabla^s u|$ and thanks to (3.2.13) we find μ-measurable functions $\theta_u, \theta_v : \Omega \to \mathbb{R}^{nM}$ with $|\theta_u| = 1 = |\theta_v|$ μ-a.e. as well as

$$\nabla^s u = \theta_u \llcorner \mu, \quad \nabla^s v = \theta_v \llcorner \mu. \tag{3.2.18}$$

Supposing that

$$\left|\nabla^s\left(\frac{u+v}{2}\right)\right|(\Omega) < |\nabla^s u|(\Omega), \tag{3.2.19}$$

we obtain by virtue of (3.2.17) and $u = v$ a.e. on $\Omega - D$

$$\begin{aligned}K\left[\frac{u+v}{2}\right] &= \int\limits_\Omega \Phi(|\nabla^a u|)dx + \overline{c}\left|\nabla^s\left(\frac{u+v}{2}\right)\right|(\Omega) + \int\limits_{\Omega - D} |u-f|^2 dx \\ &< K[u]\end{aligned}$$

which again contradicts the K-minimality of u. Consequently we have in place of (3.2.19)

$$\left|\int\limits_\Omega \frac{1}{2}(\theta_u + \theta_v)d\mu\right| = \mu(\Omega),$$

thus
$$\mu(\Omega) \le \frac{1}{2}\int_\Omega |\theta_u + \theta_v| d\mu \le \frac{1}{2}\int_\Omega (|\theta_u| + |\theta_v|) d\mu = \mu(\Omega),$$
which gives
$$|\theta_u + \theta_v| = |\theta_u| + |\theta_v| \quad \mu - \text{a.e.}$$
Therefore we may write
$$\theta_u = \widetilde{\lambda}\theta_v$$
where $\widetilde{\lambda}$ is a non-negative and μ-measurable function. Recalling $|\theta_u| = 1 = |\theta_v|$ we conclude $\widetilde{\lambda} = 1$, which means $\theta_u = \theta_v$ μ-a.e. Taking into account (3.2.18) it then follows $\nabla^s u = \nabla^s v$ and using (3.2.17) we obtain $\nabla u = \nabla v$. Recalling $u = v$ a.e. on $\Omega - D$ and quoting [7], Proposition 3.2, p.118 (recall (3.1.3)) we finally have $u = v$ a.e. on Ω and in conclusion
$$\sup_\Omega |u| \le L$$
completing the proof of Theorem 3.1.4. □

For the sake of completeness let us look at the corresponding maximum principle in the scalar case $M = 1$. Here we basically follow the procedure carried out in [21], proof of Theorem 2. However, we cannot directly quote Theorem 2 in this reference since this result was given in another setting. The claim follows if we can show that any solution $u \in BV(\Omega) \cap L^2(\Omega - D)$ of
$$K[w] = \hat{K}[w] + \frac{\lambda}{2}\int_{\Omega - D} |w - f|^2 dx \to \min \text{ in } BV(\Omega) \cap L^2(\Omega - D)$$
with
$$\hat{K}[w] := \int_\Omega F(\nabla^a w) dx + \int_\Omega F^\infty\left(\frac{\nabla^s w}{|\nabla^s w|}\right) d|\nabla^s w|,$$
satisfies $0 \le u(x) \le 1$ a.e. in Ω. In the following we sketch the proof of the validity of $u \ge 0$ a.e. in Ω for any K-minimizer u: setting $\psi(t) := \max\{0, t\}$, $t \in \mathbb{R}$, and recalling the chain rule for real-valued functions (see equation (2.2.2) in Section 2.2.1) we get $v := \psi \circ u \in BV(\Omega)$ together with
$$\nabla v = \psi'(u)\nabla^a u \llcorner \mathcal{L}^n + (\psi(u^+) - \psi(u^-))\nu_u \mathcal{H}^{n-1} \llcorner J_u + \psi'(\widetilde{u})\nabla^c u.$$
Here, our notation follows the terminology of Section 2.2.1 in this thesis. From $0 \le f(x) \le 1$ a.e. on $\Omega - D$ it is immediate that $|\psi(w) - f| \le |w - f|$ a.e. on $\Omega - D$ for any $w \in BV(\Omega) \cap L^2(\Omega - D)$. Moreover, using the arguments of [21], proof of Theorem 2, where we start with (26) in this reference we get
$$\hat{K}[v] = \hat{K}[u], \tag{3.2.20}$$
$$\int_{\Omega - D} |v - f|^2 dx = \int_{\Omega - D} |u - f|^2 dx, \tag{3.2.21}$$

where we crucially use the K-minimality of u. Clearly, (3.2.21) gives $v = u$ a.e. on $\Omega - D$. Following again the arguments as applied in [21], proof of Theorem 2, starting with (31) in this reference we can derive $\nabla v = \nabla u$ a.e. on Ω from the identity (3.2.20). Quoting [7], Proposition 3.2, p.118, again we conclude $u = v$ a.e. on Ω and therewith $u \geq 0$ a.e. on Ω. For deriving $u \leq 1$ a.e. on Ω we replace $\psi(t)$ through $\widetilde{\psi}(t) := \min\{t, 1\}, t \in \mathbb{R}$, and argue in the same manner as above. This completes the proof of Theorem 3.1.5 as well. $\square$

3.3 Partial $C^{1,\frac{1}{2}}$-regularity of generalized minimizers. Proof of Theorem 3.1.9

Let us assume the validity of the hypotheses of Theorem 3.1.9 and denote by $u \in BV(\Omega)^M \cap L^2(\Omega - D)^M$ a generalized I-minimizer which, by Theorem 3.1.4, is of class $L^\infty(\Omega)^M$ as well. For proving Theorem 3.1.9 we benefit from Corollary 3.3 in [86] choosing $p = 2$ in this reference. In order to justify the application of this corollary to our setting we first formulate a proposition which shows that we are actually in the situation of [86].

Proposition 3.3.1
Under the hypotheses of Theorem 3.1.9 it holds:

(a) for all $P \in \mathbb{R}^{nM}$ the density F satisfies (H1)-(H4) in [86] (see Section 2 in this reference);

(b) setting $g : \Omega \times \mathbb{R}^M \to \mathbb{R},\ g(x, y) := \frac{\lambda}{2}\mathbb{1}_{\Omega - D}|y - f(x)|^2$ the following statements hold true:

(i) g is a Borel function;

(ii) g satisfies a Hölder condition in the following sense: for a positive constant C we have the estimate

$$|g(x, y_1) - g(x, y_2)| \leq C(|f(x)| + |y_1| + |y_2|)|y_2 - y_1| \tag{3.3.1}$$

for all $x \in \Omega, y_1, y_2 \in \mathbb{R}^M$.

Proof of Proposition 3.3.1. In accordance with the hypotheses of Theorem 3.1.9 we state that the density F satisfies (3.1.5), (3.1.6) as well as (3.1.7). For proving assertion (a) we note that on account of (3.1.7) we deal with the non-degenerate case. Quoting Remark 2.6 in [86] we then choose $p = 2$ in this reference and as a consequence (H2), (H3) as well as (H4) in [86] correspond to the requirement that F is of class $C^2(\mathbb{R}^{nM})$ with $D^2F(P) > 0$ for all $P \in \mathbb{R}^{nM}$. Thus, F satisfies (H2)-(H4) by recalling (3.1.7). Furthermore, F fulfills (H1) since F is (strictly) convex on $\mathbb{R}^{nM}$ (see (3.1.7) again) and of linear growth.
In order to verify the statements of part (b) we first remark that assertion (b), (i) is immediate whereas a straight-forward calculation shows that (3.3.1) holds. $\square$

Recalling (3.1.4) (note that we identify f with its zero-extension to Ω) and $u \in L^\infty(\Omega)^M$ (see Theorem 3.1.4) it follows

$$|u|, |f| \in L^{n,\alpha n}(\Omega) \tag{3.3.2}$$

for any number $\alpha \in (0,1]$ in the context of the so-called Morrey spaces (see, e.g., [86], Definition 4.9). Note that the choice $\alpha = 1$ in (3.3.2) is optimal since it holds $L^{n,\alpha n}(\Omega) = \{0\}$ for $\alpha > 1$.
Hence, all hypotheses from Corollary 3.3 in [86] are satisfied with optimal value $\alpha = 1$ and it follows that for each generalized I-minimizer u from the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$ there exists an open subset Ω_0^u of Ω such that $u \in C^{1,\frac{1}{2}}(\Omega_0^u)^M$ together with $\mathcal{L}^n(\Omega - \Omega_0^u) = 0$. This proves Theorem 3.1.9. □

Remark 3.3.2
We conjecture that the choice of the Hölder exponent $\gamma = \frac{1}{2}$ is not optimal although in the context of Corollary 3.3 in [86], this choice seems to be optimal. Moreover, w.r.t. the results of [9] we are not sure whether one actually needs the structure condition (3.1.8) *for the density F in order to prove almost everywhere regularity of arbitrary K-minimizers. However, due to the presence of the data fitting term, it is not possible to refer to , e.g., Theorem 1.1 in [9] and adding some obvious modifications.*

3.4 Existence and uniqueness of weak minimizers in $W^{1,1}$ for $\mu \in (1,2)$ Proof of Theorem 3.1.15

Let us assume the validity of the hypotheses of Theorem 3.1.15. In particular we emphasize that we will crucially make use of the requirement $\mu \in (1,2)$ imposed on the ellipticity parameter $\mu > 1$. First of all, we pursue the common strategy and approximate our original problem "$I \to \min$ in $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$" by a sequence of more regular problems. Precisely we are going to regularize the energy density F as already done in the proof of Theorem 2.1.6 (see Section 2.4), i.e., for fixed $\delta \in (0,1]$ we consider the problem

$$I_\delta[w] := \int_\Omega F_\delta(\nabla w)dx + \frac{\lambda}{2}\int_\Omega |w - f|^2 dx \to \min \quad \text{in } W^{1,2}(\Omega)^M, \tag{3.4.1}$$

where $F_\delta(P) := \frac{\delta}{2}|P|^2 + F(P)$, $P \in \mathbb{R}^{nM}$. As already proven in Section 2.4, problem (3.4.1) admits a unique solution u_δ of class $W^{1,2}(\Omega)^M$ with suitable convergence properties for $\delta \to 0$. A first collection of uniform (in δ) estimates of u_δ can be found in Lemma 2.4.1 from Section 2.4. Under the (much) stronger hypotheses of Theorem 3.1.15 we now state a maximum principle being in the spirit of of Theorem 3.1.4 as well as a useful local higher weak differentiability property of u_δ. A proof of this lemma can be found in the appendix of this thesis (see Lemma 7.1.1).

Lemma 3.4.1
The unique I_δ-minimizer u_δ satisfies

(a) $\sup\limits_{\Omega} |u_\delta| \leq \sup\limits_{\Omega - D} |f|$,

(b) $u_\delta \in W^{2,2}_{loc}(\Omega)^M$.

Remark 3.4.2
The reader should note that Lemma 3.4.1 remains valid under weaker hypotheses on F. Furthermore, in the scalar case $M = 1$, we can show $0 \leq u_\delta \leq 1$ a.e. on Ω provided that we assume w.l.o.g. that $f : \Omega - D \to \mathbb{R}$ takes almost all of its values in the interval $[0, 1]$.

In order to prove Theorem 3.1.15 we proceed similiar to the proof of Theorem 1.3 in [24]. Thanks to the restriction $\mu \in (1, 2)$ it is possible to show the following uniform (in δ) statement of u_δ which will be of crucial meaning during the further proof.

Lemma 3.4.3
It holds $u_\delta \in W^{1,2}_{loc}(\Omega)^M$ uniformly in δ.

Proof of Lemma 3.4.3. First of all we observe that u_δ solves the Euler equation

$$\int\limits_{\Omega} DF_\delta(\nabla u_\delta) : \nabla\varphi dx + \lambda \int\limits_{\Omega - D} (u_\delta - f) \cdot \varphi dx = 0 \tag{3.4.2}$$

for all $\varphi \in C^\infty_0(\Omega)^M$. Quoting Lemma 3.4.1, u_δ is of class $W^{2,2}_{\mathrm{loc}}(\Omega)^M$. Furthermore, since $|D^2 F_\delta|$ is bounded, $DF_\delta(\nabla u_\delta)$ is of class $W^{1,2}_{\mathrm{loc}}(\Omega)^{nM}$ having partial derivatives

$$\partial_\gamma(DF_\delta(\nabla u_\delta)) = D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \cdot) \quad \text{a.e. on } \Omega$$

where $\gamma \in \{1, \dots, n\}$. Observing that $\partial_\gamma \varphi$ serves as an admissible choice in the Euler equation (3.4.2) and performing an integration by parts we arrive at the differentiated Euler equation

$$\int\limits_{\Omega} D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla\varphi) dx = \lambda \int\limits_{\Omega - D} (u_\delta - f) \cdot \partial_\gamma \varphi dx \tag{3.4.3}$$

for all $\varphi \in C^\infty_0(\Omega)^M$ and by approximation, (3.4.3) remains valid for functions $\varphi \in W^{1,2}(\Omega)^M$ having compact support in Ω.

Next, we fix a point $x_0 \in \Omega$, a radius $R > 0$ such that $B_{2R}(x_0) \Subset \Omega$ and let $\eta \in C^\infty_0(B_{2R}(x_0))$ with $\eta \equiv 1$ on $B_R(x_0)$, $0 \leq \eta \leq 1$ and $|\nabla\eta| \leq \frac{c}{R}$.

As a consequence, $\varphi := \eta^2 \partial_\gamma u_\delta$ is admissible in (3.4.3) and it follows (from now on summation w.r.t. γ)

$$\begin{aligned} I_0 &:= \int\limits_{B_{2R}(x_0)} D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 dx \\ &= -2 \int\limits_{B_{2R}(x_0)} D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \eta)\eta dx \\ &+ \lambda \int\limits_{B_{2R}(x_0)-D} (u_\delta - f)\cdot \partial_\gamma \varphi dx =: I_1 + \lambda I_2. \end{aligned} \tag{3.4.4}$$

Considering I_1 we get by using the Cauchy-Schwarz inequality and Young's inequality ($\varepsilon > 0$)

$$|I_1| \le \varepsilon I_0 + c\varepsilon^{-1} \int\limits_{B_{2R}(x_0)} D^2 F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \otimes \nabla\eta, \partial_\gamma u_\delta \otimes \nabla \eta)dx. \tag{3.4.5}$$

Studying I_2 an integration by parts gives

$$\begin{aligned} I_2 = &\int\limits_{B_{2R}(x_0)} (u_\delta - f)\cdot \partial_\gamma(\eta^2 \partial_\gamma u_\delta)dx - \int\limits_{B_{2R}(x_0)\cap D} (u_\delta - f)\cdot \partial_\gamma(\eta^2\partial_\gamma u_\delta)dx \\ = &- \int\limits_{B_{2R}(x_0)} |\nabla u_\delta|^2 \eta^2 dx - \int\limits_{B_{2R}(x_0)} f\cdot \partial_\gamma(\eta^2 \partial_\gamma u_\delta)dx \\ &- \int\limits_{B_{2R}(x_0)\cap D} (u_\delta - f)\cdot \partial_\gamma(\eta^2 \partial_\gamma u_\delta)dx. \end{aligned} \tag{3.4.6}$$

Putting together (3.4.5) as well as (3.4.6) and absorbing terms (we choose $\varepsilon > 0$ sufficiently small), (3.4.4) turns into (recall (3.1.4) and Lemma 3.4.1)

$$\begin{aligned} &\int\limits_{B_{2R}(x_0)} D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 dx + \lambda \int\limits_{B_{2R}(x_0)} |\nabla u_\delta|^2 \eta^2 dx \\ &\le c \int\limits_{B_{2R}(x_0)} D^2 F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \otimes \nabla \eta, \partial_\gamma u_\delta \otimes \nabla\eta)dx + c\int\limits_{B_{2R}(x_0)} \eta^2 |\nabla^2 u_\delta| dx \\ &+ c\int\limits_{B_{2R}(x_0)} |\nabla u_\delta| \eta |\nabla \eta| dx. \end{aligned} \tag{3.4.7}$$

From (3.4.7) it follows by using (3.1.7)

$$\int\limits_{B_{2R}(x_0)} \eta^2 \frac{|\nabla^2 u_\delta|^2}{(1+|\nabla u_\delta|^2)^{\frac{\mu}{2}}} dx + c\lambda \int\limits_{B_{2R}(x_0)} |\nabla u_\delta|^2 \eta^2 dx$$

$$\leq c \int_{B_{2R}(x_0)} [(1+|\nabla u_\delta|^2)^{-\frac{1}{2}} + \delta]|\nabla u_\delta|^2 |\nabla \eta|^2 dx + c \int_{B_{2R}(x_0)} \eta^2 |\nabla^2 u_\delta| dx$$
$$+ c \int_{B_{2R}(x_0)} |\nabla u_\delta| \eta |\nabla \eta| dx$$
$$\leq c(R) + c \int_{B_{2R}(x_0)} \eta^2 |\nabla^2 u_\delta| dx,$$

where the last inequality is a consequence of the uniform estimate $I_\delta[u_\delta] \leq I[0]$ (recall Lemma 2.4.1) and of the linear growth of F.
Next, Young's inequality ($\varepsilon > 0$) implies

$$c \int_{B_{2R}(x_0)} \eta^2 |\nabla^2 u_\delta| dx$$
$$\leq \varepsilon \int_{B_{2R}(x_0)} \eta^2 \frac{|\nabla^2 u_\delta|^2}{(1+|\nabla u_\delta|^2)^{\frac{\mu}{2}}} dx + c\varepsilon^{-1} \int_{B_{2R}(x_0)} \eta^2 (1+|\nabla u_\delta|^2)^{\frac{\mu}{2}} dx.$$

Absorbing terms by choosing $\varepsilon > 0$ sufficiently small we have

$$\begin{aligned} &\int_{B_{2R}(x_0)} \eta^2 \frac{|\nabla^2 u_\delta|^2}{(1+|\nabla u_\delta|^2)^{\frac{\mu}{2}}} dx + c \int_{B_{2R}(x_0)} |\nabla u_\delta|^2 \eta^2 dx \\ &\leq c(R) + c \int_{B_{2R}(x_0)} \eta^2 (1+|\nabla u_\delta|^2)^{\frac{\mu}{2}} dx. \end{aligned} \tag{3.4.8}$$

Recalling $\mu < 2$ at this point and setting $p := \frac{2}{\mu} > 1$ as well as $q := \frac{2}{2-\mu} > 1$ we get by exploiting Young's inequality ($\varepsilon > 0$) once again (note $\frac{1}{p} + \frac{1}{q} = 1$)

$$\begin{aligned} &c \int_{B_{2R}(x_0)} \eta^2 (1+|\nabla u_\delta|^2)^{\frac{\mu}{2}} dx \\ &\leq \varepsilon \int_{B_{2R}(x_0)} \eta^2 (1+|\nabla u_\delta|^2) dx + c\varepsilon^{\frac{\mu}{\mu-2}} \int_{B_{2R}(x_0)} \eta^2 dx. \end{aligned} \tag{3.4.9}$$

Choosing $\varepsilon > 0$ sufficiently small again in (3.4.9) it finally follows from (3.4.8) (recall $\eta \equiv 1$ on $B_R(x_0)$)

$$\int_{B_R(x_0)} \frac{|\nabla^2 u_\delta|^2}{(1+|\nabla u_\delta|^2)^{\frac{\mu}{2}}} dx + \int_{B_R(x_0)} |\nabla u_\delta|^2 dx \leq c(R), \tag{3.4.10}$$

where $c(R)$ represents a local constant independent of δ. This proves the claim of Lemma 3.4.3 after using a covering argument. □

Remark 3.4.4
Setting $\varphi_\delta := (1+|\nabla u_\delta|)^{1-\frac{\mu}{2}}$ *we get from* (3.4.10)

$$\int\limits_{B_R(x_0)} |\nabla\varphi_\delta|^2 dx \le c(R) \tag{3.4.11}$$

with a local constant not depending on δ. *Using a covering argument we obtain* $\varphi_\delta \in W^{1,2}_{loc}(\Omega)$ *uniformly in* δ. *If* $n=2$, *an application of Sobolev's embedding theorem then implies* $\varphi_\delta \in L^p_{loc}(\Omega)$ *uniformly in* δ *for any finite* p *which means* $\nabla u_\delta \in L^p_{loc}(\Omega)^{nM}$ *uniformly in* δ *for any finite* p. *In case* $n \ge 3$, *Sobolev's embedding theorem merely gives* $\nabla u_\delta \in L^q_{loc}(\Omega)^{nM}$ *uniformly in* δ *for any* $1 \le q \le \frac{2n}{n-2}$.

Having the auxiliary result of Lemma 3.4.3 at hand we now proceed with the proof of Theorem 3.1.15: quoting Theorem 2.1.6 (see Section 2.4) there exists a function $\overline{u} \in BV(\Omega)^M$ with $u_\delta \to: \overline{u}$ in $L^1(\Omega)^M$ (and a.e. on Ω) at least for a subsequence (not relabeled). Thanks to Lemma 3.4.3 we may conclude $\overline{u} \in W^{1,2}_{\mathrm{loc}}(\Omega)^M$, i.e. it actually holds $\overline{u} \in W^{1,1}(\Omega)^M$ by remarking that $BV(\Omega)^M \cap W^{1,2}_{\mathrm{loc}}(\Omega)^M$ is a subspace of $W^{1,1}(\Omega)^M$. Further we can arrange $u_\delta \rightharpoonup \overline{u}$ in $L^2(\Omega-D)^M$ by passing to another subsequence (recall the uniform estimate $I_\delta[u_\delta] \le I[0]$ once again), thus $\overline{u} \in W^{1,1}(\Omega)^M \cap L^2(\Omega-D)^M$, i.e., $I[\overline{u}]$ is well-defined.

Now, our goal is to show that $\overline{u}$ is I-minimizing: based on the convergence $u_\delta \rightharpoonup \overline{u}$ in $W^{1,2}_{\mathrm{loc}}(\Omega)^M$ for a subsequence we obtain

$$\int\limits_{\omega\cap(\Omega-D)} |\overline{u}-f|^2 dx \le \liminf_{\delta\to 0} \int\limits_{\omega\cap(\Omega-D)} |u_\delta - f|^2 dx \tag{3.4.12}$$

as $\delta \to 0$ for compact subregions ω of Ω.
Moreover, it holds $\nabla u_\delta \rightharpoonup \nabla\overline{u}$ in $L^2_{\mathrm{loc}}(\Omega)^{nM}$ and obviously, this convergence is also valid in $L^1_{\mathrm{loc}}(\Omega)^{nM}$. Quoting well-known results about (weak) lower semicontinuity (see, e.g., [57], Theorem 2.3, p.18 or [2]) we may deduce

$$\int\limits_\omega F(\nabla\overline{u})dx \le \liminf_{\delta\to 0}\int\limits_\omega F(\nabla u_\delta)dx. \tag{3.4.13}$$

Combining (3.4.12) and (3.4.13) it follows

$$\begin{aligned}&\int\limits_\omega F(\nabla\overline{u})dx + \int\limits_{\omega\cap(\Omega-D)} |\overline{u}-f|^2 dx\\ &\le \liminf_{\delta\to 0}\Bigg[\int\limits_\omega F(\nabla u_\delta)dx + \int\limits_{\omega\cap(\Omega-D)} |u_\delta - f|^2 dx\Bigg].\end{aligned}$$

Considering a compact exhaustion of Ω and using $\mathbb{1}_\omega \to \mathbb{1}_\Omega$ a.e. on Ω we immediately get

$$I[\overline{u}] \le \liminf_{\delta\to 0} I[u_\delta]. \tag{3.4.14}$$

Thanks to the I_δ-minimality of u_δ in $W^{1,2}(\Omega)^M$ it follows from (3.4.14)

$$I[\overline{u}] \le \liminf_{\delta\to 0} I[u_\delta] \le \liminf_{\delta\to 0} I_\delta[u_\delta] \le \liminf_{\delta\to 0} I_\delta[v] = I[v]$$

where $v \in W^{1,2}(\Omega)^M$.
Applying Lemma 2.2.4 we can find a sequence $(v_k) \subset W^{1,2}(\Omega)^M$ satisfying $v_k \to w$ in $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$. As a consequence we finally get $I[\overline{u}] \le I[w]$ for all $w \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$, i.e,. $\overline{u}$ is an I-minimizer being of class $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$.

To face the uniqueness problem, let $\widetilde{u}$ denote a second I-minimizing function from the space $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$. By strict convexity it holds $\nabla\overline{u} = \nabla\widetilde{u}$ a.e. on Ω together with $\overline{u} = \widetilde{u}$ a.e. on $\Omega - D$. Thus, it follows $\overline{u} = \widetilde{u} + c$ a.e. on Ω for a suitable constant c. Quoting (3.1.3) we directly conclude $c = 0$.

For proving the maximum principle we use Lemma 3.4.1 (a), together with the a.e.-convergence $u_\delta \to \overline{u}$ on Ω. Besides we get $\overline{u} \in W^{1,2}_{\mathrm{loc}}(\Omega)^M$ by construction. This completes the proof of Theorem 3.1.15. □

Remark 3.4.5
Note that for the unique I-minimizer $\overline{u} \in W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$ we additionally obtain $\nabla\overline{u} \in L^q_{loc}(\Omega)^{nM}$ for any $1 \le q \le \frac{2n}{n-2}$ for $n \ge 3$ by applying Sobolev's embedding theorem (recall $\overline{u} \in W^{1,2}_{loc}(\Omega)^M$).

Remark 3.4.6
Based on the uniqueness of u_δ and $\overline{u}$ we can state that it holds

$$\begin{aligned} I_\delta[u_\delta] &\to I[\overline{u}], \\ \delta \int_\Omega |\nabla u_\delta|^2 dx &\to 0, \\ u_\delta &\to \overline{u} \quad \text{in } L^1(\Omega)^M, \\ u_\delta &\rightharpoondown \overline{u} \quad \text{in } W^{1,2}_{loc}(\Omega)^M \end{aligned}$$

as $\delta \to 0$ not only for a subsequence.

3.5 Full interior $C^{1,\alpha}$-regularity of the $W^{1,1}$-minimizer. Proof of Theorem 3.1.19

Let us assume the validity of the hypotheses of Theorem 3.1.19. For proving Theorem 3.1.19 we first discuss the scalar case $M = 1$ (see Section 3.5.1 below), where we can avoid using the additional conditions (3.1.15) and (3.1.16) imposed on F. Taken the arguments in the scalar case as a basis we investigate the vector-valued case afterwards (see Section 3.5.2 below).

3.5.1 Proof of Theorem 3.1.19: the scalar case

Let us suppose the validity of (3.1.3) and assume that we are given a function $f : \Omega - D \to \mathbb{R}$ taking its values in the closed interval $[0, 1]$. As already mentioned in Remark 3.1.7, $f(x)$ can be seen as a measure for the intensity of the grey level at $x \in \Omega - D$. Moreover, the density F shall fulfill (3.1.5)–(3.1.7) for some $\mu \in (1, 2)$. For the sake of a better overview we will split this section into four parts: regularization and local uniform apriori L^p-estimates, Caccioppoli-type inequality, De Giorgi-type iteration and the conclusions.

Step 1. Regularization and local uniform apriori L^p-estimates

We consider the regularization as given in the proof of Theorem 3.1.15 (see Section 3.3) and in the proof of Theorem 2.1.6 (see Section 2.4), respectively. Thus, with $\delta \in (0, 1]$ being fixed, we denote by u_δ the unique minimizer of

$$I_\delta[w] := \frac{\delta}{2} \int_\Omega |\nabla w|^2 dx + I[w], \quad w \in W^{1,2}(\Omega).$$

In accordance with Lemma 3.4.1 and Remark 3.4.2, respectively, we can state

$$0 \le u_\delta \le 1 \quad \text{a.e. on } \Omega, \tag{3.5.1}$$

$$u_\delta \in W^{2,2}_{\mathrm{loc}}(\Omega). \tag{3.5.2}$$

Quoting elliptic regularity theory (see, e.g., [62]) we additionally obtain (for a proof we refer to the appendix of this thesis)

$$u_\delta \in C^{1,\alpha}(\Omega) \quad \text{for any } \alpha \in (0, 1), \tag{3.5.3}$$

which implies the important apriori information $u_\delta \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.
Once again we remember the following uniform estimates (recall $I_\delta[u_\delta] \le I[0]$ and $F_\delta(P) = \frac{\delta}{2}|P|^2 + F(P), P \in \mathbb{R}^n$)

$$\int_\Omega F_\delta(\nabla u_\delta) dx \le c_1, \qquad \int_{\Omega - D} (u_\delta - f)^2 dx \le c_2 \tag{3.5.4}$$

for some positive, real numbers c_1, c_2. Further u_δ solves the Euler equation

$$\int_\Omega DF_\delta(\nabla u_\delta) \cdot \nabla \varphi dx + \lambda \int_{\Omega - D} (u_\delta - f)\varphi dx = 0 \tag{3.5.5}$$

for all $\varphi \in W^{1,2}(\Omega)$ that are compactly supported in Ω.
The first step is mainly devoted to the derivation of local uniform (in δ) L^p-estimates of ∇u_δ for any finite exponent $p \in (1, \infty)$. This result will serve as an important tool for carrying out a De Giorgi-type iteration in step 3 of this proof.

Lemma 3.5.1
For any $1 < p < \infty$ and for any $\omega \Subset \Omega$ there is a constant $c(p, \omega)$, which in particular does not depend on δ, such that

$$\|\nabla u_\delta\|_{L^p(\omega)} \le c(p, \omega) < \infty. \tag{3.5.6}$$

Proof of Lemma 3.5.1. At first we will establish a variant of Caccioppoli's inequality being only valid for $\mu \in (1, 2)$. Here, we use arguments as already applied in [17], Lemma 4.19 (i), pp.108, where we include the inpainting quantity that leads to severe difficulties. This version of Caccioppoli's inequality will be crucially used when performing an iteration argument afterwards which finally gives uniform L^p_{loc}-gradient bounds of u_δ for any finite p.

Lemma 3.5.2
For any $s_0 \ge 0$ there exists a real number $c > 0$ such that for all $\eta \in C_0^\infty(\Omega)$ satisfying $0 \le \eta \le 1$ and for any $\delta \in (0, 1)$ it holds

$$\begin{aligned}
&\int_\Omega |\nabla^2 u_\delta|^2 \Gamma_\delta^{s_0 - \frac{\mu}{2}} \eta^2 dx + \delta \int_\Omega |\nabla^2 u_\delta|^2 \Gamma_\delta^{s_0} \eta^2 dx \\
&\le c \int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \eta^2 \Gamma_\delta^{s_0} dx \\
&\le c \int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \nabla \eta, \partial_\gamma u_\delta \nabla \eta) \Gamma_\delta^{s_0} dx + c \int_\Omega |\nabla \eta|^2 \Gamma_\delta^{s_0} dx \\
&+ c \int_\Omega \eta^2 \Gamma_\delta^{s_0} dx,
\end{aligned} \tag{3.5.7}$$

where we have set $\Gamma_\delta := 1 + |\nabla u_\delta|^2$ and c, in particular, is independent of δ.

Proof of Lemma 3.5.2. We start by noting that the first inequality follows from (3.1.7). For proving the second inequality we fix some number $s_0 > 0$ (for the case $s_0 = 0$ we refer to Lemma 3.4.3). As already seen at the beginning of the proof of Theorem 3.1.15 the Euler equation (3.5.5) yields

$$\int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla \psi) dx = \lambda \int_{\Omega - D} (u_\delta - f) \partial_\gamma \psi dx \tag{3.5.8}$$

for all $\psi \in W^{1,2}(\Omega)$ with compact support in Ω.

With η as given above and by quoting $u_\delta \in W^{2,2}_{\mathrm{loc}}(\Omega) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega)$ (see (3.5.2) and (3.5.3)), $\psi = \eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}$ is admissible in (3.5.8) (recall the product and the chain rule for Sobolev functions). We then get (from now on summation w.r.t.

$\gamma \in \{1, \dots, n\}$)

$$\begin{aligned}
&\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 \Gamma_\delta^{s_0} dx \\
&+ s_0 \int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \nabla \Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2 dx \\
&= -2\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \nabla\eta)\eta \Gamma_\delta^{s_0} dx \\
&+ \lambda \int_{\Omega - D} (u_\delta - f)\partial_\gamma(\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx.
\end{aligned} \tag{3.5.9}$$

For the last integral on the r.h.s. of (3.5.9), we obtain

$$\begin{aligned}
&\int_{\Omega - D} (u_\delta - f)\partial_\gamma(\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx \\
&= \int_\Omega (u_\delta - f)\partial_\gamma(\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx - \int_{\Omega\cap D} (u_\delta - f)\partial_\gamma(\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx \\
&= -\int_\Omega |\nabla u_\delta|^2 \eta^2 \Gamma_\delta^{s_0} dx - \int_\Omega f \partial_\gamma(\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx \\
&- \int_{\Omega\cap D} (u_\delta - f)\partial_\gamma(\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx,
\end{aligned} \tag{3.5.10}$$

where the last equality follows by performing an integration by parts.

Moreover we have

$$\begin{aligned}
&s_0 \int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \nabla \Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2 dx \\
&= \frac{s_0}{2} \int_\Omega D^2F_\delta(\nabla u_\delta)(\nabla \Gamma_\delta, \nabla \Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2 dx.
\end{aligned} \tag{3.5.11}$$

Incorporating (3.5.10) and (3.5.11) in (3.5.9), it follows

$$\begin{aligned}
&\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 \Gamma_\delta^{s_0} dx \\
&+ \frac{s_0}{2} \int_\Omega D^2F_\delta(\nabla u_\delta)(\nabla \Gamma_\delta, \nabla \Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2 dx + \lambda \int_\Omega |\nabla u_\delta|^2 \eta^2 \Gamma_\delta^{s_0} dx \\
&= -2\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \nabla\eta)\eta \Gamma_\delta^{s_0} dx \\
&- \lambda \int_\Omega f \partial_\gamma(\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx - \lambda \int_{\Omega\cap D} (u_\delta - f)\partial_\gamma(\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx.
\end{aligned} \tag{3.5.12}$$

An application of the inequality of Cauchy-Schwarz to the bilinear form $D^2F_\delta(\nabla u_\delta)$ and using Young's inequality ($\varepsilon > 0$) subsequently, it holds

$$\begin{aligned}
&2\left|\int\limits_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \nabla\eta)\eta\Gamma_\delta^{s_0}\,dx\right| \\
&\leq 2\int\limits_\Omega \left(D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta, \partial_\gamma\nabla u_\delta)\right)^{\frac{1}{2}}\left(D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta\nabla\eta, \partial_\gamma u_\delta\nabla\eta)\right)^{\frac{1}{2}}\eta\Gamma_\delta^{s_0}\,dx \\
&\leq \varepsilon\int\limits_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta, \partial_\gamma\nabla u_\delta)\eta^2\Gamma_\delta^{s_0}\,dx \\
&+ 4\varepsilon^{-1}\int\limits_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta\nabla\eta, \partial_\gamma u_\delta\nabla\eta)\Gamma_\delta^{s_0}\,dx.
\end{aligned}$$

Recalling $0 \leq u_\delta \leq 1$, $0 \leq f \leq 1$ a.e. and absorbing terms by choosing $\varepsilon = \frac{1}{2}$, we see that

$$\begin{aligned}
&\int\limits_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta, \partial_\gamma\nabla u_\delta)\eta^2\Gamma_\delta^{s_0}\,dx + 2\lambda\int\limits_\Omega |\nabla u_\delta|^2\eta^2\Gamma_\delta^{s_0}\,dx \\
&+ s_0\int\limits_\Omega D^2F_\delta(\nabla u_\delta)(\nabla\Gamma_\delta, \nabla\Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2\,dx \\
&\leq c\int\limits_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta\nabla\eta, \partial_\gamma u_\delta\nabla\eta)\Gamma_\delta^{s_0}\,dx + c\int\limits_\Omega \eta|\nabla\eta||\nabla u_\delta|\Gamma_\delta^{s_0}\,dx \qquad (3.5.13)\\
&+ c\int\limits_\Omega \eta^2|\nabla^2 u_\delta|\Gamma_\delta^{s_0}\,dx + c(s_0)\int\limits_\Omega \eta^2|\nabla u_\delta||\nabla\Gamma_\delta|\Gamma_\delta^{s_0-1}\,dx \\
&=: c\int\limits_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta\nabla\eta, \partial_\gamma u_\delta\nabla\eta)\Gamma_\delta^{s_0}\,dx + \sum_{j=1}^{3} I_j.
\end{aligned}$$

Starting with I_1 we again use Young's inequality (we choose $\varepsilon := \lambda$) and get

$$I_1 \leq \lambda\int\limits_\Omega \eta^2|\nabla u_\delta|^2\Gamma_\delta^{s_0}\,dx + c\int\limits_\Omega |\nabla\eta|^2\Gamma_\delta^{s_0}\,dx. \tag{3.5.14}$$

For I_3 we obtain by noting $|\nabla\Gamma_\delta| \leq c|\nabla u_\delta||\nabla^2 u_\delta|$

$$I_3 \leq c(s_0)\int\limits_\Omega \eta^2|\nabla^2 u_\delta|\Gamma_\delta^{s_0}\,dx. \tag{3.5.15}$$

As a consequence of (3.5.15) the quantity $I_2 + I_3$ is bounded by the r.h.s. of (3.5.15) and another application of Young's inequality ($\varepsilon > 0$) to this integral leads to

$$
\begin{aligned}
&c(s_0)\int_\Omega \eta^2|\nabla^2 u_\delta|\Gamma_\delta^{s_0}\,dx\\
&\le c(s_0)\int_\Omega \Big[\varepsilon\eta^2\Gamma_\delta^{-\frac{\mu}{2}}|\nabla^2 u_\delta|^2\Gamma_\delta^{s_0}+\varepsilon^{-1}\eta^2\Gamma_\delta^{s_0+\frac{\mu}{2}}\Big]dx\\
&\le c(s_0)\int_\Omega \Big[c\varepsilon D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta,\partial_\gamma\nabla u_\delta)\eta^2\Gamma_\delta^{s_0}+\varepsilon^{-1}\eta^2\Gamma_\delta^{s_0+\frac{\mu}{2}}\Big]dx,
\end{aligned}
\tag{3.5.16}
$$

where we used (3.1.7) in the last step.

Inserting (3.5.14) and (3.5.16) in (3.5.13) it follows by absorbing terms (we choose $\varepsilon>0$ sufficiently small)

$$
\begin{aligned}
&\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta,\partial_\gamma\nabla u_\delta)\eta^2\Gamma_\delta^{s_0}\,dx\\
&+c(s_0)\int_\Omega D^2F_\delta(\nabla u_\delta)(\nabla\Gamma_\delta,\nabla\Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2\,dx+c\int_\Omega|\nabla u_\delta|^2\eta^2\Gamma_\delta^{s_0}\,dx\\
&\le c\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta\nabla\eta,\partial_\gamma u_\delta\nabla\eta)\Gamma_\delta^{s_0}\,dx+c\int_\Omega|\nabla\eta|^2\Gamma_\delta^{s_0}\,dx\\
&+c\int_\Omega\eta^2\Gamma_\delta^{s_0+\frac{\mu}{2}}\,dx.
\end{aligned}
\tag{3.5.17}
$$

Now, we investigate the last integral on the right-hand side of (3.5.17). Recalling our assumption $\mu<2$ and setting $p:=\frac{2}{\mu}>1$ as well as $q:=\frac{2}{2-\mu}>1$ we get by using Young's inequality ($\varepsilon>0$) one more time (observe $\frac{1}{p}+\frac{1}{q}=1$)

$$
c\int_\Omega\eta^2\Gamma_\delta^{s_0+\frac{\mu}{2}}\,dx\le\varepsilon\int_\Omega\eta^2\Gamma_\delta^{s_0+1}\,dx+c\varepsilon^{\frac{\mu}{\mu-2}}\int_\Omega\eta^2\Gamma_\delta^{s_0}\,dx.
$$

Hence, absorbing terms, (3.5.17) yields

$$
\begin{aligned}
&\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta,\partial_\gamma\nabla u_\delta)\eta^2\Gamma_\delta^{s_0}\,dx\\
&+c\int_\Omega D^2F_\delta(\nabla u_\delta)(\nabla\Gamma_\delta,\nabla\Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2\,dx+c\int_\Omega|\nabla u_\delta|^2\eta^2\Gamma_\delta^{s_0}\,dx\\
&\le c\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta\nabla\eta,\partial_\gamma u_\delta\nabla\eta)\Gamma_\delta^{s_0}\,dx+c\int_\Omega|\nabla\eta|^2\Gamma_\delta^{s_0}\,dx\\
&+c\int_\Omega\eta^2\Gamma_\delta^{s_0}\,dx,
\end{aligned}
\tag{3.5.18}
$$

where c, in particular, does not depend on δ. Neglecting the non-negative second and the non-negative third integral on the l.h.s. of (3.5.18) we immediately get the desired variant of Caccioppoli's inequality (3.5.7). $\square$

Next, we are going to establish the local uniform (in δ) apriori L^p-estimates of ∇u_δ for any finite exponent p, where the variant of Caccioppoli's inequality stated in Lemma 3.5.2 plays a crucial role. To become more precise we adopt the iteration argument from [17], pp.116 to our situation at hand which means that we involve the data fitting term in our calculations.

Initially we fix a ball $B_{R_0}(x_0) \Subset \Omega$ where $R_0 > 0$ denotes a real number being sufficiently small. Next, we assume that there is a real number $\alpha_0 \geq 0$ such that

$$\int\limits_{B_{R_0}(x_0)} \Gamma_\delta^{\alpha_0+\frac{1}{2}} dx + \delta \int\limits_{B_{R_0}(x_0)} \Gamma_\delta^{\alpha_0+1} dx \leq c := c(R_0, \alpha_0), \tag{3.5.19}$$

where c in particular is a finite constant independent of δ. Note that (3.5.19) is valid for $\alpha_0 = 0$ since we have

$$\begin{aligned}
&\int\limits_{B_{R_0}(x_0)} \Gamma_\delta^{\frac{1}{2}} dx + \delta \int\limits_{B_{R_0}(x_0)} \Gamma_\delta dx \\
&\leq \int\limits_{B_{R_0}(x_0)} (1 + |\nabla u_\delta|) dx + \delta \int\limits_{B_{R_0}(x_0)} \Gamma_\delta dx \\
&\leq c \int\limits_{B_{R_0}(x_0)} [1 + F_\delta(\nabla u_\delta)] dx \\
&\leq c(R_0),
\end{aligned} \tag{3.5.20}$$

where the last inequality follows from the fact that $\int\limits_\Omega F_\delta(\nabla u_\delta) dx$ is uniformly bounded in δ (compare (3.5.4)).

Now, we set $\alpha := \alpha_0 + 2 - \mu$ and choose $\varphi = \eta^2 \Gamma_\delta^\alpha u_\delta$ where $\eta \in C_0^\infty(B_{R_0}(x_0))$ satisfies $0 \leq \eta \leq 1, \eta \equiv 1$ on $B_{R_0/2}(x_0)$ and $|\nabla \eta| \leq \frac{c}{R_0}$. Inserting φ into (3.5.5) we obtain

$$\begin{aligned}
0 &= \int\limits_{B_{R_0}(x_0)} DF_\delta(\nabla u_\delta)\cdot\nabla(\eta^2\Gamma_\delta^\alpha u_\delta)dx + \lambda \int\limits_{B_{R_0}(x_0)-D} (u_\delta - f)\eta^2\Gamma_\delta^\alpha u_\delta dx \\
&= \int\limits_{B_{R_0}(x_0)} DF(\nabla u_\delta)\cdot\nabla(\eta^2\Gamma_\delta^\alpha u_\delta)dx + \delta \int\limits_{B_{R_0}(x_0)} \nabla u_\delta\cdot\nabla(\eta^2\Gamma_\delta^\alpha u_\delta)dx \\
&+ \lambda \int\limits_{B_{R_0}(x_0)-D} (u_\delta - f)\eta^2\Gamma_\delta^\alpha u_\delta dx \\
&= \int\limits_{B_{R_0}(x_0)} DF(\nabla u_\delta)\cdot\nabla u_\delta\,\eta^2\Gamma_\delta^\alpha dx + 2\int\limits_{B_{R_0}(x_0)} DF(\nabla u_\delta)\cdot\nabla\eta\,\Gamma_\delta^\alpha\eta u_\delta dx \\
&+ \alpha \int\limits_{B_{R_0}(x_0)} DF(\nabla u_\delta)\cdot\nabla\Gamma_\delta\,\eta^2 u_\delta\Gamma_\delta^{\alpha-1}dx + \delta\int\limits_{B_{R_0}(x_0)} |\nabla u_\delta|^2\eta^2\Gamma_\delta^\alpha dx \\
&+ 2\delta \int\limits_{B_{R_0}(x_0)} \nabla u_\delta\cdot\nabla\eta\,\Gamma_\delta^\alpha u_\delta\eta dx + \delta\alpha\int\limits_{B_{R_0}(x_0)} \nabla u_\delta\cdot\nabla\Gamma_\delta\,\eta^2\Gamma_\delta^{\alpha-1}u_\delta dx \\
&+ \lambda \int\limits_{B_{R_0}(x_0)-D} (u_\delta - f)\eta^2\Gamma_\delta^\alpha u_\delta dx.
\end{aligned} \tag{3.5.21}$$

Recalling again $0 \le u_\delta \le 1$, $0 \le f \le 1$ a.e. and using the boundedness of DF (see (3.1.6)) we get from (3.5.21) by exploiting $|\nabla u_\delta| \le \Gamma_\delta^{\frac{1}{2}}$ as well as $|\nabla\Gamma_\delta| \le c|\nabla u_\delta||\nabla^2 u_\delta|$

$$\begin{aligned}
&\int\limits_{B_{R_0}(x_0)} DF(\nabla u_\delta)\cdot\nabla u_\delta\eta^2\Gamma_\delta^\alpha dx + \delta\int\limits_{B_{R_0}(x_0)} |\nabla u_\delta|^2\eta^2\Gamma_\delta^\alpha dx \\
&\le c\int\limits_{B_{R_0}(x_0)} |\nabla\eta|\eta\Gamma_\delta^\alpha dx + c(\alpha)\int\limits_{B_{R_0}(x_0)} |\nabla^2 u_\delta|\eta^2\Gamma_\delta^{\alpha-\frac{1}{2}}dx \\
&+ c\delta\int\limits_{B_{R_0}(x_0)} |\nabla\eta|\Gamma_\delta^{\alpha+\frac{1}{2}}\eta dx + c(\alpha)\delta\int\limits_{B_{R_0}(x_0)} |\nabla^2 u_\delta|\eta^2\Gamma_\delta^\alpha dx \\
&+ c\int\limits_{B_{R_0}(x_0)} \eta^2\Gamma_\delta^\alpha dx.
\end{aligned} \tag{3.5.22}$$

In view of Lemma 3.1.3 (i) we may estimate the l.h.s. of (3.5.22) as follows

$$\begin{aligned}
&\int\limits_{B_{R_0}(x_0)} DF(\nabla u_\delta)\cdot\nabla u_\delta\eta^2\Gamma_\delta^\alpha dx + \delta\int\limits_{B_{R_0}(x_0)} |\nabla u_\delta|^2\eta^2\Gamma_\delta^\alpha dx \\
&\ge \nu_1\int\limits_{B_{R_0}(x_0)} \eta^2\Gamma_\delta^{\alpha+\frac{1}{2}}dx - c\int\limits_{B_{R_0}(x_0)} \eta^2\Gamma_\delta^\alpha dx
\end{aligned}$$

$$+ \delta \int\limits_{B_{R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha+1} dx - \delta \int\limits_{B_{R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha} dx.$$

Fixing $\varepsilon > 0$ and using Young's inequality, we have for the r.h.s. of (3.5.22)

$$\begin{aligned}
\text{r.h.s.} \leq\; & c\varepsilon \int\limits_{B_{R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha+\frac{1}{2}} dx + c\varepsilon^{-1} \int\limits_{B_{R_0}(x_0)} |\nabla \eta|^2 \Gamma_\delta^{\alpha-\frac{1}{2}} dx \\
& + c\varepsilon \int\limits_{B_{R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha+\frac{1}{2}} dx + c\varepsilon^{-1} \int\limits_{B_{R_0}(x_0)} |\nabla^2 u_\delta|^2 \eta^2 \Gamma_\delta^{\alpha-\frac{3}{2}} dx \\
& + c\delta\varepsilon \int\limits_{B_{R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha+1} dx + c\delta\varepsilon^{-1} \int\limits_{B_{R_0}(x_0)} |\nabla \eta|^2 \Gamma_\delta^{\alpha} dx \\
& + c\delta\varepsilon \int\limits_{B_{R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha+1} dx + c\delta\varepsilon^{-1} \int\limits_{B_{R_0}(x_0)} |\nabla^2 u_\delta|^2 \eta^2 \Gamma_\delta^{\alpha-1} dx \\
& + c \int\limits_{B_{R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha} dx.
\end{aligned}$$

Hence, by absorbing terms (choose $\varepsilon > 0$ sufficiently small), (3.5.22) implies

$$\begin{aligned}
& \int\limits_{B_{R_0/2}(x_0)} \Gamma_\delta^{\alpha+\frac{1}{2}} dx + \delta \int\limits_{B_{R_0/2}(x_0)} \Gamma_\delta^{\alpha+1} dx \\
& \leq c\Bigg[\int\limits_{B_{R_0}(x_0)} |\nabla \eta|^2 \Gamma_\delta^{\alpha-\frac{1}{2}} dx + \int\limits_{B_{R_0}(x_0)} |\nabla^2 u_\delta|^2 \eta^2 \Gamma_\delta^{\alpha-\frac{3}{2}} dx \\
& + \int\limits_{B_{R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha} dx \Bigg] \\
& + c\delta\Bigg[\int\limits_{B_{R_0}(x_0)} |\nabla \eta|^2 \Gamma_\delta^{\alpha} dx + \int\limits_{B_{R_0}(x_0)} |\nabla^2 u_\delta|^2 \eta^2 \Gamma_\delta^{\alpha-1} dx \\
& + \int\limits_{B_{R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha} dx \Bigg] \\
& =: c \sum_{j=1}^{3} I_j + c\delta \sum_{j=4}^{6} I_j.
\end{aligned} \tag{3.5.23}$$

Next, we want to discuss the integrals on the r.h.s. of (3.5.23) in detail. Starting with I_1, we recall that by definition of α it holds $\alpha - \frac{1}{2} = \alpha_0 + \frac{3}{2} - \mu \leq \alpha_0 + \frac{1}{2}$. As a consequence it follows

$$\int\limits_{B_{R_0}(x_0)} |\nabla \eta|^2 \Gamma_\delta^{\alpha-\frac{1}{2}} dx \leq c(R_0) \int\limits_{B_{R_0}(x_0)} \Gamma_\delta^{\alpha_0+\frac{1}{2}} dx \leq c$$

on account of (3.5.19) where c does not depend on δ.

Since we may assume w.l.o.g. that $\mu \geq \frac{3}{2}$ it holds $\alpha \leq \alpha_0 + \frac{1}{2}$. Hence, an upper bound for $I_3, \delta I_4$ and δI_6, which is not depending on δ, can easily be found on account of (3.5.19).

Studying I_2 we state that by definition of α we have $\alpha + \frac{\mu}{2} - \frac{3}{2} \leq \alpha_0$ and since $\alpha_0 \geq 0$, Lemma 3.5.2 and (3.1.7) give

$$\begin{aligned}
I_2 &\leq \int\limits_{B_{R_0}(x_0)} |\nabla^2 u_\delta|^2 \eta^2 \Gamma_\delta^{\alpha_0 - \frac{\mu}{2}} dx \\
&\leq c \int\limits_{B_{R_0}(x_0)} D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \eta^2 \Gamma_\delta^{\alpha_0} dx \\
&\leq c \int\limits_{B_{R_0}(x_0)} D^2 F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \nabla \eta, \partial_\gamma u_\delta \nabla \eta) \Gamma_\delta^{\alpha_0} dx \\
&+ c \int\limits_{B_{R_0}(x_0)} |\nabla \eta|^2 \Gamma_\delta^{\alpha_0} dx + c \int\limits_{B_{R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha_0} dx \\
&\leq c(R_0) \int\limits_{B_{R_0}(x_0)} \left[\Gamma_\delta^{-\frac{1}{2}} + \delta\right] \Gamma_\delta^{1+\alpha_0} dx + c(R_0) \int\limits_{B_{R_0}(x_0)} \Gamma_\delta^{\alpha_0 + \frac{1}{2}} dx \\
&+ c \int\limits_{B_{R_0}(x_0)} \Gamma_\delta^{\alpha_0 + \frac{1}{2}} dx \\
&\leq c(R_0, \alpha_0)
\end{aligned}$$

where the last inequality holds in accordance with (3.5.19), thus we have found an upper bound of I_2 not depending on δ.

Proceeding with δI_5 we distinguish between two cases whereby we first assume that $\alpha \leq 1$. It follows on account of (3.1.7) and Lemma 3.5.2

$$\begin{aligned}
\delta I_5 &\leq \delta \int\limits_{B_{R_0}(x_0)} |\nabla^2 u_\delta|^2 \eta^2 dx \\
&\leq c \int\limits_{B_{R_0}(x_0)} D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \eta^2 dx \\
&\leq c \int\limits_{B_{R_0}(x_0)} D^2 F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \nabla \eta, \partial_\gamma u_\delta \nabla \eta) dx + c(R_0) \\
&\leq c(R_0) \int\limits_{B_{R_0}(x_0)} \left[\Gamma_\delta^{-\frac{1}{2}} + \delta\right] \Gamma_\delta dx + c(R_0) \\
&\leq c(R_0)
\end{aligned}$$

where the last inequality is valid by taking into account (3.5.19) and (3.5.20), respectively. In particular, the upper bound of I_5 does not depend on δ in this case.

Assuming $\alpha > 1$ now, (3.1.7) and Lemma 3.5.2 yield

$$\begin{aligned}
\delta I_5 &\le c \int\limits_{B_{R_0}(x_0)} D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 \Gamma_\delta^{\alpha-1} dx \\
&\le c \int\limits_{B_{R_0}(x_0)} D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \nabla \eta, \partial_\gamma u_\delta \nabla \eta)\Gamma_\delta^{\alpha-1} dx + c \int\limits_{B_{R_0}(x_0)} |\nabla \eta|^2 \Gamma_\delta^{\alpha-1} dx \\
&+ c \int\limits_{B_{R_0}(x_0)} \eta^2 \Gamma_\delta^{\alpha-1} dx \\
&\le c(R_0) \int\limits_{B_{R_0}(x_0)} \Big[\Gamma_\delta^{-\frac{1}{2}} + \delta\Big]\Gamma_\delta^{\alpha} dx + c(R_0) \int\limits_{B_{R_0}(x_0)} \Gamma_\delta^{\alpha-1} dx + c \int\limits_{B_{R_0}(x_0)} \Gamma_\delta^{\alpha-1} dx \\
&\le c(R_0, \alpha_0).
\end{aligned}$$

where once more we took into account (3.5.19) by exploiting $\alpha - \frac{1}{2} \le \alpha_0 + \frac{1}{2}$. Summarizing, we have proved that δI_5 is bounded from above by a constant being independent of δ.

Altogether, by means of (3.5.23) we established the following statement: suppose that (3.5.19) holds for some given $R_0 > 0$ and $\alpha_0 \ge 0$. Then there is a constant which is not depending on δ with

$$\int\limits_{B_{R_0/2}(x_0)} \Gamma_\delta^{\alpha_0+2-\mu+\frac{1}{2}} dx + \delta \int\limits_{B_{R_0/2}(x_0)} \Gamma_\delta^{\alpha_0+2-\mu+1} dx \le c. \tag{3.5.24}$$

We now prove by induction that for any $m \in \mathbb{N}$ there is a constant $c(m) > 0$, independent of δ, such that for all $\delta \in (0,1)$

$$\int\limits_{B_{R_0/2^m}(x_0)} \Gamma_\delta^{m(2-\mu)+\frac{1}{2}} dx + \delta \int\limits_{B_{R_0/2^m}(x_0)} \Gamma_\delta^{m(2-\mu)+1} dx \le c \tag{3.5.25}$$

Since (3.5.19) holds for $\alpha_0 = 0$, it follows that $\alpha_0 = 0$ is also an admissible choice in (3.5.24). Thus, (3.5.25) extends to $m = 1$.
Next we assume by induction that (3.5.25) is true for some $m \in \mathbb{N}$. As a consequence, $\alpha_0 = m(2-\mu)$ serves as an admissible choice in (3.5.19) and (3.5.24) leads to

$$\begin{aligned}
&\int\limits_{B_{R_0/2^{m+1}}(x_0)} \Gamma_\delta^{(m+1)(2-\mu)+\frac{1}{2}} dx + \delta \int\limits_{B_{R_0/2^{m+1}}(x_0)} \Gamma_\delta^{(m+1)(2-\mu)+1} dx \\
\le &\int\limits_{B_{R_0/2}(x_0)} \Gamma_\delta^{\alpha_0+2-\mu+\frac{1}{2}} dx + \delta \int\limits_{B_{R_0/2}(x_0)} \Gamma_\delta^{\alpha_0+2-\mu+1} dx
\end{aligned}$$

$$\leq c$$

where we like to stress that c does not depend on δ. Hence, (3.5.25) remains valid for any choice of $m \in \mathbb{N}$.

Next, we let $\omega \Subset \Omega$ and denote by $p \in (1, \infty)$ some number. Then there exists another number $m = m(p) \in \mathbb{N}$ with $p \leq 1 + 2m(2 - \mu)$ and a finite number of balls $B_{R_i}(x_i) \Subset \Omega$ $(i = 1, \dots, M)$ such that

$$\omega \Subset \bigcup_{i=1}^{M} B_{\rho_i}(x_i) \subset \Omega \quad \text{where } \rho_i := \frac{R_i}{2^m}.$$

Making use of (3.5.25) we infer

$$\|\nabla u_\delta\|_{p,\omega}^p \leq \sum_{i=1}^{M} \int_{B_{\rho_i}(x_i)} \Gamma_\delta^{\frac{p}{2}} dx \leq \sum_{i=1}^{M} \int_{B_{\rho_i}(x_i)} \Gamma_\delta^{m(2-\mu)+\frac{1}{2}} dx \leq c(p, \omega),$$

where the local constant $c(p, \omega)$ in particular is independent of δ. This proves the local uniform p-integrability of ∇u_δ w.r.t. δ for any finite exponent p and therewith Lemma 3.5.1. □

Remark 3.5.3
Considering the case $n = 2$, we directly obtain Lemma 3.5.1 by quoting Remark 3.4.4.

Step 2. Caccioppoli-type inequality

As the second step we are going to establish another Caccioppoli-type inequality which in particular is valid for any $\mu > 1$. This variant of Caccioppoli's inequality acts as an important tool during the De Giorgi-type iteration which will be carried out in the subsequent third step. We emphasize that in case $n = 2$, this type of Caccioppoli's inequality has already been established in [27], Lemma 3, and that the arguments for deducing this inequality do not rely on the dimension n. Nevertheless, we give the proof once again for the sake of completeness. Originally, the variant of Caccioppoli's inequality is used from [17], Lemma 4.19 (ii), pp.108, in connection with "free problems". Our major effort consists in incorporating the inpainting quantity which apriori causes some difficulties.
Initially, we introduce some notation. We fix a point $x_0 \in \Omega$ and consider radii $0 < r < R < R_0$ with $B_{R_0}(x_0) \Subset \Omega$.
Moreover, we let $A_\delta(k, R) := \{x \in B_R(x_0) : \Gamma_\delta > k\}$ where $k > 0$ and Γ_δ denotes the function from Lemma 3.5.2. Further we consider $\eta \in C_0^\infty(B_R(x_0))$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_r(x_0)$ and $|\nabla \eta| \leq \frac{c}{R-r}$. Finally, for functions $v : \Omega \to \mathbb{R}$ we denote $\max\{v, 0\}$ by v^+. We then have the following variant of Caccioppoli's inequality, which, in fact, is valid for any $\mu > 1$.

Lemma 3.5.4
With the previous notation and in particular for any $\mu > 1$ it holds

$$\begin{aligned}
&\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{-\mu}{2}} |\nabla\Gamma_\delta|^2 \eta^2 dx \\
&\leq c \int\limits_{A_\delta(k,R)} |D^2 F_\delta(\nabla u_\delta)||\nabla\eta|^2 (\Gamma_\delta - k)^2 dx + c \int\limits_{A_\delta(k,R)} \eta^2 |\nabla u_\delta|^{2+\mu} dx \\
&+ c \int\limits_{A_\delta(k,R)} \eta |\nabla\eta||\nabla u_\delta|^3 dx \\
&\leq \frac{c}{(R-r)^2} \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\nu}{2}} dx
\end{aligned} \tag{3.5.26}$$

where $\nu := \max\{4, 2+\mu\}$ and c denotes a suitable positive constant independent of δ, r and R.

Remark 3.5.5
As already outlined in [27], Remark 5, the choice of ν in Lemma 3.5.4 is not optimal. In fact, if we regularize the density F by virtue of

$$F_{\delta,q}(P) := \frac{\delta}{q}|P|^q + F(P), \quad P \in \mathbb{R}^n,$$

with $q > 1$ being sufficiently close to 1, we may choose any $\nu > \max\{3, 2+\mu\} = 2+\mu$.

Proof of Lemma 3.5.4. We note that the second inequality follows from the first since w.l.o.g. we may assume $R_0 \leq 1$ implying $1 < \frac{1}{R-r}$, and $k \geq 2$, i.e. $\Gamma_\delta - k \leq \Gamma_\delta \leq c|\nabla u_\delta|^2 \leq c\Gamma_\delta$ on $A_\delta(k,R)$. We also recall $\mu > 1$, thus $3 < 2+\mu$ and finally we use the following inequalities on $A_\delta(k,R)$

$$\begin{aligned}
|D^2 F(\nabla u_\delta)|(\Gamma_\delta - k)^2 &\leq c|\nabla u_\delta|^3, \\
\delta(\Gamma_\delta - k)^2 &\leq c|\nabla u_\delta|^4.
\end{aligned}$$

Now we prove the first inequality in (3.5.26): as already seen in the proof of Theorem 3.1.15 we can state

$$\int\limits_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla\varphi) dx = \int\limits_{\Omega - D} \lambda(u_\delta - f)\partial_\gamma \varphi dx$$

for all $\varphi \in W^{1,2}(\Omega)$ with compact support in Ω.
Observing that $\varphi = \eta^2 \partial_\gamma u_\delta (\Gamma_\delta - k)^+$ is admissible we get (from now on sum-

mation w.r.t. $\gamma \in \{1, \dots, n\}$)

$$\begin{aligned}
&\int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)(\Gamma_\delta - k)\eta^2 dx \\
&+ \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \nabla \Gamma_\delta)\eta^2 dx \\
&+ 2 \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \nabla \eta)\eta(\Gamma_\delta - k) dx \\
&= \int\limits_{B_R(x_0)-D} \lambda(u_\delta - f)\partial_\gamma[\eta^2 \partial_\gamma u_\delta (\Gamma_\delta - k)^+] dx.
\end{aligned} \tag{3.5.27}$$

For the second integral on the l.h.s. it holds

$$\begin{aligned}
&\int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u \nabla \Gamma_\delta)\eta^2 dx \\
&= \frac{1}{2} \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \Gamma_\delta, \nabla \Gamma_\delta)\eta^2 dx.
\end{aligned} \tag{3.5.28}$$

In accordance with (3.5.28) we also have for the third integral on the l.h.s. of (3.5.27)

$$\begin{aligned}
&\int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \nabla \eta)\eta(\Gamma_\delta - k) dx \\
&= \frac{1}{2} \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \eta, \nabla \Gamma_\delta)\eta(\Gamma_\delta - k) dx.
\end{aligned} \tag{3.5.29}$$

Summarizing, (3.5.27)-(3.5.29) imply with the help of the Cauchy-Schwarz inequality applied to the bilinear form $D^2F_\delta(\nabla u_\delta)$ and after the use of Young's inequality ($\varepsilon > 0$)

$$\begin{aligned}
&\int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)(\Gamma_\delta - k)\eta^2 dx \\
&+ \frac{1}{2} \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \Gamma_\delta, \nabla \Gamma_\delta)\eta^2 dx \\
&\leq \varepsilon \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \Gamma_\delta, \nabla \Gamma_\delta)\eta^2 dx \\
&+ \varepsilon^{-1} \int\limits_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla \eta, \nabla \eta)(\Gamma_\delta - k)^2 dx \\
&+ \int\limits_{B_R(x_0)-D} \lambda(u_\delta - f)\partial_\gamma[\eta^2 \partial_\gamma u_\delta (\Gamma_\delta - k)^+] dx.
\end{aligned} \tag{3.5.30}$$

In what follows, we concentrate on the last integral on the r.h.s. of (3.5.30), which is denoted by I_1. Recalling $0 \le u_\delta \le 1$, $0 \le f \le 1$ a.e., we get

$$\begin{aligned} I_1 \le c \int_{A_\delta(k,R)} |\nabla\eta|\eta|\nabla u_\delta|(\Gamma_\delta - k)dx + c \int_{A_\delta(k,R)} \eta^2|\nabla^2 u_\delta|(\Gamma_\delta - k)dx \\ + c \int_{A_\delta(k,R)} \eta^2|\nabla u_\delta||\nabla\Gamma_\delta|dx. \end{aligned} \tag{3.5.31}$$

Another application of Young's inequality ($\varepsilon > 0$) gives

$$\begin{aligned} \int_{A_\delta(k,R)} \eta^2|\nabla^2 u_\delta|(\Gamma_\delta - k)dx \le \varepsilon \int_{A_\delta(k,R)} \eta^2|\nabla^2 u_\delta|^2(\Gamma_\delta - k)\Gamma_\delta^{\frac{-\mu}{2}}\,dx \\ + \varepsilon^{-1} \int_{A_\delta(k,R)} \eta^2(\Gamma_\delta - k)\Gamma_\delta^{\frac{\mu}{2}}\,dx \end{aligned} \tag{3.5.32}$$

as well as

$$\begin{aligned} &\int_{A_\delta(k,R)} \eta^2|\nabla u_\delta||\nabla\Gamma_\delta|dx \\ &\le \varepsilon \int_{A_\delta(k,R)} \eta^2|\nabla\Gamma_\delta|^2\Gamma_\delta^{\frac{-\mu}{2}}\,dx + \varepsilon^{-1} \int_{A_\delta(k,R)} \eta^2|\nabla u_\delta|^2\Gamma_\delta^{\frac{\mu}{2}}\,dx. \end{aligned} \tag{3.5.33}$$

Recalling $k \ge 2$ we have $|\nabla u_\delta| \ge 1$ on $A_\delta(k,R)$ and therefore $\Gamma_\delta \le c|\nabla u_\delta|^2$ on $A_\delta(k,R)$. Incorporating (3.5.32) and (3.5.33) in (3.5.31) we get

$$\begin{aligned} I_1 \le c \int_{A_\delta(k,R)} |\nabla\eta|\eta|\nabla u_\delta|^3 dx + c\varepsilon \int_{A_\delta(k,R)} \eta^2|\nabla^2 u_\delta|^2(\Gamma_\delta - k)\Gamma_\delta^{\frac{-\mu}{2}}\,dx \\ + c\varepsilon \int_{A_\delta(k,R)} \eta^2|\nabla\Gamma_\delta|^2\Gamma_\delta^{\frac{-\mu}{2}}\,dx + c\varepsilon^{-1} \int_{A_\delta(k,R)} \eta^2|\nabla u_\delta|^{2+\mu}dx. \end{aligned} \tag{3.5.34}$$

Connecting (3.5.34) with (3.5.30) it follows after absorbing terms (we choose $\varepsilon > 0$ sufficiently small) und using (3.1.7)

$$\begin{aligned} &\int_{A_\delta(k,R)} \eta^2|\nabla^2 u_\delta|^2(\Gamma_\delta - k)\Gamma_\delta^{\frac{-\mu}{2}}\,dx \\ &+ \int_{A_\delta(k,R)} \eta^2|\nabla\Gamma_\delta|^2\Gamma_\delta^{\frac{-\mu}{2}}\,dx \\ &\le c \int_{A_\delta(k,R)} D^2F_\delta(\nabla u_\delta)(\nabla\eta,\nabla\eta)(\Gamma_\delta - k)^2 dx \\ &+ c \int_{A_\delta(k,R)} |\nabla\eta|\eta|\nabla u_\delta|^3 dx + c \int_{A_\delta(k,R)} \eta^2|\nabla u_\delta|^{2+\mu}dx, \end{aligned} \tag{3.5.35}$$

which proves the first inequality in (3.5.26) by neglecting the nonnegative first integral on the l.h.s. of (3.5.35). □

Step 3. De Giorgi-type iteration

The third step is devoted to the derivation of local uniform (in δ) apriori gradient bounds of u_δ. In this context the variant of Caccioppoli's inequality that has been deduced in the second step as well as the well-known Lemma of Stampacchia (see, e.g., [91], Lemma 5.1, p.219 or [71], Lemma B.1, p. 63) act as essential tools when performing a De Giorgi-type iteration.
Actually we are going to prove a De Giorgi-type lemma that provides a sufficient condition in order to close the gap between local uniform $\overline{p}$-integrability of the gradients for a certain exponent $\overline{p}$ and local uniform apriori gradient bounds. Hence, concerning future problems or applications, respectively, it might be of interest to take note of this sufficient condition that is formulated in the following

Lemma 3.5.6
Suppose that v_δ is a sequence of class $W^{2,2}_{loc}(\Omega)$ and that we are given real numbers $\overline{p}, \nu > 3$, $\mu > 1$ satisfying

$$\frac{\mu+\nu}{2} n < \overline{p}.$$

Moreover, suppose that we have a uniform constant $c > 0$ (with $\Gamma_\delta := 1 + |\nabla v_\delta|^2$ and $A_\delta(k,R), r, R, R_0, \eta$ as before) such that it holds

$$\int\limits_{A_\delta(k,R)} \Gamma_\delta^{-\frac{\mu}{2}} |\nabla \Gamma_\delta|^2 \eta^2 dx \le \frac{c}{(R-r)^2} \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\nu}{2}} dx \tag{3.5.36}$$

and assume in addition that ∇v_δ is locally $\overline{p}$-integrable uniformly in δ, i.e.

$$\sup_\delta \int\limits_{\Omega'} |\nabla v_\delta|^{\overline{p}} dx = c(\overline{p}, \Omega') < \infty, \tag{3.5.37}$$

where $\Omega' \Subset \Omega$. Then it holds $\nabla v_\delta \in L^\infty_{loc}(\Omega)^n$ uniformly in δ.

Having the inpainting model at hand, we may even use (3.5.6) for any finite p but as already mentioned in [27], a replacement of condition (3.5.37) from Lemma 3.5.6 by (3.5.6) does not simplify the following proof in an essential way. Precisely we can state an immediate conclusion of Lemma 3.5.6.

Proposition 3.5.7
Suppose that $n \ge 2$, $\mu \in (1,2)$, and that v_δ denotes the approximating sequence from Lemma 3.4.1 to the inpainting model under consideration. Then we have local uniform (in δ) apriori gradient bounds for v_δ.

Remark 3.5.8
Note that Lemma 3.5.6 has already been established in [27] (compare Lemma 4 in this reference) in the case $n = 2$ where the essenial arguments that have been given during the proof of Lemma 4 in this reference actually do not rely on the dimension $n \geq 2$. In fact, more or less, we can adapt the entire proof given in [27] where the only effort consists in adjusting the exponents to the case of arbitrary dimension n.

Remark 3.5.9
Considering the inpainting model with $n \geq 3$, assuming $\mu \in (1, 2)$ and denoting by u_δ the approximating sequence from Lemma 3.4.1, we could show that apriori, we have local uniform (in δ) $L^{\overline{p}}$-estimates of ∇u_δ for all $1 \leq \overline{p} \leq \frac{2n}{n-2}$ (see Remark 3.4.4). Consulting Lemma 3.5.6 now, it turns out that this initial local uniform starting integrability of ∇u_δ is not sufficient in order to derive uniform local apriori gradient bounds by citing Lemma 3.5.6 since we have to require $\overline{p} > 2n$ at least (notice that Lemma 3.5.4 provides a variant of Caccioppoli's inequality being in the spirit of (3.5.36)*). Consequently, we had to show higher local uniform (in δ) $\overline{p}$-integrability of ∇u_δ at least up to the fixed exponent $\frac{\mu+\nu}{2}n + \varepsilon$ with $\varepsilon > 0$ sufficiently small before quoting Lemma 3.5.6 for getting uniform (in δ) local apriori gradient bounds for u_δ. In fact, in Lemma 3.5.1, we could even show local uniform (in δ) $\overline{p}$-integrability of ∇u_δ for any finite exponent $\overline{p}$.*

Proof of Lemma 3.5.6. For proving Lemma 3.5.6 we adopt techniques as already applied in [27], proof of Lemma 4, and note that the following De Giorgi-iteration represents a substantial refinement of the one carried out in [17], Theorem 4.28, pp.119.

As in [27], we primarily establish a technical proposition being of pure algebraic nature. Its proof is given in the appendix of this thesis.

Proposition 3.5.10
Consider real numbers $\overline{p}, \nu > 3, \mu > 1$ with

$$\frac{\mu + \nu}{2} n < \overline{p}. \tag{3.5.38}$$

Then, there exist real numbers $s_1, s_2, s_3 > 1$ such that

(i) $2\dfrac{s_1}{s_1 - 1} < \overline{p}$, *(ii)* $\dfrac{1}{s_1}\dfrac{n}{n-1} > 1$,

(iii) $\mu\dfrac{s_2}{s_2 - 1} < \overline{p}$, *(iv)* $\nu\dfrac{s_3}{s_3 - 1} < \overline{p}$,

(v) $\dfrac{1}{2}\dfrac{n}{n-1}\left(\dfrac{1}{s_3} + \dfrac{1}{s_2}\right) > 1.$

Now, we start proving Lemma 3.5.6. Recalling the previous notation and applying Sobolev's inequality we get

$$\int\limits_{A_\delta(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1}} dx \le \int\limits_{B_R(x_0)} (\eta(\Gamma_\delta - k)^+)^{\frac{n}{n-1}} dx$$
$$\le c\Big(\int\limits_{B_R(x_0)} |\nabla[\eta(\Gamma_\delta - k)^+]| dx\Big)^{\frac{n}{n-1}}.$$

Moreover it holds

$$\begin{aligned} c\Big(\int\limits_{B_R(x_0)} |\nabla[\eta(\Gamma_\delta - k)^+]| dx\Big)^{\frac{n}{n-1}} &= c\Big(\int\limits_{A_\delta(k,R)} |\nabla[\eta(\Gamma_\delta - k)]| dx\Big)^{\frac{n}{n-1}} \\ &\le c\Big(\int\limits_{A_\delta(k,R)} |\nabla\eta|(\Gamma_\delta - k) dx\Big)^{\frac{n}{n-1}} \\ &\quad + c\Big(\int\limits_{A_\delta(k,R)} \eta|\nabla\Gamma_\delta| dx\Big)^{\frac{n}{n-1}} \\ &=: c\Big[I_1^{\frac{n}{n-1}} + I_2^{\frac{n}{n-1}}\Big], \end{aligned}$$

hence

$$\int\limits_{A_\delta(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1}} dx \le c\Big[I_1^{\frac{n}{n-1}} + I_2^{\frac{n}{n-1}}\Big]. \tag{3.5.39}$$

At this point, we are going to use the algebraic Proposition 3.5.10 with the same parameters as given in Lemma 3.5.6. Since we may assume the validity of (3.5.38) we get existence of real numbers $s_i > 1$, $i = 1, 2, 3$ fulfilling the claims (i)–(v) of Proposition 3.5.10.
Incorporating (3.5.37) we obtain $\Gamma_\delta - k \in L^{\frac{\overline{p}}{2}}(B_R(x_0))$ uniformly in δ. In accordance with Proposition 3.5.10, (i) and (3.5.37), we may therefore conclude $\Gamma_\delta - k \in L^{\frac{s_1}{s_1-1}}(B_R(x_0))$ uniformly in δ. By using Hölder's inequality it follows

$$\begin{aligned} I_1^{\frac{n}{n-1}} &= \Big(\int\limits_{A_\delta(k,R)} |\nabla\eta|(\Gamma_\delta - k) dx\Big)^{\frac{n}{n-1}} \\ &\le \frac{c}{(R-r)^{\frac{n}{n-1}}} (\mathcal{L}^n(A_\delta(k,R)))^{\frac{n}{n-1}\frac{1}{s_1}} \Big(\int\limits_{A_\delta(k,R)} (\Gamma_\delta - k)^{\frac{s_1}{s_1-1}} dx\Big)^{\frac{n}{n-1}\frac{s_1-1}{s_1}} \\ &\le \frac{c}{(R-r)^{\frac{n}{n-1}}} (\mathcal{L}^n(A_\delta(k,R)))^{\frac{n}{n-1}\frac{1}{s_1}} \end{aligned}$$

and since on account of Proposition 3.5.10, (ii), there exists a real number $\overline{\beta} := \frac{n}{n-1}\frac{1}{s_1} > 1$, we see

$$I_1^{\frac{n}{n-1}} \le \frac{c}{(R-r)^{\frac{n}{n-1}}} (\mathcal{L}^n(A_\delta(k,R)))^{\overline{\beta}}. \tag{3.5.40}$$

Next we discuss I_2: applying Hölder's inequality and the special type of Caccioppoli's inequality stated in (3.5.36) we get

$$\begin{aligned} I_2^{\frac{n}{n-1}} &\le \Bigg[\int\limits_{A_\delta(k,R)} \eta^2 |\nabla \Gamma_\delta|^2 \Gamma_\delta^{\frac{-\mu}{2}}\, dx \Bigg]^{\frac{1}{2}\frac{n}{n-1}} \Bigg[\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\mu}{2}}\, dx \Bigg]^{\frac{1}{2}\frac{n}{n-1}} \\ &\le \Bigg[\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\mu}{2}}\, dx \Bigg]^{\frac{1}{2}\frac{n}{n-1}} \Bigg[\frac{c}{(R-r)^2} \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\nu}{2}}\, dx \Bigg]^{\frac{1}{2}\frac{n}{n-1}} . \end{aligned} \tag{3.5.41}$$

Quoting (3.5.37) once again we have $\Gamma_\delta^{\frac{\mu}{2}} \in L^{\frac{\overline{p}}{\mu}}(B_R(x_0))$ uniformly in δ and by using Proposition 3.5.10, (iii) and (3.5.37), we get $\Gamma_\delta^{\frac{\mu}{2}} \in L^{\frac{s_2}{s_2-1}}(B_R(x_0))$ uniformly in δ. With Hölder's inequality we may estimate

$$\begin{aligned} &\Bigg[\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\mu}{2}}\, dx \Bigg]^{\frac{1}{2}\frac{n}{n-1}} \\ &\le \Bigg(\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\mu s_2}{2(s_2-1)}}\, dx \Bigg)^{\frac{1}{2}\frac{n}{n-1}\frac{s_2-1}{s_2}} \mathcal{L}^n(A_\delta(k,R))^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s_2}} \\ &\le c\mathcal{L}^n(A_\delta(k,R))^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s_2}} . \end{aligned} \tag{3.5.42}$$

Furthermore, by means of (3.5.37), it follows $\Gamma_\delta^{\frac{\nu}{2}} \in L^{\frac{\overline{p}}{\nu}}(B_R(x_0))$ uniformly in δ. Taking Proposition 3.5.10, (iv) and (3.5.37) into account it holds $\Gamma_\delta^{\frac{\nu}{2}} \in L^{\frac{s_3}{s_3-1}}(B_R(x_0))$ uniformly in δ and Hölder's inequality implies

$$\begin{aligned} \Bigg[\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\nu}{2}}\, dx \Bigg]^{\frac{1}{2}\frac{n}{n-1}} &\le \Bigg[\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\nu}{2}\frac{s_3}{s_3-1}}\, dx \Bigg]^{\frac{1}{2}\frac{n}{n-1}\frac{s_3-1}{s_3}} \mathcal{L}^n(A_\delta(k,R))^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s_3}} \\ &\le c\mathcal{L}^n(A_\delta(k,R))^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s_3}} . \end{aligned} \tag{3.5.43}$$

Putting (3.5.41) - (3.5.43) together and using Proposition 3.5.10, (v) we infer the existence of a real number $\widetilde{\beta} := \frac{1}{2}\frac{n}{n-1}(\frac{1}{s_2} + \frac{1}{s_3}) > 1$ such that

$$I_2^{\frac{n}{n-1}} \le \frac{c}{(R-r)^{\frac{n}{n-1}}} \mathcal{L}^n(A_\delta(k,R))^{\widetilde{\beta}} . \tag{3.5.44}$$

Assuming w.l.o.g. that $\mathcal{L}^n(A_\delta(k,R)) < 1$, (3.5.39), (3.5.40) and (3.5.44) imply the existence of a real number $\beta > 1$ with

$$\int\limits_{A_\delta(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1}}\, dx \le \frac{c}{(R-r)^{\frac{n}{n-1}}} \mathcal{L}^n(A_\delta(k,R))^{\beta} . \tag{3.5.45}$$

Now we define the following quantities for $k \ge 2$ and $r < R$:

$$\tau_\delta(k,r) := \int\limits_{A_\delta(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1}}\, dx, \quad a_\delta(k,r) := \mathcal{L}^n(A_\delta(k,r)).$$

Furthermore, suppose that two real numbers h, k with $h > k > 2$ are given, i.e. we have $\frac{\Gamma_\delta - k}{h-k} \geq 1$ on $A_\delta(h, R)$. Consequently it holds

$$a_\delta(h, R) \leq \int\limits_{A_\delta(h,R)} (\Gamma_\delta - k)^{\frac{n}{n-1}}(h-k)^{-\frac{n}{n-1}}dx,$$

thus

$$a_\delta(h, R) \leq \frac{1}{(h-k)^{\frac{n}{n-1}}}\tau_\delta(k, R). \tag{3.5.46}$$

From (3.5.45) and (3.5.46) it follows

$$\tau_\delta(h, r) \leq \frac{c}{(R-r)^\gamma (h-k)^\alpha}\tau_\delta(k, R)^\beta \tag{3.5.47}$$

where

$$\gamma := \frac{n}{n-1} > 0,\ \alpha := \frac{n}{n-1}\beta > 0,\ \beta > 1. \tag{3.5.48}$$

Having (3.5.47) and (3.5.48) at hand we may apply Stampacchia's well-known lemma and obtain local uniform (in δ) apriori gradient bounds of v_δ. To become more precise, an application of Stampacchia's lemma guarantees existence of a positive quantity d_δ such that for all $\sigma \in (0, 1)$ we get

$$\tau_\delta(d_\delta + k_0, R_0 - \sigma R_0) = 0$$

with

$$d_\delta^\alpha = \frac{2^{\frac{(\alpha+\beta)\beta}{\beta-1}}C}{\sigma^\gamma R_0^\gamma}[\tau_\delta(k_0, R_0)]^{\beta-1} \leq d^\alpha,$$

where d is a constant not depending on δ since we may use (3.5.37) (recall $\overline{p} > 2n$). Choosing $k_0 = 2$ and $\sigma = \frac{1}{2}$ we infer

$$0 = \tau_\delta(d_\delta + 2, R_0/2) \geq \tau_\delta(d + 2, R_0/2) \geq 0,$$

i.e. it holds

$$\tau_\delta(d + 2, R_0/2) = 0. \tag{3.5.49}$$

Condition (3.5.49) finally leads to the uniform estimate

$$|\nabla v_\delta| \leq c$$

a.e. on $B_{R_0/2}(x_0)$ for all $\delta \in (0, 1)$ where c in particular is independent of δ since $\Gamma_\delta \leq d + 2$ a.e. on $B_{R_0/2}(x_0)$.

Using a covering argument, we finally get

$$\|\nabla v_\delta\|_{L^\infty(\omega)} \leq c(\omega)$$

for all $\omega \Subset \Omega$ and $\delta \in (0, 1)$, i.e. v_δ is locally uniformly Lipschitz continuous with Lipschitz constant $c(\omega) > 0$. This completes the proof of Lemma 3.5.6. $\square$

Step 4. Conclusions

Taking the assumption $\mu \in (1, 2)$ into account, an application of Proposition 3.5.7 gives $\nabla u_\delta \in L^\infty_{\mathrm{loc}}(\Omega)^n$ uniformly in δ. Quoting Remark 3.4.6 we know $u_\delta \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and since u_δ is locally uniformly (in δ) Lipschitz continuous, we may apply Arzelà-Ascoli's theorem to get $u \in C^{0,1}(\Omega)$.

In a final step in the proof of Theorem 3.1.19 in the scalar case we are going to close the gap between local Lipschitz continuity of u and Hölder continuous first partial derivatives of u in Ω.

Letting $g := \lambda \mathbb{1}_{\Omega - D}(u - f)$ we recall ($\gamma \in \{1, \dots, n\}$) the equation

$$\int_\Omega DF(\nabla u) \cdot \partial_\gamma \nabla \psi dx = - \int_\Omega g \partial_\gamma \psi dx \tag{3.5.50}$$

valid for all $\psi \in C_0^\infty(\Omega)$. Using the information that u is Lipschitz continuous, we may argue with the standard difference quotient technique to establish $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$. Thus, we get after performing an integration by parts in (3.5.50)

$$\int_\Omega D^2F(\nabla u)(\partial_\gamma \nabla u, \nabla \psi) dx = \int_\Omega g \partial_\gamma \psi dx.$$

for all $\psi \in C_0^\infty(\Omega)$. Setting $v := \partial_\gamma u$, we then see

$$\int_\Omega D^2F(\nabla u)(\nabla v, \nabla \psi) dx = \int_\Omega g \partial_\gamma \psi dx.$$

where the coefficients $a_{\alpha\beta}(x) := \frac{\partial^2 F}{\partial p_\alpha \partial p_\beta}(\nabla u)$ are strictly elliptic and bounded on any subdomain $\omega \Subset \Omega$ (this fact follows from (3.1.7) and from the local Lipschitz continuity of u). Finally, [62], Theorem 8.22, p.200 ensures interior Hölder continuity of v and therefore of $\partial_\gamma u$ for all $\gamma \in \{1, \dots, n\}$, i.e., u has locally Hölder continuous first partial derivatives in Ω. This completes the proof of Theorem 3.1.19 in the scalar case. □

3.5.2 Proof of Theorem 3.1.19: the vectorial case

Throughout the entire section we assume that the hypotheses of Theorem 3.1.19 hold with $M > 1$ and denote by $u \in W^{1,1}(\Omega)^M \cap L^\infty(\Omega)^M$ the unique I-minimizer from Theorem 3.1.15. As already done in the scalar case, the following proof is divided into four parts: regularization and local uniform apriori L^p-estimates, Caccioppoli-type inequality, DeGiorgi-type iteration and the conclusions.

Step 1. Regularization and local uniform apriori L^p-estimates

We apply the regularization from Theorem 3.1.15, i.e., we consider the functional (recall $F_\delta(P) := \frac{\delta}{2}|P|^2 + F(P)$, $P \in \mathbb{R}^{nM}$)

$$I_\delta[w] := \int\limits_\Omega F_\delta(\nabla w)dx + \frac{\lambda}{2}\int\limits_{\Omega - D} |w-f|^2 dx, \quad w \in W^{1,2}(\Omega)^M, \tag{3.5.51}$$

and denote by $u_\delta \in W^{1,2}(\Omega)^M$ its unique minimizer which solves the Euler equation

$$\int\limits_\Omega DF_\delta(\nabla u_\delta) : \nabla\varphi dx + \lambda \int\limits_{\Omega - D} (u_\delta - f)\cdot \varphi dx = 0 \tag{3.5.52}$$

for all $\varphi \in C_0^\infty(\Omega)^M$.
Further, we make use of the following regularity and convergence properties of u_δ:

$$\sup_\Omega |u_\delta| \le \sup_{\Omega - D} |f|, \tag{3.5.53}$$

$$u_\delta \in W^{2,2}_{\mathrm{loc}}(\Omega)^M, \tag{3.5.54}$$

$$u_\delta \in W^{1,\infty}_{\mathrm{loc}}(\Omega)^M, \tag{3.5.55}$$

$$u_\delta \to u \quad \text{in } L^1(\Omega)^M. \tag{3.5.56}$$

Remark 3.5.11
The properties (3.5.53) *and* (3.5.54) *have already been stated in Lemma 3.4.1. A proof of* (3.5.53)–(3.5.55) *will be postponed to the appendix of this thesis. For* (3.5.56) *we refer to Remark 3.4.6.*

Now let us prove that we have local uniform apriori L^p-estimates of ∇u_δ in the vectorial setting as well.

Lemma 3.5.12
For any $1 < p < \infty$ and for any $\omega \Subset \Omega$ there is a constant $c(p,\omega)$, which in particular does not depend on δ, such that

$$\|\nabla u_\delta\|_{L^p(\omega)} \le c(p,\omega) < \infty. \tag{3.5.57}$$

Proof of Lemma 3.5.12. As in Section 3.5.1 we first derive a suitable variant of Caccioppoli's inequality being only valid for $\mu \in (1,2)$. It is worth mentioning that this variant of Caccioppoli's inequality is the vectorial analogon of the one being stated in Lemma 3.5.2.

Lemma 3.5.13
For any $s_0 \geq 0$ there exists a real number $c > 0$ such that for all $\eta \in C_0^\infty(\Omega)$ satisfying $0 \leq \eta \leq 1$ and for any $\delta \in (0,1)$ it holds

$$
\begin{aligned}
&\int_\Omega |\nabla^2 u_\delta|^2 \Gamma_\delta^{s_0 - \frac{\mu}{2}} \eta^2 dx + \delta \int_\Omega |\nabla^2 u_\delta|^2 \Gamma_\delta^{s_0} \eta^2 dx \\
&\leq \int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \eta^2 \Gamma_\delta^{s_0} dx \\
&\leq c \int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \otimes \nabla \eta, \partial_\gamma u_\delta \otimes \nabla \eta) \Gamma_\delta^{s_0} dx + c \int_\Omega |\nabla \eta|^2 \Gamma_\delta^{s_0} dx \\
&+ c \int_\Omega \eta^2 \Gamma_\delta^{s_0} dx,
\end{aligned}
\tag{3.5.58}
$$

where we again have set $\Gamma_\delta := 1 + |\nabla u_\delta|^2$ and the positive constant c, in particular, is independent of δ.

Proof of Lemma 3.5.13. Note that the first inequality follows from (3.1.7) and that for the case $s_0 = 0$ we refer to Lemma 3.4.3, i.e., we assume $s_0 > 0$ in the following. For proving the second inequality in (3.5.58) we may essentially follow the arguments as given in the proof of Lemma 3.5.2 with minor adjustments. The Euler equation (3.5.52) as usual yields ($\gamma \in \{1, \dots, n\}$)

$$
\int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla \varphi) dx = \lambda \int_{\Omega - D} (u_\delta - f) \cdot \partial_\gamma \varphi dx. \tag{3.5.59}
$$

being valid for all $\varphi \in W^{1,2}(\Omega)^M$ with compact support in Ω. From (3.5.54) and (3.5.55) we see that $\varphi = \eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}$ is admissible in the differentiated Euler equation (3.5.59) and we arrive at (from now on summation w.r.t. γ)

$$
\begin{aligned}
&\int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \eta^2 \Gamma_\delta^{s_0} dx \\
&+ s_0 \int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \Gamma_\delta) \Gamma_\delta^{s_0 - 1} \eta^2 dx \\
&= -2 \int_\Omega D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \eta) \eta \Gamma_\delta^{s_0} dx \\
&+ \lambda \int_{\Omega - D} (u_\delta - f) \cdot \partial_\gamma (\eta^2 \partial_\gamma u_\delta \Gamma_\delta^{s_0}) dx.
\end{aligned}
\tag{3.5.60}
$$

Studying the last integral on the l.h.s. of (3.5.60) we obtain by using the structure condition (3.1.15)

$$
\begin{aligned}
&D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \Gamma_\delta) \\
&= \frac{1}{2} D^2 F_\delta(\nabla u_\delta)(e_j \otimes \nabla \Gamma_\delta, e_j \otimes \nabla \Gamma_\delta) \geq 0 \quad \text{a.e.,}
\end{aligned}
\tag{3.5.61}
$$

where e_j denotes the j^{th} coordinate vector.
For any $\varepsilon > 0$ we obtain for the first integral on the r.h.s. of (3.5.60) using Young's inequality

$$\begin{aligned}
&2\Big|\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \nabla\eta)\eta\Gamma_\delta^{s_0}\,dx\Big| \\
&\leq \varepsilon \int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2\Gamma_\delta^{s_0}\,dx \qquad (3.5.62)\\
&+ 4\varepsilon^{-1}\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \otimes \nabla\eta, \partial_\gamma u_\delta \otimes \nabla\eta)\Gamma_\delta^{s_0}\,dx.
\end{aligned}$$

Finally, let us discuss the last integral on the r.h.s. of (3.5.60). Using the boundedness of f and u_δ (see (3.1.4) as well as (3.5.53)), an integration by parts gives

$$\begin{aligned}
&\lambda \int_{\Omega - D} (u_\delta - f)\cdot \partial_\gamma(\eta^2\partial_\gamma u_\delta \Gamma_\delta^{s_0})\,dx \\
&= -\lambda\int_\Omega |\nabla u_\delta|^2\eta^2\Gamma_\delta^{s_0}\,dx - \lambda\int_\Omega f\cdot\partial_\gamma(\eta^2\partial_\gamma u_\delta\Gamma_\delta^{s_0})\,dx \\
&- \lambda\int_{\Omega\cap D}(u_\delta - f)\cdot\partial_\gamma(\eta^2\partial_\gamma u_\delta\Gamma_\delta^{s_0})\,dx \qquad (3.5.63)\\
&\leq -\lambda\int_\Omega |\nabla u_\delta|^2\eta^2\Gamma_\delta^{s_0}\,dx + c\int_\Omega |\nabla\eta||\nabla u_\delta|\eta\Gamma_\delta^{s_0}\,dx \\
&+ c(s_0)\int_\Omega |\nabla^2 u_\delta|\eta^2\Gamma_\delta^{s_0}\,dx.
\end{aligned}$$

Applying Young's inequality for a given $\varepsilon > 0$ and quoting (3.5.61), (3.5.62) as well as (3.5.63), (3.5.60) turns into

$$\begin{aligned}
&\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta, \partial_\gamma\nabla u_\delta)\eta^2\Gamma_\delta^{s_0}\,dx + \lambda\int_\Omega |\nabla u_\delta|^2\eta^2\Gamma_\delta^{s_0}\,dx \\
&+ \frac{s_0}{2}\int_\Omega D^2F_\delta(\nabla u_\delta)(e_j\otimes\nabla\Gamma_\delta, e_j\otimes\nabla\Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2\,dx \\
&\leq \varepsilon\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta, \partial_\gamma\nabla u_\delta)\eta^2\Gamma_\delta^{s_0}\,dx \\
&+ 4\varepsilon^{-1}\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta\otimes\nabla\eta, \partial_\gamma u_\delta\otimes\nabla\eta)\Gamma_\delta^{s_0}\,dx \\
&+ \varepsilon\int_\Omega \eta^2|\nabla u_\delta|^2\Gamma_\delta^{s_0}\,dx + c\varepsilon^{-1}\int_\Omega |\nabla\eta|^2\Gamma_\delta^{s_0}\,dx \\
&+ c\varepsilon\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma\nabla u_\delta, \partial_\gamma\nabla u_\delta)\eta^2\Gamma_\delta^{s_0}\,dx + c\varepsilon^{-1}\int_\Omega \eta^2\Gamma_\delta^{s_0+\frac{\mu}{2}}\,dx.
\end{aligned}$$

Absorbing terms by choosing $\varepsilon > 0$ sufficiently small, we then obtain

$$\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2\Gamma_\delta^{s_0} dx + c\int_\Omega |\nabla u_\delta|^2\eta^2\Gamma_\delta^{s_0} dx$$
$$+ c\frac{s_0}{2}\int_\Omega D^2F_\delta(\nabla u_\delta)(e_j \otimes \nabla\Gamma_\delta, e_j \otimes \nabla\Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2 dx$$
$$\leq c\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \otimes \nabla\eta, \partial_\gamma u_\delta \otimes \nabla\eta)\Gamma_\delta^{s_0} dx$$
$$+ c\int_\Omega |\nabla\eta|^2\Gamma_\delta^{s_0} dx + c\int_\Omega \eta^2\Gamma_\delta^{s_0+\frac{\mu}{2}} dx.$$

Recalling $\mu < 2$ we may apply Young's inequality one more time to see

$$\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2\Gamma_\delta^{s_0} dx + c\int_\Omega |\nabla u_\delta|^2\eta^2\Gamma_\delta^{s_0} dx$$
$$+ c\frac{s_0}{2}\int_\Omega D^2F_\delta(\nabla u_\delta)(e_j \otimes \nabla\Gamma_\delta, e_j \otimes \nabla\Gamma_\delta)\Gamma_\delta^{s_0-1}\eta^2 dx$$
$$\leq c\int_\Omega D^2F_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \otimes \nabla\eta, \partial_\gamma u_\delta \otimes \nabla\eta)\Gamma_\delta^{s_0} dx$$
$$+ c\int_\Omega |\nabla\eta|^2\Gamma_\delta^{s_0} dx + c\int_\Omega \eta^2\Gamma_\delta^{s_0} dx.$$

This gives the desired inequality (3.5.58) after neglecting the non-negative second and third integral on the l.h.s. □

Now, by means of Lemma 3.5.13, the same iteration procedure as already carried out in the scalar case $M = 1$ (compare the arguments after the proof of Lemma 3.5.2) can be performed for proving Lemma 3.5.12 and therewith local uniform (in δ) apriori L^p-gradient bounds for ∇u_δ. □

Step 2. Caccioppoli-type inequality

In this step we provide another variant of Caccioppoli's inequality which acts as a crucial tool when carrying out a De Giorgi-type iteration. With the notation that has already been introduced in the scalar case $M = 1$ (compare the remarks before Lemma 3.5.4) the following Caccioppoli-type inequality can be established by following the lines of the proof of Lemma 3.5.4. It is worth remarking that we have to make use of the structure condition (3.1.15) imposed on the density F.

Lemma 3.5.14
With the previous notation and under the assumptions of Lemma 3.5.12, in particular for any $\mu > 1$, we have the following variant of Caccioppoli's inequality

$$\begin{aligned}
&\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{-\mu}{2}} |\nabla\Gamma_\delta|^2\eta^2 dx \\
&\leq c \int\limits_{A_\delta(k,R)} |D^2F_\delta(\nabla u_\delta)||\nabla\eta|^2(\Gamma_\delta - k)^2 dx + c \int\limits_{A_\delta(k,R)} \eta^2|\nabla u_\delta|^{2+\mu}dx \\
&+ c \int\limits_{A_\delta(k,R)} \eta|\nabla\eta||\nabla u_\delta|^3 dx \\
&\leq \frac{c}{(R-r)^2} \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\nu}{2}} dx
\end{aligned}$$

where $\nu := \max\{4, 2+\mu\}$ and for a suitable positive constant c independent of δ, r and R.

Remark 3.5.15
Note that the choice of ν is not optimal. We refer to Remark 3.5.5.

<u>*Step 3. De Giorgi-type iteration*</u>

The third step is devoted to the derivation of local uniform (in δ) apriori gradient bounds for u_δ. As already done in the scalar case $M = 1$ (see Lemma 3.5.6) we formulate a De Giorgi-type lemma that provides a sufficient condition in order to close the gap between local uniform $\overline{p}$-integrability of the gradients for a certain exponent $\overline{p}$ and local uniform apriori gradient bounds. A proof of this lemma can be deduced by using the arguments from the proof of Lemma 3.5.6.

Lemma 3.5.16
Suppose that v_δ is a sequence of class $W^{2,2}_{loc}(\Omega)^M$ and that we are given real numbers $\overline{p}, \nu > 3$, $\mu > 1$ satisfying

$$\frac{\mu+\nu}{2} n < \overline{p}. \tag{3.5.64}$$

Moreover, suppose that we have a uniform constant $c > 0$ (with $\Gamma_\delta := 1 + |\nabla v_\delta|^2$ and $A_\delta(k,R), r, R, R_0, \eta$ as usual) such that it holds

$$\int\limits_{A_\delta(k,R)} \Gamma_\delta^{-\frac{\mu}{2}} |\nabla\Gamma_\delta|^2\eta^2 dx \leq \frac{c}{(R-r)^2} \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\nu}{2}} dx \tag{3.5.65}$$

and assume in addition that ∇v_δ is locally $\overline{p}$-integrable uniformly in δ, i.e.

$$\sup_\delta \int\limits_{\Omega'} |\nabla v_\delta|^{\overline{p}} dx = c(\overline{p}, \Omega') < \infty, \tag{3.5.66}$$

where $\Omega' \Subset \Omega$. Then it holds $\nabla v_\delta \in L^\infty_{loc}(\Omega)^{nM}$ uniformly in δ.

Step 4. Conclusions

Recalling the important restriction $\mu \in (1,2)$ we may apply Lemma 3.5.16 to the unique I_δ-minimizer u_δ: in fact, Lemma 3.5.12 gives local uniform (in δ) L^p-estimates of ∇u_δ for any finite exponent $p > 1$ (note that we assume $\mu \in (1,2)$), i.e., (3.5.66) from Lemma 3.5.16 is valid for any finite $\overline{p} > 1$. Due to Lemma 3.5.14 we further have a variant of Caccioppoli's inequality at hand as required in (3.5.65) from Lemma 3.5.16 with $\nu = 4$ (recall $\mu \in (1,2)$ again). Finally, (3.5.64) from Lemma 3.5.16 is trivially satisfied.
As a consequence, Lemma 3.5.16 provides local uniform (in δ) apriori gradient bounds of u_δ, i.e., u_δ is locally uniformly (in δ) Lipschitz continuous. With the help of the convergence property (3.5.56) from above we apply Arzelà-Ascoli's theorem to get $u \in C^{0,1}(\Omega)^M$, where we recall that u denotes the solution from Theorem 3.1.15.

In the last step of the proof of Theorem 3.1.19 we are going to close the gap between local Lipschitz continuity of u and local Hölder continuity of ∇u in Ω. In contrast to the previous three steps it is not possible to benefit from the procedure given in the scalar case (see Section 3.5.1, "Step 4. Conclusions") since the arguments that have been presented there rely on an application of the well-known theory of DeGiorgi, Moser and Nash (compare, e.g., [62], Theorem 8.22, p.200).

For overcoming this problem, our goal is to prove that u is a local minimizer of a properly defined autonomous isotropic variational problem for which full interior $C^{1,\alpha}$-regularity of all its local minimizers is established.
To become more precise, we now let $\omega \Subset \Omega$ be arbitrary and set $K := K(\omega) := \|\nabla u\|_{L^\infty(\omega)}$. Following [78] we modify our density F and consider the auxiliary integrand $\widetilde{F} : \mathbb{R}^{nM} \to [0,\infty)$

$$\widetilde{F}(P) := \nu + 1 + F(P) + (|P|^2 - 4K^2)^{+3},$$

for some constant $\nu > 0$.
Some straight forward calculations show $\widetilde{F} \in C^2(\mathbb{R}^{nM})$ and that $\widetilde{F}$ satisfies the following isotropic growth and ellipticity conditions (among other things, the Hölder condition (3.1.16) imposed on D^2F is crucially used)

$$\begin{aligned} \nu_1(1+|P|^2)^2|Q|^2 \le D^2\widetilde{F}(P)(Q,Q) \le \nu_2(1+|P|^2)^2||Q|^2, \\ |D^2\widetilde{F}(P) - D^2\widetilde{F}(Q)| \le \nu_3(1+|P|^2+|Q|^2)^{2-\lambda/2}|P-Q|^\lambda, \end{aligned}$$

for all $P, Q \in \mathbb{R}^{nM}$ with some positive constants ν_1, ν_2, ν_3 and λ. Due to (3.1.15) it additionally holds $\widetilde{F}(P) = \widetilde{g}(|P|^2)$ for a function $\widetilde{g} \in C^2([0,\infty),[0,\infty))$.

In the next step we consider the functional

$$\widetilde{I}[w] := \int_\Omega \widetilde{F}(\nabla w)dx + \frac{\lambda}{2}\int_{\Omega - D} |w-f|^2 dx$$

which is well-defined for functions $w \in W^{1,6}(\Omega)^M$.
In particular, we see that $u \in C^{0,1}(\Omega)^M \subset W^{1,6}_{\mathrm{loc}}(\Omega)^M$ is a solution of the Euler equation (recall $g := \lambda \mathbb{1}_{\Omega - D}(u-f)$)

$$\int_\Omega D\widetilde{F}(\nabla u) : \nabla\varphi dx + \int_\Omega g\cdot\varphi dx = 0 \tag{3.5.67}$$

for all $\varphi \in C_0^\infty(\Omega)^M$ since $D\widetilde{F}(\nabla u) = DF(\nabla u)$ a.e. on the set $\omega := \mathrm{spt}\ \varphi \Subset \Omega$ (recall $|\nabla u| \leq K$ a.e. on ω). Thus, u is a local $\widetilde{I}$-minimizer but for getting full interior $C^{1,\alpha}$-regularity of local $\widetilde{I}$-minimizers we cannot directly quote classical references as e.g. Giaquinta/Modica [59] or Uhlenbeck [99] due to the presence of the data fitting term in Euler's equation (3.5.67). However, it is worth remarking that there exist generalizations of the results of Uhlenbeck [99] or Giaquinta/Modica [59] in the sense that a suitable right-hand side in Euler's equation is considered (see, e.g., [69] and [96]) but unfortunately, we cannot find a rigorous quotation which exactly covers our situation from (3.5.67).
Thus, we decide to give an own proof of full interior $C^{1,\alpha}$-regularity of local $\widetilde{I}$-minimizers in the fourth chapter of this thesis (see the proof of Theorem 4.1.7 in Section 4.5) where we note that we will actually prove $C^{1,\alpha}$-regularity of local minimizers of a properly defined class of non-autonomous isotropic variational problems. Nonetheless, we can apply Theorem 4.1.7 to our (autonomous) setting choosing $t = 6$ therein and we obtain that the local $\widetilde{I}$-minimizer u is of class $C^{1,\alpha}(\Omega)^M$ for any $\alpha \in (0,1)$. This completes the proof of Theorem 3.1.19 in the vectorial setting $M > 1$ as well. $\square$

Chapter 4

A modified TV-regularization: the non-autonomous case

4.1 The basic setup and statement of the main results

In this chapter we are concerned with a class of non-autonomous minimization problems that can be used in image processing. Suppose that we are given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n, n \geq 2$, as well as an observed image being corrupted by noise stemming from transmission or measuring errors. As usual, this image will be described through a $\mathcal{L}^n$-measurable function $f : \Omega \to \mathbb{R}^M$ for which we require

$$f \in L^\infty(\Omega)^M. \tag{4.1.1}$$

We then introduce the problem (as usual, $\lambda > 0$ denotes a regularization parameter)

$$\begin{aligned} I[w] := \int_\Omega |w - f|^2 dx + \lambda \int_\Omega F(x, \nabla w) dx \to \min \\ \text{in } W^{1,1}(\Omega)^M \cap L^2(\Omega)^M, \end{aligned} \tag{4.1.2}$$

for a given density $F : \overline{\Omega} \times \mathbb{R}^{nM} \to [0, \infty)$ being of class $C^2(\overline{\Omega} \times \mathbb{R}^{nM})$, being of uniform (in x) linear growth w.r.t. the second variable, and which satisfies some appropriate uniform (in x) ellipticity conditions w.r.t. the second variable that will be specified later (see Remark 4.1.5). At this point we recall the continuous embedding $W^{1,1}(\Omega)^M \hookrightarrow L^{\frac{n}{n-1}}(\Omega)^M$, which means that the additional constraint $w \in L^2(\Omega)^M$ is automatically satisfied for functions $w \in W^{1,1}(\Omega)^M$ if $n = 2$. Note that the study of problem (4.1.2), at least in the scalar case $M = 1$ together with $n = 2$, has been suggested in [28], where the authors worked with a particular choice of the density $F(x, P)$.

Taking the results from [28] as a basis, this chapter is devoted to the discussion of problem (4.1.2) for any dimension n together with arbitrary codimension M. Among justifying existence of generalized solutions of problem (4.1.2) (w.r.t. a suitable relaxation) we are particularly concerned with the derivation of regularity properties of such minimizers.
In general, when investigating regularity of (local) minimizers of non-autonomous variational problems we have to be careful in the sense that we cannot expect that the results that we obtain for the autonomous case extend to the non-autonomous setting as well. For instance, let us consider the following example of a functional ($1 < p < q < \infty$) being of non-standard growth (see [77], Section 5)

$$\widetilde{J}[w] := \int_{\Omega} \Big(|\nabla w|^p + a(x)|\nabla w|^q \Big) dx.$$

Here, $a : \Omega \to [0, \infty)$ denotes a function satisfying $0 \leq a(x) \leq M$ for a positive constant M independent of x. Moreover, the integrand $\mathcal{F}(x, Z) := |Z|^p + a(x)|Z|^q$ with $x \in \Omega$ and $Z \in \mathbb{R}^{nM}$ is of anisotropic (p, q)-growth ($1 < p < q < \infty$) w.r.t. to the second argument, i.e., it holds

$$c_1|Z|^p - c_2 \leq \mathcal{F}(x, Z) \leq c_3|Z|^q + c_4 \tag{4.1.3}$$

with suitable uniform (in x) constants $c_1, c_3 > 0$ and $c_2, c_4 \in \mathbb{R}$. As outlined in [77], Section 6.5, the presence of the function $a(x)$ crucially influences the global growth behavior of the integrand $\mathcal{F}$: in fact, when keeping x fixed and varying Z, it becomes evident that the integrand $\mathcal{F}$ satisfies some standard growth conditions while $\mathcal{F}$ globally satisfies a non-standard (p, q)-growth condition in the sense of (4.1.3).
In general, when investigating the regularity of local minimizers of functionals

$$J[w] := \int_{\Omega} H(\cdot, \nabla w) dx$$

with density $H : \overline{\Omega} \times \mathbb{R}^{nM}$ of class C^2 and which is supposed to satisfy a (p, q)-growth condition in the sense of (4.1.3), the initial question arises whether local minimizers being apriori of class $W^{1,p}_{\mathrm{loc}}(\Omega)^M$ in fact are of class $W^{1,q}_{\mathrm{loc}}(\Omega)^M$. As stated in [77], Section 6.5, and as illustrated in the above example, in contrast to the case $p = q$, the situation crucially changes since the presence of the independent variable x cannot be treated as a perturbation anymore (see, e.g., [43], where local perturbation methods are used in the case $p = q$). However, under some additional hypotheses on H (among other things, $D_P H(x, P)$ is supposed to be α-Hölder continuous w.r.t. x), Esposito, Leonetti and Mingione have stated in [51] a sufficient condition for deriving local q-integrability of a local J-minimizer u: if the condition (recall that α denotes the Hölder exponent of $D_P H(x, P)$)

$$\frac{q}{p} < \frac{n + \alpha}{n}$$

is satisfied, then a sufficient requirement for local higher integrability is given by

$$\mathcal{L}(u, B_R) = 0 \quad \text{for all balls } B_R \Subset \Omega.$$

Here, $\mathcal{L}$ denotes the so-called Lavrentiev gap functional relative to the given energy defined through

$$\mathcal{L} = \inf_{u_0+W_0^{1,p}(B_R)^M} J - \inf_{u_0+W_0^{1,q}(B_R)^M} J$$

on a ball $B_R \Subset \Omega$ and with boundary data $u_0 \in W^{1,p}(B_R)^M$. Conversely, if the above sufficient condition is not true, then local minimizers are irregular. Furthermore, Esposito, Leonetti and Mingione have stated some counterexamples w.r.t. the irregularity of local minimizers that show the sharpness of their results. In contrast, if we consider autonomous anisotropic variational problems where the density H satisfies the same assumptions as above, then the weaker sufficient condition

$$\frac{q}{p} < \frac{n+2}{n}$$

for deriving regularity of local minimizers was established by M. Bildhauer and M. Fuchs in [18]. Thus, if we study regularity of local minimizers of anisotropic variational problems it actually makes a difference if we involve a smooth x-dependence for the density H or not.
For more details concerning the regularity of (local) minimizers of non-autonomous anisotropic variational integrals we mention, without being complete, the contributions of Bildhauer and Fuchs [20], Breit [32] and Esposito, Leonetti and Mingione [51] as well as the references quoted therein. Finally we remark that non-autonomous anisotropic energies apply in physics and in the theory of electrorheological fluids.

After this short excursion to non-autonomous anisotropic variational problems we return to the investigation of the problem (4.1.2) and note that such a class of problems is not covered by the contributions to non-autonomous variational problems with (p, q)-growth due to the requirement $p > 1$ there.
From the point of view of image processing, the minimization of the appropriate relaxation of the functional I from (4.1.2) to the space $BV(\Omega)^M$ describes a non-autonomous modification of the well-known TV-regularization. As already elucidated in the third chapter of this thesis (compare (3.1.17)), a special (autonomous) modification of the TV-regularization is of particular interest: for a fixed number $\mu > 1$ we define the following family of densities

$$F_\mu(P) := \int_0^{\sqrt{\varepsilon+|P|^2}} \int_0^s (1+r)^{-\mu} dr ds, \quad P \in \mathbb{R}^{nM}, \tag{4.1.4}$$

and observe that F_μ approximates the TV-density $|P|$ as follows (see (3.1.18))

$$\lim_{\mu\to\infty} (\mu-1)F_\mu(P) = \sqrt{\varepsilon+|P|^2}, \quad \varepsilon > 0. \tag{4.1.5}$$

Here, the free parameter $\varepsilon > 0$ provides some additional flexibility when approximating the TV-density. The above approximation property suggests to seek solutions of the following (autonomous) modification of the TV-regularization:

$$\lambda \int_\Omega F_\mu(\nabla w) + \int_\Omega |w-f|^2 dx \to \min \text{ in } BV(\Omega)^M \cap L^2(\Omega)^M.$$

Here, we recall the continuous embedding $BV(\Omega)^M \hookrightarrow L^{\frac{n}{n-1}}(\Omega)^M$, which means that the additional constraint $w \in L^2(\Omega)^M$ is automatically satisfied for functions $w \in BV(\Omega)^M$, provided $n = 2$. It is worth remarking that in contrast to the standard TV-model, the above model has the clear advantage of a comprehensive existence and regularity theory of BV-minimizers. Here, we refer to the previous chapters of this thesis, where we note that all results obviously extend to the case $D = \emptyset$ which corresponds to "pure denoising of images".

Up to this point, the exponent $\mu > 1$ arising in the integrand F_μ of the above functional is a fixed real number and we conjecture that this aspect could be too inflexible for certain applications in image processing. As suggested in [28] we therefore study densities $F_{\mu(x)}$ with variable exponents $\mu(x)$ generating functionals of linear growth in what follows. This concept provides more flexibility in the sense that it is possible to work with different values of μ on prescribed subregions of Ω. Thus, we expect that solutions show a different degree of regularity on prescribed zones of Ω.

After the above preparations we now fix our setup and specify our assumptions: assume that we are given a function $\mu = \mu(x)$ of class $C^2(\overline{\Omega})$ satisfying

$$\mu(x) \in (1, \infty), \quad x \in \overline{\Omega}. \tag{4.1.6}$$

Now we define $F_{\mu(x)}(P)$ in accordance with (4.1.4) and particularly obtain the validity of the formula (4.1.5) for each $x \in \overline{\Omega}$. Setting $F(x, P) := F_{\mu(x)}(P)$ in (4.1.2) we then look at the problem

$$\begin{aligned} J[w] := \int_\Omega |w-f|^2 dx + \lambda \int_\Omega F_{\mu(x)}(\nabla w) dx \to \min \\ \text{in } W^{1,1}(\Omega)^M \cap L^2(\Omega)^M, \end{aligned} \tag{4.1.7}$$

where for notational and technical simplicity we assume w.l.o.g. $\lambda = 1$ in our following discussions. In general, we cannot expect solvability of the problem (4.1.7) since the Sobolev space $W^{1,1}(\Omega)^M$ is not reflexive. Nonetheless we can consider a suitable relaxation of the functional J from (4.1.7) to the more adequate space $BV(\Omega)^M \cap L^2(\Omega)^M$: from (4.1.6) we first derive existence of some numbers $1 < \mu_0 \le \mu_1 < \infty$ such that

$$\mu(x) \in [\mu_0, \mu_1], \quad x \in \overline{\Omega}. \tag{4.1.8}$$

Afterwards, by using (4.1.8), it is not hard to check that the density $F_{\mu(x)}(P)$ satisfies the requirements (i)–(iv) stated on p. 312 in [7] and in accordance with [7], Theorem 5.54, p. 312, we let for $w \in BV(\Omega)^M \cap L^2(\Omega)^M$

$$\overline{K}[w] := \int_\Omega |w-f|^2 dx + \int_\Omega F_{\mu(x)}(\nabla^a w) dx + \int_\Omega F^\infty_{\mu(x)}\Big(\frac{\nabla^s w}{|\nabla^s w|}\Big) d|\nabla^s w|,$$

where -as usual- $F^\infty_{\mu(x)}(P)$ denotes the recession function of $F_{\mu(x)}(P)$ defined via

$$F^\infty_{\mu(x)}(P) := \lim_{t\to\infty} \frac{F_{\mu(x)}(tP)}{t}.$$

In our particular case, it holds $F^\infty_{\mu(x)}(P) = \frac{|P|}{\mu(x)-1}$ which can be derived after performing some standard calculations. Summarizing, we consider the following non-autonomous variational problem as a model of pure denoising with energies of linear growth involving variable exponents

$$\begin{aligned} K[w] := \int_\Omega |w-f|^2 dx + \int_\Omega F_{\mu(x)}(\nabla^a w)dx + \int_\Omega \frac{1}{\mu(x)-1} d|\nabla^s w| \\ \to \min \quad \text{in } BV(\Omega)^M \cap L^2(\Omega)^M. \end{aligned} \tag{4.1.9}$$

Obviously, we can take problem (4.1.9) as a (non-autonomous) modification of TV-image inpainting as well by replacing the data fitting term in (4.1.9) through $\int_{\Omega-D} |w-f|^2 dx$ where D denotes the inpainting region satisfying $0 < \mathcal{L}^n(D) < \mathcal{L}^n(\Omega)$. However, one essential motivation to require $D = \emptyset$ throughout this chapter is that we get uniqueness of BV-solutions of problem (4.1.9). As we will see in the course of this chapter, the uniqueness of BV-solutions of problem (4.1.9) essentially simplifies the discussions about the regularity behavior of the BV-minimizer.

At this point we state a theorem which is concerned with problem (4.1.9): first we will prove solvability of problem (4.1.9), where the unique K-minimizer additionally satisfies a maximum principle. In part (b) we justify that the unique K-minimizer can be seen as a generalized minimizer of the original functional J from (4.1.7) while in part (c) we show that the set of generalized minimizers just contains the unique K-minimizer.

Theorem 4.1.1
Suppose that we are given a $\mathcal{L}^n$-measurable function $f : \Omega \to \mathbb{R}^M$ fulfilling (4.1.1) and let $\mu \in C^2(\overline{\Omega})$ satisfy (4.1.6). We then have:

(a) the problem (4.1.9) admits a unique solution u and this solution satisfies

$$\sup_\Omega |u| \le \sup_\Omega |f|;$$

(b) $\inf_{W^{1,1}(\Omega)^M \cap L^2(\Omega)^M} I = \inf_{BV(\Omega)^M \cap L^2(\Omega)^M} K$*;*

(c) let $\mathcal{M}$ denote the set of all $L^1(\Omega)^M$-cluster points of I-minimizing sequences from the space $W^{1,1}(\Omega)^M \cap L^2(\Omega)^M$. It then holds $\mathcal{M} = \{u\}$.

Remark 4.1.2
Considering the scalar case $M = 1$ and assuming that we are given an observed

image being described through a $\mathcal{L}^n$-measurable function $f : \Omega \to [0,1]$ we can establish a maximum principle for the unique BV-minimizer u as seen in the third chapter of this thesis (compare Theorem 3.1.5). To become more precise, we then can show that it holds $0 \le u(x) \le 1$ for a.a. $x \in \Omega$.

Now, we discuss another approach to problem (4.1.7) which is motivated by the dual formulation of variational problems in the theory of perfect plasticity (we refer to [54] for a survey). As outlined in the second chapter of this thesis, one essential motivation for the study of dual problems is the uniqueness of solutions, where in many applications the dual problem turns out as a maximization problem for a physically significant quantity.

As before, we suppose the validity of (4.1.1) and let $\mu \in C^2(\overline{\Omega})$ satisfy (4.1.6). In accordance with [49], we define the Lagrangian

$$l(v,\varkappa) := \int_\Omega [\varkappa : \nabla v - F^*(x,\varkappa)]dx + \int_\Omega |v-f|^2 dx \qquad (4.1.10)$$

for all $(v,\varkappa) \in (W^{1,1}(\Omega)^M \cap L^2(\Omega)^M, L^\infty(\Omega)^{nM})$. In this context, the function $\Phi^* : \overline{\Omega} \times \mathbb{R}^{nM} \to \overline{\mathbb{R}}$

$$F^*(x,Q) := \sup_{P \in \mathbb{R}^{nM}} [P : Q - F_{\mu(x)}(P)], \quad Q \in \mathbb{R}^{nM},$$

denotes the conjugate function of $F_{\mu(x)}(P)$ w.r.t. to the second variable. In view of [49], Proposition 2.1, p. 271, we get the following representation for $P \in L^1(\Omega)^{nM}$

$$\int_\Omega F_{\mu(x)}(P)dx = \sup_{\varkappa \in L^\infty(\Omega)^{nM}} \int_\Omega [\varkappa : P - F^*(x,\varkappa)]dx. \qquad (4.1.11)$$

From (4.1.11) we can derive another formula for the functional J from (4.1.7). To become more precise we obtain

$$J[v] = \sup_{\varkappa \in L^\infty(\Omega)^{nM}} l(v,\varkappa), \quad v \in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M$$

and now introduce the dual functional

$$R : L^\infty(\Omega)^{nM} \to [-\infty,\infty]$$
$$R[\varkappa] := \inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M} l(v,\varkappa).$$

Thus, the dual problem associated to (4.1.7) reads: to maximize R among all functions $\varkappa \in L^\infty(\Omega)^{nM}$.

After the above preparations we give our results on the dual variational problem associated to problem (4.1.7).

Theorem 4.1.3
Suppose that we are given a $\mathcal{L}^n$-measurable function $f : \Omega \to \mathbb{R}^M$ fulfilling (4.1.1) *and let $\mu \in C^2(\overline{\Omega})$ satisfy* (4.1.6)*. It then holds:*

(a) the dual problem

$$R \to max \quad in\ L^\infty(\Omega)^{nM} \tag{4.1.12}$$

with R from above admits at least one solution. Moreover, the inf-sup relation (with J from (4.1.7)*)*

$$\inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M} J[v] = \sup_{\sigma \in L^\infty(\Omega)^{nM}} R[\sigma]$$

is valid;

(b) in fact, the dual problem (4.1.12) *admits a unique solution σ and we further get the validity of the duality formula*

$$\sigma = \nabla_P F_{\mu(\cdot)}(\nabla^a u) \quad a.e.\ on\ \Omega,$$

where u denotes the unique K-minimizer from the space $BV(\Omega)^M \cap L^2(\Omega)^M$;

(c) with u from (b) we consider any J-minimizing sequence (u_m) from the space $W^{1,1}(\Omega)^M \cap L^2(\Omega)^M$. We then have

$$u_m \to u \quad in\ L^2(\Omega)^M.$$

At this stage we discuss regularity of the unique K-minimizer u from Theorem 4.1.1. At least in the scalar case $M = 1$ together with $n = 2$ it was shown in [28], Theorem 4 (ii), that $u \in C^{1,\alpha}(\Omega_2)$ for any $\alpha \in (0,1)$, where

$$\Omega_2 := \{x \in \Omega : 1 < \mu(x) < 2\} \tag{4.1.13}$$

denotes an appropriate subregion of Ω. In contrast, on prescribed subregions of Ω with large values of $\mu(x)$, the BV-minimizer u might show an irregular behavior.
Now, we are concerned with two questions:

- Can we extend the result $u \in C^{1,\alpha}(\Omega_2)$ stated in [28] in the scalar case $M = 1$ to any dimension $n \geq 2$?
- Is it possible to establish full interior $C^{1,\alpha}$-regularity of u in the vectorial case $M > 1$ with arbitrary dimension $n \geq 2$ as well?

It will turn out that we can give a positive answer to both questions where we crucially benefit from the arguments carried out in the autonomous setting in the third chapter of this thesis.

Theorem 4.1.4
Suppose that we are given a $\mathcal{L}^n$-measurable function $f : \Omega \to \mathbb{R}^M$ satisfying (4.1.1) *and let $\mu \in C^2(\overline{\Omega})$ fulfill* (4.1.6)*. We then have $u \in C^{1,\alpha}(\Omega_2)^M$ for any $\alpha \in (0,1)$, where the set Ω_2 (possibly empty depending on the choice of μ) is defined as in* (4.1.13)*.*

Remark 4.1.5
Note that by performing some straight forward calculations we can show that our model density $F(x,P) := F_{\mu(x)}(P)$ satisfies the following set of assumptions: there are uniform (in x) constants $\nu_0, \dots, \nu_3, K > 0$ and $\gamma \in (0,1)$, such that for all $x \in \overline{\Omega}$, for all $P, Q \in \mathbb{R}^{nM}$ and for $\gamma \in \{1, \dots, n\}$

(i) $\Phi_{\mu(x)}(P) = g(x, |P|^2), \quad g \in C^2(\overline{\Omega} \times \mathbb{R}^{nM})$;

(ii) $|\nabla_P F(x,P)| \le \nu_0$;

(iii) $\nu_1(1+|P|)^{-\mu(x)}|Q|^2 \le D_P^2 F(x,P)(Q,Q) \le \nu_2(1+|P|)^{-1}|Q|^2$;

(iv) $|\partial_\gamma \nabla_P F(x,P)| \le \nu_3$;

(v) $|D_P^2 F(x,P) - D_P^2 F(x,Q)| \le K|P-Q|^\gamma$;

(vi) the variational integrand $F = F(x,P)$ is of linear growth in P, uniformly w.r.t. x, i.e.

$$a|P| - b \le F(x,P) \le A|P|$$

where the constants $a, A > 0$, $b \in \mathbb{R}$, do not depend on x.

We remark that we can extend all results of this chapter to general integrands $F(x,P)$ satisfying the above conditions. Note that the theorems of this chapter partially remain true under much weaker assumptions on $F(x,P)$. In fact, we merely need the entire range of the conditions (i)–(vi) for proving full interior $C^{1,\alpha}$-regularity of the unique K-minimizer (with K from (4.1.9)*) in the vectorial case $M > 1$.*

Remark 4.1.6
In extension of the analysis started in the joint article with J. Müller [81], we can study a (more general) non-autonomous modification of the TV-regularization which means that for a fixed real number $\zeta > 1$ we seek minimizers of the functional

$$J_\zeta[w] := \int_\Omega F_{\mu(x)}(\nabla w)dx + \frac{\lambda}{\zeta}\int_\Omega |w-f|^\zeta dx$$

among functions from the space $W^{1,1}(\Omega)^M \cap L^\zeta(\Omega)^M$. Choosing $\zeta = 2$ we are in the same setting as discussed in this chapter. The corresponding relaxed version *of the functional J_ζ formulated on the space $BV(\Omega)^M$ is then given by*

$$K_\zeta[w] := \int_\Omega F_{\mu(x)}(\nabla^a w)dx + \int_\Omega \frac{1}{\mu(x)-1} d|\nabla^s w| \\ + \frac{\lambda}{\zeta}\int_\Omega |w-f|^\zeta dx, \tag{4.1.14}$$

and (4.1.14) *is well-defined for functions $w \in BV(\Omega)^M \cap L^\zeta(\Omega)^M$. Now we can state that we can transfer all results from Theorem 4.1.1 and Theorem 4.1.4 to this (slightly) more general setting whereas Theorem 4.1.3 (a) and (b) remain true as well. We further note that Theorem 4.1.1 partially holds true for the limit case $\zeta = 1$.*

We finish this chapter by establishing a full interior $C^{1,\alpha}$-regularity result for local minimizers of a properly defined class of non-autonomous isotropic variational problems, where we initially recall our setup and state our hypotheses: as usual, $\Omega \subset \mathbb{R}^n$ denotes a bounded Lipschitz domain and $D \subset \Omega$ is a $\mathcal{L}^n$-measurable subset satisfying

$$0 \le \mathcal{L}^n(D) < \mathcal{L}^n(\Omega). \tag{4.1.15}$$

Moreover, $f : \Omega - D \to \mathbb{R}^M$, $M \ge 1$, is a given $\mathcal{L}^n$-measurable function for which we assume

$$f \in L^\infty(\Omega - D)^M. \tag{4.1.16}$$

Furthermore, we suppose that we are given a function $H : \overline{\Omega} \times \mathbb{R}^{nM} \to [0,\infty)$ of class C^2 satisfying the following set of uniform (in x) isotropic ellipticity and growth conditions for some given $2 \le t < \infty$, for all $P, U \in \mathbb{R}^{nM}$ and with positive constants λ, Λ, c

$$\lambda(1+|P|^2)^{\frac{t-2}{2}}|U|^2 \le D_P^2 H(x,P)(U,U) \le \Lambda(1+|P|^2)^{\frac{t-2}{2}}|U|^2, \tag{4.1.17}$$

$$|\nabla_x \nabla_P H(x,P)| \le c(1+|P|^2)^{\frac{t-1}{2}}, \tag{4.1.18}$$

$$H(x,P) = h(x,|P|^2). \tag{4.1.19}$$

Here, $h : \overline{\Omega} \times \mathbb{R}^{nM} \to [0,\infty)$ is a function of class C^2. Moreover we suppose that for $\nu > 0$ and all $P, Q \in \mathbb{R}^{nM}$ it holds

$$|D_P^2 H(x,P) - D_P^2 H(x,Q)| \le c(1+|P|^2+|Q|^2)^{\frac{t-2-\nu}{2}}|P-Q|^\nu. \tag{4.1.20}$$

for a positive constant c independent of x.
In addition, we denote by $u \in W^{1,t}_{\mathrm{loc}}(\Omega)^M$ a local minimizer of the functional ($\lambda > 0$ denotes a regularization parameter)

$$I[w,\Omega] := \int_\Omega H(x,\nabla w)dx + \frac{\lambda}{2}\int_{\Omega-D} |w-f|^2 dx, \tag{4.1.21}$$

where we additionally assume $u \in L^{\infty}_{\mathrm{loc}}(\Omega)^M$.
Here, a function $u \in W^{1,t}_{\mathrm{loc}}(\Omega)^M$ is called a local minimizer of the functional I iff

$$I[u, \Omega'] < \infty \quad \text{and} \quad I[u, \Omega'] \le I[v, \Omega']$$

for any $\Omega' \Subset \Omega$ and for all $v \in W^{1,t}_{\mathrm{loc}}(\Omega)^M$ with $\mathrm{spt}(u - v) \subset \Omega'$.

Now we claim the following regularity statement:

Theorem 4.1.7
With the notation and assumptions from above we suppose that $u \in W^{1,t}_{loc}(\Omega)^M \cap L^{\infty}_{loc}(\Omega)^M$ ($t \ge 2$) denotes a local minimizer of the functional I from (4.1.21). *It then holds $u \in C^{1,\alpha}(\Omega)^M$.*

Remark 4.1.8
We stress that Theorem 4.1.7 is an essential tool in our discussions (we refer to the proof of Theorem 3.1.19 in the vectorial setting and to the proof of Theorem 4.1.4).

Remark 4.1.9

- *Neglecting the quantity $\int\limits_{\Omega - D} |w - f|^2 dx$, Theorem 4.1.7 has already been formulated and proven in [20], Lemma 2.7. As a consequence, our major effort consists in adjusting the arguments given in the proof of Lemma 2.7 in [20] to the second integral of* (4.1.21).
- *Looking at the proof of Theorem 4.1.7 it becomes evident that the arguments extend to the autonomous situation as well.*

Remark 4.1.10
We conjecture, that it is possible to establish Theorem 4.1.7 in the subquadratic case $1 < t < 2$ as well. However, due to the presence of the data fitting term, an adaption of the arguments of, e.g., [3] leads to some problems. Nonetheless, for our investigations, we merely need Theorem 4.1.7 in the case $t \ge 2$.

Finally the fourth chapter is organized as follows: in Section 4.2 we give a proof of Theorem 4.1.1. Section 4.3 is devoted to the investigation of the dual problem associated to (4.1.7). The aim of Section 4.4 is to show Theorem 4.1.4, whereas in Section 4.5, we prove Theorem 4.1.7.

4.2 Weak minimizers. Proof of Theorem 4.1.1

Assuming the validity of the hypotheses of Theorem 4.1.1 we first give a short proof of the following auxiliary result providing the continuity of (the relaxed variant of) the fidelity term occuring in the functional defined in formula (4.1.9).

Lemma 4.2.1
For $w \in BV(\Omega)^M$ let

$$\widetilde{K}[w] := \int_\Omega F_{\mu(x)}(\nabla^a w)dx + \int_\Omega \frac{1}{\mu(x)-1} d|\nabla^s w|.$$

(a) Suppose that $w_k, w \in BV(\Omega)^M$ with $w_k \to w$ in $L^1(\Omega)^M$. We then have

$$\widetilde{K}[w] \leq \liminf_{k\to\infty} \widetilde{K}[w_k].$$

(b) If we know in addition

$$\int_\Omega \sqrt{1+|\nabla w_k|^2} \to \int_\Omega \sqrt{1+|\nabla w|^2}$$

it then holds

$$\lim_{k\to\infty} \widetilde{K}[w_k] = \widetilde{K}[w].$$

Proof of Lemma 4.2.1. For part (a) we refer to [7], Theorem 5.54, p. 312 and the remarks on p.313.

For showing assertion (b), we use Reschetnyak's continuity theorem (see, e.g., [9], Proposition 2.2) and the version stated in [90], Theorem 1.3, in the euclidean setting, respectively. Following [60] (we refer to [15], Remark 2.5 as well) we homogenize $F_{\mu(x)}(P)$, i.e., we consider

$$\begin{aligned}&\widetilde{F} : \overline{\Omega} \times [0,\infty) \times \mathbb{R}^{nM} \to [0,\infty)\\ &\qquad (x,t,P) \mapsto \widetilde{F}(x,t,P) := \begin{cases} tF_{\mu(x)}(\frac{P}{t}), & t>0\\ \frac{|P|}{\mu(x)-1}, & t=0\end{cases},\end{aligned}$$

where we note that $\widetilde{F}$ is continuous and bounded on $\overline{\Omega} \times S^{nM}$. Here, S^{nM} denotes the nM-dimensional unit sphere in $\mathbb{R}^{nM+1}$. As the next step we look at the following sequence of tensor-valued Radon measures ($\mathcal{M}_b(\Omega)^{1+nM}$ denoting the space of $\mathbb{R}^{1+nM}$-valued measures on Ω with finite total variation)

$$\mu_k := (\mathcal{L}^n, \nabla w_k) \in \mathcal{M}_b(\Omega)^{1+nM}$$

and observe $|\mu_k|(\Omega) \to |\mu|(\Omega) := |(\mathcal{L}^n, \nabla w)|(\Omega)$ by the additional assumption stated in part (b). Clearly we have $\mu_k \overset{*}{\rightharpoondown} \mu$ in $\mathcal{M}_b(\Omega)^{1+nM}$ on the other hand (see [7], Proposition 3.13, p. 125) and from [90], Theorem 1.3, we conclude

$$\begin{aligned}&\lim_{k\to\infty} \int_\Omega \widetilde{F}\Big(x, \frac{(\mathcal{L}^n, \nabla w_k)}{|(\mathcal{L}^n, \nabla w_k)|}\Big) d|(\mathcal{L}^n, \nabla w_k)|\\ &= \int_\Omega \widetilde{F}\Big(x, \frac{(\mathcal{L}^n, \nabla w)}{|(\mathcal{L}^n, \nabla w)|}\Big) d|(\mathcal{L}^n, \nabla w)|.\end{aligned} \tag{4.2.1}$$

Note that (4.2.1) just gives

$$\lim_{k\to\infty} \widetilde{K}[w_k] = \widetilde{K}[w]. \tag{4.2.2}$$

For justifying that the representation stated in (4.2.1) implies (4.2.2) we can use the Lebesgue decomposition of the tensor-valued Radon measure ∇w, Radon-Nikodým's theorem and the definition of $\widetilde{F}$ from above (compare, e.g., [15], Remark 2.5). This completes the proof of Lemma 4.2.1. □

Now, let us proceed by proving Theorem 4.1.1, where it is to be noted that by strict convexity of $w \mapsto \int_\Omega |w-f|^2 dx$, it is clear that problem (4.1.9) admits at most one solution. To justify that problem (4.1.9) has at least one solution we denote by (w_m) an arbitrary K-minimizing sequence from the space $BV(\Omega)^M \cap L^2(\Omega)^M$ and state that due the uniform (w.r.t. x) linear growth of F (recall (4.1.8) as well) it holds

$$\sup_m |\nabla w_m|(\Omega) < \infty. \tag{4.2.3}$$

We further obtain

$$\sup_m \|w_m\|_{L^2(\Omega)} < \infty. \tag{4.2.4}$$

By BV-compactness it follows existence of a function $u \in BV(\Omega)^M$ such that $w_m \to u$ in $L^1(\Omega)^M$ and a.e. on Ω up to a subsequence. Moreover, (4.2.4) together with Fatou's lemma gives $u \in L^2(\Omega)^M$ implying $u \in BV(\Omega)^M \cap L^2(\Omega)^M$. Hence, $K[u]$ is well-defined.
Combining Lemma 4.2.1 (a) with Fatou's lemma it follows

$$K[u] \le \liminf_{m\to\infty} K[w_m] = \inf_{BV(\Omega)^M \cap L^2(\Omega)^M} K, \tag{4.2.5}$$

implying that u is K-minimizing. The maximum principle

$$\sup_\Omega |u| \le \sup_\Omega |f|$$

for u can be derived by arguing in the same manner as in the proof of Theorem 3.1.4 in Section 3.2 of this thesis. Note that the proof essentially simplifies by taking into account the uniqueness of the K-minimizer.

For proving part (b) we proceed exactly as in the proof of Theorem 2.1.1 (c) from Section 2.3 (we set $D = \emptyset$ therein): quoting Lemma 2.2.6 in Section 2.2.2 of this thesis (choose $D = \emptyset$ and $q = 2$) there is a sequence (u_m) of class $C^\infty(\overline{\Omega})^M$ satisfying

$$u_m \to u \quad \text{in } L^2(\Omega)^M,$$
$$\int_\Omega \sqrt{1+|\nabla u_m|^2}dx \to \int_\Omega \sqrt{1+|\nabla u|^2}.$$

An application of Lemma 4.2.1 (b) then gives continuity of the functional K w.r.t. to convergence "$u_m \to u$" which yields part (b).

To establish assertion (c) we follow the lines of the proof of Theorem 2.1.1 (d) (we set $D = \emptyset$ therein). Once again we then make use of the continuity of the functional K w.r.t. the convergence "$u_m \to u$" as stated above. Uniqueness of the minimizer finally implies $\mathcal{M} = \{u\}$. □

4.3 The dual problem. Proof of Theorem 4.1.3

Assuming the validity of the hypotheses of Theorem 4.1.3 we adapt the procedure as already been given in the proof of Theorem 2.1.6 in the autonomous setting and make some minor adjustments.
We approximate our original problem (4.1.7) by a sequence of more regular problems admitting regular solutions with appropriate convergence properties. To become more precise, for fixed $\delta \in (0,1]$, we denote by $u_\delta \in W^{1,2}(\Omega)^M$ the unique solution of the problem (with J from (4.1.7))

$$J_\delta[w] := \frac{\delta}{2}\int_\Omega |\nabla w|^2 dx + J[w] \to \min \quad \text{in } W^{1,2}(\Omega)^M \tag{4.3.1}$$

and state that u_δ satisfies

$$\sup_\Omega |u_\delta| \le \sup_\Omega |f|, \tag{4.3.2}$$

which is an easy consequence of Theorem 3.1.4 from the third chapter of this thesis.
Due to the uniform estimate $J_\delta[u_\delta] \le J_\delta[0] = J[0]$ we further get

$$\sup_\delta \|\nabla u_\delta\|_{L^1(\Omega)} < \infty, \tag{4.3.3}$$

$$\sup_\delta \delta \int_\Omega |\nabla u_\delta|^2 dx < \infty, \tag{4.3.4}$$

where (4.3.3) is a consequence of the uniform (in x) linear growth of $F_{\mu(x)}(P)$.
Next we let

$$\tau_\delta := \nabla_P F_{\mu(x)}(\nabla u_\delta) \quad \text{and} \quad \sigma_\delta := \delta \nabla u_\delta + \tau_\delta,$$

and see that (4.3.4) implies

$$\|\delta \nabla u_\delta\|_{L^2(\Omega)} \to 0 \quad \text{as } \delta \to 0, \tag{4.3.5}$$

whereas the uniform (w.r.t. δ) boundedness of τ_δ gives

$$\sup_\delta \|\tau_\delta\|_{L^\infty(\Omega)} < \infty. \tag{4.3.6}$$

After passing to a suitable subsequence $\delta \to 0$ we get from (4.3.5) and (4.3.6)

$$\sigma_\delta \rightharpoondown: \sigma \quad \text{in } L^2(\Omega)^{nM} \quad \text{and} \quad \tau_\delta \overset{*}{\rightharpoondown}: \tau \quad \text{in } L^\infty(\Omega)^{nM} \tag{4.3.7}$$

which yields $\sigma = \tau$ by means of (4.3.5).
Using (4.3.3) and (4.3.2) we may assume by BV-compactness

$$u_\delta \to: \overline{u} \quad \text{in } L^1(\Omega)^M \quad \text{and a.e.,} \tag{4.3.8}$$

for a function $\overline{u} \in BV(\Omega)^M$ which is of class $L^\infty(\Omega)^M$ as well. As a consequence of uniqueness we note that (4.3.3), (4.3.4) and (4.3.8) hold for a particular sequence $\delta \to 0$.
Now, we prove that σ is a solution of the dual problem: u_δ solves the Euler equation (compare (4.3.1))

$$\int\limits_\Omega \tau_\delta : \nabla\varphi dx + \delta \int\limits_\Omega \nabla u_\delta : \nabla\varphi dx + 2 \int\limits_\Omega (u_\delta - f) \cdot \varphi dx = 0 \tag{4.3.9}$$

for all $\varphi \in W^{1,2}(\Omega)^M$.
With the help of the duality formula $\tau_\delta : \nabla u_\delta - F^*(\cdot, \tau_\delta) = F_{\mu(\cdot)}(\nabla u_\delta)$ being valid for all $x \in \Omega$ (remember the definition of the conjugate function $F^*(x, P)$ to $F_{\mu(x)}(P)$) we obtain

$$I_\delta[u_\delta] = \frac{\delta}{2} \int\limits_\Omega |\nabla u_\delta|^2 dx + \int\limits_\Omega [\tau_\delta : \nabla u_\delta - F^*(x, \tau_\delta)] dx + \int\limits_\Omega |u_\delta - f|^2 dx.$$

Inserting the admissible function $\varphi = u_\delta$ in Euler's equation (4.3.9) we find

$$\begin{aligned} I_\delta[u_\delta] = &-\frac{\delta}{2} \int\limits_\Omega |\nabla u_\delta|^2 dx + \int\limits_\Omega (-F^*(x, \tau_\delta)) dx - \int\limits_\Omega |u_\delta|^2 dx \\ &+ \int\limits_\Omega |f|^2 dx. \end{aligned} \tag{4.3.10}$$

Next we let $v \in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M$ and observe (recall (4.1.7) and the definition of the dual functional R)

$$I[v] = \sup_{\varkappa \in L^\infty(\Omega)^{nM}} l(v, \varkappa) \geq l(v, \rho) \geq \inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M} l(v, \rho) = R[\rho]$$

for any $\rho \in L^\infty(\Omega)^{nM}$, which yields

$$\sup_{\rho \in L^\infty(\Omega)^{nM}} R[\rho] \leq \inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M} I[v].$$

Using (4.3.10) we then obtain

$$\begin{aligned} \sup_{\kappa \in L^\infty(\Omega)^{nM}} R[\kappa] &\leq \inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M} I[v] \\ &\leq -\frac{\delta}{2} \int\limits_\Omega |\nabla u_\delta|^2 dx + \int\limits_\Omega (-F^*(x, \tau_\delta)) dx - \int\limits_\Omega |u_\delta|^2 dx \\ &\quad + \int\limits_\Omega |f|^2 dx. \end{aligned} \tag{4.3.11}$$

By dominated convergence (recall (4.3.2) and (4.3.8)) we obtain (as $\delta \to 0$)

$$\int_\Omega |u_\delta|^2 dx \to \int_\Omega |\overline{u}|^2 dx,$$

while upper semicontinuity of $\int_\Omega (-F^*(x,\cdot))dx$ w.r.t. weak-$*$ convergence shows

$$\limsup_{\delta\to 0} \int_\Omega (-F^*(x,\tau_\delta))dx \le \int_\Omega (-F^*(x,\tau))dx.$$

Dropping the term $-\frac{\delta}{2}\int_\Omega |\nabla u_\delta|^2 dx$ for the moment and passing to the limit $\delta \to 0$, (4.3.11) turns into

$$\begin{aligned} \sup_{L^\infty(\Omega)^{nM}} R &\le \inf_{W^{1,1}(\Omega)^M \cap L^2(\Omega)^M} I \\ &\le \int_\Omega (-F^*(x,\tau))dx - \int_\Omega |\overline{u}|^2 dx + \int_\Omega |f|^2 dx. \end{aligned} \tag{4.3.12}$$

Passing to the limit $\delta \to 0$ in Euler's equation (4.3.9) we find (recall (4.3.5), (4.3.7) and (4.3.8))

$$\int_\Omega \tau : \nabla\varphi dx + 2\int_\Omega (\overline{u} - f)\cdot \varphi dx = 0 \tag{4.3.13}$$

for any $\varphi \in W^{1,2}(\Omega)^M$.

By performing standard approximation arguments (we moreover refer to Lemma 2.2.4 choosing $D = \emptyset$, $p = 1$ and $q = 2$), we obtain that (4.3.13) extends to $\varphi \in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M$.

Besides it holds

$$\begin{aligned} R[\tau] :&= \inf_{v\in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M} l(v,\tau) \\ &= \int_\Omega (-F^*(x,\tau))dx \\ &\quad + \inf_{v\in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M} \Big[\int_\Omega \tau : \nabla v dx + \int_\Omega |v-f|^2 dx\Big] \\ &= \int_\Omega (-F^*(x,\tau))dx \\ &\quad + \inf_{v\in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M} \Big[-2\int_\Omega (\overline{u}-f)\cdot v dx + \int_\Omega |v-f|^2 dx\Big] \\ &= \int_\Omega (-F^*(x,\tau))dx \\ &\quad + \inf_{v\in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M} \Big[\int_\Omega |\overline{u}-v|^2 dx + \int_\Omega |f|^2 dx - \int_\Omega |\overline{u}|^2 dx\Big]. \end{aligned}$$

Here, we used (4.3.13) with the admissible choice $\varphi = v$ and the quadratic structure of the data fitting term as well. Hence,

$$R[\tau] \geq \int_\Omega (-F^*(x,\tau))dx + \int_\Omega |f|^2 dx - \int_\Omega |\overline{u}|^2 dx$$

and (4.3.12) implies

$$\sup_{L^\infty(\Omega)^{nM}} R \leq \inf_{W^{1,1}(\Omega)^M \cap L^2(\Omega)^M} I \leq R[\tau].$$

As a consequence, τ is R-maximizing and the inf-sup relation is valid. Thus, assertion (a) of Theorem 4.1.3 is proven.
As byproducts we further have shown (recall (4.3.11))

$$\delta \int_\Omega |\nabla u_\delta|^2 dx \to 0, \tag{4.3.14}$$

$$(u_\delta) \text{ is an } I - \text{minimizing sequence.} \tag{4.3.15}$$

at least for a subsequence $\delta_m \to 0$. From (4.3.15) and Theorem 4.1.1 (c), it then follows that $\overline{u} \in BV(\Omega)^M \cap L^\infty(\Omega)^M$ is K-minimizing. By uniqueness we get $\overline{u} = u$ a.e. on Ω, where u denotes the K-minimizer whose existence is guaranteed by Theorem 4.1.1.

A proof of assertion (c) can be deduced by adapting the arguments from the proof of Theorem 2.1.6, (c), choosing $D = \emptyset$ therein.

As the last step we want to establish statement (b), where we adopt the procedure as given in the proof of Theorem 2.1.7 making some minor adjustments: we recall that $u \in BV(\Omega)^M \cap L^\infty(\Omega)^M$ denotes the unique K-minimizer and remember the Lebesgue decomposition $\nabla u = \nabla^a u \llcorner \mathcal{L}^n + \nabla^s u$ of the tensor-valued Radon measure ∇u. We then assert

Lemma 4.3.1
The tensor $\rho := \nabla_P F_{\mu(\cdot)}(\nabla^a u)$ is a maximizer of the dual functional R.

Proof of Lemma 4.3.1. Recalling $\rho \in L^\infty(\Omega)^M$ and remembering the definition of the dual functional

$$R[\varkappa] = \inf_{v \in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M} l(v,\varkappa), \quad \varkappa \in L^\infty(\Omega)^{nM},$$

where the Lagrangian $l(v,\varkappa)$ is given as in (4.1.10), it follows that $R[\rho]$ is well-defined. We further obtain for $v \in W^{1,1}(\Omega)^M \cap L^2(\Omega)^M$ (recall that $F^*(x,P)$

denotes the conjugate function to $F_{\mu(x)}(P)$ w.r.t. the P-variable)

$$\begin{aligned} l(v,\rho) &= \int_\Omega [\nabla_P F_{\mu(x)}(\nabla^a u) : \nabla v - F^*(x, \nabla_P F_{\mu(x)}(\nabla^a u))]dx \\ &+ \int_\Omega |v-f|^2 dx \\ &= \int_\Omega F_{\mu(x)}(\nabla^a u)dx + \int_\Omega (\nabla v - \nabla^a u) : \nabla_P F_{\mu(x)}(\nabla^a u)dx \\ &+ \int_\Omega |v-f|^2 dx, \end{aligned} \tag{4.3.16}$$

where the important formula $F_{\mu(x)}(P) + F^*(x, \nabla_P F_{\mu(x)}(P)) = P : \nabla_P F_{\mu(x)}(P)$, $P \in \mathbb{R}^{nM}$, being valid for all $x \in \Omega$ has been used.
The K-minimality of u moreover gives (note that $\nabla^s(u+tv) = \nabla^s u$ holds for the singular part of the measures)

$$0 = \frac{d}{dt_{|0}} K[u+tv] = \int_\Omega \nabla_P F_{\mu(x)}(\nabla^a u) : \nabla v dx + 2\int_\Omega v \cdot (u-f)dx. \tag{4.3.17}$$

Clearly, we have $\nabla(u+tu) = (1+t)\nabla u$ and by using the K-minimality of u once again we get

$$\begin{aligned} 0 &= \frac{d}{dt_{|0}} K[u+tu] \\ &= \int_\Omega \nabla_P F_{\mu(x)}(\nabla^a u) : \nabla^a u dx + \int_\Omega \frac{1}{\mu(x)-1} d|\nabla^s u| \\ &+ 2\int_\Omega u \cdot (u-f)dx. \end{aligned} \tag{4.3.18}$$

Inserting (4.3.17) and (4.3.18) into (4.3.16) we see

$$\begin{aligned} l(v,\rho) = &\int_\Omega F_{\mu(x)}(\nabla^a u)dx + \int_\Omega \frac{1}{\mu(x)-1} d|\nabla^s u| \\ &+ \int_\Omega |u-f|^2 dx + \int_\Omega |u-v|^2 dx. \end{aligned} \tag{4.3.19}$$

Clearly, we have the validity of the formula $-2v\cdot(u-f) + 2u\cdot(u-f) + |v-f|^2 = |u-f|^2 + |u-v|^2$ a.e. on Ω which, in combination with (4.3.19), implies

$$l(v,\rho) \geq K[u]$$

Thus, by recalling the definition of the dual functional R, it follows

$$R[\rho] \geq K[u]$$

and this proves Lemma 4.3.1 after quoting part (a) of Theorem 4.1.3 (note that u is K-minimal). $\square$

Now let us show uniqueness of the dual solution σ: we first note that $P \mapsto F^*(x,P)$ is strictly convex on the set $\{P \in \mathbb{R}^{nM}, F^*(x,P) < \infty\}$ for all $x \in \Omega$ which can be justified by adopting the arguments of the proof of Theorem 1.4 ii) in [25]. But then, σ is unique which can be proven by arguing in the same manner as in the proof of Theorem 2.1.6, (b) (or in the proof of Theorem 1.4 in [25]).
Using Lemma 4.3.1, we then get the validity of the duality formula

$$\sigma = \nabla_P F_{\mu(\cdot)}(\nabla^a u) \quad \text{a.e. on } \Omega,$$

which completes the proof of Theorem 4.1.3. □

4.4 $C^{1,\alpha}$-regularity of the unique BV-minimizer. Proof of Theorem 4.1.4

Assuming the validity of the hypotheses of Theorem 4.1.4 and denoting by $u \in BV(\Omega)^M \cap L^\infty(\Omega)^M$ the unique K-minimizer with K from (4.1.9) we consider the same regularization as introduced in Section 4.3: for fixed $\delta \in (0,1]$ we denote by $u_\delta \in W^{1,2}(\Omega)^M$ the unique solution of the problem

$$J_\delta[w] = \int\limits_\Omega F_\delta(x, \nabla w)dx + \int\limits_\Omega |w - f|^2 dx \to \min \quad \text{in } W^{1,2}(\Omega)^M, \tag{4.4.1}$$

where we recall $F_\delta(x,P) := \frac{\delta}{2}|P|^2 + F_{\mu(x)}(P)$ for any $x \in \overline{\Omega}$ and $P \in \mathbb{R}^{nM}$. Furthermore, u_δ is uniformly bounded w.r.t. δ (see (4.3.2)) and we have shown that it holds $u_\delta \to u$ in $L^1(\Omega)^M$ and a.e. up to a subsequence (compare (4.3.15)). In the following lemma we state some regularity properties of u_δ that serve as essential tools in the further proof. A remark to a proof of this lemma can be found in the appendix of this thesis (see Lemma 7.1.1 and Remark 7.1.4, respectively)

Lemma 4.4.1
It holds $u_\delta \in W^{2,2}_{loc}(\Omega)^M \cap W^{1,\infty}_{loc}(\Omega)^M$.

Furthermore, u_δ solves the Euler equation (see (4.3.9))

$$\int\limits_\Omega \nabla_P F_\delta(x, \nabla u_\delta) : \nabla\varphi dx + 2\int\limits_\Omega (u_\delta - f)\cdot \varphi dx = 0 \tag{4.4.2}$$

for all $\varphi \in C_0^\infty(\Omega)^M$. The corresponding differentiated variant of Euler's equation (4.4.2) then reads as (recall the uniform (in x) boundedness of $|D_P^2 F_\delta|$ and

$u_\delta \in W^{2,2}_{\mathrm{loc}}(\Omega)^M$)

$$
\begin{aligned}
&\int_\Omega D_P^2 F_\delta(x,\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla\varphi)dx + \int_\Omega (\partial_\gamma \nabla_P F_\delta)(x,\nabla u_\delta) : \nabla\varphi dx \\
&= 2\int_\Omega (u_\delta - f)\cdot \partial_\gamma \varphi dx
\end{aligned}
\tag{4.4.3}
$$

for all $\varphi \in W^{1,2}(\Omega)^M$ with compact support in Ω.
The proof of Theorem 4.1.4 is divided into three parts: local uniform (in δ) L^p-estimates of ∇u_δ for all finite $p > 1$, local uniform (in δ) gradient bounds, and the conclusions.
Within these three steps we use the same arguments as already used in the autonomous setting (compare the proof of Theorem 3.1.19 in the third chapter), where we only apply minor adjustments.

Step 1. Local uniform L^p-estimates of ∇u_δ

We claim the following statement and note that we crucially will make use of the uniform (in x) growth and ellipticity conditions of our density $F_{\mu(x)}(P)$.

Lemma 4.4.2
For any $1 < p < \infty$ and for any $\omega \Subset \Omega_2$ there is a constant $c(p,\omega)$, which is not depending on δ, such that

$$\|\nabla u_\delta\|_{L^p(\omega)} \le c(p,\omega) < \infty. \tag{4.4.4}$$

Proof of Lemma 4.4.2. As already seen in the second step of the proof of Theorem 3.1.19 in Section 3.4, we initially prove an appropriate variant of Caccioppoli's inequality which acts as an important tool during the iteration procedure that we apply afterwards to establish uniform L^p_{loc}-gradient bounds of u_δ on Ω_2.

Lemma 4.4.3
Fix a ball $B_r(x_0) \Subset \Omega_2$. Then for any $s_0 \ge 0$ there exists a real number $c > 0$ such that for all $\eta \in C_0^\infty(B_r(x_0))$ satisfying $0 \le \eta \le 1$ and for any $\delta \in (0,1)$ it holds

$$
\begin{aligned}
&\int_{B_r(x_0)} |\nabla^2 u_\delta|^2 \Gamma_\delta^{s_0 - \frac{\mu}{2}} \eta^2 dx + \delta \int_{B_r(x_0)} |\nabla^2 u_\delta|^2 \Gamma_\delta^{s_0}\eta^2 dx \\
&\le \int_{B_r(x_0)} D_P^2 F_\delta(x,\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 \Gamma_\delta^{s_0} dx \\
&\le c \int_{B_r(x_0)} D_P^2 F_\delta(x,\nabla u_\delta)(\partial_\gamma u_\delta \otimes \nabla\eta, \partial_\gamma u_\delta \otimes \nabla \eta)\Gamma_\delta^{s_0} dx \\
&+ c\int_{B_r(x_0)} |\nabla\eta|^2 \Gamma_\delta^{s_0} dx + c\int_{B_r(x_0)} \eta^2 \Gamma_\delta^{s_0} dx,
\end{aligned}
\tag{4.4.5}
$$

where we have set $\Gamma_\delta := 1 + |\nabla u_\delta|^2$ *and* c *is a positive constant which, in particular, is independent of* δ.

Remark 4.4.4
Based on the requirement $\mu \in C^2(\overline{\Omega})$ *with* μ *satisfying* $\mu(x) \in [\mu_0, \mu_1]$ *for suitable numbers* $1 < \mu_0 \leq \mu_1 < \infty$ *we particularly get* $\mu_1 < 2$ *on each compact subset* ω *of* Ω_2.

Proof of Lemma 4.4.3. Note that the first inequality follows after using the uniform (in x) ellipticity condition of our density $F_{\mu(x)}(P)$. Now, we fix some number $s_0 \geq 0$ and a ball $B_r(x_0) \Subset \Omega_2$. With η given above and by quoting Lemma 4.4.1, the function $\varphi = \eta^2 \Gamma_\delta^{s_0} \partial_\gamma u_\delta$ with $\gamma \in \{1, \ldots, n\}$ is an admissible choice in equation (4.4.3) and it follows (from now on summation w.r.t. $\gamma \in \{1, \ldots, n\}$)

$$\begin{aligned}
&\int\limits_{B_r(x_0)} D_P^2 F_\delta(x, \nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 \Gamma_\delta^{s_0} dx + 2 \int\limits_{B_r(x_0)} |\nabla u_\delta|^2 \eta^2 \Gamma_\delta^{s_0} dx \\
&+ s_0 \int\limits_{B_r(x_0)} D_P^2 F_\delta(x, \nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial u_\delta \otimes \nabla \Gamma_\delta)\Gamma_\delta^{s_0 - 1}\eta^2 dx \\
&= - \int\limits_{B_r(x_0)} (\partial_\gamma \nabla_P F_\delta)(x, \nabla u_\delta) : \partial_\gamma \nabla u_\delta \eta^2 \Gamma_\delta^{s_0} dx \\
&- 2 \int\limits_{B_r(x_0)} (\partial_\gamma \nabla_P F_\delta)(x, \nabla u_\delta) : \nabla \eta \otimes \partial_\gamma u_\delta \Gamma_\delta^{s_0} \eta dx \\
&- s_0 \int\limits_{B_r(x_0)} (\partial_\gamma \nabla_P F_\delta)(x, \nabla u_\delta) : \partial_\gamma \nabla u_\delta \eta^2 \Gamma_\delta^{s_0} dx \\
&- 2 \int\limits_{B_r(x_0)} D_P^2 F_\delta(x, \nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla \eta \otimes \partial_\gamma u_\delta)\eta \Gamma_\delta^{s_0} dx \\
&- 2 \int\limits_{B_r(x_0)} f \cdot \partial_\gamma \varphi dx \\
&=: \sum_{i=1}^{5} I_i.
\end{aligned} \tag{4.4.6}$$

Now, we crucially use the (uniform) boundedness of the quantity $|\partial_\gamma \nabla_P F_{\mu(x)}(P)|$ to discuss I_1, I_2 as well as I_3 whereas for I_4 and I_5 we argue in the same manner as already seen in the proof of Lemma 3.5.2 and Lemma 3.5.13, respectively. From (4.4.6) we thus get after using Young's inequality and absorbing terms

($c > 0$ denotes a positive constant being independent of δ)

$$\begin{aligned}
&\int\limits_{B_r(x_0)} D_P^2 F_\delta(x,\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\eta^2 \Gamma_\delta^{s_0} dx + c \int\limits_{B_r(x_0)} |\nabla u_\delta|^2 \eta^2 \Gamma_\delta^{s_0} dx \\
&\leq c \int\limits_{B_r(x_0)} D_P^2 F_\delta(x,\nabla u_\delta)(\partial_\gamma u_\delta \otimes \nabla\eta, \partial_\gamma u_\delta \otimes \nabla\eta)\Gamma_\delta^{s_0} dx \\
&+ c \int\limits_{B_r(x_0)} \eta^2 \Gamma_\delta^{s_0+\frac{\mu_1}{2}} dx + c \int\limits_{B_r(x_0)} |\nabla\eta|^2 \Gamma_\delta^{s_0} dx.
\end{aligned} \tag{4.4.7}$$

Recalling Remark 4.4.4 we have $\mu_1 < 2$ on $B_r(x_0) \Subset \Omega_2$ and another application of Young's inequality gives for any $\varepsilon > 0$

$$c \int\limits_{B_r(x_0)} \eta^2 \Gamma_\delta^{s_0+\frac{\mu_1}{2}} dx \leq \varepsilon \int\limits_{B_r(x_0)} \eta^2 \Gamma_\delta^{s_0+1} dx + c\varepsilon^{\frac{\mu_1}{\mu_1 - 2}} \int\limits_{B_r(x_0)} \eta^2 \Gamma_\delta^{s_0} dx.$$

Absorbing terms by choosing $\varepsilon > 0$ sufficiently small we finally obtain the desired second inequality in (4.4.5) after neglecting the non-negative second integral on the l.h.s. of (4.4.7). □

As the next step we use exactly the same iteration argument as already seen in the proof of Theorem 3.1.19 in the scalar case (see "Step 1. Regularization and local uniform apriori L^p-estimates "). Here, the variant of Caccioppoli's inequality (4.4.5) is a crucial tool and it is worth remarking that we again benefit from the uniform (in x) growth and ellipticity conditions of our involved density. Furthermore we use the fact that it holds (note that $c > 0$ is a uniform constant and $x_0 \in \Omega$)

$$\nabla_P F_{\mu(x_0)}(P) : P \geq c|P| \int\limits_0^{|P|} (1+\rho^2)^{-\frac{\mu_1}{2}} d\rho,$$

which implies that we have the validity of Lemma 3.1.3 (a) on $B_r(x_0)$ (we choose r sufficiently small).
Thus, we get $\nabla u_\delta \in L^p_{\mathrm{loc}}(\Omega_2)^{nM}$ uniformly in δ for any finite $p > 1$ and Lemma 4.4.2 is proven. □

Step 2. Local uniform apriori gradient bounds.

In this step we first derive another variant of Caccioppoli's inequality which, in particular, is valid for all $\mu(x) \in [\mu_0, \mu_1]$ with appropriate numbers $1 < \mu_0 \leq \mu_1 < \infty$ (recall (4.1.8)). This inequality takes a crucial part when performing DeGiorgi-type arguments in the third step below.
Let us first introduce some notation: we fix a point $x_0 \in \Omega$ and consider radii

$0 < r < R < R_0$ with $B_{R_0}(x_0) \Subset \Omega$.
Moreover, we let for $k > 0$

$$A_\delta(k, R) := \{x \in B_R(x_0) : \Gamma_\delta > k\}.$$

Further we consider a suitable cut-off function $\eta \in C_0^\infty(B_R(x_0))$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_r(x_0)$ and $|\nabla\eta| \leq \frac{c}{R-r}$. Finally, for functions $v : \Omega \to \mathbb{R}$ we denote $\max\{v, 0\}$ by v^+. The following variant of Caccioppoli's inequality then can be established where we emphasize that the choice of the parameter ν is not optimal (see Remark 3.5.5).

Lemma 4.4.5
With the previous notation, in particular, for any $x \in \Omega$, it holds

$$\begin{aligned}
&\int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{-\mu_1}{2}} |\nabla\Gamma_\delta|^2 \eta^2 dx \\
&\leq c \int\limits_{A_\delta(k,R)} |D_P^2 F_\delta(x, \nabla u_\delta)||\nabla\eta|^2 (\Gamma_\delta - k)^2 dx + c \int\limits_{A_\delta(k,R)} \eta^2 |\nabla u_\delta|^{2+\mu_1} dx \\
&+ c \int\limits_{A_\delta(k,R)} \eta|\nabla\eta||\nabla u_\delta|^3 dx \\
&\leq \frac{c}{(R-r)^2} \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\nu}{2}} dx
\end{aligned} \tag{4.4.8}$$

where $\nu := \max\{4, 2 + \mu_1\}$ and c denotes a positive constant independent of δ, r and R.

Proof of Lemma 4.4.5. A proof of this lemma can be accomplished by inserting the same test functions in the differentiated Euler equation (4.4.3) and carrying out the same arguments as applied in the autonomous setting (compare the proof of Lemma 3.5.4 and the proof of Lemma 3.5.14, respectively). As already seen in the proof of Lemma 4.4.3, we note that the additional term occuring in the equation (4.4.3) will be estimated by using the (uniform) boundedness of the quantity $|\partial_\gamma \nabla_P F_{\mu(x)}(P)|$. We emphasize that we further benefit from the uniform (in x) growth and ellipticity conditions of the density $F_{\mu(x)}(P)$. $\square$

For proving local uniform apriori gradient bounds of u_δ we make use of the following De Giorgi-type lemma closing the gap between local uniform $\overline{p}$-integrability of the gradients for a certain exponent $\overline{p}$ and local uniform apriori gradient bounds. This lemma has already been proven in Section 3.5.1 (see the proof of Lemma 3.5.6) and we will repeat its statement below.

Lemma 4.4.6
Suppose that v_δ is a sequence of class $W^{2,2}_{loc}(\Omega)^M$ and that we are given real

numbers $\overline{p}, \nu > 3$, $\mu > 1$ satisfying

$$\frac{\mu+\nu}{2} n < \overline{p}. \tag{4.4.9}$$

Moreover, suppose that we have a uniform constant $c > 0$ (with $\Gamma_\delta := 1+|\nabla v_\delta|^2$ and $A_\delta(k,R), r, R, R_0, \eta$ as above) such that it holds

$$\int\limits_{A_\delta(k,R)} \Gamma_\delta^{-\frac{\mu}{2}} |\nabla \Gamma_\delta|^2 \eta^2 dx \leq \frac{c}{(R-r)^2} \int\limits_{A_\delta(k,R)} \Gamma_\delta^{\frac{\nu}{2}} dx \tag{4.4.10}$$

and assume in addition that ∇v_δ is locally $\overline{p}$-integrable uniformly in δ, i.e.

$$\sup_\delta \int\limits_{\Omega'} |\nabla v_\delta|^{\overline{p}} dx = c(\overline{p}, \Omega') < \infty, \tag{4.4.11}$$

where $\Omega' \Subset \Omega$. Then it holds $\nabla v_\delta \in L^\infty_{loc}(\Omega)^{nM}$ uniformly in δ.

As a consequence we can state the following conclusion.

Proposition 4.4.7
It holds $\nabla u_\delta \in L^\infty_{loc}(\Omega_2)^{nM}$ uniformly w.r.t. δ.

Proof of Proposition 4.4.7. For showing the claim we apply Lemma 4.4.6 to our setting: quoting Lemma 4.4.1 it holds $u_\delta \in W^{2,2}_{\mathrm{loc}}(\Omega)^M$. By means of Lemma 4.4.4 we obtain uniform $L^{\overline{p}}_{\mathrm{loc}}$-estimates of ∇u_δ on Ω_2 for any finite $\overline{p} > 1$. Thus, (4.4.11) from Lemma 4.4.6 with $v_\delta := u_\delta$ is satisfied for any finite $\overline{p}$. Furthermore, Lemma 4.4.5 provides a Caccioppoli-type inequality in the spirit of (4.4.10) in Lemma 4.4.6 (with a uniform constant $c > 0$) where we set $\mu := \mu_1 > 1$ and $\nu := \max\{4, 2+\mu_1\} > 3$. Recalling (4.1.8) it moreover holds $\mu_1 < \infty$, where we can even arrange $\mu_1 < 2$ on each compact subset $\Omega' \Subset \Omega_2$ (compare Remark 4.4.4). Finally, the requirement (4.4.9) from Lemma 4.4.6 is trivially fulfilled. As a consequence, Lemma 4.4.6 then provides local uniform (in δ) apriori gradient bounds for u_δ on Ω_2. □

Step 3. Conclusions

In accordance with Theorem 4.1.3 (compare (4.3.15)) we know $u_\delta \to u$ in $L^1_{\mathrm{loc}}(\Omega_2)$ as $\delta \to 0$ where u denotes the unique K-minimizer from the space $BV(\Omega)^M \cap L^\infty(\Omega)^M$. Using Proposition 4.4.7 we get that u_δ is locally uniformly (in δ) Lipschitz continuous on Ω_2. An application of Arzelá-Ascoli's theorem then yields $u \in C^{0,1}(\Omega_2)^M$.
Moreover we see that u solves the Euler equation

$$\int\limits_\Omega \nabla_P F_{\mu(x)}(\nabla u) : \nabla\varphi dx + 2\int\limits_\Omega (u-f)\cdot\varphi dx = 0 \tag{4.4.12}$$

for all $\varphi \in C_0^\infty(\Omega_2)^M$.

For proving local Hölder continuity of ∇u in Ω_2 in the scalar case $M = 1$ we can proceed as in the proof of Theorem 3.1.19 (see "Step 4. Conclusions") and quote elliptic regularity theory (see, e.g., [62], Theorem 8.22, p. 200).

For closing the gap between $u \in C^{0,1}(\Omega_2)^M$ and local Hölder continuity of ∇u on Ω_2 in the vectorial setting $M > 1$ we adopt the same procedure as already carried out in the autonomous case (see proof of Theorem 3.1.19 in the vectorial case, "Step 4. Conclusions").
To become more precise we fix a ball $B_r(x_0) \Subset \Omega_2$ and choose a positive number M such that $\sup\limits_{B_r(x_0)} |\nabla u| \leq M$. Following [78] we consider the following integrand $\widetilde{F} : \overline{\Omega} \times \mathbb{R}^{nM} \to [0, \infty)$

$$\widetilde{F}(x, P) := 1 + \nu + F_{\mu(x)}(P) + (|P|^2 - 4K^2)^{+3},$$

where $\nu > 0$ denotes a constant.
By construction of $\widetilde{F}$ we directly see that $\widetilde{F} \in C^2(\overline{\Omega} \times \mathbb{R}^{nM})$. Further, some straight forward calculations (recall the uniform (w.r.t. x) growth and ellipticity properties of $F_{\mu(x)}(P)$ and remember $\mu \in C^2(\overline{\Omega})$) show that we can establish the following estimates (note that we also use that $D_P^2 F_{\mu(x)}(P)$ satisfies a uniform (in x) Hölder condition)

$$\begin{aligned} \nu_1(1+|P|^2)^2|Q|^2 \leq D_P^2\widetilde{F}(x,P)(Q,Q) &\leq \nu_2(1+|P|^2)^2|Q|^2, \\ |D_P^2\widetilde{F}(x,P) - D_P^2\widetilde{F}(x,Q)| &\leq \nu_3(1+|P|^2+|Q|^2)^{2-\lambda/2}|P-Q|^\lambda, \\ |\nabla_x \nabla_P \widetilde{F}(x,P)| &\leq \nu_4, \end{aligned}$$

for all $P, Q \in \mathbb{R}^{nM}$, for all $x \in \overline{\Omega}$ and with some positive constants $\nu_1, \dots, \nu_4$ and λ. Furthermore we remark that it additionally holds $\widetilde{F}(x, P) = \widetilde{g}(x, |P|^2)$ for a function $\widetilde{g} \in C^2(\overline{\Omega} \times [0, \infty), [0, \infty))$.
From $\nabla_P \widetilde{F}(x, P) = \nabla_P F_{\mu(x)}(P)$ for all $|P| \leq 2K$ it follows that u is a local minimizer of the functional

$$\int_{B_r(x_0)} \widetilde{F}(x, \nabla w)dx + \int_{B_r(x_0)} |w - f|^2 dx$$

and thus of class $C^{1,\alpha}(B_r(x_0))^M$ for any $\alpha \in (0, 1)$ by Theorem 4.1.7 (we choose $t = 6$). From this information we obtain the assertion of the theorem after using a covering argument (note that we crucially need $\nabla u \in L^\infty_{\text{loc}}(\Omega_2)^{nM}$ for carrying out this argument). $\square$

4.5 $C^{1,\alpha}$-solutions to a class of non-autonomous isotropic variational problems. Proof of Theorem 3.1.19

Let us assume the validity of the hypotheses of Theorem 4.1.7. We essentially follow the arguments of Lemma 2.7 in [20] with some adjustments: first we fix a point $x_0 \in \Omega$ and let $B_R(x_0) \Subset \Omega$ with $R \leq R_0$ where the radius R_0 will be fixed later. Setting $H_0 := H(x_0, \cdot)$, the crucial observation of the entire proof is that the unique solution v of the variational problem

$$\int_{B_R(x_0)} H_0(\nabla w)dx \to \min \quad \text{in } u|_{B_R(x_0)} + W_0^{1,t}(B_R(x_0))^M$$

satisfies the Campanato estimates (3.1) and (3.2) from Theorem 3.1 of [59] where we remark that H_0 is admissible in the sense that we have the validity of the assumptions $H.1 - H.4$ from [59] (see (4.1.17) and (4.1.20)). To become more precise, the inequality (3.1) of Theorem 3.1 in [59] gives together with the minimality of H_0 and the growth of H_0

$$\|\nabla v\|^t_{L^\infty(B_{R/2})} \leq c \fint_{B_R} (1 + |\nabla v|^2)^{\frac{t}{2}} dx \leq c \fint_{B_R} (1 + |\nabla u|^2)^{\frac{t}{2}} dx. \tag{4.5.1}$$

Defining $V(\xi) := (1 + |\xi|^2)^{\frac{t-2}{4}} \xi$ and $H_t(\xi) := (1 + |\xi|^2)^{\frac{t}{2}}$, Lemma 2.3 of [67] gives

$$|\sqrt{H_t(\xi)} - \sqrt{H_t(\bar{\xi})}| \leq c|V_t(\xi) - V_t(\bar{\xi})|,$$

which implies for $\rho \leq \frac{R}{2}$

$$\begin{aligned}\int_{B_\rho} (1 + |\nabla u|^2)^{\frac{t}{2}} dx &\leq c\Big[\int_{B_\rho} (1 + |\nabla v|^2)^{\frac{t}{2}} dx + \int_{B_\rho} \Big|(1 + |\nabla u|^2)^{\frac{t}{4}} - (1 + |\nabla v|^2)^{\frac{t}{4}}\Big|^2 dx\Big] \\ &\leq c \int_{B_\rho} (1 + |\nabla v|^2)^{\frac{t}{2}} dx + c \int_{B_\rho} |V(\nabla u) - V(\nabla v)|^2 dx.\end{aligned}$$

From (4.5.1) we may then conclude

$$\begin{aligned}\int_{B_\rho} (1 + |\nabla u|^2)^{\frac{t}{2}} dx \leq & c\Big(\frac{\rho}{R}\Big)^n \int_{B_R} (1 + |\nabla u|^2)^{\frac{t}{2}} dx \\ &+ c \int_{B_\rho} |V(\nabla u) - V(\nabla v)|^2 dx.\end{aligned} \tag{4.5.2}$$

By virtue of (2.3) from [67] and (2.1) of [59] it follows

$$\int_{B_\rho} |V(\nabla u) - V(\nabla v)|^2 dx \leq c \int_{B_R} (1 + |\nabla u|^2 + |\nabla v|^2)^{\frac{t-2}{2}} |\nabla u - \nabla v|^2 dx$$

$$\leq c \int_{B_R} \underbrace{\int_0^1 (1 + |\nabla v + t(\nabla u - \nabla v)|^2)^{\frac{t-2}{2}} |\nabla u - \nabla v|^2 dt \, dx}_{(*)}.$$

Next we state that on account of (4.1.17) we have

$$\begin{aligned}
&(DH_0(\nabla u) - DH_0(\nabla v)) : (\nabla u - \nabla v) \\
&= \int_0^1 D^2 H_0(\nabla v + t(\nabla u - \nabla v))(\nabla u - \nabla v, \nabla u - \nabla v) dt \geq \lambda \, (*).
\end{aligned}$$

Taking into account the Euler equations valid for u and v as well as the growth condition (4.1.18) we get by means of the above inequality

$$\begin{aligned}
&\int_{B_R} |V(\nabla u) - V(\nabla v)|^2 dx \\
&\leq c \int_{B_R} (DH_0(\nabla u) - DH_0(\nabla v)) : (\nabla u - \nabla v) dx \\
&= \int_{B_R} (DH_0(\nabla u) - \nabla_P H(x, \nabla u)) : (\nabla u - \nabla v) dx \\
&+ \int_{B_R - D} \lambda(u - f) \cdot (v - u) dx \\
&\leq cR \int_{B_R} (1 + |\nabla u|^2)^{\frac{t-1}{2}} |\nabla u - \nabla v| dx + \int_{B_R - D} \lambda(u - f) \cdot (v - u) dx.
\end{aligned} \tag{4.5.3}$$

In the last integral on r.h.s. of (4.5.3) we use Young's and Poincaré's inequality. It follows for any $\tau > 0$ (recall $u \in W^{1,t}_{\mathrm{loc}}(\Omega)^M \cap L^\infty_{\mathrm{loc}}(\Omega)^M$ and $t \geq 2$)

$$\begin{aligned}
\int_{B_R - D} \lambda(u - f) \cdot (v - u) dx &\leq c\tau^{-1} \int_{B_R - D} |u - f|^2 dx + \tau \int_{B_R} |v - u|^2 dx \\
&\leq c\tau^{-1} R^n + c\tau R^2 \int_{B_R} |\nabla v - \nabla u|^2 dx \\
&\leq c\tau^{-1} R^n + c\tau R^2 \int_{B_R} (1 + |\nabla u|^2)^{\frac{t-2}{2}} |\nabla v - \nabla u|^2 dx \\
&\leq c\tau^{-1} R^n + c\tau R^2 \int_{B_R} |V(\nabla v) - V(\nabla u)|^2 dx
\end{aligned}$$

which by setting $\tau := \varepsilon R^{-2}$ with $\varepsilon > 0$ turns into

$$\int_{B_R - D} \lambda(u - f) \cdot (v - u) dx \leq c(\varepsilon) R^{n+2} + c\varepsilon \int_{B_R} |V(\nabla v) - V(\nabla u)|^2 dx. \tag{4.5.4}$$

Further, another application of Young's inequality gives for the first integral on the r.h.s. of (4.5.3)

$$
\begin{aligned}
& cR\int_{B_R} (1+|\nabla u|^2)^{\frac{t-1}{2}}|\nabla u-\nabla v|dx \\
& \leq c\varepsilon \int_{B_R} (1+|\nabla u|^2)^{\frac{t-2}{2}}|\nabla u-\nabla v|^2dx + c(\varepsilon)R^2\int_{B_R}(1+|\nabla u|^2)^{\frac{t}{2}}dx \\
& \leq c\varepsilon \int_{B_R} |V(\nabla u)-V(\nabla v)|^2dx + c(\varepsilon)R^2\int_{B_R}(1+|\nabla u|^2)^{\frac{t}{2}}dx.
\end{aligned}
\tag{4.5.5}
$$

Hence, incorporating (4.5.4) as well as (4.5.5) into (4.5.3) and absorbing terms by choosing $\varepsilon > 0$ sufficiently small we see

$$
\int_{B_R} |V(\nabla u)-V(\nabla v)|^2dx \leq cR^2\int_{B_R}(1+|\nabla u|^2)^{\frac{t}{2}}dx + cR^{n+2}. \tag{4.5.6}
$$

By means of (4.5.6) we now arrive at (recall (4.5.2))

$$
\int_{B_\rho}(1+|\nabla u|^2)^{\frac{t}{2}}dx \leq c\Big[\Big(\frac{\rho}{R}\Big)^n + R^2\Big]\int_{B_R}(1+|\nabla u|^2)^{\frac{t}{2}}dx + cR^{n+2} \tag{4.5.7}
$$

where we remark that (4.5.7) was just shown in the case $\rho \leq \frac{R}{2}$ (the estimate in the case $\frac{R}{2} < \rho < R$ is trivial).

As the next step we choose $\beta < n$ which may be arbitrarily close to n and with a suitable choice of R_0 we may apply [57], Lemma 2.1, p.86, to (4.5.7). Precisely, for all radii $\rho_* \leq R^* \leq R_0$ being sufficiently small we obtain

$$
\int_{B_{\rho_*}}(1+|\nabla u|^2)^{\frac{t}{2}}dx \leq c\Big[\Big(\frac{\rho_*}{R^*}\Big)^\beta\int_{B_{R^*}}(1+|\nabla u|^2)^{\frac{t}{2}}dx + c\rho_*^\beta\Big],
$$

which, for the particular choice $\rho_* = R$ and $R^* = R_0$, turns into

$$
\int_{B_R}(1+|\nabla u|^2)^{\frac{t}{2}}dx \leq c\Big[\Big(\frac{R}{R_0}\Big)^\beta\int_{B_{R_0}}(1+|\nabla u|^2)^{\frac{t}{2}}dx + cR^\beta\Big]. \tag{4.5.8}
$$

At this point we make of the crucial Campanato estimate (3.2) from [59], i.e., for some exponent $\sigma > 0$ it holds

$$
\begin{aligned}
& \fint_{B_\rho} |V(\nabla v)-(V(\nabla v))_{x_0,\rho}|^2dx \\
& \leq c\Big(\frac{\rho}{R}\Big)^\sigma \fint_{B_R} |V(\nabla v)-(V(\nabla v))_{x_0,R}|^2dx.
\end{aligned}
\tag{4.5.9}
$$

Here, for a given function w and a given radius r, the quantity $(w)_{x_0,r}$ is defined as $(w)_{x_0,r} := \fint_{B_r} w\,dx$. In accordance with [59], (5.6), the inequality (4.5.9) yields

$$\fint_{B_\rho} |V(\nabla u) - (V(\nabla u))_{x_0,\rho}|^2 dx \le c\Big(\frac{\rho}{R}\Big)^\sigma \fint_{B_R} |V(\nabla u) - (V(\nabla u))_{x_0,R}|^2 dx + \Big(\frac{R}{\rho}\Big)^n \fint_{B_R} |V(\nabla u) - V(\nabla v)|^2 dx.$$

As a consequence, (4.5.6) and (4.5.8) imply (recall $\beta < n$)

$$\begin{aligned}
&\int_{B_\rho} |V(\nabla u) - (V(\nabla u))_{x_0,\rho}|^2 dx \\
&\le c\bigg[\Big(\frac{\rho}{R}\Big)^{n+\sigma} \int_{B_R} |V(\nabla u) - (V(\nabla u))_{x_0,R}|^2 dx + R^2 \int_{B_R} (1+|\nabla u|^2)^{\frac{t}{2}} dx + cR^{n+2}\bigg] \\
&\le c\bigg[\Big(\frac{\rho}{R}\Big)^{n+\sigma} \int_{B_R} |V(\nabla u) - (V(\nabla u))_{x_0,R}|^2 dx + R^{2+\beta}\bigg]
\end{aligned}$$

Observing that

$$\Psi : \rho \mapsto \Psi(\rho) := \int_{B_\rho} |V(\nabla u) - (V(\nabla u))_{x_0,\rho}|^2 dx$$

is an increasing function and choosing $n < \beta + 2 < n + \sigma$ we may infer from [57], Lemma 2.1, p.86, that Ψ grows like $\rho^{2+\beta}$. Since $2 + \beta > n$ we get Hölder continuity of $V(\nabla u)$, in particular, ∇u is of class C^0. Taking into account the continuity of ∇u we observe that the function $w = \partial_\gamma u$ with $\gamma \in \{1, \dots, n\}$ solves an elliptic system with continuous coefficients and [57], Theorem 3.1, p.87, then provides local Hölder continuity of ∇u with exponent $\alpha \in (0, 1)$ (recall $u \in L^\infty_{\mathrm{loc}}(\Omega)^M$ as well as (4.1.16)). This completes the proof of Theorem 4.1.7. $\square$

Chapter 5

A modified TV-image inpainting method: minimization among sets with finite perimeter

5.1 Minimization among sets with finite perimeter: the classical setting

In this chapter we discuss a particular modification of the TV-image inpainting method. Briefly speaking we present a technique being specially devoted to the task of restoring images that consist only of completely black or completely white regions. In this context we adopt ideas as applied in [26] in the two-dimensional case.
Before going into details we fix our setup and state our precise assumptions: we consider a function $u : \Omega \to \mathbb{R}$ defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, where in contrast to the investigations from the previous chapters we now impose the restriction

$$u(x) \in \{0, 1\} \quad \text{a.e. on } \Omega, \tag{5.1.1}$$

which acts as an additional constraint on the restored image. Further we assume that our observed image is given through a $\mathcal{L}^n$-measurable function $f : \Omega - D \to [0, 1]$ where the $\mathcal{L}^n$-measurable subset D of Ω -as usual- denotes a possible inpainting region satisfying

$$0 \leq \mathcal{L}^n(D) < \mathcal{L}^n(\Omega). \tag{5.1.2}$$

As already mentioned, for points $x \in \Omega - D$, $f(x)$ can be interpreted as a measure for the intensity of the observed grey level. Our aim is to recover the missing part $D \to [0, 1]$ from the given data f where the apriori restriction

(5.1.1) on u represents a new feature in our forthcoming analysis.
As a starting point we look at the well-known variant of the TV-image inpainting problem

$$J[u] := \int\limits_{\Omega} |\nabla u| + \frac{\lambda}{2} \int\limits_{\Omega - D} (u - f)^2 dx \to \min.$$

Keeping in mind the constraint (5.1.1) it becomes evident that J has to be minimized on an adequate subclass of the space $BV(\Omega)$ consisting of characteristic functions. Thus, instead of minimizing J among functions we just seek minimizing sets of the functional (χ_E denoting the characteristic function of E)

$$\mathcal{F}[E] := \int\limits_{\Omega} |\nabla \chi_E| + \frac{\lambda}{2} \int\limits_{\Omega - D} (\chi_E - f)^2 dx, \tag{5.1.3}$$

being well-defined for all Borel sets $E \subset \Omega$ having finite perimeter in Ω, i.e., it holds $P(E, \Omega) := \int\limits_{\Omega} |\nabla \chi_E| < \infty$ and E is also called a Caccioppoli set in this context. For more details concerning sets of finite perimeter and the behavior of characteristic functions of such sets we refer the reader to [7] and [63].

Remark 5.1.1
Quoting [63], Proposition 3.1, p.42, we know that for a Borel set E there exists a Borel set $\widetilde{E}$ being equivalent to E (i.e. they may differ only by a set of $\mathcal{L}^n$-measure zero) and satisfying

$$0 < \mathcal{L}^n(\overline{E} \cap B_r(x)) < \omega_n r^n \quad \text{for all } x \in \partial \widetilde{E} \text{ and all } r > 0, \tag{5.1.4}$$

where ω_n denotes the volume of the unit ball. As a matter of fact we can modify Caccioppoli sets E on sets with $\mathcal{L}^n$-measure zero without changing the perimeter which means that we are concerned with equivalence classes of sets. As a consequence we assume in what follows that (5.1.4) *is valid for any set being under our consideration.*

After the above preparations we state our results on the existence of $\mathcal{F}$-minimizing sets having finite perimeter and on their regularity properties. To become more precise we take a more detailed look at the smoothness properties of the boundary part $\partial F \cap \Omega$ of an arbitrary $\mathcal{F}$-minimizing set and additionally prove a geometric statement about the intersection $\partial F \cap B_r(x)$ for a suitable ball $B_r(x) \subset \mathrm{Int}(D)$ provided that $\mathrm{Int}(D) \neq \emptyset$. In fact we can essentially adopt the arguments as already carried out in [26], Theorem 1, in the two-dimensional setting. However, for the reader's convenience, we give a proof of Theorem 5.1.2 without referring to the proof of Theorem 1 in [26].

Theorem 5.1.2
Suppose that D satisfies (5.1.2) *and consider a $\mathcal{L}^n$-measurable function $f : \Omega - D \to [0, 1]$ as well as the corresponding functional $\mathcal{F}$ from* (5.1.3). *Then it holds*

(a) There exists a set E of finite perimeter in Ω such that

$$\mathcal{F}[E] \leq \mathcal{F}[G]$$

for any Caccioppoli set $G \subset \Omega$.

(b) If $n \leq 7$ the following assertions hold true:

(i) the boundary part $\partial F \cap \Omega$ of any $\mathcal{F}$-minimizer F represents a C^1-hypersurface;

(ii) suppose that Int$(D) \neq \emptyset$. If E is a $\mathcal{F}$-minimizing set and if $x \in \partial E$ belongs to Int(D), then for an appropriate ball $B_r(x) \subset$ Int(D) the intersection $\partial E \cap B_r(x)$ is contained in a $(n-1)$-dimensional hyperplane.

Remark 5.1.3
Note that the regularity and geometric statement formulated in Theorem 5.1.2, part (b), (i) and part (b), (ii), respectively, crucially depend on the dimension n. In fact, quoting the regularity theory of minimal surfaces in $\mathbb{R}^n$, we have to expect singularities if $n \geq 8$. This has the reason that no singular minimal cones C in $\mathbb{R}^n$ (that are not hyperplanes) can exist if $n \leq 7$ (see, e.g., [63], and combine Theorem 9.10 and Theorem 10.10 therein) and since a minimal set E in $\mathbb{R}^n$ can only have singularities if there exist minimal cones in $\mathbb{R}^n$ having singularities, it is not surprising that part (b) of Theorem 5.1.2 is only valid if $n \leq 7$.
However, if $n \geq 8$, we indeed expect the existence of singularities but in accordance with the theory of the so-called reduced boundary $\partial^* E$, which is a particular subset of ∂E, the possible singularities of E occur in the set $(\partial E - \partial^* E) \cap \Omega$ and we can estimate its size and its Hausdorff dimension, respectively. To become more precise, if $n \geq 8$, we have to modify the statements of Theorem 5.1.2, (b), as follows

(i) The reduced boundary part $\partial E^* \cap \Omega$ of any $\mathcal{F}$-minimizer E is a C^1-hypersurface and $\mathcal{H}^s[(\partial E - \partial E^*) \cap \Omega] = 0$ for all $s > n - 8$.

(ii) Suppose that Int$(D) \neq \emptyset$. If E is a $\mathcal{F}$-minimizing set and if $x \in \partial E^*$ belongs to Int(D), then for an appropriate ball $B_r(x) \subset$ Int(D) the intersection $\partial E^* \cap B_r(x)$ is contained in a $(n-1)$-dimensional hyperplane.

Note that a definition of the reduced boundary ∂E^* of a Caccioppoli set E is provided in Remark 5.1.5 stated after the proof of Theorem 5.1.2. For more details concerning the regularity of minimal surfaces in $\mathbb{R}^n$ we refer the reader to [63].

Remark 5.1.4
As already elucidated in [26], Remark 1, we do not know if the statements (b), (i), and (b), (ii), of Theorem 5.1.2 are realistic in the context of image

inpainting. In fact, from the analytical point of view, these results seem to be nice.

Let us now come to the

Proof of Theorem 5.1.2. Let us assume the validity of the hypotheses of Theorem 5.1.2. For verifying part (a) we let (E_n) denote a $\mathcal{F}$-minimizing sequence of Caccioppoli sets and observe that

$$\sup_{n\in\mathbb{N}}\left[\int_\Omega |\nabla\chi_{E_n}| + \int_{\Omega-D}(\chi_{E_n}-f)^2dx\right] < \infty. \tag{5.1.5}$$

By BV-compactness (recall $\chi_{E_n}\in BV(\Omega)$ since E_n is Caccioppoli set for all $n\in\mathbb{N}$) we obtain existence of $u\in L^1(\Omega)$ together with the convergence

$$\chi_{E_n} \to: u \quad \text{in } L^1(\Omega) \text{ and a.e. on } \Omega \tag{5.1.6}$$

being valid at least for a subsequence (not relabeled).
As a consequence, lower semicontinuity of the total variation implies

$$\int_\Omega |\nabla u| \le \liminf_{n\to\infty}\int_\Omega |\nabla\chi_{E_n}| \tag{5.1.7}$$

and therewith $u\in BV(\Omega)$ whereas the a.e.-convergence from (5.1.6) directly yields $u(x)\in\{0,1\}$ a.e. on Ω. Besides we see (as $n\to\infty$)

$$\int_{\Omega-D}(\chi_{E_n}-f)^2dx \to \int_{\Omega-D}(u-f)^2dx \tag{5.1.8}$$

by dominated convergence. Letting $E := \{x\in\Omega, u(x)=1\}$, then $u=\chi_E$ and from (5.1.7) as well as (5.1.8) we further conclude $\mathcal{F}(E)\le\liminf\limits_{n\to\infty}\mathcal{F}(E_n)$ showing the $\mathcal{F}$-minimality of the set E. This proves assertion (a) of Theorem 5.1.2.

For verifying part (i) of assertion (b) we fix an arbitrary $\mathcal{F}$-minimizing set F. Here, the main idea is to apply the regularity results valid for almost minimal boundaries (see [93], Section 1.9), where we first introduce the terminology of almost minimal boundaries as done in [93], Section 1.5: consider a Caccioppoli set $\widetilde{F}$ with

$$F\Delta\widetilde{F} := (F-\widetilde{F})\cup(\widetilde{F}-F)\Subset B_r(x) \tag{5.1.9}$$

for a ball $B_r(x)\Subset\Omega$. We then call the boundary of a Caccioppoli set F almost minimal in Ω if for any $A\Subset\Omega$ there exists a number $R\in(0,\operatorname{dist}(A,\partial\Omega))$ and a non-decreasing function $\alpha:(0,R)\to[0,\infty)$ satisfying $\lim\limits_{r\to0^+}\alpha(r)=0$ such that

$$|\nabla\chi_F|(B_r(x)) \le |\nabla\chi_{\widetilde{F}}|(B_r(x)) + \alpha(r)r^{n-1} \tag{5.1.10}$$

for any $x \in A, r \in (0,R)$ and any $\widetilde{F}$ satisfying condition (5.1.9).
So, let us consider a Caccioppoli set $\widetilde{F}$ satisfying (5.1.9) for a ball $B_r(x) \Subset \Omega$. Since F is $\mathcal{F}$-minimal we see (recall $0 \le f \le 1$ a.e)

$$\begin{aligned}
&\int\limits_{B_r(x)} |\nabla \chi_F| \\
&\le \int\limits_{B_r(x)} |\nabla \chi_{\widetilde{F}}| + \frac{\lambda}{2} \int\limits_{(\Omega - D)\cap B_r(x)} \Big[(\chi_{\widetilde{F}} - f)^2 - (\chi_F - f)^2\Big] dx \\
&\le \int\limits_{B_r(x)} |\nabla \chi_{\widetilde{F}}| + \frac{\lambda}{2}\mathcal{L}^n(B_r(x)) \\
&= \int\limits_{B_r(x)} |\nabla \chi_{\widetilde{F}}| + \frac{\lambda}{2}\mathcal{L}^n(B_1(x))r^n.
\end{aligned} \tag{5.1.11}$$

Setting $\alpha(r) := \frac{\lambda}{2}\mathcal{L}^n(B_1(x))r$ it is obvious that $\alpha(r)$ serves as an admissible choice in (5.1.10). Summarizing, (5.1.11) shows that the boundary part $\partial F \cap \Omega$ of each $\mathcal{F}$-minimizer F is almost minimal in the sense of (5.1.10).

Besides, $r^{-1}\alpha(r)$ is non-increasing on $(0,R)$ and $\int\limits_0^R r^{-1}\alpha^{\frac{1}{2}}(r)dr < \infty$. At this point we can quote regularity results about almost minimal boundaries in the sense of (5.1.10) (see [93], Section 1.9) to justify that $\partial F \cap \Omega$ represents a C^1-hypersurface. This completes the proof of assertion (i) of part (b) of Theorem 5.1.2.

In view of the smoothness property of the boundary part $\partial E \cap \Omega$ of $\mathcal{F}$-minimizing sets E we can complete the proof of Theorem 5.1.2: we can state

$$\int\limits_U |\nabla \chi_E| = \mathcal{H}^{n-1}(\partial E \cap U)$$

for any open set $U \Subset \Omega$.
Choosing $U \Subset \mathrm{Int}(D)$, we may conclude that E is a local minimizer of the perimeter within the set U. This proves claim (ii) of part (b) of Theorem 5.1.2. □

Remark 5.1.5
The reduced boundary $\partial^* E$ of a Caccioppoli set E is a particular subset of ∂E and is defined as follows: a point x belongs to the reduced boundary $\partial^* E$ of a Caccioppoli set E if (compare [63], Definition 3.3, p.43)

- $\int\limits_{B_\rho(x)} |\nabla \chi_E| > 0$ *for all* $\rho > 0$,

- *the limit* $\nu(x) = \lim_{\rho \to 0} \nu_\rho(x)$ *exists, where*

$$\nu_\rho(x) = \frac{\int\limits_{B_\rho(x)} \nabla \chi_E}{\int\limits_{B_\rho(x)} |\nabla \chi_E|}$$

and

- $|\nu(x)| = 1$.

Using Besicovitch's theorem of the differentation of measures (see, e.g., [7], Theorem 2.22, p. 54) we get that $\nu(x)$ *exists together with* $|\nu(x)| = 1$ *for* $|\nabla \chi_E|$*-a.e.* $x \in \mathbb{R}^n$*. Furthermore, it holds* $\nabla \chi_E = \nu |\nabla \chi_E|$.
Note that the concept of the reduced boundary of a Caccioppoli set plays a crucial role when investigating regularity of the boundary of minimizing sets. For instance, the reduced boundary $\partial^* E$ *of a minimizing set* E *is analytic, where possible singularities of* ∂E *must occur in* $\partial E - \partial^* E$ *(see, e.g., [63], Chapter 8). Thus, one challenging problem is to estimate the size of the singular set* $\partial E - \partial^* E$.

5.2 Minimization among sets with finite perimeter including a volume constraint

In this section we discuss a problem arising in the above context, namely the inclusion of a "volume constraint". This idea has been suggested by M. Bildhauer and M. Fuchs in [26], Extension 3, and to become more precise we suppose that we are faced with the problem of restoring the observed image $f : \Omega - D \to \mathbb{R}$ taking only values in $\{0, 1\}$ on the entire domain Ω where we merely may use a given amount of black color for instance.

Recalling the defintion of the functional $\mathcal{F}$ via ((recall (5.1.3))

$$\mathcal{F}[E] := P(E, \Omega) + \frac{\lambda}{2} \int\limits_{\Omega - D} (\chi_E - f)^2 dx \tag{5.2.1}$$

we consider the minimization problem $\mathcal{F}[E] \to \min$ among all Caccioppoli sets E in Ω satisfying

$$\mathcal{L}^n(E) = m. \tag{5.2.2}$$

where $n \geq 2$ and $m \in (0, \mathcal{L}^n(\Omega))$ denotes a fixed number. In addition to the question of existence of minimizing sets satisfying the volume constraint it might also be of interest to study the analytical and topological properties of such minimizing sets subjected to the volume constraint.
Before stating our theorem concerning existence and regularity of minimizing

sets satisfying the volume constraint we introduce some notation where we essentially follow the article of Qinglan [103] (see Section 4 in this reference).

For any $\sigma \in (0,1)$ we define the set

$$\mathcal{F}_\sigma := \{G \subset \Omega : P(G,\Omega) < \infty, \mathcal{L}^n(G) = \sigma \mathcal{L}^n(\Omega)\}. \tag{5.2.3}$$

Moreover, for any set $G \in \mathcal{F}_\sigma$, a quasi perimeter of G is of the form

$$\mathcal{T}(G) = P(G,\Omega) + \mathcal{G}(G)$$

where $\mathcal{G}$ is a lower semicontinuous functional on $\mathcal{F}_\sigma$ satisfying the following estimate

$$\mathcal{G}(A) \leq \mathcal{G}(B) + C\mathcal{L}^n(A\Delta B)^\beta \tag{5.2.4}$$

for any $A, B \in \mathcal{F}_\sigma$, for a constant $C > 0$ and for a number $\beta > 1 - \frac{1}{n}$ for any $n \in \mathbb{N}$.

Thus, by setting $m := \sigma\mathcal{L}^n(\Omega)$ and $\mathcal{G}(G) := \frac{\lambda}{2} \int\limits_{\Omega - D} (\chi_G - f)^2 dx$ we observe that the problem of minimizing the functional $\mathcal{T}$ among all sets G in $\mathcal{F}_\sigma$ is equivalent to minimizing the functional $\mathcal{F}$ from (5.2.1) among all Caccioppoli sets E in Ω satisfying a volume constraint in the spirit of (5.2.2).

After these preparations we give a lemma showing that the functional $\mathcal{F}$ from (5.2.1) describes a quasi perimeter of E for any $E \in \mathcal{F}_\sigma$.

Lemma 5.2.1
For any $E \in \mathcal{F}_\sigma$ with $\mathcal{F}_\sigma$ from (5.2.3)*, the functional $\mathcal{F}$ from* (5.2.1) *is a quasi perimeter of E in the sense of the definition given above.*

Proof of Lemma 5.2.1. We fix $E \in \mathcal{F}_\sigma$. Clearly, $\mathcal{G}$ is lower semicontinuous on $\mathcal{F}_\sigma$. In order to verify that $\mathcal{G}$ satisfies the estimate (5.2.4) we fix $A, B \in \mathcal{F}_\sigma$ and obtain after carrying out some standard estimates (recall $\chi_{A\Delta B} = |\chi_A - \chi_B|$ as well as $0 \leq f \leq 1$ a.e.)

$$\frac{\lambda}{2} \int\limits_{\Omega - D} (\chi_A - f)^2 dx - \frac{\lambda}{2} \int\limits_{\Omega - D} (\chi_B - f)^2 dx \leq C\mathcal{L}^n(A\Delta B)$$

with constant $C := \frac{\lambda}{2} > 0$. This completes the proof of Lemma 5.2.1. $\square$

Now we state our main result in this section.

Theorem 5.2.2
Suppose that D *satisfies* (5.1.2) *and consider a* $\mathcal{L}^n$*-measurable function* $f : \Omega - D \to [0,1]$. *It then holds:*

(a) for any $\sigma \in (0,1)$ *the problem*

$$\mathcal{F} \to min \quad in\ \mathcal{F}_\sigma$$

admits at least one solution;

(b) if $n \leq 7$ *then the boundary* ∂E *of any* $\mathcal{F}$*-minimizer* E *in* $\mathcal{F}_\sigma$ *is a* $(n-1)$*-dimensional* $C^{1,\frac{1}{2}}$*-hypersurface in* Ω.

Proof of Theorem 5.2.2. The existence of a $\mathcal{F}$-minimizing Caccioppoli set E can be derived by following the lines of the proof of Theorem 5.1.2, where we directly obtain $E \in \mathcal{F}_\sigma$ by construction.
For proving part (b) we combine Lemma 5.2.1 and Theorem 4.5 in [103]. This completes the proof of Theorem 5.2.2. □

Remark 5.2.3
Note that the analytical and topological behavior of minimizing sets satisfying the volume constraint again crucially depends on the dimension n. *If* $n \geq 8$ *we then have to modify the statement of part (b) in the following way: the reduced boundary* $\partial^* E$ *of any* $\mathcal{F}$*-minimizer* E *in* $\mathcal{F}_\sigma$ *is a* $(n-1)$*-dimensional* $C^{1,\frac{1}{2}}$*-hypersurface in* Ω*, and moreover it holds* $dim((\partial E - \partial^* E) \cap \Omega) \leq n - 8$.

Chapter 6

Final remarks and some extensions

The aim of this thesis was to develop and to provide a comprehensive and complete existence and regularity theory for minimizers of the following modification of the TV-image inpainting method for any dimension $n \geq 2$ together with arbitrary codimension $M \geq 1$. This model consists in the minimization of the following functional

$$I[w] := \int_{\Omega} F(\nabla w) + \frac{\lambda}{2} \int_{\Omega - D} |w - f|^2 dx \tag{6.0.1}$$

on the space $BV(\Omega)^M \cap L^2(\Omega - D)^M$.
Here, $\lambda > 0$ denotes a positive regularization parameter, f is at least of class $L^2(\Omega - D)^M$ and $F \in C^1(\mathbb{R}^{nM})$ is a strictly convex density being of linear growth.
For the sake of completeness we recap the main results of this thesis:

- The problem "$I \to \min$" with I from (6.0.1) is solvable in $BV(\Omega)^M \cap L^2(\Omega - D)^M$ where solutions of this problem can be seen as generalized minimizers (w.r.t. a suitable relaxation) of the original problem "$I \to \min$ in $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$". For two minima $u, \widetilde{u}$ we can moreover state $u = \widetilde{u}$ a.e. on $\Omega - D$ whereas $\nabla^a u = \nabla^a \widetilde{u}$ a.e. on Ω.
  This result is not very surprising but the novelty in this context was that we imposed rather mild assumptions on the data (see Remark 2.1.2) which caused severe problems during our analysis. In particular, we did not assume a structure condition on F in the spirit of, e.g., (3.1.8). From the point of view of applications, this might seem somewhat artificial but from the analytical point of view, it is interesting to drop this condition. As a byproduct we proved the density of smooth functions in spaces like $BV(\Omega)^M \cap L^2(\Omega - D)^M$ which, according to our knowledge, is not known in literature yet.

- The dual problem associated to "$I \to \min$ in $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$" admits a unique solution $\sigma \in L^\infty(\Omega)^{nM}$. Moreover, the inf-sup relation holds and we have the validity of the duality formula

$$\sigma = DF(\nabla^a u) \quad \text{a.e. on } \Omega.$$

  Additionally, each I-minimizing sequence converges strongly in $L^2(\Omega - D)^M$ to the unique restriction of any generalized minimizer to the set $\Omega - D$.
  Here, we particularly mention the uniqueness of the dual solution, the validity of the duality formula on the entire domain Ω and the described compactness property of I-minimizing sequences as new contributions. However, from the theoretical point of view, it might be interesting if our results (in particular the uniqueness of the dual solution) can be established under even weaker assumptions on F as well. To become more precise we want to drop the condition of continuously differentiability of F which leads to the necessity of working with the subdifferential of the convex function F (see, e.g., (2.1.15)).

- Assume $f \in L^\infty(\Omega - D)^M$. Suppose further that $F \in C^2(\mathbb{R}^{nM})$ satisfies the above assumptions, the condition of μ-ellipticity for some $\mu \in (1, 2)$ and that some suitable structure conditions in the vectorial case $M > 1$ (plus a Hölder condition on D^2F) are imposed. We then could prove solvability of the problem "$I \to \min$ in $W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M$", where we even get uniqueness of the corresponding I-minimizer u and $u \in L^\infty(\Omega)^M$. Moreover, it holds $u \in C^{1,\alpha}(\Omega)^M$ for any $\alpha \in (0, 1)$. For large values of μ, precisely for $\mu \geq 2$, we could show that for each generalized minimizer $\widetilde{u}$ there exists an open subset $\Omega_0^{\widetilde{u}} \subset \Omega$ of full $\mathcal{L}^n$-measure such that $u \in C^{1,\frac{1}{2}}(\Omega_0^{\widetilde{u}})^M$ with $\mathcal{L}^n(\Omega - \Omega_0^{\widetilde{u}}) = 0$.
  The above full interior $C^{1,\alpha}$-regularity result under the condition that $\mu < 2$ is in accordance with the result from the joint article with M. Bildhauer and M. Fuchs [27], where the same regularity degree for u has been derived in the scalar case together with $n = 2$. We think, this result is optimal in the presence of the inpainting quantity. However we conjecture that the partial $C^{1,\frac{1}{2}}$-regularity result is not optimal. Nevertheless, due to the presence of the inpainting quantity, it is not possible to refer, e.g., to [9], by adding some obvious modifications.

After this review of the essential results of this thesis we want to discuss some extensions of the presented material.

With f as above and for a strictly convex density F of linear growth, a natural extension of the problem (6.0.1) is the following problem, which seems to be of interest in the context of higher order denoising of images (see, e.g., [83])

$$J[w] := \int_\Omega |w - f|^2 dx + \int_\Omega F(\nabla^2 w) \to \min \ BV^2(\Omega)^M \cap L^2(\Omega)^M. \qquad (6.0.2)$$

Here, the space $BV^2(\Omega)^M$ is defined as the space of all $W^{1,1}(\Omega)^M$-functions for which the second order distributional derivative $\nabla^2 u$ is a tensor-valued Radon measure of finite total variation. In what follows we briefly sketch the proof of solvability of the problem (6.0.2): fixing a minimizing sequence (u_k) from the space $BV^2(\Omega)^M \cap L^2(\Omega)^M$ we directly get the uniform information

$$\sup_k \int_\Omega |u_k|^2 dx < \infty,$$
$$\sup_k |\nabla^2 u_k|(\Omega) < \infty.$$

Using the well-known Poincaré-inequality for functions from the space $W^{2,1}(\Omega)^M$ (see, e.g., [104], Lemma 4.2.2, p. 183, choosing $m = 2, p = 1$ and $\varepsilon = 1$ therein) and quoting standard approximation results for functions of class $BV^2(\Omega)^M$ (see [45]) we can state the following variant of Poincaré's inequality ($C = C(n, \Omega)$ denotes a positive constant)

$$\|\nabla w\|_{L^1(\Omega)} \le C[\|w\|_{L^1(\Omega)} + |\nabla^2 w|(\Omega)].$$

Combining this inequality with the above uniform estimates we directly get

$$\sup_k \|\nabla u_k\|_{L^1(\Omega)} < \infty$$

and we have compactness of (u_k) in $BV^2(\Omega)^M$, i.e., there exists a function $u \in BV^2(\Omega)^M$ for which we have $u_k \to: u$ in $W^{1,1}(\Omega)^M$ and a.e. up to a subsequence (see [45]). Fatou's lemma further implies $u \in L^2(\Omega)^M$ and thus, $J[u]$ is well-defined.
Applying lower semicontinuity of $|\nabla^2 u|(\Omega)$ w.r.t. L^1-convergence (see, e.g., [68]) in combination with Fatou's lemma it follows that u is a solution of (6.0.2). Note that by adapting the approximation procedure for functions of class $BV^2(\Omega)^M$ we can further show that u can be seen as a generalized minimizer of the problem

$$\begin{aligned} I[w] := \int_\Omega |w - f|^2 dx + \int_\Omega F(\nabla^2 w) dx \to \min \\ \text{in } W^{2,1}(\Omega)^M \cap L^2(\Omega)^M, \end{aligned} \tag{6.0.3}$$

where among other things we use Reshetnyak's continuity theorem (see, e.g., [9], Proposition 2.2).
Note that in view of the continuous embedding (see [45], Theorem 3.1)

$$BV^2(\Omega)^M \hookrightarrow L^q(\Omega)^M \quad \text{for all } q \le \frac{n}{n-2},$$

we can study the problem (6.0.2) in the entire space $BV^2(\Omega)^M$ provided that $n \le 4$. Hence, in case $n > 4$, the requirement "$w \in L^2(\Omega)^M$" acts as an additional constraint.

Clearly we can adjust the problem (6.0.2) to obtain a modification of the higher order total variation image inpainting model, i.e., we then seek minimizers of the functional

$$\widetilde{J}[w] := \int_{\Omega - D} |w - f|^2 dx + \int_{\Omega} F(\nabla^2 w), \quad w \in BV^2(\Omega)^M \cap L^2(\Omega - D)^M.$$

In this setting, existence of at least one minimizer from the space $BV^2(\Omega)^M \cap L^2(\Omega - D)^M$ can be derived by quoting (more or less) standard arguments. However, the justification that minimizers of the above functional can be seen as generalized minimizers of the functional

$$\widetilde{I}[w] := \int_{\Omega - D} |w - f|^2 dx + \int_{\Omega} F(\nabla^2 w) dx, \quad w \in W^{2,1}(\Omega)^M \cap L^2(\Omega - D)^M$$

is not immediate and requires the proof of new density results of smooth functions in spaces like $BV^2(\Omega)^M \cap L^2(\Omega - D)^M$ (we refer the reader to [80], where such density results are established).

As before, an alternative approach is to pass to the dual problem associated to "$\widetilde{I} \to \min$ in $W^{2,1}(\Omega)^M \cap L^2(\Omega - D)^M$". Here, among solvability of the dual problem and the validity of the inf-sup relation, it might be of interest to derive uniqueness of the dual solution and to establish a duality formula relating the dual solution to the generalized $\widetilde{I}$-minimizers of class BV.

Another interesting question arising in the above context is to derive smoothness properties of generalized $\widetilde{I}$-minimizers. Fixing a real number $\mu > 1$ and taking into account our model integrand $F(P) := \Phi_\mu(P)$ with $\Phi_\mu(P)$ from (3.1.11) we conjecture that it is possible to show at least partial $C^{2,\beta}$-regularity of generalized minimizers for all $\mu > 1$. Besides we think that it might be of interest to investigate whether we obtain full interior $C^{2,\alpha}$-regularity for generalized minimizer on condition that $\mu \in (1, 2)$ (for any dimension n with arbitrary codimension M).
Considering the scalar case together with $n = 2$, another interesting idea is to investigate the regularity behavior of $\widetilde{I}$-minimizers under the condition that F is of subquadratic growth, i.e., $F(\nabla^2 w) = (1 + |\nabla^2 w|^2)^{\frac{p}{2}}$ with $1 < p < 2$. In this setting it is reasonable to minimize $\widetilde{I}$ in the energy class $\mathcal{K} := \{w \in W^{2,p}(\Omega),\ 0 \le w(x) \le 1\}$, where we remark that in accordance with the continuity of the Sobolev embedding $W^{2,p}(\Omega) \hookrightarrow C^0(\Omega)$, the class $\mathcal{K}$ is non-empty. Now, it might be of interest to discuss the regularity properties of $\widetilde{I}$-minimizers being of class $\mathcal{K}$.

From the numerical point of view the above models of higher order denoising are not very nice to handle. For that reason we propose to consider the following alternative model (note that the dimension n and the codimension M are arbitrary) and consider exponents $p, q, s \in (1, \infty)$ and positive regularization parameters $\alpha_1, \alpha_2 > 0$ (we further assume $f \in L^s(\Omega)^M$)

$$E(u,v) := \int\limits_\Omega |u-f|^s dx + \alpha_1 \int\limits_\Omega |\nabla u - v|^q dx + \alpha_2 \int\limits_\Omega |\nabla v|^p dx \to \min \qquad (6.0.4)$$
$$\text{in the energy class } Y,$$

where

$$Y := \{(u,v) \in W^{1,q}(\Omega)^M \times W^{1,p}(\Omega)^{nM} : \ u \in L^s(\Omega)^M, v \in L^q(\Omega)^{nM}\}.$$

Note that according to the values of p, q and s, we might have $Y = W^{1,q}(\Omega)^M \times W^{1,p}(\Omega)^{nM}$.
Now it is interesting to discuss existence and regularity of solutions of the problem (6.0.4). In fact, strict convexity of the corresponding terms shows that (6.0.4) has at most one solution.
Clearly we can involve linear growth models as well. This means that we study solvability of the problem

$$\int\limits_\Omega |u-f|^s dx + \alpha_1 \int\limits_\Omega |\nabla u - v|^q dx + \alpha_2 \int\limits_\Omega |\nabla v| \to \min$$
$$\text{in } W^{1,q}(\Omega)^M \times BV(\Omega)^{nM},$$

where more generally we can replace the rather unpleasant total variation $|\nabla v|$ through $\Phi(|\nabla v|)$ with a function $\Phi : [0,\infty) \to [0,\infty)$ being (strictly) convex, (strictly) increasing and of linear growth in a suitable sense. Fixing a real number $\mu > 1$, a natural choice in this context would be $\Phi(t) := \Phi_\mu(t)$ with $\Phi_\mu(t)$ from (3.1.11). Choosing $\Phi(t) := \Phi_\mu(t)$ it follows that the above problem has at most one solution $(\widetilde{u}, \widetilde{v}) \in W^{1,q}(\Omega)^M \times BV(\Omega)^{nM}$.
Of course we can also look at “linear coupling“ , which means that we seek minimizers of the functional

$$\int\limits_\Omega |u-f|^s dx + \alpha_1 \int\limits_\Omega |\nabla u - v| + \alpha_2 \int\limits_\Omega |\nabla v| \to \min$$
$$\text{in } BV(\Omega)^M \times BV(\Omega)^{nM},$$

where, according to the situation at hand, it might be reasonable to replace the total variation through a function being in the spirit of Φ or Φ_μ from above. Note that the above models are also applicable in the context of image inpainting.

Finally we discuss an extension of the model that has been presented in the fifth chapter where from now on we consider the scalar case $M = 1$ as well as the context of pure denoising of images. With Ω and $f : \Omega \to [0,1]$ as above we first consider the classical TV-image inpainting method leading to the minimization of the functional ($\lambda > 0$- as usual- denotes a free regularization parameter)

$$J[w] := \int\limits_\Omega |\nabla w| + \frac{\lambda}{2} \int\limits_\Omega (w-f)^2 dx, \quad w \in BV(\Omega) \cap L^2(\Omega). \qquad (6.0.5)$$

As already elucidated in the introduction of this thesis we can derive existence of a unique minimizer $u \in BV(\Omega)$ satisfying $0 \le u(x) \le 1$ a.e. on Ω. As a consequence, u might have singularities as jumps on the edges of the generated image. However, in accordance with the decomposition of the vector-valued Radon measure $\nabla u = \nabla^a u \llcorner \mathcal{L}^n + \nabla^j u + \nabla^c u$ (see, e.g., [7], pp. 184 or Section 2.2 of this thesis), we apriori cannot exclude other forms of singularities due to the presence of the Cantor part $\nabla^c u$. As a consequence, the idea is to minimize the above functional in those subclasses of the space $BV(\Omega)$ where we automatically have $\nabla^c u = 0$.

A natural approach in this context is to introduce the space of special functions of bounded variation $SBV(\Omega)$ which just contains all functions of class $BV(\Omega)$ for which the Cantor part of its distributional derivative vanishes (for more details concerning special functions of bounded variation we refer the reader to [7], pp.212). However, an analytical problem arises if we minimize J among all functions of class $SBV(\Omega)$ since J-minimizing sequences (w_k) from the space $SBV(\Omega)$ in general have no compactness properties. An idea to overcome this difficulty is to replace J through a functional of the type

$$\hat{J}[w] := \int\limits_{\Omega} |\nabla^a w|^p dx + \mathcal{H}^{n-1}(S_w) + \frac{\lambda}{2} \int\limits_{\Omega} (w-f)^2 dx, \quad w \in SBV(\Omega) \cap L^2(\Omega),$$

where $p > 1$ is a fixed exponent and the set S_w -as usual- denotes the approximate discontinuity set of w (see Section 2.2 of this thesis). We can apply [7], Theorem 4.8, p.216, for getting compactness of $\hat{J}$-minimizing sequences (w_k) in the appropriate space $SBV(\Omega)$: first we note that $\hat{J}$-minimizing sequences (w_k) from the space $SBV(\Omega) \cap L^2(\Omega)$ can be chosen in such a way that

$$0 \le w_k(x) \le 1 \quad \text{a.e. on } \Omega. \tag{6.0.6}$$

Note that (6.0.6) can be proven by following the arguments of the proof of Theorem 3.1.4, where we briefly show that w.l.o.g. we may assume $w_k \le 1$ a.e. on Ω: with $\psi(t) := \min\{t, 1\}$ for $t \in \mathbb{R}$ we consider $v_k := \psi \circ w_k$ which is clearly of class $BV(\Omega)$. Furthermore, v_k is of class $SBV(\Omega)$ since by applying the chain rule for real-valued BV-functions (see equation (2.2.2) in this thesis), the singular part $\nabla^s v_k$ of the vector-valued Radon measure ∇v_k is concentrated on the set J_{u_k} (recall $u_k \in SBV(\Omega)$), i.e., on the set of approximate jump points of u_k, which by Federer-Vol'pert's theorem is countably $\mathcal{H}^{n-1}$-rectifiable (see, e.g., [7], Theorem 3.87, p. 178). In particular, J_{u_k} is then σ-finite w.r.t. $\mathcal{H}^{n-1}$ and by using [7], Proposition 4.2, p. 213, we get that $v_k \in SBV(\Omega)$.
Furthermore we have $|v_k - f| \le |w_k - f|$ a.e. on $\Omega - D$. Recalling the inequality $|\nabla v_k| \le \mathrm{Lip}(\psi)|\nabla w_k| = |\nabla w_k|$ for the measures $|\nabla v_k|$ and $|\nabla w_k|$ (compare Lemma 2.2.1 in this thesis) we can quote arguments as already used in the proof of Theorem 3.1.4 and Theorem 3.1.5, respectively, to infer

$$\hat{J}[v_k] \le \hat{J}[w_k].$$

Thus, w.l.o.g., we may assume $w_k \le 1$ a.e. on Ω for any $\hat{J}$-minimizing sequence (w_k). With $\widetilde{\psi}(t) := \max\{0, t\}, t \in \mathbb{R}$, we can use analogous arguments as applied

above to justify that w.l.o.g. we may assume $0 \le w_k$ a.e. on Ω as well.
As a consequence we derive weak-$*$ convergence in $BV(\Omega)$ of a subsequence of (w_k) to a function $u \in SBV(\Omega)$ which means (see [7], Proposition 3.13, p.125) $w_k \to: u$ in $L^1(\Omega)$ and a.e. up to a subsequence as $k \to \infty$ where we additionally get $0 \le u(x) \le 1$ a.e. on Ω. In view of the weak-$*$ convergence of w_k to u in $BV(\Omega)$ (up to a subsequence) we then get by applying [7], Theorem 4.7, p.216 (choose $\varphi(t) := t^p$ and $\theta \equiv 1$) and Fatou's lemma

$$\begin{aligned}\hat{J}[u] &\le \liminf_{k\to\infty} \int_\Omega |\nabla^a w_k|^p dx + \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{w_k}) + \frac{\lambda}{2} \liminf_{k\to\infty} \int_\Omega (w_k - f)^2 dx \\ &\le \liminf_{k\to\infty} \left[\int_\Omega |\nabla^a w_k|^p dx + \mathcal{H}^{n-1}(J_{w_k}) + \frac{\lambda}{2} \int_\Omega (w_k - f)^2 dx \right] \\ &= \liminf_{k\to\infty} \hat{J}[w_k].\end{aligned}$$

Thus, $u \in SBV(\Omega)$ is a $\hat{J}$-minimizer satisfying $0 \le u(x) \le 1$ a.e. on Ω. From the point of view of image processing, $u(x)$ can be interpreted as a measure for the intensity of the grey level where possible points of discontinuity are performed by jumps.

Another way to force that the Cantor part of the distributional derivative of any BV-minimizer of the functional J from (6.0.5) vanishes is to minimize J among all bounded piecewise constant functions $w : \Omega \to \mathbb{R}$, i.e., in the subspace $PC(\Omega) \cap L^\infty(\Omega)$ of $BV(\Omega)$. The space $PC(\Omega)$ contains all special functions of bounded variation whose family of level sets is a Caccioppoli partition, i.e., $u \in PC(\Omega)$ if there exists a Caccioppoli partition $(E_i)_{i\in\mathbb{N}}$ of Ω and a map $t : \mathbb{N} \to \mathbb{R}$ with (see [7], Definition 4.21, p.231)

$$u = \sum_{i\in\mathbb{N}} t_i \chi_{E_i}.$$

Note that we can replace $\mathbb{N}$ by any countably infinite index set I.
Going back to the problem of minimizing J among bounded piecewise constant functions, a similar problem arises as already seen in the setting of SBV-functions since J-minimizing sequences apriori have no compactness properties in $PC(\Omega)$. However, replacing J by the functional

$$\overline{J}[w] := \mathcal{H}^{n-1}(S_w) + \frac{\lambda}{2} \int_\Omega (w - f)^2 dx, \quad w \in PC(\Omega) \cap L^\infty(\Omega)$$

it follows the following uniform estimate for $\overline{J}$-minimizing sequences (w_k)

$$\sup_k \left[\|w_k\|_{L^\infty} + \mathcal{H}^{n-1}(S_{w_k}) \right] < \infty,$$

which, in view of [7], Theorem 4.25, p. 234, gives compactness in $PC(\Omega)$. At this point we note that we can replace (w_k) by a $\overline{J}$-minimizing sequence $(\widetilde{w_k})$ with the property $0 \le \widetilde{w_k} \le 1$ a.e. on Ω since we may argue in an analogous manner as already seen in the context of SBV-functions above. Here it is worth remarking that for an arbitrary Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$ together with a given function $u : \Omega \to \mathbb{R}$ of class $PC(\Omega)$, the composition $v := \psi \circ u$ is still of class $PC(\Omega)$. This fact follows from the chain rule for real-valued BV-functions (see formula (2.2.2) in this thesis again). Thus, there exists a function $u \in PC(\Omega)$ such that $w_k \to: u$ in measure as $k \to \infty$ and up to subsequence. From this convergence we can extract another subsequence (not relabeled) with $w_k \to u$ a.e. on Ω. By using lower semicontinuity we then obtain that $u \in PC(\Omega)$ is the unique $\overline{J}$-minimizer. Furthermore we can state that $0 \le u(x) \le 1$ a.e. on Ω and that all possible singularities are jumps at the edges (for more details we refer, e.g., to [7], pp. 244).

Chapter 7

Appendix

7.1 Some auxiliary lemmata

In this section we collect and prove some auxiliary results that have been of important meaning in the course of this thesis.

7.1.1 Regularization

This section is devoted to the discussion of the regularity properties of the unique minimizer of the regularization that has been introduced in Section 2.4 of this thesis.
With Ω and D as usual we suppose that our partial observation $f : \Omega - D \to \mathbb{R}^M$ fulfills

$$f \in L^\infty(\Omega - D)^M. \tag{7.1.1}$$

We further assume that we are given a density $F : \mathbb{R}^{nM} \to [0, \infty)$ satisfying the following set of hypotheses: there are real, positive constants ν_0, ν_1, ν_2, K and $\nu \in (0, 1)$ such that for all $P, Q \in \mathbb{R}^{nM}$

$$F \in C^2(\mathbb{R}^{nM}) \tag{7.1.2}$$

$$|DF(P)| \leq \nu_0 \tag{7.1.3}$$

$$\nu_1(1 + |P|)^{-\mu}|Q|^2 \leq D^2F(P)(Q, Q) \leq \nu_2(1 + |P|)^{-1}|Q|^2, \tag{7.1.4}$$

$$F(Z) = g(|Z|^2), \quad g \in C^2([0, \infty), [0, \infty)), \tag{7.1.5}$$

$$|D^2F(P) - D^2F(Q)| \leq K|P - Q|^\nu. \tag{7.1.6}$$

We now approximate our original problem

$$I[w] = \int\limits_\Omega F(\nabla w)dx + \frac{\lambda}{2} \int\limits_{\Omega - D} |w - f|^2 dx \to \min$$

$$\text{in } W^{1,1}(\Omega)^M \cap L^2(\Omega - D)^M,$$

by the following sequence of more regular problems (we fix $\delta \in (0,1]$)

$$\begin{aligned} I_\delta[w] := \int_\Omega F_\delta(\nabla w)dx + \frac{\lambda}{2}\int_{\Omega - D} |w-f|^2 dx \to \min \\ \text{in } W^{1,2}(\Omega)^M, \end{aligned} \tag{7.1.7}$$

where

$$F_\delta(P) := \frac{\delta}{2}|P|^2 + F(P), \quad P \in \mathbb{R}^{nM}, \tag{7.1.8}$$

and denote by $u_\delta \in W^{1,2}(\Omega)^M$ the unique I_δ-minimizer for which we can show the following nice properties.

Lemma 7.1.1
For $\delta \in (0,1]$ being fixed and with Ω, D as usual we assume (7.1.1) for the data f. In addition we require that F satisfies (7.1.2)–(7.1.6) with the prescribed ellipticity parameter $\mu > 1$. For the unique solution $u_\delta \in W^{1,2}(\Omega)^M$ of problem (7.1.7) we then have

(a) $\sup\limits_\Omega |u_\delta| \le \sup\limits_{\Omega - D} |f|$,

(b) $u_\delta \in W^{2,2}_{loc}(\Omega)^M$,

(c) $u_\delta \in W^{1,\infty}_{loc}(\Omega)^M$,

(d) $u_\delta \in C^{1,\kappa}(\Omega)^M$ *for all* $\kappa \in (0,1)$.

Remark 7.1.2
In the scalar case $M = 1$, the statements of Lemma 7.1.1 hold true under much weaker assumptions on the density F (for details we refer to [24]). Considering the vectorial case $M > 1$, Lemma 7.1.1 partially holds true under weaker hypotheses on F. Precisely we merely need the entire range of hypotheses imposed on F for deriving full interior $C^{1,\kappa}$-regularity of u_δ.

Proof of Lemma 7.1.1. Assuming the hypotheses of Lemma 7.1.1 we first refer to Remark 3.4.2 in Section 3.3 of this thesis for a discussion of part (a).

For proving (b) we make use of the well-known difference quotient technique: let us fix a point $\xi \in \Omega$ and a radius $R > 0$ such that $B_{2R}(\xi) \Subset \Omega$. Moreover we let $\eta \in C^\infty_0(B_{2R}(\xi))$ satisfying $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_R(\xi)$ as well as $|\nabla \eta| \le \frac{c}{R}$ $(c > 0)$. Further $(\gamma \in \{1, \dots, n\})$

$$\Delta^h_\gamma u_\delta := \frac{u_\delta(x + he_\gamma) - u_\delta(x)}{h}$$

denotes the difference quotient of u_δ in the direction $e_\gamma \in \mathbb{R}^n$ with step width h fulfilling $0 < |h| < \operatorname{dist}(B_{2R}(\xi), \partial\Omega)$.

We start by observing that u_δ is a solution of the Euler equation

$$\int_\Omega DF_\delta(\nabla u_\delta) : \nabla\varphi dx + \lambda \int_{\Omega - D} (u_\delta - f) \cdot \varphi dx = 0$$

for all $\varphi \in W^{1,2}(\Omega)$ with compact support in Ω. Inserting the admissible function $\varphi = \Delta_\gamma^{-h}(\eta^2 \Delta_\gamma^h u_\delta)$ in the above Euler equation it follows after integrating by parts

$$\begin{aligned} &\int_{B_{2R}(\xi)} \Delta_\gamma^h DF_\delta(\nabla u_\delta) : \nabla(\eta^2 \Delta_\gamma^h u_\delta) dx \\ &= \lambda \int_{B_{2R}(\xi) - D} (u_\delta - f) \cdot \Delta_\gamma^{-h}(\eta^2 \Delta_\gamma^h u_\delta) dx \end{aligned} \tag{7.1.9}$$

In the following we concentrate on the term $\Delta_\gamma^h DF_\delta(\nabla u_\delta)$. We get

$$\begin{aligned} \Delta_\gamma^h DF_\delta(\nabla u_\delta) &= \frac{1}{h}(DF_\delta(\nabla u_\delta(x + he_\gamma)) - DF_\delta(\nabla u_\delta)) \\ &= \frac{1}{h}\int_0^1 \frac{d}{dt} DF_\delta(\nabla u_\delta + th\Delta_\gamma^h \nabla u_\delta) dt \\ &= \Big(\int_0^1 D^2F_\delta(\nabla u_\delta + th\Delta_\gamma^h \nabla u_\delta) dt\Big) \Delta_\gamma^h \nabla u_\delta \\ &=: \Big(\int_0^1 B_{h,\gamma}^{(\delta)}(t) dt\Big)(\Delta_\gamma^h \nabla u_\delta, \cdot). \end{aligned} \tag{7.1.10}$$

Incorporating (7.1.10) in (7.1.9) we have

$$\begin{aligned} &\int_{B_{2R}(\xi)} \eta^2 \Big(\int_0^1 B_{h,\gamma}^{(\delta)}(t) dt\Big)(\Delta_\gamma^h \nabla u_\delta, \Delta_\gamma^h \nabla u_\delta) dx \\ &= -2 \int_{B_{2R}(\xi)} \eta \Big(\int_0^1 B_{h,\gamma}^{(\delta)}(t) dt\Big)(\Delta_\gamma^h \nabla u_\delta, \nabla\eta \otimes \Delta_\gamma^h u_\delta) dx \\ &+ \lambda \int_{B_{2R}(\xi) - D} (u_\delta - f) \cdot \Delta_\gamma^{-h}(\eta^2 \Delta_\gamma^h u_\delta) dx. \end{aligned} \tag{7.1.11}$$

At this point we study the last integral on the r.h.s. of (7.1.11). An integration by parts leads to

$$\begin{aligned}
&\int\limits_{B_{2R}(\xi)-D} (u_\delta - f)\cdot \Delta_\gamma^{-h}(\eta^2 \Delta_\gamma^h u_\delta)dx \\
&= \int\limits_{B_{2R}(\xi)} (u_\delta - f)\cdot \Delta_\gamma^{-h}(\eta^2 \Delta_\gamma^h u_\delta)dx - \int\limits_{B_{2R}(\xi)\cap D} (u_\delta - f)\cdot \Delta_\gamma^{-h}(\eta^2 \Delta_\gamma^h u_\delta)dx \\
&= -\int\limits_{B_{2R}(\xi)} |\Delta_\gamma^h u_\delta|^2 \eta^2 dx - \int\limits_{B_{2R}(\xi)} f\cdot \Delta_\gamma^{-h}(\eta^2 \Delta_\gamma^h u_\delta)dx \\
&- \int\limits_{B_{2R}(\xi)\cap D} (u_\delta - f)\cdot \Delta_\gamma^{-h}(\eta^2 \Delta_\gamma^h u_\delta)dx.
\end{aligned}$$

Using part (a) and the boundedness of f, (7.1.11) turns into

$$\begin{aligned}
&\int\limits_{B_{2R}(\xi)} \eta^2 \Bigg(\int\limits_0^1 B_{h,\gamma}^{(\delta)}(t)dt\Bigg)(\Delta_\gamma^h \nabla u_\delta, \Delta_\gamma^h \nabla u_\delta)dx + \lambda \int\limits_{B_{2R}(\xi)} |\Delta_\gamma^h u_\delta|^2 \eta^2 dx \\
&\le -2 \int\limits_{B_{2R}(\xi)} \eta \Bigg(\int\limits_0^1 B_{h,\gamma}^{(\delta)}(t)dt\Bigg)(\Delta_\gamma^h \nabla u_\delta, \nabla\eta \otimes \Delta_\gamma^h u_\delta)dx \qquad (7.1.12) \\
&+ c \int\limits_{B_{2R}(\xi)} |\Delta_\gamma^{-h}(\eta^2 \Delta_\gamma^h u_\delta)|dx
\end{aligned}$$

The last integral on the r.h.s. of (7.1.12) can be handled via [62], Lemma 7.23, p.168, which yields after using Young's inequality ($\varepsilon > 0$)

$$\begin{aligned}
&\int\limits_{B_{2R}(\xi)} |\Delta_\gamma^{-h}(\eta^2 \Delta_\gamma^h u_\delta)|dx \\
&\le c \int\limits_{B_{2R}(\xi)} \eta |\nabla\eta| |\Delta_\gamma^h u_\delta| dx + c \int\limits_{B_{2R}(\xi)} \eta^2 |\Delta_\gamma^h \nabla u_\delta| dx \\
&\le \varepsilon \int\limits_{B_{2R}(\xi)} \eta^2 |\Delta_\gamma^h u_\delta|^2 dx + c\varepsilon^{-1} \int\limits_{B_{2R}(\xi)} |\nabla\eta|^2 dx \\
&+ \varepsilon \int\limits_{B_{2R}(\xi)} \eta^2 |\Delta_\gamma^h \nabla u_\delta|^2 dx + c\varepsilon^{-1} \int\limits_{B_{2R}(\xi)} \eta^2 dx \\
&\le c\varepsilon \int\limits_{B_{2R}(\xi)} \eta^2 \Bigg(\int\limits_0^1 B_{h,\gamma}^{(\delta)}(t)dt\Bigg)(\Delta_\gamma^h \nabla u_\delta, \Delta_\gamma^h \nabla u_\delta)dx + c\varepsilon^{-1} \int\limits_{B_{2R}(\xi)} |\nabla\eta|^2 dx \\
&+ \varepsilon \int\limits_{B_{2R}(\xi)} \eta^2 |\Delta_\gamma^h u_\delta|^2 dx + c\varepsilon^{-1} \int\limits_{B_{2R}(\xi)} \eta^2 dx.
\end{aligned}$$

where, in the last inequality, we have used (7.1.4) which yields ($\widetilde{\lambda} > 0$ denotes a real parameter which, in particular, is independent of h)

$$\Big(\int_0^1 B^{(\delta)}_{h,\gamma}(t)dt\Big)(Q,Q) = \Big(\int_0^1 D^2F_\delta(\nabla u_\delta + th\Delta^h_\gamma \nabla u_\delta)dt\Big)(Q,Q)$$
$$\geq \widetilde{\lambda}|Q|^2, \quad Q \in \mathbb{R}^{nM}.$$

Going back to (7.1.12) we use the fact that $\int_0^1 B^{(\delta)}_{h,\gamma}(t)dt$ represents a symmetric and positive definite bilinear form. An application of the inequality of Cauchy-Schwarz and then Young's inequality ($\varepsilon > 0$) afterwards gives with the help of the previous calculations from above

$$\int_{B_{2R}(\xi)} \eta^2\Big(\int_0^1 B^{(\delta)}_{h,\gamma}(t)dt\Big)(\Delta^h_\gamma\nabla u_\delta, \Delta^h_\gamma \nabla u_\delta)dx + \lambda \int_{B_{2R}(\xi)} |\Delta^h_\gamma u_\delta|^2\eta^2 dx$$
$$\leq c\varepsilon \int_{B_{2R}(\xi)} \eta^2\Big(\int_0^1 B^{(\delta)}_{h,\gamma}(t)dt\Big)(\Delta^h_\gamma\nabla u_\delta, \Delta^h_\gamma \nabla u_\delta)dx$$
$$+ c\varepsilon^{-1} \int_{B_{2R}(\xi)} \Big(\int_0^1 B^{(\delta)}_{h,\gamma}(t)dt\Big)(\nabla\eta \otimes \Delta^h_\gamma u_\delta, \nabla\eta\otimes\Delta^h_\gamma u_\delta)dx$$
$$+ \varepsilon \int_{B_{2R}(\xi)} \eta^2|\Delta^h_\gamma u_\delta|^2 dx + c\varepsilon^{-1}\int_{B_{2R}(\xi)} \eta^2 dx + c\varepsilon^{-1}\int_{B_{2R}(\xi)} |\nabla\eta|^2 dx.$$

Using the boundedness of $|D^2F_\delta|$ (see (7.1.4)) and choosing $\varepsilon > 0$ sufficiently small, we obtain after absorbing terms

$$\begin{aligned}&\int_{B_{2R}(\xi)} \eta^2\Big(\int_0^1 B^{(\delta)}_{h,\gamma}(t)dt\Big)(\Delta^h_\gamma\nabla u_\delta, \Delta^h_\gamma \nabla u_\delta)dx + c \int_{B_{2R}(\xi)} |\Delta^h_\gamma u_\delta|^2\eta^2 dx\\ &\leq \frac{c}{R^2}\int_{B_{2R}(\xi)} |\Delta^h_\gamma u_\delta|^2 dx + c(R),\end{aligned} \tag{7.1.13}$$

where the local constant c does not depend on h.
Finally we apply Lemma 7.23 in [62] on p.168 once again. We then obtain by neglecting the non-negative second term on the l.h.s. of (7.1.13) and by using (7.1.4) (recall $\eta \equiv 1$ on $B_R(\xi)$)

$$\int_{B_R(\xi)} |\Delta^h_\gamma \nabla u_\delta|^2 dx \leq \frac{c}{R^2}\int_\Omega |\nabla u_\delta|^2 dx + c(R) \leq c(\delta, R), \tag{7.1.14}$$

where the local constant $c(R,\delta)$, in particular, does not depend on h. Using a covering argument we find

$$\|\Delta^h_\gamma \nabla u_\delta\|_{L^2(\omega)} \leq c(\omega,\delta)$$

for all $\omega \Subset \Omega$, where we emphasize one more time that the constant $c(\omega, \delta)$ is independent of h. As a consequence it holds $\partial_\gamma \nabla u_\delta \in L^2_{\text{loc}}(\Omega)^{nM}$ for all $\gamma \in \{1, \dots, n\}$ and therefore $u_\delta \in W^{2,2}_{\text{loc}}(\Omega)^M$ which proves part (b).

For proving claim (c) we quote [47], Theorem 1.3, choosing $a(z) := \delta z + DF(z)$ ($z \in \mathbb{R}^{nM}$) in this reference (recall the structure condition (7.1.5) imposed on F). An alternative approach for verifying part (c) is to adopt the arguments from [54], Theorem 3.2.3, p. 173, to our situation. Among other things we then use the corresponding Campanato estimates of the Laplacian. It is worth remarking that we do not need the Hölder condition (7.1.6) imposed on F in order to prove local apriori gradient bounds for u_δ.

Now let us show assertion (d): starting with the scalar case $M = 1$ we note that standard arguments from elliptic regularity theory yield the desired result. For more details we refer the reader to the proof of Theorem 3.1.19 in this thesis (see Section 3.5.1, “Step 4. Conclusions“).
Considering the vectorial case $M > 1$ we first remark that a direct application of Theorem 4.1.7 fails since we would need the validity of the stronger Hölder condition (4.1.20) for D^2F (we recall that we merely require (7.1.6) for D^2F). However, we can argue in a similiar manner as already done in Section 3.5.2 (see the proof of Theorem 3.1.19, “Step 4. Conclusions“), i.e., we benefit from Theorem 4.1.7 in the autonomous setting: we fix $\Omega' \Subset \Omega$ and a constant $M := M(\Omega', \delta) > 0$ such that $|\nabla u_\delta(x)| \le M$ for a.a. $x \in \Omega'$. Afterwards, in view of [78], we construct an auxiliary integrand $\widetilde{F}_\delta$ of class $C^2(\mathbb{R}^{nM})$ satisfying the assumptions of Theorem 4.1.7 in the autonomous case for an appropriate choice of $t \ge 2$ and such that we have ($c > 0$ denotes a constant)

$$F_\delta(P) + c = \widetilde{F}_\delta(P) \quad \text{and} \quad DF_\delta(P) = D\widetilde{F}_\delta(P)$$

for all $P \in \mathbb{R}^{nM}$ with $|P| \le 2M$. As a consequence we may derive that u_δ is a local minimizer of an appropriate (autonomous) isotropic variational problem for which local minimizers are of class $C^{1,\kappa}(\Omega)^M$ for any $\kappa \in (0,1)$ by applying Theorem 4.1.7.
Altogether, the lemma is proven. □

Remark 7.1.3
Considering $n = 3$, it is worth mentioning that we can show that $|\nabla u_\delta|^2$ is a weak subsolution of an elliptic equation. Here we merely need the structure condition (7.1.5) and the information $u_\delta \in W^{2,2}_{loc}(\Omega)^M$. At the end, our calculations result in the inequality ($\alpha, \beta \in \{1, 2, 3\}$)

$$\begin{aligned}
&\int_\Omega \frac{1}{2}(a_{\alpha\beta} + \delta_{\alpha\beta})\partial_\alpha |\nabla u_\delta|^2 \partial_\beta \eta^2 dx \\
&\le \int_\Omega g \cdot \partial_\alpha u_\delta \partial_\beta \eta^2 dx + c \int_\Omega \eta^2 dx,
\end{aligned}$$

where $\eta \in C_0^\infty(\Omega)$ with $0 \le \eta \le 1$, $g := \mathbb{1}_{\Omega - D}(u_\delta - f) \in L^\infty(\Omega)^M$ and $a_{\alpha\beta} :=$

$\frac{\partial^2 F}{\partial p^j_\alpha \partial p^j_\beta}$. Now the theory of De Giorgi-Moser-Nash (see, e.g., [62], Theorem 8.15, p. 179) (locally) applies to the weak subsolution $|\nabla u_\delta|^2$ if the function $g \cdot \partial_\alpha u_\delta$ is of class $L^q_{loc}(\Omega)$ for some $q > n$. Indeed we merely get $g \cdot \partial_\alpha u_\delta \in L^{\frac{2n}{n-2}}_{loc}(\Omega)$ by Sobolev's embedding theorem (recall $u_\delta \in W^{2,2}_{loc}(\Omega)^M$). Thus, since the condition $q := \frac{2n}{n-2} > n$ is only valid if $n = 3$, we then can quote the local bound on $|\nabla u_\delta|$ given by the theory of De Giorgi-Moser-Nash.

Remark 7.1.4
In the non-autonomous case (compare the fourth chapter of this thesis) we used the regularization with $\delta \in (0,1]$ being fixed (recall (4.1.4) for the definition of the density $F_{\mu(x)}(P)$ for $x \in \overline{\Omega}$ and $P \in \mathbb{R}^{nM}$)

$$\begin{aligned} J_\delta[w] := \frac{\delta}{2} \int\limits_\Omega |\nabla w|^2 dx + \int\limits_\Omega F_{\mu(x)}(\nabla w) dx + \int\limits_\Omega |w - f|^2 dx \\ \to min \quad in\ W^{1,2}(\Omega)^M, \end{aligned} \tag{7.1.15}$$

where we denote by $v_\delta \in W^{1,2}(\Omega)^M$ the unique solution. Due to the uniform (in x) ellipticity and growth conditions of $F_{\mu(x)}(P)$ (compare Remark 4.1.5) we can essentially adopt the arguments from the proof of Lemma 7.1.1 in order to establish the statements of Lemma 7.1.1 for the unique J_δ-minimizer v_δ as well. For getting $v_\delta \in W^{1,\infty}_{loc}(\Omega)^M$ we can adjust the arguments from [54], Theorem 3.2.3, p. 173, to the non-autonomous situation at hand. In order to prove $v_\delta \in C^{1,\kappa}(\Omega)^M$ for all $\kappa \in (0,1)$ we proceed similar to the proof of Theorem 4.1.4 (see "Step 3. Conclusions" in Section 4.4).

7.1.2 An algebraic lemma

The last lemma which will be proven is of pure algebraic nature and has act as a technical tool during the proof of Theorem 3.1.19 and Theorem 4.1.4, respectively (compare Lemma 3.5.10).

Lemma 7.1.5
Consider real numbers $\overline{p}, \nu > 3, \mu > 1$ with

$$\frac{\mu + \nu}{2} n < \overline{p}. \tag{7.1.16}$$

Then, there exist real numbers $s_1, s_2, s_3 > 1$ such that

(i) $2\frac{s_1}{s_1 - 1} < \overline{p}$, *(ii)* $\frac{1}{s_1}\frac{n}{n-1} > 1$,
(iii) $\mu\frac{s_2}{s_2 - 1} < \overline{p}$, *(iv)* $\nu\frac{s_3}{s_3 - 1} < \overline{p}$,
(v) $\frac{1}{2}\frac{n}{n-1}\left(\frac{1}{s_3} + \frac{1}{s_2}\right) > 1$.

Proof of Lemma 7.1.5. Primarily we choose $\widetilde{p} < \overline{p}$ such that (7.1.16) still holds for $\widetilde{p}$ instead of $\overline{p}$.

Due to (7.1.16) it holds $\widetilde{p} > \nu\frac{n}{2}$. As a consequence, the statements (i) and (iv) are obvious by setting $s_1 := \frac{\widetilde{p}}{\widetilde{p}-2} > 1$ as well as $s_3 := \frac{\widetilde{p}}{\widetilde{p}-\nu} > 1$. Besides, combining $\mu > 1$ and (7.1.16) we may conclude the validity of the inequality $\widetilde{p} > 2n$. Recalling our choice of the parameter s_1 from above we immediately obtain (ii). For proving (v) we observe that we have

$$m := 2\frac{n-1}{n} - \frac{1}{s_3} = 2 - \frac{2}{n} - 1 + \frac{\nu}{\widetilde{p}} < 1 \tag{7.1.17}$$

since $\widetilde{p} > \nu\frac{n}{2}$.
Thanks to (7.1.17) we may choose $s_2 > 1$ in such a way that

$$m < \frac{1}{s_2} < 1 \tag{7.1.18}$$

and an application of (7.1.18) implies

$$\frac{1}{2}\frac{n}{n-1}\left(\frac{1}{s_3} + \frac{1}{s_2}\right) > \frac{1}{2}\frac{n}{n-1}\left(\frac{1}{s_3} + m\right) = 1$$

which shows (v).
In order to verify the last statement of Lemma 7.1.5 we claim that it holds

$$\frac{1}{s_2} < 1 - \frac{\mu}{\widetilde{p}} \tag{7.1.19}$$

and remark that the validity of (7.1.19) directly implies assertion (iii) of Lemma 7.1.5.
On account of (7.1.16) it follows

$$m - 1 + \frac{\mu}{\widetilde{p}} = \frac{\nu + \mu}{\widetilde{p}} - \frac{2}{n} < 0.$$

Thus, we get

$$m < 1 - \frac{\mu}{\widetilde{p}}$$

and consequently we may choose $s_2 > 1$ in addition to (7.1.18) in such a way that

$$m < \frac{1}{s_2} < 1 - \frac{\mu}{\widetilde{p}}.$$

This shows (7.1.19) and completes the proof of Lemma 7.1.5. □

7.2 Notation and conventions

The set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, always denotes a bounded Lipschitz domain. For $M \geq 1$, we abbreviate a given function space $X(\Omega, \mathbb{R}^M)$ by $X(\Omega)^M$ where for $M = 1$ we use the common notation $X(\Omega)$. With respect to the notation of the classical function spaces we follow the notation as introduced in [17] or [62]. In this thesis we are particularly concerned with

- Hölder spaces $C^{k,\alpha}(\Omega)^M$,
- Lebesgue spaces $L^p(\Omega)^M$,

and

- Sobolev spaces $W^{k,p}(\Omega)^M$, $W_0^{k,p}(\Omega)^M$.

For more details in relation to Sobolev spaces we refer the reader to the classical monography of Adams [4].
Moreover, for the local variants we write

- $L^p_{\mathrm{loc}}(\Omega)^M$, etc.

Now let us consider a function $w \in L^1(\Omega)^M$. Following the monographies of Ambrosio, Fusco and Pallara [7] or of Giusti [63], we say that w is a

- function of bounded variation in Ω (abbreviated: $w \in BV(\Omega)^M$),

if the distributional derivative can be represented by a finite Radon measure in Ω, i.e., for some $\mathbb{R}^{nM}$-valued measure $(\partial_\alpha w^i)^{1\le i\le M}_{1\le\alpha\le n}$ we have

$$\begin{aligned}\int_\Omega w\cdot \operatorname{div}\varphi\, dx &= \int_\Omega w^i \operatorname{div}\varphi^i dx = -\int_\Omega \varphi^i_\alpha \nabla_\alpha w^i\\ &= -\int_\Omega \varphi : \nabla w\end{aligned}$$

for any $\varphi \in C_0^\infty(\Omega)^{nM}$. Note that a smoothing argument shows that the integration by parts from above remains also for all functions of class $C_0^1(\Omega)^M$ as well.
The total variation of w is given by

$$\int_\Omega |\nabla w| = \sup_{\varphi\in C_0^1(\Omega)^{nM}, |\varphi|\le 1} \int_\Omega w\cdot\operatorname{div}\varphi\, dx,$$

and we say that w is of class $BV(\Omega)^M$ if and only if the total variation is finite.
Furthermore we have the following conventions throughout the entire thesis: $\operatorname{div}\varphi = (\sum_{\alpha=1}^{n} \partial_\alpha \varphi^i_\alpha) \in \mathbb{R}^M$, summation is always assumed w.r.t. repeated indices- for Latin indices the sum is taken over $i = 1, \dots, M$, for Greek indices this is done w.r.t. $\alpha = 1, \dots, n$. Further, the standard scalar product in $\mathbb{R}^M$ is denoted by "·" whereas the symbol ":" is reserved for the standard scalar product in $\mathbb{R}^{nM}$.
Further, derivatives will be denoted by "D" or "∇", where the precise meaning will always be evident in the context.
Last but not least we state that

- if it is necessary (and possible) we usually pass to suitable subsequences without relabeling;
- $c = c(\cdot, \dots, \cdot)$ usually denotes a constant depending only on the quantities appearing in the current context where we commonly emphasize if it is important to observe that c is a uniform constant (in a suitable sense). Besides, the same letter c will be used to label different constants.

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