Pierre Tarrago

Non-commutative generalization of some probabilistic results from representation theory

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Philippe Biane and Roland Speicher

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Präsident der Univ. Paris-Est:	Univ.-Prof. Dr. P. Tchamitchian
Dekan der Univ. des Saarlandes-NTF I:	Univ.-Prof. Dr. M. Bläser
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Berichterstatter:	Univ.-Prof. Dr. R. Speicher
	Univ.-Prof. Dr. T. Banica
	Univ.-Prof. Dr. G. Olshanski
Prüfer:	Univ.-Prof. Dr. P. Biane
	Univ.-Prof. Dr. J-Y. Thibon

Abstract

The subject of this thesis is the non-commutative generalization of some probabilistic results that occur in representation theory. The results of the thesis are divided into three different parts.

In the first part of the thesis, we classify all unitary easy quantum groups whose intertwiner spaces are described by non-crossing partitions, and develop the Weingarten calculus on these quantum groups. As an application of the previous work, we recover the results of Diaconis and Shahshahani on the unitary group and extend those results to the free unitary group.

In the second part of the thesis, we study the free wreath product. First, we study the free wreath product with the free symmetric group by giving a description of the intertwiner spaces: several probabilistic results are deduced from this description. Then, we relate the intertwiner spaces of a free wreath product with the free product of planar algebras, an object which has been defined by Bisch and Jones in [46]. This relation allows us to prove the conjecture of Banica and Bichon.

In the last part of the thesis, we prove that the minimal and the Martin boundaries of a graph introduced by Gnedin and Olshanski are the same. In order to prove this, we give some precise estimates on the uniform standard filling of a large ribbon Young diagram. This yields a positive answer to the conjecture that Bender, Helton and Richmond gave in [18].

Abstrakt

In dieser Dissertation widme ich mich der nicht-kommutativen Verallgemeinerung probabilistischer Ergebnisse aus der Darstellungstheorie. Die Dissertation besteht aus einer Einleitung und drei Teilen, die jeweils separate Veröffentlichungen darstellen.

In dem ersten Teil der Dissertation wird der Begriff von easy Quantengruppe im unitären Fall untersucht. Es wird eine Klassifikation aller unitären easy Quantengruppen in dem klassischen und freien unitären Fall gegeben. Des weiteren werden die probabilistischen Ergebnisse von [14] auf den unitären Fall ausgedehnt.

In dem zweiten Teil der Dissertation widme ich mich zunächst dem freien Kranzprodukt einer kompakten Quantengruppe mit der freien symmetrischen Gruppe. Die Darstellungstheorie solcher Kranzprodukte wird beschrieben, und verschiedene probabilistische Ergebnisse werden aus dieser Beschreibung gezogen. Dann wird eine Beziehung zwischen freien Kranzprodukten und planaren Algebren hergestellt, die zu dem Beweis einer Vermutung von Banica und Bichon führt.

In dem dritten Teil dieser Dissertation wird der Graph $\mathcal{Z}$ der Multiplikation der fundamentalen quasi-symmetrischen Basis untergesucht. Der minimale Rand dieses Graphs wurde schon von Gnedin und Olshanski identifiziert [42]. Wir beweisen jedoch, dass der minimale Rand und der Martin-Rand gleich sind. Als Nebenprodukt des Beweises erhalten wir mehrere asymptotische kombinatorische Ergebnisse bezüglich großer Ribbon-Young-Tableaus.

Résumé

Le sujet de cette thèse est la généralisation non-commutative de résultats probabilistes venant de la théorie des représentations. Les résultats obtenus se divisent en trois parties distinctes.

Dans la première partie de la thèse, le concept de groupe quantique easy est étendu au cas unitaire. Tout d'abord, nous donnons une classification de l'ensemble des groupes quantiques easy unitaires dans le cas libre et classique. Nous étendons ensuite les résultats probabilistes de [14] au cas unitaire.

La deuxième partie de la thèse est consacrée à une étude du produit en couronne libre. Dans un premier temps, nous décrivons les entrelaceurs des représentations dans le cas particulier d'un produit en couronne libre avec le groupe symétrique libre: cette description permet également d'obtenir plusieurs résultats probabilistes. Dans un deuxième temps, nous établissons un lien entre le produit en couronne libre et les algèbres planaires: ce lien mène à une preuve d'une conjecture de Banica et Bichon.

Dans la troisième partie de la thèse, nous étudions un analoque du graphe de Young qui encode la structure multiplicative des fonctions fondamentales quasi-symétriques. La frontière minimale de ce graphe a déjà été décrite par Gnedin et Olshanski [42]. Nous prouvons que la frontière minimale coïncide avec la frontière de Martin. Au cours de cette preuve, nous montrons plusieurs résultats combinatoires asymptotiques concernant les diagrammes de Young en ruban.

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Cette thèse a fait l'objet d'un soutien financier de l'Université franco-allemande (UFA).

Thesis summary

The subject of this thesis is the non-commutative generalization of some probabilistic results that occur in representation theory. The results of the thesis are divided into three parts, which are summarized here.

Weingarten calculus and free easy quantum groups Easy quantum groups have been defined in [15] as a class of orthogonal compact quantum groups whose associated intertwiners are described by set partitions. This class of compact quantum groups contains important examples of quantum groups as the classical orthogonal and symmetric groups and their free analogs, the free orthogonal and free symmetric groups (see [95, 96]). In a second paper [14], it has been possible to systematically develop the Weingarten calculus on these compact quantum groups in order to get some probabilistic results: in particular, they recovered the convergence results of Diaconis and Shahshahani (see [33]) on the orthogonal and symmetric group, and extended them to the free case. The usual gaussian and Poisson laws are replaced in the free case by the semicircular and the Marchenko-Pastur laws, their free analogs in free probability theory.
The first part of the thesis is devoted to the generalization of this framework in the unitary case. Namely the compact quantum groups are not assumed to be orthogonal anymore, but their intertwiner spaces are still described by set partitions with colors. The classical example is given by the classical unitary group whose intertwiner spaces are described by permutations (which can be seen as two-colored pair partitions) through the Schur-Weyl duality. We classify all unitary easy quantum groups whose intertwiner spaces are described by non-crossing partitions, and develop the Weingarten calculus on these quantum groups. As an application of the previous work, we recover the results of Diaconis and Shahshahani on the unitary group and extend those results to the free unitary group.

Free wreath product The free wreath product is a non-commutative analog of the classical wreath product. The free wreath product is an algebraic construction that produces a new compact quantum group from a compact quantum group and a non-commutative permutation group. This construction arises naturally in the study of quantum symmetries of lexicographical products of graphs. In the classical case, the representation theory of a wreath product is well-know (see for example [60], Part 1, Annex B) and the Haar measure has a straightforward expression. It is for example easy to prove that the fundamental character of a wreath product with the symmetric group S_n converges toward a compound Poisson law as n goes to infinity. However in the free case, the Haar state doesn't have any straightforward expression. For instance Banica and Bichon conjectured in [10] that in some cases, the fundamental character of a free wreath product is distributed as the free multiplicative convolution of the law of the two initial fundamental characters.
In the second part of the thesis, we study the free wreath product. First, we study the free wreath product with the free symmetric group by giving a description of the intertwiner spaces: several probabilistic results are deduced from this description. Then, we relate the intertwiner spaces of a free wreath product with the free product of planar algebras, an object which has been defined by Bisch and Jones in [46]. This relation allows us to express the law of the character of a free wreath product as a free multiplicative convolution of the initial laws, which proves the conjecture of Banica and Bichon.

Martin boundary of the Zig-zag lattice The ring $QSym$ of quasi-symmetric functions is a refinement of the ring of symmetric functions, in the sense that any symmetric function has a decomposition in terms of quasi-symmetric ones. An important basis of this ring is called the fundamental basis, and its elements have a monomial expansion similar to the Schur basis of the ring of symmetric functions: this expansion is indexed by semi-standard filling of ribbon Young diagrams for the fundamental basis of $QSym$ and by semi-standard filling of Young diagram for the Schur basis of Sym. The multiplication structure of the Schur basis is encoded by an important graph which is called the Young graph and denoted by $\mathcal{Y}$. This graph has many applications in the representation theory of the infinite group S_∞ and in the probabilistic behavior of some discrete processes. It has been intensively studied by Thoma, Vershik and Kerov in [85, 86, 47]. In particular they identified the minimal and Martin boundaries of $\mathcal{Y}$, and proved that the two coincide. The analog of $\mathcal{Y}$ for the fundamental basis of $QSym$ is the graph $\mathcal{Z}$ of Zigzag diagrams. This lattice has been deeply studied by Gnedin and Olshanski who identified in [42] its minimal boundary. They conjectured that the minimal and Martin boundaries also coincide on $\mathcal{Z}$.
In the last part of the thesis, we prove that the minimal and the Martin boundaries of $\mathcal{Z}$ are the same. In order to prove this, we give some precise estimates on the uniform standard filling of a large ribbon Young diagram: we prove that in a uniform filling, the fillings of distant cells become independent in a certain sense. This yields a positive answer to the conjecture that Bender, Helton and Richmond gave in [18].

Résumé de la thèse

Le sujet de cette thèse est la généralisation non-commutative de résultats probabilistes venant de la théorie des représentations. Les résultats obtenus se divisent en trois parties qui sont résumées ici.

Groupes quantiques easy et calcul de Weingarten: La théorie des représentations de certains groupes et groupes quantiques orthogonaux compacts mettent en jeu un même objet combinatoire, les partitions d'ensembles finis. Ceci est le cas pour le groupe orthogonal et le groupe symétrique, ainsi que pour le groupe orthogonal libre et le groupe symétrique libre: ces deux derniers sont des groupes quantiques qui ont été introduits par Wang [95, 96] comme version non-commutative de leurs homologues classiques. Dans [15], Banica et Speicher ont généralisé ces exemples en définissant les groupes quantiques easy. Il y a dans cette classe deux situations extrêmes: celle où le groupe quantique est un groupe classique et celle où la théorie des représentations du groupe est décrite par des partitions non-croisées. Dans ce dernier cas, le groupe quantique est dit libre. La classification de tous les groupes quantiques easy dans le cas classique et libre a été initiée par Banica et Speicher, puis complétée par Weber [15, 97]. Dans un troisième temps, Raum et Weber [71] ont réussi à classifier l'ensemble des groupes quantiques easy.
Pour un tel groupe quantique, le calcul de Weingarten [28] donne un moyen efficace de calculer les intégrales par rapport à la mesure de Haar sur le groupe quantique. Avec l'aide du calcul de Weingarten, Banica, Curran et Speicher [14] ont pu obtenir plusieurs resultats probabilistes dans le cas des groupes quantiques easy libres ou classiques: par exemple, ils ont étendu à l'ensemble de ces groupes quantiques les théorèmes asymptotiques de Diaconis et Shahshahani [33] sur les traces des groupes orthogonaux et symétriques.
Dans la première partie de la thèse, le concept de groupe quantique easy est étendu au cas unitaire. Tout d'abord, nous donnons une classification de l'ensemble des groupes quantiques easy unitaires dans le cas libre et classique. Nous étendons ensuite les résultats probabilistes de [14] au cas unitaire.

Produit en couronne libre: Le produit en couronne libre est une construction algébrique dûe à Bichon [21] qui associe un groupe quantique compact à un sous-groupe quantique du groupe symétrique libre pour créer un nouveau groupe quantique, d'une manière analogue au produit en couronne classique. Alors que la mesure de Haar d'un produit en couronne classique a une expression simple en fonction des mesures de Haar des groupes initiaux, il n'y a dans le cas libre aucun moyen d'obtenir une formulation explicite de l'état de Haar. Banica et Bichon ont conjecturé dans [10] que la loi du caractère fondamental d'un produit en couronne libre est dans certains cas la convolution multiplicative libre des lois de caractère des groupes quantiques initiaux.
La deuxième partie de la thèse est consacrée à une étude plus approfondie du produit en couronne libre. Dans un premier temps, nous décrivons les entrelaceurs des représentations dans le cas particulier d'un produit en couronne libre avec le groupe symétrique libre: cette description permet également d'obtenir plusieurs résultats probabilistes. Dans un deuxième temps, nous établissons un lien entre le produit en couronne libre et les algèbres planaires: ce lien mène à une preuve de la conjecture de Banica et Bichon précitée.

Frontière de Martin du graph $\mathcal{Z}$: Le graphe de Young est un graphe qui encode la structure multiplicative de l'anneau des fonctions symétriques dans la base de Schur [85, 86, 47]. Cet

anneau, également défini comme l'anneau commutatif universel engendré par un nombre infini et denombrable de variables, joue un rôle important dans la théorie des représentations du groupe symétrique et du groupe unitaire. En retirant la condition de commutativité dans cet anneau, on obtient un nouvel anneau non-commutatif qui a été introduit [41] comme l'anneau des fonctions symétriques non-commutatives. Un résultat fondamental est qu'on peut associer à cet anneau non-commutatif un anneau commutatif, l'anneau des fonctions quasi-symétriques, qui présente un structure combinatoire similaire à celle de l'anneau des fonctions symétriques. L'anneau des fonctions quasi-symétriques possède ainsi une base semblable à la base de Schur, la base des fonctions fondamentales quasi-symétriques.
Dans la troisième partie de la thèse, nous étudions un analoque du graphe de Young qui encode la structure multiplicative de la base des fonctions fondamentales. La frontière minimale de ce graphe a déjà été décrite par Gnedin et Olshanski [42]. Nous prouvons que la frontière minimale coïncide avec la frontière de Martin. Au cours de cette preuve, nous montrons plusieurs résultats combinatoires asymptotiques concernant les diagrammes de Young en ruban.

Zusammenfassung der Dissertation

In dieser Dissertation widme ich mich der nicht-kommutativen Verallgemeinerung probabilistischer Ergebnisse aus der Darstellungstheorie. Die Dissertation besteht aus einer Einleitung und drei Teilen, die jeweils separate Veröffentlichungen darstellen.

Easy Quantengruppen und Weingarten-Kalkül: In mehreren Fällen besitzen orthogonale Gruppen und Quantengruppen eine ähnliche Darstellungstheorie, deren kombinatorische Struktur mit Hilfe von mengentheoretischen Partitionen beschrieben wird: dies gilt zum Beispiel für die symmetrische Gruppe und die orthogonale Gruppe sowie für die freie symmetrische Quantengruppe und die freie orthogonale Quantengruppe, wobei letztere als nicht-kommutative Verallgemeinerung von ersteren von Wang [95, 96] definiert wurden. In [15] wurden easy Quantengruppen von Banica und Speicher zur Systematisierung dieses Phänomens eingeführt. Im Rahmen der easy Quantengruppen gibt es zwei extreme Situationen: diejenige, in der die easy Quantengruppe eine klassische Gruppe ist und diejenige, in der die Darstellungstheorie der easy Quantengruppe mit Hilfe von nicht-kreuzenden Partitionen beschrieben wird. In letzterem Fall wird die easy Quantengruppe frei genannt. Die Klassifikation aller klassischen und aller freien Quantengruppen wurde von Banica, Speicher und Weber[15, 97] erreicht und später für alle easy Quantengruppen von Raum und Weber [71] vollendet.
Für eine easy Quantengruppe existiert eine effiziente Methode, die Weingarten-Kalkül genannt wird [28], um Integrale bezüglich des Haarmaßes zu berechen. Mit dem Weingarten-Kalkül konnten Banica, Curran und Speicher [14] mehrere probabilistische Ergebnisse im Rahmen der easy Quantengruppen erlangen: insbesondere wurde der Grenzwertsatz von Diaconis und Shahshahani [33] bezüglich der Verteilung des fundamentalen Charakters der symmetrischen und orthogonalen Gruppen auf alle klassischen und freien easy Quantengruppen ausgedehnt.
In dem ersten Teil der Dissertation wird der Begriff von easy Quantengruppe im unitären Fall untersucht. Es wird eine Klassifikation aller unitären easy Quantengruppen in dem klassischen und freien unitären Fall gegeben. Des weiteren werden die probabilistischen Ergebnisse von [14] auf den unitären Fall ausgedehnt.

Freies Kranzprodukt: Das freie Kranzprodukt ist eine von Bichon [21] eingeführte nicht-kommutative Version des klassichen Kranzprodukts, mit Hilfe dessen eine neue Quantengruppe aus einer kompakten Quantengruppe und einer Untergruppe der freien symmetrischen Quantenruppe erzeugt wird. Während das Haarmaß für ein klassisches Kranzprodukt eine einfache Gestalt hat, gibt es für das Haarmaß eines freien Kranzprodukts keine explizite Formulierung. Banica und Bichon [10] stellten jedoch die Vermutung auf, dass die Verteilung des fundamentalen Charakters eines freien Kranzprodukts in vielen Fällen die multiplikative freie Faltung der Verteilungen der beiden originären Charaktere ist.
In dem zweiten Teil der Dissertation widme ich mich zunächst dem freien Kranzprodukt einer kompakten Quantengruppe mit der freien symmetrischen Gruppe. Die Darstellungstheorie solcher Kranzprodukte wird beschrieben, und verschiedene probabilistische Ergebnisse werden aus dieser Beschreibung gezogen. Dann wird eine Beziehung zwischen freien Kranzprodukten und planaren Algebren hergestellt, die zu dem Beweis der Vermutung von Banica und Bichon führt.

Martin-Rand des Graphs $\mathcal{Z}$: Der Young-Graph $\mathcal{Y}$ beschreibt die multiplikative Struktur des Rings der symmetrischen Funktionen in der sogenannten Schur-Basis [85, 86, 47]. Dieser

Ring ist der universelle kommutative Ring mit abzählbar unendlich vielen Variablen, der eine große Rolle in der Darstellungstheorie der symmetrischen und unitären Gruppen spielt. Wenn man die Kommutativität der Variablen wegfallen lässt, erhält man einen neuen Ring, der der Ring der nicht-kommutativen symmetrischen Funktionen genannt wird [41]. Der Punkt ist, dass man daraus trotzdem einen kommutativen Ring erzeugen kann, der ähnlich dem Ring der symmetrischen Funktionen ähnlich ist. Insbesondere gibt es in diesem neuen Ring ein Gegenstück der Schur-Basis, das die fundamentale quasi-symmetrische Basis genannt wird.
In dem dritten Teil dieser Dissertation wird der Graph $\mathcal{Z}$ der Multiplikation dieser fundamentalen quasi-symmetrischen Basis untergesucht. Der minimale Rand dieses Graphs wurde schon von Gnedin und Olshanski identifiziert [42]. Wir beweisen jedoch, dass der minimale Rand und der Martin-Rand gleich sind. Als Nebenprodukt des Beweises erhalten wir mehrere asymptotische kombinatorische Ergebnisse bezüglich großer Ribbon-Young-Tableaus.

Contents

List of Figures

Part I

Introduction

Chapter 1

Partitions and free probability

This chapter is an introduction to set partitions, a class of objects that underlies the combinatorics of free probability. We set the notations, explain how set partitions are transformed into linear morphisms, and describe their role in free probability.

1.1 Set partitions

1.1.1 Definition and notations

Definition of a set partition:

Definition 1.1. *Let $n, r \geq 1$. A set partition of n with r parts is a set p of subsets $B_1, \ldots, B_r$ of $[\![1; n]\!]$ such that $\bigcup_{i=1}^{r} B_i = \{1, \ldots, n\}$ and for $1 \leq i < j \leq r$, $B_i \cap B_j = \emptyset$.*

A set B_i in the definition above is called a block of p. A block of cardinal one is called a singleton and a block of cardinal 2 is called a pair. When no confusion is possible, a set partition of n with r parts is simply called a partition of n. The set of all set partitions of n is denoted by $P(n)$ and the number of blocks of a partition p is denoted by $b(p)$. $P(0)$ denotes the empty set. We write $i \sim_p j$ if and only if i and j are in a same block of p. This is an equivalence relation on $\{1, \ldots, n\}$. Assigning to each equivalence relation the set of its equivalence classes yields a bijection between equivalence relations of $\{1, \ldots, n\}$ and set partitions of n.
A set partition is depicted by drawing the integers 1 to n on a row, and the blocks as lines between them. Figure 1.1 is an example of such a drawing for $n = 8$ and $p = \{\{1, 3, 4\}, \{2, 7\}, \{5, 8\}, \{6\}\}$.

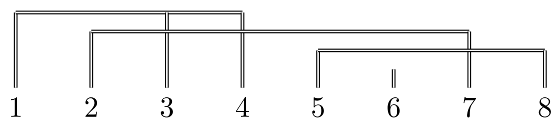


Figure 1.1: Partition $\{\{1, 3, 4\}, \{2, 7\}, \{5, 8\}, \{6\}\}$ with 4 blocks.

We distinguish several subsets of P_n:

- The set $P_2(n)$ of pair partitions: these are partitions such that all blocks are pairs.

- The set $NC(n)$ of non-crossing partitions: these are the partitions p of n such that if $1 \leq i < j < k < l \leq n$ and $i \sim_p k$ and $j \sim_p l$, then $j \sim_p k$. This means that we can draw p such that the blocks do not cross each other. For example $\{\{1, 2, 5\}, \{3, 4\}, \{6, 8\}, \{7\}\}$

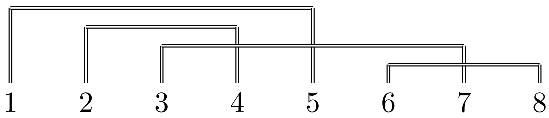


Figure 1.2: Pair partition $\{\{1,5\},\{2,4\},\{6,8\},\{3,7\}\}$ with 4 blocks.

is a non-crossing partition:

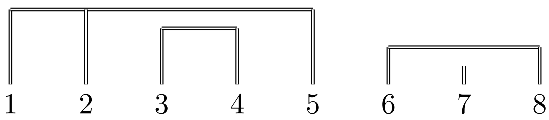


Figure 1.3: Non-crossing partition $\{\{1,2,5\},\{3,4\},\{6,8\},\{7\}\}$ with 4 blocks

- The set $NC_2(n)$ of non-crossing pair partitions: $NC_2(n)$ is the set $NC(n) \cap P_2(n)$.

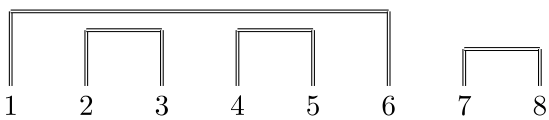


Figure 1.4: Non-crossing pair partition $\{\{1,6\},\{2,3\},\{4,5\},\{7,8\}\}$ with 4 blocks

The lattice of set partitions Let p, q be two partitions of n, One says that p refines q (denoted by $p \leq q$) if any block of p is contained in a block of q. $\leq$ yields a partial order on the set $P(n)$ (resp. $NC(n)$). One can check that for $p, q \in P(n)$ (resp. $NC(n)$), there always exist a unique supremum $p \vee q$ and infimum $p \wedge q$ of p and q in $P(n)$ (resp. $NC(n)$), yielding that $(P(n), \leq)$ and $(NC(n), \leq)$ are actually lattices.
Note that $NC(n)$ is a subset of $P(n)$ but not a sublattice, since two elements of $NC(n)$ may have a supremum in $P(n)$ that differs from the one in $NC(n)$. For example $\{\{1,3\},\{2\},\{4\}\}$ and $\{\{1\},\{3\},\{2,4\}\}$ are both in $NC(n)$; their supremum in $NC(n)$ is $\{\{1,2,3,4\}\}$ whereas their supremum in $P(n)$ is $\{\{1,3\},\{2,4\}\}$. However, for all $p, q \in NC(n)$, $p \wedge q$ is again in $NC(n)$. To distinguish both lattice, we write $\wedge_P, \vee_P$ for the supremum in $P(n)$ and $\wedge_{NC}, \vee_{NC}$ for the one in $NC(n)$ (the subscripts are omitted when there is no confusion).

Two colored set partitions Let S be a denumerable set. A $S-$coloring of $[\![1,r]\!]$ is a map $c : [\![1,r]\!] \to S$. A $S-$colored partition p of r is a partition $\tilde{p}$ of $P(r)$ together with a coloring c of $[\![1,r]\!]$. The partition $\tilde{p}$ is called the uncolored version of p. The set of S-colored partitions with a particular coloring is denoted by $P(c)$ or $P(c(1), \ldots, c(r))$. We replace P by P_2, NC or NC_2 to emphasize the shape of the partitions.
A two-colored partition p of n is a $S-$partition with $S = \{\circ, \bullet\}$, a set of cardinal 2. The integer n is thus fixed by the definition of c, and is also denoted by $|c|$. A two-colored partition corresponds to a coloring of the extreme points of the blocks of $\tilde{p}$ with elements of $\{\circ, \bullet\}$. We denote by $P^{\circ\bullet}(n)$ (resp. $P_2^{\circ\bullet}(n), NC^{\circ\bullet}(n), NC_2^{\circ\bullet}(n)$) the set of two-colored partitions (resp. pair partitions, non-crossing partitions, non-crossing pair partitions). For each map $c : [\![1;n]\!] \to \{\circ, \bullet\}$,

$P^{\circ\bullet}(c)$ denotes the set of two-colored partitions such that the coloring is given by the map c (and the same for the three other kinds of partitions).

1.1.2 Two-level partitions

Definition 1.2. *A two-level partition is a set partition p of n with a distinguished integer $l \in [\![0;n]\!]$. The integers lower than l are called the upper points and the integers greater than $l+1$ the lower points.*

For $k, l \geq 0$, the set of two-level set partitions of $k+l$ with k upper points and l lower points is denoted by $P(k,l)$ (and by $P^{\circ\bullet}(k,l)$ when the partition is colored). The subset of non-crossing two-level partitions (resp. two-level pairing, non-crossing two-level pairings) is denoted $NC(k,l)$ (resp. $P_2(k,l), NC_2(k,l)$) or $NC^{\circ\bullet}(k,l)$ (resp. $P_2^{\circ\bullet}(k,l), NC_2^{\circ\bullet}(k,l)$) depending on whether they are considered colored or not. When no confusion is possible, a two-level set partition is simply called a partition. For $c_1 : [\![1;k]\!] \to \{\circ, \bullet\}$ and $c_2 : [\![k+1;k+l]\!] \to \{\circ, \bullet\}$, we denote by $P(c_1, c_2)$ (resp. $NC(c_1, c_2)$) the set of two-level partitions in $P(k,l)$ such that the coloring of the upper points is given by c_1 and the one of the lower points by c_2.
A two-level set partition is drawn with two rows of integers, in such a way that the numbering is cyclic:

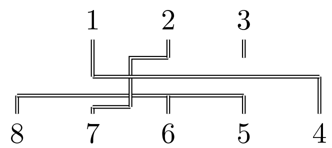


Figure 1.5: Two-level partition in $P(3,5)$ with block structure $\{\{5,6,8\},\{2,7\},\{1,4\},\{3\}\}$

The integers are omitted when they do not play any role.
Note that the lattice structure on $P(n)$ (resp. $NC(n)$) extends to the case of two-level partitions $P(k,l)$ (resp. $NC(k,l)$). In the latter case, the lattice structure is the same as the one of $P(k+l)$, forgetting the role of lower and upper points. In the case of colored partitions, the same identification is made to also give a lattice structure to the set $P^{\circ\bullet}(c)$ for each $c : [\![1, k+l]\!] \to \{\circ, \bullet\}$.

Operations on two-level colored partitions Several operations can be performed on two-level colored partitions. The easiest is to give a pictorial description of each of these operations.

- The *tensor product* of two partitions $p \in P^{\circ\bullet}(k,l)$ and $q \in P^{\circ\bullet}(k',l')$ is the partition $p \otimes q \in P^{\circ\bullet}(k+k', l+l')$ obtained by horizontal concatenation (writing p and q side by side). The first k points of the $k+k'$ upper points are connected by p to the first l of the $l+l'$ lower points, and the remaining k' upper points are connected to the remaining l' lower points by q.

- The *horizontal reflection* of a partition $p \in P^{\circ\bullet}(k,l)$ is given by the reflection of p through the horizontal axis. We also call it the *involution* of the partition p and denote it by $p^* := R_h(p)$.

- The *vertical reflection* of a partition $p \in P^{\circ\bullet}(k,l)$ is given by the reflection $R_v(p) \in P^{\circ\bullet}(k,l)$ of p through the vertical axis.

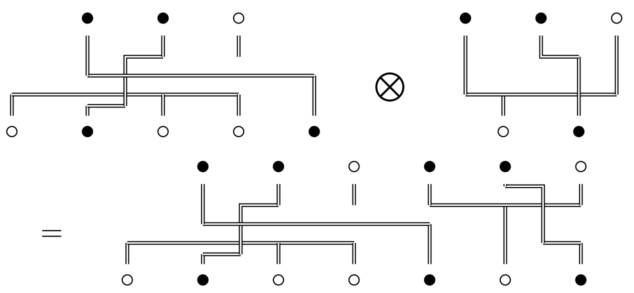


Figure 1.6: Tensor product of two partitions

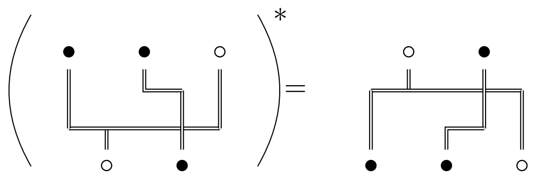


Figure 1.7: Horizontal reflexion of a partition

- The *composition* of two partitions $q \in P^{\circ\bullet}(k,l)$ and $p \in P^{\circ\bullet}(l,m)$ is the partition $pq \in P^{\circ\bullet}(k,m)$ obtained by vertical concatenation (writing p below q): First connect k upper points by q to l middle points and then connect these middle points to m lower points by p. This yields two kinds of objects : a partition, connecting k upper points with m lower points, and a certain number $rl(p,q)$ of blocks containing only middle points. The latter blocks and all the middle points l are removed. Note that we can compose two partitions $q \in P^{\circ\bullet}(k,l)$ and $p \in P^{\circ\bullet}(l',m)$ only if

  (i) the numbers l and l' coincide,

  (ii) the colorings match, i.e. the color of the j-th lower point of q coincides with the color of the j-th upper point of p, for all $1 \leq j \leq l$.

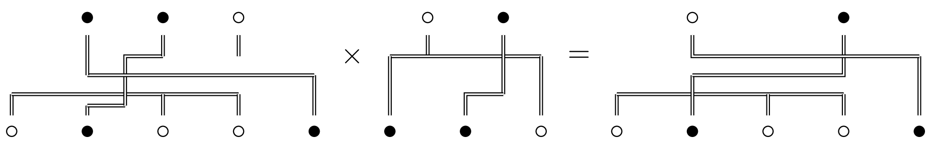


Figure 1.8: Composition of two partitions

- The *inversion of colors* of a partition $p \in P^{\circ\bullet}(k,l)$ is given by the partition $R_c(p) \in P^{\circ\bullet}(k,l)$ with same uncolored partition as p, but with all the colors inverted.

- The *verticolor reflection* of a partition p is given by $\tilde{p} := R_v R_c(p)$.

- The *rotation* of a partition: Let $p \in P^{\circ\bullet}(k,l)$ be a partition connecting k upper points with l lower points. Shifting the very left upper point to the left of the lower points and inverting its color gives rise to a partition in $P^{\circ\bullet}(k-1,l+1)$, a *rotated version* of p. Note that the point still belongs to the same block after rotation. We may also rotate the leftmost lower point to the very left of the upper line (again inverting its color), and we may as well rotate in the right hand side of the lines. In particular, for a partition

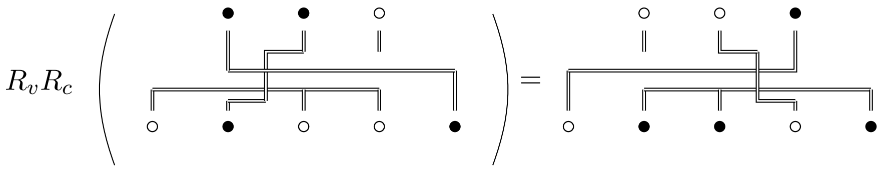


Figure 1.9: Verticolor reflection of a partition

$p \in P^{\circ\bullet}(0,l)$, we may rotate the very left point to the very right and vice versa. Such a rotation on one line does *not* change the colors of the points.

Here is a list of basic two-colored partitions that play an important role in Part II.

- The two *identity partitions* , $\in P^{\circ\bullet}(1,1)$ connects one upper point with one lower point of the same color. Note that and are *not* identity partitions.
- The *bicolored pair partitions* , $\in P^{\circ\bullet}(0,2)$ connect two lower points of different colors. We also have their horizontally reflected versions , $\in P^{\circ\bullet}(2,0)$. The *unicolored* pair partitions are , $\in P^{\circ\bullet}(0,2)$ and , $\in P^{\circ\bullet}(2,0)$.
- The *singleton partitions* , $\in P^{\circ\bullet}(0,1)$ consist of a single lower point respectively. Their reflected versions are denoted by , $\in P^{\circ\bullet}(1,0)$.
- We also have *four block partitions* like , $\in P^{\circ\bullet}(0,4)$ and , $\in P^{\circ\bullet}(2,2)$.
- All preceding examples are partitions consisting of a single block. The *crossing partition* $\in P^{\circ\bullet}(2,2)$ however consists of two blocks. It connects a white upper left point to a white lower right point, as well as a black upper right point to a black lower left point; we also have other colorings like or . These partitions are not in $NC^{\circ\bullet}(2,2)$.

Category of partitions A collection $\mathcal{C}$ of subsets $\mathcal{C}(\varepsilon,\varepsilon') \subseteq P^{\circ\bullet}(\varepsilon,\varepsilon')$ (indexed by all the words ε,ε' in $\{\circ,\bullet\}$) is a *category of partitions*, if it is closed under the tensor product, the composition and the involution, and if it contains the bicolored pair partitions and as well as the identity partitions and . We say that $\mathcal{C} \subseteq \mathcal{C}'$ if for any pair of words ε,ε' in $\{\circ,\bullet\}$, $\mathcal{C}(\varepsilon) \subseteq \mathcal{C}(\varepsilon')$ in the set-theoretic sense.
An easy check yields that the set of all partitions $P^{\circ\bullet}$, the set of all pair partitions $P_2^{\circ\bullet}$ (i.e. all blocks have length two), the set of all non-crossing partitions $NC^{\circ\bullet}$, and the set of all non-crossing pair partitions $NC_2^{\circ\bullet}$ form each of them a category of partition. Similarly let $P^{\circ\bullet}_{2,\text{altenating}}$ (resp. $NC^{\circ\bullet}_{2,\text{alternating}}$) be the set of pair partitions (resp. non-crossing pair partitions) with pairs having endpoints of opposite colors if these endpoints are on the same level and endpoints of same color if they are on different levels. Then $P^{\circ\bullet}_{2,\text{altenating}}$ and $NC^{\circ\bullet}_{2,\text{alternating}}$ are also categories of partitions. We have moreover the relation :

$$\begin{array}{ccccc} P^{\circ\bullet}_{2,\text{altenating}} & \subsetneq & P_2^{\circ\bullet} & \subsetneq & P^{\circ\bullet} \\ \cup & & \cup & & \cup \\ NC^{\circ\bullet}_{2,\text{alternating}} & \subsetneq & NC_2^{\circ\bullet} & \subsetneq & NC^{\circ\bullet} \end{array} \tag{1.1.1}$$

1.1.3 Contraction of tensor products

Kernel of a sequence of integers Let $\vec{i} = (i_1, \ldots, i_n)$ be a sequence of integers. This sequence defines an equivalence relation on $[\![1, n]\!]$ by saying that $r \sim_{\vec{i}} s$ if and only if $i_r = i_s$. The set partition associated to the relation $\sim_{\vec{i}}$ through the bijection given in Section 1.1.1 is denoted $\ker(\vec{i})$. If $\vec{i} = (i_1, \ldots, i_k), \vec{j} = (j_1, \ldots, j_l)$ are two sequences of integers of respective length k and l, we can similarly define a set partition $\ker(\vec{i}, \vec{j})$ in $P(k, l)$ by the same construction: this is the two-level partition $(\ker(j_1, \ldots, j_l, i_k, \ldots, j_1), l)$. Note that we reversed the order of the indices i for convenience in later computations. If we specifiy a coloring $\varepsilon : [\![1, k]\!] \to \{\circ, \bullet\}, \varepsilon' : [\![1, l]\!] \to \{\circ, \bullet\}$, we can assume that this partition is in $P(\varepsilon, \varepsilon')$. Here is an example of such a construction:

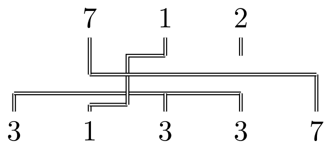


Figure 1.10: The partition $\ker((7, 1, 2), (3, 1, 3, 3, 7))$

Given a sequence of integers of length n and a partition $p \in P(n)$, we set $\delta_p(\vec{i}) = 1$ if $p \leq \ker(\vec{i})$ and $\delta_p(\vec{i}) = 0$ otherwise. Similarly if $p \in P(k, l)$ and $\vec{i}, \vec{j}$ are sequences of integers of respective length k and l, we set $\delta_p(\vec{i}, \vec{j}) = 1$ if $p \leq \ker(\vec{i}, \vec{j})$ and $\delta_p(\vec{i}, \vec{j}) = 0$ otherwise.

The maps T_p's: Let $V^\circ, V^\bullet$ be two Hilbert spaces of dimension n, and let $(e_i^\circ)_{1 \leq i \leq n}$, $(e_i^\bullet)_{1 \leq i \leq n}$ be respectively an orthonormal basis of V° and $V^\bullet$. For any word $\varepsilon = \varepsilon_1 \ldots \varepsilon_r$ in $\{\circ, \bullet\}$, the scalar product $\langle , \rangle$ on these Hilbert spaces is extended to a scalar product on the tensor product $V^\varepsilon = V^{\varepsilon_1} \otimes \cdots \otimes V^{\varepsilon_r}$ by saying that the basis $\{e_{i_1}^{\varepsilon_1} \otimes \cdots \otimes e_{i_r}^{\varepsilon_r}\}_{1 \leq i_1, \ldots, i_r \leq n}$ is othonormal.
For each partition $p \in P^{\circ\bullet}(\varepsilon, \varepsilon')$, one can define a map $T_p : V^\varepsilon \to V^{\varepsilon'}$ by the relation

$$\langle T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}), e_{j_1} \otimes \cdots \otimes e_{j_{k'}} \rangle = \delta_p(\vec{i}, \vec{j}).$$

If p is considered without colors, the same definition holds by considering tensor products of a unique vector space V of dimension n.
With this definition of the maps $T_p's$, the operations on two-level partitions defined in Paragraph 1.1.2 transpose to the usual operations on linear maps between Hilbert spaces as follows:

- $T_p \otimes T_q = T_{p \otimes q}$.
- $T_p \circ T_q = n^{rl(p,q)} T_{pq}$.
- $T_p^* = T_{R_h(p)}$.

Some linear maps can be easily expressed by the maps $T_p's$. For example T and T are respectively the identity map on V° and $V^\bullet$, T and T are the scalar products on V° and $V^\bullet$.

1.2 Free independence and set partitions

1.2.1 Non-commutative probability spaces and freeness

The free independence is a phenomenon arising in the study of non-commutative random variables. The latter are a generalization of probabilty spaces in the framework of non-commutative algebras.

Definition 1.3. *A non-commutative probability space* (A, φ) *is a unital* $*-$*algebra* A *with a linear functional* φ*, such that* $\varphi(\mathbf{1}_A) = 1$*.*

φ is called the expectation on A and is usually a trace (namely $\varphi(ab) = \varphi(ba)$ for $a, b \in A$). The joint law of $a_1, \dots, a_r \in A$ is defined as the expectation map

$$\varphi_{a_1,\dots,a_r} : \begin{cases} \mathbb{C} < X_1, \dots, X_r > & \longrightarrow \quad \mathbb{C} \\ \quad X_{i_1} \dots X_{i_p} & \mapsto \quad \varphi(a_{i_1} \dots a_{i_p} \end{cases}.$$

Example 1.4. *There are two basic examples of noncommutative probability spaces:*

- *If* $(\Omega, \mathbb{P})$ *is a classical probability space, then the algebra* $L^{\infty-}(\Omega) = \bigcup_{p\geq 1} L^p(\Omega)$ *of measurable functions having all moments finite is a noncommutative probability space, and the linear functional is given by the expectation* $\mathbb{E}$ *with respect to* $\mathbb{P}$*.*
- *Let* $(\Omega, \mathbb{P})$ *be a classical probability space and consider the algebra* $M_n \otimes L^{\infty-}(\Omega)$*. This algebra is again a noncommutative probability space with expectation given by the map* $A \mapsto \mathbb{E}(\frac{1}{n} \operatorname{Tr}(A))$*.*

When A is a C^*-algebra and a is a normal element of A (i.e $aa^* = a^*a$), the spectral theorem yields that a is an actual random variable on its spectrum, with moments given by $\{\varphi(a^k(a^*)^{k'})\}_{k,k'\geq 0}\}$. The law of a is denoted by μ_a.
If two commuting random variables a, b are independent, the knowledge of the respective laws of a and of b suffices to compute the expectation of any polynomial in a and b.
The concept of freeness is the analog of the independence of classical random variables in the setting of highly non-commutative variables. It has been introduced by Voiculescu around 1983 (see [90], see also [93] for an introduction to the subject).

Definition 1.5. *Let* (A, φ) *be a non-commutative probability space and* $A_1, \dots, A_r$ *be subalgebras of* A*.* $A_1, \dots, A_r$ *are called free (or freely independent) if for any sequence* $(a_1, \dots, a_p)$ *with* $a_i \in A_{k_i}$*,* $k_i \neq k_{i+1}$ *for* $1 \leq i \leq p-1$ *and* $\varphi(a_i) = 0$ *for* $1 \leq i \leq p$*, the relation*

$$\varphi(a_1 \dots a_p) = 0$$

holds. The variables $x_1, \dots x_r$ *are called freely independent if the algebras respectively generated by* $x_1, \dots, x_r$ *are free.*

In particular if $a_1, \dots, a_r$ are free, the data $\{\varphi(a_i^n)\}_{1\leq i\leq r, n\geq 1}$ suffices to characterize the joint law of $(a_1, \dots, a_r)$.

Example 1.6. *Originally introduced to study free products of* C^*-*algebra and free group factors, free probability has drawn hudge interests when it has been discovered by Voiculescu in [92] that free probability encodes the limit law of large matrices with independent entries.*

If (A, φ) is C^*-algebra and a_1, a_2 are two free self-adjoint elements of A, then $a_1 + a_2$ is again self-adjoint. We denote by $\mu_{a_1} \boxplus \mu_{a_2}$ the law of $a_1 + a_2$, which depends only on μ_{a_1} and μ_{a_2} by the remark above. If $a_2 \geq 0$, $a_2^{1/2} a_1 a_2^{1/2}$ is again a self-adjoint element, and we denote by $\mu_{a_1} \boxtimes \mu_{a_2}$ the law of $a_2^{1/2} a_1 a_2^{1/2}$. If φ is tracial and a_1 is also positive, $\mu_{a_1} \boxtimes \mu_{a_2}$ is also equal to $a_1^{1/2} a_2 a_1^{1/2}$.

1.2.2 Classical and free cumulants

In the classical case, the computation of the additive convolution of two independent random variables is greatly simplified by the use of the Fourier transform. In the free case, an anologuous method exists with the so-called R-transform introduced by Voiculescu in [91]. However the R-transform is a complicated object, mainly because it involves using the inverse of analytic functions with respect to the composition.
In [77], Speicher introduced a combinatorial method to compute the sum of two free random variables. It is based on the notion of free cumulants, a non-commutative analog of cumulants in classical probability. We present both classical and free cumulants at the same time, since they will both be used in following chapters.
Let $\{f_i\}_{i\geq 1}$ be a family of multilinear functionals on A such that f_i is $i-$multilinear (namely $f_i : A^{\otimes i} \to \mathbb{C}$). For π a partition of r and $a_1, \dots, a_r$ elements of A, f_π denotes the $r-$multilinear map defined by

$$f_\pi(a_1, \dots, a_r) = \prod_{B=\{i_1,\dots,i_s\}\in\pi} f_s(a_{i_1}, \dots, a_{i_s}).$$

For $i \geq 1$ and $a \in A$, we denote by $f_i(a)$ the quantity $f_i(a, \dots, a)$. The expectation φ yields such a family of multilinear maps $\{m_i\}_{i\geq 1}$ with the relation $m_i(a_1, \dots, a_i) = \varphi(a_1 \dots a_i)$; the $r-$th moment of a is $m_r(a)$.

Definition 1.7. *The classical (resp. free) cumulants of (A, φ) is the unique family of multilinear maps $\{c_i\}_{i\geq 1}$ (resp. $\{k_i\}_{i\geq 1}$) such that*

$$m_r = \sum_{\pi\in P(r)} c_\pi, \text{ (resp. } m_r = \sum_{\pi\in NC(r)} k_\pi).$$

$c_r(a)$ (resp. $k_r(a)$) is called the $r-$th cumulant (resp. $r-$th free cumulant) of a.

The existence and unicity of such families is easily proved by recurrence on r. The relation in the definition is also known as the moment-cumulant formula. By this formula, the knowledge of $\{k_i(a)\}$ is equivalent to the knowledge of the law of a in (A, φ), and the same holds with classical cumulants.
Thanks to the poset structure on the set of partitions and the set of non-crossing partitions, there exists a direct formula to express the cumulants in terms of moments. In a finite poset $(G, \leq)$, the Moebius function $\mu_G : G \times G \to \mathbb{R}$ is defined as the unique function satisfying $\sum_{g'\leq h\leq g} \mu_G(g', h) = \delta_{g,g'}$ for $g' \leq g$ and $\mu_G(h, g) = 0$ if $h \not\leq g$. Let μ_P and μ_{NC} denote respectively the Moebius function on the poset of partitions and non-crossing partitions. The following result is due to Speicher ([77]) in the non-commutative case.

Theorem 1.8. *The cumulants and free cumulants have the following expression :*

$$c_r = \sum_{\pi\in P(r)} \mu_P(\pi, \mathbf{1}_r) m_\pi, \; k_r = \sum_{\pi\in NC(r)} \mu_{NC}(\pi, \mathbf{1}_r) m_\pi.$$

In both case the Moebius function is given by an explicit formula. Let $\pi \leq \sigma$ in $P(r)$ (resp. $NC(r)$). One can easily show that the interval $[\pi, \sigma]$ is isomorphic as a poset to $P(k_1)\times\cdots\times P_{k_n})$ (resp. $NC(k_1) \times \cdots \times NC(k_n)$) for some positive integers $k_1, \dots, k_n$. Then

$$\mu_P(\pi, \sigma) = \prod_{1\leq i\leq n} (-1)^{k_i-1}(k_i - 1)!$$

for $P(r)$, and

$$\mu_{NC}(\pi,\sigma) = \prod_{1\leq i\leq n} (-1)^{k_i-1} C_{k_i-1}$$

in the non-crossing case. The formula in the non-crossing case has also been proved by Speicher in [77].
The important property of cumulants (resp. free cumulants) is that they characterize independence (resp. free independence). The free part of the following result comes from [77]:

Theorem 1.9. *Let $a_1,\dots,a_r$ be r elements of A. $a_1,\dots,a_r$ are independent (resp. free) if and only if $c_n(a_{i(1)},\dots,a_{i(n)})$ (resp. $k_n(a(i_1),\dots,a(i_n))$) vanishes for any non-constant function $i : [\![1,n]\!] \to [\![1,r]\!]$.*

Therefore if a_1 and a_2 are free then we have the simple relation $k_r(a_1+a_2) = k_r(a_1)+k_r(a_2)$ for all $r \geq 1$. A same formula involving free cumulants exists to compute the law $\mu_{a_1} \boxtimes \mu_{a_2}$ (see [66]).
In the classical setting several distributions arising as universal limit distributions have very simple expression in terms of cumulants:

- The standard Gaussian variable N, with density $d_N(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, has cumulants $c_2(N) = 1$ and $c_i(N) = 0$ for $i \neq 2$. This distribution appears as the limit distribution of $\frac{X_1+\cdots+X_n}{\sqrt{n}}$ when $n \to +\infty$ and $(X_i)_{i\geq 1}$ is a family of i.i.d centered random variables of variance 1.
A standard complex gaussian variable Z is defined as $Z = \frac{1}{\sqrt{2}}(X+iY)$, with X and Y two independent standard gaussian variables. All cumulants of Z vanish except $c_2(Z,Z^*)$ and $c_2(Z^*,Z)$ which are equal to 1.

- The Poisson variable P, with distribution $\mathbb{P}(P=n) = \frac{e^{-1}}{n!}$, has cumulants $c_i(P) = 1$ for all $i \geq 1$. This distribution is the limit distribution of $Y_1^n + \cdots + Y_n^n$ as n goes to infinity, where $(Y_i^j)_{1\leq i\leq j}$ is a family of independent variables, Y_i^j being a Bernoulli variable with law $\frac{n-1}{n}\delta_0 + \frac{1}{n}\delta_1$.

- The compound Poisson variable P_μ with original probability measure μ is defined by the formula $P_\mu = \sum_{i=1}^{P} Z_i$, where P is a Poisson variable and $(Z_i)_{i\geq 1}$ is a sequence of i.i.d $\mu-$distributed random variables (also independent from P). The cumulants of P_μ are $c_i(P_\mu) = m_i(\mu)$ for $i \geq 1$. This distribution is the limit distribution of $Z_1Y_1^n + \cdots + Z_nY_n^n$ as n goes to infinity, where $(Y_i^j)_{1\leq i\leq j}$ is distributed as before and independent from $(Z_i)_{i\geq 1}$.

In the free case, the same phenomenon arises (see [66] for a detailed exposition of each case):

- The semi-circular variable s, having density $d_s(x) = \frac{1}{2\pi}\mathbf{1}_{|x|\leq 2}\sqrt{4-x^2}$, has free cumulants $k_2(s) = 1$ and $k_i(s) = 0$ for $i \neq 2$. This distribution is the the limit distribution of $\frac{X_1+\cdots+X_n}{\sqrt{n}}$ when $n \to +\infty$ and $(X_i)_{i\geq 1}$ is a family of free identically distributed centered random variables of variance 1.
A standard circular variable c is defined by $c = \frac{1}{\sqrt{2}}(s_1+is_2)$, s_1 and s_2 being two free standard semi-circular variables. All the free cumulants of c vanish except $k_2(c,c^*)$ and $k_2(c^*,c)$ that are equal to one.

- The free Poisson variable p, with density $d_p(x) = \frac{1}{\pi x}\mathbf{1}_{0\leq x\leq 4}\sqrt{4-(2-x)^2}$ has free cumulants $k_i(p) = 1$ for all $i \geq 1$. This distribution is the limit distribution of $Y_1^n + \cdots + Y_n^n$ as n goes to infinity, where $(Y_i^j)_{1\leq i\leq j}$ is a family of free independent variables, Y_i^j being a Bernoulli variable with law $\frac{n-1}{n}\delta_0 + \frac{1}{n}\delta_1$.

- The free compound Poisson variable p_μ with original probability measure μ is defined as the limit distribution of $Z_1 Y_1^n + \cdots + Z_n Y_n^n$ as n goes to infinity, where $(Y_i^j)_{1\leq i\leq j}$ is as before and $(Z_i)_{i\geq 1}$ is a free family of $\mu-$distributed random variables such that Z_i is classicaly independent from Y_i^j for all $j \geq i$. The $i-$th free cumulant of p_μ is the $i-$th moment of μ.

There is an obvious similarity between the three classical examples and the free ones. This correspondance has lead to a systematic bijection, the Bercovici-Pata bijection, between distributions arising as a limit of sums of independent variables and the ones arising as limits of sum of free variables (see [19]).
The cumulant description of the aforementioned distributions and Theorem 1.9 yield interesting combinatorial formulae for some joint moments of free variables. Let us state for example the following result that will be used in Chapter 5:

Proposition 1.10. *Let $c_1, \dots, c_k$ be k free standard circular elements, and write $c_{-i} = c_i^*$. Then for $j_1, \dots, j_r \in \{-k, \dots, -1, 1, \dots k\}$,*

$$m_r(c_{j_1}, \dots, c_{j_r}) = \#\{p \in NC_2(j_1, \dots, j_r) | \forall \{b_1, b_2\} \in p, j_{b_1} = -j_{b_2}\}.$$

Proof. By the moment cumulant formula,

$$m_r(c_{j_1}, \dots, c_{j_r}) = \sum_{\pi \in NC(j_1, \dots, j_r)} k_\pi(c_{j_1}, \dots, c_{j_r}).$$

Since $(c_i)_{1\leq i\leq k}$ is a free family and for each $i \geq 1$, only $k_2(c_i, c_i^*)$ and $k_2(c_i^*, c_i)$ are non-zero, the result follows. ☐

Chapter 2

Probabilistic aspects of representation theory: the unitary group

In this chapter we briefly review how the representation theory of the unitary group U_n leads to interesting probabilistic results. This chapter is mainly intended for probabilists having no backgrounds on representation theory, and serves as a motivation for the non-commutative results in the following chapter. The first section presents the framework of compact groups and their associated probability space, and introduces the representation theory of the unitary group. The irreducible representations of U_n are indexed by symmetric functions; the theory of these functions is quickly reviewed in the second section. The third section is devoted to the full description of the representations of the unitary group and to the description of the Schur-Weyl duality. The fourth section introduces the Weingarten calculus and the fifth section gives some applications of this method. Finally, generalization to other groups are discussed in Section 6.

2.1 Compact groups as probability spaces

This section follows the book [26], and the reader should refer to this reference for omitted proofs.

2.1.1 Compact group and Haar measure

Definition 2.1. *A compact group G is a group that is also a compact topological space, with the property that the maps $(g_1, g_2) \mapsto g_1 g_2$ from $G \times G$ to G and $g \mapsto g^{-1}$ from G to G are continuous maps.*

In the definition above, $G \times G$ is considered with the product topology.

Example 2.2. *The group $\mathbb{U}$ of complex numbers of modulus 1 with the topology inherited from $\mathbb{C}$ is a compact group.*
More generally, the unitary group U_n consisting of matrices U in $M_n(\mathbb{C})$ which satisfy $UU^* = Id_n$ is a compact group; the same is true for any closed subgroup of U_n.

The main feature of a compact group G is the existence of an invariant probability measure on the topological space G:

Theorem 2.3. *Let G be a compact group. There exists a unique regular probability measure $\int_G$ on G such that for any measurable set $X \subseteq G$ and any $g \in G$,*

$$\int_G g.X = \int_G X.g = \int_G X$$

This probability measure is called the Haar measure.

In the statement above, $g.X$ (resp. $X.g$) denotes the set $\{gx, x \in X\}$ (resp. $\{xg, x \in X\}$). For $G = \mathbb{U}$, the Haar measure is simply the Lebesgue measure on the circle.

Remark 2.4. *If G is only assumed to be locally compact, there still exists a unique regular measure (up to a constant multiple) being invariant by left translation . However this measure is not necessarily invariant by right translation. Consider for example the group $G = \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} | x, y \in \mathbb{R}, y > 0 \right\}$, with the left invariant measure μ_L given by $d\mu_L = y^{-2}dxdy$ and the right invariant measure μ_R given by $d\mu_R = y^{-1}dxdy$. A locally compact group such that left and rignt invariant translations coincide is called unimodular.*

Any compact group is thus a natural probability space with the Haar measure. Since it is also a topological space, it carries a canonical class of measurable functions with respect to $\int_G$, namely the algebra $C(G, \mathbb{C})$ of continuous complex functions on G. If $f \in C(G, \mathbb{C})$, we denote by $\bar{f}$ the complex conjugate of this function. With this conjugation, $C(G, \mathbb{C})$ is a commutative $*-$algebra.

2.1.2 Representations of a compact group

In most cases, the Haar measure doesn't have any straightforward expression and therefore the computation of the law of any element of $C(G, \mathbb{C})$ may become cumbersome. Fortunately, the representations theory helps to better understand these random variables.

Definition 2.5. *A representation (V, ρ) of G is the data of a vector space V and a continuous morphism of groups $\rho : G \to Gl(V)$.*
A subspace $W \subseteq V$ is called invariant if for all $g \in G$, $\rho(g)(W) \subseteq W$, and $v \in V$ is called a fixed point if $\rho(g)v = v$ for all $g \in V$. A representation V is said irreductible if V is finite dimensional and has no invariant subspace apart from $\{0\}$ and V.
An intertwiner operator from (V, ρ) to (V', ρ') is a linear map $T : V \to V'$ such that, for any $g \in G$, $\rho'(g) \circ T = T \circ \rho(g)$. The space of morphisms between (V, ρ) and $(V', \rho')$is denoted $\mathrm{Mor}_G(\rho, \rho')$.
Two representations (V, ρ) and (V', ρ') are isomorph if there is an invertible map T in $\mathrm{Mor}_G(\rho, \rho')$.

From now on we will only consider finite dimensional representations and denote by $d(\rho)$ the dimension of the vector space V of a representation (V, ρ).
Given two representations (V, ρ) and (V', ρ') of G, we can construct the direct sum representation $\rho \oplus \rho'$ on $V \oplus W$ (resp. the tensor product representation $\rho \otimes \rho'$ on $V \otimes W$) by taking the direct sum of the maps ρ and ρ' (resp. tensor product of the maps ρ and ρ'):

$$\rho \oplus \rho'(g) = \rho(g) \oplus \rho'(g) \in Gl(V \oplus W), \rho \otimes \rho'(g) = \rho(g) \otimes \rho'(g) \in Gl(V \otimes W).$$

Finally if (V, ρ) is a representation of G, we can define the dual representation ρ^* on the dual V^* of V by

$$\rho^*(f)(v) = f(\rho(g^{-1})v),$$

where $f \in V^*$ and v is any vector in V.

Example 2.6. *As an example of the previous constructions, let (V, ρ) be a representation of G. It yields a representation $\tilde{\rho}$ of G on* $\mathrm{End}(V)$ *defined by*

$$\tilde{\rho}(f)(v) = \rho(g)[f(\rho(g^{-1})v)],$$

with $f \in \mathrm{End}(V)$ and $v \in V$. It is possible to prove that, as a representations of G, $\mathrm{End}(V)$ *is isomorphic to $V \otimes V^*$. Moreover the set of fixed points of* $\mathrm{End}(V)$ *under the action of G is precisely the space of intertwiners* $\mathrm{Mor}_G(\rho, \rho')$.

Let (V, ρ) be a finite dimensional representation of G. Any scalar product $\langle\langle ., . \rangle\rangle$ on V defines an average scalar product $\langle ., . \rangle_G$ on V by the formula :

$$\langle v_1, v_2 \rangle_\rho = \int_G \langle\langle \rho(g)v_1, \rho(g)v_2 \rangle\rangle dg,$$

where $v_1, v_2 \in V$.
The invariance of the Haar measure implies that $\langle ., . \rangle_\rho$ is $G-$invariant, namely

$$\langle \rho(g)v_1, \rho(g)v_2 \rangle_\rho = \langle v_1, v_2 \rangle_\rho.$$

With the latter scalar product we can prove that the irreducible representations are the building block of the representation theory of G :

Proposition 2.7. *Let (V, ρ) be a finite dimensional representation of G. Then V is the direct sum of irreducible representations of G.*

Noticing that the eigenspaces and the image of an intertwiner are invariant subspaces yields the description of the intertwiner space between irreducible representations :

Lemma 2.8 (Schur Lemma)**.** *Let $(V, \rho), (V', \rho)$ be two irreducible representations, and $T \in$* $\mathrm{Mor}(\rho, \rho')$. *Then T is either 0 or an isomorphism.*
If (V, ρ) is an irreducible representation and $f \in \mathrm{Mor}(\rho, \rho)$, then there exists a scalar $\lambda \in \mathbb{C}$ such that $f = \lambda Id$.

2.1.3 Matrix coefficients

The goal is now to construct a family of random variables which is dense in $C(G, \mathbb{C})$ and whose law with respect to the Haar measure could be theoretically computed. This family is a class of particular continuous functions based on the $G-$invariant scalar products $\langle ., . \rangle_\rho$.

Definition 2.9. *A matrix coefficient on G is a function φ on G of the form*

$$\varphi(g) = \langle \rho(g)v_1, v_2 \rangle_\rho,$$

with (V, ρ) a representation of G and $v_1, v_2 \in V$.

The name of these functions is clear if we consider an orthonormal basis $(e_1, \dots, e_d)$ of V with respect to the scalar product $\langle ., . \rangle_G$. With respect to this basis, the representation ρ is

$$\rho : g \mapsto \begin{pmatrix} \rho_{11}(g) & \dots & \rho_{1d}(g) \\ \vdots & & \vdots \\ \rho_{d1}(g) & \dots & \rho_{dd}(g) \end{pmatrix}$$

with $\rho_{ij}(g) = \langle \rho(g)e_j, e_i \rangle_\rho = \mathrm{Tr}(\rho(g)E_{ji})$, where $E_{ij}e_k = \delta_{jk}e_i$. From now on, each representation (V, ρ) is considered with a particular choice of orthonormal basis $(e_i)_{1 \leq i \leq d(\rho)}$ with respect to

an invariant scalar product on V: in this way a canonical set of matrix coefficients $\{\rho_{ij}\}_{1\leq i,j\leq d(\rho)}$ is associated to each representation (V,ρ).
Since we have defined a representation (V,ρ) as a continuous map $G \to Gl(V)$, any matrix coefficient on G is continuous.
Considering direct sums, dual and tensor products of representations shows that the sum, the conjugate and the product of matrix coefficients are again matrix coefficients. The trivial representation $g \mapsto 1$ yields the unit element of $C(G,\mathbb{C})$. Therefore the vector space $\mathcal{A}$ of matrix coefficients on G is a unital $*-$subalgebra of $C(G,\mathbb{C})$. Since any representation of G is a direct sum of irreducible representation, a basis of $\mathcal{A}$ is given by the set of matrix coefficients $\mathcal{I} = \{\rho_{ij}\}_{\substack{\rho \text{ irreductible}\\ 1\leq i,j\leq d(\rho)}}$.
It is possible to construct intertwiners from (V,ρ) to (V',ρ') by averaging on G matrix coefficients coming from these two representations. A careful study of these intertwiners yields the first following important result:

Theorem 2.10 (Schur orthogonality relations, [26] Thm 2.3, Thm 2.4)**.** *Let (V,ρ) and (V',ρ') be two non-isomorphic irreducible representations. Then the matrix coefficients are orthogonal with respect to the Haar measure. Namely for $1 \leq i,j \leq d(\rho)$ and $1 \leq k,l \leq d(\rho')$,*

$$\int_G \rho_{ij}(g)\overline{\rho'_{kl}(g)}dg = 0.$$

If (V,ρ) and (V',ρ') are isomorphic irreducible representations, we can identify their basis and in this case

$$\int_G \rho_{ij}(g)\overline{\rho_{kl}(g)}dg = \frac{1}{d(\rho)}\delta_{ik}\delta_{jl}.$$

Therefore the matrix coefficients of irreducible representations yield an orthonormal basis of $\mathcal{A}$ with respect to the Haar measure on G:

$$\mathcal{A} = \bigoplus_{(V,\rho)\text{ irred.}} \bigoplus_{1\leq i,j\leq d(\rho)} \mathbb{C}\rho_{ij}.$$

The subspace $\bigoplus_{1\leq i,j\leq d(\rho)} \mathbb{C}\rho_{ij}$ is denoted by W_ρ.
The second important Theorem is that the algebra $\mathcal{A}$ is dense in $C(G,\mathbb{C})$:

Theorem 2.11 (Peter-Weyl Theorem, Thm 4.1 in [26])**.** *The matrix coefficients are dense in $C(G,\mathbb{C})$. In particular $\mathcal{I}$ is an orthonormal basis of $L^2(G)$, the space of square-integrable functions on G.*

The proof of this Theorem is a bit evolved in the general case. However, if G is already described as a subgroup of $Gl_n(\mathbb{C})$, the proof of the density is a straightforward consequence of Stone-Weierstrass Theorem.

Example 2.12. *Let us apply these results to the unit circle $\mathbb{U}$. In this case, since the group is commutative, an irreducible representation (V,ρ) of $\mathbb{U}$ is one dimensional and thus it is just a group homomorphism $\rho : \mathbb{U} \to \mathbb{C}^\times$. Since ρ has to be continuous, there exists $n \in \mathbb{Z}$ such that $\rho = e_n$, where $e_n(z) = z^n$ for all $z \in \mathbb{U}$. Reciprocally any function of this type is indeed an irreducible representation of $\mathbb{U}$.*
From Example 2.2, the Haar measure on $\mathbb{U}$ is just the Lebesgue measure on the unit circle. Therefore by the content of this paragraph, the set of functions $\{e_n\}_{n\in\mathbb{Z}}$ is an othonormal basis of $L^2(\mathbb{U})$ with respect to the Lebesgue measure on the circle: the decomposition of any continuous

function in this basis is exactly the usual Fourier expansion.
Thus the constructions made in this paragraph are a generalization of the usual Fourier expansion on the circle to a general non-commutative compact group.

To sum up, any compact group G comes naturally with a probability measure $\int_G$, and a particular set $\mathcal{I}$ of random variables that form an orthonormal basis of $L^2(G, \int_G)$. To fully describe this probability space, we need to know the joint law of these random variables. This is equivalent to knowing the expansion of products of matrix coefficients in the basis $\mathcal{I}$. A theoretical answer to this problem will be given in Section 2.4 with the Weingarten formula.
The concrete computations are hard to achieve. A smaller space of continuous functions, the space of class functions, is easier to handle and still give interesting informations on the compact group.

2.1.4 Characters and the unitary group

Definition 2.13. *Let (V, ρ) be a representation of G. The character of ρ is the function*

$$\chi_\rho(g) = \mathrm{Tr}(\rho(g)).$$

The character is said irreducible if it is the character of an irreducible representation.
A virtual character is a function of the form $\chi_1 - \chi_2$ with χ_1 and χ_2 characters.

Since the direct sum and tensor product of representations are again representations, the set of characters is stable by addition and multiplication. Since the character of the trivial representation is the constant unit function, the set of virtual characters forms therefore a ring $Cl(G)$.
Note moreover that since the trace is invariant by conjugation,

$$\chi(hgh^{-1}) = \mathrm{Tr}(\rho(hgh^{-1})) = \mathrm{Tr}(\rho(h)\rho(g)\rho(h)^{-1}) = \mathrm{Tr}(\rho(g))$$

with $g, h \in G$. Thus by linear extension of this relation, $Cl(G)$ is a ring of functions which are constant on conjugacy classes of G. A function f on G such that $f(hgh^{-1}) = f(g)$ for all $g, h \in G$ is called a class function. Actually the vector space $\mathbb{C}Cl(G)$ spanned by the characters is dense in the space of class functions in $L^2(G)$ (see [26], Thm 2.6).
Theorem 2.10 yields the following straightforward result :

Proposition 2.14. *Let $(V, \rho), (V', \rho)$ be two representations of G. Then*

$$\int_G \chi_\rho(g)\overline{\chi_{\rho'}(g)}dg = \dim \mathrm{Mor}(\rho, \rho').$$

If ρ, ρ' are irreducible representations,

$$\int_G \chi_\rho(g)\overline{\chi_{\rho'}(g)}dg = \begin{cases} 1 & \text{if } (V,\rho) \simeq (V',\rho') \\ 0 & \text{if } (V,\rho) \not\simeq (V',\rho') \end{cases}.$$

In particular the set of irreducible characters is a basis of $Cl(G)$.

Therefore the set of irreducible characters forms a basis of the L^2-space of class functions. To compute the moments of a character with respect to the Haar measure amounts to decompose tensor products of the representation of this character into irreducible ones.

Example 2.15. *Let (V, ρ) be a representation of U_n and $U \in U_n$. Since U is a unitary matrix, it is thus diagonalizable in an orthonormal basis and there exists $P \in U_n$ and Δ a diagonal matrix with modulus one coefficients such that $U = P\Delta P^{-1}$.*
Since χ_ρ is a class function and U is diagonalizable, $\chi_\rho(U) = \chi_\rho(P\Delta P^{-1}) = \chi_\rho(\Delta)$. Therefore the value of the character on U only depends on the eigenvalues of U.
If $\sigma \in S_n$ is a permutation, the matrix $W(\sigma) = (\delta_{i\sigma(j)})_{1\leq i,j\leq n}$ is in U_n, and

$$W(\sigma)\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}W(\sigma)^{-1} = \begin{pmatrix}\lambda_{\sigma^{-1}(1)} & & \\ & \ddots & \\ & & \lambda_{\sigma_n^{-1}}\end{pmatrix}.$$

Since $\chi_\rho(W(\sigma)\Delta W(\sigma)^{-1}) = \chi_\rho(\Delta)$, the value of the character on U is a symmetric function of the eigenvalues of U.

The previous example shows that the theory of symmetric functions plays a role in the study of characters of the unitary group.

2.2 Symmetric functions

In this section we briefly review the basics of symmetric functions. Most of the results come from [60], and the reader should refer to this book for complete proofs.

2.2.1 Young diagrams

Definition 2.16. *Let $n \geq 1$. A partition λ of n, also written $\lambda \vdash n$, is a finite decreasing sequence of integers $(\lambda_1 \geq \lambda_2 \geq \lambda_r > 0)$ such that $\sum \lambda_i = n$. The length of λ is the length of the sequence of non-zero integers.*

The set of partitions of n is denoted $\mathcal{Y}_n$. For each partition λ, $m_k(\lambda)$ denotes the number of elements equal to k in $(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$. With this notation λ is also written $\lambda = 1^{m_1(\lambda)} \ldots n^{m_n(\lambda)}$.
A partition is pictorially represented by a Young diagram, which is an array of n cells with λ_1 cells on the first row, λ_2 cels on the second and so on. The Young diagram of the partition $(7, 4, 2, 1)$ is drawn in Figure 2.1.

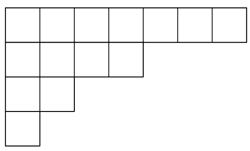


Figure 2.1: Young diagram of $(7, 4, 2, 1)$.

Definition 2.17. *A partial order is defined on $\mathcal{Y}_n$ by saying that $\lambda \leq \mu$ if and only if $l(\lambda) \geq l(\mu)$ and*

$$\lambda_1 \leq \mu_1, \lambda_1 + \lambda_2 \leq \mu_1 + \mu_2, ..., \lambda_1 + \cdots + \lambda_{l(\mu)} \leq \mu_1 + \cdots + \mu_{l(\mu)}.$$

The transpose λ^t of a partition λ is defined as the partition corresponding to the symmetry of the Young diagram of λ through the diagonal axis. For example the transpose of $(7, 4, 2, 1)$ is the partition $(4, 3, 2, 2, 1, 1, 1)$, as suggests the following picture :

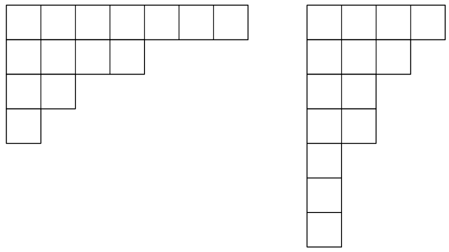


Figure 2.2: Young diagram of $(7,4,2,1)$ and its transpose.

A Young tableau T is the assignment of a positive integer to each cell of a Young diagram λ. λ is then refered as the shape of T. For $\{x_i\}_{i\geq 1}$ an infinite set of commutating variables, a monomial x^T is assigned to each Young tableau T with the formula $x^T = x_1^{\text{number of 1 in } T} x_2^{\text{number of 2 in } T} \dots$. Since the Young diagram has a finite number of cells, the aforementioned product is finite. The next figure is an example of such correspondance:

4	4	1	7	10	6	11
7	5	4	1			
8	6					
6						

Figure 2.3: Young tableau T of shape $(7,4,2,1)$ giving $x^T = x_1^2 x_4^3 x_5 x_6^3 x_7^2 x_8 x_{10} x_{11}$.

2.2.2 Symmetric functions

Definition 2.18. *A symmetric function f is a polynomial in n variables $x_1,\dots,x_n$ such that for all permutation $\sigma \in S_n$, $f(x_{\sigma(1)},\dots,x_{\sigma(n)}) = f(x_1,\dots,x_n)$.*
A rational symmetric function g is a polynomial in n variables $x_1,\dots,x_n$ and their inverse $x_1^{-1},\dots,x_n^{-1}$ such that $g(x_{\sigma(1)},\dots,x_{\sigma(n)}) = g(x_1,\dots,x_n)$.

We denote by Λ_n (resp. $\Lambda_n^{\pm}$) the ring of symmetric functions (resp. rational symmetric functions) in n variables with integer coefficients in the basis of monomials. This is a graded ring with the grading given by the degree of an homogeneous polynomial.
Let e_n be the monomial $x_1 \dots x_n$. Since any rational symmetric function has a monomial of lowest degree, any rational symmetric function g is equal to $\frac{1}{e_n^m} f$, with f a symmetric function. Therefore we will only consider symmetric functions in this subsection.
A straightforward basis of Λ_n is the so-called monomial basis, whose elements are indexed by partition λ with $l(\lambda) \leq n$. The monomial symmetric polynomial m_λ is defined as the sum of $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$ (where we set $\lambda_{r+1} = \dots = \lambda_n = 0$) and all the different monomials obtained from $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$ by permuting the indices in the variables $\{x_i\}_{1\leq i\leq n}$. For example

$$m_{(3,1,1)}(x_1,x_2,x_3) = x_1^3 x_2 x_3 + x_1 x_2^3 x_3 + x_1 x_2 x_3^2.$$

Besides this basis, there exist three bases which can be constructed with Young tableaux:

- Let $RT(\lambda)$ denote the set of tableaux of shape λ such that the integers are weakly increasing along the rows, and define

  $$h_\lambda(x_1,\dots,x_n) = \sum_{T\in RT(\lambda)} x^T.$$

  These functions are called homogeneous symmetric polynomials.

- Let $CT(\lambda)$ denote the set of tableaux of shape λ such that the integers are strictly increasing along the columns, and define
$$\tilde{e}_\lambda(x_1, \dots, x_n) = \sum_{T \in CT(\lambda)} x^T.$$
The functions $e_\lambda = \tilde{e}_{\lambda^t}$ are called the elementary symmetric polynomials.

- Let $SSYT(\lambda)$ denote the set of tableaux of shape λ such that the integers are weakly increasing along the rows and stricly increasing along the colums; such tableau is called a semi-standard Young tableau. Define
$$s_\lambda(x_1, \dots, x_n) = \sum_{T \in SSYT(\lambda)} x^T.$$
These functions are called Schur polynomials. These are the most important polynomials in the study of the representations of U_n.

The sets $\{h_\lambda\}_{l(\lambda)\leq n}$, $\{\tilde{e}_\lambda\}_{l(\lambda)\leq n}$,$\{s_\lambda\}_{l(\lambda)\leq n}$ are all bases of Λ_n. In the case of the elementary and Schur polynomials this result is straightforward: indeed after ordering the bases with respect to the order $\leq$ on partitions (as defined in 2.17), the transition matrix between $\{m_\lambda\}$ and $\{\tilde{e}_\lambda\}$ (or s_λ) is upper-triangular with 1 on the diagonal. Complete proofs and complement can be found in [60], Part I, Ch 6.
From the list of bases above, $e_\lambda = e_{\lambda_1} \dots e_{\lambda_r}$. Thus, since $\{e_{\lambda^t}\}_{l(\lambda^t)\leq n}$ is a basis of Λ_n, Λ_n can be identified with the free commutative ring $\mathbb{Z}[e_1, \dots, e_n]$, with e_r being the polynomial
$$e_r(x_1, \dots, x_n) = \sum_{i_1 < \dots < i_r} x_{i_1} \dots x_{i_r}.$$
A fifth family of symmetric functions arises by considering power sums. Namely let $p_k = \sum_{1\leq i\leq n} x_i^k$ and $p_\lambda = p_{\lambda_1} \dots p_{\lambda_r}$. Although the set $\{p_\lambda\}_{l(\lambda)\leq n}$ is not a basis of Λ_n, it is still a basis of $\mathbb{Q} \otimes \Lambda_n$.

2.2.3 Hall inner product

Projective limit The map $\Phi_n : \Lambda_n \to \Lambda_{n-1}$ defined by $\Phi_n(e_i) = \begin{cases} e_i & \text{if } i \neq n \\ 0 & \text{if } i = n \end{cases}$ is a surjective homomorphism of graded algebra from Λ_n to Λ_{n-1}. We can thus define the projective limit $\Lambda = \varprojlim \Lambda_n$.
Λ can be seen as the algebra of symmetric polynomials in an infinite denumerable set of variables $\{x_1, \dots, x_n, \dots\}$, with the grading given by the degree of homogeneous polynomials. For example the monomial symmetric polynomial $m_{(3,1,1)}$ is defined as
$$m_{(3,1,1)}(\{x_1, \dots, x_n, \dots\}) = \sum_{i_1, i_2, i_3 \text{ distinct}} x_{i_1}^3 x_{i_2} x_{i_3}.$$
The bases given for Λ_n are also bases of Λ if we drop the restriction $l(\lambda) \leq n$ on the partitions indexing elements of the bases.
An important result is that the coefficients of any expansion of an element in Λ_n in one of the bases we gave before is constant for n large enough. Therefore any algebraic result obtained in Λ on a finite set of elements can be considered as also true in Λ_n for n large enough. For example, if we write $m_\lambda m_\mu = \sum_{l(\nu)\leq n} a_{\lambda\mu}^\nu(n)$ the expansion of $m_\lambda m_\mu$ in the basis $\{m_\lambda\}_{l(\lambda)\leq n}$ of Λ_n, the coefficients $a_{\lambda\mu}^\nu(n)$ are independent of n as soon as $n \geq l(\lambda) + l(\mu)$.

Hall inner product We introduce here the Hall inner product in Λ. Note that the same construction and results exist for Λ_n.

Definition 2.19. *The Hall inner product $\langle ., . \rangle$ is the bilinear form on Λ defined by its value on the basis $\{m_\lambda\}$ and $\{h_\lambda\}$ as*

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}.$$

This bilinear form is naturally extended to $\mathbb{Q} \otimes \Lambda$.

This bilinear form is actually an inner product, as we will see in Example 2.21.
It is possible to characterize dual bases with respect to this product. If $\{x_i\}, \{y_i\}$ are two denumerable infinite sets of variables, the basis $\{m_\lambda\}$ and $\{h_\lambda\}$ are related by the Cauchy formula

$$\prod \frac{1}{1 - x_i y_j} = \sum_\lambda m_\lambda(\{x_i\}) h_\lambda(\{y_i\}).$$

The Cauchy formula implies that if f is a symmetric function, then $\langle \prod \frac{1}{1-x_i y_i}, f(\{x_i\}) \rangle = f(\{y_i\})$. The latter equality yields a proof of the following fact:

Proposition 2.20. *Two bases $\{f_\lambda\}$ and $\{g_\lambda\}$ consisting of homogeneous polynomials are dual with respect to $\langle ., . \rangle$ if and only if*

$$\sum_\lambda f_\lambda(\{x_i\}) g_\lambda(\{y_i\}) = \prod \frac{1}{1 - x_i y_j}.$$

Let us apply this proposition to the power sums basis.

Example 2.21. *If we set $z_\lambda = \prod m_k! \prod k^{m_k}$ (recall that m_k is the number of parts of λ equal to k), then*

$$\begin{aligned}
\sum_\lambda \frac{1}{z_\lambda} p_\lambda(\{x_i\}) p_\lambda(\{y_i\}) =& \sum_r \sum_{\lambda, l(\lambda)=r} \frac{1}{\prod m_k(\lambda)!} \sum_{\substack{a_1,\dots,a_r \\ b_1,\dots,b_r}} \prod \left(\frac{(x_{a_j} y_{b_j})^{\lambda_j}}{\lambda_j} \right) \\
=& \sum_r \frac{1}{r!} \left(- \sum_{i,j} \log(1 - x_i y_j)\right)^r \\
=& \exp\left(- \sum_{i,j} \log(1 - x_i y_j)\right) = \prod_{i,j} \frac{1}{1 - x_i y_j}
\end{aligned}$$

Therefore from the last proposition, the dual basis of $\{p_\lambda\}$ is $\{z_\lambda p_\lambda\}$. This implies that

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda.$$

In particular $\langle ., . \rangle$ is positive definite and symmetric.

An important result is that the Schur basis is an orthonormal basis of Λ: namely $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$. The proof is done in [60], Part I, Ch. 4. By a linear algebraic argument, an inner product on Λ has at most one orthonormal basis, up multiplication by ± 1. The Schur basis is therefore the unique orthonormal basis of Λ, and the unique graded basis such that

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_\lambda s_\lambda(\{x_i\}) s_\lambda(\{y_i\}).$$

2.3 Representation theory of the unitary group and probabilistic applications

It is now possible to describe the ring of virtual characters $Cl(U_n)$. This description gives a way to compute moments of characters with respect to the Haar measure: some important results obtained by Diaconis and Shahshahani in [33] are given in example of this machinery.

2.3.1 Irreducible representations of U_n

The content of this subsection comes from [26] Ch. 35 and 38.

Rational and polynomial representations The unitary group U_n has a fundamental representation given by its canonical embedding in $Gl_n(\mathbb{C})$. Denote by $(u_{ij})_{1\leq i,j\leq n}$ the matrix coefficients of this representation. A representation (V,ρ) of U_n is said rational (resp. polynomial) if the matrix coefficients of ρ are rational (resp. polynomial) expressions of the u_{ij}. In the next paragraph we will classify all rational and polynomial representation of U_n.
In any case the character of a continuous representation of U_n is a rational function of the eigenvalues. Indeed let (V,ρ) be a continuous representation of U_n and consider its restriction to the $n-$torus $T_n = \left\{\begin{pmatrix} e^{i\vartheta_1} & & \\ & \ddots & \\ & & e^{i\vartheta_N}\end{pmatrix}\right\}$. This yields a continuous representation $(V,\tilde\rho)$ of T_n. Since T_n is the commutative product of n different copies of $\mathbb{U}$, V decomposes in $\dim(V)$ vectors v_i, each of them being a one-dimensional continuous representation of T_n. Each one-dimensional continuous representation of T_n has the form $\tilde\rho(e^{i\vartheta_1},\dots,e^{i\vartheta_n}) = e^{i(k_1\vartheta_1+\dots+k_n\vartheta_n)}$ with $k_1,\dots,k_n\in\mathbb{Z}$; therefore the character $\chi_{\tilde\rho}(e^{i\vartheta_1},\dots,e^{i\vartheta_n})$ is a rational function of $e^{i\vartheta_1},\dots,e^{i\vartheta_n}$ with non-negative integer coefficient in front of each monomial. But we have seen in Example 2.15 that for $U\in U_n$, $\chi_\rho(U)$ is equal to $\chi_\rho(\Delta)$, with $\Delta\in T_n$ being a diagonal matrix such that $U = P\Delta P^{-1}$. Thus if (V,ρ) is a continuous representation of U_n, $\chi_\rho(U)$ is the evaluation of an element of $\Lambda_n^{\pm}$ on the eigenvalues of U. From now on the characters of U_n are thus identified with elements of $\Lambda_n^{\pm}$, and the ring of virtual characters with a subring of $\Lambda_n^{\pm}$.

Examples of rational representations We review here some basic examples of rational representations, with the identification of the associated character as an element of $\Lambda_n^{\pm}$. Recall that in order to identify the character, it suffices to consider the restriction of the representation to the $n-$torus T_n. In the sequel, $(v_1,\dots,v_n)$ denotes the canonical basis of $\mathbb{C}^n$, and for each unitary matrix U, $u=\{u_1,\dots,u_n\}$ denote its eigenvalues.

- The fundamental representation: this is the identity map on $U_n(\mathbb{C})$. Therefore the character is just the symmetric function $\sum u_i = m_1(u) = e_1(u)$.

- The determinant: the determinant $\det : U_n \to \mathbb{C}$ is a group homomorphism, and thus a represenation. By the relation between determinant and eigenvalues of a matrix, the associated character is the elementary symmetric function $e_n(u_1,\dots,u_n) = u_1\dots u_n$.

- One can generalize the previous representation by considering powers of the determinant: for $m\in\mathbb{Z}$ the map $\det^m : U\mapsto(\det(U))^m$ is again a group homomorphism, and the associated character is the symmetric function e_n^m.

- The $m-$fold exterior representation: let $1 \leq m \leq n$ and let $\bigwedge^m \mathbb{C}^n$ be the $m-$fold exterior product of $\mathbb{C}^n$. The latter is the quotient of the tensor product $(\mathbb{C}^n)^{\otimes m}$ by the relations $v_{i_1} \otimes \cdots \otimes v_{i_m} = 0$ if $i_j = i_k$ for some $j \neq k$. A basis of $\bigwedge^m \mathbb{C}^n$ is given by $\{v_{i_1} \wedge \cdots \wedge v_{i_m}\}_{i_1 < i_2 < \cdots < i_m}$. The map $(U, v_{i_1} \wedge \cdots \wedge v_{i_m}) \mapsto U(v_{i_1}) \wedge \cdots \wedge U(v_{i_m})$ gives a representation of U_n on $\bigwedge^m \mathbb{C}^n$. Considering the restriction to T_n yields that the associated character is exactly the elementary symmetric function e_m.

Note that all these representations are polynomial, except for $\det^m$ with $m < 0$.
Moreover, taking tensor products of the last example with different positive values of m yields that any symmetric function of the form $e_1^{k_1} \dots e_n^{k_n}$ with $k_1, \dots, k_r > 0$ are characters of some polynomial representations. Since $\Lambda_n = \mathbb{Z}[e_1, \dots, e_n]$, taking direct sums of these representations shows that any element of Λ_n corresponds to a virtual character.
From Section 2.2, any rational symmetric function is of the form $e_n^{-m} f$ with $f \in \Lambda_n$, and thus $Cl(U_n) \simeq \Lambda_n^{\pm}$.

Is there a continuous representation of U_n which is not rational ? It seems not clear whether there exists a continuous representation that is not rational. For example $Gl_n(\mathbb{C})$ has continuous non rational representations: consider for instance the group homomorphism $G = (g_{ij})_{1 \leq i,j \leq n} \mapsto \bar{G} = (\bar{g}_{ij})_{1 \leq i,j \leq n}$. However in the case of the unitary group the answer is negative: we have seen that the character of any continuous representation corresponds to an element of $\Lambda_n^{\pm}$. But since, from the previous paragraph, $\Lambda_n^{\pm}$ is already spanned by characters of rational representations, any continuous representation is actually rational.

Irreducible characters of U_n It remains to find which elements of $\Lambda_n^{\pm}$ correspond to irreducible characters.
Since the representations $\det^m$ are all one-dimensional, the functions e_n^m correspond to irreducible characters. Therefore let us consider only the polynomial representations. The irreducible characters can be directly obtained thanks to the Hall inner product on Λ_n. Consider the graded algebra $\mathbb{C}[u_{ij}] = \bigoplus_{d \geq 0} \mathbb{C}[u_{ij}]_d$ of polynomials in the variables u_{ij}, with $\mathbb{C}[u_{ij}]_d$ being the subspace of homogeneous polynomials of degree d.
Since the set $\{u_{ij}\}_{1 \leq i,j \leq n}$ is a set of matrix coefficients of the fundamental representation, $\mathbb{C}[u_{ij}]$ is a subalgebra of the algebra $\mathcal{A}$ of matrix coefficients of U_n. Recall that

$$\mathcal{A} = \bigoplus_{(V,\rho) \text{ irred.}} W_\rho,$$

with $W_\rho = \bigoplus_{1 \leq i,j \leq d(\rho)} \mathbb{C}\rho_{ij}$. Since $\rho_{ij}(g) = \langle \rho(g) e_i, e_j \rangle$, $U_n \times U_n$ acts on W_ρ as

$$((h, h').\rho_{ij})(g) = \langle \rho(g)(\rho(h) e_i), \rho(h^{-1}) e_j \rangle.$$

This shows that as a representation of $U_n \times U_n$, $W_\rho \simeq V_\rho \otimes V_\rho^*$. It is an easy computation to check that $V_\rho \otimes V_\rho^*$ is an irreducible representation of $U_n \times U_n$. Therefore the $U_n \times U_n$ representation $\mathcal{A}$ has a decomposition into irreducible $U_n \times U_n$ representations

$$\mathcal{A} \simeq \bigoplus_{(V,\rho) \text{ irred.}} V_\rho \otimes V_\rho^*.$$

Since $\mathbb{C}[u_{ij}]_d$ is invariant under the action of $U_n \times U_n$, it has a unique decomposition into $U_n \times U_n$ irreducible representations. The polynomial form of these representations yields that

$$\mathbb{C}[u_{ij}] = \bigoplus_{(V,\rho) \text{ poly. irred.}} W_\rho \simeq= \bigoplus_{(V,\rho) \text{ poly. irred.}} V_\rho \otimes V_\rho^*.$$

Looking at the trace of the action of ($\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$, $\begin{pmatrix} y_1^{-1} & & \\ & \ddots & \\ & & y_n^{-1} \end{pmatrix}$) on both sides of the equaliyy above yields

$$\prod_{1\leq i,j\leq n} \frac{1}{1-x_iy_j} = \sum_{\lambda, l(\lambda)\leq n} f_\lambda(x_1,\dots,x_n)f_\lambda(y_1,\dots,y_n).$$

Remark 2.22. *There is obviously a convergence problem here, since the x_i, y_i have modulus one as eigenvalues of U_n. However this equality between functions should be considered as an equality between each homogeneous component, and the value of the whole serie doesn't play any role.*

From Section 2.2.3 we deduce that, with the appropriate labelling, $f_\lambda = s_\lambda$. This yields the following result:

Theorem 2.23. *The ring of virtual characters of U_n is isomorphic to $\Lambda_n^\pm$, and the basis of irreducible characters is given by the set $\{e_n^{-m}s_\lambda\}_{\substack{m\geq 0\\ l(\lambda)\leq n}}$.*
The algebra $\mathbb{C}Cl_{pol}(U_n)$ of polynomial characters is isomorphic to $\mathbb{C}\otimes\Lambda_n$, and the basis of irreducible polynomial characters is given by the Schur basis.
Through the isomorphism $\Phi : \mathbb{C}Cl_{pol}(U_n) \mapsto \mathbb{C}\otimes\Lambda_n$ the L^2-scalar product with respect to the Haar measure on U_n yields the Hall inner product on Λ_n.

2.3.2 Probabilistic applications

The random variables χ_λ Let us label by λ the irreducible representation whose character is given by s_λ through the map Φ in Theorem 2.23, and denote by χ_λ the associated character. Then for n distinct partitions $\lambda^1,\dots,\lambda^r$, the joint law of $(\chi_{\lambda^1},\dots,\chi_{\lambda^r})$ can be explicitly computed. Indeed the product formula on the Schur basis in Λ_n is given by the Littlewood-Richardson coefficients $\{c^\nu_{\lambda\mu}\}$ as

$$s_\lambda s_\mu = \sum_{l(\nu)\leq n} c^\nu_{\lambda\mu} s_{nu}. \tag{2.3.1}$$

These coefficients have a combinatorial nature (see [60], Ch.9), which allows to algorithmically compute them. Let us write $M_\lambda = (c^\nu_{\lambda\mu})_{\mu\nu}$ the matrix of the multiplication by s_λ in the basis $\{s_\mu\}_{l(\mu)\leq n}$. Then for $m_1,\dots,m_r,n_1,\dots,n_r>0$,

$$\int_{U_n} \left((\chi_{\lambda_1}^{m_1}\dots\chi_{\lambda_r}^{m_r})\overline{(\chi_{\lambda_1}^{n_1}\dots\chi_{\lambda^r}^{n_r})}\right) = \left(M_{\lambda^1}^{m_1}\dots M_{\lambda_r}^{m_r}(M_{\lambda_r}^t)^{n_r}\dots(M_{\lambda_1}^t)^{n_1}\right)_{(0)(0)},$$

where (0) is the empty partition corresponding to the constant function $\mathbf{1}$. However, the Littlewood-Richardson coefficients are hard to compute and the previous formula is difficult to deal with.

Diaconis-Shahshahani results Other kinds of formulae can occur by expanding characters in different bases and evaluating the Hall inner product in these bases. Let us look at the power sum basis, and write $X_i = \mathrm{Tr}(U^i)$ for $i\geq 1$. Diagonalizing U shows that the random variable X_k correspond to the power sum p_k through the map Φ. Therefore if $a_1,\dots,a_r,b_1,\dots,b_r\geq 0$,

$$\int_{U_n} (X_1^{a_1}\dots X_r^{a_r})\overline{\left(X_1^{b_1}\dots X_r^{p_r}\right)} = \langle p_\mu, p_\nu\rangle_{\Lambda_n}.$$

with $\mu = 1^{a_1} \dots r^{a_r}$ and $\nu = 1^{b_1} \dots r^{b_r}$. For $n \geq (\sum a_i i) \vee (\sum b_i i)$, the expansion of p_μ, p_ν in the Schur basis is independent of n, and thus the evaluation of the Hall inner product on p_μ and p_ν is the same in Λ_n and Λ for n large enough. Therefore in this case, from the computation of the Hall inner product for the power sum basis in Section 2.2.3,

$$\int_{U_n} (X_1^{a_1} \dots X_r^{a_r}) \overline{\left(X_1^{b_1} \dots X_r^{p_r}\right)} = \delta_{\mu\nu} z_\mu.$$

Since $z_\mu = \prod i^{a_i} a_i!$, this yields for $n \geq (\sum i a_i) \vee (i \sum b_i)$

$$\int_{U_n} (X_1^{a_1} \dots X_r^{a_r}) \overline{\left(X_1^{b_1} \dots X_r^{p_r}\right)} = \prod \int_{U_n} X_i^{a_i} \overline{X_i^{b_i}},$$

with $\int_{U_n} X_i^{a_i} \overline{X_i^{b_i}} = \delta_{a_i b_i} i^{a_i} a_i!$.
The latter are exactly the moments of a symmetric Gaussian complex variable with mean 0 and variance i (see 1.2.2). This is the content of the following Theorem of Diaconis and Shahshahani:

Theorem 2.24 ([33]). *As n goes to $+\infty$, the random vector $(\mathrm{Tr}(U^i))_{i \geq 1}$ converges in moments to a family of independent symmetric complex gaussian variables $(Z_i)_{i \geq 1}$, such that Z_i has mean 0 and variance i.*

By using the representation theory of the symmetric group S_n, Diaconis and Evans also computed in [32] the value of the variance of $\mathrm{Tr}(U^i)$ for all $n \geq 1$ and found that $\int_{U_n} \mathrm{Tr}(U^i)\overline{\mathrm{Tr}(U^j)} = \delta_{ij}(i \wedge n)$. This allowed them to extend this convergence to all symmetric functions having certain Fourier expansions. They could also prove that the convergence of these random variables is stronger than the convergence in moments.

Concrete realization of the Hall inner product The Hall inner product, abstractly defined on the bases $\{m_\lambda, h_\lambda\}$ of Λ_n, can be concretely defined as the inner product of a L^2-space. Indeed, the Haar measure on U_n yields a probability measure on the torus T_n (identified with $[0, 2\pi]^n$) as the pushforwards measure through the map sending U to its eigenvalues. Some care is needed because of the ordering of the eigenvalues, but eventually this yields the existence of a measure dm on T_n, invariant under the action of S_n, such that for μ, ν with $l(\mu), l(\nu) \leq n$,

$$\int_{T_n} s_\mu(e^{i\vartheta_1}, \dots, e^{i\vartheta_n}) \overline{s_\nu(e^{i\vartheta_1}, \dots, e^{i\vartheta_n})} dm(\vartheta_1, \dots, \vartheta_m) = \delta_{\mu\nu}.$$

The density of the measure dm with respect to the Lebesgue measure can be explicitly computed with the Weyl integration formula (see [98]):

Theorem 2.25. *The density of dm with respect to the Lebesgue measure is given by*

$$dm(\vartheta_1, \dots, \vartheta_n) = \frac{1}{(2\pi)^n n!} \prod_{j<k} |e^{i\vartheta_k} e^{i\vartheta_j}|^2.$$

2.4 Weingarten calculus for U_n

In the last section we have used representation theory to obtained probabilistic results on class functions. The goal is to extend this approach to any element of $C(U_n, \mathbb{C})$. This has been done by Collins in [28], and more generally by Collins and Sniady in [29]. The content of this section comes mainly from [29]. This section is particularly exhaustive, since the generalization of the Weingarten formula to quantum groups is the main motivation of the first part of this thesis.

We have seen that the algebra of polynomial matrix coefficients is exactly $\mathbb{C}[u_{ij}]$. From Section 2.3.1, rational representations of U_n are tensor products of polynomial representations with a one dimensional representation of the form $\det^{-m} = (\overline{\det})^m$ for $m \geq 0$. Since $\det \in \mathbb{C}[u_{ij}]$, the algebra of matrix coefficients is therefore $\mathbb{C}([u_{ij}, \bar{u}_{ij}]$.
The main goal is to compute the integral

$$I_{\vec{i},\vec{i'},\vec{j},\vec{j'}} = \int_{U_n} u_{i_1j_1}\dots u_{i_dj_d}\bar{u}_{i'_1j'_1}\dots\bar{u}_{i'_{d'}j'_{d'}} \tag{2.4.1}$$

for $\vec{i},\vec{j} \in [\![1;n]\!]^d$ and $\vec{i'},\vec{j'} \in [\![1;n]\!]^{d'}$. Note that we can always assume that $d = d'$, since otherwise by invariance of the Haar measure by a scalar rotation,

$$\begin{aligned}\int_{U_n} u_{i_1j_1}\dots u_{i_nj_n}\bar{u}_{i'_1j'_1}\dots\bar{u}_{i'_{d'}j'_{d'}} =& \int_{U_n}(zu_{i_1j_1})\dots(zu_{i_nj_n})(\bar{z}\bar{u}_{i'_1j'_1})\dots(\bar{z}\bar{u}_{i'_{d'}j'_{d'}})\\ =& z^{d-d'}\int_{U_n} u_{i_1j_1}\dots u_{i_nj_n}\bar{u}_{i'_1j'_1}\dots\bar{u}_{i'_{d'}j'_{d'}} = 0.\end{aligned}$$

2.4.1 The method

Weingarten calculus is based on the following observation: if G is a compact group and (V,ρ) is a representation of G, then $p_\rho : v \mapsto \int_G \rho(g)vdg$ is the orthonal projection (with respect to the invariant scalar product) on the vector space of fixed points of (V,ρ). The latter is a consequence of the invariance of the Haar measure by left multiplication and of Theorem 2.10.
The idea is thus to consider the integral (2.4.1) as the average of an endomorphism of $(\mathbb{C}^n)^{\otimes d}$ with respect to the action of U_n on $\mathrm{End}((\mathbb{C}^n)^{\otimes d})$. From the previous phenomenon, this average is a fixed point of $\mathrm{End}((\mathbb{C}^n)^{\otimes d})$, and thus an intertwiner of the representation of U_n on $(\mathbb{C}^n)^{\otimes d}$. Relating intertwiners of $(\mathbb{C}^n)^{\otimes d}$ with a particular action of the symmetric group S_d gives then a combinatorial formula for the integral (2.4.1).

Expressing integrals as elements of $\mathrm{Mor}_{U_n}((\mathbb{C}^n)^{\otimes d})$ Let us denote by E_{ij} the matrix $(\delta_{ri}\delta_{sj})_{1\leq r,s\leq n}$ in $M_n(\mathbb{C})$ (the latter is identified with $\mathrm{End}(\mathbb{C}^n)$ through the action on the canonical basis). Then $E_{j_1j'_1}\otimes\dots\otimes E_{j_dj'_d} \in \mathrm{End}((\mathbb{C}^n)^{\otimes d})$. U_n acts on $(\mathbb{C}^n)^{\otimes d}$ by the $d-$fold tensor product of the fundamental representation; thus as in Example 2.6, U_n acts on $\mathrm{End}((\mathbb{C}^n)^{\otimes d})$ by conjugation. For $U \in U_n$,

$$UE_{j_1j'_1}U^*\otimes\dots\otimes UE_{j_dj'_d}U^* = M_{j_1j'_1}\otimes\dots\otimes M_{j_dj'_d},$$

with $M_{j_rj'_r}$ being the matrix

$$\begin{pmatrix} U_{1j_r}\bar{U}_{1j'_r} & \dots & U_{1j_r}\bar{U}_{nj'_r}\\ \vdots & \ddots & \\ U_{nj_r}\bar{U}_{1j'_r} & & U_{nj_r}\bar{U}_{nj'_r}\end{pmatrix}.$$

Thus $\mathrm{Tr}(M_{j_rj'_r}E_{i'_ri_r}) = U_{i_rj_r}\bar{U}_{i'_rj'_r}$ and

$$\mathrm{Tr}((M_{j_1j'_1}\otimes\dots\otimes M_{j_dj'_d})(E_{i'_1i_1}\otimes\dots\otimes E_{i'_di_d})) = U_{i_1j_1}\dots U_{i_dj_d}\bar{U}_{i'_1j'_1}\dots\bar{U}_{i'_dj'_d}.$$

Integrating with respect to the Haar measure yields

$$I_{\vec{i},\vec{i'},\vec{j},\vec{j'}} = \mathrm{Tr}((\int UE_{j_1j'_1}U^*\otimes\dots\otimes UE_{j_dj'_d}U^*dU)(E_{i'_1i_1}\otimes\dots\otimes E_{i'_di_d})).$$

To emphasize the geometric aspect of the right hand-side, this equality can be written as

$$I_{\vec{i},\vec{i}',\vec{j},\vec{j}'} = \langle p_{\mathrm{End}((\mathbb{C}^n)^{\otimes d})}(E_{j_1 j'_1} \otimes \cdots \otimes E_{j_d j'_d}), E_{i_1 i'_1} \otimes \cdots \otimes E_{i_d i'_d})\rangle,$$

with $\langle , \rangle$ being the invariant scalar product $(A, B) \mapsto \mathrm{Tr}(AB^*)$ in $\mathrm{End}((\mathbb{C}^n)^{\otimes d})$. Since $p_{\mathrm{End}((\mathbb{C}^n)^{\otimes d})}$ is a projector, the latter quantity is the same as $\langle p_{\mathrm{End}((\mathbb{C}^n)^{\otimes d})}(E_{j_1 j'_1}\otimes\cdots\otimes E_{j_d j'_d}), p_{\mathrm{End}((\mathbb{C}^n)^{\otimes d})}(E_{i_1 i'_1}\otimes \cdots\otimes E'_{i_d i_d})\rangle$. From Example 2.6, the space of fixed points of $\mathrm{End}((\mathbb{C}^n)^{\otimes d})$ is exactly $\mathrm{Mor}_{U_n}((\mathbb{C}^n)^{\otimes d}, (\mathbb{C}^n)^{\otimes d})$. Therefore evaluating the above scalar product requires a good description of $\mathrm{Mor}_{U_n}((\mathbb{C}^n)^{\otimes d}, (\mathbb{C}^n)^{\otimes d})$.

2.4.2 Schur-Weyl duality

Let S_d denote the symmetric group of order d. S_d is a finite group with cardinal $d!$. By a result of Young (see [60] Part I, Ch. 7), the irreducible representations of S_d are indexed by the integer partitions of d. The irreducible representation corresponding to λ is also called the Specht module of the partition λ and denoted by S_λ.
The representation theory of S_d has a very rich combinatorial structure. Looking at the representations of S_d for several d yields in particular a link between the theory of representation of S_d and symmetric functions :

Theorem 2.26 (Frobenius character formula, [60] p.114)**.** *Let $\mu \vdash d$ and $\nu \vdash d$ be two partitions. The value of the irreducible character of S_μ on a permutation σ with cycle decomposition ν is*

$$\chi_\mu(\sigma) = \langle s_\mu, p_\nu\rangle, \tag{2.4.2}$$

where the scalar product on the right hand side is the Hall inner product on Λ.

S_d acts also on $(\mathbb{C}^n)^{\otimes d}$ by permuting the entries of the tensor product. Namely for $\sigma \in S_d$, the representation $((\mathbb{C}^n)^{\otimes d}, w)$ is defined by

$$w(\sigma)(v_1 \otimes \cdots \otimes v_d) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)}.$$

This action commutes with the action of U_n on each component of the tensor product and thus $w(\sigma) \in \mathrm{Mor}_{U_n}((\mathbb{C}^n)^{\otimes d}, (\mathbb{C}^n)^{\otimes d})$. Moreover

$$\langle E_{j_1 j'_1} \otimes \cdots \otimes E_{j_d j'_d}, w(\sigma)\rangle = \delta_{j_1 \sigma(j'_1)} \ldots \delta_{j_d \sigma(j'_d)}.$$

Therefore we know a particular subset of $\mathrm{Mor}_{U_n}((\mathbb{C}^n)^{\otimes d}, (\mathbb{C}^n)^{\otimes d})$, namely the set $\{w(\sigma)\}_{\sigma\in S_d}$, for which the scalar product with $E_{j_1 j'_1} \otimes \cdots \otimes E_{j_d j'_d}$ is particularly simple. The question is to know whether the knowledge of all these scalar products is enough to reconstruct $p_{\mathrm{End}((\mathbb{C}^n)^{\otimes d})}(E_{j_1 j'_1} \otimes \cdots\otimes E_{j_d j'_d})$. The answer is positive if $\{w(\sigma)\}_{\sigma\in S_d}$ spans the vector space $\mathrm{Mor}_{U_n}((\mathbb{C}^n)^{\otimes d}, (\mathbb{C}^n)^{\otimes d})$. This is exactly the content of the Schur Weyl duality.

Theorem 2.27 (Schur-Weyl duality, [26], Ch.36)**.** *As a $U_n \times S_d$-representation, $(\mathbb{C}^n)^{\otimes d}$ decomposes as*

$$(\mathbb{C}^n)^{\otimes d} = \bigoplus_{\lambda, l(\lambda)\leq n} V_\lambda \otimes S_\lambda.$$

In particular, the action w of S^d yields a surjective map

$$w : \mathbb{C}[S_d] \to \mathrm{Mor}_{U_n}((\mathbb{C}^n)^{\otimes d}, (\mathbb{C}^n)^{\otimes d})$$

which restricts to an isomorphism

$$\tilde{w} = \bigoplus_{\lambda, l(\lambda)\leq n} M_{d(\lambda)}(\mathbb{C}) \to \mathrm{Mor}_{U_n}((\mathbb{C}^n)^{\otimes d}, (\mathbb{C}^n)^{\otimes d}),$$

where $\mathbb{C}S_d$ is identified with $\bigoplus_{\lambda\vdash d} M_{d(\lambda)}(\mathbb{C})$ as a semi-simple algebra.

The decomposition of $\mathbb{C}S_d$ as a direct sum of matrix algebras indexed by irreducible representations of S_d is the content of Artin-Wedderburn's Theorem (see [74], Section 1.10).
We will give here a proof of Theorem 2.27 based on the Frobenius character formula. Note however that its apparent simplicity is misleading, since the Frobenius character formula is a nontrivial result.

Proof. The action of S_d and U_n commutes, yielding an action φ of $S_d \times U_n$ on $(\mathbb{C}^n)^{\otimes d}$. We can thus decompose $(\mathbb{C}^n)^{\otimes d} = \bigoplus_{\rho \text{ irred. of } U_n} V_\rho \otimes W_\rho$, where V_ρ the irreducible representation of U_n with character s_ρ and W_ρ a representation of S_d.
Moreover one can prove (see [57], Section 2) that if $\mu = (\mu_1 \geq \cdots \geq \mu_r)$ is the cycle decomposition of σ and $e^{i\vartheta_1}, \ldots, e^{i\vartheta_n}$ are the eigenvalues of $U \in U_n$, then

$$\chi_\varphi(\sigma, U) = \operatorname{Tr}(U^{\mu_1}) \operatorname{Tr}(U^{\mu_2}) \ldots \operatorname{Tr}(U^{\mu_r}) = p_\mu(e^{i\vartheta_1}, \ldots, e^{i\vartheta_n}). \tag{2.4.3}$$

For example if σ is just the cycle $[1, \ldots, d]$,

$$\begin{aligned} \chi_\varphi(\sigma, U) &= \sum_{1 \leq i_1, \ldots, i_d \leq n} (Ue_{i_2} \otimes \cdots \otimes Ue_{i_d} \otimes Ue_{i_1}, e_{i_1} \otimes \cdots \otimes e_{i_d}) \\ &= \sum_{1 \leq i_1, \ldots, i_d \leq n} U_{i_1 i_2} \ldots U_{i_{n-1} i_n} U_{i_n i_1} = \operatorname{Tr}(U^d). \end{aligned}$$

Since $\chi_\varphi(\sigma, U) = \sum_{(V,\rho) \text{ irred of } U_n} s_\rho(U) \chi_{W_\rho}(\sigma)$ and the Schur functions $\{s_\rho\}_{l(\rho) \leq n}$ form an orthonormal basis of the class functions in $L^2(U_n, \int_{U_n})$,

$$\chi_{W_\rho}(\sigma) = \int_{U_n} \chi_\varphi(\sigma, U) \overline{s_\rho(U)} dU.$$

Therefore $\chi_{W_\rho}(\sigma) = \int_{U_n} p_\mu(U) \overline{s_\rho(U)} dU = \langle p_\mu, s_\rho \rangle_{\Lambda_n}$. Since $l(\rho) \leq n$, $\langle p_\mu, s_\rho \rangle_{\Lambda_n} = \langle p_\mu, s_\rho \rangle_{\Lambda}$. Thus by the Frobenius character formula (2.4.2), $\chi_{W_\rho}(\sigma) = \chi_\rho$ and $W_\rho \simeq S_\rho$, the Specht module of the partition ρ.

□

To summurize, the purpose is to evaluate the scalar product $\langle p_{\mathrm{End}((\mathbb{C}^n)^{\otimes d}}(A), p_{\mathrm{End}((\mathbb{C}^n)^{\otimes d})}(B) \rangle$, with A, B two elements of $\mathrm{End}((\mathbb{C}^n)^{\otimes d})$. To each element A, one can associate the function f_A on S_d defined by $f_A(\sigma) = \langle A, w(\sigma) \rangle$; by the Schur-Weyl duality, the intertwiner space of $(\mathbb{C}^n)^{\otimes d}$ is spanned by $\{w(\sigma)\}_{\sigma \in S_d}$ and thus the data of f_A and f_B is enough to compute $\langle p_{\mathrm{End}((\mathbb{C}^d)^{\otimes d}}(A), p_{\mathrm{End}((\mathbb{C}^n)^{\otimes d})}(B) \rangle$. The matter is therefore to relate exactly $\langle p_{\mathrm{End}((\mathbb{C}^d)^{\otimes d}}(A), p_{\mathrm{End}((\mathbb{C}^n)^{\otimes d})}(B) \rangle$ to $(f_A, f_B)_{L^2(S^d)}$.

2.4.3 Convolution algebra

Convolution algebra Let G be a compact group. We have seen in Section 2 that $L^2(G, \int_G) = \bigoplus_{(V,\rho \text{ irred})} \bigoplus_{1 \leq i,j \leq d(\rho)} \rho_{ij}$.
Let (V, ρ) be an irreducible representation. We identify $\mathrm{End}(V)$ with $M_{d(\rho)}(\mathbb{C})$ through the particular orthogonal basis $(e_1, \ldots, e_{d(\rho)})$ chosen in Section 2.1.3. Thus there is a linear map $\Phi_\rho : M_{d(\rho)}(\mathbb{C}) \to L^2(G, \int_G)$ sending E_{ij} to $d(\rho)\rho_{ij}$, and this linear map is an isomorphism onto the vector space C_ρ of matrix coefficients of the irreducible representation ρ. Φ_ρ maps $A \in M_{d(\rho)}(\mathbb{C})$ to

$$g \mapsto d(\rho) \operatorname{Tr}(\rho(g) A^*).$$

However $M_{d(\rho)}(\mathbb{C})$ has a richer structure given by the matrix multiplication, and by isomorphism this structure transposes to the vector space $\bigoplus_{1 \leq i,j \leq d(\rho)} \rho_{ij}$.

Definition 2.28. *The convolution algebra on G is the $*-$algebra $(C(G,\mathbb{C}),*)$, with the product $*$ given by*

$$f_1 * f_2(g) = \int_G f_1(h) f_2(h^{-1}g)dh.$$

The involution f^* is given by

$$f^*(g) = \bar{f}(g^{-1}).$$

There exists a state ε on this algebra which is defined by the formula $f \mapsto f(e)$, where e is the unit in G.

Note in particular that $\varepsilon(f_1 * f_2^*) = \int_G f_1(h)\bar{f}_2(h)dh = \langle f_1, f_2\rangle_{L^2(G,\int_G)}$.
By Schur orthogonality's Theorem, if (e_i) (resp. (f_i)) is the chosen basis of (V,ρ) (resp. (V',ρ')),

$$\begin{aligned}\Phi_\rho(E^\rho_{ij})\Phi_{\rho'}(E^{\rho'}_{kl}) = d(\rho)d(\rho')\rho_{ij} * \rho'_{kl}(g) =& d(\rho)d(\rho')\int_G \langle\rho(h)e_j, e_i\rangle\langle\rho'(h^{-1}g)f_l, f_k\rangle dh\\ =& d(\rho)d(\rho')\int_G \langle\rho(h)e_j, e_i\rangle\overline{\langle\rho'(h)f_k, \rho(g)f_l\rangle}dh\\ =& d(\rho)\delta_{\rho\rho'}\delta_{kl}\rho_{il} = \Phi_\rho(E^\rho_{ij}E^\rho_{kl}).\end{aligned}$$

Since $\Phi_\rho(E^*_{ij}) = \Phi_\rho(E_{ji}) = d(\rho)\rho_{ji}$ and $\rho_{ji} = \langle\rho(g)e_i, e_j\rangle = \overline{\langle\rho(g^{-1})e_j, e_i\rangle}$ yields also $\Phi_\rho(E^*_{ij}) = \Phi_\rho(E_{ij})^*$, Φ_ρ is a $*-$algebra isomorphism. With this isomorphism, the scalar product given by the trace on $M_{d(\rho)}(\mathbb{C})$ gives the scalar product $\frac{1}{d(\rho)}\langle .,.\rangle$ on C_ρ.

Fourier transform of a representation Let (W,ϑ) be a finite dimensional representation of G, and let $\mathcal{A}_\vartheta$ be the matrix algebra generated by $\{\vartheta(g)\}_{g\in G}$. Since W is finite dimensional, $W = \bigoplus_{(V,\rho)\text{ irred}} V^{\oplus r_\rho} \simeq \bigoplus_{(V,\rho)} \mathbb{C}^{r_\rho}\otimes V$, with $\sum r_\rho < \infty$. Since G doesn't act on the left of each tensor product, $A \in \mathcal{A}_\vartheta$ has the form

$$A = \bigoplus_{(V,\rho)\text{ irred}} Id_{\mathbb{C}^{r_\rho}} \otimes A_\rho, \tag{2.4.4}$$

with $A_\rho \in M_{d(\rho)}(\mathbb{C})$.

Definition 2.29. *The Fourier transform of $A \in \mathrm{End}(W)$ is the function $f_A \in C(G,\mathbb{C})$ defined by*

$$f_A(g) = \mathrm{Tr}(\vartheta(g)A^*).$$

Since on $W \simeq \bigoplus_{(V,\rho)\text{ irred.}} \mathbb{C}^{r_\rho}\otimes V$, $\vartheta(g)$ has the form $\vartheta(g) = \bigoplus_{(V,\rho)\text{ irred.}} Id_{r_\rho}\otimes\rho(g)$, for $A \in \mathcal{A}_\vartheta$

$$f_A(g) = \sum_{(V,\rho)\text{ irred.}} r_\rho \,\mathrm{Tr}(A^*_\rho\rho(g)) = \sum_{(V,\rho)} \frac{r_\rho}{d(\rho)}\Phi(A_\rho),$$

where A has been decomposed as in (2.4.4).
Let $B \in \mathcal{A}_\vartheta$ be another operator with the expansion $B = \bigoplus_{(V,\rho)\text{ irred}} Id_{\mathbb{C}^{r_\rho}}\otimes B_\rho$. Then on one hand,

$$\mathrm{Tr}(AB^*) = \sum_{(V,\rho)} r_\rho \,\mathrm{Tr}(A_\rho B^*_\rho) = \sum_{(V,\rho)} \frac{r_\rho}{d(\rho)}\langle\Phi(A_\rho),\Phi(B_\rho)\rangle,$$

and on the other hand in $C(G,\mathbb{C})$,

$$\langle f_A, f_B\rangle_{L^2(G)} = \sum_{(V,\rho)} (\frac{r_\rho}{d(\rho)})^2\langle\Phi(A_\rho),\Phi(B_\rho)\rangle.$$

Thus if we denote by P_ϑ the operator multiplying each element of C_ρ by $\frac{d(\rho)}{r(\rho)}$, then

$$\mathrm{Tr}(AB^*) = \langle P_\vartheta f_A, f_B\rangle = (P_\vartheta * f_A * f_B^*)(e).$$

Characters in the convolution algebra Since P_ϑ is diagonal on each space C_ρ, it lies in the center of $(C(G,\mathbb{C}), *)$. Note that for $g, g' \in G$;

$$f(gg') = \int_G \delta_{g^{-1}}(h) f(h^{-1}g')dh = (\delta_{g^{-1}} * f)(g')$$

and

$$(f * \delta_{g^{-1}})(g') = \int_G f(h)\delta_{g^{-1}}(h^{-1}g')dh = f(g'g).$$

Thus a function f lies in the center $\mathcal{Z}$ of $(C(G,\mathbb{C}), *)$ if and only if for all $g, g' \in G$, $f(gg') = f(g'g)$. This means that the center of the convolution algebra coincides with the space of class functions, and has a basis consisting in the irreducible characters $\{\chi_\rho\}_{\rho \text{ irred}}$.
Moreover if ρ is an irreductible representation, then by the Schur orthogonality's Theorem,

$$\chi_\rho * \rho'_{ij}(g) = \frac{1}{d(\rho)}\delta_{\rho,\rho'}\rho'_{ij}.$$

Thanks to this formula, we can express the operator P_ϑ above as :

$$P_\vartheta = \sum_{(V,\rho) \text{ irred}} \frac{d(\rho)^2}{r_\rho}\chi_\rho.$$

Weingarten Calculus for U_n Let Wg denotes the function $P_{(\mathbb{C}^n)^{\otimes d}}$. Applying the result of the last paragraph to the representations of S_d on $(\mathbb{C}^n)^{\otimes d}$ in order to compute $\langle p_{\mathrm{End}((\mathbb{C}^n)^{\otimes d})}(E_{j_1 j'_1} \otimes \cdots \otimes E_{j_d j'_d}), E_{i'_1 i_1} \otimes \cdots \otimes E_{i'_d i_d})\rangle$ yields

$$\begin{aligned} I_{\vec{i},\vec{i}',\vec{j},\vec{j}'} =& \langle p_{\mathrm{End}((\mathbb{C}^n)^{\otimes d})}(E_{j_1 j'_1} \otimes \cdots \otimes E_{j_d j'_d}), p_{\mathrm{End}((\mathbb{C}^n)^{\otimes d})}(E_{i_1 i'_1} \otimes \cdots \otimes E_{i_d i'_d}))\rangle \\ =& (Wg * f_{E_{j_1 j'_1} \otimes \cdots \otimes E_{j_d j'_d}} * f^*_{E_{i_1 i'_1} \otimes \cdots \otimes E_{i_d i'_d}})(e) \end{aligned}$$

Computing the last product in the convolution algebra gives the Weingarten formula obtained by Collins in[28]:

Theorem 2.30.

$$I_{\vec{i},\vec{i}',\vec{j},\vec{j}'} = \sum_{\sigma,\tau \in S_d} \delta_{j_1 \sigma(j'_1)} \ldots \delta_{j_d \sigma(j'_d)} \delta_{i_1 \tau(i'_1)} \ldots \delta_{i_d \tau(j'_d)} Wg(\sigma\tau^{-1}).$$

with $Wg(\sigma) = \frac{1}{(d!)^2} \sum_{\lambda \vdash d, l(\lambda) \leq n} \frac{d(\lambda)^2}{\dim V_\lambda} \chi_\lambda(\sigma)$.

2.5 Application of the Weingarten calculus

We review here some applications of the Weingarten calculus. The main motivation of [28] for developing the Weingarten calculus was to compute the coefficients of the so-called Itzykson-Zuber integrals $(z, X, Y) \mapsto \int_{U_n} \exp(nz\,\mathrm{Tr}(XUYU^*)dU$. However we will only give results concerning asymptotic freeness and second order freeness, since the latter involve free probability. One should refer to [28] for more details on the asymptotic expansion of the Itzykson-Zuber integrals.

2.5.1 Asymptotic of the Weingarten function and pair partitions

All the results of this section rely on the asymptotic value of the Weingarten function. Suppose from now on that $n \geq d$. In this case, $Wg(\sigma) = \sum_{\lambda \vdash d} \frac{d(\lambda)^2}{\dim V_\lambda} \chi_\lambda(\sigma)$ and Wg is precisely the inverse of the character $\chi_{(\mathbb{C}^n)^{\otimes d}}$ in the convolution algebra S_d. Applying expression (2.4.3) to $\sigma \times Id$ yields

$$\chi_{(\mathbb{C}^n)^{\otimes d}} = n^{c(\sigma)},$$

where $c(\sigma)$ is the number of cycles of σ. Since $\chi_{(\mathbb{C}^n)^{\otimes d}}$ is polynomial in n, it is expected that Wg is rational in n. Actually, Collins proved in [28] that

$$Wg(\sigma) = \sum_{k \geq 0} a_k(\sigma) n^{-k-(2d-c(\sigma))},$$

and that

$$a_0(\sigma) = (d!)^2 \prod_{\mu_i} l_{\mu_i - 1}(-1)^{\mu_i - 1},$$

where μ is the partition coming from the cycle decomposition of σ and $l_k = \frac{(2k)!}{k!(k+1)!}$ is the $k-$th Catalan number. There exists also a combinatorial description of the other coefficients a_k.
A simpler proof for the expression of a_0, based on Biane's algebra (see [20]), has been given by Collins and Sniady in [29]. Note in particular that the expression of a_0 is a particular value of the Moebius function of the lattice of non-crossing partitions (see Section 1.2.2).
The higher order term in the expansion is given by the value of Wg on the identity, with $Wg(\sigma) = (d!)^2 n^{-d} \delta_{e\sigma}(1 + O(n^{-1}))$. An independent proof of this first order expansion can be given using a scalar product on $\mathbb{C}S_d$.
Let $(.,.)$ be the scalar product defined on $\mathbb{C}S_d$ by the formula $(\sigma, \tau) = n^{c(\sigma^{-1}\tau)}$ and let G_{nd} be the scalar product matrix $((\sigma,\tau))_{\sigma,\tau \in S_d}$. With these notations $(G_{nd})_{\sigma\tau} = \chi_{(\mathbb{C}^n)^{\otimes d}}(\sigma^{-1}\tau)$. Thus if we set $G_{nd}^{-1} = (a_\sigma(\tau))_{\sigma,\tau \in S_d}$, then the functions a_τ have to satisfy the relation

$$\sum_\mu a_\sigma(\mu) \chi_{(\mathbb{C}^n)^{\otimes d}}(\mu^{-1}\tau) = \delta_{\sigma\tau}.$$

In the convolution algebra of S_d, this means that $a_\sigma * \chi_{(\mathbb{C}^n)^{\otimes d}} = \frac{1}{d!}\delta_\sigma$. Therefore $a_\sigma = \frac{1}{d!}\delta_\sigma * Wg$ and

$$(G_{nd}^{-1})_{\sigma\tau} = a_\sigma(\tau) = \frac{1}{d!} \sum_h \frac{1}{d!} \delta_\sigma(h) Wg(h^{-1}\tau) = \frac{1}{(d!)^2} Wg(\sigma^{-1}\tau).$$

Thus $(G_{nd}^{-1})_{\sigma\tau} = \frac{1}{(d!)^2} Wg(\sigma^{-1}\tau)$. On the other hand since $c(\sigma\tau^{-1}) < d$ for $\sigma\tau^{-1} \neq e$, $G_{nd} = n^d(Id + o(n^{-1}))$. Inverting G_{nd} in the latter first order expansion yields $(G_{nd}^{-1}) = n^{-d}(Id + o(n^{-1}))$, which gives the first order expansion of the Weingarten formula.
The method using the Gram-Schmidt matrix of the scalar product on the intertwiner spaces of U_n will be generalized to a large class of quantum groups in Chapter 5.

2.5.2 Asymptotic freeness of unitary invariant random matrices

Second order freeness A second-order probability space intends to capture both expectations and fluctuations of non-commutative random variables. Second-order probability spaces and second-order freeness have been introduced by Mingo and Speicher in [64] to express the flucutations of large random matrices. Unless specified otherwise, all the content of this subsection comes from [64].

Definition 2.31. *A second-order probability space is the data of a probability space (A, φ) with a bilinear functional $\tilde{\varphi} : A \times A \to \mathbb{C}$ which is tracial in both arguments and such that $\tilde{\varphi}(., \mathbf{1}_A) = \tilde{\varphi}(\mathbf{1}_A, .) = 0$.*

The natural construction of a second order probability space is made by considering an algebra A together with a linear map $\vartheta : A \to L^{\infty-}(\Omega)$ sending $\mathbf{1}_A$ to the constant function $\mathbf{1}_\Omega$, and such that $\vartheta(a^*) = \overline{\vartheta(a)}$. Then (A, ϑ) yields a second-order probability space with the maps $\varphi, \tilde{\varphi}$ defined by

$$\varphi(a) = \mathbb{E}(\vartheta(a)), \tilde{\varphi}(a, b) = \mathrm{Cov}(\vartheta(a), \vartheta(b)).$$

Note that ϑ does not need to be an algebra homomorphism. Actually if ϑ is an algebra homomorphism, the map $\tilde{\varphi}$ doesn't give further information than φ, since in this case

$$Cov(\vartheta(a), \vartheta(b)) = \mathbb{E}(\vartheta(ab)) - \mathbb{E}(\vartheta(a))\mathbb{E}(\vartheta(b)) = \varphi(ab) - \varphi(a)\varphi(b).$$

Example 2.32. *Let $A = M_n \otimes L^{\infty-}(\Omega)$. The trace maps any random matrix to a random variable, and therefore from the discussion above, $\big(A, \mathbb{E}(\frac{1}{n}\operatorname{Tr}(.)), \mathrm{Cov}(\frac{1}{n}\operatorname{Tr}(.), \frac{1}{n}\operatorname{Tr}(.))\big)$ is a second-order probability space.*

Definition 2.33. *Let $(A, \varphi, \tilde{\varphi})$ be a second-order probability space and $A_1, \dots, A_r$ be subalgebras of A. $A_1, \dots, A_r$ are called second-order free if they are free and if for all centered elements $a_1, \dots, a_p, b_1, \dots b_{p'}$ with $a_i \in A_{k_i}, b_j \in A_{k'_j}$, $k_i \neq k_{i+1}, k'_j \neq k'_{j+1}$, $k_p \neq k_1$ and $k'_{p'} \neq k'_1$,*

- *if $p = p' = 1$ and $k_1 \neq k'_1$, $\tilde{\varphi}(a_1, b_1) = 0$.*
- *otherwise*
$$\tilde{\varphi}(a_1 \dots a_p, b_{p'} \dots b_1) = \delta_{pp'} \sum_{i=0}^{p-1} \varphi(a_1 b_{1+i}) \dots \varphi(a_p b_{p+i}),$$
where the indices are understood modulo p.

As for freeness, second order freeness allows to recover $\tilde{\varphi}$ from the value of φ and $\tilde{\varphi}$ on each subalgebra A_i.

Second-order limit distribution Let $\{A_n\}_{n\geq 1}$ be a family of unital $*-$algebras and let $\{\vartheta_n\}_{n\geq 1}$ be a family of linear maps $\vartheta_n : A_n \to L^{\infty-}(\Omega, \mathbb{C})$ with $\vartheta_n(a^*) = \overline{\vartheta_n(a)}$ for any $a \in A_n$. A sequence $((a_1^n, \dots, a_p^n))_{n\geq 1}$ of $p-$tuples $(a_1^n, \dots, a_p^n)$ in A_n has a second-order limit distribution $(\varphi, \tilde{\varphi})$ if and only if

- the family $\big(\vartheta_n(P(a_1^n, \dots, a_p^n))\big)_{P \in \mathbb{C}<X_i>_{1\leq i\leq p}}$ converges in moment to a family of complex gaussian variables $(\vartheta(P))_{P\in\mathbb{C}<X_i>_{1\leq i\leq p}}$. The expectation of $\vartheta(P)$ is given by a functional $\varphi : \mathbb{C} < X_i > \to \mathbb{C}$ and covariances are given by a bilinear functional $\tilde{\varphi} : \mathbb{C} < X_i > \times \mathbb{C} < X_i > \longrightarrow \mathbb{C}$.
- the space $\mathbb{C} < X_1, \dots, X_p >$ with the functional φ and $\tilde{\varphi}$ is a second-order probability space.

Note that $\mathbb{C} < X_1, \dots, X_p >$ denotes here the $*-$algebra of noncommutative polynomials in X_i, X_i^*. We will write $X_i^* = X_i^{-1}$ in the sequel.

Example 2.34. *To illustrate the meaning of a second-order limit distribution, let us consider the convergence result of Diaconis and Shahshahani in Theorem 2.24. Let $\mathbb{C}[U_n]$ be the unital algebra of polynomials in $U(n)$, with $U(n)$ being a unitary matrix chosen randomly according to the Haar* *measure on U_n, and let $\vartheta_n : \mathbb{C}[U_n] \to L^{\infty-}(U_n, \int_{U_n})$ define by $\vartheta_n(B) = \mathrm{Tr}(B) - \frac{n-1}{n}\mathbb{E}(\mathrm{Tr}(B))$. Then from Theorem 2.24, $U(n)$ has a second-order limit distribution $(\mathbb{C}[X], \varphi, \tilde{\varphi})$ with $\varphi(X^i) = 0$ and $\tilde{\varphi}(X^i, X^j) = \delta_{i(-j)}i$.*

Definition 2.35. *Let $\{A_n\}_{n\geq 1}$ be a family of unital algebras and let $\{\vartheta_n\}$ be a family of linear maps $\vartheta_n : A_n \mapsto L^{\infty-}(\Omega, \mathbb{C})$. A couple of sequences $((a_1^n, \ldots, a_p^n), (b_1, \ldots, b_q^n))_{n\geq 1}$ is asymptotically second order free if and only if the sequence of $p+q$ elements $((a_1^n, \ldots, a_p^n, b_1, \ldots, b_q^n))_{n\geq 1}$ has a second order limit distribution $(\mathbb{C} < X_1, \ldots, X_p, Y_1, \ldots, Y_q >, \varphi, \tilde{\varphi})$ such that $\mathbb{C} < X_i >$ and $\mathbb{C} < Y_i >$ are second order free in $(\mathbb{C} < X_1, \ldots, X_p, Y_1, \ldots, Y_q >, \varphi, \tilde{\varphi})$.*

Second order freeness for random matrices The two following theorems are two striking applications of the asymptotic computation of the Weingarten formula. The first one is directly based on the evalutation of integrals of type (2.4.1) with the Weingarten calculus :

Theorem 2.36. *[[63],[62]] For each $n \geq 1$ let $U_n(1), \ldots, U_n(p)$ be p independent Haar-distributed unitary matrices of dimension n. Let $A_n = \mathbb{C}[U_n(1), \overline{U}_n(1), \ldots, U_n(p), \overline{U}_n(p)]$ with $\vartheta_n = \mathrm{Tr}(.) - \frac{n-1}{n}\int \mathrm{Tr}(.)$.*
Then the family $(U_n(1), \overline{U}_n(1), \ldots, U_n(p), \overline{U}_n(p))$ is asymptotically second order free and the second order distribution $(\mathbb{C}[X_i], \varphi, \tilde{\varphi})$ (resp. $\mathbb{C}[Y_i]$) of each $U_n(i)$ (resp. $\overline{U}_n(i)$) is given by

$$\varphi(X_i^k) = 0, \tilde{\varphi}(X_i^k, X_i^{k'}) = \delta_{k,-k'}k.$$

A generalization of this result is given for the free unitary group in Chapter 5.
The second theorem should be seen as a generalization of the asymptotic freeness result of Voiculescu on independent random matrices in [92]. We consider here (A_n, ϑ_n) as the $*-$algebra of random matrices with the $*-$linear map $\vartheta_n = \mathrm{Tr}(.) - \frac{n-1}{n}\mathbb{E}(\mathrm{Tr}(.))$. An $n-$dimensional random matrix A is said unitarily invariant if the law of A is the same as the law of UAU^* for all $U \in U_n$.

Theorem 2.37. *[[63]] Let $([A_n^1, \ldots, A_n^p])_{n\geq 1}$ and $([B_n^1, \ldots, B_n^q])_{n\geq 1}$ be two sequences of random matrices, each of them having a second order limit distribution. Suppose moreover that the entries of $[A_n^1, \ldots, A_n^p]$ are independent to the ones of $[B_n^1, \ldots, B_n^q]$, and that the law of each B_n^i is unitarily invariant.*
Then $([A_n^1, \ldots, A_n^p])_{n\geq 1}$ and $([B_n^1, \ldots, B_n^q])_{n\geq 1}$ are asymptotically second order free.

The Weingarten calculs is the cornerstone of the proof. Indeed, since the law of each B_n^i is unitarily invariant, any expectation of products of B_n^i results in elements of $\mathrm{Mor}_{U_n}((\mathbb{C}^n)^{\otimes d}, (\mathbb{C}^n)^{\otimes d})$. Therefore computing expectations of traces of products of these matrices with the ones of $[A_n^1, \ldots, A_n^p]$ yields projections on the space of intertwiners, and thus the use of the machinery of Section 5.

2.6 Generalization to other groups: the Tannaka-Krein duality

We have seen that the computation of an integral with respect to the Haar measure was done through the following procedure :

1. Relate the integral to the scalar product of the projection of two operators on the space of intertwiners of the group.

2. Find a spanning set of the space of intertwiners, on which orthogonal projections have a straightforward expression.

3. Use the properties of the space of intertwiners to recover the scalar product from the orthogonal projections on each element of the spanning set.

Although the first step is a straightforward, the second and the third ones depend heavily on the group and may not be possible. Fortunately there exist some groups for which the second and third steps are still feasible. This will lead to define a category of groups whose intertwiners have a nice combinatorial description (see Section 3.2 and [15]). This combinatorial description is greatly simplified by the rich structure underlying the intertwiner spaces. This structure is given by the Tannaka-Krein duality.

2.6.1 Tannaka-Krein duality for compact matrix group

The Tannaka-Krein duality describes the operations which exist on intertwiner spaces, and establish a bijection between compact groups and collections of spaces stable under these operations. We will only give the result in the case of a matrix compact group but a similar result exists for general compact groups.
Let $G \subseteq U_n$ be a compact subgroup of U_n. There exists a natural family or representations of G, indexed by words in $\{\circ, \bullet\}$. Namely let (V°, ρ°) be the fundamental representation of G given by the identity morphism $\rho^\circ : G \to U_n$ and let $(V^\bullet, \rho^\bullet)$ be its dual representation given by the morphism $\rho^\bullet((g_{ij})_{1\leq i,j\leq n}) = (\bar{g}_{ij})_{\leq i,j\leq n}$. Note that as vector spaces, $V^\circ \simeq V^\bullet \simeq \mathbb{C}^n$. Let $(e_i)_{1\leq i\leq n}$ be a basis of V° which is orthogonal with respect to the invariant scalar product. The dual basis in $V^\bullet$ is denoted by $(\bar{e}_i)_{1\leq i\leq n}$ and the pairing between both bases is denoted by $\langle ., . \rangle$. Note that $\langle ., . \rangle$ is an intertwiner from $V^\circ \otimes V^\bullet$ to the trivial representation $V^\emptyset \simeq \mathbb{C}$.
Taking tensor products of these two representations yields the existence, for any finite word $\varepsilon = \varepsilon_1 \ldots, \varepsilon_r$ in $\{\circ, \bullet\}$, of a representation $(V^\varepsilon, \rho^\varepsilon)$ with

$$V^\varepsilon = V^{\varepsilon_1} \otimes \cdots \otimes V^{\varepsilon_r}, \rho^\varepsilon = \rho^{\varepsilon_1} \otimes \cdots \otimes \rho^{\varepsilon_r}.$$

Let us denote by $\mathrm{Mor}_G(\varepsilon, \varepsilon^{'})$ the vector space of intertwiners from $(V^\varepsilon, \rho^\varepsilon)$ to $(V^{\varepsilon'}, \rho^{\varepsilon'})$. Thus for any $G \subseteq U_n$, $\mathrm{Mor}_G(\varepsilon, \varepsilon')$ is a vector subspace of $\mathcal{L}(V^\varepsilon, V^{\varepsilon'})$, the space of linear maps from V^ε to $V^{\varepsilon'}$.

Remark 2.38. *The collection of vector spaces* $\{\mathrm{Mor}_G(\varepsilon, \varepsilon')\}$ *satisfies several properties :*

- $Id_{V^\circ} \in \mathrm{Mor}(\circ, \circ), Id_{V^\bullet} \in \mathrm{Mor}(\bullet, \bullet),\ \langle ., . \rangle \in \mathrm{Mor}(\circ\bullet, \emptyset), \langle ., . \rangle \in \mathrm{Mor}(\bullet\circ, \emptyset)$.
- *If* $T_1 \in \mathrm{Mor}(\varepsilon_1, \varepsilon_2), T_2 \in \mathrm{Mor}(\varepsilon_2, \varepsilon_3)$*, then* $T_2 \circ T_1 \in \mathrm{Mor}(\varepsilon_1, \varepsilon_3)$.
- *If* $T_1 \in \mathrm{Mor}(\varepsilon_1, \varepsilon_3), T_2 \in \mathrm{Mor}(\varepsilon_2, \varepsilon_4)$*, then* $T_1 \otimes T_2 \in \mathrm{Mor}(\varepsilon_1\varepsilon_2, \varepsilon_3\varepsilon_4)$.
- *If* $T \in \mathrm{Mor}(\varepsilon_1, \varepsilon_2)$*,* $T^* \in \mathrm{Mor}(\varepsilon_2, \varepsilon_1)$.

Of course, since the functions on a compact group form a commutative algebra,

$$T : e_i \otimes e_j \mapsto e_j \otimes e_i \text{ is in } \mathrm{Mor}(\circ\circ, \circ\circ). \tag{2.6.1}$$

All these properties are straightforward deductions of Section 1.2.

Example 2.39. *For* U_n*, the space of intertwiners* $\mathrm{Mor}_{U_n}(\varepsilon_1, \varepsilon_2)$ *is exactly the vector space spanned by all* T_p *for* $p \in P_{2,alternating}(\varepsilon_1, \varepsilon_2)$ *(as defined in Section 1.1.3). If* $\varepsilon = \varepsilon' = \circ^d$*. Each allowed pair partition is encoding a permutation of* S_d*. In particular the scalar product on permutations considered in Section* 5.1 *is the scalar product between the maps* T_p *for* p *in* $P_{2,alternating}(\circ^d, \circ^d)$*.*

The striking fact is that the collection $\{\mathrm{Mor}_G(\varepsilon,\varepsilon')\}_{\varepsilon,\varepsilon'}$ completely characterizes G: this is the content of Tannaka's Theorem. Note that the actual statement of the following Theorem is a bit more evolved (refer to [81] for an exact statement):

Theorem 2.40 ([81])**.** *G can be reconstructed from the data of $\{\mathrm{Mor}_G(\varepsilon,\varepsilon')\}_{\varepsilon,\varepsilon'}$. In particular $\{\mathrm{Mor}_G(\varepsilon,\varepsilon')\}_{\varepsilon,\varepsilon'}$ completely characterizes G.*

Krein's Theorem gives an answer to the dual question: to find all collections of vector spaces $\{H(\varepsilon,\varepsilon')\}$ being the collection of intertwiners of any compact subgroup of U_n. Of course such a collection has to satisfy the conditions (2.38). Krein's Theorem asserts that these conditions are enough:

Theorem 2.41 ([51])**.** *Let $\{H(\varepsilon,\varepsilon')\}$ be a collection of vector spaces such that $H(\varepsilon,\varepsilon') \subseteq \mathcal{L}(V^{\varepsilon},V^{\varepsilon'})$. If $\{H(\varepsilon,\varepsilon')\}$ fulfills the four conditions* (2.38) *and the commutativity relation* (2.6.1)*, then there exists a compact subgroup G of U_n such that for all ε,ε',*

$$H(\varepsilon,\varepsilon') = \mathrm{Mor}_G(\varepsilon,\varepsilon').$$

By Tannaka's Theorem, the compact group G coming from Krein's Theorem is uniquely determined by $\{H(\varepsilon,\varepsilon')\}$.

2.6.2 Other groups with intertwiners described by set partition

In this thesis we are mainly interested in groups (and later quantum groups) whose associated intertwiner spaces are spanned by maps T_p's as in the case of U_n.

Compact classical groups The orthogonal group O_n is the group of matrices $O \in Gl_n(\mathbb{R})$ such that $OO^t = Id$, and the symplectic group Sp_n is the group of matrices $T \in U_{2n}$ such that $TJT^t = J$, with $J = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}$.
In both cases, the fundamental representation and the dual ones are isomorphic: the isomorphisms are given by the map Id in the orthogonal case and by the map J in the symplectic case. Therefore it is enough to specify the description of the intertwiner spaces $\mathrm{Mor}_G(\varepsilon,\varepsilon')$ in the case $\varepsilon = \circ^k$ and $\varepsilon' = \circ^{k'}$. Let us simply denote these vector spaces by $\mathrm{Mor}_G(k,k')$.
The intertwiners of the compact classical groups O_n and Sp_n are also described by pair partitions. Refer to Section 1.1.3 for the definition of the maps $T_p's$ for a given set partition and a Hilbert space V. In the case of O_n,

$$\mathrm{Mor}_{O_n}(k,k') = \langle T_p \rangle_{p\in P_2(k,k')}.$$

Moreover $\{T_p\}_{p\in P_2(k,k')}\}$ is a basis of $\mathrm{Mor}_{O_n}(k,k')$ for $n \geq k+k'$. A same result holds for Sp_n, but it is necessary to adapt the maps T_p's to the non-degenerate bilinear form given by J. Using this description of the interwiners, it has been shown by Diaconis and Shahshahani in [33] that the random vectors $(\mathrm{Tr}(O_n^k))_{k\geq 1}$ and $(\mathrm{Tr}(T_n^k))_{k\geq 1}$ converge in moments respectively to a gaussian vector $(o_k)_{k\geq 1}$ and $(t_k)_{k\geq 1}$, with covariance matrices $\mathbb{E}(o_k o_{k'}) = \mathbb{E}(t_k t_{k'}) = \delta_{kk'} k$ and expectations $\mathbb{E}(o_k) = \delta_{k \text{ even}}$ and $\mathbb{E}(t_k) = -\delta_{k \text{ even}}$.
In [29], Collins and Sniady used the Weingarten calculus to compute the Haar integral of arbitrary polynomials in the coefficients of the fundamental and dual representations of these two groups. In particular the same results as in Section 5 exist in the orthogonal and symplectic case.

Symmetric group A permutation σ_n of S_n can be embedded into O_n by considering its permutation action on the vector space $\mathbb{C}^n$: this action simply permutes the elements of the canonical basis $(e_i)_{1\leq i\leq n}$. This yields a representation $(v(\sigma)_{ij})_{1\leq i,j\leq n}$ of S_n called the fundamental representation. Thus S_n can be seen as a compact matrix group through this fundamental representation v.

As for the orthogonal group, the fundamental representation is isomorphic to the dual representation, and therefore it is enough to describe the interwiner spaces for $\varepsilon = \circ^k, \varepsilon' = \circ^{k'}$. The description is once again achieved by using the maps $\{T_p\}$ for general partitions. Indeed the spaces of intertwiners are $\mathrm{Mor}_{S_n}(k,k') = \langle T_p \rangle_{p\in P(k,k')}$, and the set $\{T_p\}_{p\in P(k,k')}$ is a basis of $\mathrm{Mor}_{S_n}(k,k')$ for $n \geq k+k'$ (see [44]). If $n \leq k+k'$, a basis is given by restricting to the set $\{T_p\}$ where p is a partition having less than n blocks.

Diaconis ans Shahshahani proved in [33] the convergence in moments of the random vector $(\mathrm{Tr}(\sigma_n^k))_{k\geq 1}$ toward a vector of independent random variables $(s_k)_{k\geq 1}$, s_k having a Poisson distribution with parameter $\frac{1}{k}$. Their proof of the result doesn't use the description of the intertwiner spaces. A proof involving this description has been done by Banica, Curran and Speicher in [14].

As it was already said in Section 4.2, symmetric functions play also an important role in the representation theory of S_n. Indeed irreducible representations S_ν of S_n are indexed by Young diagrams ν with n cells. Note first that there is a natural inclusion $S_l \times S_m \subseteq S_n$ for $l+m=n$. Therefore an irreducible representation ρ of S_n is not necessarily an irreducible representation of $S_l \times S_m$ and the decomposition of ρ into irreducible representations of $S_l \times S_m$ is given by the multiplicative structure of the ring of symmetric functions in the Schur basis: namely if $\lambda \vdash l, \mu \vdash m$ and $\nu \vdash n$, there is a decomposition

$$S_\nu = \bigoplus_{\substack{\lambda\vdash l\\ \mu\vdash m}} (S_\lambda \otimes S_\mu)^{\bigoplus c^\nu_{\lambda\mu}},$$

where $c^\nu_{\lambda\mu}$ are the Littlewood-Richardson coefficients (see (2.3.1) and [60], Part I, Ch.9 for their precise definition).

The Weingarten calculus for the symmetric group is not as much developed as the one for the classical Lie groups. We have seen in Section 5 that the precision of the Weingarten calculus is given by the ability to invert a Gram-Schmidt matrix. In the unitary case, this was greatly simplified by the Schur-Weyl duality and the well-known representation theory of the different symmetric groups S_k, for $k \geq 1$. The same method applies also to the orthogonal and symplectic case. However in the symmetric case, the Schur-Weyl theory involves another family of algebras, namely the partition algebras $P_k(n)$ for $n, k \geq 1$. The understanding of the algebraic properties of this family is recent (see the work of Halverson and Ram in [44] for example), and therefore the Weingarten calculus has still not been fully achieved in this setting.

Wreath product with S_n Let G be a classical group and $n \geq 1$. Then S_n acts on G^n by the automorphisms

$$s : \sigma \in S_n \mapsto s(\sigma).(g_1,\ldots,g_n) = (g_{\sigma^{-1}(1)},\ldots,g_{\sigma^{-1}(n)}). \tag{$*$}$$

Definition 2.42. *The wreath product between G and S_n, denoted $G \wr S_n$, is the semi-direct product of G^n and S_n, where S_n acts on G^n by the action $(*)$. In other words,*

$$G \wr S_n = \{((g_1,\ldots,g_n),\sigma), g_i \in G, \sigma \in S_n\},$$

with the product

$$((g_1,\ldots,g_n),\sigma)\cdot((g'_1,\ldots,g'_n),\mu) = ((g_1 g'_{\sigma^{-1}(1)},\ldots,g_n g'_{\sigma^{-1}(n)}),\sigma\mu).$$

If G is a matrix compact group, there is an equivalent way to define $G \wr S_n$. Namely let $(u_{kl}(g))_{1\leq k,l\leq p}$ be the fundamental representation of G. Then $G \wr S_n$ can be defined as the subgroup of $U_{n\times p}$ consisting of the matrices $\left\{(v_{ij}(\sigma)u_{kl}(g_i))_{\substack{1\leq i,j\leq n\\ 1\leq k,l\leq p}}\right\}$ for $\sigma \in S_n$ and $g_1,\ldots,g_n \in G$.
If G is a compact group, $G \wr S_n$ is compact as well and thus there exists a Haar measure on $G \wr S_n$. It is easy to see that $G \wr S_n$ is isomorphic to $G\times\cdots\times G\times S_n$ as a measure space and that the Haar measure on $G \wr S_n$ is given by $d\lambda_{G\wr S_n} = \bigotimes_i dg_i \otimes d\sigma$, where d_g denotes the Haar measure on G and $d\sigma$ the normalized counting measure on S_n.
Let $G \subseteq U_m$. In this case $G \wr S_n \subseteq U_n \otimes U_m$, and by the Tannaka-Krein duality, the description of $G \wr S_n$ is completely given by the data of $\mathrm{Mor}_{G\wr S_n}(\varepsilon,\varepsilon')$ for all words ε,ε' in $\{\circ,\bullet\}$. Actually it is a straightforward computation to express $\mathrm{Mor}_{G\wr S_n}(\varepsilon,\varepsilon')$ in terms of $\mathrm{Mor}_{S_n}(\varepsilon,\varepsilon')$ and $\{\mathrm{Mor}_G(\varepsilon,\varepsilon'\}_{\varepsilon,\varepsilon'}$: an element of $\mathrm{Mor}_{G\wr S_n}(\varepsilon,\varepsilon')$ is given by a partition $p\in P(\varepsilon,\varepsilon')$ together with an element of $\mathrm{Mor}_G(\varepsilon_B,\varepsilon'_B)$ for each block B of p, ε_B and ε'_B being respectively the restriction of ε and ε' to the elements in B (see Chapter 6 for more details on the subject).
As consequence, we get the convergence in law of $\mathrm{Tr}(u_{G\wr S_n})$ toward a compound Poisson distribution with initial law $\mathrm{Tr}(u_G)$.
Note that the irreducible representations of $G \wr S_n$ are described by generalization of Schur functions. Refer to [60], Part I, Appendix B for an exposition in the case of a wreath product $G \wr S_n$ with G a finite group.

Remark 2.43. *For two sets X, Y denote by $\mathcal{F}(X,Y)$ the set of maps from X to Y. The wreath product is a more general construction than the one presented here. Let G be a group. For any set X and group F acting on X, the wreath product $G \wr_X F$ is the set $\mathcal{F}(X,G)\times F$ with the product $\star$ defined as follows: for $h,h'\in\mathcal{F}(X,G)$ and $f,f'\in F$, $(h,f)\star(h',f') = (\tilde{h}, ff')$, where for $x\in X$,*

$$\tilde{h}(x) = h(x)h'(f^{-1}(x)),$$

with the product on the right hand side being done in G.
In Chapter 6, we will study this more general wreath product for G a compact group, X a finite set and F a permutation group of X. In this case the construction is exactly the same as in the case of the symmetric group, and $G \wr_X F$ is compact.

Chapter 3

Compact quantum group

In this chapter we introduce the notion of compact matrix quantum group and give the non-commutative version of the results of last chapter. This lead to an overview of the results obtained in the thesis.

3.1 Noncommutative spaces and quantum groups

3.1.1 What is a compact quantum group ?

Non-commutative spaces The notion of quantum groups fits into the more general framework of noncommutative spaces. The starting idea is that most properties of a classical object, like a topological or a measurable space, can be seen through the algebra of functions on this object. Thus by a considering noncommutative generalization of these algebras, it is possible to define noncommutative analogs to the classical objects.

Example 3.1 (Historical example)**.** *The most trivial example is the one of complex functions on a unique point. Classically this space is just $\mathbb{C}$, with multiplication given by the canonical one on $\mathbb{C}$. The noncommutative generalization is obtained by replacing $\mathbb{C}$ by the algebra $M_n(\mathbb{C})$ of $n-$dimensional matrices. This is exactly what Heisenberg, Born and Jordan (see [24]) did when replacing the orbital position x and momentum p of an electron by two matrices X and P (which were infinite dimensional in this case).*
This example can be transposes to the case of $\mathbb{C}$-valued functions on r points. In this case the algebra $\mathbb{C}^r$ with the pointwise multiplication turns into a matrix algebra $\bigoplus_{i=1}^r M_{n_i}(\mathbb{C})$. Note that the classical algebra $\mathbb{C}^r$ coincides with the center of $\bigoplus_{i=1}^r M_{n_i}(\mathbb{C})$.

In the previous example there is no particular interest in defining topological or measurable noncommutative spaces, since the classical space is a finite set. The correct approach to the definition of functions on a noncommutative topological space is the one of C^*-algebras:

Definition 3.2. *A C^*-algebra A is a $*-$algebra over $\mathbb{C}$ with a norm $\|.\|$ such that*

- *A is complete with respect to $\|.\|$.*
- *for all $x, y \in A$, $(xy)^* = y^*x^*$.*
- *for all $x, y \in A$, $\|xy\| \leq \|x\|\|y\|$ and $\|x^*x\| = \|x\|^2$.*

.

This definition is the most natural one for two reasons. First, the algebra of complex functions on a locally compact Hausdorff space, considered with the $\|.\|_\infty-$norm, is a commutative C^*-algebra. Actually any commutative C^*-algebra is of this form:

Theorem 3.3 (Gelfand's Theorem)**.** *Let A be a commutative C^*-algebra. There exists a locally compact Hausdorff space X such that A is isomorphic to $C^0(X)$, the algebra of complex functions on X vanishing at infinity.*
If A is unital, then X is compact.

Secondly, if A is finite dimensional, we recover the construction made in Example 3.1:

Theorem 3.4 (Artin-Wedderdurn's Theorem)**.** *Let A be a finite dimensional C^*-algebra. Then there exist $r > 0$ and $n_1, \dots, n_r > 0$ such that*

$$A \simeq \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}).$$

If X, Y are compact spaces, a continuous map φ from X to Y yields a C^*-morphism $\Phi : C(Y) \to C(X)$ defined by $\Phi(f)(x) = f(\varphi(x))$. If φ is an injective map (resp. surjective, bijective), then Φ is a surjective map (resp. injective, resp. invertible). Therefore C^*-morphisms encode continuous maps between non-commutative topological spaces.

Remark 3.5 (Where are the points in a noncommutative space ?)**.** *Even if the right way to see a noncommutative space is to consider the functions defined on it, it is still possible to recover a topological space from a general C^*-algebra. If A is commutative, we have seen that A isomorphic to $C^0(X)$ for a locally compact Hausdorff space X. In this case one can show that any irreducible (continuous) representation of A is of the form $ev_x : a \mapsto a(x)$ for an element $x \in X$.*
Similarly if A is a general C^*-algebra, we define the spectrum $Spec(A)$ as the set of equivalence classes of continuous representations of the C^*-algebra. It is possible to define a topology on $Spec(A)$ such that in the commutative case, $Spec(C^0(X)) \simeq X$. In Example 3.1, this yields as expected that $Spec(A)$ is a discrete space with r elements. Therefore formally, evaluating $a \in A$ on $x \in Spec(A)$ is taking the image of a in the irreducible representation x.
This point of view is however often limited, since in many cases, $Spec(A)$ is just a point.

It is also possible to define noncommutative measurable spaces. This yields the notion of von Neumann algebra, which won't be explained here (refer to [80]).

Compact quantum group Following the dual approach to the study of spaces, we want to translate the axioms of a compact group G at the level of the continous functions on G, in order to construct noncommutative analogs.
If $(X, \bullet)$ is a compact Hausdorff space with a continuous semigroup structure $\bullet : X \times X \to X$, the algebra of continuous functions on X inherits an additionnal structure. Namely it is possible to define the map

$$\Delta : \begin{cases} C(X) & \longrightarrow \qquad C(X \times X) \\ \quad f & \mapsto \quad (x, x') \mapsto f(x \bullet x') \end{cases}.$$

By Arzela-Ascoli Theorem, $C(X \times X) \simeq C(X) \otimes C(X)$ (where $C(X) \otimes C(X)$ is the norm completion of the algebraic tensor product). Since $(fg)(x \bullet x') = f(x \bullet x')g(x \bullet x')$ and $\bar{f}(x \bullet x') = \overline{f(x \bullet x')}$, Δ is a $*-$homomorphism from $C(X)$ to $C(X) \otimes C(X)$. Moreover the associativity of the product on X yields the relation:

$$\begin{aligned}(\Delta \otimes Id) \circ \Delta(f))(x_1, x_2, x_3) &= \Delta(f)((x_1 \bullet x_2), x_3) = f((x_1 \bullet x_2) \bullet x_3)) \\ &= f(x_1 \bullet (x_2 \bullet x_3)) = (Id \otimes \Delta) \circ \Delta(f)(x_1, x_2, x_3).\end{aligned}$$

A map Δ satisfying the relation $(Id \otimes \Delta) \circ \Delta = (\Delta \otimes Id) \circ \Delta$ is called coassociative.
Let us consider the maps $\varphi : (x, x') \mapsto (x \bullet x', x')$ and $\varphi' : (x, x') \mapsto (x, x \bullet x')$. If X is a group, these maps are homeomorphisms of topological spaces. By duality φ and φ' yield on $C(X) \otimes C(X)$ the maps $\Phi(f, g) = \Delta(f)(1 \otimes g)$ and $\Phi'(f, g) = (f \otimes 1)\Delta(g)$. Since φ and φ' are injective, Φ and Φ' are surjective maps, and therefore $\{(f \otimes 1)\Delta(g)\}_{f,g\in C(X)}$ and $\{\Delta(f)(1 \otimes g)\}_{f,g\in C(X)}$ are dense in $C(X) \otimes C(X)$.
Reciprocally, if these sets are dense, this means that the maps Φ, Φ' are surjective, and thus the maps φ, φ' are injective. But this is equivalent to the left and right cancellation property for the compact semigroup X, and therefore X is actually a group.
The C^*-algebra $C(G)$ of functions on a compact group G is therefore a commutative unital C^*-algebra with an associative coproduct $\Delta : C(G) \to C(G) \otimes C(G)$, and such that the sets $\{(f \otimes 1)\Delta(g)\}_{f,g\in C(G)}$ and $\{\Delta(f)(1 \otimes g)\}_{f,g\in C(G)}$ are dense in $C(G) \otimes C(G)$. This motivates the following definition, which has been introduced by Woronowicz:

Definition 3.6 (Woronowicz,[99])**.** *A compact quantum group is a unital C^*-algebra A with a coassociative C^*-morphism $\Delta : A \to A \otimes A$ such that $\Delta(A)(1 \otimes A) = (A \otimes 1)\Delta(A) = A \otimes A$.*

The C^*-algebra is often denoted $C(G)$ to emphasize its quantum group nature, even if there is no concrete underlying space G.
$(C(H), \Delta')$ is a quantum subgroup of $(C(G), \Delta)$ if there is a sujective C^*-morphism $\Phi : C(G) \to C(H)$ such that $(\Phi \otimes \Phi)\Delta = \Delta'\Phi$. If Φ is an isomorphism, then H and G are called isomorphic.
As for C^*-algebras, a commutative compact quantum group is a classical group in the following sense :

Proposition 3.7. *Let A be a compact quantum group. If A is commutative, then there exists a compact group G such that $A \simeq C(G)$.*

3.1.2 Representation theory

In this subsection we will introduce the representation theory of a compact quantum group for finite dimensional representations. The content of this subsection comes from [99].

Haar state Since the purpose is to extend probabilistic results from the classical group to the quantum case, one need a natural probability space on compact quantum groups. In the classical setting, this probability space was given by the Haar measure $\int_G$. This probability measure is the unique to satisfy the relations $\int_G f(gh)dg = \int_G f(hg)dg = \int_G f(g)dg$ for all continuous functions f on G; equivalently, for any regular probability measure μ on G and any function $f \in C(G, \mathbb{C})$, $\int_{G\times G} f(gh)dgd\mu(h) = \int_{G\times G} f(hg)dgd\mu(h) = \int_G f(g)dg$.
By the Riesz representation theorem, there is a bijection between regular signed finite measures on G and bounded linear functionals on $C(G, \mathbb{C})$. This bijection restricts to a bijection between regular probability measures μ on G and positive linear functionals l on $C(G, \mathbb{C})$ such that $l(\mathbf{1}) = 1$. Positiveness means that $l(f) \geq 0$ is $f \geq 0$ on G; such positive linear functional l with $l(\mathbf{1}) = 1$ is called a state on $C(G, \mathbb{C})$. If we use the dual approach of the last subsection, the Haar measure corresponds to the unique state $\int_G$ on $C(G, \mathbb{C})$ satisfying the relations :

$$(h \otimes l)\Delta = (l \otimes h)\Delta = h, \tag{3.1.1}$$

for any other state l on $C(G, \mathbb{C})$.
In the quantum framework, we don't have access to the space but only to the functions defined on it. Therefore we can not define measures, but only states: a state ω on a unital C^*-algebra A is a linear functional which is positive, in the sense that $\omega(aa^*) \geq 0$ for any $a \in A$, and such

that $\omega(\mathbf{1}) = 1$.
One of the major results deduced from the axioms of a compact quantum group $C(G)$ is the existence of a state on $C(G)$ satisfying the relations (3.1.1).

Theorem 3.8 ([99])**.** *Let $C(G)$ be a compact quantum group. There exists a unique state h on $C(G)$ such that for any bounded linear functional φ on $C(G)$,*

$$(h \otimes \varphi)\Delta = (\varphi \otimes h)\Delta = h.$$

Therefore as in the classical case, a compact quantum group becomes naturally a noncommutative probability space with the Haar state h. We will mainly be interested in the behavior of elements of $C(G)$ with respect to this Haar state. The example of U_n showed us that the representation theory of the group plays an important role in the computation of expectations with respect to this Haar state. Fortunately it is also possible to build a representation theory of a compact quantum group, and this representation theory is approximately the same as in the classical case.

Finite dimensional representations A finite-dimensional representation of a classical compact group is a finite-dimensional vector space V together with a continuous map $\rho : G \mapsto \mathrm{End}(V)$, such that $\rho(gg') = \rho(g)\rho(g')$ for all $g, g' \in G$ and $\rho(e) = Id_V$. If the dimension of V is n, the space of functions from G to $\mathrm{End}(V)$ is isomorphic to the space $\mathrm{End}(V) \otimes C(G, \mathbb{C})$; thus the previous definition is equivalent to the data of a vector space V together with an element ρ in $\mathrm{End}(V) \otimes C(G, \mathbb{C})$ satisfying

$$(Id \otimes \Delta) \circ \rho = \rho_{12}\rho_{13},$$

where $(a \otimes b)_{12} = a \otimes b \otimes \mathbf{1}_{C(G)}$ and $(a \otimes b)_{13} = a \otimes \mathbf{1}_{C(G)} \otimes b$.
Applying ρ to a vector $v \in V$ yields an element in $V \otimes C(G, \mathbb{C})$, and the image of a subspace W of V is a subspace of $V \otimes C(G, \mathbb{C})$.
This yields the following definition in the quantum case :

Definition 3.9. *Let $(C(G), \Delta)$ be a compact quantum group. A finite dimensional representation of $C(G)$ is a finite-dimensional vector space V with an element $\alpha \in \mathrm{End}(V) \otimes C(G)$ such that*

$$(\alpha \otimes Id) \circ \alpha = (Id \otimes \Delta) \circ \alpha,$$

as maps from V to $V \otimes C(G) \otimes C(G)$.
An intertwiner from (V, α) to (V', α') is a linear map $T : V \to V'$ such that

$$\alpha' \circ T = (T \otimes Id) \circ \alpha.$$

The vector space of intertwiners from (V, α) to (V', α') is denoted $\mathrm{Mor}_G(\alpha, \alpha')$.

Since V is finite dimensional, we can express α in a basis $(e_i)_{1 \leq i \leq n}$ of V. This yields a matrix $(u_{ij})_{1 \leq i,j \leq}$ in $M_n(C(G))$ such that

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

Reciprocally any matrix in $M_n(C(G))$ satisfiying the above relations yields a finite dimensional representation of $C(G)$.

We can take tensor products, direct sums and dual of finite dimensional representations $(u_{ij})_{1\le i,j\le n}$ and $(v_{kl})_{1\le k,l\le m}$ by considering the following usual operations on $M_n(C(G)), M_m(C(G))$:

$$u \otimes v = (u_{ij}v_{kl})_{\substack{1\le i,j\le n\\ 1\le k,l\le m}} \in M_{mn}(C(G)), u \oplus v = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in M_{n+m}(C(G))$$

and $\bar{u} = (u_{ij}^*)_{1\le i,j\le n}$.
The representation is called non-degenerate if $(u_{ij})_{1\le i,j\le n}$ is invertible in $M_n(C(G))$, and two representations are said equivalent if there is an invertible intertwiner from one to the other.
As in the classical case, an invariant subspace of (V,α) is a subspace $W \subseteq V$ such that $\alpha(W) \subseteq W \otimes C(G)$ and a fixed vector is a an element v of V such that $\alpha(v) = v \otimes \mathbf{1}_{C(G)}$. A representation (V,α) is called irreducible if there is no invariant subspace except $\{0\}$ and V.
Thanks to the Haar state, if (V,α) is non-degenerate, it is still possible to defined a scalar product $(.,.)$ on V which is invariant with respect to α: namely $(.,.)$ satisfies

$$(\alpha(e_i), \alpha(e_k)) = \sum (e_j, e_l) \otimes u_{ji}u_{lk}^* = (e_i, e_k) \otimes \mathbf{1}.$$

To obtain this scalar product it suffices to take any scalar product $\langle .,. \rangle$ on V, and to average $\langle .,. \rangle$ with respect to the Haar state:

$$(e_i, e_k) = \sum \langle e_j, e_l \rangle \otimes h(u_{ji}u_{kl}^*).$$

Therefore any non-degenerate representation (V,α) has a basis $\mathcal{B}$ such that the matrix u of α in $\mathcal{B}$ verifies $uu^* = u^*u = Id$, where $(u^*)_{ij} = u_{ji}^*$. Such a matrix is called unitary. The main difference with the classical case is that the conjugate matrix defined by $\bar{u}_{ij} = u_{ij}^*$ is not necessarily unitary as well. However we can show that the representation associated to $\bar{u}$ has good properties, namely: it is non-degenerate (resp. irreducible) if u is non-degenerate (resp.irreducible) and the representation associated to $\bar{u}$ doesn't depend, up to equivalence of representations, to the choice of a matrix u for (V,α). A compact quantum group such that for any unitary representation u, $\bar{u}$ is also unitary, is called a compact quantum group of Kac type.

Matrix quantum group We have seen in Chapter 2 that the situation is much simpler when the group is already described as a subgroup of U_n for some integer $n \ge 1$. In particular the Peter-Weyl Theorem, which is a deep result in the general case, has a much simpler proof in this case.

Definition 3.10. *A compact matrix quantum group is a triple $(A, (u_{ij})_{1\le i,j\le n})$ such that :*

- *A is a C^*-algebra.*
- *The $*-$algebra generated by $\{u_{ij}\}_{1\le i,j\le n}$ is dense in A.*
- *The map $\Phi : u_{ij} \mapsto \sum u_{ik} \otimes u_{ij}$ extends to a C^*-homorphism from A to $A \otimes A$.*
- *The matrices $u = (u_{ij})_{1\le i,j\le n}$ and $\bar{u} = (u_{ij}^*)_{1\le i,j\le n}$ are invertible in $M_n(A)$.*

u can always be chosen unitary, up to equivalence of representations. Since $\bar{u}$ is non-degenerate, there exists a matrix $F \in Gl_n(\mathbb{C})$ such that $F\bar{u}F^{-1} = (u^t)^{-1}$ (F is the matrix encoding the invariant scalar product on the representation of $\bar{u}$).
One can prove that a matrix compact quantum group is actually a compact quantum group.

Therefore a compact matrix quantum group is just a compact quantum group $(C(G), \Delta)$ with a particular representation $u = (u_{i,j})_{1 \le i,j \le n}$ whose coeffficients generate all the C^*-algera $C(G)$. A compact matrix quantum group is of Kac type if F can be chosen equal to the identity. In this case we have

$$u^* u = uu^* = \bar{u}u^t = u^t \bar{u} = Id.$$

Remark 3.11. *This definition is very convenient to define new compact matrix quantum group. Namely it suffices to specify relations among abstract variables $\{u_{ij}, u_{ij}^*\}_{1 \le i,j \le n}$ that are compatible with the coproduct defined above, and then to construct the universal C^*-algebra having these relations.*
An important example is given by the free unitary quantum group U_n^+, introduced by Wang in [95]. The C^*-algebra is the universal C^*- algebra generated by n^2 elements $u = (u_{ij})_{1 \le i,j \le n}$ satisfying the relations $u^*u = uu^* = \bar{u}u^t = u^t\bar{u} = Id$. Since any other compact matrix quantum group of Kac type has also to fulfill these relations, U_n^+ can be seen as the biggest compact matrix quantum group of Kac type of dimension n.
If we add the commutation relations $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all $1 \le i,j,k,l \le n$, the resulting C^*-algebra is commutative and corresponds therefore to a compact matrix group. Actually this group is U_n, the unitary group of dimension n.

3.1.3 Tannaka-Krein Duality

We will present here the Tannaka-Krein duality in the framework of compact quantum groups. This duality extends the Tannaka-Krein duality of Chapter 2 to compact quantum groups.

Peter-Weyl Theorem and Schur's orthogonality Theorem The similarity in the representation theory of the classical and the quantum groups extend to the two majors Theorem of the first section of Chapter 2. Let $C(G)$ be a compact quantum group. A matrix coefficient of $C(G)$ is an element $u_{ij} \in C(G)$ coming from a finite-dimensional representation of G. The vector subspace of matrix coefficients of $C(G)$ is a $*-$algebra for the same reasons as in the classical case. This $*-$algebra is denoted by $C(G)_0$.

Theorem 3.12. *Let $C(G)$ be a compact quantum group. The $*-$algebra $C(G)_0$ is dense in $C(G)$.*

The Schur's orthogonality Theorem is in the quantum case is analogous to the one in the classical case. However one needs to modify a bit the orthogonality relations, because the dual of a unitary representation is not necessarily also unitary.
Let $\{(u_{ij}^\alpha)_{1 \le i,j \le d(\alpha)}\}_{\alpha \text{ irred}}$ be the set of equivalence classes of irreducible representations (written in an orthonormal basis with respect to the invariant scalar product). We have seen that for each irreducible representation u^α, the dual representation $\bar{u}^\alpha$ is not necessarily unitary but always nondegenerate, and therefore there exists $F^\alpha \in M_{d(\alpha)}(\mathbb{C})$ such that $F^\alpha (u^\alpha)^t (F^\alpha)^{-1} \bar{u}^\alpha = Id_{Mn(C(G))}$.
Let $L^2(C(G), h)$ be the completion of $C(G)$ with respect to the scalar product $(a, b) \mapsto h(ab^*)$.

Theorem 3.13 (Schur's orthogonality Theorem)**.** *The set $\{(u_{ij}^\alpha)_{1 \le i,j \le d(\alpha)}\}_{\alpha \text{ irred}}$ forms a basis of $L^2(C(G), h)$, and for α, β irreducible representations, $1 \le i,j \le d(\alpha)$ and $1 \le k,l \le d(\beta)$,*

$$h((u_{ip}^\beta)^* u_{jq}^\alpha) = \delta_{\alpha\beta}\delta_{pq}\frac{(Q_\alpha)_{ji}}{D_\alpha},$$

with Q_α, D_α only depending on F^α. Similarly

$$h(u^\beta_{jq}(u^\alpha_{ip})^*) = \delta_{\alpha\beta}\delta_{ij}\frac{(Q_\alpha^{-1})_{pq}}{D_\alpha}.$$

If G is of Kac type, $h((u^\beta_{ip})^* u^\alpha_{jq}) = \delta_{\alpha\beta}\delta_{pq}\delta_{ij}\frac{1}{d(\alpha)}$ and the Haar state is a trace:

$$h(xy) = h(yx) \text{ for all } x, y \in C(G, h).$$

In particular in the case of a compact quantum group of Kac type, the situation is very close to the one of classical compact groups.

Tannaka-Krein duality In the last part of the previous chapter, it has been shown that a classical compact group is essentially the same as a collection of vector spaces of linear maps stable under certain operations and that most of the properties of the group could be seen on this collection of spaces. The similarity between classical compact groups and compact quantum groups continues here, since the same kind of alternative description exists for a compact quantum group.
The natural framework to describe the representation theory of a compact quantum group is the one of concrete monoidal W^*-category with dual. We won't introduce the basics of category theory here and an interested reader should refer to [59] to get precise statements and theoretical explanation of the formalism introduced here. The definition is given from an abstract point of view, but keeping in mind the representations of a compact quantum group makes it more concrete.

Definition 3.14 ([100]). *A concrete monoidal W^*-category (or CMW^*-category) $\mathcal{C}$ is a monoid R together with a family of finite dimensional Hilbert spaces $\{H_r\}_{r\in R}$ such that $H_r \otimes H_s = H_{rs}$ ($(H_r \otimes H_s) \otimes H_t$ is canonically identified with $H_r \otimes (H_s \otimes H_t)$), and a family of vector spaces $\mathrm{Mor}(r, s) \subseteq \mathcal{L}(H_r, H_s)$ with the following properties :*

- $Id_{H_r} \in \mathrm{Mor}(r, r)$
- *If $T \in \mathrm{Mor}(r, r')$ and $T' \in \mathrm{Mor}(r', r'')$ then $T'T \in \mathrm{Mor}(r, r'')$.*
- *If $T \in \mathrm{Mor}(r, r')$ then $T^* \in \mathrm{Mor}(r', r)$.*
- *If $T \in \mathrm{Mor}(r, r'')$ and $T' \in \mathrm{Mor}(r', r^{(3)})$ then $T \otimes T' \in \mathrm{Mor}(rr', r''r^{(3)})$.*
- $H_e = \mathbb{C}$

$\mathcal{C}$ is called complete if moreover

- *For any $r \in R$ and Hilbert space H such that there exists a unitary operator $V : H_r \to H$, $H = H_s$ for some $s \in R$, and $V \in \mathrm{Mor}(r, s)$.*
- *For any projector $p \in \mathrm{Mor}(r, r)$, there exists $s \in R$ such that $H_s = pH_r$ and the embedding $i : H_s \to H_r$ is in $\mathrm{Mor}(s, r)$.*
- *For any r, r', there exists $s \in R$ such that $H_r \oplus H_{r'} = H_s$ and the canonical inclusion $H_r \to H_s$ and $H_{r'} \to H_s$ are respectively in $\mathrm{Mor}(r, s)$ and $\mathrm{Mor}(r', s)$.*

Example 3.15 ([100]). *The set of finite dimensional representations of a compact quantum group $(C(G), \Delta)$ together with the spaces of intertwiners between them is a complete CMW^*-category denoted $\mathrm{Rep}\, G$. Moreover H is a quantum subgroup of G if and only if $\mathrm{Rep}\, G$ is a subcategory of $\mathrm{Rep}\, H$.*

Note that we didn't formalize the fact that the dual of a finite dimensional representation is equivalent to a unitary representation. In the context of CMW^*-category, this is equivalent to the following definition :

Definition 3.16. *Let $r \in R$ and H_r the associated Hilbert space with basis $\langle e_i \rangle_{1 \leq i \leq n}$. r has a complex conjugate $\bar{r} \in R$ if there is an invertible antilinear map $j : H_r \to H_{\bar{r}}$ such that $\sum e_i \otimes j(e_i) \in \mathrm{Mor}(e, r\bar{r})$ and $\sum j^{-1}(e_i) \otimes e_i \in \mathrm{Mor}(e, \bar{r}r)$.*
A CMW^*-category such that any r has a complex conjugate $\bar{r}$ is called a CMW^*-category with conjugates.

For example, a finite dimensional representation u^α of a compact quantum group has a dual $\bar{u}^\alpha$, and the map j is given by the matrix F^α as constructed in Paragraph 3.1.3.
In the classical case, it was easy to compare the categories of representations of two compact groups G and G' since we only considered matrix compact groups of fixed dimension. Thus we just had to compare the intertwiner spaces $\mathrm{Mor}_G(\varepsilon, \varepsilon')$ and $\mathrm{Mor}_{G'}(\varepsilon, \varepsilon')$ for all $\varepsilon, \varepsilon'$. In the broader case of CMW^*-category, we still have to be able to compare these objects. This leads to the following definition:

Definition 3.17. *Let $\mathcal{C} = (R, \{H_r\}_{r \in R}, \{\mathrm{Mor}(r, r')\}_{r,r' \in R})$ and $\mathcal{C}' = (S, \{K_s\}_{s \in S}, \{\mathrm{Mor}(s, s')\}_{s,s' \in S})$ be two CMW^*-categories.*
$\mathcal{C}$ and $\mathcal{C}'$ are unitarily monoidally equivalent if there is a monoid morphism $\mathcal{F} : R \mapsto S$ and for each r, r' a vector space isomorphism $\mathcal{F} : \mathrm{Mor}(r, r') \to \mathrm{Mor}(\mathcal{F}(r), \mathcal{F}(r'))$ such that :

- *for all $s \in S$, there exists $r \in R$ such that $\mathrm{Mor}(\mathcal{F}(r), s)$ contains a unitary operator.*
- *$\mathcal{F}$ respects the operations on $\mathcal{C}$ and $\mathcal{C}'$: namely if $T \in \mathrm{Mor}(r, r'), T' \in \mathrm{Mor}(r', r''), T'' \in \mathrm{Mor}(r'', r^{(3)})$, then $\mathcal{F}(T^*) = \mathcal{F}(T)^*, \mathcal{F}(T'T) = \mathcal{F}(T')\mathcal{F}(T)$ and $\mathcal{F}(T \otimes T'') = \mathcal{F}(T) \otimes \mathcal{F}(T')$.*

$\mathcal{C}$ and $\mathcal{C}'$ are unitarily isomorphic if they are unitarily monoidally equivalent, and moreover there exists for each $r \in R$ a unitary operator $F_r : H_r \to K_{\mathcal{F}(r)}$ such that:

- *$F(rr') = F(r) \otimes F(r')$.*
- *If $T \in \mathrm{Mor}(r, r')$, $F(r') \circ T = \mathcal{F}(T) \circ F(r)$.*

This means that two CMW^*-categories are unitarily monoidally equivalent if they have the same structure (composition, tensor products, decomposition into simple pieces,...), but the concrete realization on Hilbert spaces are different; they are unitarily isomorphic if even the realization of these structure on Hilbert spaces is the same. In the latter case the two CMW^*-categories should be considered as being the same object.
We can now state the Tannaka-Krein duality in the compact quantum group case. This duality has been discovered and proved by Woronowicz in [100].

Theorem 3.18 (Tannaka-Krein's Duality)**.** *Let $\mathcal{C}$ be a complete CMW^*-category with conjugate. There exists a compact quantum group $(C(G), \Delta)$ such that $\mathcal{C}$ is unitarily isomorphic to $\mathrm{Rep}\, G$. Moreover if $(C(H), \Delta')$ is another compact quantum group such that $\mathcal{C}$ is unitarily isomorphic to $\mathrm{Rep}\, H$, then H is isomorphic to G.*

There exist compact quantum group whose categories of representations are unitarily monoidally equivalent but not unitarily isomorphic. In the latter case the representations are still similar, and for example two compact quantum groups with unitarily monoidally equivalent categories of representations have isomorphic fusion rings.
In the case of a matrix compact quantum group of Kac type, the situtation is much simpler. Let $n \geq 1$. We are using here the notations of Section 2.6.1.

Theorem 3.19 (Tannaka-Krein's duality in the matrix case). *Let $\{H(\varepsilon,\varepsilon')\}$ be a collection of vector spaces such that $H(\varepsilon,\varepsilon') \subseteq \mathcal{L}(V^{\varepsilon}, V^{\varepsilon'})$. If $\{H(\varepsilon,\varepsilon')\}$ fulfills the four conditions* (2.38)*, then there exists a compact subgroup $(C(G),\Delta)$ of U_n^+ such that for all ε,ε',*

$$H(\varepsilon,\varepsilon') = \mathrm{Mor}_G(\varepsilon,\varepsilon').$$

Moreover the matrix compact quantum group is uniquely determined by the data of $\{H(\varepsilon,\varepsilon')\}$.

Example 3.20. *By the stability results 1.1.3 of Chapter* 1*, for each category of partition $\mathcal{C}$ and $n \geq 1$, the maps T_p's give a collection of vector spaces $\{\mathcal{C}_n(\varepsilon,\varepsilon')\}$ that fulfills the four conditions* (2.38)*. Therefore for each catergory of partition $\mathcal{C}$ and $n \geq 1$, there is a quantum subgroup of U_n^+ whose representation theory is encoded by $\mathcal{C}$.*

3.2 Unitary easy quantum groups

3.2.1 Easy quantum groups

Free versions of the classical groups and their associated categories of representation We have seen in Chapter 2, Section 6 that we could associate categories of partition to certain classical groups. This correspondance is summarized by the following list :

U_n	$P^{\circ\bullet}_{2,\text{alternating}}$
O_n	$P^{\circ\bullet}_2$
S_n	$P^{\circ\bullet}$

Figure 3.1: Correspondance between classical groups and categories of partitions through the map $p \mapsto T_p$

Note that the size n of the group is reflected through the map $p \mapsto T_p$ by the choice of the dimension of V° and $V^\bullet$ (which is theoretically speaking a concrete realization of the corresponding category of partition).
The relation $S_n \subseteq O_n \subseteq U_n$ corresponds to the inverse relation $P^{\circ\bullet}_{2,\text{alternating}} \subseteq P^{\circ\bullet}_2 \subseteq P^{\circ\bullet}$, as it was predicted in Example 3.15. In Chapter 1, we have seen the existence of non-crossing analogs $NC^{\circ\bullet}, NC^{\circ\bullet}_2, NC^{\circ\bullet}_{2,alternating}$ of these categories of partitions. By Example 3.20, each category of partition $\mathcal{C}$ yields a subgroup of U_n^+ for each $n \geq 1$. Actually the quantum groups corresponding to the three categories of non-crossing partitions aforementioned have already been introduced by Wang in [95] and [96], and the correspondance with categories of partitions has been proved by Banica in [3],[5] and [4]. In each case the construction is done by using Remark 3.11:

- $NC^{\circ\bullet}_{2,\text{alternating}}$ corresponds to the free unitary group itself U_n^+.
- $NC^{\circ\bullet}_2$ corresponds to the free orthogonal group O_n^+. This quantum group is the quantum subgroup of U_n^+ defined by the relation $u^*_{ij} = u_{ij}$. This is the biggest compact matrix quantum group $(C(G),(u_{ij})_{1\leq i,j\leq n})$ such that all the u'_{ij}s are self-adjoint.
- $NC^{\circ\bullet}$ corresponds to the free symmetric group S_n^+. This quantum group is the quantum subgroup of O_n^+ defined by the relation $\sum_j u_{ij} = \sum_i u_{ij} = \mathbf{1}$ and $u_{ij}u_{ik} = \delta_{jk}u_{ij}$ for all $1 \leq i,j,k \leq n$.

If we add the relations $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all $1 \leq i,j,k,l \leq n$, U_n^+, O_n^+ and S_n^+ become respectively U_n, O_n, S_n. Thus we have the following inclusion relations:

$$\begin{array}{ccccc} S_n^+ & \subsetneq & O_n^+ & \subsetneq & U_n^+ \\ \cup & & \cup & & \cup \\ S_n & \subsetneq & O_n & \subsetneq & U_n \end{array}$$

Definition of easy quantum groups The diagram above is the starting point to define the class of unitary easy quantum groups. This class has been first introduced in the orthogonal case by Speicher and Banica in [15], and then extended to the unitary case in an unpublished article of Banica, Curran and Speicher ([83]).

Definition 3.21. *A compact matrix quantum group G with $S_n \subseteq G \subseteq U_n^+$ is called* easy*, if there is a category of partitions $\mathcal{C} \subseteq P^{\circ\bullet}$ such that for every words ε and ε' in $\{\circ, \bullet\}$, the space of intertwiners $\mathrm{Mor}_G(\varepsilon, \varepsilon')$ is spanned by all linear maps T_p where p is in $\mathcal{C}^{\circ\bullet}(\varepsilon, \varepsilon')$. An easy quantum group G is called* orthogonal easy quantum group*, if $G \subseteq O_n^+$.*

Refer to Chapter 1, Section 1 for the definition of the map T_p for p a two-colored partition. As we will see in Chapter 4, an easy quantum group is orthogonal if and only if belongs to the associated category of partitions.

Example 3.22. *Let us consider the set $\mathcal{H}$ of all partition $p \in P^{\circ\bullet}$ such that each block of p has an even number of elements. $\mathcal{H}$ is a category of partitions. This category corresponds to the hyperoctahedral group $H_n = Z_2 \wr S_n$. This is a subgroup of O_n, and $C(H_n)$ is defined by taking the quotient of $C(O_n)$ by the relations $u_{ij}u_{ik} = u_{ji}u_{ki} = 0$ for all $1 \leq i, j, k \leq n$ with $j \neq k$.*
Once again, we can define the same with non-crossing partitions, yielding the free hyperoctahedral quantum group introduced by [11]. This quantum group is the subgroup of O_n^+ defined by imposing the same relations as above.

The natural question is to find all the compact matrix quantum groups that are easy; this question is equivalent to the classification of all categories of partitions. From a probabilistic point of view, an answer to this question is interesting because for such quantum groups, we expect that the Weingarten calculus may have simpler combinatorial expressions, as this is the case in Chapter 2 for the classical groups. There exists also a method to study the representation theory of easy quantum groups, see [40].
The classification of all orthogonal easy quantum groups has been done in a serie of papers [15],[97],[71]. In this classification, there are two particularly simple cases: the case where the easy quantum group is a classical group and the case where the category of partitions associated to the quantum group is a category of non-crossing partitions. The classification in these both cases has been done in [15, 97]. In-between the situation is much harder to handle with, since there is an uncountable set of such easy quantum groups (see [71]); we should stress nonetheless that the situation becomes simple again when restricting to easy quantum groups between O_n and O_n^+. In the latter case there is only one such quantum group, namely the half-liberated orthogonal group O_n^* (see [15]).

3.2.2 Free easy quantum groups

Following the last comment, we focus particularly on the easy quantum groups described by non-crossing partitions:

Definition 3.23. *A* free easy quantum group *is an easy quantum group G_n such that the corresponding category of partitions $\mathcal{C}$ is a subcategory of $NC^{\circ\bullet}$ (or equivalently $S_n^+ \subseteq G_n$).*

The terminology goes back to Wang's papers [95, 96]; see also [11] or [39]. For example the three quantum groups S_n^+, O_n^+ and U_n^+ are all free easy quantum groups. Several other free easy quantum groups have already been discovered.

The orthogonal case As we said before the classification of all free easy quantum groups has been done in the orthogonal case by Banica, Speicher and Weber in [15, 97].

Proposition 3.24. *Let G_n be an orthogonal free easy quantum group. Then G_n coincides with one of the following quantum groups.*

- O_n^+: $u_{ij} = u_{ij}^*$, *u orthogonal, i.e.* $\sum_k u_{ik}u_{jk} = \sum_k u_{ki}u_{kj} = \delta_{ij}$.
- H_n^+: $u_{ij} = u_{ij}^*$, *u orthogonal,* $u_{ik}u_{jk} = u_{ki}u_{kj} = 0$ *if* $i \neq j$.
- $S_n'^+$: $u_{ij} = u_{ij}^*$, *u orthogonal,* $u_{ik}u_{jk} = u_{ki}u_{kj} = 0$ *if* $i \neq j$, $\sum_k u_{ik} = \sum_k u_{kj}$ *for all* i, j.
- S_n^+: $u_{ij} = u_{ij}^* = u_{ij}^2$, *u orthogonal,* $\sum_k u_{ik} = \sum_k u_{kj} = 1$ *for all* i, j.
- $B_n^{\#+}$: $u_{ij} = u_{ij}^*$, *u orthogonal,* $\sum_k u_{ik} = \sum_k u_{kj}$ *for all* i, j.
- $B_n'^+$: $u_{ij} = u_{ij}^*$, *u orthogonal,* $r := \sum_k u_{ik} = \sum_k u_{kj}$ *for all* i, j, $u_{ij}r = ru_{ij}$.
- B_n^+: $u_{ij} = u_{ij}^*$, *u orthogonal,* $\sum_k u_{ik} = \sum_k u_{kj} = 1$ *for all* i, j.

Note that the quantum groups B_n^+, $B_n^{\#+}$ (with a different notation) and $S_n'^+$ appeared first in [15], and $B_n'^+$ was discovered in [97].
Moreover the category of partitions of each of these quantum groups can be explicitly described. For example $\mathcal{B}^{\#}$, the category of partitions associated to $B_n^{\#+}$, is the category of non-crossing partitions whose blocks are only pairs and singleton, and such that there is an even number of singletons between two elements of a same pair. We won't describe all of these categories, since a more general result will be given in Chapter 4.

Banica and Vergnioux's quantum reflection groups H_n^{s+} The quantum reflection groups H_n^{s+} were first defined by Banica and Vergnioux in [16] and studied by Banica, Belinschi, Capitaine and Collins in [9].

Definition 3.25. *Given $n, s \in \mathbb{N}$, the* quantum reflection group H_n^{s+} *is given by the universal C^*-algebra generated by elements u_{ij}, $1 \leq i, j \leq n$ subject to the conditions:*

- $u = (u_{ij})$ *and* $\bar{u} = (u_{ij}^*)$ *are unitaries*
- *all u_{ij} are partial isometries (i.e. $u_{ij}u_{ij}^*u_{ij} = u_{ij}$) and the projections $u_{ij}^*u_{ij}$ and $u_{ij}u_{ij}^*$ coincide*
- $u_{ij}^s = u_{ij}u_{ij}^*$

We define $H_n^{\infty+}$ by omitting the third of the above conditions.

Note that $H_n^{1+} = S_n^+$ and $H_n^{2+} = H_n^+$. Furthermore, $H_n^{s+} = \widehat{\mathbb{Z}_s} \wr_* S_n^+$ where $\wr_*$ denotes Bichon's free wreath product [21] and $\mathbb{Z}_s$ is shorthand for the cyclic group $\mathbb{Z}/s\mathbb{Z}$. Moreover, the quotient of the above C^*-algebras by the commutator ideal yields $C(H_n^s)$, where $H_n^s = \mathbb{Z}_s \wr S_n$. The quantum reflection groups have also been studied in [16] , [9] and [55].

Proposition 3.26. *Let $s \in \mathbb{N} \cup \{\infty\}$. The quantum reflection group H_n^{s+} is easy and the corresponding category of partition is $\mathcal{H}^{s,s}$, the category of non-crossing partitions such that each block has the same number of black and white points modulo s.*

This serie of quantum groups will be further studied in Chapter 5 and 6, since they are also free wreath products.

Banica, Bichon, Capitaine and Collins's $H^{\#,+}$ The quantum group $H^{\#,+}$ has been defined in [9] as an auxiliary object.

Definition 3.27. *The quantum group $H^{\#,+}$ is given by the universal C^*-algebra generated by u_{ij} such that u and $\bar{u}$ are unitaries and:*

$$u_{ik}u_{jk}^* = u_{ki}u_{kj}^* = u_{ik}^*u_{jk} = u_{ki}^*u_{kj} = 0, \qquad \text{whenever } i \neq j.$$

It has been proven in [8] that

$$H^{\#,+}$$

is free easy with category $\mathcal{H}^{\#}$, the category of non-crossing partitions with blocks having an even number of elements with alternating colors.

3.2.3 Overview of the results

Let us review the results that are obtained in Part II. The results of Part II are from a joint work with Moritz Weber.

Classification of free easy quantum groups The first result is a classification of all free easy quantum groups. This classification is done in two steps. The first step is the classification of all the categories of non-crossing two-colored partitions, which can be summarized as follows :

Theorem 3.28 (*Ch*.3, Th.4.41 and Th.4.42)**.** *There exist five denumerable families of categories of two-colored non-crossing two-colored partitions :*

- $\mathcal{U}$
- $\mathcal{O}^k$ *for* $d \in 2\mathbb{N}$
- $\mathcal{H}^{\#}$ *and* $\mathcal{H}^{d,k}$ *for* $d|k, k \geq 2$.
- $\mathcal{B}^{r,d,k}$ *for* $d|k, k \geq 1$ *and* $r \in \{*, 0, d/2\}$ *($r = d/2$ is possible only if d is even).*
- $\mathcal{S}^{d,k}$ *for* $d|k, k \geq 1$.

In each case these categories have a combinatorial description.
In a second step we identify the compact quantum groups that corresponds to each of these categories of partitions. This identification is greatly simplified by the fact that a lot of free easy quantum groups have already been identified. The essential tool of the remaining part of this identification is the introduction of two algebraic operations, the tensor and the free complexifications denoted respectively $\tilde{\times}$ and $\tilde{*}$) by Z_d. The tensor complexification has been already considered in an unpublished draft [83], and the free complexification with $\mathbb{Z}$ has been first introduced in [8]. This yields the following classification :

Theorem 3.29 (*Ch*.3, Th. 4.57)**.** *For each $n \geq 1$, the following correspondance holds between categories of partitions and unitary easy quantum groups:*

1. *the category $\mathcal{U}$ corresponds to the free unitary quantum group U_n^+. the category $\mathcal{O}^k$ corresponds to $O_n^+ \tilde{\times} Z_k$.*
2. *the category $\mathcal{H}^{k,d}$ corresponds to $(Z_d \wr_* S_n^+)\tilde{\times} Z_k$. $Z_d \wr_* S_n^+$ is also denoted $H_n^{d,+}$ and has been introduced by Banica and Vergnioux in [16].*

3. *the category $\mathcal{H}^{\#}$ corresponds to $H_n^{+\#} = H_n^+ \tilde{*} Z_2$.*

4. *the category $\mathcal{B}^{k,d,*}$ (resp $\mathcal{B}^{k,d,0}$), corresponds to $(C_n^+ \tilde{*} Z_d) \tilde{\times} Z_k)$ (reps. $(B_n^+ \tilde{*} Z_d) \tilde{\times} Z_k)$).*

5. *the category $\mathcal{B}^{k,d,d/2}$ corresponds to $\tilde{C}_n^{+,d} \tilde{\times} Z_k$.*

6. *the category $\mathcal{S}^{k,d}$ corresponds to $(S_n^+ \tilde{*} Z_d) \tilde{\times} Z_k)$.*

As a corollary, we obtain also all the unitary easy quantum groups that are classical groups (see $Ch.3$, Th 4.61).

Weingarten calculus on free easy quantum groups We have seen in Chapter 2, Section 3 that for u_n the fundamental representation of U_n, the family $(\mathrm{Tr}(u_n^k)_{k\geq 1})$ converges in law toward a family of independent complex gaussian variables $(u_k)_{k\geq 1}$ such that u_k has variance k; since this result holds also for O_n and S_n with different limit distributions, it is expected that the result can be generalized to all easy quantum groups. In [14], Banica, Curran and Speicher proved that the same phenomenon holds for orthogonal free easy quantum groups, with limit distributions involving free semicircular and free Poisson distributions.
In Chapter 5, we will extend this result to all free easy quantum groups. The main tool is the Weingarten formula, which takes a simpler expression for easy quantum groups. In the second part of Chapter 5, we prove that the second-order freeness for the unitary group (see Chapter 2, Th.2.36) has a natural analog in the free case: namely the family of traces of arbitrary reduced products of $u, u^t, \bar{u}, u^*$ converges in distribution to a family of circular variables.

3.3 Noncommutative permutations and free wreath product

The free wreath product is an algebraic construction that generalizes the usual wreath product between permutation groups and compact groups. It has been introduced by Bichon in [21] as a way to encode the quantum symmetries of a finite product of graphs.

3.3.1 Free wreath product

In the classical case, a permutation group is a subgroup of S_n for some $n \geq 1$. The natural extension in the noncommutative case yields the following definition:

Definition 3.30. *A non-commutative permutation group $F = (C(F), (v_{ij})_{1\leq i,j\leq n})$ is a quantum subgroup of S_n^+. The non-commutative permutation group F is said irreducible if* $\dim \mathrm{Mor}_F(0,1) =$ 1.

From an algebraic point of view, this means that F is a compact matrix quantum group whose fundamental representation matrix v satisfies at least the following relations:

$$v_{ij}^* = v_{ij}, \sum_j v_{ij} = \sum_i v_{ij} = \mathbf{1}, v_{ij}v_{ik} = \delta_{jk}v_{ij},$$

for all $1 \leq i,j,k \leq n$. From a representation theoretic point of view, $\mathrm{Rep}(S_n^+)$ is a subcategory of $\mathrm{Rep}(F)$, and therefore all the maps T_p with $p \in NC^{\circ\bullet}$ are also intertwiners of F.
The free wreath product is the generalisation of the construction Chapter 2, Section 6.2.

Definition 3.31 (Bichon)**.** *([21, Definition 2.2]) Let $G = (C(G), \Delta)$ be a compact quantum group and $F = (C(F), (v_{ij})_{1\leq i,j\leq n})$ be a non-commutative permutation group. Let $\nu_i : C(G) \to C(G)^{*n}$ be the canonical inclusion of $C(G)$ as the $i-th$ copie in the free product $C(G)^{*n}$, $i =$*

$1, \ldots, n$.
The free wreath product of G by F is the quotient of the C^*-algebra $C(G)^{*N} * C(F)$ by the two-sided ideal generated by the elements

$$\nu_k(a)v_{ki} - v_{ki}\nu_k(a), \quad 1 \leq i, k \leq n, \quad a \in C(G).$$

It is denoted by $C(G) *_w C(F)$.

It has been proved in [21] that there exists a coproduct Δ on $C(G) *_w C(F)$ such that $(C(G) *_w C(F), \Delta)$ is a compact quantum group. This coproduct is defined as

$$\Delta(v_{ij}) = \sum_{k=1}^{n} v_{ik} \otimes v_{kj}, \forall i, j \in \{1, \ldots, n\}$$

and

$$\Delta(\nu_i(a)) = \sum_{k=1}^{n} \nu_i(a_{(1)})v_{ik} \otimes \nu_k(a_{(2)}),$$

where $\Delta_G(a) = \sum a^{(1)} \otimes a^{(2)}$ is the value of the coproduct Δ_G on a with the Seedler notations. We denote by $G \wr_* F$ the quantum group $(C(G) *_w C(F), \Delta)$.
Suppose that G is a compact matrix quantum group with fundamental representation $(u_{kl})_{1\leq k,l\leq m}$. In this case $G \wr_* F$ is also a compact matrix quantum group with a fundamental representation $(w_{ij,kl}))_{\substack{1\leq i,j\leq n \\ 1\leq k,l\leq m}}$ defined by $w_{ij,kl} = v_{ij}u^i_{kl}$, where u^i_{kl} denotes the element $\nu_i(u_{kl})$.

Quantum symmetries of a graph The free wreath product construction is a natural construction when considering quantum symmetries of graphs. Let $\mathcal{G}$ be a finite graph with n vertices $\{1, \ldots, n\}$ and adjacency matrix $d_{\mathcal{G}} = (d_{ij})_{1\leq i,j\leq n}$. We suppose that $\mathcal{G}$ doesn't have any loop. A symmetry of $\mathcal{G}$ is a permutation σ of $\{v_1, \ldots, v_n\}$ such that $d_{\sigma(i)\sigma(j)} = d_{ij}$ for all $1 \leq i, j \leq n$. The set of symmetries of $\mathcal{G}$ forms a subgroup of S_n called the symmetry group of $\mathcal{G}$ and denoted by $S(\mathcal{G})$. From a dual point of view, $C(S(\mathcal{G})) = C(S_n)/\langle v d_{\mathcal{G}} = d_{\mathcal{G}} v\rangle$, where v is the fundamental matrix $(v_{ij})_{1\leq i,j\leq n}$ of S_n defined in 2.6.2.

Definition 3.32 ([10])**.** *The quantum symmetry group of $\mathcal{G}$ is the matrix quantum subgroup $(A(\mathcal{G}), (v_{ij})_{1\leq i,j\leq n})$ of $(C(S_n^+, (u_{ij})_{1\leq i,j\leq n})$ defined as follows: $A(\mathcal{G})$ is the quotient of $C(S_n^+)$ by the relation $ud = du$ and v_{ij} is the image of u_{ij} in this quotient.*

There is a natural operation on graphs yielding free wreath products on the level of the quantum symmetry group.

Definition 3.33. *Let $\mathcal{F}, \mathcal{G}$ be two graphs without loop, with vertices indexed respectively by $[\![1; n]\!]$ and $[\![1, m]\!]$, and respective adjacency matrices c and d. The lexicographical product $\mathcal{G} \circ \mathcal{F}$ is the graph with vertices indexed by $[\![1, m]\!] \times [\![1, n]\!]$, and adjacendy matrix*

$$D_{ij,kl} = \begin{cases} d_{kl} & \text{if } i = j \\ c_{ij} & \text{if } i \neq j \end{cases}$$

Figure 3.2 is an example of such construction, with the lexicographical product of a segment with a square.
By a result of [27], $A(\mathcal{G}) \wr_* A(\mathcal{F}) \subseteq A(\mathcal{G} \circ \mathcal{F})$, with equality if and only if $(\mathcal{G}, \mathcal{F})$ respects the conditions of Sabidussi (see [73] for a description of this conditions). Note that the same result holds also in the classical case : namely $S(\mathcal{G}) \wr S(\mathcal{F}) \subseteq S(\mathcal{G} \circ \mathcal{F})$, with equality if and only if $(\mathcal{G}, \mathcal{F})$ respects the conditions of Sabidussi.

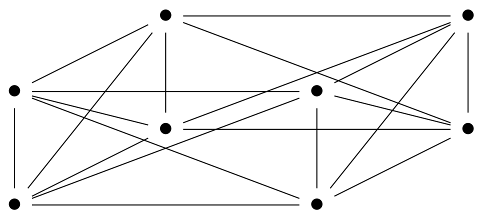


Figure 3.2: Lexicographical product $\mathcal{G} \circ \mathcal{F}$, with $\mathcal{G}$ a segment and $\mathcal{F}$ a square.

Free product formulae An important invariant of a quantum permutation group F is the law of the character χ_F of the fundamental representation $(v_{ij})_{1\leq i,j\leq n}$ under the Haar measure. Since F is a quantum permutation group, F is a subgroup of O_n^+ and therefore χ is a self-adjoint element. Thus, although $C(F)$ is a non-commutative C^*-algebra, χ is a well-defined random variable on $\mathbb{R}$, whose associated measure is denoted by $\mu(F)$. From the representation theory of F, the $k-$th moment c_k of F is exactly $\dim(\mathrm{Fix}_F(k))$, where $\mathrm{Fix}_F(k)$ denotes the vector space of invariant vectors of the $k-$th tensor power representation $v^{\otimes k}$.
By considering several examples of lexicographical products of graph and extended versions of this constructions, Banica and Bichon have been lead to the following conjecture in [10]:

Conjecture 3.34. *Let $\mathcal{F}$ and $\mathcal{G}$ be two graphs such that $S(\mathcal{F})$ and $S(\mathcal{G})$ are respectively transitive on $\mathcal{F}$ and $\mathcal{G}$. Then*
$$\mu(A(\mathcal{G}) \wr_* A(\mathcal{F}) = \mu(A(\mathcal{G})) \boxtimes \mu(A(\mathcal{F})).$$

Recall that $\boxtimes$ is the free multiplicative convolution as defined in Section 1.2.1. The condition that $S(\mathcal{F})$ is transitive on $\mathcal{F}$ means that for all $i, j \in \mathcal{F}$, there exists $\sigma \in S(\mathcal{F})$ such that $\sigma(i) = j$. Equivalently this means that $\dim(\mathrm{Fix}_{S(\mathcal{F})}(1)) = \dim(\mathrm{Fix}_{A(\mathcal{F})}(1)) = 1$.
This conjecture comes from the theory of planar algebras and a similar result exists by taking a free product of certain planar algebras.

3.3.2 Planar algebra

A planar algebra is a collection of vector spaces which is stable under a set of transformations indexed by planar diagrams. They have been introduced by Jones in [46] in order to give a diagrammatic approach to the study of subfactors of finite factors.

Planar tangles A planar tangle P of degree $k \geq 0$ consists of the following objects:

- A disk D_0 of $\mathbb{R}^2$, called the outer disk.
- Some disjoint disks $D_1, \ldots, D_n$ in the interior of D_0 which are called the inner disks.
- For each $0 \leq i \leq n$, a finite subset $S_i \in \partial D_i$ of cardinal $2k_i$ (such that $k_0 = k$) with a particular element $i_* \in S_i$. The elements of S_i are called the distinguished points of D_i and numbered counterclockwise starting from i_*. k_i is called the degree of the inner disk D_i.
- A finite set of disjoint smooth curves $\{\gamma_j\}_{1\leq j\leq r}$ such that each $\mathring{\gamma}_j$ lies in the interior of $D_0 \setminus \bigcup_{i\geq 1} D_i$ and such that $\bigcup_{1\leq j\leq r} \partial\gamma_j = \bigcup_{0\leq i\leq n} S_i$; it is also required that each curve meets a disk boundary orthogonally, and that its endpoints have opposite (resp. same) parity if they both belong to inner disks or both belong to the outer disk (resp. one belongs to an inner disk and the other one to the outer disk).

- A region of P is a connected component of $D_0 \setminus (\bigcup_{i\geq 1} D_i \cup (\bigcup \gamma_j))$. Give a chessboard shading on the regions of P in such a way that the interval components of type $(2i+1, 2i)$ are boundaries of shaded regions.

In the above description, closed curves are allowed among the set $\{\gamma_i\}$. On each disk, the labelling of the distinguished points is fixed by the choice of a particular distinguished point i_*: this choice is pictorially represented by adding a mark $*$ on the interval component directly preceeding this point. An example of planar tangle is given in Figure 3.3.

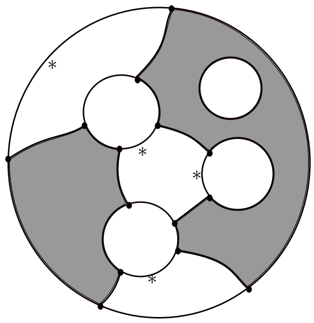


Figure 3.3: Planar tangle of degree 4

Planar tangles can be composed in the following way: suppose that T_1 and T_2 are tangles of respective degree k and k', and that T_1 has an interior disk D of degree k'. Plugging T_2 inside D in such way that the marked interval of the exterior disk of T_2 coincides with the marked interval of D, and then erasing the boundary of the exterior disk of T_2 (except the distinguished points of the exterior disk of T_2, which become usual points in the resulting picture) yields a new planar tangle $T_1 \circ_D T_2$. An example of such a gluing is given in Figure 3.4.

Planar algebra A planar algebra is a family of finite-dimensional vector spaces $\{V_k\}_{k\in\mathbb{Z}_{>0}\cup\{+,-\}}$ together with an action of the planar tangles. Namely, each planar tangle T of degree k yields a linear maps $Z(T) : \bigotimes_{\substack{D \text{ internal} \\ \text{disk of } T}} V_{k_D} \to V_k$ with the compatibility condition

$$Z(T)(v_{D_1} \otimes \cdots \otimes Z(S)(v_{D_j}) \otimes \cdots \otimes v_{D_r}) = Z(T \circ_{D_j} S)(v_{D_1} \otimes \cdots \otimes v_{D_j} \otimes \cdots \otimes v_{D_r}),$$

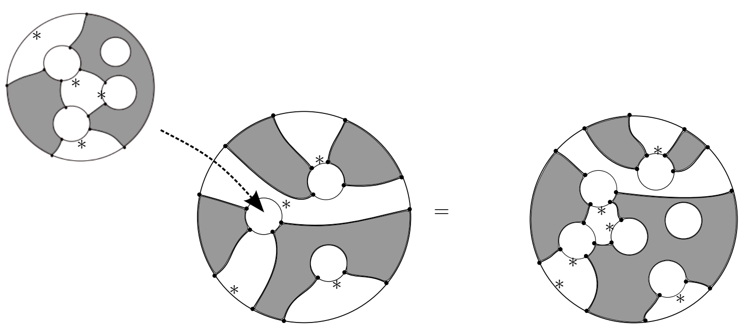


Figure 3.4: Composition of planar tangles

for any planar tangles T, S such that the degree of S is the degree of the internal disc D_j of T and for all vectors $v_{D_i} \in V_{k_i}$ for $i \neq j$ and $v_{D_j} \in \bigotimes_{\substack{D \text{ internal} \\ \text{disk of } S}} V_{k_D}$.
The planar algebra structure yields several natural operations on $\{V_k\}$. In particular each vector space V_k becomes an algebra with the action of the planar tangle of Figure 3.5.

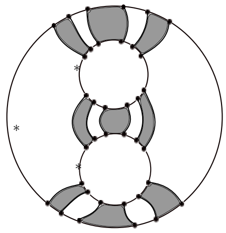


Figure 3.5: : Multiplication tangle of degree $\mathcal{P}_6$.

Under certain assumptions, each vector space V_k is a semi-simple $*-$algebra and $\bigcup V_k$ is a tower of algebras with a common trace: such planar algebra is called a subfactor planar algebra (see [46]). One of the major results on planar algebras relates the dimension of each vector space V_k of a subfactor planar algebra with the cardinality of some paths on a graph:

Theorem (Jones,[46])**.** *Let $\mathcal{P}$ be a subfactor planar algebra. There exists a bipartite graph $G_{\mathcal{P}}$ with root vertex $*$ such that:*

$$\dim \mathcal{P}_k = \#\{ \textit{ walk of length } 2k \textit{ on } G_{\mathcal{P}} \textit{ starting and ending at } * \}.$$

The random variable μ_P whose $k-$th moment is $\dim \mathcal{P}_k$ is called the spectral measure of the planar algebra $\mathcal{P}$.

3.3.3 Overview of the results

The main motivation is the proof of Conjecture 3.34. Aiming this proof, we obtained several results on the intertwiner spaces of a free wreath product.

Free wreath product with S_n^+ (joint work with François Lemeux) In Chapter 6, we study the free wreath product of a compact quantum group with the free symmetric group S_n^+. In particular we describe completely the intertwiner spaces in this case, which yields some useful expressions for the Weingarten matrix of a free wreath product with S_n^+. As a byproduct of these results, an asymptotic formula has been found for the law of the characters of $\mathbb{G} \wr_* S_n^+$, where $\mathbb{G}$ is a fixed compact matrix quantum group and n goes to $+\infty$.
Let $t \in (0,1]$ and let $\mathbb{G}$ be a matrix compact quantum group of Kac type and dimension r. Denote by $\chi_{\mathbb{G}}$ the law of the character of its fundamental representation. Let $(\mathbb{G}\wr_* S_n^+, (w_{ij,kl})_{1\leq i,j\leq r, 1\leq k,l\leq n})$ be the matrix quantum group $\mathbb{G} \wr_* S_n^+$ with its fundamental representation w. For $1 \leq s \leq n$, denote by $\chi_w(s)$ the truncated character $\chi_w(s) = \sum_{\substack{1\leq i\leq r \\ 1\leq k\leq s}} w_{ii,kk}$.

Theorem 3.35. *With respect to the Haar measure, if $s \sim tn$ for n going to infinity,*

$$\chi_{\mathbb{G}\wr_* S_n^+}(s) \to \mathcal{P}_t(\chi_{\mathbb{G}}),$$

where $\mathcal{P}_t(\chi_{\mathbb{G}})$ is the free compound Poisson with parameter t and original law $\chi_{\mathbb{G}}$.

Free wreath product and planar algebras (joint work with Jonas Wahl) In Chapter 7, we establish a link between the free wreath product $G \wr_* F$ of two irreducible non-commutative permutation groups and the free product of two planar algebras. The free product of two planar algebras is a construction which has been done by Bisch and Jones in [23], in order to deal with chains of inclusions of subfactors. In [6], Banica proved that the intertwiner spaces of any irreducible non-commutative permutation group F is a planar algebra $\mathcal{P}(V)$.
Using the study of the case $G \wr_* S_n^+$, we prove that the intertwiner spaces of a free wreath product $G \wr_* F$ is the free product of the planar algebras $\mathcal{P}(F)$ and $\mathcal{P}(G)$. Using the results of [23], we deduce a positive (and more general) answer to Conjecture 3.34:

Theorem 3.36 ([10], Conj 3.1)**.** *Let F and G be two non-commutative permutation groups such that $\dim_F \operatorname{Mor}(0,1) = \dim_G \operatorname{Mor}(0,1) = 1$. Then*

$$\mu(F \wr_* G) = \mu(F) \boxtimes \mu(G),$$

where $\mu(\mathbb{G})$ denotes the law (with respect to the Haar measure) of the character of the fundamental representation of a matrix compact quantum group $\mathbb{G}$.

Using a result of Landau from [54] and the link between Boolean and free independence (see [17]), we also give a combinatorial proof to the fact that the spectral measure of a free product of irreducible planar algebras is the free multiplicative convolution of the spectral measures of the initial planar algebras (a result which is already proven in [23]).

3.4 Free fusion rings and non-commutative symmetric functions

We have seen in Chapter 2 that symmetric functions play an important role in the representation theory of classical easy groups : they encode the fusion rules of U_n and O_n by expressing the characters in terms of the eigenvalues of the matrices (Chapter 2, Section 3), and they describe the induction and restriction operations on the irreducible respresentations of S_n (see Chapter 2, Section 6.2).
For free easy guantum groups, the analog of the symmetric functions has not been found yet. On the other hand, a non-commutative analog of the ring of symmetric functions called $NSym$ has been introduced in [41]. It appears that the fusion rules of free easy quantum groups share a common pattern which the multiplicative structure of the ring $NSym$ in a particular basis.

3.4.1 Free fusion ring

The notion of free fusion ring has been introduced by Banica and Vergnioux in [16] to describe the fusion rules of free hyperoctahedral groups.
Let R be a set with an involution $r \mapsto \bar{r}$ and a product $* : R \times R \to R \cup \{\emptyset\}$. The involution and the product are extended to the set $F(R)$ of words in R with the formulae

$$\overline{r_1 \dots r_k} = \overline{r_k} \dots \overline{r_1},$$

and

$$(r_1 \dots r_k) * (s_1 \dots s_l) = r_1 \dots r_{k-1}(r_k * s_l) s_2 \dots s_l,$$

with the conventrion that $r_k \star s_1 = \emptyset$ implies that $(r_1 \dots r_k) * (s_1 \dots s_l) = \emptyset$.

Definition 3.37. *For $(R, *, -)$ has above, the free fusion ring with underlying set $(R, *, -)$ is the algebra whose basis is the words in R with the product*

$$x \otimes y = \sum_{x = vz, y = \bar{z}w} vw + v * w. \tag{3.4.1}$$

This algebra is denoted $F(R, *, -)$.

Most of the free easy quantum groups have fusion rules following a free fusion ring:

- For U_n^+, the ring of characters is a free fusion ring with $R = \mathbb{Z}_2$, $\bar{r} = 1 - r$ and $r * s = \emptyset$.
- For O_n^+, the ring of characters is a free fusion ring with $R = \{1\}$, $\bar{1} = 1$ and $1 * 1 = \emptyset$.
- For $H_n^{s,+}$, the ring of characters is a free fusion ring with $R = \mathbb{Z}_s$, $\bar{r} = -r$ and $r * s = r + s$.

Freslon proved in [38] that the fusion ring of any free easy quantum group whose category of partitions is stable by removing block is a free fusion ring.
An alternative formula can be used to describe the multiplication in a free fusion ring. Indeed (3.4.1) is equivalent to the recurrence formula

$$\begin{aligned} r_1 \dots r_n \otimes s_1 \dots s_m = r_1 \dots r_n s_1 \dots s_m + r_1 \dots r_{n-1}(r_n * s_1) s_2 \dots s_m \\ + \delta_{s_1 = \overline{r_n}} r_1 \dots r_{n-1} \otimes s_2 \dots s_m. \end{aligned}$$

The latter formula will be generalized in the last section of Chapter 6 to describe the fusion rules of general free wreath products with S_n^+. With this formula, Lemeux proved in [55] that a free fusion ring is isomorphic to the free ring $\mathbb{Z}\langle R \rangle$.

3.4.2 Noncommutative symmetric functions

The content of this subsection comes mainly from [84].

The Hopf algebra of symmetric functions In Chapter 2, the ring of symmetric functions Λ has been defined as the algebra of symmetric polynomials in an infinite number of commuting variables $(X_1, \dots, X_n, \dots)$. The fundamental theorem of symmetric polynomials yields that Λ is isomorphic, as a graded algebra, to the graded algebra of polynomials $\mathbb{Z}[e_1, \dots, e_n, \dots]$ with $\deg(e_i) = i$.
It turns out that Λ has the structure of a Hopf algebra. Indeed let $(Y_1, \dots, Y_n, \dots)$ be another infinite family of variables and let $P \in \Lambda$. Then $P(\{X_i\}, \{Y_i\})$ has a decomposition

$$P(\{X_i\}, \{Y_i) = \sum_{i=1}^{r} P_i(\{X_i\}) \tilde{P}_i(\{Y_i\}),$$

for some $r \geq 1$ and $P_i, \tilde{P}_i \in \Lambda$. The formula $\Delta(P) = \sum P_i \otimes \tilde{P}_i$ defines a coproduct Δ on Λ. Since $(PQ)(\{X_i\}, \{Y_i\}) = P(\{X_i\}, \{Y_i\}) Q(\{X_i\}, \{Y_i\})$, this coproduct is an algebra morphism, turning Λ into an Hopf algebra. The counit ε of Λ is given by $\varepsilon(e_i) = 0, \varepsilon(1) = 1$ and the antipode is given by the involutive automorphism $\omega(e_i) = (-1)^i h_i$.
The coproduct takes simple expressions on the basis of elementary functions and power sums, as we have

$$\Delta(e_n) = \sum_{i=0}^{n} e_i \otimes e_{n-i}, \Delta(p_n) = 1 \otimes p_n + p_n \otimes 1.$$

The Hall inner product defined in Chapter 2, Section 2 plays a particular role in this framework, since Λ is a self-dual Hopf algebra with respect to this inner product. Namely

$$\langle PQ, R \rangle = \langle P \otimes Q, \Delta R \rangle, \tag{3.4.2}$$

for any $P, Q, R \in \Lambda$, and where $\langle ., . \rangle$ is canonically extended to the algebraic tensor product $\Lambda \otimes \Lambda$.

Noncommutative symmetric function The ring of non-commutative symmetric functions **NSym** is the non-commutative version of Λ obtained by removing the commutation relation on the e_i's.

Definition 3.38. **NSym** *is the graded free ring* $\mathbb{Z}\langle S_1, \ldots, S_n, \ldots\rangle$ *with the grading* $\deg(S_i) = i$ *and the comultiplication* $\Delta(S_n) = \sum S_i \otimes S_{n-i}$.

It is still possible to define an involution ω in such a way that **NSym** is a Hopf-algebra. Since the elements S_i don't commute anymore, the basis of **NSym** is not indexed by integers partitions but by sequences of integers: a sequence I of integers $(i_1, \ldots, i_r)$ such that $\sum i_j = n$ is called a composition of n of length r and denoted by $I \vdash n$. The set of all composition of n is denoted by $Comp(n)$. For example $\{S_I\}_{I\vdash n}$ is a basis of the degree n subspace of **NSym**. The multiplication structure on S_I is given by the simple expression $S_I S_J = S_{I.J}$, where $I.J$ is the concatenation of the sequences I and J.
An other important basis is given by the ribbon Schur functions $\{R_I\}$. Define a partial order $\leq$ on $Comp(n)$ by the relation $I \leq J$ if and only if $J = (j_1, \ldots, j_r)$ and $I = (j_1 + \cdots + j_{s_1}, j_{s_1+1} + \cdots + j_{s_1+s_2}, \ldots, j_{r-s_t} + \cdots + j_r)$, where $s \vdash r$. The Moebius function on $(Comp(n), \leq)$ is the function $\mu(J, I) = \delta_{J\leq I}(-1)^{l(I)-l(J)}$, where $l(I)$ is the length of the composition I. The ribbon Schur function R_I is defined in the same way as the free cumulant in Chapter 1:

$$R_I = \sum_{J\leq I} \mu(J, I) S_J.$$

Equivalently $S_I = \sum_{J\leq I} R_J$. The multiplication in the basis R_I is given by the formula $R_I R_J = R_{I.J} + R_{I\triangleright J}$, where $(i_1, \ldots, i_r) \triangleright (j_1, \ldots, j_s) = (i_1, \ldots, i_{r-1}, i_r + j_1, j_2, \ldots, j_s)$. Note in particular that this formula yields that **NSym** is a free fusion ring without involution.
The commutative quotient of **NSym** is isomorphic to Λ through the map $S_i \mapsto e_i$. By this map, R_I is mapped to the ribbon Schur function r_I, which is a particular skew Schur function (see [60], Part I, Ch.5).

Quasi-symmetric functions Unlike the Hopf-algebra of symmetric functions, **NSym** is not self-dual and therefore there is no inner product such that the relation 3.4.2 holds. However it is still possible to construct the dual Hopf-algebra of **NSym**. For each composition $I = (i_1, \ldots, i_r)$ define the polynomial $M_I(\{X_i\})$ in the commuting variables $\{X_i\}_{i\geq 1}$ as

$$M_I(X) = \sum_{j_1<j_2<\cdots<j_r} x_{j_1}^{i_1} x_{j_2}^{i_2} \ldots x_{j_r}^{i_r}.$$

Definition 3.39. *The ring generated by the polynomials* M_I *over* $\mathbb{Z}$ *is called the ring of quasi-symmetric functions. This ring is denoted by* $QSym$.

The product of M_J and M_K has a decomposition in $\{M_I\}$ with integers coefficients and thus $\{M_I\}$ is a basis of $QSym$. As for the ring of symmetric functions, a coproduct γ is defined on $QSym$ by decomposing a quasi-symmetric function in the variables $\{X_i\} \cup \{Y_i\}$ into a product of quasi-symmetric functions respectively in the variables $\{X_i\}$ and in $\{Y_i\}$. This coproduct turns $QSym$ into a Hopf algebra which is the dual Hopf algebra of **NSym**. With the pairing $\langle,\rangle : QSym \times \mathbf{NSym} \to \mathbb{R}$ defined by the formula $\langle M_I, S_J\rangle = \delta_{IJ}$, the following equalities hold:

$$\langle f \otimes g, \Delta(P)\rangle = \langle fg, P\rangle,$$

$$\langle \gamma(f), P \otimes Q\rangle = \langle f, PQ\rangle,$$

for all $f, g \in QSym$ and $P, Q \in \mathbf{NSym}$. The definition of the basis $\{R_I\}$ of $\mathbf{NSym}$ in terms of $\{S_I\}$ yields that the dual basis of $\{R_I\}$ is the set of quasi-symmetric function F_I with

$$F_I = \sum_{J \geq I} M_J.$$

The basis $\{F_I\}$ is called the basis of fundamental quasi-symmetric functions. Each F_I has a decomposition in terms of monomials similar to the decomposition of classical Schur functions with the semi-standard filling of Young diagram. In the case of F_I, the decomposition is described by the semi-standard filling of ribbon Young diagram :

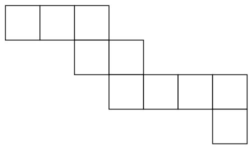


Figure 3.6: Skew Young tableau associated to the composition $I = (3, 2, 4, 1)$.

3.4.3 Overview of the results

The results obtained in Part IV are somehow independent of Part II and Part III, apart from the relation with free fusion rings.

NSym as a probability space At the end of Chapter 6, the ring $\mathbf{NSym}$ is embedded in the ring of characters of the free wreath product $H_n^{+,\infty}$. Through this embedding the ribbon Schur functions are characters of certain irreducible representations of $H_n^{+,\infty}$. Therefore the Haar measure on $H_n^{+,\infty}$ turns the ribbon Schur functions into noncommutative random variables exactly as in the case of U_n with the ring of symmetric functions. The law of the ribbon Schur functions $\{R_{(n)}\}_{n\geq 1}$ can be described thanks to the fusion rules of $H_n^{+,\infty}$ found by Banica and Vergnioux in [16]. This leads to the following result :

Theorem 3.40. *There exists a injective algebra morphism $\Phi : \mathbf{NSym} \to Cl(H_n^{+,\infty})$. For any ribbon Schur function R_I, $\Phi(R_I)$ is an irreducible character.*
The family of random variables $(\Phi(R_{(n)}))_{n\geq 1}$ with respect to the Haar measure is distributed as $(sz^n s)_{n\geq 1}$, where s is a semi-circular variable of variance 1 and z is a uniform measure on the the unit circle and free from s.

The Martin boundary of Zigzag diagrams Since $\{F_I\}$ is a basis of $QSym$, for any composition J the product $F_J F_{(1)}$ has a decomposition in the basis $\{F_I\}$. A combinatorial argument shows that the coefficients in this decomposition are 0 or 1.
Therefore we can construct a graph whose vertices are the compositions, and such that there is an edge between I and J if and only if the coefficient of F_J in the product $F_I F_{(1)}$ is equal to 1. This graph is called the graph of Zigzag diagrams, and denoted by $\mathcal{Z}$. It has been deeply studied by Olshanski and Gnedin, who have identified in [42] the minimal boundary of the graph (see Chapter 9 for a detailed exposition of the different boundaries of a graph): this boundary is a measured space which encodes the behavior of directed random walks on the graph.
An analogous graph exists by considering the Schur basis of Sym. This graph, denoted by $\mathcal{Y}$, has vertices indexed by Young diagrams and edges between s_λ and s_μ if and only if the coefficient of s_μ in $s_{(1)}s_\lambda$ is non-zero. This graph has played an important role in the study of certain irreducible representations of S_∞ (see [86]), the group of permutations of $\mathbb{N}$ with finite

support. Through its study it has been shown in [86] that the Poisson boundary of $\mathcal{Y}$ coincides with its Martin boundary. The Martin boundary is a geometrical boundary which comes from a compactification of the graph $\mathcal{Y}$.
In Chapter 9 we prove an analogous result for $\mathcal{Z}$:

Theorem 3.41. *The Martin boundary of $\mathcal{Z}$ coincides with its Poisson boundary.*

Note that in general the Martin boundary is a subset of the Poisson boundary. The result of Theorem 3.41 had been conjectured by Olshanski and Gnedin in [42]. In order to prove this fact, we obtain some estimates on the filling of large ribbon Young diagrams in Chapter 8.
Finally in Chapter 9, Section 7, we establish a link between paths on $\mathcal{Y}$ and paths on $\mathcal{Z}$, and we give a central limit theorem for the shape of the descent pattern of a large permutation.

Part II

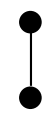
Unitary easy quantum groups

Chapter 4

Classification of categories of non-crossing colored partitions

We have seen in Chapter 3 that categories of two-colored partitions yield matrix compact quantum groups whose intertwiners have a simple description : the use of the Weingarten formula is thus greatly simplified on these particular quantum groups. It is therefore interesting to find all the possible categories of two colored partitions. This task seems very difficult in the general case; however, if we assume that the partitions have to be non-crossing, the situation is much easier.
This chapter is devoted to the classification of all categories of non-crossing two-colored partitions and to the construction of their associated unitary easy quantum group. The full classification is given by Theorem 4.41 and 4.42 in Section 7. The list of associated easy quantum group is given in Theorem 4.57. As a corollary, we also obtain in Section 9 the classification of all classical easy groups.

4.1 Categories of two-colored partitions and first results

Recall that from Chapter 1, a category of partitions is a collection $\mathcal{C}$ of subsets $\mathcal{C}(k,l) \subseteq P^{\circ\bullet}(k,l)$ (for all $k,l \in \mathbb{N}_0$) is a *category of partitions*, if it is closed under the tensor product, the composition and the involution, and if it contains the bicolored pair partitions  and  as well as the identity partitions  and .

If $\mathcal{C}$ is the smallest category of partitions containing the partitions $p_1, \dots, p_n$, we write $\mathcal{C} = \langle p_1, \dots, p_n \rangle$ and say that $\mathcal{C}$ is *generated* by $p_1, \dots, p_n$.

Lemma 4.1. *Let $\mathcal{C} \subseteq P^{\circ\bullet}$ be a category of partitions.*

(a) $\mathcal{C}$ is closed under rotation and verticolor reflection.

(b) If $p \in P^{\circ\bullet}(k,l)$ is a partition in $\mathcal{C}$, we can erase *two neighbouring points of p if they have different (!) colors, i.e. if the j-th and the $(j+1)$-th of the lower points have inverse colors, then the partition $p' \in P^{\circ\bullet}(k,l-2)$ is in $\mathcal{C}$ which is obtained from p by first connecting the blocks to which the j-th and the $(j+1)$-th lower points belong respectively, and then erasing these two points. We may also erase neighbouring points of inverse colors on the upper line.*

(c) Let $p_1 \otimes p_2 \in \mathcal{C}$. Then $p_1 \otimes \tilde{p}_1 \in \mathcal{C}$ and $p_2 \otimes \tilde{p}_2 \in \mathcal{C}$. Note that we do not have $p_1 \in \mathcal{C}$ or $p_2 \in \mathcal{C}$ in general.

Proof. (a) Firstly, let $p \in P^{\circ\bullet}(k,l)$ be a partition and let the first of the k upper points be black. Let $r \in P^{\circ\bullet}(k-1,k-1)$ be the tensor product of the identity partitions and with the same color pattern as the latter $k-1$ upper points of p. The composition ($\otimes\, p$)($\otimes\, r$) yields a partition $p' \in P^{\circ\bullet}(k-1,l+1)$ which conincides with the partition obtained from p when rotating the first upper points to the row of lower points. See also [15, Lem 2.7]. If now $p \in \mathcal{C}$, then also $p' \in \mathcal{C}$ since all partitions we used are in the category. Similarly we prove the other cases of rotation.

Secondly, if $p \in P^{\circ\bullet}(k,l)$ is in $\mathcal{C}$, then also $p^* \in P^{\circ\bullet}(l,k)$ is in $\mathcal{C}$ by the definition of a category. Rotating the k upper points to below and the lower points to the upper line yields $\tilde{p}$, which is in $\mathcal{C}$.

(b) Compose p with $r_1 \otimes$ $\otimes\, r_2$ or $r_1 \otimes$ $\otimes\, r_2$, where r_1 and r_2 are suitable tensor products of the identity partitions and .

(c) By (a), we have $\tilde{p}_2 \otimes \tilde{p}_1 \in \mathcal{C}$ and thus $p_1 \otimes p_2 \otimes \tilde{p}_2 \otimes \tilde{p}_1 \in \mathcal{C}$. Using (b), we infer $p_1 \otimes \tilde{p}_1 \in \mathcal{C}$ and likewise $p_2 \otimes \tilde{p}_2 \in \mathcal{C}$ (using rotation). □

Tensor product, composition, involution, and the operations of the preceding lemma are called the *category operations.*

4.1.1 Special operations on partitions

The category operations may be performed in any category of partitions. Other procedures are allowed if and only if certain key partitions are contained in the category.

Lemma 4.2. *Let $\mathcal{C}$ be a category of partitions and let $p \in P^{\circ\bullet}(0,l)$ be a partition without upper points.*

(a) If $\otimes$ $\in \mathcal{C}$, then $\mathcal{C}$ is closed under permutation of colors, i.e. if $p \in \mathcal{C}$, then $p' \in \mathcal{C}$, where p' is obtained from p by some permutation of the colors of the points (without changing the strings connecting the points).

(b) If $\otimes$ $\in \mathcal{C}$, then $\mathcal{C}$ is closed under permutation of colors of neighbouring *singletons. Furthermore, we may disconnect any point from a block and turn it into a singleton.*

(c) If $\in \mathcal{C}$, then $\mathcal{C}$ is closed under permutation of colors of neighbouring *points belonging to the same block. Furthermore, we may connect neighbouring blocks.*

(d) If $\in \mathcal{C}$, we have no permutation of colors in general, and we may only connect neighbouring blocks if they meet at two points with inverse colors.

(e) If $\in \mathcal{C}$, then $\mathcal{C}$ is closed under arbitrary positioning of singletons, i.e. if $p \in \mathcal{C}$, then $p' \in \mathcal{C}$, where p' is obtained from p by shifting blocks of length one to other positions.

(f) If $\in \mathcal{C}$, then we may swap a singleton with a neighbour point of inverse color. This procedure inverts both colors. In other words, if $p = XabY \in \mathcal{C}$ where b is a singleton and a is a point of color inverse to b, then $p' = Xb^{-1}a^{-1}Y \in \mathcal{C}$.

Proof. (a) By rotation, the partitions $\otimes$ and $\otimes$ are in $\mathcal{C}$. Note that the partitions and themselves are *not* necessarily contained in $\mathcal{C}$. Composing p with partitions $r_1 \otimes$ $\otimes$ $\otimes\, r_2$

where the r_i are given by suitable tensor products of the identity partitions $\overset{\circ}{\underset{\circ}{|}}$ and $\overset{\bullet}{\underset{\bullet}{|}}$ yields a transposition of the colors of the points of p.

(b) Using rotation, we infer $\underset{\circ}{\uparrow} \otimes \underset{\circ}{\uparrow} \otimes \underset{\bullet}{\uparrow} \otimes \underset{\bullet}{\uparrow} \in \mathcal{C}$. Analoguous to (a), we see that we may permute the colors of neighbouring singletons. Furthermore, rotations of $\underset{\circ}{\uparrow} \otimes \underset{\bullet}{\uparrow}$ yield partitions $\overset{\circ}{\underset{\circ}{\downarrow\uparrow}}$ and $\overset{\bullet}{\underset{\bullet}{\downarrow\uparrow}}$ in $P^{\circ\bullet}(1,1)$ consisting of two disconnected points of the same color. Composing a partition with suitable tensor products of the identity partitions $\overset{\circ}{\underset{\circ}{|}}$, $\overset{\bullet}{\underset{\bullet}{|}}$ and the partitions $\overset{\circ}{\underset{\circ}{\downarrow\uparrow}}$, $\overset{\bullet}{\underset{\bullet}{\downarrow\uparrow}}$ yields a partition where some points are disconnected from their blocks (without changing the color).

(c) Again, similar to (a), we infer that $\mathcal{C}$ is closed under permutation of colors of neighbouring points belonging to the same block, using $\underset{\circ\circ\bullet\bullet}{\sqcap\!\sqcap\!\sqcap} \in \mathcal{C}$. Since we then also have $\underset{\circ\bullet\circ\bullet}{\sqcap\!\sqcap\!\sqcap} \in \mathcal{C}$, the partitions $\overset{\circ\bullet}{\underset{\circ\bullet}{\sqcup\!\sqcap}}$, $\overset{\bullet\circ}{\underset{\bullet\circ}{\sqcup\!\sqcap}}$, $\overset{\circ\circ}{\underset{\circ\circ}{\sqcup\!\sqcap}}$ and $\overset{\bullet\bullet}{\underset{\bullet\bullet}{\sqcup\!\sqcap}}$ are all in $\mathcal{C}$ by rotation. Composing with them effects that some neighbouring blocks are connected.

(d) We argue as in (c), but we may only use $\overset{\circ\bullet}{\underset{\circ\bullet}{\sqcup\!\sqcap}}$ and $\overset{\bullet\circ}{\underset{\bullet\circ}{\sqcup\!\sqcap}}$.

(e) Check that $\underset{\circ}{\uparrow}\,\overset{\circ}{\underset{\circ}{|}}\,\overset{\circ}{\downarrow}$, $\underset{\bullet}{\uparrow}\,\overset{\bullet}{\underset{\bullet}{|}}\,\overset{\bullet}{\downarrow}$, $\underset{\circ}{\uparrow}\,\overset{\bullet}{\underset{\bullet}{|}}\,\overset{\circ}{\downarrow}$, $\underset{\bullet}{\uparrow}\,\overset{\circ}{\underset{\circ}{|}}\,\overset{\bullet}{\downarrow}$ etc are in $\mathcal{C}$ using rotation and verticolor reflection.

(f) Use $\underset{\circ}{\uparrow}\,\overset{\circ}{\underset{\bullet}{|}}\,\overset{\bullet}{\downarrow}$ or $\underset{\bullet}{\uparrow}\,\overset{\bullet}{\underset{\circ}{|}}\,\overset{\circ}{\downarrow}$ etc. $\square$

We formulated the above lemma only for partitions having no upper points, but the statements may be extended to arbitrary partitions $p \in P^{\circ\bullet}(k,l)$. We then have to take into account that the colors are inverted whenever they are rotated from the upper line to the lower line or the converse.

4.1.2 The non- (or one-) colored case

Let us end this section with a comparison to the case of categories of non-colored partitions, which were studied in [15] and in other articles and which were completely classified in [71]. For the classification in the noncrossing case, see [15] and [97]. Recall that there are exactly seven categories, given by:

$$\begin{array}{ccccccc}
\langle \uparrow \rangle & \supseteq & \langle {\scriptstyle\uparrow}\sqcap{\scriptstyle\uparrow} \rangle & \supseteq & \langle \uparrow \otimes \uparrow \rangle & \supseteq & \langle \emptyset \rangle = NC_2 \\
\cup\!\!| & & \cup\!\!| & & & & \cup\!\!| \\
\langle \uparrow, \sqcap\!\sqcap\!\sqcap \rangle = NC & \supseteq & \langle \uparrow \otimes \uparrow, \sqcap\!\sqcap\!\sqcap \rangle & & \supseteq & & \langle \sqcap\!\sqcap\!\sqcap \rangle
\end{array}$$

By $P(k,l)$ we denote the set of *non-colored partitions* where all points have no color. Likewise we use the notations P for all non-colored partitions and NC for all non-colored noncrossing partitions. Categories of non-colored partitions are defined like categories of two-colored partitions when forgetting all colors, see for instance [15] or [71]. The key link between non-colored categories and two-colored categories is given by the partition $\underset{\circ\circ}{\sqcap}$ as may be seen in the next proposition. Note that $\overset{\circ}{\underset{\bullet}{|}}$ and $\overset{\bullet}{\underset{\circ}{|}}$ are rotated versions of $\underset{\circ\circ}{\sqcap}$. Composing a partition p with these partitions, we can change the colors of the points of p to every possible color pattern. Hence, categories containing $\underset{\circ\circ}{\sqcap}$ are non-colored categories, in this sense.

Let $\Psi : P^{\circ\bullet} \to P$ be the map given by forgetting the colors of a two-colored partition. For a set $\mathcal{C} \subseteq P$, we denote by $\Psi^{-1}(\mathcal{C}) \subseteq P^{\circ\bullet}$ its preimage under Ψ.

Proposition 4.3. *(a) Let $\mathcal{C} \subseteq P$ be a category of non-colored partitions. Then $\Psi^{-1}(\mathcal{C}) \subseteq P^{\circ\bullet}$ is a category of two-colored partitions containing the unicolored pair partition $\underset{\circ\,\circ}{\sqcap}$ (or equivalently $\underset{\bullet\,\bullet}{\sqcap}$).*

(b) Let $\mathcal{C} \subseteq P^{\circ\bullet}$ be a category of two-colored partitions containing the unicolored pair partition $\underset{\circ\,\circ}{\sqcap}$ (or equivalently $\underset{\bullet\,\bullet}{\sqcap}$). Then $\Psi(\mathcal{C}) \subseteq P$ is a category of non-colored partitions and $\Psi^{-1}(\Psi(\mathcal{C})) = \mathcal{C}$.

Hence, there is a one-to-one correspondence between categories of non-colored partitions and categories of two-colored partitions containing $\underset{\circ\,\circ}{\sqcap}$.

Proof. (a) It is easy to see from the definition that $\Psi^{-1}(\mathcal{C})$ is a category of partitions. Furthermore, $\underset{\circ\,\circ}{\sqcap} \in \Psi^{-1}(\mathcal{C})$ since $\Psi(\underset{\circ\,\circ}{\sqcap}) = \sqcap \in \mathcal{C}$.

(b) It is easy to see that $\Psi(\mathcal{C})$ is closed under tensor product and involution and that it contains the pair partition $\sqcap$ and the identity partition $|$. The composition is a bit more subtle. If $p, q \in \Psi(\mathcal{C})$, their composition is in $\Psi(\mathcal{C})$ only if we can lift p and q to partitions in $\mathcal{C}$ whose color patterns allow the composition in $P^{\circ\bullet}$. But since $\overset{\circ}{\underset{\bullet}{|}}$ and $\overset{\bullet}{\underset{\circ}{|}}$ are in $\mathcal{C}$ (by rotation), we can do so: If $p \in \Psi(\mathcal{C})$, there is a partition $p_0 \in \mathcal{C}$ such that $\Psi(p_0) = p$. Composing it wih $\overset{\circ}{\underset{\bullet}{|}}$ and $\overset{\bullet}{\underset{\circ}{|}}$, we may assume that all points of p_0 are white. Now, $\Psi(\mathcal{C})$ is closed under composition since $\mathcal{C}$ is. Similarly, we prove $\Psi^{-1}(\Psi(\mathcal{C})) \subseteq \mathcal{C}$ using $\overset{\circ}{\underset{\bullet}{|}}$ and $\overset{\bullet}{\underset{\circ}{|}}$. $\square$

4.2 Dividing the categories into cases

The classification of categories of noncrossing partitions is given by a detailed case study which we will now prepare.

4.2.1 The cases $\mathcal{O}$, $\mathcal{H}$, $\mathcal{S}$ and $\mathcal{B}$

The first division into cases is given by the sizes of blocks. The next lemma is formulated for arbitrary categories of partitions (not necessarily noncrossing ones).

Lemma 4.4. *Let $\mathcal{C} \subseteq P^{\circ\bullet}$ be a category of partitions.*

(a) If $\uparrow_{\circ} \otimes \uparrow_{\bullet} \notin \mathcal{C}$, then all blocks of partitions $p \in \mathcal{C}$ have length at least two.

(b) If $\underset{\circ\bullet\circ\bullet}{\sqcap\!\sqcap\!\sqcap} \notin \mathcal{C}$, then all blocks of partitions $p \in \mathcal{C}$ have length at most two.

Proof. (a) Let $p \in \mathcal{C}$ be a partition containing a block of size one. By rotation and possibly verticolor reflection, it is of the form $\uparrow_{\circ} \otimes q$, with no upper points. By Lemma 4.1(b) we have $\uparrow_{\circ} \otimes \uparrow_{\bullet} \in \mathcal{C}$.

(b) Let $p \in \mathcal{C}$ be a partition containing a block of size at least three. By rotation, it is of the form $p = a^{\varepsilon_1} X_1 a^{\varepsilon_2} X_2 a^{\varepsilon_3} X_3$ with no upper points, where the points a^{ε_i} belong to the same block, and $\varepsilon_i \in \{1, -1\}$ depending on the color. The subwords X_1, X_2 and X_3 are possibly connected to the block on the a^{ε_i}. By verticolor reflection, we infer that the following partition is in $\mathcal{C}$.

$$p \otimes \tilde{p} = a^{\varepsilon_1} X_1 a^{\varepsilon_2} X_2 a^{\varepsilon_3} X_3 \otimes \tilde{X}_3 b^{-\varepsilon_3} \tilde{X}_2 b^{-\varepsilon_2} \tilde{X}_1 b^{-\varepsilon_1}$$

By 4.1(b), we obtain that a partition $q := a^{\varepsilon_1} X_1' a^{\varepsilon_2} a^{-\varepsilon_2} \tilde{X}_1' a^{-\varepsilon_1}$ is in $\mathcal{C}$. Note that while the blocks on a^{ε_i} and $b^{-\varepsilon_i}$ are *not* connected in $p \otimes \tilde{p}$, the points a^{ε_i} and $a^{-\varepsilon_i}$ *are* connected in q

due to the procedure as described in Lemma 4.1(b). Using rotation, we infer that the partion $a^{-\varepsilon_1}a^{\varepsilon_1}X_1'a^{\varepsilon_2}a^{-\varepsilon_2}\tilde{X}_1'$ is in $\mathcal{C}$. Again, tensoring it with its verticolor reflected version and using Lemma 4.1(b), we obtain $a^{-1}aa^{-1}a \in \mathcal{C}$ which implies $\in \mathcal{C}$. $\square$

Note that unlike in the non-colored case, $\otimes \notin \mathcal{C}$ does *not* imply that all blocks have even size. Consider for instance $\langle \rangle$.

Definition 4.5. *Let $\mathcal{C} \subseteq P^{\circ\bullet}$ be a category of partitions. We say that:*

- *$\mathcal{C}$ is in* case $\mathcal{O}$, *if $\otimes \notin \mathcal{C}$ and $\notin \mathcal{C}$.*
- *$\mathcal{C}$ is in* case $\mathcal{B}$, *if $\otimes \in \mathcal{C}$ and $\notin \mathcal{C}$.*
- *$\mathcal{C}$ is in* case $\mathcal{H}$, *if $\otimes \notin \mathcal{C}$ and $\in \mathcal{C}$.*
- *$\mathcal{C}$ is in* case $\mathcal{S}$, *if $\otimes \in \mathcal{C}$ and $\in \mathcal{C}$.*

4.2.2 Global and local colorization

It is convenient to study categories $\mathcal{C} \subseteq NC^{\circ\bullet}$ case by case according to the above definition. According to Lemma 4.2(a), we divide each of these cases into two subcases: Those categories $\mathcal{C}$ containing $\otimes$ behave very differently from those not containing this partition.

Definition 4.6. *A category of partitions $\mathcal{C} \subseteq P^{\circ\bullet}$ is*

- globally colorized, *if $\otimes \in \mathcal{C}$*
- *and* locally colorized *if $\otimes \notin \mathcal{C}$.*

By Lemma 4.2, we may permute the colors of the points of partitions in globally colorized categories. Hence the coloring of partitions turns out to be of a global nature – the difference between the number of white and black points is the only number that matters for the coloring of a partition in such categories.

4.2.3 The global parameter $k(\mathcal{C})$

Studying categories of noncrossing partitions case by case, we will use certain global and local (color) parameters.

Definition 4.7. *Let $p \in P^{\circ\bullet}$.*

- *Denote by $c_\circ(p)$ the sum of the number of white points on the lower line of p and the black points on the upper line.*
- *Denote by $c_\bullet(p)$ the sum of the number of black points on the lower line of p and the white points on the upper line.*
- *Define $c : P^{\circ\bullet} \to \mathbb{Z}$ by $c(p) := c_\circ(p) - c_\bullet(p)$.*

We will mainly consider partitions $p \in P^{\circ\bullet}(0, l)$ with no upper points. In this case $c_\circ$ is counting the white points whereas $c_\bullet$ is counting the black points of a partition. Note that rotating black points from the upper line to the lower line turns them into white points. In this sense, $c_\circ$ counts black points on the upper line as white points on the lower line.

Definition 4.8. *Let $\mathcal{C}$ be a category of partitions. We set $k(\mathcal{C})$ as the minimum of all numbers $c(p)$ such that $c(p) > 0$ and $p \in \mathcal{C}$, if such a partition exists in $\mathcal{C}$. Otherwise $k(\mathcal{C}) := 0$. The parameter $k(\mathcal{C})$ is called the* degree of reflection *of $\mathcal{C}$. It is the* global parameter *of $\mathcal{C}$.*

Note that we always find a partition p in $\mathcal{C}$ such that $c(p) = 0$; take for instance $p =$ . In the next lemma, we show that the map $c : P^{\circ\bullet} \to \mathbb{Z}$ behaves well with respect to the category operations. In particular, if there exists a partition $p \in \mathcal{C}$ such that $c(p) < 0$, then $\tilde{p} \in \mathcal{C}$ and $c(\tilde{p}) = -c(p) > 0$.

Lemma 4.9. *For the map $c : P^{\circ\bullet} \to \mathbb{Z}$ the following holds true.*

(a) $c(p \otimes q) = c(p) + c(q)$

(b) $c(pq) = c(p) + c(q)$

(c) $c(p^*) = -c(p)$

(d) $c(p') = c(p)$, *if p' is obtained from p by rotation.*

(e) $c(\tilde{p}) = -c(p)$

Proof. From the definition it is clear that (a), (c), (d) and (e) hold. To see the invariance under composition, let w_1 be the number of upper white points of q, and b_1 be the number of upper black points. Let w_2 be the number of lower white points of q and likewise b_2 for the black points. Since p and q are composable, the numbers w_2 and b_2 also count the number of upper white and upper black points of p, respectively. Finally, let w_3 and b_3 be the number of lower white and black points of p respectively. We thus have:

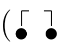
$$\begin{aligned} c(q) &= c_\circ(q) - c_\bullet(q) = (w_2 + b_1) - (w_1 + b_2) \\ c(p) &= (w_3 + b_2) - (w_2 + b_3) \\ c(pq) &= (w_3 + b_1) - (w_1 + b_3) \end{aligned}$$

This implies $c(pq) = c(p) + c(q)$. $\square$

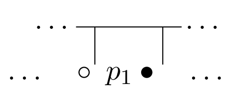
The global parameter $k(\mathcal{C})$ gives rise to a complete description of all possible numbers $c(p)$ of a category $\mathcal{C}$.

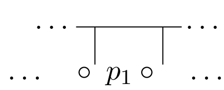
Proposition 4.10. *Let $\mathcal{C} \subseteq P^{\circ\bullet}$ be a category of partitions and let $k := k(\mathcal{C}) \in \mathbb{N}_0$. Then $c(p) \in k\mathbb{Z}$ for all partitions $p \in \mathcal{C}$.*

Proof. The statement is obvious for $k = 0$ by definition and Lemma 4.9(e), thus we may assume $k > 0$. Let $p \in \mathcal{C}$, such that $c(p) \neq 0$. By Lemma 4.9(e), we may restrict to $c(p) > 0$. Now, assume that there is a number $m \in \mathbb{N}_0$ such that $km < c(p) < k(m+1)$. By the definition of $k(\mathcal{C})$, there is a partition $q \in \mathcal{C}$ such that $c(q) = k$. Put $r := \tilde{q}^{\otimes m} \otimes p$. Then $r \in \mathcal{C}$ and $c(r) = -mc(q) + c(p)$. Hence $0 < c(r) < k$ which contradicts the definition of $k(\mathcal{C})$. $\square$

4.2.4 The local parameters $d(\mathcal{C})$ and $K^{\mathcal{C}}$()

We also have some local parameters. The idea is to determine possible numbers $c(p_1)$ of sub-partitions p_1 between two legs of a block of a partition $p \in \mathcal{C}$. The situation when these two legs have the same color behaves quite differently from the case of equally colored legs.

$p =$ 

$p =$ 

By rotation, we can reduce it to the following situation.

Definition 4.11.

- *Let $p \in NC^{\circ\bullet}(0,l)$ be a partition with no upper points. Assume that p can be decomposed as $p = p_1 \otimes p_2$. If the first and the last point of p_2 belong to the same block and if p_2 has length at least two, we say that $p = p_1 \otimes p_2$ is in* nest decomposed form.
- *Let $NDF(\lceil_{\circ}\ \rceil_{\bullet})$ be the set of all partitions $p = p_1 \otimes p_2$ in nest decomposed form such that the first point of p_2 is white and the last one is black. By $NDF^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$ we denote the intersection of $NDF(\lceil_{\circ}\ \rceil_{\bullet})$ and $\mathcal{C} \subseteq P^{\circ\bullet}$. Likewise we use the notations $NDF^{\mathcal{C}}(\lceil_{\bullet}\ \rceil_{\circ})$, $NDF^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\circ})$ and $NDF^{\mathcal{C}}(\lceil_{\bullet}\ \rceil_{\bullet})$ for the three other cases.*
- *By $K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$ we denote the set of all numbers $c(p_1) \in \mathbb{Z}$ such that $p = p_1 \otimes p_2 \in NDF^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$. Likewise we define $K^{\mathcal{C}}(\lceil_{\bullet}\ \rceil_{\circ})$, $K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\circ})$ and $K^{\mathcal{C}}(\lceil_{\bullet}\ \rceil_{\bullet})$.*
- *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a category of noncrossing partitions. We define the following* local parameter *$d(\mathcal{C})$. If $K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$ contains a number $t > 0$, we put $d(\mathcal{C})$ as the minimum of those numbers. Otherwise $d(\mathcal{C}) := 0$.*

Example 4.12. (a) The partition $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}$ is in $NDF(\lceil_{\bullet}\ \rceil_{\bullet})$ with $c(p_1) = 2$, whereas $\sqcap_{\bullet\bullet} \otimes \sqcap_{\circ\circ}$ is in $NDF(\lceil_{\circ}\ \rceil_{\circ})$ with $c(p_1) = -2$.

(b) The partition $p := \sqcap_{\circ\circ} \otimes \sqcap_{\circ\circ} \otimes \sqcap_{\circ\circ}$ is in nest decomposed form, where $p_1 = \sqcap_{\circ\circ} \otimes \sqcap_{\circ\circ}$ and $p_2 = \sqcap_{\circ\circ}$. It is in $NDF(\lceil_{\circ}\ \rceil_{\circ})$ with $c(p_1) = 4$. The partition $q := \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \otimes \sqcap_{\circ\circ}$ is in $NDF(\lceil_{\circ}\ \rceil_{\circ})$ with the decomposition $q_1 := \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}$ and $q_2 := \sqcap_{\circ\circ}$.

When working on the classfication of noncrossing categories, we will be interested in the sets $K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$, $K^{\mathcal{C}}(\lceil_{\bullet}\ \rceil_{\circ})$, $K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\circ})$ and $K^{\mathcal{C}}(\lceil_{\bullet}\ \rceil_{\bullet})$ as local parameters. In the remainder of this subsection, we shall prove that we always have $K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet}) = K^{\mathcal{C}}(\lceil_{\bullet}\ \rceil_{\circ}) = d\mathbb{Z}$ for $d = d(\mathcal{C})$ and $K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\circ}) = -K^{\mathcal{C}}(\lceil_{\bullet}\ \rceil_{\bullet})$. Thus, the study of these local parameters boils down to knowing $d(\mathcal{C})$ and $K^{\mathcal{C}}(\lceil_{\bullet}\ \rceil_{\bullet})$. Note that $\sqcap_{\circ\bullet} \otimes \sqcap_{\circ\bullet}$ is in $NDF^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$, hence $K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$ always contain the zero. As for $K^{\mathcal{C}}(\lceil_{\bullet}\ \rceil_{\bullet})$ the situation is a bit more complicated – for instance, it could be empty.

Proposition 4.13. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a category of noncrossing partitions. Then, $K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet}) = d\mathbb{Z}$ for $d = d(\mathcal{C})$. Furthermore, if $k(\mathcal{C}) \neq 0$, then $d(\mathcal{C}) \neq 0$ and $d(\mathcal{C})$ is a divisor of $k(\mathcal{C})$.*

Proof. Let $s, t \in K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$. We only need to show that $s + t$ and $-s$ are in $K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$. Let $p = p_1 \otimes p_2$ and $q = q_1 \otimes q_2$ be two partitions in $NDF^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$. Firstly, we have $p_1 \otimes q_1 \otimes q_2 \otimes p_2 \in \mathcal{C}$ using rotation. Using the pair partition $\sqcup^{\bullet\circ}$ and composition, we can connect the last point of q_2 with the first point of p_2. This erases these two points and we obtain a partition $p_1 \otimes q_1 \otimes r \in \mathcal{C}$ in nest decomposed form such that the first point of r is white and the last one is black. Thus, $c(p_1) + c(q_1) \in K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$. Secondly, we have $\tilde{p}_2 \otimes \tilde{p}_1 \in \mathcal{C}$, which yields $\tilde{p}_1 \otimes \tilde{p}_2 \in \mathcal{C}$ by rotation. Now, this is a partition in nest decomposed form, such that the first point of $\tilde{p}_2$ is white and the last one is black. Thus, $-c(p_1) = c(\tilde{p}_1) \in K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$.

We deduce $K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet}) = d\mathbb{Z}$ using the Euklidean algorithm. Furthermore, note that $k(\mathcal{C}) \in K^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$, since $p \otimes \sqcap_{\circ\bullet} \in NDF^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$ for all $p \in \mathcal{C}$. Hence, $d(\mathcal{C}) \neq 0$ whenever $k(\mathcal{C}) \neq 0$ and also $k(\mathcal{C})$ is a multiple of $d(\mathcal{C})$. □

Definition 4.14. *By b_s we denote the partition in $P^{\circ\bullet}(0,s)$ consisting of a single block of length s such that all points are white, hence $b_2 = \sqcap_{\circ\circ}$, $b_3 = \sqcap_{\circ\circ\circ}$, etc.*

The next lemma is of quite technical nature, but it will be needed in this subsection as well as in the remainder of this chapter several times.

Lemma 4.15. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a category of noncrossing partitions.*

(a) Let $\uparrow_\circ \otimes \uparrow_\bullet \in \mathcal{C}$. We then have $\uparrow_\circ^{\otimes s}\ \sqcap\!\left(\circ\ \uparrow_\bullet^{\otimes s}\ \bullet\right) \in \mathcal{C}$ for all $s \in K^{\mathcal{C}}(\sqcap_{\circ\bullet})$, $s \geq 0$. Likewise $\uparrow_\circ^{\otimes s}\ \sqcap\!\left(\bullet\ \uparrow_\bullet^{\otimes s}\ \circ\right) \in \mathcal{C}$ for all $s \in K^{\mathcal{C}}(\sqcap_{\bullet\circ})$, $s \geq 0$.

More generally, let $q \in \{\sqcap_{\circ\circ}, \sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}, \sqcap_{\bullet\bullet}\}$ and let $s \in K^{\mathcal{C}}(q)$ (not necessarily $s \geq 0$). Then $\uparrow_\circ^{\otimes s}\ a^{\varepsilon_1}\ \uparrow_\circ^{\otimes -(s+c(q))}\ a^{\varepsilon_2} \in \mathcal{C}$, where a^{ε_1} and a^{ε_2} form the pair block q. Here, we use the notation $\uparrow_\circ^{\otimes s} = \uparrow_\bullet^{\otimes -s}$ for $s < 0$.

(b) Let $\sqcap_{\circ\circ\bullet\bullet} \in \mathcal{C}$ and let $s \in K^{\mathcal{C}}(q)$ for some $q \in \{\sqcap_{\circ\circ}, \sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}, \sqcap_{\bullet\bullet}\}$. Then $b_s \otimes \tilde{b}_s \in \mathcal{C}$. Here $b_{-s} = \tilde{b}_s$ if $s < 0$.

(c) Let $p \in \mathcal{C}$ be a partition such that two neighbouring points have the same color and belong to the same block. Then $\sqcap_{\circ\circ\bullet\bullet} \in \mathcal{C}$ or $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \in \mathcal{C}$. In particular, if $\uparrow_\circ \otimes \uparrow_\bullet \notin \mathcal{C}$ and $NDF^{\mathcal{C}}(\sqcap_{\bullet\bullet}) \neq \emptyset$, then $\sqcap_{\circ\circ\bullet\bullet} \in \mathcal{C}$ or $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \in \mathcal{C}$. The same holds true, if $\uparrow_\circ \otimes \uparrow_\bullet \notin \mathcal{C}$ and if $\mathcal{C}$ contains a partition $p_1 \otimes p_2$ such that $c(p_1) \neq 0$.

Proof. (a) Let $q \in \{\sqcap_{\circ\circ}, \sqcap_{\circ\bullet}, \sqcap_{\bullet\circ}, \sqcap_{\bullet\bullet}\}$ and let $s \in K^{\mathcal{C}}(q)$. Let $p = p_1 \otimes p_2$ be a partition in $NDF^{\mathcal{C}}(q)$ such that $c(p_1) = s$. Using the pair partitions and composing p_1 with the rotated versions $|^{\circ}_{\circ}$ and $|^{\bullet}_{\bullet}$ of $\uparrow_\circ \otimes \uparrow_\bullet$, we may assume $p_1 = \uparrow_\circ^{\otimes s}$. We proceed in the same way for the points between the first and the last point of p_2 and we infer that $\uparrow_\circ^{\otimes s}\ a^{\varepsilon_1}\ \uparrow_\circ^{\otimes \alpha}\ a^{\varepsilon_2}$ is in $\mathcal{C}$ for some $\alpha \in \mathbb{Z}$. Here a^{ε_1} and a^{ε_2} form the pair block q. Now, disconnecting these two points using the partitions $|^{\circ}_{\circ}$ and $|^{\bullet}_{\bullet}$ again, we deduce $\uparrow_\circ^{\otimes s} \otimes \uparrow_\circ^{\otimes \alpha + c(q)} \in \mathcal{C}$. Using verticolor reflection and composition, this implies:

$$\uparrow_\circ^{\otimes s}\ a^{\varepsilon_1}\ \uparrow_\circ^{\otimes \alpha}\ \uparrow_\circ^{\otimes -s}\ \uparrow_\circ^{\otimes -\alpha - c(q)}\ a^{\varepsilon_2} \in \mathcal{C}$$

Using the pair partitions, we obtain $\uparrow_\circ^{\otimes s}\ a^{\varepsilon_1}\ \uparrow_\circ^{\otimes -s-c(q)}\ a^{\varepsilon_2} \in \mathcal{C}$.

(b) Let $p = p_1 \otimes p_2 \in \mathcal{C}$ be a partition in nest decomposed form such that $c(p_1) = s$. Assume $s > 0$. Using the pair partitions and Lemma 4.2, we may connect all blocks in p_1 and we may erase all of its black points such that finally $b_s \otimes p_2 \in \mathcal{C}$. By Lemma 4.1, $b_s \otimes \tilde{b}_s \in \mathcal{C}$; likewise if $s < 0$.

(c) Let $p \in \mathcal{C}$ be a partition of the form $p = aaX$, where aa are two points of the same color belonging to the same block. If this block is of size two, we have $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \in \mathcal{C}$ by Lemma 4.1. Otherwise, we consider $p \otimes \tilde{p} \in \mathcal{C}$ and using the pair partition, we infer $\sqcap_{\circ\circ\bullet\bullet} \in \mathcal{C}$.

Now, if $\mathcal{C}$ contains a partition $p_1 \otimes p_2$ such that $c(p_1) > 0$, we use the pair partition to erase all black points in p_1. We obtain a partition $p_1' \otimes p_2 \in \mathcal{C}$ such that all points of p_1' are white. If $\uparrow_\circ \otimes \uparrow_\bullet \notin \mathcal{C}$, all blocks in p_1' have size at least two. Since p_1' is noncrossing, we find two neighbouring points belonging to the same block. By rotation, this yields a partition $p = aaX$ as above; likewise for $c(p_1) < 0$.

Finally, if $NDF^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\bullet}) \neq \emptyset$, then there is a $p = p_1 \otimes p_2 \in \mathcal{C}$ in nest decomposed form such that the first and the last point of p_2 are black. If $c(p_1) = 0$, we erase p_1 using the pair partitions and we obtain $p_2 \in \mathcal{C}$. By rotation, p_2 is of the form aaX. If $c(p_1) \neq 0$ we use the argument above. $\square$

In globally colorized categories, the local parameters $d(\mathcal{C})$ and $K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\bullet})$ are trivial. In particular, we show $K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet}) = K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\circ})$ and $K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\circ}) = -K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\bullet})$ in the special case of globally colored categories.

Lemma 4.16. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a globally colorized category of noncrossing partitions. The following holds true.*

(a) The sets $K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet})$, $K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\circ})$, $K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\circ})$ and $K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\bullet})$ coincide and are equal to $d\mathbb{Z}$ for $d = d(\mathcal{C})$.

(b) We have $d(\mathcal{C}) \in \{1, 2\}$ with $d(\mathcal{C}) = 1$ if and only if $\uparrow_{\circ}\sqcap(\circ\uparrow_{\bullet}\bullet) \in \mathcal{C}$.

Proof. (a) Let $p = p_1 \otimes p_2 \in \mathcal{C}$ be a partition in nest decomposed form. Thus, $p_2 = a^{\varepsilon_1} X a^{\varepsilon_2}$, where a^{ε_1} and a^{ε_2} belong to the same block, and X is some subpartition (possibly connected to a^{ε_1} and a^{ε_2}). Then, the following partition is in $\mathcal{C}$, by composition.

$$p' := p_1 \otimes a^{\varepsilon_1} \sqcap_{\circ\circ} \sqcap_{\bullet\bullet} X a^{\varepsilon_2}$$

Using permutation of colors, we may change the colors of a^{ε_1} and a^{ε_2} arbitrarily. Moreover, $K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet}) = d\mathbb{Z}$ by Proposition 4.13.

(b) By Example 4.12, we have $2 \in K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\bullet})$, which by (a) yields $2 \in K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet})$. Thus, $d(\mathcal{C}) \in \{1, 2\}$. If $\uparrow_{\circ}\sqcap(\circ\uparrow_{\bullet}\bullet) \in \mathcal{C}$, then $d(\mathcal{C}) = 1$ by definition. Conversely, let $d(\mathcal{C}) = 1$. We thus find a partition $\uparrow_{\circ} \otimes p_2$ in $\mathcal{C}$. By Lemma 4.1, we deduce $\uparrow_{\circ} \otimes \uparrow_{\bullet} \in \mathcal{C}$, which by Lemma 4.15 implies $\uparrow_{\circ}\sqcap(\circ\uparrow_{\bullet}\bullet) \in \mathcal{C}$. $\square$

We finally prove that our local parameters are given only by $d(\mathcal{C})$ and $K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\bullet})$.

Proposition 4.17. *We have $K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet}) = K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\circ}) = d\mathbb{Z}$ with $d = d(\mathcal{C})$ for all categories $\mathcal{C} \subseteq NC^{\circ\bullet}$. Furthermore, $K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\circ}) = -K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\bullet})$.*

Proof. Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a category of noncrossing partitions. If $p_1 \otimes p_2 \in \mathcal{C}$ is a partition in nest decomposed form such that the first and the last point of p_2 have the same color, then $\tilde{p}_1 \otimes \tilde{p}_2$ is in $\mathcal{C}$ proving $K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\circ}) = -K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\bullet})$. Next, we prove $K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet}) = K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\circ})$.

Case 1. Let $\uparrow_{\circ} \otimes \uparrow_{\bullet} \in \mathcal{C}$. We have $\uparrow_{\circ}^{\otimes s}\sqcap(\circ\,\uparrow_{\bullet}^{\otimes s}\,\bullet) \in \mathcal{C}$ if and only if $\uparrow_{\circ}^{\otimes s}\sqcap(\bullet\,\uparrow_{\bullet}^{\otimes s}\,\circ) \in \mathcal{C}$, using verticolor reflection and rotation. This proves $K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet}) = K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\circ})$ using Lemma 4.15.

Case 2. Let $\uparrow_{\circ} \otimes \uparrow_{\bullet} \notin \mathcal{C}$ and let $NDF^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\bullet}) \neq \emptyset$. By Lemma 4.15 we have $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \in \mathcal{C}$ or $\sqcap\sqcap\sqcap_{\circ\circ\bullet\bullet} \in \mathcal{C}$. In the first case, the sets $K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet})$ and $K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\circ})$ coincide by Lemma 4.16. In the second case, we have to show that $b_s \otimes \tilde{b}_s \in \mathcal{C}$ implies $s \in K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet})$ and $s \in K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\circ})$. By Lemma 4.15 this will finish the proof.

So, let $b_s \otimes \tilde{b_s} \in \mathcal{C}$. By rotation, we have $(\tilde{b_s})^* \otimes \tilde{b_s} \in \mathcal{C}$ which is a partition in $P^{\circ\bullet}(s,s)$. Now, compose $b_s \otimes \tilde{b}_s \otimes \sqcap_{\bullet\circ}$ with $\left(|^{\circ}_{\circ}\right)^{\otimes s} \otimes |^{\bullet}_{\bullet} \otimes (\tilde{b}_s)^* \otimes \tilde{b}_s \otimes |^{\circ}_{\circ}$. This yields $b_s \otimes a\tilde{b}_s a^{-1} \in \mathcal{C}$, where a is black and a^{-1} is white forming the pair block $\sqcap_{\bullet\circ}$. Hence $s \in K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\circ})$. Similarly, we obtain $b_s \otimes a^{-1}\tilde{b}_s a \in \mathcal{C}$ and $s \in K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet})$.

Case 3. Let $\uparrow_{\circ} \otimes \uparrow_{\bullet} \notin \mathcal{C}$ and let $NDF^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\bullet}) = \emptyset$. Then also $NDF^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\circ}) = \emptyset$. Let $p_1 \otimes p_2 \in \mathcal{C}$ be a partition in nest decomposed form such that the first and the last point of p_2 have inverse colors. Assume $c(p_1) > 0$. Using the pair partitions, we may erase all black points in p_1 and we obtain a partition $p_1' \otimes p_2$ such that all points in p_1' are white. Since all blocks in p_1' have size at least two by Lemma 4.4, we obtain a partition in $NDF^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\circ})$ up to rotation which is a contradiction. Likewise we show that $c(p_1)$ cannot be strictly negative. We conclude that $c(p_1)$ is zero and hence $K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet}) = \{0\}$ and $K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet}) = \{0\}$. Thus $K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet}) = K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet})$. $\square$

4.2.5 Summary of the strategy for the classification

We now have all tools at hand for the classification of categories $\mathcal{C} \subseteq NC^{\circ\bullet}$ of noncrossing partitions. The general strategy is as follows.

- We study the cases $\mathcal{O}, \mathcal{H}, \mathcal{S}$ and $\mathcal{B}$ (see Definition 4.5) step by step subdividing them again into the local and the global colorization (see Definition 4.6) respectively.
- In each of these cases we first **determine** all possible global and local **parameters** $k(\mathcal{C})$, $d(\mathcal{C})$ and $K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\bullet})$.
- We then **find characteristic sample partitions** which somehow represent these parameters.
- Next, we isolate sets of partitions M depending on the possible values of the parameters and we prove $M \subseteq \langle p_1, \ldots, p_n \rangle$, where $p_1, \ldots, p_n$ are the sample partitions.
- Finally, we prove that **these are all possible categories** in the considered case. To do so, if $\mathcal{C}$ has parameters k, d and $K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\bullet})$, we have $\langle p_1, \ldots, p_n \rangle \subseteq \mathcal{C}$. On the other hand, $\mathcal{C} \subseteq M$, which proves $\mathcal{C} = \langle p_1, \ldots, p_n \rangle = M$.

4.3 Case $\mathcal{O}$

Let us first consider the case $\mathcal{O}$, i.e. the case of categories $\mathcal{C} \subseteq NC^{\circ\bullet}$ of noncrossing partitions such that $\uparrow_{\circ} \otimes \uparrow_{\bullet} \notin \mathcal{C}$ and $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \notin \mathcal{C}$. By Lemma 4.4, $\mathcal{C}$ is a subset of the set $NC_2^{\circ\bullet}$ of all noncrossing pair partitions.

4.3.1 Determining the parameters

Proposition 4.18. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a category of noncrossing partitions in case $\mathcal{O}$.*

(a) If $\mathcal{C}$ is globally colorized, then $d(\mathcal{C}) = 2$ and $k(\mathcal{C}) \in 2\mathbb{N}_0$.

(b) If $\mathcal{C}$ is locally colorized, then $d(\mathcal{C}) = k(\mathcal{C}) = 0$ and $K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\bullet}) = \emptyset$, i.e.:

$$K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\bullet}) = K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\circ}) = \{0\}, \qquad K^{\mathcal{C}}(\ulcorner_{\circ}\,\urcorner_{\circ}) = K^{\mathcal{C}}(\ulcorner_{\bullet}\,\urcorner_{\bullet}) = \emptyset$$

Proof. (a) By Lemma 4.16, $d(\mathcal{C}) = 2$ and hence $k(\mathcal{C}) \in 2\mathbb{N}_0$, using Proposition 4.13.

(b) Let $p \in \mathcal{C}$ be a partition having no upper points. We prove that all blocks connect a white point with a black point from which the assertion easily follows. Assume that there is a block V connecting two white points. We choose V such that all blocks nested into it (if there are any) connect a white point with a black point. Using the pair partitions, we erase all those blocks and we end up with a partition in $\mathcal{C}$ such that two neighbouring points have the same color and belong to the same block. By Lemma 4.15(c), this is a contradiction. $\square$

4.3.2 Finding partitions realizing the parameters

Lemma 4.19. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a globally colorized category of noncrossing partitions in case $\mathcal{O}$ such that $k = k(\mathcal{C}) \neq 0$. Then $\sqcap_{\circ\circ}^{\otimes\frac{k}{2}} \in \mathcal{C}$.*

Proof. We find a partition $p \in \mathcal{C}$ having no upper points such that $c(p) = k$, i.e. there are $x + k$ white points and x black points in p. Now, the partition $\sqcap_{\circ\bullet}^{\otimes\frac{k}{2}} \otimes p$ is in $\mathcal{C}$. By Lemma 4.2, we may permute the colors of this partition, and we infer that $\sqcap_{\circ\circ}^{\otimes\frac{k}{2}} \otimes p'$ is in $\mathcal{C}$ for some partition p' with $c(p') = 0$. Using the pair partitions $\sqcap_{\circ\bullet}$ and $\sqcap_{\bullet\circ}$ to erase p', we infer that $\sqcap_{\circ\circ}^{\otimes\frac{k}{2}} \in \mathcal{C}$. $\square$

4.3.3 Description of natural categories in case $\mathcal{O}$

The preceding subsection shows that there are natural categories $\langle \sqcap_{\circ\circ}^{\otimes\frac{k}{2}}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$ in case $\mathcal{O}$. Let us describe these categories and also the "empty category" $\langle \emptyset \rangle = \langle \sqcap_{\circ\bullet}, \sqcap_{\bullet\circ} \rangle$. (Recall that the partitions $\sqcap_{\circ\bullet}$ and $\sqcap_{\bullet\circ}$ are always contained in a category and that we omit to write them down explicitely as generators.)

Proposition 4.20. *We have the following natural categories of partitions in case $\mathcal{O}$.*

(a) The category $\mathcal{O}_{\mathrm{loc}} := \langle \emptyset \rangle$ consists of all noncrossing pair partitions such that each block connects a white point with a black point, when the partition is rotated such that it has no upper points.

(b) Let $k \in 2\mathbb{N}_0$. Then $\mathcal{O}_{\mathrm{glob}}(k) := \langle \sqcap_{\circ\circ}^{\otimes\frac{k}{2}}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$ coincides with $\{p \in NC_2^{\circ\bullet} \mid c(p) \in k\mathbb{Z}\}$. Here, $\sqcap_{\circ\circ}^{\otimes\frac{k}{2}} = \emptyset$ if $k = 0$.

In particular, all these categories are pairwise different.

Proof. (a) We may construct all partitions from the assertion using $\sqcap_{\circ\bullet}$, $\sqcap_{\bullet\circ}$ and the category operations due to a simple inductive argument. Assume that p has $m + 1$ blocks. Since p is noncrossing, it contains at least one block $q = \sqcap_{\circ\bullet}$ or $q = \sqcap_{\bullet\circ}$. Removing q yields a partition which is in $\langle \emptyset \rangle$ by induction hypothesis. Composing this partition with $r_1 \otimes q \otimes r_2$ where r_i are suitable tensor products of the identity partitions $|^{\circ}_{\circ}$ and $|^{\bullet}_{\bullet}$ yields the partition p which is hence in $\langle \emptyset \rangle$. Conversely, the set of all noncrossing pair partitions with the block rule of the assertion forms a category of partitions, hence containing $\langle \emptyset \rangle$.

(b) Let $p \in NC_2^{\circ\bullet}(0, l)$ be a partition with no upper points such that $c(p) = km \geq 0$, for some $m \in \mathbb{N}_0$. Let $p' \in NC_2^{\circ\bullet}(0, l)$ be the partition obtained from p by replacing each unicolored block

$\sqcap_{\circ\circ}$ or $\sqcap_{\bullet\bullet}$ by $\sqcap_{\circ\bullet}$. Then, p' is a partition in $\langle\emptyset\rangle \subseteq \langle \sqcap_{\circ\circ}^{\otimes\frac{k}{2}}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}\rangle$ by (a) and $c(p') = 0$. This implies that $p' \otimes \sqcap_{\circ\circ}^{\otimes\frac{km}{2}}$ is in $\langle \sqcap_{\circ\circ}^{\otimes\frac{k}{2}}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}\rangle$, too, with $c(p' \otimes \sqcap_{\circ\circ}^{\otimes\frac{km}{2}}) = c(p)$ by Lemma 4.9. Hence, permutation of colors yields that $p \otimes \sqcap_{\circ\bullet}^{\otimes\frac{km}{2}}$ is in $\langle \sqcap_{\circ\circ}^{\otimes\frac{k}{2}}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}\rangle$. Using the pair partition, we infer $p \in \langle \sqcap_{\circ\circ}^{\otimes\frac{k}{2}}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}\rangle$.

Conversely, the set $\{p \in NC_2^{\circ\bullet} \mid c(p) \in k\mathbb{Z}\}$ is a category of partitions due to Lemma 4.9 containing $\sqcap_{\circ\circ}^{\otimes\frac{k}{2}}$ and $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}$. $\square$

4.3.4 Classification in the case $\mathcal{O}$

We are now ready to prove our first classification theorem.

Theorem 4.21. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a category of noncrossing partitions in case $\mathcal{O}$. Then $\mathcal{C}$ coincides with one of the following categories.*

(a) If $\mathcal{C}$ is globally colorized, then $\mathcal{C} = \mathcal{O}_{\mathrm{glob}}(k) = \langle \sqcap_{\circ\circ}^{\otimes\frac{k}{2}}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}\rangle$ for $k = k(\mathcal{C}) \in 2\mathbb{N}_0$.

(b) If $\mathcal{C}$ is locally colorized, then $\mathcal{C} = \mathcal{O}_{\mathrm{loc}} = \langle\emptyset\rangle$.

Proof. (a) By Propositions 4.10 and 4.20, we have that $\mathcal{C}$ is contained in $\langle \sqcap_{\circ\circ}^{\otimes\frac{k}{2}}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}\rangle$. Conversely, $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \in \mathcal{C}$ by Definition 4.6 and $\sqcap_{\circ\circ}^{\otimes\frac{k}{2}} \in \mathcal{C}$ by Lemma 4.19.

(b) Let $\mathcal{C}$ be locally colorized and let $p \in \mathcal{C}$ be a partition with no upper points. Then, each block of p connects a white point to a black point, see the proof of Proposition 4.18. By Proposition 4.20, $\mathcal{C}$ is contained in $\langle\emptyset\rangle$, hence they coincide. $\square$

Remark 4.22. *For $k = 2$, the category $\langle \sqcap_{\circ\circ}^{\otimes\frac{k}{2}}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}\rangle$ coincides with the non-colored category of partitions $\langle\sqcap\rangle$ in the sense of Proposition 4.3.*

4.4 Case $\mathcal{H}$

We now turn to the case $\mathcal{H}$, i.e. to categories $\mathcal{C} \subseteq NC^{\circ\bullet}$ such that $\uparrow_{\circ} \otimes \uparrow_{\bullet} \notin \mathcal{C}$ but $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$. By Lemma 4.4, no blocks of size one occur in any partition considered in this section. Recall, that due to Lemma 4.2(d), we may connect neighbouring blocks of partitions in $\mathcal{C}$, if the blocks meet at two points with inverse colors.

4.4.1 Determining the parameters

Proposition 4.23. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a category of noncrossing partitions in case $\mathcal{H}$.*

(a) If $\mathcal{C}$ is globally colorized, then $d(\mathcal{C}) = 2$ and $k(\mathcal{C}) \in 2\mathbb{N}_0$.

(b) If $\mathcal{C}$ is locally colorized, then

(i) either $K^{\mathcal{C}}(\ulcorner_{\bullet}\ \urcorner_{\bullet}) = \emptyset$ and $k(\mathcal{C}) = d(\mathcal{C}) = 0$,

(ii) or $k(\mathcal{C}), d(\mathcal{C}) \in \mathbb{N}_0 \backslash \{1, 2\}$ and $K^{\mathcal{C}}(\ulcorner_{\bullet}\ \urcorner_{\bullet}) = d\mathbb{Z}$ for $d = d(\mathcal{C})$, i.e. in this case:

$$K^{\mathcal{C}}(\ulcorner_{\circ}\ \urcorner_{\bullet}) = K^{\mathcal{C}}(\ulcorner_{\bullet}\ \urcorner_{\circ}) = K^{\mathcal{C}}(\ulcorner_{\circ}\ \urcorner_{\circ}) = K^{\mathcal{C}}(\ulcorner_{\bullet}\ \urcorner_{\bullet}) = d\mathbb{Z}$$

Moreover, we have $K^{\mathcal{C}}(\bullet\;\bullet) \neq \emptyset$ *if and only if* $\sqcap\!\sqcap\!\sqcap_{\circ\circ\bullet\bullet} \in \mathcal{C}$.

Proof. (a) This is analogue to Proposition 4.18.

(b) We first prove that $K^{\mathcal{C}}(\bullet\;\bullet) \neq \emptyset$ holds if and only if $\sqcap\!\sqcap\!\sqcap_{\circ\circ\bullet\bullet} \in \mathcal{C}$. If $\sqcap\!\sqcap\!\sqcap_{\circ\circ\bullet\bullet} \in \mathcal{C}$, then $\sqcap_{\circ\bullet} \otimes \sqcap\!\sqcap\!\sqcap_{\bullet\circ\circ\bullet} \in \mathcal{C}$ from which we deduce $K^{\mathcal{C}}(\bullet\;\bullet) \neq \emptyset$. Conversely, if $K^{\mathcal{C}}(\bullet\;\bullet) \neq \emptyset$, then $NDF^{\mathcal{C}}(\bullet\;\bullet) \neq \emptyset$ from which we deduce $\sqcap\!\sqcap\!\sqcap_{\circ\circ\bullet\bullet} \in \mathcal{C}$ by Lemma 4.4(c).

Next, we prove that a locally colorized category can only be in the cases (i) or (ii) of the assertion.

Case 1: $K^{\mathcal{C}}(\bullet\;\bullet) = \emptyset$. Assume $d(\mathcal{C}) \neq 0$. Then there is a partition $p = p_1 \otimes p_2 \in \mathcal{C}$ such that $c(p_1) \neq 0$, which by Lemma 4.15 implies $\sqcap\!\sqcap\!\sqcap_{\circ\circ\bullet\bullet} \in \mathcal{C}$, a contradiction. Thus, we have $d(\mathcal{C}) = 0$ which implies $k(\mathcal{C}) = 0$ by Proposition 4.13.

Case 1: $K^{\mathcal{C}}(\bullet\;\bullet) \neq \emptyset$. Assume $d(\mathcal{C}) = 2$. We find a partition $p = p_1 \otimes p_2 \in \mathcal{C}$ such that $c(p_1) = 2$. Using the pair partition, we infer $\uparrow_\circ \otimes \uparrow_\circ \otimes p_2 \in \mathcal{C}$ or $\sqcap_{\circ\circ} \otimes p_2 \in \mathcal{C}$. By Lemma 4.1, we have $\uparrow_\circ \otimes \uparrow_\bullet \in \mathcal{C}$ or $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \in \mathcal{C}$, both is a contradiction. Similarly, $d(\mathcal{C}) = 1$ implies $\uparrow_\circ \otimes p_2 \in \mathcal{C}$ for some partition p_2, and the cases $k(\mathcal{C}) \in \{1, 2\}$ can be excluded analoguously (with $p_2 = \emptyset$).

It remains to show that $K^{\mathcal{C}}(\bullet\;\bullet) = d\mathbb{Z}$ for $d = d(\mathcal{C})$. Since $K^{\mathcal{C}}(\bullet\;\bullet) \neq \emptyset$, we have $\sqcap\!\sqcap\!\sqcap_{\circ\circ\bullet\bullet} \in \mathcal{C}$. Thus, for $0 \neq s \in K^{\mathcal{C}}(\circ\;\bullet)$, we infer $b_s \otimes \tilde{b}_s \in \mathcal{C}$ by Lemma 4.15. This proves $s \in K^{\mathcal{C}}(\bullet\;\bullet)$. Moreover, we have $0 \in K^{\mathcal{C}}(\bullet\;\bullet)$ since a rotated version of $b_s \otimes \tilde{b}_s$ yields a partition $p_2 \in \mathcal{C}$ such that the first and the last point are black and belong to the same block. Then $\sqcap_{\circ\bullet} \otimes p_2$ is in $\mathcal{C}$ proving $0 \in K^{\mathcal{C}}(\bullet\;\bullet)$. We conclude $K^{\mathcal{C}}(\circ\;\bullet) \subseteq K^{\mathcal{C}}(\bullet\;\bullet)$. Conversely, let $0 \neq s \in K^{\mathcal{C}}(\bullet\;\bullet)$. By Lemma 4.15, we have $b_s \otimes \tilde{b}_s \in \mathcal{C}$. Like in the proof of Proposition 4.17, we obtain a partition $b_s \otimes a\tilde{b}_s a^{-1} \in \mathcal{C}$, where a and a^{-1} is a pair block $\sqcap_{\circ\bullet}$. This proves $s \in K^{\mathcal{C}}(\circ\;\bullet)$ and hence $d\mathbb{Z} = K^{\mathcal{C}}(\bullet\;\bullet)$. □

4.4.2 Finding partitions realizing the parameters

Lemma 4.24. *Let* $\mathcal{C} \subseteq NC^{\circ\bullet}$ *be a category of noncrossing partitions in case* $\mathcal{H}$.

(a) If $k = k(\mathcal{C}) \neq 0$*, then* $b_k \in \mathcal{C}$.

(b) If $d = d(\mathcal{C}) \neq 0$*, then* $b_d \otimes \tilde{b}_d \in \mathcal{C}$.

Proof. (a) By Proposition 4.23, we have $\sqcap\!\sqcap\!\sqcap_{\circ\circ\bullet\bullet} \in \mathcal{C}$ as soon as $k \neq 0$. Let $p \in \mathcal{C}$ be a partition with no upper points such that $c(p) = k$. Using the pair partition, we may assume that p is a partition on k white points. Using Lemma 4.2, we may connect all its blocks, ending up with $b_k \in \mathcal{C}$.

(b) Similar to (a), we deduce $b_d \otimes p_2 \in \mathcal{C}$ from the existence of a partition $p_1 \otimes p_2 \in \mathcal{C}$ with $c(p_1) = d$. By Lemma 4.1, we have $b_d \otimes \tilde{b}_d \in \mathcal{C}$. □

4.4.3 Description of natural categories

Motivated by Lemma 4.24, we want to describe the categories $\langle b_k, b_d \otimes \tilde{b}_d, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \rangle$. Note that for $k \geq 2$ or $d \geq 2$, we may always construct the partition $\sqcap\!\sqcap\!\sqcap_{\circ\circ\bullet\bullet}$ inside the category (Lemma 4.15). Due to Proposition 4.23, this is a natural generator indeed, so we add it in the following lemma also for the cases $k = d = 0$ and treat the case $\langle \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \rangle$ separately.

Proposition 4.25. *We have the following natural categories in case $\mathcal{H}$.*

(a) The category $\mathcal{H}'_{\mathrm{loc}} := \langle \circ\bullet\circ\bullet \rangle$ consists of all noncrossing partitions such that each block is of even length connecting white and black points in an alternating way, when the partition is rotated such that it has no upper points.

(b) Let $k, d \in \mathbb{N}_0 \backslash \{1\}$ be such that d is a divisor of k, if $k \neq 0$, and denote $b_0 := \emptyset$. Denote by $\mathcal{H}_{\mathrm{loc}}(k, d)$ the set of all partitions $p \in NC^{\circ\bullet}$ such that

(i) all blocks have length at least two,

(ii) $c(p) \in k\mathbb{Z}$,

(iii) if $p_1 \otimes p_2$ is any rotated version of p in nest decomposed form, then $c(p_1) \in d\mathbb{Z}$.

We have $\mathcal{H}_{\mathrm{loc}}(k, d) \subseteq \langle b_k, b_d \otimes \tilde{b}_d, \circ\circ\bullet\bullet, \circ\bullet\circ\bullet \rangle$.

Proof. (a) By Lemma 4.2(d) we may deduce $\circ\bullet\circ\bullet\circ\bullet \in \langle \circ\bullet\circ\bullet \rangle$ from $\circ\bullet\circ\bullet \otimes \circ\bullet \in \langle \circ\bullet\circ\bullet \rangle$. Iteratively, we may construct all one block partitions of even length connecting white and black points in an alternating way, and using the category operations, we may construct all partitions of the assertion. Conversely, the set of all noncrossing partitions with the above block rule forms a category of partitions containing $\circ\bullet\circ\bullet$.

(b) Denote the category $\langle b_k, b_d \otimes \tilde{b}_d, \circ\circ\bullet\bullet, \circ\bullet\circ\bullet \rangle$ by $\mathcal{D}$. Let $p \in \mathcal{H}_{\mathrm{loc}}(k, d)$. We prove $p \in \mathcal{D}$ by induction on the number m of blocks of p. Since $\mathcal{D}$ is closed under rotation, we may assume that p has no upper points. For $m = 1$, note that $\iota_l \otimes b_k^{\otimes t} \in \mathcal{D}$ by (a), where $\iota_l \in P^{\circ\bullet}(0, 2l)$ consists of a single block on $2l$ points with alternating colors, and $t \geq 0$. Using Lemma 4.2(c), we infer that all one block partitions with $c(p) \in k\mathbb{Z}$ are in $\mathcal{D}$.

Let $m > 1$. By rotation and since p is noncrossing, p is of the form $p = p_1 \otimes p_2$ such that p_2 consists only of one block. Since we are in case $\mathcal{H}$, p_2 has length at least two and thus p is in nest decomposed form. Thus $c(p_1) \in d\mathbb{Z}$ and hence also $c(p_2) = c(p) - c(p_1) \in d\mathbb{Z}$. Assume $c(p_1) = ds$ with some $s \geq 0$, by verticolor reflection. Let p_1' be the partition obtained from $p_1 \otimes \tilde{b}_d^{\otimes s}$ by connecting all points of $\tilde{b}_d^{\otimes s}$ to the last point of p_1. Then p_1' is a partition with $m - 1$ blocks and $c(p_1') = c(p_1) - ds = 0$. Furthermore, any nest decomposed form $q_1' \otimes q_2'$ of p_1' yields a nest decomposed form $q_1 \otimes q_2$ of p such that $c(q_1') \in c(q_1) + d\mathbb{Z} \subseteq d\mathbb{Z}$ because $c(p_2) \in d\mathbb{Z}$ and $c(\tilde{b}_d^{\otimes s}) \in d\mathbb{Z}$. Hence, $p_1' \in \mathcal{H}_{\mathrm{loc}}(k, d)$ and by induction hypothesis, $p_1' \in \mathcal{D}$. Composing it with $r \otimes (\tilde{b}_d^{\,*})^{\otimes s} \otimes (\tilde{b}_d)^{\otimes s}$, where r is a suitable tensor product of the identity partitions, yields $p_1 \otimes \tilde{b}_d^{\otimes s} \in \mathcal{D}$. Similary $b_d^{\otimes s} \otimes p_2 \in \mathcal{D}$, since the partition p_2' obtained from connecting all blocks of $b_d^{\otimes s} \otimes p_2$ is a one block partition with $c(p_2') = c(p)$. We conclude that $p_1 \otimes \tilde{b}_d^{\otimes s} \otimes b_d^{\otimes s} \otimes p_2 \in \mathcal{D}$ and using the pair partitions, we obtain $p = p_1 \otimes p_2 \in \mathcal{D}$. $\square$

4.4.4 Classification in the case $\mathcal{H}$

Theorem 4.26. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a category of noncrossing partitions in case $\mathcal{H}$. Then $\mathcal{C}$ coincides with one of the following categories.*

(i) If $\mathcal{C}$ is globally colorized, then $\mathcal{C} = \mathcal{H}_{\mathrm{glob}}(k) := \langle b_k, \circ\bullet\circ\bullet, \circ\circ \otimes \bullet\bullet \rangle$ for $k = k(\mathcal{C}) \in 2\mathbb{N}_0$.

(ii) If $\mathcal{C}$ is locally colorized, and $K^{\mathcal{C}}(\circ\ \circ) = \emptyset$, then $\mathcal{C} = \mathcal{H}'_{\mathrm{loc}} = \langle \circ\bullet\circ\bullet \rangle$. Otherwise, we have $\mathcal{C} = \langle b_k, b_d \otimes \tilde{b}_d, \circ\circ\bullet\bullet, \circ\bullet\circ\bullet \rangle$ for $k = k(\mathcal{C}) \in \mathbb{N}_0 \backslash \{1, 2\}$ and $d = d(\mathcal{C}) \in \mathbb{N}_0 \backslash \{1, 2\}$. We use the notation $b_0 = \emptyset$.

Proof. (i) Using Lemma 4.24, we know $\langle b_k, \ , \ \otimes \ \rangle \subseteq \mathcal{C}$. For the converse inclusion, let $p \in \mathcal{C}$. Then, $c(p) \in k\mathbb{Z}$ by Proposition 4.10. Furthermore, if $p_1 \otimes p_2$ is any rotated version of p in nest decomposed form, then $c(p_1) \in 2\mathbb{Z}$ by Lemma 4.16 and Proposition 4.23. By Proposition 4.25, we infer $p \in \mathcal{H}_{\mathrm{loc}}(k,2) \subseteq \langle b_k, b_2 \otimes \tilde{b}_2, \ , \ \rangle$. Since $b_2 \otimes \tilde{b}_2 = \ \otimes \ $, we infer $\mathcal{C} = \langle b_k, \ , \ \otimes \ \rangle$.

(ii) Let $\mathcal{C}$ be locally colorized and let $k := k(\mathcal{C})$ and $d := d(\mathcal{C})$.

Case 1. Let $K^{\mathcal{C}}(\ ) = \emptyset$. Since $\mathcal{C}$ is in case $\mathcal{H}$, we have $\langle \ \rangle \subseteq \mathcal{C}$. Conversely, let $p \in \mathcal{C}$ be a partition without upper points. Assume that there is a block of p which does not connect white and black points in an alternating way. Using rotation, we can bring p in nest decomposed form $p = p_1 \otimes p_2$ such that the first and the last point of p_2 have the same color. This contradicts $K^{\mathcal{C}}(\ ) = K^{\mathcal{C}}(\ ) = \emptyset$. We conclude that each block of $p \in \mathcal{C}$ connects white and black points in an alternating way. Furthermore, they are of even length. Otherwise, the first and the last point would have the same color, and again we would find an example of a partition in $NDF^{\mathcal{C}}(\ )$ or in $NDF^{\mathcal{C}}(\ )$. By Proposition 4.25, we infer $p \in \langle \ \rangle$ and hence $\mathcal{C} = \langle \ \rangle$.

Case 2. Let $K^{\mathcal{C}}(\ ) \neq \emptyset$. Then $\langle b_k, b_d \otimes \tilde{b}_d, \ , \ \rangle \subseteq \mathcal{C}$ by Proposition 4.23 and Lemma 4.24. Conversely, let $p \in \mathcal{C}$. By Lemma 4.10, we have $c(p) \in k\mathbb{Z}$. Furthermore, if $p' = p_1 \otimes p_2$ is any rotated version of p in nest decomposed form, we have $c(p_1) \in d\mathbb{Z}$ by Proposition 4.23. We thus have $p \in \mathcal{H}_{\mathrm{loc}}(k,d)$, which by Proposition 4.25 yields $p \in \langle b_k, b_d \otimes \tilde{b}_d, \ , \ \rangle$. This shows $\langle b_k, b_d \otimes \tilde{b}_d, \ , \ \rangle = \mathcal{C}$. $\square$

Corollary 4.27. *We have $\mathcal{H}_{\mathrm{loc}}(k,d) = \langle b_k, b_d \otimes \tilde{b}_d, \ , \ \rangle$ in Proposition 4.25. In particular, all these categories are pairwise different.*

Proof. In the above theorem, we showed $\langle b_k, b_d \otimes \tilde{b}_d, \ , \ \rangle \subseteq \mathcal{C} \subseteq \mathcal{H}_{\mathrm{loc}}(k,d)$ whenever $\mathcal{C}$ is a locally colorized category in case $\mathcal{H}$ with $k = k(\mathcal{C})$, $d = d(\mathcal{C})$ and $K^{\mathcal{C}}(\ ) \neq \emptyset$. Together with $\mathcal{H}_{\mathrm{loc}}(k,d) \subseteq \langle b_k, b_d \otimes \tilde{b}_d, \ , \ \rangle$ of Proposition 4.25, we have equality here. Moreover, it can easily be seen that the sets $\mathcal{H}_{\mathrm{loc}}(k,d)$ are distinct. $\square$

Remark 4.28. *(a) One can show that the categories $\langle b_k, \ , \ \otimes \ \rangle$ are given by the set of all partitions $p \in NC^{\circ\bullet}$ such that $c(p) \in k\mathbb{Z}$ and all blocks of p have even length.*

(b) The non-colored case $\langle \sqcap\sqcap \rangle$ is obtained from $\langle b_k, \ , \ \otimes \ \rangle$ for $k = 2$ in the sense of Proposition 4.3.

4.5 Case $\mathcal{S}$

We now consider the case $\mathcal{S}$, i.e. categories $\mathcal{C} \subseteq NC^{\circ\bullet}$ such that $\ $ and $\ \otimes \ $ are in $\mathcal{C}$.

4.5.1 Determining the parameters

Proposition 4.29. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a category of noncrossing partitions in case $\mathcal{S}$.*

(a) We always have $\ \in \mathcal{C}$.

(b) If $\mathcal{C}$ is globally colorized, then $d(\mathcal{C}) = 1$ and $k(\mathcal{C}) \in \mathbb{N}_0$. Moreover, $\ \in \mathcal{C}$.

(c) If $\mathcal{C}$ is locally colorized, then $k(\mathcal{C}), d(\mathcal{C}) \in \mathbb{N}_0\backslash\{1\}$ and $K^{\mathcal{C}}(\sqcap_{\bullet\bullet}) = K^{\mathcal{C}}(\sqcap_{\circ\bullet}) + 1 = d\mathbb{Z} + 1$ for $d = d(\mathcal{C})$, i.e.:

$$K^{\mathcal{C}}(\sqcap_{\circ\bullet}) = K^{\mathcal{C}}(\sqcap_{\bullet\circ}) = d\mathbb{Z}, \qquad K^{\mathcal{C}}(\sqcap_{\bullet\bullet}) = d\mathbb{Z}+1, \qquad K^{\mathcal{C}}(\sqcap_{\circ\circ}) = d\mathbb{Z}-1$$

Moreover, $\sqcap\!\sqcap\!\sqcap_{\circ\circ\bullet\bullet} \notin \mathcal{C}$.

Proof. (a) By Lemma 4.2, we may disconnect the white points from $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}$.

(b) By (a) and using color permutation, we have $\uparrow_\circ \sqcap_{\circ\uparrow_\bullet\bullet} \in \mathcal{C}$, thus $d(\mathcal{C}) = 1$.

(c) If $k(\mathcal{C}) = 1$, then $\uparrow_\circ \in \mathcal{C}$ which allows us to erase arbitrary points of partitions in $\mathcal{C}$. Thus $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \in \mathcal{C}$ implies $\sqcap_{\circ\circ} \in \mathcal{C}$ which is a contradiction to $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \notin \mathcal{C}$. If $d(\mathcal{C}) = 1$, then $\uparrow_\circ \sqcap_{\circ\uparrow_\bullet\bullet} \in \mathcal{C}$ by Lemma 4.15(a). Using (a) and Lemma 4.2(e), we infer $\uparrow_\circ \otimes \uparrow_\circ \otimes \sqcap_{\bullet\bullet} \in \mathcal{C}$ which implies $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \in \mathcal{C}$ by Lemma 4.1, a contradiction.

Now, let $s \in K^{\mathcal{C}}(\sqcap_{\bullet\bullet})$, $s > 0$. We find a partition $p_1 \otimes p_2 \in \mathcal{C}$ in nest decomposed form such that the first and the last point of p_2 are black, and $c(p_1) = s$. We may assume that $p_1 = \uparrow_\circ^{\otimes s}$ using Lemma 4.2(b) and the pair partitions. Swaping one of the white singletons of p_1 with the first point of p_2 (by (a) and Lemma 4.2(f)) yields a partition $\uparrow_\circ^{\otimes s-1} \otimes p_2' \in \mathcal{C}$ in nest decomposed form such that the first and the last point of p_2' have inverse colors. This shows $s-1 \in K^{\mathcal{C}}(\sqcap_{\circ\bullet})$. We proceed similarly in the case $s < 0$ and we deduce $K^{\mathcal{C}}(\sqcap_{\bullet\bullet}) \subseteq K^{\mathcal{C}}(\sqcap_{\circ\bullet}) + 1$. Note that $0 \notin K^{\mathcal{C}}(\sqcap_{\bullet\bullet})$ by Lemma 4.15(c). Conversely, if $s+1 \in K^{\mathcal{C}}(\sqcap_{\circ\bullet}) + 1$, $s \geq 0$, we find a partition $\uparrow_\circ^{\otimes s} \otimes p_2 \in \mathcal{C}$ in nest decomposed form such that the first point of p_2 is white and the last point is black. Then, $\uparrow_\circ^{\otimes s} \otimes \uparrow_\circ \otimes \uparrow_\bullet \otimes p_2 \in \mathcal{C}$ and using the partition $\uparrow_\circ \sqcap_{\bullet\uparrow_\circ\bullet}$, we infer $s+1 \in K^{\mathcal{C}}(\sqcap_{\bullet\bullet})$. We conclude $K^{\mathcal{C}}(\sqcap_{\bullet\bullet}) = K^{\mathcal{C}}(\sqcap_{\circ\bullet}) + 1$ and $K^{\mathcal{C}}(\sqcap_{\circ\circ}) = -K^{\mathcal{C}}(\sqcap_{\bullet\bullet}) = K^{\mathcal{C}}(\sqcap_{\circ\bullet}) - 1$.

Finally, if $\sqcap\!\sqcap\!\sqcap_{\circ\circ\bullet\bullet} \in \mathcal{C}$, then also $\sqcap_{\circ\circ} \otimes \uparrow_\bullet \otimes \uparrow_\bullet \in \mathcal{C}$, which implies $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \in \mathcal{C}$ by Lemma 4.2(b). □

4.5.2 Finding partitions realizing the parameters

Lemma 4.30. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a category in case $\mathcal{S}$.*

(a) If $k = k(\mathcal{C}) \neq 0$, then $\uparrow_\circ^{\otimes k} \in \mathcal{C}$.

(b) If $d = d(\mathcal{C}) \neq 0$, then $\uparrow_\circ^{\otimes d} \sqcap_{\circ\,\uparrow_\bullet^{\otimes d}\,\bullet} \in \mathcal{C}$.

Proof. (a) We find a partition $p \in \mathcal{C}$ such that $c(p) = k$. Using the pair partition we erase all black points and using $\uparrow_\circ \otimes \uparrow_\bullet$ we know $\uparrow_\circ^{\otimes k} \in \mathcal{C}$ by Lemma 4.2.

(b) This follows from Lemma 4.15(a). □

4.5.3 Description of natural categories

Proposition 4.31. *We have the following natural categories in case $\mathcal{S}$. Let $k, d \in \mathbb{N}_0$ such that d is a divisor of k, if $k \neq 0$. Denote by $\mathcal{S}_{\text{loc}}(k,d)$ the set of all partitions $p \in NC^{\circ\bullet}$ such that*

(i) $c(p) \in k\mathbb{Z}$,

(ii) if $p_1 \otimes p_2$ is any rotated version of p in nest decomposed form such that the first and the last point of p_2

... have inverse colors, then $c(p_1) \in d\mathbb{Z}$,

... both are black, then $c(p_1) \in d\mathbb{Z} + 1$,

... both are white, then $-c(p_1) \in d\mathbb{Z} + 1$.

We have $\mathcal{S}_{\text{loc}}(k,d) \subseteq \langle \uparrow_\circ^{\otimes k},\ \uparrow_\circ^{\otimes d}\,\ulcorner\!\circ\,\uparrow_\bullet^{\otimes d}\,\bullet\!\urcorner,\ \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet},\ \uparrow_\circ \otimes \uparrow_\bullet \rangle$.

Proof. Denote $\langle \uparrow_\circ^{\otimes k},\ \uparrow_\circ^{\otimes d}\,\ulcorner\!\circ\,\uparrow_\bullet^{\otimes d}\,\bullet\!\urcorner,\ \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet},\ \uparrow_\circ \otimes \uparrow_\bullet \rangle$ by $\mathcal{D}$ and let $p \in \mathcal{S}_{\text{loc}}(k,d)$. We give a proof by induction on the number $m(p)$ of those blocks of p which have length greater or equal two. By rotation, we may always assume that p has no upper points.

Case 1. Let $m(p) = 0$, i.e. p consists only of singletons. Since $c(p) \in k\mathbb{Z}$, we have up to permutation of the colors (see Lemma 4.2(b)) $p = \uparrow_\circ^{\otimes kt} \otimes \left(\uparrow_\circ \otimes \uparrow_\bullet\right)^{\otimes w}$ for some number w. Hence $p \in \mathcal{D}$.

Case 2. Let $m(p) = 1$. Using rotation, p is of the following form:

$$p = a^{\varepsilon_1} X_1 a^{\varepsilon_2} X_2 \ldots a^{\varepsilon_l} X_l$$

Here, the points a^{ε_i} form a block of length $l \geq 2$, and the X_i are some tensor products of the singletons $\uparrow_\circ$ and $\uparrow_\bullet$. If now all points a^{ε_i} had alternating colors, we could first argue that the partition $a^{\varepsilon_1} \ldots a^{\varepsilon_l}$ is in $\langle \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \rangle \subseteq \mathcal{D}$ and then insert the tensor products X_i of singletons between the legs using $\uparrow_\circ^{\otimes k}$ and $\uparrow_\circ^{\otimes d}\,\ulcorner\!\circ\,\uparrow_\bullet^{\otimes d}\,\bullet\!\urcorner$. Unfortunately, the alternating coloring is not always the case. We therefore construct a partition p' involving some "correction points". It will be of the form:

$$p' = A_1' X_1' A_2' X_2' \ldots A_l' X_l'$$

The construction of p' is as follows. If a^{ε_i} and $a^{\varepsilon_{i+1}}$ have different colors, then up to permutation of the colors, X_i is of the form $X_i = \uparrow_\circ^{\otimes dt_i} \otimes \left(\uparrow_\circ \otimes \uparrow_\bullet\right)^{\otimes w_i}$ for some $w_i \in \mathbb{N}_0$, $t_i \in \mathbb{Z}$, by condition (ii) of $\mathcal{S}_{\text{loc}}(k,d)$. We put $A_{i+1}' := a^{\varepsilon_{i+1}}$ and $X_i' := X_i$. On the other hand, if a^{ε_i} and $a^{\varepsilon_{i+1}}$ both are black, then $X_i = \uparrow_\circ^{\otimes dt_i+1} \otimes \left(\uparrow_\circ \otimes \uparrow_\bullet\right)^{\otimes w_i}$ up to permutation, and we put $A_{i+1}' := a^{-\varepsilon_{i+1}} a^{\varepsilon_{i+1}}$, $X_i' := X_i \otimes \uparrow_\bullet$, $w_i' := w_i + 1$. If a^{ε_i} and $a^{\varepsilon_{i+1}}$ both are white, then $X_i' := X_i \otimes \uparrow_\circ$ instead. Finally, we put $A_1' := a^{\varepsilon_1}$ and $X_l' := X_l$ if a^{ε_1} and a^{ε_l} have inverse colors and $A_1' := a^{-\varepsilon_l} a^{\varepsilon_1}$, $X_l' := X_l \otimes \uparrow_\bullet$ or $X_l' := X_l \otimes \uparrow_\circ$ otherwise.

Now, the partition $q_1 := A_1' A_2' \ldots A_l'$ consists only of one block of even length with alternating colors, by construction. By Lemma 4.25, it is contained in $\langle \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \rangle \subseteq \mathcal{D}$.

Let q_2 be the partition obtained from q_1 by inserting subpartitions $(\uparrow_\circ \otimes \uparrow_\bullet)^{\otimes w_i'}$ between A_i' and A_{i+1}', and $(\uparrow_\circ \otimes \uparrow_\bullet)^{\otimes w_l'}$ after A_l'. Since q_2 can be obtained from q_1 using the category operations, we have $q_2 \in \mathcal{D}$. Moreover, $c(p') = c(p)$ by construction and $c(p) = \sum_i dt_i \in k\mathbb{Z}$. Let $q_3 := \uparrow_\circ^{\otimes c(p)}$, hence $q_2 \otimes q_3$ is in $\mathcal{D}$. Since $\uparrow_\circ^{\otimes d}\,\ulcorner\!\circ\,\uparrow_\bullet^{\otimes d}\,\bullet\!\urcorner \in \mathcal{D}$, we may use Lemma 4.2 to shift the partitions $\uparrow_\circ^{\otimes dt_i}$ (or $\uparrow_\bullet^{\otimes dt_i}$ resp.) at the right positions, and we infer $p' \in \mathcal{D}$. Using the pair partitions, we finally erase all extra points in p' together with the "correction singletons" and we deduce $p \in \mathcal{D}$.

Case 3. Let $m(p) > 1$. By rotation, p can be brought in nest decomposed form $p = p_1 \otimes p_2$ such that $m(p_2) = 1$. Such a decomposition exists since p is noncrossing.

Case 3a. If the first and the last point of p_2 have inverse colors, we have $c(p_1) \in d\mathbb{Z}$ by condition (ii). We may assume $c(p_1) \geq 0$, i.e. $c(p_1) = ds$ for some $s \in \mathbb{N}_0$. Then, the partition $p_1' := p_1 \otimes \uparrow_\bullet^{\otimes ds}$ satisfies $c(p_1') = 0$ and $m(p_1') = m(p_1) = m(p) - 1$. As conditions (ii) and (iii) are fulfilled for p_1', we infer $p_1' \in \mathcal{D}$ by induction hypothesis. By Case 2, we also have $p_2' := \uparrow_\circ^{\otimes ds} \otimes p_2 \in \mathcal{D}$, since $c(p_2') = c(p_2) + c(p_1) = c(p) \in k\mathbb{Z}$. Thus, we obtain $p_1' \otimes p_2' \in \mathcal{D}$ and hence $p = p_1 \otimes p_2 \in \mathcal{D}$ using the pair partitions.

Case 3b. On the other hand, if both the first and the last point of p_2 are black, we consider $p_1' := p_1 \otimes \uparrow_\bullet$. Furthermore, we consider the partition p_2' obtained from p_2 by changing the color of its first point from black to white and we insert a black singleton to the right of this first point. Then, $p_1' \otimes p_2'$ is a partition in nest decomposed form fulfilling the conditions (i), (ii) and (iii). By Case 3a, we infer $p_1' \otimes p_2' \in \mathcal{D}$. Since $\uparrow_\circ \sqcap_{\bullet \uparrow_\circ \bullet} \in \mathcal{D}$, we may use Lemma 4.2 to infer that $p_1 \otimes \uparrow_\bullet \otimes \uparrow_\circ \otimes p_2$ is in $\mathcal{D}$, which yields $p \in \mathcal{D}$. We proceed in the same way, if the first and the last point of p_2 are white. $\square$

4.5.4 Classification in the case $\mathcal{S}$

Theorem 4.32. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a category of noncrossing partitions in case $\mathcal{S}$. Then $\mathcal{C}$ coincides with one of the following categories.*

(i) If $\mathcal{C}$ is globally colorized, then $\mathcal{C} = \mathcal{S}_{\mathrm{glob}}(k) := \langle \uparrow_\circ^{\otimes k}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \uparrow_\circ \otimes \uparrow_\bullet, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$ for $k = k(\mathcal{C}) \in \mathbb{N}_0$.

(ii) If $\mathcal{C}$ is locally colorized, then $\mathcal{C} = \langle \uparrow_\circ^{\otimes k}, \uparrow_\circ^{\otimes d} \sqcap_{\circ\, \uparrow_\bullet^{\otimes d}\, \bullet}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \uparrow_\circ \otimes \uparrow_\bullet \rangle$ for $k = k(\mathcal{C}) \in \mathbb{N}_0 \backslash \{1\}$ and $d = d(\mathcal{C}) \in \mathbb{N}_0 \backslash \{1\}$.

Proof. (i) By Lemma 4.30, we have $\uparrow_\circ^{\otimes k} \in \mathcal{C}$ for $k = k(\mathcal{C})$. Thus $\langle \uparrow_\circ^{\otimes k}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \uparrow_\circ \otimes \uparrow_\bullet, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle \subseteq \mathcal{C}$. Conversely, let $p \in \mathcal{C}$. Then $c(p) \in k\mathbb{Z}$ by Proposition 4.10. Furthermore, $d = d(\mathcal{C}) = 1$ by Proposition 4.29, hence $p \in \mathcal{S}_{\mathrm{loc}}(k, 1) \subseteq \langle \uparrow_\circ^{\otimes k}, \uparrow_\circ \sqcap_{\circ \uparrow_\bullet \bullet}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \uparrow_\circ \otimes \uparrow_\bullet \rangle$ by Proposition 4.31. We have $\langle \uparrow_\circ^{\otimes k}, \uparrow_\circ \sqcap_{\circ \uparrow_\bullet \bullet}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \uparrow_\circ \otimes \uparrow_\bullet \rangle = \langle \uparrow_\circ^{\otimes k}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \uparrow_\circ \otimes \uparrow_\bullet, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$.

(ii) Let $\mathcal{C}$ be locally colorized. Using Lemma 4.30, we infer $\langle \uparrow_\circ^{\otimes k}, \uparrow_\circ^{\otimes d} \sqcap_{\circ\, \uparrow_\bullet^{\otimes d}\, \bullet}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \uparrow_\circ \otimes \uparrow_\bullet \rangle \subseteq \mathcal{C}$ for $k = k(\mathcal{C})$ and $d = d(\mathcal{C})$. Conversely, let $p \in \mathcal{C}$. Then $c(p) \in k\mathbb{Z}$ by Proposition 4.10. Let $p_1 \otimes p_2$ be a rotated version of p in nest decomposed form. If the first and the last point of p_2 have inverse colors, then $c(p_1) \in d\mathbb{Z}$ by Proposition 4.13. If the first and the last point of p_2 both are black, we have to prove $s - 1 \in d\mathbb{Z}$ for $s := c(p_1)$.

Assume that $s > 0$. Using the pair partitions and Lemma 4.2, we may assume that p_1 is of the form $p_1 = \uparrow_\circ^{\otimes s}$, hence $\uparrow_\circ^{\otimes s} \otimes p_2 \in \mathcal{C}$. We have $s \neq 0$, since otherwise $p_2 \in \mathcal{C}$ and rotation would yield a partition such that two neighbouring points have the same color and belong to the same block. By Lemma 4.15(c) and Proposition 4.29 this would lead to a contradiction.

We thus have $s \geq 1$. Since $\uparrow_\circ \sqcap_{\bullet \uparrow_\circ \bullet} \in \mathcal{C}$ by Proposition 4.29, we may shift one of the white singletons to the right hand side of the first point of p_2, which inverts the colors of these two points. We infer that the partition $\uparrow_{\circ}^{\otimes s-1} \otimes p_2'$ is in $\mathcal{C}$, where p_2' is in nest decomposed form such that the first and the last point have inverse colors. By Proposition 4.13, we thus have $s-1 \in d\mathbb{Z}$.

As for $s<0$, we have $\uparrow_{\bullet}^{\otimes -s} \otimes p_2 \in \mathcal{C}$. By composition, we infer that also $\uparrow_{\bullet}^{\otimes -s} \otimes \uparrow_{\bullet} \otimes \uparrow_{\circ} \otimes p_2 \in \mathcal{C}$. Again, shifting the white singleton to the right hand side of the first point of p_2 yields $\uparrow_{\bullet}^{\otimes -s+1} \otimes p_2' \in \mathcal{C}$ where the first and the last point of p_2' have inverse colors belonging to the same block. Thus, $-s+1 \in d\mathbb{Z}$ and hence $s-1 \in d\mathbb{Z}$.

A similar proof shows that $s+1 \in d\mathbb{Z}$ if the first and the last point of p_2 are white. We thus have $p \in \mathcal{S}_{\mathrm{loc}}(k,d)$ and by Proposition 4.31, we deduce $p \in \langle \uparrow_{\circ}^{\otimes k}, \uparrow_{\circ}^{\otimes d}\sqcap_{\circ\bullet}\uparrow_{\bullet}^{\otimes d}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \uparrow_{\circ} \otimes \uparrow_{\bullet}\rangle$. This shows $\mathcal{C} = \langle \uparrow_{\circ}^{\otimes k}, \uparrow_{\circ}^{\otimes d}\sqcap_{\circ\bullet}\uparrow_{\bullet}^{\otimes d}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \uparrow_{\circ} \otimes \uparrow_{\bullet}\rangle$. ☐

Corollary 4.33. *We have $\mathcal{S}_{\mathrm{loc}}(k,d) = \langle \uparrow_{\circ}^{\otimes k}, \uparrow_{\circ}^{\otimes d}\sqcap_{\circ\bullet}\uparrow_{\bullet}^{\otimes d}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \uparrow_{\circ} \otimes \uparrow_{\bullet}\rangle$ in Proposition 4.31. In particular, all these categories are pairwise different.*

Remark 4.34. *The non-colored case $\langle \uparrow \otimes \uparrow, \sqcap\!\sqcap\!\sqcap\rangle$ is obtained from $\langle \uparrow_{\circ}^{\otimes k}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \uparrow_{\circ} \otimes \uparrow_{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}\rangle$ for $k=2$, whereas $\langle \uparrow, \sqcap\!\sqcap\!\sqcap\rangle$ is the case $k=1$ (see Proposition 4.3).*

4.6 Case $\mathcal{B}$

Finally, we turn to the case $\mathcal{B}$, i.e. to categories $\mathcal{C} \subseteq NC^{\circ\bullet}$ such that $\sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet} \notin \mathcal{C}$ and $\uparrow_{\circ} \otimes \uparrow_{\bullet} \in \mathcal{C}$, i.e. all blocks of partitions $p \in \mathcal{C}$ have length at most two (Lemma 4.4). Like in the non-colored case, this is the most complicated situation, as we can already see when investigating which parameters can occur.

4.6.1 Determining the parameters

Proposition 4.35. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a category of noncrossing partitions in case $\mathcal{B}$.*

(a) If $\mathcal{C}$ is globally colorized, then the cases $d(\mathcal{C})=1$ and $d(\mathcal{C})=2$ can occur.

(b) If $\mathcal{C}$ is locally colorized, then

(i) either $K^{\mathcal{C}}(\sqcap_{\bullet\bullet}) = \emptyset$ and $k(\mathcal{C}), d(\mathcal{C}) \in \mathbb{N}_0$,

(ii) or $K^{\mathcal{C}}(\sqcap_{\bullet\bullet}) = d\mathbb{Z} + (r+1)$ for $r := r(\mathcal{C}) := \min\{s \geq 1 \mid s \in K^{\mathcal{C}}(\sqcap_{\bullet\bullet})\} - 1$ and $k(\mathcal{C}) \in \mathbb{N}_0\backslash\{1\}$, $d = d(\mathcal{C}) \in \mathbb{N}_0\backslash\{1\}$. Furthermore, $r=0$ or $r=\frac{d}{2}$; and $r \neq 1$. Thus:

$$K^{\mathcal{C}}(\sqcap_{\circ\bullet}) = K^{\mathcal{C}}(\sqcap_{\bullet\circ}) = d\mathbb{Z}, \qquad K^{\mathcal{C}}(\sqcap_{\bullet\bullet}) = d\mathbb{Z}+(r+1), \qquad K^{\mathcal{C}}(\sqcap_{\circ\circ}) = d\mathbb{Z}-(r+1)$$

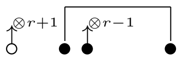
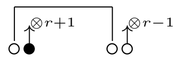
Proof. (a) This follows directly from Lemma 4.16.

(b) Let $K^{\mathcal{C}}(\sqcap_{\bullet\bullet}) \neq \emptyset$. It is clear that $k(\mathcal{C}) \neq 1$, since $\uparrow_{\circ} \in \mathcal{C}$ (see Lemma 4.36) would imply $\sqcap_{\bullet\bullet} \in \mathcal{C}$ as $K^{\mathcal{C}}(\sqcap_{\bullet\bullet}) \neq \emptyset$. Next, observe that we have $-(s-2) \in K^{\mathcal{C}}(\sqcap_{\bullet\bullet})$ whenever $s \in K^{\mathcal{C}}(\sqcap_{\bullet\bullet})$. Indeed, by Lemma 4.15, we have (recall the notation $\uparrow_{\circ}^{\otimes s} = \uparrow_{\bullet}^{\otimes -s}$, if $s<0$):

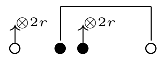
$$\uparrow_{\circ}^{\otimes s}\sqcap_{\bullet}\uparrow_{\bullet}^{\otimes s-2}\sqcap_{\bullet} \in \mathcal{C}$$

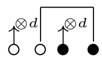
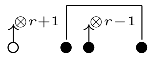
Rotation yields $-(s-2) \in K^{\mathcal{C}}(\sqcap_{\bullet\bullet})$. We observe that $0 \notin K^{\mathcal{C}}(\sqcap_{\bullet\bullet})$ and $2 \notin K^{\mathcal{C}}(\sqcap_{\bullet\bullet})$, since otherwise  $\in \mathcal{C}$, in contradiction to $\mathcal{C}$ being locally colorized. Thus $r \neq 1$. Moreover, we deduce that there is always a number $s \in K^{\mathcal{C}}(\sqcap_{\bullet\bullet})$ such that $s \geq 1$, and hence the number r is well-defined.

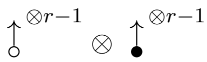
Like above, we have the following two partitions in $\mathcal{C}$, the latter one being obtained from the first one by verticolor reflection and rotation:

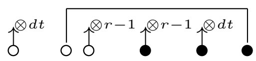
 $\in \mathcal{C}$ and  $\in \mathcal{C}$

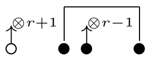
Forming the tensor product of these two partitions and composing it with a suitable tensor product of  and the identity partitions, we infer:

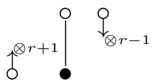
 $\in \mathcal{C}$

Thus, $2r \in K^{\mathcal{C}}(\sqcap_{\circ\bullet}) = d\mathbb{Z}$. Let $r \neq 0$. Then $d \neq 0$. We now prove $2r = d$. Put $r_0 := r+1$. Assume $2r = ds$ for some $s \geq 2$. Then $2d \leq ds = 2r$, hence $d < r_0$. Thus, $r' := r_0 - d$ is a number $0 < r' < r_0$. Using the partition  $\in \mathcal{C}$ (which is in $\mathcal{C}$ by Lemma 4.15), we can shift exactly like in the proof of Lemma 4.2(e) d of the $r+1$ white singletons of  from the outside of the pair to the inside. This yields a partition showing that $r' \in K^{\mathcal{C}}(\sqcap_{\bullet\bullet})$ in contradiction to the minimality of r_0. We conclude $2r = d$ if $r \neq 0$.

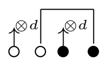
It remains to show that $K^{\mathcal{C}}(\sqcap_{\bullet\bullet}) = d\mathbb{Z} + r_0$. Let $t \in \mathbb{Z}$. Since $dt \in d\mathbb{Z} = K^{\mathcal{C}}(\sqcap_{\circ\bullet})$ and  $\in \mathcal{C}$, we have (by Lemma 4.15):

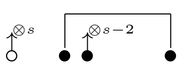
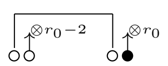
$p :=$  $\in \mathcal{C}$

Furthermore, the following partition is in $\mathcal{C}$ since it is a rotated version of :

$q :=$ 

Now, composing p with q we may shift $r-1$ white singletons from the inside of the pair of p to the outside, by which their number increases to $r+1$ white singletons. Furthermore, the color of the first point of the pair changes from white to black. We conclude that $dt + (r+1)$ is in $K^{\mathcal{C}}(\sqcap_{\bullet\bullet})$, which proves $d\mathbb{Z} + r_0 \subseteq K^{\mathcal{C}}(\sqcap_{\bullet\bullet})$.

Conversely, let $r_0 > 1$. Thus $r > 0$ and hence $d = 2r > 0$. Let $s \in K^{\mathcal{C}}(\sqcap_{\bullet\bullet})$ and assume $s \notin d\mathbb{Z} + r_0$. We have $s + d\mathbb{Z} \subseteq K^{\mathcal{C}}(\sqcap_{\bullet\bullet})$, since the partition  $\in \mathcal{C}$ allows us to shift clusters of d singletons of the same color to arbitrary positions. We may thus assume $r_0 < s < r_0 + d$ which implies $0 < s - r_0 < d$. Like above, we have the following two partitions in $\mathcal{C}$ using verticolor reflection:

 $\in \mathcal{C}$ and  $\in \mathcal{C}$

Composing the tensor product of them with yields $s - r_0 \in K^{\mathcal{C}}( )= d\mathbb{Z}$ in contradication to $0 < s - r_0 < d$.

In the case $r_0 = 1$, we have $\in \mathcal{C}$. If now $s \in K^{\mathcal{C}}( )$, then $s - 1 \in K^{\mathcal{C}}( ) = d\mathbb{Z}$ again by applying to $^{\otimes s}$ $^{\otimes s-2}$.

Finally, if $d = 1$, then $\in \mathcal{C}$ and $r = 0$ implies $\in \mathcal{C}$. Hence $\otimes$ $\in \mathcal{C}$, a contradiction. □

4.6.2 Finding partitions realizing the parameters

Lemma 4.36. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a category in case $\mathcal{B}$.*

(a) If $k = k(\mathcal{C}) \neq 0$, then $^{\otimes k} \in \mathcal{C}$.

(b) If $d = d(\mathcal{C}) \neq 0$, then $^{\otimes d}$ $^{\otimes d}$ $\in \mathcal{C}$.

(c) If $K^{\mathcal{C}}( ) \neq \emptyset$ and $r(\mathcal{C}) = 0$, then $\in \mathcal{C}$; if $r = r(\mathcal{C}) \neq 0$, then $^{\otimes r+1}$ $^{\otimes r-1}$ $\in \mathcal{C}$.

Proof. (a) Using $\otimes$ $\in \mathcal{C}$, we may disconnect any points from their blocks. Thus, we may assume that a partition $p \in \mathcal{C}$ with $c(p) = k$ is of the form $p = {}^{\otimes k}$.

(b)&(c) This is Lemma 4.15(a). □

4.6.3 Description of natural categories

Proposition 4.37. *We have the following natural categories in case $\mathcal{B}$.*

(a) The category $\langle \otimes \rangle$ consists of all noncrossing partitions $p \in NC^{\circ\bullet}$ such that when p is rotated to a partition having no upper points

(i) all blocks have size one or two,

(ii) the blocks of size two connect a black point and a white point,

(iii) the number of black singletons and the number of white singletons between two legs of every pair coincide, and on the global level, too.

(b) Let $k, d \in \mathbb{N}_0$ be such that d is a divisor of k, if $k \neq 0$. Let $r \in \{0, \frac{d}{2}\} \backslash \{1\}$. Denote by $\mathcal{B}'_{\mathrm{loc}}(k, d, r)$ the set of all noncrossing partitions $p \in NC^{\circ\bullet}$ such that

(i) all blocks have size one or two,

(ii) $c(p) \in k\mathbb{Z}$,

(iii) if $p_1 \otimes p_2$ is any rotated version of p in nest decomposed form such that the first and the last point of p_2

... have inverse colors, then $c(p_1) \in d\mathbb{Z}$,

... both are black, then $c(p_1) \in d\mathbb{Z} + r + 1$,

... both are white, then $-c(p_1) \in d\mathbb{Z} + r + 1$.

We have $\mathcal{B}'_{\mathrm{loc}}(k, d, r) \subseteq \langle {}^{\otimes k}, {}^{\otimes d} {}^{\otimes d}, {}^{\otimes r+1} {}^{\otimes r-1}, \otimes \rangle$.

(c) *Denote by* $\mathcal{B}_{\mathrm{loc}}(k,d)$ *the set defined as* $\mathcal{B'}_{\mathrm{loc}}(k,d,r)$*, but with the additional condition that all blocks of* p *of size two are of the form* $\sqcap_{\circ\bullet}$ *or* $\sqcap_{\bullet\circ}$ *when being rotated to one line. We then have* $\mathcal{B}_{\mathrm{loc}}(k,d) \subseteq \langle\, \overset{\uparrow}{\circ}^{\otimes k},\ \overset{\uparrow}{\circ}^{\otimes d}\sqcap_{\circ\,\overset{\uparrow}{\bullet}^{\otimes d}\,\bullet},\ \overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet}\,\rangle$.

Proof. (a) Denote the set of all partitions $p \in NC^{\circ\bullet}$ with (i), (ii) and (iii) by $\mathcal{E}$. It is easy to see that $\mathcal{E}$ is a category of partitions containing $\overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet}$. So, we only need to prove $p \in \langle\, \overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet}\,\rangle$ for all $p \in \mathcal{E}$. We do so by induction on the number m of blocks of size two of p. If $m = 0$, then p consists of l white singletons and l black singletons, for some $l \in \mathbb{N}$. Hence, it is of the form $(\overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet})^{\otimes l} \in \langle\, \overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet}\,\rangle$ up to permutation of colors (see also Lemma 4.2). If $m = 1$, the partition p is of the form $p = XaYa^{-1}$ up to rotation, where X and Y are tensor products of singletons and a and a^{-1} form a pair block $\sqcap_{\circ\bullet}$ or $\sqcap_{\bullet\circ}$. By assumption, the number of white singletons in Y and the number of black singletons coincide, hence $Y \in \langle\, \overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet}\,\rangle$ by case $m = 0$. Likewise $X \in \langle\, \overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet}\,\rangle$, by the assumption on the global color distribution of the singletons. We infer $p \in \langle\, \overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet}\,\rangle$.

If $m > 1$, we can write $p = p_1 \otimes p_2$ in nest decomposed form up to rotation, where p_2 consists of one pair block and some singletons. Then $p_2 \in \langle\, \overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet}\,\rangle$ by case $m = 1$ and $p_1 \in \langle\, \overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet}\,\rangle$ by the induction hypothesis. Thus $p = p_1 \otimes p_2 \in \langle\, \overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet}\,\rangle$.

(b) Let $p \in \mathcal{B'}_{\mathrm{loc}}(k,d,r)$. Denote the category $\langle\, \overset{\uparrow}{\circ}^{\otimes k},\ \overset{\uparrow}{\circ}^{\otimes d}\sqcap_{\circ\,\overset{\uparrow}{\bullet}^{\otimes d}\,\bullet},\ \overset{\uparrow}{\circ}^{\otimes r+1}\sqcap_{\bullet\,\overset{\uparrow}{\bullet}^{\otimes r-1}\,\bullet},\ \overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet}\,\rangle$ by $\mathcal{D}$. We prove $p \in \mathcal{D}$ by induction on the number m of blocks of p of size two.

Case 1. Let $m = 0$. Up to rotation and permutation of the colors, p is of the form $p = \overset{\uparrow}{\circ}^{\otimes ks} \otimes (\overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet})^{\otimes w}$ for some $w \geq 0$ and $c(p) = ks \in k\mathbb{Z}$. Hence $p \in \langle\, \overset{\uparrow}{\circ}^{\otimes k},\ \overset{\uparrow}{\circ}\otimes\overset{\uparrow}{\bullet}\,\rangle \subseteq \mathcal{D}$.

Case 2. Let $m = 1$. Up to rotation, p is of the form $p = p_1 \otimes a^{\varepsilon_1} p_2^0 a^{\varepsilon_2}$ where a^{ε_1} and a^{ε_2} form a pair block, and p_1 and p_2^0 consist only of singletons respectively.

Case 2a. If the pair on a^{ε_1} and a^{ε_2} is of the form $\sqcap_{\circ\bullet}$ or $\sqcap_{\bullet\circ}$, then $c(p_1) \in d\mathbb{Z}$. Consider $p_1' := p_1 \otimes \overset{\uparrow}{\circ}^{-c(p_1)}$. Then $p_1' \in \mathcal{D}$ by Case 1 since $c(p_1') = 0$. Furthermore, $p_2^{0'} := \overset{\uparrow}{\circ}^{\otimes c(p_1)} \otimes p_2^0$ is in $\mathcal{D}$, again by Case 1 because $c(p_2^{0'}) = c(p_1) + c(p_2^0) = c(p) \in k\mathbb{Z}$. Therefore, the partition $p_1 \otimes \overset{\uparrow}{\circ}^{-c(p_1)} \otimes a^{\varepsilon_1} \overset{\uparrow}{\circ}^{\otimes c(p_1)} p_2^0 a^{\varepsilon_2}$ is in $\mathcal{D}$. Since $c(p_1) \in d\mathbb{Z}$, we use the partition $\overset{\uparrow}{\circ}^{\otimes d}\sqcap_{\circ\,\overset{\uparrow}{\bullet}^{\otimes d}\,\bullet} \in \mathcal{D}$ to shift $c(p_1)$ singletons from inside the pair to the outside. Thus, $p_1 \otimes \overset{\uparrow}{\circ}^{-c(p_1)} \otimes \overset{\uparrow}{\circ}^{\otimes c(p_1)} \otimes a^{\varepsilon_1} p_2^0 a^{\varepsilon_2} \in \mathcal{D}$ from which we infer $p \in \mathcal{D}$ using the pair partitions.

Case 2b. If the pair on a^{ε_1} and a^{ε_2} is of the form $\sqcap_{\bullet\bullet}$, then $c(p_1) \in d\mathbb{Z} + (r+1)$. Assume $c(p_1) = ds + r + 1$ for some $s \geq 0$. Let p_1' be the partition obtained from p_1 by removing $c(p_1)$ white singletons. Then $p_1' \in \mathcal{D}$ by Case 1 since $c(p_1') = 0$. Furthermore, let p_2' be obtained from $p_2^0 \otimes \overset{\uparrow}{\circ}^{\otimes ds}$ by adding $r - 1$ white singletons. Then $p_2' \in \mathcal{D}$ since $c(p_2') = c(p_2^0) + ds + (r-1) = c(p) \in k\mathbb{Z}$. Finally, consider the partition $p_1' \otimes \overset{\uparrow}{\circ}^{\otimes r+1}\sqcap_{\bullet\,\overset{\uparrow}{\bullet}^{\otimes r-1}\,\bullet}$ composed with p_2' in such a way that p_2' is placed between the legs of the pair $\sqcap_{\bullet\bullet}$. The resulting partition is in $\mathcal{D}$ and using $\overset{\uparrow}{\circ}^{\otimes d}\sqcap_{\circ\,\overset{\uparrow}{\bullet}^{\otimes d}\,\bullet}$ we can shift ds white singletons from inside the pair to the outside. Up to permutation of colors of the singletons and using the pair partitions, this yields p which is hence in $\mathcal{D}$. We proceed in a similar way for $s < 0$ and likewise in the case that a^{ε_1} and a^{ε_2} form a pair $\sqcap_{\circ\circ}$.

Case 3. Let $m > 1$. Up to rotation, p is in nest decomposed form $p = p_1 \otimes p_2$ such that p_2

contains only one block of size two. Then, $p_1' := p_1 \otimes \uparrow_{\circ}^{\otimes c(p_2)}$ is in $\mathcal{D}$ by induction hypothesis, since $c(p_1') = c(p)$. Likewise $p_2' := \uparrow_{\circ}^{\otimes -c(p_2)} \otimes p_2$ is in $\mathcal{D}$ by Case 2. Hence $p_1 \otimes \uparrow_{\circ}^{\otimes c(p_2)} \otimes \uparrow_{\circ}^{\otimes -c(p_2)} \otimes p_2 \in \mathcal{D}$ from which we deduce $p \in \mathcal{D}$.

(c) Note that in the proof of (b) we used the partition $\uparrow_{\circ}^{\otimes r+1}\,\sqcap_{\bullet\bullet}(\uparrow_{\bullet}^{\otimes r-1})$ only when blocks $\sqcap_{\circ\circ}$ or $\sqcap_{\bullet\bullet}$ where involved. Hence, $p \in \langle \uparrow_{\circ}^{\otimes k}, \uparrow_{\circ}^{\otimes d}\,\sqcap_{\circ\bullet}(\uparrow_{\bullet}^{\otimes d}), \uparrow_{\circ} \otimes \uparrow_{\bullet} \rangle$ if all blocks of p of size two are of the form $\sqcap_{\circ\bullet}$ or $\sqcap_{\bullet\circ}$. □

4.6.4 Classification in the case $\mathcal{B}$

Theorem 4.38. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a category of noncrossing partitions in case $\mathcal{B}$. Then $\mathcal{C}$ coincides with one of the following categories.*

(a) If $\mathcal{C}$ is globally colorized and

... if $d(\mathcal{C}) = 2$, then $\mathcal{C} = \mathcal{B}_{\mathrm{glob}}(k) := \langle \uparrow_{\circ}^{\otimes k}, \uparrow_{\circ} \otimes \uparrow_{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$ for $k = k(\mathcal{C}) \in 2\mathbb{N}_0$,

... if $d(\mathcal{C}) = 1$, then $\mathcal{C} = \mathcal{B}'_{\mathrm{glob}}(k) := \langle \uparrow_{\circ}^{\otimes k}, \uparrow_{\circ}\,\sqcap_{\circ\bullet}(\uparrow_{\bullet}), \uparrow_{\circ} \otimes \uparrow_{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$ for $k = k(\mathcal{C}) \in \mathbb{N}_0$.

(b) If $\mathcal{C}$ is locally colorized and

... if $K^{\mathcal{C}}(\lceil_{\bullet}\ \rceil_{\bullet}) = \emptyset$, then $\mathcal{C} = \langle \uparrow_{\circ}^{\otimes k}, \uparrow_{\circ}^{\otimes d}\,\sqcap_{\circ\bullet}(\uparrow_{\bullet}^{\otimes d}), \uparrow_{\circ} \otimes \uparrow_{\bullet} \rangle$ for $k = k(\mathcal{C})$ and $d = d(\mathcal{C}) \in \mathbb{N}_0$,

... if $K^{\mathcal{C}}(\lceil_{\bullet}\ \rceil_{\bullet}) \neq \emptyset$, then $\mathcal{C} = \langle \uparrow_{\circ}^{\otimes k}, \uparrow_{\circ}^{\otimes d}\,\sqcap_{\circ\bullet}(\uparrow_{\bullet}^{\otimes d}), \uparrow_{\circ}^{\otimes r+1}\,\sqcap_{\bullet\bullet}(\uparrow_{\bullet}^{\otimes r-1}), \uparrow_{\circ} \otimes \uparrow_{\bullet} \rangle$ for $k = k(\mathcal{C}) \in \mathbb{N}_0 \backslash \{1\}$, $d = d(\mathcal{C}) \in \mathbb{N}_0 \backslash \{1\}$ and $r = r(\mathcal{C}) = \frac{d}{2} \neq 1$ or $r(\mathcal{C}) = 0$.

Proof. (a) If $k(\mathcal{C}) \neq 0$, then $\uparrow_{\circ}^{\otimes k} \in \mathcal{C}$ by Lemma 4.36. Hence $\langle \uparrow_{\circ}^{\otimes k}, \uparrow_{\circ} \otimes \uparrow_{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle \subseteq \mathcal{C}$.

Case 1. Let $d(\mathcal{C}) = 2$. Let $p \in \mathcal{C}$ be a partition without upper points such that $c(p) \geq 0$. Then $c(p) = ks$ for some $s \in \mathbb{N}_0$ by Proposition 4.10. The number of points between two legs of a pair of p is even, because $c(p_1) \in 2\mathbb{Z}$ for all $p_1 \otimes p_2 \in NDF^{\mathcal{C}}(\lceil_{\circ}\ \rceil_{\bullet})$ by Proposition 4.13. Consider $p' := p \otimes \uparrow_{\bullet}^{\otimes ks}$. Let p'' be the partition obtained from p' by replacing the colors of the points by the alternating color pattern white-black-white-black-etc. Then, all pair blocks are of the form $\sqcap_{\circ\bullet}$ or $\sqcap_{\bullet\circ}$, because there is an even number of points between two legs of a pair. Thus, $p'' \in \langle \uparrow_{\circ} \otimes \uparrow_{\bullet} \rangle \subseteq \langle \uparrow_{\circ}^{\otimes k}, \uparrow_{\circ} \otimes \uparrow_{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$ by Proposition 4.37. Using permutation of colors, we infer $p' \in \mathcal{C}$ since $c(p') = c(p'') = 0$. This implies $p' \otimes \uparrow_{\circ}^{\otimes ks} \in \mathcal{C}$ from which we deduce $p \in \mathcal{C}$ using the pair partitions.

Case 2. If $d(\mathcal{C}) = 1$, we have $\uparrow_{\circ}\,\sqcap_{\circ\bullet}(\uparrow_{\bullet}) \in \mathcal{C}$ by Lemma 4.16. Hence, $\langle \uparrow_{\circ}^{\otimes k}, \uparrow_{\circ}\,\sqcap_{\circ\bullet}(\uparrow_{\bullet}), \uparrow_{\circ} \otimes \uparrow_{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle \subseteq \mathcal{C}$. Conversely, let $p \in \mathcal{C}$. Then $c(p) \in k\mathbb{Z}$ by Proposition 4.10 and thus $p \in \mathcal{B}'_{\mathrm{loc}}(k, 1, 0) \subseteq \langle \uparrow_{\circ}^{\otimes k}, \uparrow_{\circ}\,\sqcap_{\circ\bullet}(\uparrow_{\bullet}), \uparrow_{\circ}\,\sqcap_{\bullet\bullet}(\uparrow_{\circ}), \uparrow_{\circ} \otimes \uparrow_{\bullet} \rangle$ by Proposition 4.37. But this category contains $\sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet}$ since we may shift the singletons in $\uparrow_{\circ}\,\sqcap_{\bullet\bullet}(\uparrow_{\circ})$ arbitrarily, and hence it coincides with $\langle \uparrow_{\circ}^{\otimes k}, \uparrow_{\circ}\,\sqcap_{\circ\bullet}(\uparrow_{\bullet}), \uparrow_{\circ} \otimes \uparrow_{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$.

(b) Let $\mathcal{C}$ be locally colorized. For $k = k(\mathcal{C})$ and $d = d(\mathcal{C})$, we have $\uparrow_{\circ}^{\otimes k} \in \mathcal{C}$ and $\uparrow_{\circ}^{\otimes d}\,\sqcap_{\circ\bullet}(\uparrow_{\bullet}^{\otimes d}) \in \mathcal{C}$ by Lemma 4.36.

Case 1. Let $K^{\mathcal{C}}(\sqcap_{\bullet\bullet}) = \emptyset$. Then $\langle \overset{\uparrow}{\circ}^{\otimes k}, \overset{\uparrow}{\circ}^{\otimes d} \circ \, \overset{\uparrow}{\bullet}^{\otimes d} \bullet, \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\bullet} \rangle \subseteq \mathcal{C}$. Conversely, let $p \in \mathcal{C}$. Then $c(p) \in k\mathbb{Z}$ by Proposition 4.10 and $c(p_1) \in d\mathbb{Z}$ for all $p_1 \otimes p_2$ in nest decomposed form by Proposition 4.13. Furthermore, all blocks of size two are of the form $\sqcap_{\circ\bullet}$ or $\sqcap_{\bullet\circ}$ when being rotated to one line, since $K^{\mathcal{C}}(\sqcap_{\bullet\bullet}) = K^{\mathcal{C}}(\sqcap_{\circ\circ}) = \emptyset$. Thus $p \in \mathcal{B}_{\mathrm{loc}}(k,d)$ which implies $p \in \langle \overset{\uparrow}{\circ}^{\otimes k}, \overset{\uparrow}{\circ}^{\otimes d} \circ \, \overset{\uparrow}{\bullet}^{\otimes d} \bullet, \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\bullet} \rangle$ by Proposition 4.37.

Case 2. Let $K^{\mathcal{C}}(\sqcap_{\bullet\bullet}) \neq \emptyset$. Then $\langle \overset{\uparrow}{\circ}^{\otimes k}, \overset{\uparrow}{\circ}^{\otimes d} \circ \, \overset{\uparrow}{\bullet}^{\otimes d} \bullet, \overset{\uparrow}{\circ}^{\otimes r+1} \bullet \, \overset{\uparrow}{\bullet}^{\otimes r-1} \bullet, \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\bullet} \rangle \subseteq \mathcal{C}$ for r as in Proposition 4.35. Again, we use Proposition 4.37 to finish the proof. □

Corollary 4.39. *We have* $\mathcal{B}'_{\mathrm{loc}}(k,d,r) = \langle \overset{\uparrow}{\circ}^{\otimes k}, \overset{\uparrow}{\circ}^{\otimes d} \circ \, \overset{\uparrow}{\bullet}^{\otimes d} \bullet, \overset{\uparrow}{\circ}^{\otimes r+1} \bullet \, \overset{\uparrow}{\bullet}^{\otimes r-1} \bullet, \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\bullet} \rangle$ *and* $\mathcal{B}_{\mathrm{loc}}(k,d) = \langle \overset{\uparrow}{\circ}^{\otimes k}, \overset{\uparrow}{\circ}^{\otimes d} \circ \, \overset{\uparrow}{\bullet}^{\otimes d} \bullet, \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\bullet} \rangle$ *in Proposition 4.37. In particular, these categories are pairwise different.*

Remark 4.40. *(a) If* $r \neq 0$*, then* $\overset{\uparrow}{\circ}^{\otimes r+1} \bullet \, \overset{\uparrow}{\bullet}^{\otimes r-1} \bullet \in \mathcal{C}$ *implies* $\overset{\uparrow}{\circ}^{\otimes d} \circ \, \overset{\uparrow}{\bullet}^{\otimes d} \bullet \in \mathcal{C}$*, see the proof of Proposition 4.35.*

(b) The non-colored case $\langle \uparrow \otimes \uparrow \rangle$ *is obtained from* $\langle \overset{\uparrow}{\circ}^{\otimes k}, \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$ *for* $k = 2$*, whereas* $\langle \uparrow \rangle$ *is given by* $k = 1$*. The category* $\langle \sqcap\!\!\sqcap \rangle$ *in turn coincides with* $\langle \overset{\uparrow}{\circ}^{\otimes k}, \overset{\uparrow}{\circ} \circ \, \overset{\uparrow}{\bullet} \bullet, \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$ *for the case* $k = 2$ *(see Proposition 4.3).*

4.7 Main result: Summary of the noncrossing case

We finally classified all categories $\mathcal{C} \subseteq NC^{\circ\bullet}$ of noncrossing (two-colored) partitions. This constitutes the main result of our chapter. Here is an overview on the results split into the globally colorized case and the locally colorized case. For the convenience of the reader we recall that the definition of a category of partitions may be found in Section 1.1.2, the cases $\mathcal{O}, \mathcal{H}, \mathcal{S}$ and $\mathcal{B}$ are defined in Definition 4.5, globally and locally colorization is given in Definition 4.6, the partition b_k is defined in Definition 4.14 whereas the operation $p \mapsto \tilde{p}$ is the map giving the same partition with inversion of colors, as defined in Section 1.1.2, and the classification theorems are Theorems 4.21, 4.26, 4.32, and 4.38.

Theorem 4.41. *Let* $\mathcal{C} \subseteq NC^{\circ\bullet}$ *be a* globally colorized *category of noncrossing partitions. Then it coincides with one of the following categories.*

Case $\mathcal{O}$*:* $\mathcal{O}_{\mathrm{glob}}(k) = \langle \sqcap_{\circ\circ}^{\otimes \frac{k}{2}}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$ *for* $k \in 2\mathbb{N}_0$

Case $\mathcal{H}$*:* $\mathcal{H}_{\mathrm{glob}}(k) = \langle b_k, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$ *for* $k \in 2\mathbb{N}_0$

Case $\mathcal{S}$*:* $\mathcal{S}_{\mathrm{glob}}(k) = \langle \overset{\uparrow}{\circ}^{\otimes k}, \sqcap\!\sqcap\!\sqcap_{\circ\bullet\circ\bullet}, \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$ *for* $k \in \mathbb{N}_0$

Case $\mathcal{B}$*:* $\mathcal{B}_{\mathrm{glob}}(k) = \langle \overset{\uparrow}{\circ}^{\otimes k}, \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$ *for* $k \in 2\mathbb{N}_0$

or $\mathcal{B}'_{\mathrm{glob}}(k) = \langle \overset{\uparrow}{\circ}^{\otimes k}, \overset{\uparrow}{\circ} \circ \, \overset{\uparrow}{\bullet} \bullet, \overset{\uparrow}{\circ} \otimes \overset{\uparrow}{\bullet}, \sqcap_{\circ\circ} \otimes \sqcap_{\bullet\bullet} \rangle$ *for* $k \in \mathbb{N}_0$

Theorem 4.42. *Let $\mathcal{C} \subseteq NC^{\circ\bullet}$ be a* locally colorized *category of noncrossing partitions. Then it is of the following form:*

Case $\mathcal{O}$: $\mathcal{O}_{\mathrm{loc}} = \langle \emptyset \rangle$

Case $\mathcal{H}$: $\mathcal{H}'_{\mathrm{loc}} = \langle \ \rangle$

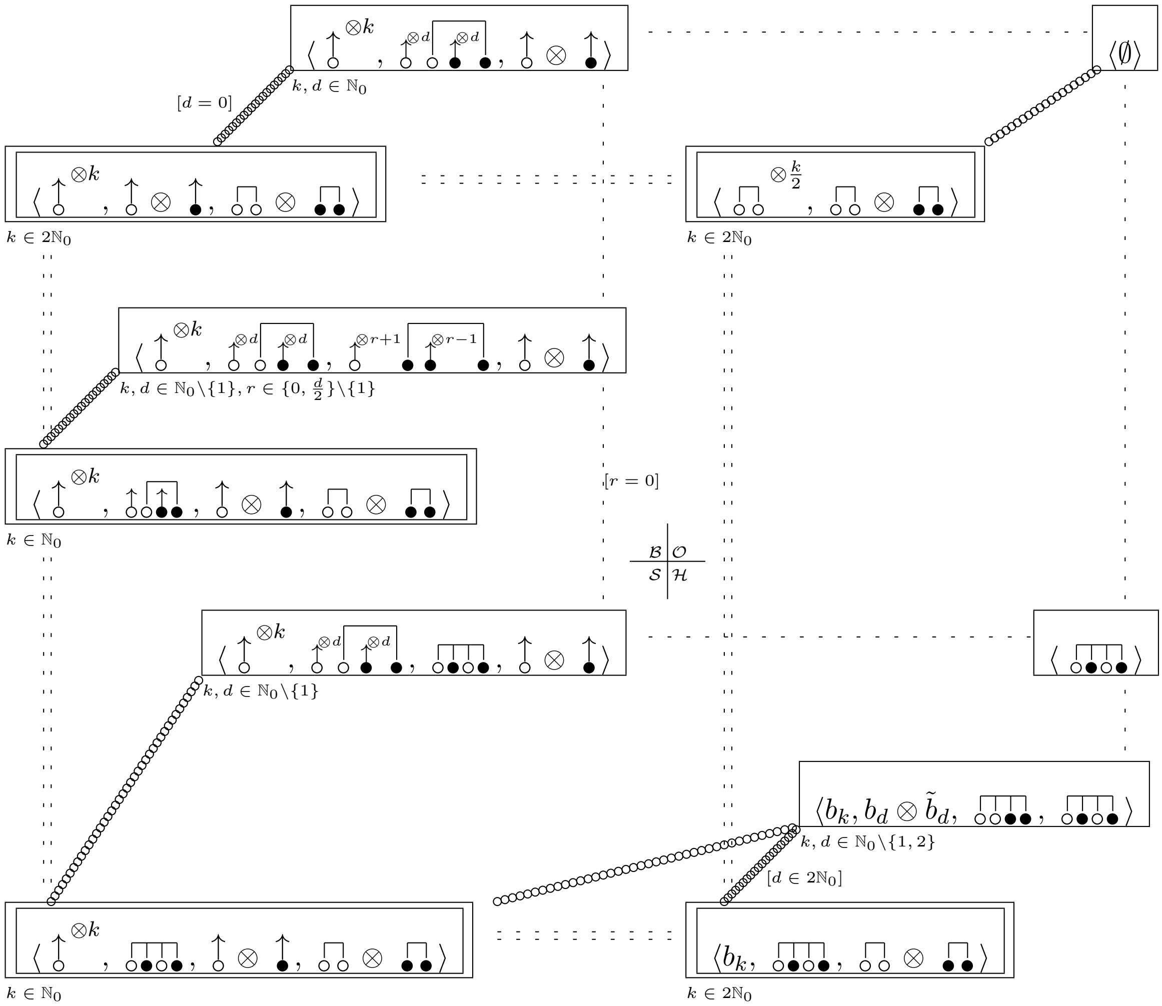
or $\mathcal{H}_{\mathrm{loc}}(k,d) = \langle b_k, b_d \otimes \tilde{b}_d, \ , \ \rangle$ *for* $k, d \in \mathbb{N}_0 \backslash \{1,2\}$, $d|k$

Case $\mathcal{S}$: $\mathcal{S}_{\mathrm{loc}}(k,d) = \langle \ ^{\otimes k}, \ ^{\otimes d} \ ^{\otimes d}, \ , \ \otimes \ \rangle$ *for* $k, d \in \mathbb{N}_0 \backslash \{1\}$, $d|k$

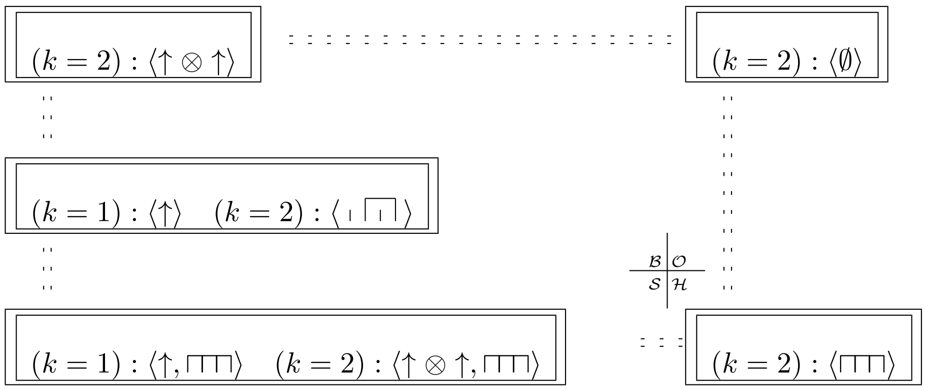
Case $\mathcal{B}$: $\mathcal{B}_{\mathrm{loc}}(k,d) = \langle \ ^{\otimes k}, \ ^{\otimes d} \ ^{\otimes d}, \ \otimes \ \rangle$ *for* $k, d \in \mathbb{N}_0$, $d|k$

or $\mathcal{B}'_{\mathrm{loc}}(k,d,r) = \langle \ ^{\otimes k}, \ ^{\otimes d} \ ^{\otimes d}, \ ^{\otimes r+1} \ ^{\otimes r-1}, \ \otimes \ \rangle$ *for* $k, d \in \mathbb{N}_0 \backslash \{1\}$, $r \in \{0, \frac{d}{2}\} \backslash \{1\}$, $d|k$

Here is a graphical overview of all categories of two-colored noncrossing partitions. The single framed categories are the locally colorized ones whose inclusions are indicated by single dashed lines (inclusions from top to bottom and from right to left, for fixed parameters k and d). Constraints for inclusions are marked in brackets. The double framed categories are the globally colorized ones with inclusion pattern according to the double dahed lines. The locally colorized categories are contained in the globally colorized ones according to the diagonal chain lines. In our graphic, we also included a cross marking the areas of the cases $\mathcal{B}$, $\mathcal{O}$, $\mathcal{S}$ and $\mathcal{H}$.



We also give the corresponding graphic in the orthogonal case ($\sqcap$ $\in \mathcal{C}$), for comparison.



Remark 4.43. *The constraints on the parameters k, d and r in the above theorems can be understood by the fact that we have the following equalities.*

- $\mathcal{H}_{\mathrm{glob}}(k) = \mathcal{H}_{\mathrm{loc}}(k, 2)$ *and* $\mathcal{H}_{\mathrm{glob}}(2m+1) = \mathcal{S}_{\mathrm{glob}}(2m+1)$

- $\mathcal{S}_{\mathrm{glob}}(k) = \mathcal{S}_{\mathrm{loc}}(k,1) = \mathcal{H}_{\mathrm{loc}}(k,1)$
- $\mathcal{B}_{\mathrm{glob}}(k) = \mathcal{B}'_{\mathrm{loc}}(k,2m,1)$ *and* $\mathcal{B}_{\mathrm{glob}}(2m+1) = \mathcal{B}'_{\mathrm{glob}}(2m+1)$
- $\mathcal{B}'_{\mathrm{glob}}(k) = \mathcal{B}'_{\mathrm{loc}}(k,1,0) = \mathcal{B}'_{\mathrm{loc}}(k,2m+1,1)$

4.8 C^*-algebraic relations associated to partitions

We can associate C^*-algebras to categories of partitions by associating relations to partitions. This is the main step in the direction to defining unitary easy quantum groups.

Definition 4.44. *Let $p \in P^{\circ\bullet}(k,l)$ and let $\alpha = (\alpha_1,\dots,\alpha_k)$ and $\beta = (\beta_1,\dots,\beta_l)$ be multi indices. We decorate the upper points of p with α and the lower ones with β. If now for every block of p all of the corresponding indices coincide, we put $\delta_p(\alpha,\beta) := 1$; otherwise $\delta_p(\alpha,\beta) := 0$.*

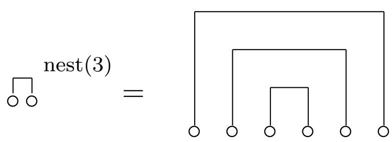
Definition 4.45. *Let $n \in \mathbb{N}$ and let A be a C^*-algebra generated by n^2 elements u_{ij}, $1 \leq i,j \leq n$. Let $p \in P^{\circ\bullet}(k,l)$ be a partition and let $r = (r_1,\dots,r_k) \in \{\circ,\bullet\}^k$ be its upper color pattern and $s = (s_1,\dots,s_l) \in \{\circ,\bullet\}^l$ be its lower color pattern. We put $u_{ij}^{\circ} := u_{ij}$ and $u_{ij}^{\bullet} := u_{ij}^*$.*

We say that the generators u_{ij} fulfill the relations $R(p)$, *if for all $\beta_1,\dots,\beta_l \in \{1,\dots,n\}$ and for all $i_1,\dots,i_k \in \{1,\dots,n\}$, we have:*

$$\sum_{\alpha_1,\dots,\alpha_k=1}^{n} \delta_p(\alpha,\beta) u_{\alpha_1 i_1}^{r_1} \dots u_{\alpha_k i_k}^{r_k} = \sum_{\gamma_1,\dots,\gamma_l=1}^{n} \delta_p(i,\gamma) u_{\beta_1\gamma_1}^{s_1} \dots u_{\beta_l\gamma_l}^{s_l}$$

The left-hand side of the equation is $\delta_p(0,\beta)$ if $k=0$ and analogous for the right-hand side.

Using this definition, we can give a list of relations associated to partitions that appeared throughout the classification of categories of noncrossing partitions. We denote by u the matrix $u = (u_{ij})_{1\leq i,j\leq n}$, and $\bar{u} = (u_{ij}^*)$. Furthermore, $\mathrm{rot}_t(p) \in P^{\circ\bullet}(t,k)$ denotes the partitions obtained from $p \in P^{\circ\bullet}(0,k+t)$ by rotating the last t points to the upper line. If we simply write $\mathrm{rot}(p)$, we do not specify which of the points are rotated. It is often more convenient to consider the relations of a partition in some rotated form rather than of the partition itself. By $\sqcap_{\circ\circ}^{\mathrm{nest}(k)}$ we denote the partition obtained from nesting the partition $\sqcap_{\circ\circ}$ k-times into itself, i.e.:



Note that $\sqcap_{\circ\circ}^{\otimes k} \in \mathcal{C}$ if and only if $\sqcap_{\circ\circ}^{\mathrm{nest}(k)} \in \mathcal{C}$, since $\sqcap_{\circ\bullet}^{\otimes k}$ and $\sqcap_{\circ\bullet}^{\mathrm{nest}(k)}$ are in any category (then use Lemma 4.2(a)). The next relations can directly be derived from Definition 4.45.

$$R(\sqcap_{\circ\bullet}) : \sum_k u_{ik}u_{jk}^* = \delta_{ij}, \text{ i.e. } uu^* = 1$$

$$R(\sqcap_{\bullet\circ}) : \sum_k u_{ik}^*u_{jk} = \delta_{ij}, \text{ i.e. } u^*u = 1$$

$$R(\ ) : \sum_k u_{ki}u_{kj}^* = \delta_{ij}, \text{ i.e. } (\bar{u})^*\bar{u} = 1$$

$$R(\ ) : \sum_k u_{ki}^*u_{kj} = \delta_{ij}, \text{ i.e. } \bar{u}(\bar{u})^* = 1$$

$$R(\ ) = R(\ ) : u_{ij}u_{kl} = u_{kl}u_{ij}$$

$$R(\ ) = R(\ ) : u_{ij}u_{kl}^* = u_{kl}^*u_{ij}$$

$$R(\ ) = R(\ ) = R(\mathrm{rot}_1(\ )) : u_{ij} = u_{ij}^*, \text{ i.e. } u = \bar{u}$$

$$R(\mathrm{rot}_k(\ ^{\mathrm{nest}(k)})) : u_{i_1j_1}\dots u_{i_kj_k} = u_{i_1j_1}^*\dots u_{i_kj_k}^*$$

$$R(\ \otimes\ ) = R(\mathrm{rot}(\ \otimes\ )) : u_{ij}^*u_{kl} = u_{ij}u_{kl}^*$$

$$R(\mathrm{rot}_1(\uparrow\otimes\uparrow)) : \left(\sum_k u_{kj}\right) = \left(\sum_l u_{il}\right)$$

$$R(\mathrm{rot}_t(\uparrow^{\otimes s+t})) : \left(\sum_{k_1} u_{k_1j_1}\right)\dots\left(\sum_{k_s} u_{k_sj_s}\right) = \left(\sum_{l_1} u_{i_1l_1}^*\right)\dots\left(\sum_{l_t} u_{i_tl_t}^*\right)$$

$$R(\uparrow^{\otimes d}) : \left(\sum_{k_1} u_{k_1j_1}\right)\dots\left(\sum_{k_d} u_{k_dj_d}\right) = 1$$

$$R(\ ) = R(\mathrm{rot}_2(\ )) : u_{ki}u_{kj}^* = u_{ik}u_{jk}^* = 0 \textit{ if } i \neq j$$

$$R(\ ) : u_{ki}^*u_{kj} = u_{ik}^*u_{jk} = 0 \textit{ if } i \neq j$$

$$R(\ ) = R(\mathrm{rot}_2(\ )) : u_{ki}u_{kj} = u_{ik}u_{jk} = 0 \textit{ if } i \neq j$$

$$R(\mathrm{rot}_d(b_d\otimes\tilde{b}_d)) : \sum_k \delta_{i_1=\dots=i_d}u_{kj_1}\dots u_{kj_d} = \sum_l \delta_{j_1=\dots=j_d}u_{i_1l}\dots u_{i_dl}$$

$$R(b_k) : \sum_l u_{i_1l}\dots u_{i_kl} = \delta_{i_1=\dots=i_k}$$

$$R(\mathrm{rot}_t(b_{s+t})) : \delta_{i_1=\dots=i_s}u_{i_1j_1}\dots u_{i_1j_s} = \delta_{j_1=\dots=j_t}u_{i_1j_1}^*\dots u_{i_tj_1}^*$$

$$R(\mathrm{rot}_{d+1}(\uparrow^{\otimes d}\ \uparrow^{\otimes d})) : u_{ij}\left(\sum_{k_1} u_{k_1j_1}\right)\dots\left(\sum_{k_d} u_{k_dj_d}\right) = \left(\sum_{l_1} u_{i_1l_1}\right)\dots\left(\sum_{l_d} u_{i_dl_d}\right)u_{ij}$$

$$R(\mathrm{rot}_r\ \uparrow^{\otimes r+1}\ \uparrow^{\otimes r-1})) : u_{ij}\left(\sum_{k_1} u_{k_1j_1}\right)\dots\left(\sum_{k_{r-1}} u_{k_{r-1}j_{r-1}}\right) = \left(\sum_{l_1} u_{i_1l_1}\right)\dots\left(\sum_{l_{r+1}} u_{i_{r+1}l_{r+1}}\right)u_{ij}^*$$

4.9 Free unitary easy group

4.9.1 Definition of C_n^+

In the sequel, the following quantum group will play an important role. It is a kind of a non-orthogonal version of B_n^+. It has been introduced in an unpublished paper of Banica, Curran and Speicher (see [83]).

Definition 4.46. *Let C_n^+ be the quantum group given by the universal C^*-algebra generated by u_{ij} such that u and $\bar{u}$ are unitaries and $\sum_k u_{ik} = \sum_k u_{kj} = 1$ for all i, j.*

Again, it can be read off directly from the relations in Section 4.8 that C_n^+ is free easy with category $\mathcal{B}_{\text{loc}}(1, 0)$.

4.9.2 Free and tensor complexifications with $\mathbb{Z}_d$

In [95], Wang proved the existence of a comultiplication on the free product as well as on the tensor product of the C^*-algebras associated to quantum groups. More precisely, let G and H be two compact (matrix) quantum groups with comultiplications Δ_G resp. Δ_H. Let $C(G)\Box C(H)$ either be the untail free product $C(G) * C(H)$ of the two C^*-algebras or the maximal tensor product $C(G)\otimes_{\max}C(H)$. Denote by $\iota_{C(G)}$ the embedding of $C(G)$ into $C(G)\Box C(H)$ and likewise by $\iota_{C(G)\Box C(G)}$ the embedding of $C(G) \otimes_{\min} C(G)$ into $(C(G)\Box C(H)) \otimes_{\min} (C(G)\Box C(H))$.

Proposition 4.47. *Given two compact (matrix) quantum groups G and H, there is always a comultiplication Δ on $C(G)\Box C(H)$ for $\Box \in \{*, \otimes_{\max}\}$ such that:*

$$\Delta \circ \iota_{C(G)} = \iota_{C(G)\Box C(G)} \circ \Delta_G \qquad \text{and} \qquad \Delta \circ \iota_{C(H)} = \iota_{C(H)\Box C(H)} \circ \Delta_H$$

As a consequence, one can define the free product and the direct product of compact matrix quantum groups. The fundamental corepresentation is then given by the direct sum of these representations, thus by $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, where u and v are the matrices of generators for G resp. H. We now define another kind of free resp. tensor product of two compact matrix quantum groups. Recall that for unital C^*-algebras A and B, the maximal tensor product $A \otimes_{\max} B$ can be seen as the universal C^*-algebra generated by elements $a \in A$ (with the relations of A) and $b \in B$ (with the relations of B) such that all such a and b commute. We thus simply write ab for elements $a \otimes b$.

Definition 4.48. *Let (G, u) and (H, v) be two compact matrix quantum groups with u of size n and v of size m.*

(a) The glued free product $G\tilde{*}H$ of G and H *is given by the C^*-subalgebra $C^*(u_{ij}v_{kl}, 1 \leq i, j \leq n, 1 \leq k, l \leq m) \subseteq C(G) * C(H)$.*

(b) The glued direct product $G\tilde{\times}H$ of G and H *is given by the C^*-subalgebra $C^*(u_{ij}v_{kl}, 1 \leq i, j \leq n, 1 \leq k, l \leq m) \subseteq C(G) \otimes_{\max} C(H)$.*

As a simple consequence of Wang's result, the glued free product and the glued direct product are again compact matrix quantum groups.

Corollary 4.49. *The C^*-subalgebra $C^*(u_{ij}v_{kl}, 1 \leq i, j \leq n, 1 \leq k, l \leq m)$ of $C(G)\Box C(H)$, $\Box \in \{*, \otimes_{\max}\}$ admits a comultiplication $\Delta(u_{ij}v_{kl}) = \Delta_G(u_{ij})\Delta_H(v_{kl})$.*

Proof. Restriction of the comultiplication Δ of Proposition 4.47 yields the result. $\Box$

To a discrete group Γ, we associate the universal C^*-algebra $C^*(\Gamma)$ generated by unitaries u_g, $g \in \Gamma$ with $u_g u_h = u_{gh}$, $u_g^* = u_{g^{-1}}$. It is well known that the comultiplication $\Delta(u_g) = u_g \otimes u_g$ turns it into a compact quantum group denoted by $\widehat{\Gamma}$.

Corollary 4.50. *Let Γ be a discrete group generated by a single element g_0, and denote by z the generator u_{g_0} of $C^*(\Gamma)$. Let (G, u) be a compact matrix quantum group. Then $G\tilde{*}\widehat{\Gamma}$ and $G\tilde{\times}\widehat{\Gamma}$ are compact matrix quantum groups given by $C^*(u_{ij}z)$ in $C(G) * C^*(\Gamma)$ resp. in $C(G) \otimes_{\max} C^*(\Gamma)$ and $\Delta(u_{ij}z) = \sum_k u_{ik}z \otimes u_{kj}z$.*

Proof. Since Γ is generated by a single element, $(C^*(\Gamma), z)$ is a compact matrix quantum group of size 1. Using Corollary 4.49 we obtain the result. □

As before, denote by $\mathbb{Z}_d$ the cyclic group $\mathbb{Z}_d := \mathbb{Z}/d\mathbb{Z}$.

Definition 4.51. *Let G be a compact matrix quantum group.*

(a) The quantum group $G\tilde{*}\widehat{\mathbb{Z}_d}$ is called the free d-complexification of G *and $G\tilde{*}\widehat{\mathbb{Z}}$ is called the* free complexification.

(b) The quantum group $G\tilde{\times}\widehat{\mathbb{Z}_d}$ is called the tensor d-complexification of G *and $G\tilde{\times}\widehat{\mathbb{Z}}$ is called the* tensor complexification.

The above definition is a generalization of Banica's free complexification [8].

---

[8]: $H_N^{\#}$, the free complexified of H_N^+, and $S_N^{\#}$, the free complexified of S_N^+. These two groups are free quantum groups corresponding respectively to the categories $\mathcal{H}^{\#}$ and $\mathcal{S}^{0,0}$.

Remark 4.52. *The commutative complexification shouldn't be confused with the traditional complexification of Lie groups. In our case it is just the commutative counterpart of the free complexification introduced by Banica in[8]. This commutative complexification also appeared in the unpublished paper of Banica, Curran and Speicher [83]. Note that for a classical group G, the commutative $d-$complexification is just the product of G with $\mathbb{Z}_d$.*

The description of the free quantum groups is done in two steps. We first deal with the local parameters, and then with the global parameter.

4.9.3 Local colorization

In order to achieve the description of all free quantum groups, we have to describe the meaning of the local parameters d, r for the cases $\mathcal{S}$, $\mathcal{B}$ and $\mathcal{H}$. In the case $\mathcal{H}$, this meaning has been already interpreted in [16]. The free complexification gives an interpretation in the cases $\mathcal{S}$, and $\mathcal{B}$ for $r = 0$.

Proposition 4.53. *The following correspondance holds between quantum groups and categories of partitions :*

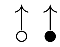
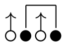
- *$\mathcal{S}^{d,d}$ is the category of partitions corresponding to $S^{+,+d}$, the free $d-$complexification of S^+.*
- *$\mathcal{B}^{d,d}$ is the category of partitions corresponding to $C^{+,+d}$*

- $\mathcal{B}^{d,d,0}$ *is the category of partitions corresponding to* $B^{+,+d}$

Proof. For these three families, the proof follows exactly the same pattern. We sketch the proof and detail it for the case $\mathcal{S}$, the latter being the one needing more arguments.

1. Let $n \geq 1$. We indentify the category of partition $\mathcal{S}^{d,d}$ with its image as linear maps on tensor products of $\mathbb{C}^n$ through the construction of Section 1.1.3. Denote by G_n the quantum group of dimension n corresponding to the category of partition $\mathcal{S}^{d,d}$, $\mathcal{T}$ the Woronowicz tensor category associated to the quantum group $S_n^{+,+d}$. The goal is to prove that $G_n = S_n^{+,+d}$.

2. In order to prove that $G_n = S_n^{d,+d}$, it suffices to prove that on one hand $\mathcal{S}^{d,d} \subseteq \mathcal{T}$ (yielding $S_n^{+,+d} \subseteq G_n$), and that on the other hand there exist a C^*- morphism sending the fundamental matrix of $S_n^{+,+d}$ on the one of G_n. The first step is just a verification that is left to the reader.

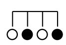
3. Write $G_n = (v_{ij})_{1 \leq i,j \leq n}$, $S_n^{+,+d} = (u_{ij})_{1 \leq i,j \leq n}$. We shall construct a surjective C^*-homorphism $\Phi : C(S_n^{+,+d}) \to C(G_n)$ sending $u_{ij}z$ on v_{ij}.
Recall that $C(S_n^{+,+d})$ is the C^*-subalgebra of $C(S_n^+) * \mathbb{C}\mathbb{Z}_d$ generated by $\{u_{ij}z\}_{1 \leq i,j \leq n}$. Since $\sum_j v_{ij}$ is independent of i (thank to the partition ), and $(\sum_j v_{ij})^d = 1$, there is a map from $C(\mathbb{Z}_d)$ to $C(G_n)$ sending z to $\sum_i v_{ij}$. Set $\tilde{s}_{ij} = v_{ij}(\sum_i v_{ij})^{-1}$. The partition  implies that $(\sum_i v_{ij})^{-1} v_{ij} = v_{ij}^*(\sum_i v_{ij})$, and thus

$$\begin{aligned}\tilde{s}_{ij}^* = (\sum_i v_{ij})v_{ij}^* =& (\sum_i v_{ij})v_{ij}^*(\sum_i v_{ij})(\sum_i v_{ij})^{-1}\\ =& v_{ij}(\sum_i v_{ij})^{-1} = \tilde{s}_{ij}\end{aligned}$$

.

Thus the matrix $(\tilde{s}_{ij})$ is an orthogonal matrix fulfilling the relations

$$\sum \tilde{s}_{ij} = 1$$

The expression of $\tilde{s}_{ij}$ together with the fact that if $k \neq l$, $v_{ik}v_{il}^* = 0$ (implied by the presence of the partition ) yields

$$k \neq j \Rightarrow \tilde{s}_{ij}\tilde{s}_{ik} = 0.$$

Due to these relations, there exists a C^*-homomorphism from $C(S_n^+)$ to $C(G_n)$ sending u_{ij} to $\tilde{s}_{ij}$. By the universality property, there exists a C^*-homorphism Φ from $C(S_n^+) * \mathbb{C}\mathbb{Z}_d$ to $C(G_n)$ sending z to $\sum v_{ij}$ and u_{ij} to $v_{ij}(\sum v_{ij})^{-1}$. This homorphism sends thus $u_{ij}z$ to v_{ij}, which concludes the proof.

The proof for $B^{+,+d}$ is the same except that we don't need to prove the relation $k \neq j \Rightarrow \tilde{s}_{ij}\tilde{s}_{ik} = 0$. The one for $C^{+,+d}$ is the same as the latter except that we don't have to prove the self-adjointness of $\tilde{s}_{ij}$. □

4.9.4 The family $\mathcal{B}^{+,d,d/2-1}$

The free unitary group U_n^+ has been first constructed in [95] as a free 0−complexification of O_n^+. This construction gives also an alternative description of the C^*−algebra underlying C^+. Let $O_n^+ = (o_{ij})_{1\leq i,j\leq n}$ be the orthogonal group of dimension n. The following alternative description of C^+ holds:

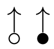
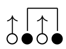
Proposition 4.54. *Let $C_n^+ = (u_{ij})_{1\leq i,j\leq n}$. Let $\mathbb{C}Z_d$ be the group algebra of $\mathbb{Z}/d\mathbb{Z}$ generated by the element z, with $d \in 2\mathbb{N}$ non zero. Then we can write $u_{ij} = \tilde{o}_{ij} z^{d/2}$ with :*

- *$\tilde{o}_{ij}$ is the image of o_{ij} through the projection on $C(O_n^+) * \mathbb{C}Z_d \diagup \langle \sum_j o_{ij} = z^{d/2}\rangle$*
- *$C(C^+)$ is defined as the C^*−subalgebra of $C(O^+) * \mathbb{C}Z_d \diagup \langle \sum_j o_{ij} = z^{d/2}\rangle$ generated by the elements $\tilde{o}_{ij} z^{d/2}$*

Proof. Note first that as a compact quantum group, $C_n^+ \simeq U_{n-1}^+ \oplus 1$: indeed let $F \in \mathcal{U}_n$ sending e_n to $\sum e_i$. Then $(v_{ij})_{1\leq i,j\leq n} = F^*UF$ is again a unitary quantum group. Moreover the condition $\sum u_{ij} = 1$ translates into the condition $v_{in} = v_{nj} = 0$ for all $i, j < n$. Thus C_n^+ is a quantum subgroup of $F(U_{n-1}^+ \oplus 1)F^*$. Since the intertwiners of C_n^+ are also intertwiners of $F(U_{n-1}^+ \oplus 1)F^*$, we deduce that $C_n^+ = F(U_{n-1}^+ \oplus 1)F^*$. Moreover the C^*−algebra defined as $C(O_n^+ - *\mathbb{C}Z_d \diagup \langle \sum_j o_{ij} = z^{-d/2}\rangle$, is isomorph to the C^*−algebra of $O_{n-1}^+ \oplus \varepsilon$ (with $\varepsilon = z^{d/2}$). Indeed the former is exatcly the C^*−algebra of the compact quantum group $B_n^{+\#}$ as described in [97], and $B_n^{+\#}$ has been shown by Raum in [70, Thm 4.1] to be isomorph to $O_{n-1}^+ \oplus \varepsilon$. Since, from a result of Banica, $(o_{ij}\varepsilon)_{1\leq i,j\leq n-1}$ is isomorph to U_{n-1}^+, $\tilde{O}_n^+ \varepsilon \simeq C_n^+$. □

Let us denote $\tilde{C}_n^{+,d}$ the quantum subgroup of $C^{+,+d}$ generated by the matrix $(\tilde{o}_{ij} z^{d/2+1})_{1\leq i,j\leq n}$ (with the same notations we gave in the latter proposition).

Proposition 4.55. *If d is even, $\mathcal{B}^{d,d,d/2}$ is the category of partition corresponding to $\tilde{C}_n^{+,d}$.*

Proof. The pattern of the proof is the same as for the free complexification.
Let $n \geq 1$. We denote $G_n = (v_{ij})_{1\leq i,j\leq n})$ the quantum group of dimension n corresponding to the category of partition $\mathcal{B}^{d,d,d/2-1}$, $\mathcal{T}$ the Woronowicz tensor category associated to the quantum group $\tilde{C}_n^{+,d}$. Note first that $\mathcal{B}^{d,d,d/2-1} \subseteq \mathcal{T}$, so that $\tilde{C}_n^{+,d} \subseteq G_n$.
Let us show that there exists a C^*−morphism from $C(O_n^+) * \mathbb{C}\mathbb{Z}_d)$ to $C(G_n)$ sending $o_{ij} z^{d+1}$ to v_{ij}. Since $(\sum_j v_{ij})^d = 1$, there exists a C^*−morphism sending z on $\sum_j v_{ij}$. Thank to the intertwiner , the sum $\sum_j v_{ij}$ is independent from i. Moreover the intertwiner associated with  implies that

$$v_{ij}(\sum_j v_{ij})^{d/2-1} = (\sum_j v_{ij})^{d/2+1} v_{ij}^*$$

and thus since $(\sum_j v_{ij})^{d/2-1} = ((\sum_j v_{ij})^{d/2+1})^*$, $v_{ij}(\sum_j v_{ij})^{d/2-1}$ is self-adjoint. Let $\tilde{o}_{ij} = v_{ij}(\sum_j v_{ij})^{d/2-1}$. The matrix $(\tilde{o}_{ij})_{1\leq i,j\leq n}$ contains self-adjoint elements and

$$\sum_i \tilde{o}_{ij}\tilde{o}_{ik} = \sum \tilde{o}_{ji}\tilde{o}_{ki} = \delta_{jk}\mathbf{1}.$$

Thus there exist a C^*−morphism from $C(O_n^+)$ to $C(G_n)$ sending o_{ij} to $\tilde{o}_{ij}$. By universal property there exists a C^*−morphism Φ between $C(O_n^+) * \mathbb{C}\mathbb{Z}_d$ extending the two latter morphisms. In particular we have $\Phi(o_{ij} z^{d/2+1}) = v_{ij}$. By construction $\Phi(\sum o_{ij} - z^{d/2}) = 0$, and thus the morphism factorizes through $C(\tilde{C}_n^{+,d})$. □

4.9.5 Global colorization

It remains to interpret the global parameter k. For each words r in $\circ, \bullet$, let $c(r)$ denote the quantity $\# \circ - \# \bullet$. The following result holds for every matrix compact quantum groups:

Proposition 4.56. *Let $G = (u_{ij})_{1 \leq i,j \leq n}$ be a compact matrix quantum group, $\mathcal{C}(G) = \{\mathrm{Mor}_G(r, r')\}_{r,r'}$ the set of intertwiners associated to this matrix compact quantum group by the Tannaka-Krein duality. Then*

$$\mathcal{C}(G^d) = \{\mathrm{Mor}_G(r, r')\}_{\substack{r,r' \\ c(r)-c(r')=0[d]}}$$

Proof. Let $(u_{ij})_{1 \leq i,j \leq n}$ be the fundamental matrix of G, and $(v_{ij})_{1 \leq i,j \leq n}$ be the one of G^d. Since G is a quantum subgroup of G^d, for any word r, r' in $\circ, \bullet$,

$$\mathrm{Mor}_{G^d}(r, r') \subseteq \mathrm{Mor}_G(r, r')$$

By duality, it suffices to consider only the case $r' = \emptyset$. Let $\{e_i\}$ be a basis of the fundamental representation u of G. Let $X = \sum \lambda_{\vec{i}} e_{i_1}^{r_1} \otimes \cdots \otimes e_{i_t}^{r_t}$ be a vector in $\mathrm{Mor}(u^{\otimes r}, \mathbf{1})$. We note $\vec{i}$ for the tuple $(i_1, \ldots, i_r)$, and $u_{\vec{i}\vec{j}}$ for the product $u_{i_1 j_1}^{r_1} \ldots u_{i_t j_t}^{r_t}$. Then

$$u_G^{\otimes r}(X) = \sum_{\vec{i}, \vec{j}} \lambda_{\vec{i}} e_{j_1}^{r_1} \otimes \cdots \otimes e_{j_t}^{r_t} \otimes u_{\vec{j}\vec{i}} = X \otimes \mathbf{1}_{C(G)}$$

And thus $\sum_{\vec{i}} \lambda_{\vec{i}} u_{\vec{j}\vec{i}} = \lambda_{\vec{j}} \mathbf{1}_{C(G)}$. Applying this equality together with the expression $v_{ij} = u_{ij} z_d$ yields

$$v^r(X) = \sum_{\vec{i}, \vec{j}} \lambda_{\vec{i}} e_{j_1}^{r_1} \otimes \cdots \otimes e_{j_t}^{r_t} \otimes u_{\vec{j}\vec{i}} z^{c(r)} = X \otimes z^{c(r)}$$

Finally X is invariant under the action of G^d if and only if $c(r) = 0[d]$. $\square$

Summary of the classification of free quantum groups

We can conclude by summarizing the previous description in the following theorems :

Theorem 4.57. *The following correspondance holds between categories of partitions and unitary easy quantum groups :*

1. *the category $\mathcal{U}$ corresponds to the free unitary quantum group U^+.*
2. *the category $\mathcal{H}^{k,d}$ corresponds to the commutative $k-$complexification of $Z_d \wr S^+$.*
3. *the category $\mathcal{H}^{\#}$ corresponds to the free complexification of H^+.*
4. *the category $\mathcal{B}^{k,d}$ (resp $\mathcal{B}^{k,d,0}$), corresponds to the commutative $k-$complexification of the free $d-$complexification of C^+ (reps. B^+).*
5. *the category $\mathcal{B}^{k,d,d/2}$ corresponds to the commutative $k-$complexification of $\tilde{C}^{+,d}$.*
6. *the category $\mathcal{S}^{k,d}$ corresponds to the commutative $k-$complexification of the free $d-$complexification of S^+.*

Proof. The free quantum groups corresponding to $\mathcal{U}, \mathcal{O}, \mathcal{H}^{d,d}, \mathcal{H}^{\#}, \mathcal{B}^{d,d}, \mathcal{B}^{d,d,d/2}, \mathcal{B}^{d,d,0}$ and $\mathcal{S}^{d,d}$ have been already indetified in previous paragraph. Proposition 4.56 concludes the proof. $\square$

4.10 The group case and unitary easy groups

For general categories $\mathcal{C} \subseteq P^{\circ\bullet}$ of two-colored partitions, there are two natural extreme cases. The first is the one of noncrossing partitions, completely classified in the preceding sections. The second is the one containing the crossing partitions , , and , which allow us to permute the points of a partition in an arbitrary way (without changing their colors). It is easy to see that one of these four partitions is in a category if and only if all are (by verticolor reflection and rotation).

Definition 4.58. *A category of two colored partitions $\mathcal{C}$ is in the* group case *if one (and hence all) of the partitions , , and is in $\mathcal{C}$.*

The name "group case" refers to the situation when a quantum group is associated to a category of partitions (see Section 3.2). If $\mathcal{C}$ is in the group case, the associated quantum group is in fact a group.

The classification of all categories in the group case follows directly from the classification of all categories of noncrossing partitions and the following lemma.

Lemma 4.59. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories of two-colored partitions.*

(a) Then $\mathcal{C} \cap \mathcal{D}$ is again a category of partitions.

(b) Let $\mathcal{C}$ be in the group case and put $\mathcal{C}_0 := \mathcal{C} \cap NC^{\circ\bullet}$. Then $\mathcal{C} = \langle \mathcal{C}_0, \ \rangle$.

Proof. (a) This follows directly from the definition of a category.

(b) Let $p \in \mathcal{C}$. Using the four kinds of crossing partitions of Definition 4.58, we may permute the points of p such that we obtain a noncrossing partition p'. Since this can be done in $\mathcal{C}$, we have $p' \in \mathcal{C}_0 \subseteq \langle \mathcal{C}_0, \ \rangle$. Thus, we can also reconstruct p in $\langle \mathcal{C}_0, \ \rangle$ doing all these operations backwards, so $\mathcal{C} \subseteq \langle \mathcal{C}_0, \ \rangle$. We deduce that equality holds. $\square$

For each category of partition $\mathcal{C}$, denote by $\mathcal{C}_c = \langle \mathcal{C}, \ \rangle$. Thus the latter Lemma says that any category of partition in the group case is of the form $\mathcal{C}_c$ for a category of non-crossing partition.

Theorem 4.60. *The categories in the group case are the following.*

- $\mathcal{O}_{\text{grp,glob}}(k) = \langle \ ^{\otimes \frac{k}{2}}, \ \otimes \ , \ \rangle$ *for* $k \in 2\mathbb{N}_0$
- $\mathcal{O}_{\text{grp,loc}} := \langle \ \rangle$
- $\mathcal{H}_{\text{grp,glob}}(k) = \langle b_k, \ , \ \otimes \ , \ \rangle$ *for* $k \in 2\mathbb{N}_0$
- $\mathcal{H}_{\text{grp,loc}}(k,d) = \langle b_k, b_d \otimes \tilde{b}_d, \ , \ , \ \rangle$ *for* $k, d \in \mathbb{N}_0 \backslash \{1, 2\}$, $d|k$
- $\mathcal{S}_{\text{grp,glob}}(k) = \langle \ ^{\otimes k}, \ , \ \otimes \ , \ \otimes \ , \ \rangle$ *for* $k \in \mathbb{N}_0$

- $\mathcal{S}_{\text{grp,loc}}(k) = \langle \;{}^{\otimes k},\; ,\; \otimes \;,\; \rangle$ *for* $k \in \mathbb{N}_0 \backslash \{1\}$
- $\mathcal{B}_{\text{grp,glob}}(k) = \langle \;{}^{\otimes k},\; \otimes \;,\; \otimes \;,\; \rangle$ *for* $k \in 2\mathbb{N}_0$
- $\mathcal{B}_{\text{grp,loc}}(k) = \langle \;{}^{\otimes k},\; \otimes \;,\; \rangle$ *for* $k \in \mathbb{N}_0$

Proof. The crossing partition permutes the points of a partition. Applying this partitions on the generators of the categories of partitions in Section 4.7 yields that $\mathcal{H}_c^{\#} = \mathcal{H}_c^{0,0}$. Furthermore for the same reasons, $\mathcal{S}_c^{d,k} = \tilde{\mathcal{S}}_c^k$, $\mathcal{B}_c^{k,d} = \mathcal{B}_c^k$ and $\mathcal{B}_c^{k,d,0} = \mathcal{B}_c^{k,d,d/2} = \tilde{\mathcal{B}}_c^k$. Apart from these equalities, the commutative image of the categories of noncrossing partitions are different, yielding the result. □

If $\mathcal{C}$ is a category containing the crossing partitions and , then the C^*-algebra associated to it is commutative (see the relations in Section 4.8). Hence, the associated quantum groups are in fact groups. They are listed in the next theorem.

Theorem 4.61. *The groups corresponding to the catgories in the group case are the following :*

- $\mathcal{U}_c$ *corresponds to the unitary group* U_n.
- $\mathcal{O}_c^k$ *corresponds to the* $k-$*complexification of the orthogonal group* O_n.
- $\mathcal{H}_c^{k,d}$ *corresponds to the* $k-$*complexification of the wreath product* $\mathcal{Z}_d \wr S_n$.
- $\tilde{\mathcal{S}}_c^k$ *corresponds to the* $k-$*complexification of the permutation group* S_n.
- $\tilde{\mathcal{B}}_c^k$ *corresponds to the* $k-$*complexification of the group* $\left(\begin{pmatrix} O_{n-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$.
- $\mathcal{B}_c^k$ *corresponds to the* $k-$*complexification of the group* $\left(\begin{pmatrix} U_{n-1} & 0 \\ 0 & 1 \end{pmatrix}\right)$.

Remark 4.62. *Note that* $\left(\begin{pmatrix} U_{n-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \simeq U_{n-1}$ *and* $\left(\begin{pmatrix} O_{n-1} & 0 \\ 0 & 1 \end{pmatrix}\right) \simeq O_{n-1}$. *The category are still different because we singled out different fundamental representations of the same group.*

Chapter 5

Stochastics on the free unitary easy groups

In this chapter we develop the Weingarten calculus for unitary easy quantum groups. This consists mainly in a rewriting of the classical method of Collins and Sniady (see Chapter 2 and [28, 29]) for a general unitary easy quantum groups. In the orthogonal case, this has already been done by Collins and Banica in [12] and by Banica, Curran and Speicher in [14]. As an application of this generalization, we recover the results of Diaconis and Shahshahani on the unitary group (see Chapter 2, Section 3 and [33]) and the second order freeness result of Mingo and Speicher (see Chapter 2, Section 5 and [63]), and extend both result to the free unitary group.

5.1 Weingarten calculus for easy quantum groups

In this section we introduce the Weingarten calculus for unitary easy quantum group, using the formalism developed by Banica and Speicher in [15]. Throughout this section n is a fixed positive integer and $(C(G),(u_{ij})_{1\leq i,j\leq n})$ is a unitary easy quantum group, with associated category of partition $\mathcal{C}$. By Chapter 4, this implies that the vector space $\mathrm{Fix}_G(\varepsilon)$ of invariant vectors of V^ε under the action of G is given by $\langle T_p\rangle_{p\in\mathcal{C}(\varepsilon)}$, and $\{T_p\}_{p\in\mathcal{C}(\varepsilon)}$ is a basis of $\mathrm{Fix}_G(\varepsilon)$ if $n\geq|\varepsilon|$. Therefore we can express integrals of polynomials in u_{ij},u_{ij}^* using the vectors $T_p's$, as in Chapter 2. We will start by giving a general formula to emphasize the geometric aspect of the Weingarten calculus, then we will specify this result to the unitary easy quantum groups. In the following expression we formally write $u_{ij}^\circ=u_{ij}$ and $u_{ij}^\bullet=u_{ij}^*$:

Proposition 5.1. *Let $(C(G),(u_{ij})_{1\leq i,j\leq n})$ be a compact matrix quantum group. Let $1\leq r$ and let ε be a word in $\circ,\bullet$ of length r. Suppose that $\{f_i\}_{1\leq i\leq s}$ is a basis of $\mathrm{Fix}_G(\varepsilon)$. Then for each couple of sequences $1\leq i_1,\dots,i_r\leq n, 1\leq j_1,\dots,j_r\leq n$,*

$$\int_G u_{i_1j_1}^{\varepsilon_1}\dots u_{i_rj_r}^{\varepsilon_r}=\langle p_{\mathrm{Fix}_G(\varepsilon)}(X_{j_1}^{\varepsilon_1}\otimes\dots\otimes X_{j_r}^{\varepsilon_r}),X_{i_1}^{\varepsilon_1}\otimes\dots\otimes X_{i_r}^{\varepsilon_r}\rangle,$$

$p_{\mathrm{Fix}_G(\varepsilon)}$ being the orthogonal projection on $\mathrm{Fix}_G(\varepsilon)$.

Proof. On one hand,

$$\alpha_\varepsilon(X_{j_1}^{\varepsilon_1}\otimes\dots\otimes X_{j_r}^{\varepsilon_r})=\sum_{1\leq i_1,\dots,i_r\leq n}X_{i_1}\otimes\dots\otimes X_{i_r}\otimes u_{i_1j_1}^{\varepsilon_1}\dots u_{i_rj_r}^{\varepsilon_r},$$

and thus

$$(Id\otimes\int_G)\alpha_\varepsilon(X_{j_1}^{\varepsilon_1}\otimes\cdots\otimes X_{j_r}^{\varepsilon_r})=\sum(\int_G u_{i_1j_1}^{\varepsilon_1}\dots u_{i_rj_r}^{\varepsilon_r})X_{i_1}^{\varepsilon_1}\otimes\cdots\otimes X_{i_r}^{\varepsilon_r}. \tag{5.1.1}$$

On the other hand, V^ε decomposes into irreducible representations of $C(G)$ as $V^\varepsilon=\mathrm{Fix}_G(\varepsilon)+\bigoplus_{\omega\text{ irred},\omega\neq\mathbf{1}}V_\omega^{\bigoplus n_\omega}$, and this decomposition is orthogonal. If $v\in\mathrm{Fix}_G(\varepsilon)$, $(Id\otimes\int_G)\alpha_\varepsilon(v)=(Id\otimes\int_G)(v\otimes\mathbf{1}_{C(G)})=v$. If $v\in V_\omega$ with $\omega\neq\mathbf{1}$, by Schur othogonality $(Id\otimes\int_G)\alpha_\varepsilon(v)=0$. Thus $(Id\otimes\int_G)\alpha_\varepsilon$ is an othogonal projection onto the vector space $\mathrm{Fix}_G(\varepsilon)$.
By (5.1.1), $\int_G u_{i_1j_1}^{\varepsilon_1}\dots u_{i_rj_r}^{\varepsilon_r}$ is the coordinate $(i_1,\dots,i_r)$ of the orthogonal projection of $X_{j_1}^{\varepsilon_1}\otimes\cdots\otimes X_{j_r}^{\varepsilon_r}$ on $\mathrm{Fix}_G(\varepsilon)$.

□

In order to evaluate $\langle p_{\mathrm{Fix}_G(\varepsilon)}(X_{j_1}^{\varepsilon_1}\otimes\cdots\otimes X_{j_r}^{\varepsilon_r}),X_{i_1}^{\varepsilon_1}\otimes\cdots\otimes X_{i_r}^{\varepsilon_r}\rangle$, a general framework is given by the Gram-Schmidt orthogonalization:

Proposition 5.2 (Gram-Schmidt orthogonalization). *Let $(V,\langle.,.\rangle)$ be a Hilbert space and W a finite dimensional vector subspace of V. Suppose that $(e_i)_{1\leq i\leq s}$ is a basis of W, and let $x,y\in V$. The orthogonal projection p_W on W is given by the expression:*

$$\langle p_W(x),y\rangle=\sum_{i,j=1}^s\langle x,e_i\rangle\langle e_j,y\rangle K^{-1}(i,j),$$

with $K(i,j)=\langle e_i,e_j\rangle$ the Gram-Schmidt matrix of the basis $(e_i)_{1\leq i\leq s}$.

Proof. K is the matrix of the scalar product $\langle.,.\rangle$ in the basis $(e_i)_{1\leq i\leq s}$. Let $\Lambda=(\lambda_i)_{1\leq i\leq s}$ and $\Xi=(\xi_i)_{1\leq i\leq s}$ be two vectors of $\mathbb{C}^s$ such that $x=h_1+\sum\lambda_ie_i$ and $y=h_2+\sum\mu_ie_i$ with $h_1,h_2\perp W$. Then

$$\langle p_W(x),y\rangle=\langle p_W(x),p_W(y)\rangle=\Lambda^tK\Xi.$$

On the other hand setting $\tilde\Lambda=(\langle e_i,x\rangle)_{1\leq i\leq s}$ and $\tilde\Xi=(\langle e_i,y\rangle)_{1\leq i\leq s}$ yields that $K\Lambda=\tilde\Lambda$ and $K\Xi=\tilde\Xi$. Therefore, since K is invertible,

$$\langle p_W(x),y\rangle=\tilde\Lambda^tK^{-1}\tilde\Xi.$$

□

The combination of the two previous results gives a general Weingarten formula for compact matrix quantum groups :

Theorem 5.3 (Weingarten Formula). *Let $(C(G),(u_{ij})_{1\leq i,j\leq n})$ be a compact matrix quantum group. Let $\varepsilon=\varepsilon_1\dots\varepsilon_r$ be a word in $\circ,\bullet$, and $1\leq i_1,\dots,i_r\leq n$, $1\leq j_1,\dots,j_n\leq n$ be two sequences of integers. Suppose that $\{f_a\}_{1\leq a\leq s}$ is a basis of $\mathrm{Fix}_G(\varepsilon)$; then*

$$\int_G u_{i_1j_1}^{\varepsilon_1}\dots u_{i_rj_r}^{\varepsilon_r}=\sum_{1\leq a,b\leq s}\langle X_{i_1}\otimes\cdots\otimes X_{i_r},f_b\rangle\langle f_a,X_{j_1}^{\varepsilon_1}\otimes\cdots\otimes X_{j_r}\rangle Wg_G(f_a,f_b),$$

where $Wg_G=K_G^{-1}$, K_G being the Gram-Schmidt matrix $K_G(f_a,f_b)=\langle f_a,f_b\rangle_{V^\varepsilon}$.

As it was already said in Chapter 2, this Theorem is really useful provided the set $(f_a)_{1\leq a\leq s}$ has an explicit expression relatively to the basis $(X_{i_1}^{\varepsilon_1}\otimes\cdots\otimes X_{i_r}^{\varepsilon_r})_{1\leq i_1,\dots,i_r\leq n}$ and the matrix Wg_G can be easily computed - or at least presents good approximations. The first condition is fulfilled in the case of a unitary easy quantum group, since an explicit basis is given by the vectors T_p's:

Corollary 5.4. *Let G_n be a unitary easy quantum group with associated category of partitions $\mathcal{C}$. With the notations above, if $n \geq |\varepsilon|$ (or $n \geq 4$ if $\mathcal{C}$ is non-crossing), then*

$$\int_{G_n} u_{i_1 j_1}^{\varepsilon_1} \dots u_{i_r j_r}^{\varepsilon_r} = \sum_{\substack{p,q \in \mathcal{C} \\ p \leq \ker(\vec{i}), q \leq \ker(\vec{j})}} Wg_{G_n}(p,q),$$

where $Wg_{G_n} = K_{G_n}^{-1}(p,q)$ with $K_{G_n}(p,q) = n^{b(p \vee q)}$.

Proof. The condition $n \geq |\varepsilon|$ or $n \geq 4$ if $\mathcal{C}$ is non-crossing yields that the set $\{T_p\}_{p \in \mathcal{C}(\varepsilon)}$ is a basis of $\mathrm{Fix}_G(\varepsilon)$. Since $T_p = \sum_{p \leq \ker(\vec{i})} X_{i_1} \otimes \dots \otimes X_{i_r}$, the scalar product of Theorem 5.3 is $\langle T_p, X_{i_1} \otimes \dots \otimes X_{i_r} \rangle = \delta_{p \leq \ker(\vec{i})}$. The corollary is thus a direct application of this Theorem and the fact that $\langle T_p, T_q \rangle = n^{b(p \vee q)}$. □

The main problem remains the computation of the matrix Wg_{G_n}. We have seen in Chapter 2 that in the case of U_n, the computation of Wg_{U_n} is already complicated. In the general case no explicit expression of this matrix has been found for a unitary easy quantum group; however a first order asymptotic of Wg_{G_n} allows to get several asymptotic probabilistic results as n goes to $+\infty$.

Proposition 5.5 ([12])**.** *As n goes to $+\infty$,*

$$Wg_{G_n}(p,q) = (-1)^{\delta_{p \neq q}} n^{b(p \vee q) - b(p) - b(q)} (1 + O(1/n)).$$

Proof. For n large enough, $\{T_p\}$ is a basis of $\mathrm{Fix}_{G_n}(\varepsilon)$. Let K_{G_n} be the Gram-Schmidt matrix of the basis $\{T_p\}_{p \in \mathcal{C}}$. From the expression of T_p, $K_{G_n}(p,q) = n^{b(p \vee q)}$. Since $p \leq p \vee q$, $b(p \vee q) \leq b(p)$; for the same reasons, $b(p \vee q) \leq b(q)$ and thus $b(p \vee q) \leq \frac{b(p)+b(q)}{2}$. Moreover if $p \neq q$, the inequality becomes $b(p \vee q) \leq \frac{b(p)+b(q)}{2} - 1/2$.
Let Δ be the diagonal matrix defined by $\Delta_{pq} = \delta_{pq} n^{b(p)}$. By the previous inequality

$$(\Delta^{-1/2} K_{G_n} \Delta^{-1/2})_{pq} = n^{b(p \vee q) - \frac{b(p)+b(q)}{2}} = Id + M,$$

where $M = O(1/\sqrt{n})$. Thus $(Id + M)^{-1} = (Id - M)(1 + O(\sqrt{n}))$ and $K_{G_n}^{-1} = \Delta^{-1/2}(Id - M)\Delta^{-1/2}(1 + O(\sqrt{n}))$. This yields

$$(K_{G_n}^{-1})_{pq} = (-1)^{\delta_{p \neq q}} n^{b(p \vee q) - b(p) - b(q)} (1 + O(1/\sqrt{n})).$$

□

5.2 Diaconis-Shahshahani results in the free case

The first application of the Weingarten calculus is a computation of the asymptotic law of the family $(\mathrm{Tr}(u^k))_{k \geq 1}$, where u is the fundamental matrix of a free unitary quantum group G_n and n goes to $+\infty$. For all easy unitary classical groups, this has already been done by Diaconis and Shahshahani in [33] (see Chapter 2, Section 3), or is a direct consequences of their results. In the free orthogonal case, this has been done in [14].
Recall that the list of free unitary easy groups is the following (we put the associated category of partitions in parenthesis) :

- U_n^+ ($\mathcal{U}$),

- $O_n^+ \tilde{\times} \mathbb{Z}_k$ $(\mathcal{O}^k)$,
- $(H_n \wr_* Z_d) \tilde{\times} \mathbb{Z}_k$ $(\mathcal{H}^{d,k})$,
- $H^{\#}$ $(\mathcal{H}^{\#})$,
- $(B_n \tilde{*} \mathbb{Z}_d) \tilde{\times} \mathbb{Z}_k$ $(\mathcal{B}^{d,k})$,
- $(C_n \tilde{*} \mathbb{Z}_d) \tilde{\times} \mathbb{Z}_k$ $(\mathcal{B}^{d,k,0})$,
- $\tilde{C}_n^d \tilde{\times} \mathbb{Z}_k$ $(\mathcal{B}^{d,k,d/2)})$,
- $(S_n^+ \tilde{*} \mathbb{Z}_d) \tilde{\times} \mathbb{Z}_k$ $(\mathcal{S}^{d,k})$

The reader should refer to Chapter 4 for a detailed description of these quantum groups. We will prove the following result:

Theorem 5.6. *Let $(C(G_n), (u_{ij})_{1 \leq i,j \leq n})$ be one of the quantum groups above, with category of partition $\mathcal{C}$. As n goes to $+\infty$, the family $(\mathrm{Tr}(u^k))_{k \geq 1}$ converges in moment with respect to $\int_{G_n}$ to a family of random variables $(u_k(\mathcal{C}))_{k \geq 1}$. The law of $(u_k(\mathcal{C}))_{k \geq 1}$ depends on $\mathcal{C}$ and is explicitly described in Section 2.2.*

The proof of the Theorem is achieved in two parts. The first part is a proof of the convergence in law, which is a generalization of the proof of Theorem 2.5 in [14]. This proof gives also a combinatorial formula for the moments of the limit law. In the second part, this combinatorial formula is used to describe the law of the family $(u_k(\mathcal{C}))_{k \geq 1}$.
For $k > 0$, $\mathrm{Tr}(u^k)^* = \mathrm{Tr}(u^{-k})$, and thus the goal is therefore to prove the existence of the asymptotic moment

$$m_{G,k_1,\ldots,k_r} := \lim_{n \to +\infty} \int \mathrm{Tr}(u^{k_1}) \ldots \mathrm{Tr}(u^{k_r})^{\varepsilon_r},$$

with $k_1, \ldots, k_r$ non-zero integers. It suffices to only consider the cases without tensor complexification: indeed the Haar state on the tensor complexification $H \tilde{\times} \mathbb{Z}_k$ is the classical independence convolution of the Haar state on (H, u) and the one on $(\mathbb{Z}_k, z)$. The law of $\mathrm{Tr}((uz)^r)$ can thus be deduced from the law of $\mathrm{Tr}(u^r)$ with the equality $\mathrm{Tr}((uz)^r) \simeq_{law} \mathrm{Tr}(u^r) \otimes z^r$.

The asymptotic trace moment formula

We detailed here the generalization of a result of Banica, Curran and Speicher that relates $m_{G,\varepsilon,k_1,\ldots,k_r}$ with the cardinal of a set of partitions. The result was originally given in the framework of orthogonal easy group; since the statement and the proof for in unitary case are identical, so we present only this latter version.
Let $r \in \mathbb{Z}_+$ and denote by $\mathbb{Z}_*^r$ the set of sequences $\vec{j} = (j_1, \ldots, j_r)$ of r non-zero integers. To each sequence $\vec{j}$ of $\mathbb{Z}_*^r$ we associate the following objects:

- A vector $\vec{k} = (k_1, \ldots, k_r)$ of positive integers with $k_s = |j_s|$. This is the absolute part of $\vec{j}$. We set $k = \sum k_i$.
- A word $\varepsilon(\vec{j})$ in $\{\circ, \bullet\}$ of length r by the condition that $\varepsilon_s(\vec{j}) = \circ$ if and only if $j_s > 0$.
- A word $w(\vec{j})$ in $\{\circ, \bullet\}$ of length k such that $w_s(\vec{j}) = \circ$ if and only if $\sum_{a=1}^{t-1} k_a + 1 \leq s \leq \sum_{a=1}^{t} k_a$ and $\varepsilon(t) = \circ$.

- A permutation $\gamma_{\vec{j}}$ described by the cycle decomposition

$$\gamma_{\vec{j}} = (1, \dots, k_1)^{\varepsilon_1(\vec{j})}(k_1+1, \dots, k_1+k_2)^{\varepsilon_2(\vec{j})} \dots (k-k_r+1, \dots, k)^{\varepsilon_j(\vec{k})},$$

  with the convention that for a cycle τ, $\tau^\circ = \tau$ and $\tau^\bullet = \tau^{-1}$.

- A two colored partition $p_{\vec{j}} \in P(0, w(\vec{j}))$ with $a \sim_{p_{\vec{j}}} b$ if and only if there exists $1 \leq t \leq r$ such that $\sum_{s=1}^{t-1} k_s + 1 \leq a, b \leq \sum_{s=1}^{t} k_s$.From the definition of $w(\vec{j})$ and the one of $p_{\vec{j}}$, the partition $p_{\vec{j}}$ consists in r interval blocks $\{B_1, \dots, B_r\}$ such that B_i has cardinal k_i and all elements of B_i have the color $\varepsilon_i(\vec{j})$.

The dependence on $\vec{j}$ of the latter object is omitted when the situation is clear.

Remark 5.7. *Any $p \in P(0, w(\vec{j}))$ yields a partition $\tilde{p}$ in $P(0, \varepsilon_{\vec{j}})$ by considering the set $\{B_i\}$ of blocks of $p_{\vec{j}}$ with the lexicographical order and the relation $B_i \sim_{\tilde{p}} B_j$ if B_i and B_j are in the same block of $p \vee p_{\vec{j}}$. Since $p_{\vec{j}}$ is non crossing, p non-crossing implies that $p \vee p_{\vec{j}}$ is also non-crossing and therefore that $\tilde{p}$ is non-crossing.*

For $p \in P^{\circ\bullet}(k, 0)$ and $\sigma \in S_k$, $\sigma(p)$ denotes the unique partition such that $\sigma(i) \sim_{\sigma(p)} \sigma(j)$ if and only if $i \sim_p j$, the colors being also permuted by σ. For example $\gamma_{\vec{j}}(w_{\vec{j}}) = w_{\vec{j}}$. Moreover we take the convention that for $x \in C(G)$, $x^\circ = x$ and $x^\bullet = x^*$.
Theorem 2.5 of [14] extends to the unitary case as follows:

Theorem 5.8. *Let $G = (u_{ij})_{1 \leq i,j \leq n}$ be an easy compact group with $\mathcal{C}$ its associated category of partitions, and $\vec{j} \in \mathbb{Z}_*^r$. With the same notations as before,*

$$\int Tr(u^{j_1}) \dots Tr(u^{j_r}) = \#\left\{p \in \mathcal{C}(w_{\vec{j}}) | p = \gamma_{\vec{j}}(p)\right\} + O(1/\sqrt{n}). \tag{5.2.1}$$

Proof. The proof is a direct computation. Let $I = \int_G Tr(u^{j_1}) \dots Tr(u^{j_r})$. Then

$$\begin{aligned} I &= \sum_{1 \leq i_1, \dots, i_k \leq n} \int_G (u_{i_1 i_2} u_{i_2 i_3} \dots u_{i_{k_1} i_1})^{\varepsilon_1} \dots (u_{i_{k-k_r+1} i_{k-k_r+2}} \dots u_{i_k i_{k-k_r+1}})^{\varepsilon_r} \\ &= \sum_{1 \leq i_1, \dots, i_k \leq n} \int_G u^{\varepsilon_1}_{i_1 i_{\gamma(1)}} \dots u^{\varepsilon_1}_{i_{k_1} i_{\gamma(k_1)}} u^{\varepsilon_2}_{i_{k_1+1} i_{\gamma(k_1+1)}} \dots u^{\varepsilon_r}_{i_k i_{\gamma(k)}}. \end{aligned}$$

Applying the Weingarten formula to the latter expression yields

$$\begin{aligned} I &= \sum_{1 \leq i_1, \dots, i_k \leq n} \left(\sum_{\substack{p,q \in \mathcal{C}(w_{\vec{j}}) \\ \ker((i_j)_{1 \leq j \leq k}) \leq p, \ker((i_{\gamma(j)})_{1 \leq j \leq k}) \leq q}} Wg_{G_n}(p,q) \right) \\ &= \sum_{1 \leq i_1 \dots, i_k \leq n} \sum_{\substack{p,q \in \mathcal{C}(w_{\vec{j}}) \\ \ker((i_j)_{1 \leq j \leq k}) \leq p \vee \gamma(q)}} Wg_{G_n}(p,q) \\ &= \sum_{p,q \in \mathcal{C}(w_{\vec{j}})} n^{b(p \vee \gamma(q))} Wg_{G_n}(p,q), \end{aligned}$$

and thanks to the asymptotic formula of Proposition 5.5 we get

$$I = \sum_{p,q \in \mathcal{C}(w_{\vec{j}})} n^{b(p\vee\gamma(q))} n^{b(p\vee q)-b(p)-b(q)} (1 + O(1/\sqrt{n})).$$

The coefficient $n^{b(p\vee\gamma(q))+b(p\vee q)-b(p)-b(q)}$ doesn't vanish as n goes to $+\infty$ if and only if $b(p \vee \gamma(q)) + b(p\vee q) - b(p) - b(q) \geq 0$. But $b(p\vee q) \leq \min(b(p), b(q))$ and since $b(q) = b(\gamma(q))$, $b(p\vee\gamma(q)) \leq \min(b(p), b(q))$. Therefore the positivity condition requires $b(p \vee q) = b(p)$, $b(p) = b(q) = b(\gamma(q))$, and $b(p\vee\gamma(q)) = b(p)$. The first two equalities yield $p = q$ and then the last two equalities yield $p = \gamma(p)$; if the latter conditions are fullfilled then $n^{b(p\vee\gamma(q))+b(p\vee q)-b(p)-b(q)} = 1$. Therefore,

$$I = \#\{p \in \mathcal{C}(w), p = \gamma(p)\} + O(1/\sqrt{n}).$$

$\square$

Remark 5.9. *Note that the latter Theorem actually yields that for any easy quantum groups, without any conditions on the crossings, the family* $(\mathrm{Tr}(u), \mathrm{Tr}(u^2), \dots)$ *converges in law to a random vector* $(u_1, u_2, \dots)$.

The remaining part of the method of [14] to describe the law of $(u_1, u_2, \dots)$ doesn't apply here, since most of the categories of partitions considered in the present situation are not block stable (in particular Proposition 3.1 of [14] doesn't hold anymore). Therefore the goal is to better understand the set $\{p \in \mathcal{C}(w_{\vec{j}}), p = \gamma(p)\}$. Let us first simplify the condition $p = \gamma(p)$:

Lemma 5.10. *There is an equivalence between the condition* $p = \gamma_{\vec{j}}(p)$ *and* $\gamma_{\vec{j}}(p) \leq p$.

Proof. Clearly $p = \gamma(p)$ yields $\gamma(p) \leq p$. Suppose that $\gamma(p) \leq p$. This means that if $\gamma(i) \sim_{\gamma(p)} \gamma(j)$, then $\gamma(i) \sim_p \gamma(j)$. By definition $\gamma(i) \sim_{\gamma(p)} \gamma(j)$ if and only if $i \sim_p j$. Therefore if $\gamma(p) \leq p$, then $i \sim_p j$ yields $\gamma(i) \sim_p \gamma(j)$. Since the permutation group S_k is finite, there exists d such that $\gamma^d = \gamma^{-1}$. Thus iterating the latter implication yields also that if $i \sim_p j$, then $\gamma^{-1}(i) \sim_p \gamma^{-1}(j)$. Therefore if $i \sim_p j$, $i = \gamma(\gamma^{-1}(i)) \sim_{\gamma(p)} \gamma(\gamma^{-1}(j)) = j$. Thus $p \leq \gamma(p)$, and $p = \gamma(p)$. $\square$

In particular checking that $i \sim_p j$ implies $\gamma(i) \sim_p \gamma(j)$ is sufficient to know whether $p = \gamma(p)$. In the free case, we will see that the condition $p = \gamma(p)$ is very strong.
Let us first define a pairing between two blocks :

Definition 5.11. *Let* p *be a partition of* k *and* B_1, B_2 *be two disjoint intervals of* $[\![1, k]\!]$ *corresponding respectively to* $[\![i_1, i_1 + t]\!]$ *and* $[\![i_2 - t, i_2]\!]$ *with* $i_1 < i_2$. *We say that* B_1 *and* B_2 *are block paired by* p *if, for* $0 \leq s \leq t$, *the only element of* $[\![1, k]\!]$ *linked to* $i_1 + s$ *by* p *is* $i_2 - s$.

The condition $i_1 < i_2$ ensures that the pairing is non-crossing. For $1 \leq u \leq k$, denote by B_u the blocks of $p_{\vec{j}}$ containing u. The rigidity of the condition $p = \gamma_{\vec{j}}(p)$ in the free case appears as follows:

Lemma 5.12. *Let* $\vec{j} \in \mathbb{Z}_*^r$ *and let* $p \in NC(w_{\vec{j}})$. *Then* $\gamma_{\vec{j}}(p) = p$ *if and only if for all* $1 \leq a, b \leq r$ *with* $B_a \sim_{\tilde{p}} B_b$, *one of the following conditions hold:*

- $a = b$ *and* B_a *contains only singletons of* p.
- B_a *and* B_b *are contained in a same block of* p.
- B_a *and* B_b *are block paired. In this case* $|B_a| = |B_b|$ *and if* B_a *has more than three points,* B_a *and* B_b *have opposite colors.*

Proof. Suppose that $p = \gamma(p)$ and let $1 \leq a, b \leq r$ with $B_a \sim_{\tilde{p}} B_b$. Thus there exists $u \in B_a, v \in B_b$ with $u \sim_p v$.

- Suppose that $a = b$ and u is a singleton, $u = v$. The property $p = \gamma(p)$ yields that all elements of the orbit of u under γ are singletons. This orbit is precisely B_a, which proves the first point.

- Suppose that $a = b$ and $u \neq v$. We can assume without loss of generality that B_a is colored $\circ$, with $u < v$. Then there exists $m \geq 0$ such that $v = \gamma^m(u)$. If $m = \pm 1[k_a]$, iterating γ yields that for all $t \geq 1$, $\gamma^t(u) \sim_p \gamma^{t-1}(u) \cdots \sim_p u$ and thus B_a is in a same block of p. Otherwise, $p = \gamma(p)$ implies $\gamma(u) \sim_p \gamma^{m+1}(u)$. The block B_a is colored $\circ$ and $m \notin \{1, -1\}$, thus $u < \gamma(u) < v$, and $\gamma^{m+1}(u)$ is either greater than v or lower than u. Since p is non-crossing, this requires $u \sim_p \gamma(u)$ or $u \sim_p \gamma^{-1}(u)$. In both cases, iterating the equality $p = \gamma(p)$ yields that for all $t \geq 1$, $\gamma^t(u) \sim_p \gamma^{t-1}(u) \sim_p \cdots \sim_p \gamma(u) \sim_p u$. Therefore B_a is contained in a block of p.
  Suppose that $a \neq b$ and that u, v are in a block of p that contains a third element t, and let $1 \leq x \leq r$ be such that B_x is the block of $p_{\vec{j}}$ containing t. After relabelling if necessary, we can suppose that $u < v < t$. If $b = x$, from the previous paragraph, all elements of B_b are in a same block of p. The equality $p = \gamma(p)$ yields that any element of B_a is linked by p to an element of B_b, and thus B_a and $B_b = B_x$ lie in a same block of p.
  Since $\gamma(p) = p$, if B_a is a singleton then every element in the orbit B_b of b is connected to u by p, and the same holds for B_x. Therefore B_a, B_b and B_x are in the same block of p.
  Suppose that $b \neq x$ and B_a is not a singleton. The latter conditions imply that $\gamma(u) \neq u$ and $B_b \neq B_x$. The condition $\gamma(p) = p$ yields that $\gamma(u)$ has to be connected to an element v' of B_b and t' of B_x. Since any element of B_b is lower than the elements of B_x, $v' < t$, and $v < t'$. Since p is non-crossing, if $u < \gamma(u)$, the inequality $u < \gamma(u) < v < t'$ yields that $\gamma(u) \sim_p u$. For the same reasons, if $\gamma(u) < u$, the inequality $\gamma(u) < u < v' < t$ yields $\gamma(u) \sim_p u$. Thus iterating the equality $p = \gamma(p)$ yields that for all $m \geq 1$, $\gamma^m(u) \sim_p u$, and B_a is contained in a block of p. Once again by the equality $\gamma(p) = p$, any element of B_b or B_x is linked to an element of B_a through p and B_a, B_b, B_x are in a same block of p.

- Suppose that $B_a \neq B_b$ and that u and v are in a pair. We can assume that $u < v$. Since $\gamma(p) = p$, $\gamma^m(u)$ is paired with $\gamma^m(v)$. Thus any element of the orbit of u is paired with an element of the orbit of v, and conversely. Therefore necessarily $|B_a| = |B_b|$ and there is a bijective map $\varphi : B_a \to B_b$ sending an element x of B_a to the unique element y of B_b such that $u \sim_p v$. Since p is noncrossing, if $x_1 < x_2$ in B_a then $\varphi(x_1) > \varphi(x_2)$. Thus the map φ is decreasing and this shows that there is a unique way to pair elements of B_a with elements of B_b, which is the block pairing of B_a with B_b. Suppose that B_a has more than two elements, and let x be the first element of B_a, y the last one of B_b. By the previous reasoning, $x \sim_p y$ and $x + 1 \sim_p y - 1$. Thus if $x + 1 = \gamma(x)$ then $y - 1 = \gamma(y)$ and if $x + 1 = \gamma^{-1}(x)$ then $y - 1 = \gamma^{-1}(y)$. In any case if B_a has more than three elements, the cycles of γ have opposite direction on B_a and B_b, and therefore B_a and B_b have opposite colors.

Conversely let p be a partition which satisfies the conditions of the lemma. From Lemma 5.2 it suffices to prove that for all $1 \leq u \leq v \leq k$, $u \sim_p v$ implies $\gamma(u) \sim_p \gamma(v)$. From the conditions of the Lemma, either B_a contains only singletons or B_a and B_b are in the same block of p or B_a and B_b are block paired. If B_a contains only singletons, $u = v$ and $\gamma(u) \sim_p \gamma(v)$. Since B_a is the orbit of u and B_b the orbit of v, if B_a and B_b are in a same block of p then $\gamma(u) \sim_p \gamma(v)$.
Suppose that B_a and B_b are block paired and let c be the cardinal of B_a (which is also the

one of B_b). There exists $1 \leq i_1 \leq i_2 \leq k$ and $s \geq 0$ such that $u = i_1 + s$, $v = i_2 - s$ and for all $0 \leq x \leq c-1$ $i_1 + x \sim_p i_2 - x$. If $c = 1$, B_a and B_b are singletons and therefore $\gamma(u) = u, \gamma(v) = v$ and $\gamma(u) \sim_p \gamma(v)$. If $c = 2$, whatever the color of B_a is, $\gamma(u) = i_1 + (1-s)$ and $\gamma(v) = i_2 - (1-s)$; thus once again $\gamma(u) \sim_p \gamma(v)$. If $c > 3$, from the condition of the Lemma, B_a and B_b have opposite colors. We can assume without loss of generality that B_a is colored $\circ$ and B_b is colored $\bullet$. Thus $\gamma(u) = i_1 + s + 1$ and $\gamma(v) = i_2 - s - 1$ (resp. $\gamma(u) = i_1$ and $\gamma(v) = i_2$) if $s < c-1$ (resp. $s = c-1$). In any case $\gamma(u) \sim_p \gamma(v)$. $\square$

For p a partition, let us denote by $\mathcal{B}_{p,2}$ the set of blocks of p of cardinal less than 2, and for each $b \in \mathcal{B}_{p,2}$, let b_1 and b_2 be respectively the first and second elements of b, with the convention that $b_1 = b_2$ if b is a singleton. For $\vec{j} \in \mathbb{Z}_*^r$, the set of non-crossing partitions of r with point coloring $c(i) = j_i$ is denoted $NC(j_1, \dots, j_r)$. Following the latter Lemma, we introduce the following set of colored partitions :

Definition 5.13. *Let $r \geq 1$. A tracial partition is a partition in $NC(j_1, \dots, j_r)$ with an additional block coloring $\star_p : \mathcal{B}_{p,2} :\to \{1, *\}$ with the condition that for $b \in \mathcal{B}_{p,2}$, $\star(b) = 1$ if $\min(k_{b_1}, k_{b_2}|) = 1$ or if b is a pair with $\sup(k_{b_1}, k_{b_2}) > 2$ and $j_{b_1} + j_{b_2} \neq 0$.*

The set of tracial partitions in $NC(j_1, \dots, j_r)$ is denoted $\mathcal{P}(\vec{j})$.
For convenience we extend the pair coloring $\star_p$ to a coloring of all the blocks by setting $\star_p(b) = 1$ if b is a block not belonging to $\mathcal{B}_{p,2}$.
We define now a map $\Phi : \mathcal{P}_{\vec{j}} \to P(w_{\vec{j}})$ as follows: let p be a tracial partition in $\mathcal{P}(\vec{j})$ and let $1 \leq s \leq t \leq r$ with $s \sim_p t$. If s is a singleton colored $*$, all elements of B_s are singletons of $\Phi(p)$. If s and t are in a pair b of p colored $*$ then B_s is block paired with B_t in $\Phi(p)$. In all other cases, all elements of B_s and B_t are in a same block of $\Phi(p)$.

Proposition 5.14. *The map Φ is injective and $\Phi(\mathcal{P}_{\vec{j}}) = \{p \in NC(w_{\vec{j}}), \gamma_{\vec{j}}(p) = p\}$. In particular, if G_n is an easy unitary free group with associated category of partitions $\mathcal{C}$, then for n large enough*

$$m_G(\vec{j}) = \#\Phi^{-1}(\mathcal{C}(w_{\vec{j}})).$$

Proof. Suppose that p, q are two distinct tracial partitions of $\mathcal{P}_{\vec{j}}$. Then either p, q have different block structures, or they have the same block structure and the block colorings differ. Suppose that they have a different block structure : by symmetry we can assume that there exists $1 \leq s_1 < s_2 \leq r$ with s_1 and s_2 in a same block b of p, but $s_1 \not\sim_q s_2$. In any case from the construction of Φ, the first element of B_{s_1} is linked to the last element of B_{s_2} through $\Phi(p)$ but not through q. Therefore $\Phi(p) \neq \Phi(q)$.
Suppose that the block structure of p and q is the same but the block colorings differ. By symmetry we can assume that there exists a block $b = \{b_1, b_2\}$ such that $\star_p(b) = 1$ and $\star_q(b) = *$. But from the definition of Φ this means that all elements of B_{b_1} are in the same block of $\Phi(p)$, but they are each in distinct blocks of $\Phi(q)$. Therefore $\Phi(p) \neq \Phi(q)$ and Φ is injective.
By construction, $\Phi(p)$ verifies the conditions of Lemma 5.12 and thus $\gamma(\Phi(p)) = p$.
Conversely, if $\gamma(q) = q$, q verifies the conditions of Lemma 5.12. We construct $p \in \mathcal{P}_{\vec{j}}$ as follows: The block structure of p is $\tilde{q} \in P(\varepsilon_{\vec{j}})$. Recall that the element i of $\tilde{q}$ corresponds to the block B_i of $p_{\vec{j}}$. Set $j_i = |B_i|$ if $(\varepsilon_{\vec{j}})_i = \circ$, and $j_i = -|B_i|$ if $(\varepsilon_{\vec{j}})_i = \bullet$. Let $b = \{b_1 < b_2\}$ be a pairing in $\tilde{q}$ with $\min(k_{b_1}, k_{b_2}) > 1$. Since $\gamma(q) = q$, Lemma 5.12 yields that either B_{b_1} and B_{b_2} are in a same block of q, or B_{b_1} and B_{b_2} are block paired. In the first case , set $\star(b) = 1$ and in the second case set $\star(b) = *$ (note that if $k_{b_1} > 2$ then Lemma 5.12 enforces $j_{b_1} = -j_{b_2}$). Similarly if i is a singleton of $\tilde{q}$, from Lemma 5.12 either B_i contains only singletons, or it is a block of q of cardinal greater than 2. In the first case set $\star(\{i\}) = *$ and the other case set $\star(\{i\}) = 1$. Set $\star(b) = 1$ for all other singletons and pair. This gives a tracial partition $p \in \mathcal{P}(\vec{j})$ such that

$\Phi(p) = q$.
The last part of the proposition is straightforward. $\square$

The law of the family $(u_k(\mathcal{C})_{k\geq 1}$ is therefore given by the cardinal of $\Phi^{-1}(\mathcal{C}(w_{\vec{j}}))$ for various $\vec{j}$.

Description of the law of $(u_k(\mathcal{C}))_{k\geq 1}$

As for the classification of categories in Chapter 4, we shall deal separately with the five cases $\mathcal{U}, \mathcal{O}, \mathcal{H}, \mathcal{B}$ and $\mathcal{S}$.

Partial partitions Let us first give a combinatorial tool:

Definition 5.15. *A partial partition (p, S) of n is a set $S = \{1 \leq i_1 < \cdots < i_s \leq n\}$ together with a partition p of S (identified with $[\![1, s]\!]$ by the natural order on integers).*

This is equivalent to the data of a set $\{B_1, \ldots, B_r\}$ of disjoint subsets of $[\![1, n]\!]$, with $S = \bigcup B_i$. The natural order on S yields a canonical bijection between the set of partial permutation $\{(p, S)\}_{p\in P(S)}$ and $P(s)$, and thus the order on $P(s)$ gives an order on the set of partial permutations $\{(p, S)\}_{p\in P(S)}$. (p, S) is called a non-crossing partial partition if p is a non-crossing partition, view as an element of $P(s)$; two partial partitions (p, S) and (p', S^c) yield a partition $(p, S) \vee (p', S^c)$ of $[\![1, n]\!]$ simply by considering the reunion of the set of blocks of p and the one of p'. If (p, S) is a non-crossing partial partition of S, the Kreweras complement $kr(p)$ of p is the biggest partial partition (p, S^c) such that $p \vee p'$ is non-crossing.

Lemma 5.16. *Let (p, S) be a partial partition of n. Then $kr(p)$ is the partial partition with support S^c defined by*

$$i \sim_{kr(p)} j \Leftrightarrow k \not\sim_{(p,S)} l, \text{ for all } k \in [\![i, j]\!] \cap S, l \in S \setminus [\![i, j]\!]$$

Proof. Since $(p, S) \vee (kr(p), S^c)$ is noncrossing the direct implication holds.
Suppose that for all $k \in [\![i, j]\!] \cap S, l \in S \setminus [\![i, j]\!]$, $k \not\sim_{(p,S)} l$; if π is any noncrossing partition of S^c such that $(p, S) \vee (\pi, S^c)$ is non crossing, then $\tilde{\pi}$ obtained from π by the reunion of the block containing i and the one containing j is again noncrossing, and $(p, S) \vee (\pi, S^c)$ is again noncrossing. Thus by maximality of $kr(p)$, $i \sim_{(kr(p),S^c)} j$. $\square$

We denote by $NC_p(n)$ (resp. $NC_p(\vec{j})$) the set of non-crossing partial partitions of n (resp. non-crossing partition of n with a coloring $\vec{j}$ on $[\![1, n]\!]$). The set of non-crossing partial partitions with support S is denoted $NC(S)$.

Case $\mathcal{U}$ There is a unique category of partition in the case $\mathcal{U}$, consisting only in pairings with endpoints of different colors.

Lemma 5.17. *Let $\vec{j} \in \mathbb{Z}_*^r$. $\Phi^{-1}(\mathcal{U}(w_{\vec{j}}))$ is isomorphic to the set of non-crossing pairings in $NC_2(j_1, \ldots, j_r)$ such that the endpoints of a pair have opposite colors.*

Proof. Let $\vec{j} \in \mathbb{Z}_*^r$ and $p \in \mathcal{P}_{\vec{j}}$ such that $\Phi(p) \in \mathcal{U}(w_{\vec{j}})$. If there exists a block b of p such that $\star(b) = 1$, this means that all sets B_i with $i \in b$ are in a same block of $\Phi(p)$. Since $\mathcal{U}$ consists only in pairings, $|\bigcup_i B_i| = \sum_{i\in b} k_i = 2$. Thus either b is a singleton $\{i\}$ with $k_i = 2$, or b is a pairing $\{i_1, i_2\}$ with $k_{i_1} = k_{i_2} = 1$. The first case is not possible because both elements of B_i have the same color, and the second case is possible only if $j_{i_1} = -j_{i_2}$.
If $\star(b) = *$, b contains two elements i_1 and i_2 with $k_{i_1} = k_{i_2}$. But B_{i_1} and B_{i_2} are block paired

through $\Phi(p)$. Thus the condition $\Phi(p) \in \mathcal{U}(w_{\vec{j}})$ yields that $j_{i_1} = -j_{i_2}$ (excluding only the case $j_{i_1} = j_{i_2} = \pm 2$).
Finally p has only pairing $\{i_1, i_2\}$ between elements of opposite colors, which are colored $*$ if $k_{i_1} \geq 2$ and colored 1 if $k_{i_1} = 1$. $\square$

Thus we can express the law of $(u_1(\mathcal{U}), u_2(\mathcal{U}), \dots)$ as follows :

Proposition 5.18. *$(u_1(\mathcal{U}), u_2(\mathcal{U}), \dots)$ is a family of free independent variables, each of them being a circular variable with mean 0 and variance 1.*

Proof. Let $(c_1, c_2, \dots)$ be a family of free independent circular variables of variance 1 and mean 0 and set $c_{-i} = c_i^*$. The moment-cumulant formula shows that for $\vec{j} \in \mathbb{Z}_*^r$, $m(c_{j_1}, \dots, c_{j_r})$ is exactly the number of non-crossing pairings in $NC(j_1, \dots, j_r)$ such that the endpoints of each pair have opposite colors (see also Chapter 1, Prop. 1.10). $\square$

Case $\mathcal{O}$ Since we only consider the categories without tensor complexification, there is only one category to study, namely the category $\mathcal{O}^2$ of all pairings. This case has already been done in [14]. We give the proof here for the sake completeness.

Lemma 5.19. *Let $\vec{j} \in \mathbb{Z}_*^r$. $\Phi^{-1}(\mathcal{O}^2(w_{\vec{j}}))$ is isomorphic to the set of non-crossing partitions of $NC(j_1, \dots, j_r)$ consisting in pairs $b = \{i_1, i_2\}$ with either $j_{i_1} = -j_{i_2}$ or $j_{i_1} = j_{i_2} \in \{-2, -1, 1, 2\}$ and singletons $\{i\}$ with $j_i = \pm 2$.*

Proof. Let $p \in \mathcal{P}_{\vec{j}}$ such that $\Phi(p) \in \mathcal{O}^2(w_{\vec{j}})$, and let b be a block in p. The same reasoning as in the $\mathcal{U}-$case shows that if $\star_p(b) = 1$, $|\bigcup_{i \in b} B_i| = 2$. Therefore in this case b is a singleton $\{i\}$ with $k_i = 2$ or a pair $\{i_1, i_2\}$ with $j_{i_1}, j_{i_2} \in \{-1, 1\}$.
Any pair $\{i_1, i_2\}$ colored $*$ is allowed. For $k_{i_1} > 2$, a block colored $*$ has endpoints of opposite colors, which concludes the proof. $\square$

Thus $(u_1, u_2, \dots)$ is again a family of free (semi-)circular elements, the proof being the same as in the $\mathcal{U}$-case.

Proposition 5.20. *$(u_1, u_2, \dots)$ is a family of free variables such that u_k is a circular variable with covariance 1 and mean 0 if $k \geq 3$, a semi-circular variable with variance 1 and mean 0 if $k = 1$ and a semi-circular variable with variance 1 and mean 1 if $k = 2$.*

Case $\mathcal{H}$ Let $d \geq 2$. $\mathcal{H}^{d,d}$ consists in non-crossing two-colored partitions such that each block has the same numbers of black and white points modulo d. $\mathcal{H}^{\#}$ is the category of non-crossing partitions whose blocks have an even number of elements with endpoints having alternating colors.

Lemma 5.21. *Let $d \geq 3$. $\Phi^{-1}(\mathcal{H}^{d,d})$ is the set of non-crossing partitions of $\mathcal{P}_{\vec{j}}$ such that blocks $b = \{i_1, \dots, i_s\}$ are either pair colored $\star$ with endpoints of opposite colors or block colored 1 with $\sum_{t=1}^{s} j_{i_t} \equiv 0[d]$.*
If $d = 2$, the conditions are the same except that any pair colored $*$ is allowed (which adds only the case where the two endpoints have the color ± 2).
$\Phi^{-1}(\mathcal{H}^{\#})$ is the set of partitions of $\mathcal{P}_{\vec{j}}$ such that blocks colored 1 have an even number of elements and have endpoints colored ± 1 with alternating signs, and blocks colored $*$ are pairs with opposite colors.

Proof. $\mathcal{H}^{d,d}$ is the category of partitions whose blocks have the same number of white and black points modulo d. Let $p \in \mathcal{P}_{\vec{j}}$ such that $\Phi(p) \in \mathcal{H}^{d,d}(w_{\vec{j}})$. If b is a singleton of p colored $*$, all the elements of B_b are singletons in $\Phi(p)$, which is impossible since $d \geq 2$. If $b = \{b_1, b_2\}$ is a pair colored $*$, B_{b_1} is block-paired to B_{b_2} in $\Phi(p)$. If $j_{b_1} = -j_{b_2}$, the pairs between B_{b_1} and B_{b_2} have endpoints of different colors and thus verify the condition of $\mathcal{H}(d,d)$. If $j_{b_1} = j_{b_2}$, then $j_{b_1} = j_{b_2} = \pm 2$ and the pairs between B_{b_1} and B_{b_2} have endpoints with the same color; this is possible only if $d = 2$.
If $b = \{i_1, \dots, i_s\}$ is a block of p colored 1, then $B = B_{i_1} \cup B_{i_2} \cup \dots B_{i_s}$ is a block of $\Phi(p)$. The difference between the number of white and black points in B is exactly $\sum_{t=1}^{s} j_{i_t}$, thus the latter quantity has to be zero modulo d in order to have $\Phi(p) \in \mathcal{H}^{d,d}$.
Let us consider the category $\mathcal{H}^{+}$ and let $p \in \mathcal{P}_{\vec{j}}$ such that $\Phi(p) \in \mathcal{H}^{\#}$. A block $b = \{i_1, \dots, i_s\}$ colored 1 with an endpoint i_t such that $k_{i_t} \geq 2$ yields in $\Phi(p)$ a block with at least two consecutive endpoints of the same color; therefore, from the description of $\mathcal{H}^{\#}$, a block b of p colored 1 has only endpoints colored ± 1, with two consecutive endpoints having opposite colors. Since any element of b is colored ± 1, b yields a block with an even number of elements in $\Phi(p)$ if and only if b has an even number of elements. The description of $\mathcal{H}^{\#}$ yields that pairs must also have endpoints with opposite colors. Thus the only blocks colored $*$ allowed are the pairs with endpoints of opposite color.

□

The description of $\{u_1(\mathcal{C}), u_2(\mathcal{C}) \dots\}$ is the following:

Proposition 5.22. *Let $d \geq 2$. $(u_i(\mathcal{H}^{d,d}))_{i \geq 1}$ has the same joint law as $(sz^i s + c_i)_{i \geq 1}$, where :*

- *$(c_i)_{i \geq 1}$ is a family of free random variables with $c_1 = 0$, c_2 a circular variable (resp. semi-circular variable) of variance 1 and mean 0 for $d \geq 3$ (resp. $d = 2$) and c_i is a circular variable of variance 1 and mean 0 for $i \geq 3$.*

- *s and z are two free variables, also free from $(c_i)_{i \geq 1}$, s being a semi-circular variable of variance 1 and mean 0 and z a uniform variable on the complex $d-$roots of unity.*

$(u_i(\mathcal{H}^{+}))_{i \geq 1}$ is distributed as $u_i = c_i$ for $i \geq 2$, $u_1 = pz$, with c_k a family of free circular variables of variance 1 and mean 0, p a free poisson variable of mean 1, free from $\{c_k\}_{k \geq 2}$, and z a Bernoulli variable of mean 0 and variance 1 free from p and $\{c_k\}_{k \geq 2}$.

Proof. Let $(v_i)_{i \geq 1}$ be a random vector with $v_i = sz^i s + c_i$, s, z and $(c_i)_{i \geq 1}$ being as in the statement of the proposition. Let $(j_1, \dots, j_r) \in \mathbb{Z}_*^r$ and set $Y_i = sz^i s$. The moment cumulant formula yields

$$m_{\vec{v}}(j_1, \dots, j_r) = \sum_{(p,S) \in NC_p(\vec{j})} k_p(\vec{c}) m_{kr(p)}(\vec{Y}).$$

From the law of $\{c_i\}_{i \geq 1}$, $k_p(\vec{c}) \in \{0, 1\}$ and $k_p(\vec{c})$ is non-zero if and only if (p, S) is a pair partition with each block $\{a, b\}$ satisfying $j_a \neq \pm 1$ and such that $j_a = -j_b$ or, if $d = 2$, $j_a = j_b = \pm 2$. Let us denote by $\mathcal{P}(S)$ the set of such partial partitions.
For $i_1, \dots, i_t \geq 1$, $m_r(sz^{i_1} ssz^{i_2} s \dots sz^{i_t} s) = m(z^{i_1} s^2 z^{i_2} s^2 \dots z^{i_t} s^2)$. Since s^2 is a free Poisson distribution of variance 1, the moment-cumulant formula yields $m(z^{i_1} s^2 z^{i_2} s^2 \dots z^{i_t} s^2) = \#\{p \in NC(i_1, \dots, i_t), p \text{ is } d\text{−balanced}\}$, where p is said $d-$balanced if each block $\{x_1 < \dots < x_q\}$ satisfies the condition $\prod_{1 \leq \alpha \leq q} z^{i_{x_\alpha}} = 1$. The latter condition is equivalent to $\sum_{\alpha=1}^{q} i^{x_\alpha} = 0[d]$.

Thus

$$\begin{aligned}m_{\vec{v}}(j_1,\dots,j_r) = &\sum_{(p,S)\in NC_p(\vec{j})} k_p(\vec{c})\#\{\pi \le kr(p), \pi \text{ is } d-\text{balanced}\}\\ =&\#\{p\in\mathcal{P}(S), q\in NC(S^c)|p\vee q\in NC(\vec{j}), q \text{ } d\text{-balanced}\}.\end{aligned}$$

The latter set is exactly the set of non-crossing partitions of $NC(j_1,\dots,j_r)$ with blocks either colored 1 (the ones coming from (q,S^c)) and being $d-$balanced, or colored $*$ (the ones coming from (p,S)) with the endpoints satisfying the conditions of Lemma 5.21.Therefore

$$m_{\vec{v}}(j_1,\dots,j_r) = m(u_{j_1}(\mathcal{H}^{d,d}),\dots,u_{j_r}(\mathcal{H}^{d,d})).$$

The same proof yields the law of $\{u_k(\mathcal{H})^{\#}\}_{k\ge1}$. □

Case $\mathcal{B}$ In the case $\mathcal{B}$ there are three different families, depending on the value of the parameter $r\in\{*;0;d/2\}$. If $p\in NC(\vec{j})$ and $b=\{i_1,i_2\}$ is a pair of p, denote by $c(b)$ the sum of colors between i_1 and i_2. Namely $c(b)=\sum_{s=i_1+1}^{i_2-1} j_s$.

Lemma 5.23. *$\Phi^{-1}(\mathcal{B}^{d,r}(w_{\vec{j}}))$ is non-empty only if $\sum j_i = 0[d]$. If $\sum j_i = 0[d]$, $\Phi^{-1}(\mathcal{B}^{d,r}(w_{\vec{j}}))$ is the set of partitions p in $\mathcal{P}(\vec{j})$ such that:*

1. *p contains only singletons $\{i\}$ (which are colored $*$ if $k_i>2$), pairs colored $*$ or pairs with endpoints colored ±1.*
2. *If b is a pair of p with endpoints of opposite colors, $c(b)\equiv 0[d]$.*
3. *Let $r\in\{0,d/2\}$. If b is a pair of p with both endpoints colored either 1 or 2 (resp. -1 or -2), then $c(b)\equiv r-1[d]$ (resp. $c(b)=r+1[d]$). If $r\neq 1[d]$ (in particular $d\notin\{1,2\}$), p has no singleton $\{i\}$ colored 1 with $k_i=2$. If $d\notin\{1,2\}$, p has no pair $\{b_1,b_2\}$ with $j_{b_1}=j_{b_2}=\pm2$.*
4. *if $r=*$, all pairs have endpoints of opposite color and p has no singleton $\{i\}$ with $k_i=2$.*

Recall that since $p\in\mathcal{P}(\vec{j})$, a pair $\{b_1,b_2\}$ colored $*$ has endpoints of the same color only if $k_{b_1}=2$.

Proof. From Chapter 4, $\mathcal{B}^{d,r}(w_{\vec{j}})$ is non-empty only if $\sum w_{\vec{j}}(s)=\sum j_i = 0[d]$. If $\mathcal{B}^{d,r}(w_{\vec{j}})$ is non-empty, it consists in partitions q having the following properties:

1. q contains only singletons and pairs.
2. For any pair b of q having endpoints of opposite colors, $c(b)\equiv 0[d]$.
3. If $r\in\{0,d/2\}$, and b is a pair having black endpoints (resp. white endpoints), $c(b)\equiv r+1[d]$ (resp. $c(b)\equiv r-1[d]$).
4. If $r=*$, any pair of q has endpoints of opposite colors.

Suppose that $\sum j_i = 0[d]$. We shall characterize the set of partitions p of $\mathcal{P}_{\vec{j}}$ such that $\Phi(p)$ satisfies condition $1)-4)$.

1. From the definition of Φ, a block b of p yields a block B in $\Phi(p)$ having more than three elements if and only $b = \{i_1, \dots, i_s\}$ is a block colored 1 such that $|\bigcup B_{i_s}| > 2$. The only blocks colored 1 with $|\bigcup B_{i_s}| \leq 2$ are the singleton $\{i\}$ with $k_i \leq 2$ and the pairs $\{b_1, b_2\}$ with $k_{b_1} = k_{b_2} = 1$. Thus condition 1) on $\Phi(p)$ is equivalent to the first condition of the Lemma on p. From now on by a pair $\{b_1, b_2\}$ of p, we mean a pair colored 1 if $k_{b_1} = 1$ and colored $*$ otherwise: in any case this pair yields a block pairing in $\Phi(p)$.

2. An element of p colored j_i expands in k_i singletons of color ε_i in $\Phi(p)$. Moreover a pair $b = \{b_1, b_2\}$ of p yields a block pairing in $\Phi(p)$: if the endpoints of b have opposite colors, each pair of the block pairing has also endpoints of opposite colors. Therefore the condition $c(\tilde{b}) \equiv 0[d]$ for all pairs $\tilde{b}$ of the block pairing between B_{b_1} and B_{b_2} is equivalent to the condition $c(b) \equiv 0[d]$.

3. Let $r \in \{0, d/2\}$. The only blocks of p yielding pairs with endpoints of the same colors in $\Phi(p)$ are the pairs $\{b_1, b_2\}$ with $j_{b_1} = j_{b_2} \in [\![-2, 2]\!]$ and the singletons $\{i\}$ colored 1 (which implies $k_i = 2$ from the first point).
A pair $b = \{b_1, b_2\}$ with $j_{b_1} = j_{b_2} = \pm 2$ yields a block pairing in $\Phi(p)$ consisting in two pairs $\{a_1, a_2\}$ and $\{a_1 + 1, a_2 - 1\}$. Let us assume without loss of generality that $j_{b_1} = 2$. Then $a_1, a_1 + 1, a_2 - 1, a_2$ are colored $\circ$. Since $\{a_1, a_2\}$ is a pair with endpoints colored $\circ$, the third condition of $\mathcal{B}^{d,r}$ yields that $\sum_{i=a_1+1}^{a_2-1} w(i) = r - 1[d]$. But $\{a_1 + 1, a_2 - 1\}$ is also a pair with endpoints colored $\circ$. Therefore $\sum_{i=a_1+2}^{a_2-2} w(i) = r - 1[d]$. This implies that $w_{a_1+1} + w_{a_2-1} = 0[d]$. Since $a_1 + 1$ and $a_2 - 1$ are colored $\circ$, $w_{a_1+1} + w_{a_2-1} = 2$. Thus d has to be equal to 1 or 2. Since $\sum_{i=a_1+2}^{a_2-2} w(i) = \sum_{i=b_1+1}^{b_2-1} j_i$, the condition $\sum_{i=a_1+2}^{a_2-2} w(i) = r[d]$ is equivalent to the condition $c(b) = r[d]$.
A singleton $\{i\}$ colored 1 with $k_i = 2$ yields a pair $\tilde{b} = \{b_1, b_1 + 1\}$ with endpoints of the same color in $\Phi(p)$. $c(\tilde{b}) = 0$ means that $r + 1 = 0[d]$ or $r - 1 = 0[d]$. Since $r \in \{0, d/2\}$, either $d = 1$, or $r = 1$ and $d = 2$. In these two cases we have also $c(b) = 0 = r + 1[d]$.
In any other cases, $b = \{b_1, b_2\}$ has endpoints of the same color if $c(b_1) = c(b_2) = \pm 1$. Let us assume that $c(b_1) = 1$. Thus b yields a pair $\tilde{b} = \{a_1, a_2\}$ in $\Phi(p)$ with endpoints colored $\circ$. Since $\sum_{a_1+1}^{a_2-1} w(i) = \sum_{i=b_1+1}^{i=b_2-1} j_i$, the condition $c(\tilde{b}) = r - 1[d]$ is equivalent to $c(b) = r - 1[d]$.

4. A block b of p yields in $\Phi(p)$ pairs having endpoints of different colors if and only if b is a pair having endpoints of opposite colors: thus if $r = *$, then $\Phi(p)$ satisfies the fourth property of $\mathcal{B}^{d,*}$ if and only if all pairs of p have endpoints of opposite colors.

□

The latter lemma yields the law of $\{u_i(\mathcal{B}^{d,r})\}$:

Proposition 5.24. *1. Let $r = *$. The family $\{u_i(\mathcal{B}^{d,*})\}_{i \geq 1}$ is distributed as $u_i = c_i + z^i$, with $\{c_i\}_{i \geq 1}$ a family of circular variables of mean 0 and variance 1, and z a variable free from $\{c_i\}_{i \geq 1}$ and distributed uniformly on the $d-$th roots of unity.*

2. Let $r = 0$, $d \geq 3$. The family $\{u_i(\mathcal{B}^{d,0})\}_{i \geq 1}$ is distributed as $u_i = c_i + z^i$ for $i \geq 2$, $u_1 = c_1 z$, where $\{c_i\}_{i \geq 2}$ is a family of circular variables of mean 0 and variance 1, z is a variable free from $\{c_i\}_{i \geq 2}$ and distributed uniformly on the $d-$th roots of unity, and c_1 is a free semi-circular variable of variance 1 and mean 0. If $d = 2$, the distribution is the same except that $u_2 = c_2 z + 1$ with c_2 a semi-circular variable of variance 1 and mean 0, free from the other variables.

3. *Let $r = d/2$. The family $\{u_i(\mathcal{B}^{d,d/2})\}_{i\geq 1}$ is distributed as $u_i = c_i + z^i$ for $i \geq 2$ and $u_1 = c_1 z^{d/2+1} + z$, with $\{c_i\}_{i\geq 2}$ a family of circular variables of mean 0 and variance 1, z a variable free from $\{c_i\}_{i\geq 1}$ and distributed uniformly on the $d-$th roots of unity, and c_1 a free semi-circular variable of mean 0 and variance 1. If $d \in \{1,2\}$, the law is the same except that $u_2 = c_2 + 1$, with c_2 is a semi-circular variable of mean 1 and variance 1 free from the other variables.*

Proof. Let $r \in \{*, 0, d/2\}$, and let $\{t_k\}_{k\geq 1}$ be a family of variables with the expected law. Set $t_{-k} = t_k^*, c_{-k} = c_k^*$. Thus we have to prove that the mixed moment $m(t_{j_1}, \ldots, t_{j_s})$ is exactly the number of partitions of $\mathcal{P}_{\vec{j}}$ satisfying the conditions of Lemma 5.23. On one hand, expanding the product in $t_{j_1} \ldots t_{j_s}$ yields:

$$m(t_{j_1}, \ldots, t_{j_s}) = \sum_{S \subseteq [\![1,s]\!]} m(\vec{\omega}),$$

where $\omega_i = z^{j_i}$ if $i \notin S$ and $\omega_i = c_{j_i}$ if $i \in S$ (or $c_{j_i} z^{r+1}$ if $t_{j_i} = c_{j_i} z^{r+1} + z^{j_i}$, $z^{r-1} c_{j_i}$ if $t_{j_i} = z^{r-1} c_{j_i} + z^{j_i}$).
If $r = *$, we formally set $r+1 = r-1 = 0$. Let us define a sequence $\tilde{\omega}$ of length $3s$ and a coloring $\vec{h}$ of $[\![1, 3s]\!]$ from ω as follows : for $1 \leq i \leq s$,

- $\tilde{\omega}(3i-1) = c_{j_i}$ and $h(3i-1) = 0$ if $i \in S$, $\tilde{\omega}(3i-1) = z^{j_i}$ and $h(3i-1) = j_i$ else,
- $\tilde{\omega}(3i-2) = z^{r-1}$ and $h(3i-2) = r-1$ if $t_i = c_{-1}$ (or $t_i = c_{-2}$ and $d \in \{1,2\}$), $\tilde{\omega}(3i-2) = 1$ and $h(3i-2) = 0$ else,
- $\tilde{\omega}(3i) = z^{r+1}$ and $h(3i) = r+1$ if $t_i = c_1$ (or $t_i = c_2$ and $d \in \{1,2\}$), $\tilde{\omega}(3i) = 1$ and $h(3i) = 0$ else.

Let $\tilde{S}$ be the subset $\{3i-1 | 1 \leq i \leq s, t_i = c_{j_i}\}$ of $[\![1, 3s]\!]$. Since $\{c_i\}$ and z are free, the moment-cumulant formula yields

$$m(\vec{\omega}) = \sum_{(\tilde{p}, \tilde{S})} k_{(\tilde{p}, \tilde{S})}(\tilde{\omega}) m_{kr(\tilde{p}, \tilde{S})}(\omega),$$

where $\tilde{p}$ is a non-crossing partial partition whose support is $\tilde{S}$. Since $S \simeq \tilde{S}$, there is a bijection $p \mapsto \tilde{p}$ between partial partitions of $[\![1, s]\!]$ with support S and partial partitions of $[\![1, 3s]\!]$ with support $\tilde{S}$, and thus

$$m(\vec{\omega}) = \sum_{p \in NC(S)} k_{(\tilde{p}, \tilde{S})}(\tilde{\omega}) m_{kr(\tilde{p}, \tilde{S})}(\omega).$$

The elements outside $\tilde{S}$ consist only in powers of z, thus a block B of $kr(\tilde{p})$ yields a moment 1 if $\prod_{i \in B} \tilde{\omega}(i) = 1$ (namely if $\sum_{i \in B} h(i) = 0[d]$), and 0 else. Since $\{c_i\}$ is a collection of free semi-circular or circular variables, $k(\tilde{p}) \in \{0, 1\}$ and $k(\tilde{p})$ is zero if $\tilde{p}$ has blocks with more than three elements or blocks containing different c_i. Thus

$$m(t_1, \ldots, t_s) = \#\{(p, S) | k_{(\tilde{p}, \tilde{S})}(\tilde{\omega}) = 1, \forall B \in kr(\tilde{p}), \sum_{i \in B} h_i = 0[d]\}.$$

The condition $\forall B \in kr(\tilde{p}), \sum_{i \in B} h(i) = 0[d]$ is equivalent to the condition that $\sum h(i) = 0[d]$ and that for any pair $\tilde{b} = \{\tilde{b}_1 < \tilde{b}_2\}$ in $\tilde{p}$, $c(\tilde{b}) = 0[d]$. Moreover

- If $\tilde{b}_1$ and $\tilde{b}_2$ have opposite colors, then $h(\tilde{b}_1 + 1) = -h(\tilde{b}_2 - 1)^{-1}$. Thus $c(\tilde{b}) = 0[d]$ if and only if $\sum_{\tilde{b}_1 + 1 < i < \tilde{b}_2 - 1, i \in \tilde{S}^c} h(i) = 0[d]$.

- If $\tilde{b}_1$ and $\tilde{b}_2$ are colored white (resp. black), $h(\tilde{b}_1+1) = r+1$ and $h(\tilde{b}_2-1) = 1$ (resp. $h(\tilde{b}_1+1) = 1$ and $h(\tilde{b}_2-1) = r-1$). Thus $c(\tilde{b}) = 0[d]$ is equivalent to $\sum_{\tilde{b}_1+1<i<\tilde{b}_2-1, i\in\tilde{S}^c} h(i) = r-1$ if $\tilde{b}_1$ and $\tilde{b}_2$ are white and to $\sum_{\tilde{b}_1+1<i<\tilde{b}_2-1, i\in\tilde{S}^c} h(i) = r+1$ if $\tilde{b}_1$ and $\tilde{b}_2$ are black.

It remains to show that $\sum_{\tilde{b}_1+1<i<\tilde{b}_2-1, i\in\tilde{S}^c} h(i) = \sum_{b_1<i<b_2} j_i$, where $\{b_1, b_2\}$ is the pair of p yielding $\tilde{b}$ in $\tilde{p}$. An element i in S^c colored j_i yields in $(\tilde{p}, \tilde{S})$ three elements $3i, 3i+1, 3i+2$ colored respectively $0, j_i, 0$: so the total contribution is j_i. An element i is a singleton of (p, S) only if $j_i = \pm 2$ and $r = 1, d \in \{1, 2\}$. It contributes in $\tilde{p}$ to three elements $3i, 3i+1, 3i+2$ colored 0, whose sum is exactly j_i modulo 1 or 2.
A pair $\{b_1, b_2\}$ of (p, S) with endpoints of opposite colors contributes to one pair, and four singletons, two of them colored 0, and two having opposite colors, yielding a null contribution. A pair $\{b_1, b_2\}$ with endpoints of the same color only occurs if $j_{b_1} \in \{\pm 1, \pm 2\}$ and $r \in \{0, d/2\}$. If $j_{b_1} > 0$, this pair contributes in $\tilde{p}$ to one pair and four singletons, two of them colored 0 and two of them colored $r+1$. Thus the total contribution is $2r+2$; but $r \in \{0, d/2\}$ and thus $2r+2 = 2[d]$. If $j_{b_1} =< 0$, the same reasoning yields a contribution equal to -2 modulo d. If $j_{b_1} = \pm 1$, this contribution is equal to $j_{b_1} + j_{b_2}$. If $j_{b_1} = \pm 2$, $d \in \{1, 2\}$ and once again the contribution is equal to $j_{b_1} + j_2$.
To sum up, $(\tilde{p}, \tilde{S})$ verifies the conditions if and only if $\sum j_i = 0$ and (p, S) is such that $c(b) = 0[d]$ if b is a pair with endpoints of different colors and $c(b) = r-1[d]$ (resp. $c(b) = r+1[d]$) if b is a pair with white (resp. black) endpoints. Filling (p, S) with singletons on elements of S^c gives exactly the partitions satisfying the conditions of Lemma 5.23. □

Case $\mathcal{S}$ The computation for the category $\mathcal{S}^d$ is a simpler version of the computation for the category $\mathcal{B}^{d,r}$. For $(c_1, \ldots, c_r)$ a sequence of integers (for example an element of $\mathbb{Z}_*^r$ or a word in $\circ, \bullet$ with the usual substitution $\circ \leftrightarrow 1, \bullet \leftrightarrow -1$), we set $c(s,t) = \sum_{s<x<t} c_x$.

Lemma 5.25. *$\Phi^{-1}(\mathcal{S}^d(w_{\vec{j}}))$ is the set of partitions p in $\mathcal{P}(\vec{j})$ with:*

1. *If $d \geq 3$, blocks of p colored 1 contain only elements colored ± 1 and blocks colored $*$ have endpoints of opposite colors. If $d = 2$ the same holds except that all pairs colored $*$ are allowed.*

2. *If i_1, i_2 are two consecutive elements of the same block and have positive (resp. negative) color, $j(i_1, i_2) \equiv -1[d]$ (resp. $+1$). If $i_1 \sim_p i_2$ and $j_{i_1} = -j_{i_2}$, then $j(i_1, i_2) \equiv 0[d]$.*

Proof. The category $\mathcal{S}^d(w_{\vec{j}})$ is the set of non-crossing partitions p having the following property: for i_1 and i_2 two consecutive elements of a block, $w_{\vec{j}}(i_1, i_2) \equiv 0[d]$ (resp. $w_{\vec{j}}(i_1, i_2) \equiv -1[d]$, resp. $w_{\vec{j}}(i_1, i_2) \equiv 1[d]$) if $\varepsilon(i_1) = -\varepsilon(i_2)$ (resp. $\varepsilon(i_1) = \varepsilon(i_2) = 1$, $\varepsilon(i_1) = \varepsilon(i_2) = -1$). If $d = 1$, this just means that any non-crossing partition is allowed, and thus $\Phi^{-1}(\mathcal{S}^1(w_{\vec{j}})) = \Phi^{-1}(NC(\vec{j})) = \mathcal{P}(\vec{j})$. If $d \geq 2$, the same proof as the one of Lemma 5.23 yields the result. □

The proof of the law of $(u_i(\mathcal{S}^d))_{i\geq 1}$ is similar to the one of Proposition 5.24, so we only state the result:

Proposition 5.26. *Let $d \geq 3$. The family $(u_i(\mathcal{S}^d))_{i\geq 1}$ is distributed as $u_i = c_i + z^i$ for $i \geq 2$, with $\{c_i\}_{i\geq 2}$ a family of free circular variables of variance 1 and mean 0 and $u_1 = pz$ with p a free Poisson distribution of mean 1 free from $\{c_i\}_{i\geq 2}$ and z a uniform variable on the $d-$roots of unity, free from p and $\{c_i\}_{i\geq 2}$.*
If $d = 2$, the law is the same as before, except that $u_2 = c_2 z + 1$ with c_2 a semi-circular variable of variance 1 and mean 0.
If $d = 1$, for all $i \geq 1$, $u_i = p + c_i$, with p a free poisson variable of mean 1, $\{c_i\}_{i\geq 3}$ a family

of free circular variables withs mean 0 *and variance* 1, c_2 *a semi-circular variable with mean* 1 *and variance* 1 *and* $c_1 = 0$.

5.3 Second-order freeness for the free unitary group

In [62], Cor. 15, the results of Diaconis and Shahshahani on the unitary group have been extended to the situation where traces of arbitrary cyclically reduced products of $u, \bar{u}, u^t, u^*$ were considered (see also [69]). We give here a similar result for the free unitary group. The method used in the present chapter differs from the one of [62] in order to take the non-commutativity of U_n^+ into account. However both rely on the Weingarten formula, and the proof of the present section can be used to recover the results of [62]. In the case of U_n^+, the non-commutativity allows to consider the law of any reduced words in $u, \bar{u}, u^t, u^*$, instead of only considering cyclically reduced words. We present the result in the classical and free case.
To state the result and detail the proof, we shall use the notations introduced by Radulescu in [69]. Let $\mathbb{F}_2$ be the free group with two generators a and b. Any element f of $\mathbb{F}_2$ admits a unique representation as a reduced word $f_1 f_2 \dots f_r$ in a, b, a^{-1} and b^{-1}. f is called cyclically reduced if $f_r \neq f_1^{-1}$.
For $n \geq 1$, we associate to each $f \in F_2$ the variable $X_f(n) = Tr(u^{f_1} \dots u^{f_r})$, where u is the fundamental matrix of U_n^+ or U_n and $u^a = u, u^{a^{-1}} = u^*, u^b = \bar{u}, u^{b^{-1}} = u^t$.
With these notations we have a main theorem describing the asymptotice laws of $\{X_f(n)\}_{f \in \mathbb{F}_2}$ for U_n^+ and U_n. From now on, we always assume that elements of $\mathbb{F}_2$ are cyclically reduced in the classical case.

Theorem 5.27. *In the free case (resp. classical case), when n tends to infinity, the collection of variables* $\{X_f(n)\}$ *converges in law to a circular (resp. gaussian) system of free (resp. independent) variables* $\{\tilde{X}_f\}$ *whose covariance matrix can be explicitly described.*

The description of the covariance matrix is given in Proposition 5.36.
Throughout this section, $r \geq 1$ is fixed, and we use the convention $r + 1 := 1$.

5.3.1 Bidiagrams associated to $X_f(n)$

The proof of Theorem 5.27 relies on some properties of a combinatorial object called bidiagram. We present here this object and prove some useful combinatorial results.

Bidiagrams Let $k \geq 1$. We define an involution $i \mapsto \bar{i}$ on $\{1, \dots, 2k\}$ by the formula $\bar{i} = 2k - i + 1$.
We define p_k as the partition of $P(k,k)$ defined by $p_k = \{\{i, \bar{i}\}, 1 \leq i \leq k\}$ and for $1 \leq i \leq k$, we denote by S_i the subset $\{i, \bar{i}\}$.

Definition 5.28. *Let* $k \geq 1$. *A cyclic partition* p *of* $P(k,k)$ *is a two level pair partition such that* i *is paired either with* $i+1, i-1, \overline{i-1}$ *or* $\overline{i+1}$, *and such that* $p \vee_P p_k$ *is the one block partition* $\{[\![1, 2k]\!]\}$.
A tracial diagram $\mathcal{D} \in P(k,k)$ *is a tensor product of cyclic partitions* $p_1 \otimes \dots \otimes p_r$ *such that* $p_i \in P(k_i, k_i)$ *with* $\sum k_i = k$.

We write $\mathcal{D} = p_1 \otimes \dots \otimes p_r$ to emphasize the unique decomposition into cyclic partitions. The conditions in the definition of a cyclic partition is equivalent to the fact that for any $1 \leq i \leq k$,

p links exactly one element of S_i with an element of S_{i+1}, where $i+1$ is understood modulo k. The same definition holds for colored cyclic partitions and colored tracial diagrams. A two-colored tracial diagram is represented in Figure 5.1.

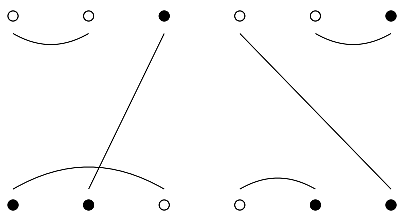


Figure 5.1: Tracial diagram consisting in two cyclic partitions

We can now define a bidiagram :

Definition 5.29. *Let $\varepsilon_1, \varepsilon_2$ be two words in $\circ, \bullet$. A bidiagram is the data of a tracial diagram $\mathcal{D} \in P(\varepsilon_1, \varepsilon_2)$ and two partitions p, q such that $p \in P(\varepsilon_1)$ and $q \in P(\varepsilon_2)$.*

A bidiagram is written $(p|\mathcal{D}|q)$. An example is shown in Figure 5.2.

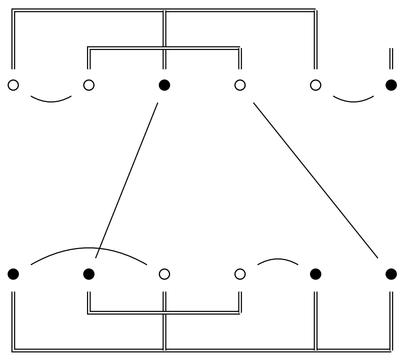


Figure 5.2: Two-colored bidiagram

The natural bijection between $P(\varepsilon_1)$ and $P(\varepsilon_1, \emptyset)$, and between $P(\varepsilon_2)$ and $P(\varepsilon_2, \emptyset)$ identifies (p, q) with an element of $P(\varepsilon_1, \varepsilon_2)$ through the composition $pR_h(q)$ (recall the definition of R_h in Chapter 1). In particular the blocks of $b(p|\mathcal{D}|q)$ are defined as the blocks of the partition $\mathcal{D} \vee pR_h(q)$. We denote also by $b(p|\mathcal{D}|q)$ the number of blocks of $\mathcal{D} \vee pR_h(q)$.

Diagrams coming from $\{X_f(n)\}$: A cyclic partition $\mathcal{D}_f$, independent of n, is associated to each variable $X_f(n)$ in the following way :

- Let $f \in \mathbb{F}_2$. We can write the reduced form of f as

$$f = x_{\eta(1)}^{\varepsilon(1)} x_{\eta(2)}^{\varepsilon(2)} \dots x_{\eta(r)}^{\varepsilon(r)} = \prod_{\rightarrow} x_{\eta(i)}^{\varepsilon(i)},$$

  where r is the length of the reduced word, $\prod_{\rightarrow}$ stands as an ordered product, and $x_{\eta_i}^{\varepsilon_i} \in \{a, b, a^{-1}, b^{-1}\}$ according to the following rule :

$$\begin{array}{|c|c|c|} \hline \eta \backslash \varepsilon & 1 & -1 \\ \hline 1 & a & b \\ \hline -1 & b^{-1} & a^{-1} \\ \hline \end{array} \tag{5.3.1}$$

The fact that the above expression comes from a reduced word of f implies that $\varepsilon_i\varepsilon_{i+1} + \eta_i\eta_{i+1} \neq -2$ for $1 \leq i \leq r-1$. If f is cyclically reduced, the same holds for $i = r$. The words $\eta(1)\dots\eta(r)$ and $\varepsilon(1)\dots\varepsilon(r)$ are denoted $\eta(f)$ and $\varepsilon(f)$.

- The partition $\mathcal{D}_f$ in $P(\varepsilon, \varepsilon)$ is constructed by pairing, for $1 \leq i \leq r$, exactly one element of S_i with one element of S_{i+1} with the rule:

$\eta(i+1)\backslash\eta(i)$	1	-1
1	$(i, \overline{i+1})$	$(\bar{i}, i+1)$
-1	$(i, i+1)$	$(\bar{i}, \overline{i+1})$

(5.3.2)

For example the cyclic partition associated to $ab^2a^{-2}b$ is drawn in Figure 5.3:

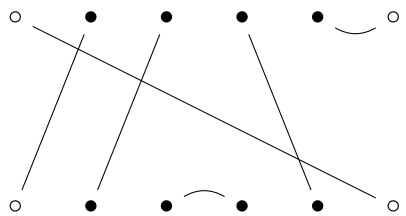


Figure 5.3: The cyclic partition $\mathcal{D}_{ab^2a^{-2}b}$

The rules (5.3.1) and (5.3.2) yield that the partition $\mathcal{D}_f$ is indeed a cyclic partition. Thus if $f^1, \dots, f^s \in \mathbb{F}_2$, $\mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^s}$ is a tracial diagram. Actually one can prove that any tracial diagram such that the coloring is the same on each set S_i can be written $\mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^s}$ for some $f^1, \dots, f^s \in \mathbb{F}_2$.
We will need some basic facts on tracial diagrams. They are summed up in the following Lemma :

Lemma 5.30. *Let $f = f_1 \dots f_r \in \mathbb{F}_2$ and let D_f be the associated cyclic partition:*

- *For $1 \leq i \leq r-1$ two consecutive elements i and $i+1$ (or $\bar{i}$ and $\overline{i+1}$) can be paired only if they have the same color. If f is cyclically reduced, 1 is paired with r (resp. $\bar{1}$ is paired with $\bar{r}$) only if both elements have the same color.*

- *There is the same number of pairs between elements of $\{1, \dots, r\}$ and between elements of $\{\bar{1}, \dots, \bar{r}\}$.*

Proof.
- Since f is a reduced word, the rule (5.3.1) yields that for $1 \leq i \leq r-1$, $\eta(i)\eta(i+1)$ and $\varepsilon(i)\varepsilon(i+1)$ are not both equal to -1. Therefore by (5.3.2), this means that two consecutive points of different colors cannot be paired. If f is cyclically reduced, the same is truc for $i = r$ and $i + 1 = 1$ (or $\bar{r}$ and $\bar{1}$).

- By the rule (5.3.2), $\bar{i}$ is paired with $\overline{i+1}$ if and only if $\eta(i+1) = 1$ and $\eta(i) = -1$, yielding $\eta(i+1) - \eta(i) = 2$; i is paired with $i+1$ if and only if $\eta(i+1) = -1$ and $\eta(i) = 1$, yielding $\eta(i+1) - \eta(i) = -2$. In all other cases, $\eta(i) = \eta(i+1)$ and thus $\eta(i) - \eta(i+1) = 0$. Since $\sum_{i=1}^{r} \eta(i+1) - \eta(i) = 0$, the number of pairs inside $\{1, \dots, r\}$ and inside $\{\bar{1}, \dots, \bar{r}\}$ are the same.

$\square$

A pair containing two points of the same row is called an horizontal strip. The other ones are called vertical strips.

Bidiagrams with pair partitions Since the intertwiners of U_n^+ and U_n are described by pair partitions, we will use bidiagrams $(p|\mathcal{D}|q)$ with p, q being pairings; as in the previous section, the use of the Weingarten formula requires bounds on the number of blocks of $(p|\mathcal{D}|q)$. Recall that $\mathcal{U}$ (resp. $\mathcal{U}_{class}$) denotes the set of non-crossing pairings (resp. pairings) with pairs having endpoints of opposite colors.

Proposition 5.31. *Let $f^1, \dots, f^s \in \mathbb{F}_2$ and $p, q \in \mathcal{U}_{class}(\varepsilon(f^1)\dots\varepsilon(f^s))$. Let $\mathcal{D} = \mathcal{D}(f^1) \otimes \dots \otimes \mathcal{D}(f^s)$ We suppose that p and q are in $\mathcal{U}$ if at least one f^i is not cyclically reduced. Then*

$$b(p|\mathcal{D}|q) \leq \frac{b(p) + b(q)}{2}.$$

In the case of non cyclically reduced terms, the statement needs a preliminary combinatorial result. From the definition of a bidiagram, a pair of $(p|\mathcal{D}|q)$ is a set $\{b_1, b_2\}$ in $\{1, \dots, k, \bar{1}, \dots, \bar{k}\}$ such that $b_1 \sim_{pR_h(q)} b_2$ and $b_1 \sim_{\mathcal{D}} b_2$. Since partitions $\mathcal{D}_{f^i}$ are disconnected from each other, b_1 and b_2 must be in a same block $\mathcal{D}_{f^i}$ to fulfill the condition $b_1 \sim_{\mathcal{D}} b_2$. The notations in the following Lemma are the same as in 5.31.

Lemma 5.32. *Suppose that p, q are in $\mathcal{U}(\varepsilon(f^1)\dots\varepsilon(f^s))$. For $1 \leq i \leq s$, there is at most one pair t_i in $\mathcal{D}(f^i)$ which is also a block of $(p|\mathcal{D}|q)$. If this pair exists, there is a block of $(p|\mathcal{D}|q)$ containing at least 6 elements, among which at least 4 are in $\mathcal{D}(f^i)$.*

Proof. Let $1 \leq i \leq s$ and suppose that such a t_i exists in D_{f^i}, with endpoints $(a_i < b_i)$. By a rotation of all the partitions, we can suppose that $i = 1$.
Since t_1 is a block of $(p|\mathcal{D}|q)$, t_1 is also block of p or q. By symmetry we can assume that $t_1 \in p$ and $a_1, a_2 \in \{1, \dots, r\}$. a_1 and b_1 being in the same row, they are linked by a horizontal edge in $\mathcal{D}_{f^1}$. Since $\mathcal{U} = \langle \sqcap_{\circ\bullet}, \sqcap_{\bullet\circ} \rangle$, the two endpoints have opposite colors; therefore 5.30 yields that $a_1 = 1$ and $b_1 = r$ in $\mathcal{D}(f^1)$. Thus, since p is non crossing, any pair of p having an endpoint in $\{2, r-1\}$ must have its other endpoint in $\{2, r-1\}$. This implies that $\mathcal{D}_{f^1}$ has as many points of both colors. Let us denote by $h_{u,w}$ (resp. $h_{u,b}, h_{d,w}, h_{d,b}$) the number of upper white (resp. upper black, lower white, lower black) points belonging to an horizontal strip of $\mathcal{D}_{f^1}$. Then we have :

- By 5.30, $h_{u,b} + h_{u,w} = h_{d,w} + h_{d,b}$.
- By 5.30, since in the upper part all horizontal strips except (a_1, b_1) link points of the same color, $h_{u,b}$ and $h_{u,w}$ are odd; in the lower part all horizontal strips link points of the same color, thus $h_{d,b}$ and $h_{d,w}$ are even.

Suppose that there is a pair $t' = \{a, b\}$ in p with a in an horizontal strip $\{a, a'\}$ and b in a vertical strip $\{b, b'\}$; a' and b' cannot be in the same pair of p or q, and thus t' is in a block of $(p|\mathcal{D}|q)$ with more than 6 elements among which at least 4 elements are in $\mathcal{D}(f^1)$.
Suppose that there is no pair $t \in p$ with one endpoint in an horizontal strip of $\mathcal{D}(f^1)$, and the other endpoint in a vertical strip. This means that we must have in the upper row as many horizontal strips with endpoints colored black and horizontal stips with endpoints colored white. Thus $h_{u,b} = h_{u,w}$. Since $h_{u,b} + h_{u,w} = h_{d,w} + h_{d,b}$, a parity argument yields that either $h_{d,b} > h_{u,b} = h_{u,w} > h_{d,w}$, either $h_{d,w} > h_{u,b} = h_{u,w} > h_{d,b}$. By a color symmetry we can assume that we are in the first case and $h_{d,b} + h_{u,b} > h_{u,w} + h_{d,w}$. Since there are as many points of both colors in $\mathcal{D}_{f^1}$, the number of white points linked by a vertical edge is strictly greater than the number of black points linked by a vertical edge. This implies that there must be a pair $t' \in p$ whose endpoints are linked to vertical edges with white lower endpoints. Since these two lower endpoints have the same color, they cannot be in the same pair of q and each of them are in

different pairs of q; therefore t' is in a block of $(p|\mathcal{D}|q)$ with more than 6 elements, among which at least 4 elements are in $\mathcal{D}(f^1)$.

$\square$

Let us now prove Proposition 5.31:

Proof. Let l_i be the length of the reduced word of f^i. If one of the pair partitions is crossing, from the hypothesis of the Proposition, all words are cyclically reduced. Therefore there is no horizontal strips between points of different colors. Since p is in $\mathcal{U}$ or $\mathcal{U}_{class}$, each pair of p has endpoints of opposite colors; thus endpoints of a pair of p cannot be in the same pair of $\mathcal{D}$. In particular each block of $(p|\mathcal{D}|q)$ contains at least two pairs from p or q, and $b(p|\mathcal{D}|q) \leq \frac{b(p)+b(q)}{2}$. From now on we assume that p, q are non-crossing.
By the previous lemma there is at most one block of $(p|\mathcal{D}_{f^1} \otimes \cdots \otimes \mathcal{D}_{f^r}|q)$ having two vertices in each $\mathcal{D}(f^i)$: denote by $B_{i(1)}, \ldots, B_{i(k)}$ the blocks of $(p|\mathcal{D}_{f^1} \otimes \cdots \otimes \mathcal{D}_{f^r}|q)$ having only two vertices (with $B_{i(j)}$ lying in $\mathcal{D}_{f^{i(j)}}$). By the same lemma, one can associate to each block $B_{i(j)}$ a block $\tilde{B}_{i(j)}$ having at least 6 vertices and at least four vertices in $\mathcal{D}_{i(j)}$. Some blocks $\tilde{B}_{i(j)}$ may be the same: for $1 \leq s, t \leq k$, write $s \sim t$ if $\tilde{B}_{i(s)} = \tilde{B}_{i(t)}$. This gives a partition $A_1 \amalg \cdots \amalg A_l$ of $\{1, 2, \ldots, k\}$ and a bijective map $\varphi : [\![1, l]\!] \to \{\tilde{B}_{i(j)}\}_{1 \leq i \leq k}$ sending s to the block $\tilde{B}_{i(j)}$ such that $j \in A_s$. By Lemma 5.32, $\varphi(s)$ has at least $\min(6, 4|A(s)|)$ elements. Moreover

$$\begin{aligned}
\sum_{j=1}^{k} |B_{i(j)}| + \sum_{j=1}^{l} |\varphi(j)| =& 2k + \sum_{j, A_j \text{ singleton}} |\varphi(j)| + \sum_{j, |A(j)| \geq 2} |\varphi(j)| \\
\geq& 2k + 6|\{j, A(j) \text{ singleton}\}| + \sum_{j, |A(j)| \geq 2} 4|A(j)| \\
\geq& 2k + \left(2|\{j, A(j) \text{ singleton}\}| + 2 \sum_{j, |A(j)| \geq 2} |A(j)| \right) \\
& + 4|\{j, A(j) \text{ singleton}\}| + 2 \sum_{j, |A(j)| \geq 2} |A(j)| \\
\geq& 4k + 4|\{j, A(j) \text{ singleton}\}| + 4|\{j, A(j) \geq 2\}| \\
\geq& 4(k + l).
\end{aligned}$$

On one hand, since p and q are pair partitions, $b(p) = b(q) = \frac{1}{2}(\sum_{i=1}^{r} l_i)$. On the other hand,

$$\sum_{B \in \{\text{blocks of } (p|\mathcal{D}_{f^1} \otimes \cdots \otimes \mathcal{D}_{f^r}|q)\}} |B| = \sum_{i=1}^{r} 2l_i.$$

But from the previous computation,

$$\begin{aligned}
\sum_{\substack{B \text{ block of} \\ (p|\mathcal{D}_{f^1} \otimes \cdots \otimes \mathcal{D}_{f^r}|q)}} |B| =& \sum_{j=1}^{k} |B_{i(j)}| + \sum_{j=1}^{l} |\varphi(j)| + \sum_{\substack{B \text{ other block of} \\ (p|\mathcal{D}_{f^1} \otimes \cdots \otimes \mathcal{D}_{f^r}|q)}} |B| \\
\geq& 4(k + l) + 4|\{\text{other block of } (p|\mathcal{D}_{f^1} \otimes \cdots \otimes \mathcal{D}_{f^r}|q)\}| \\
\geq& 4b(p|\mathcal{D}_{f^1} \otimes \cdots \otimes \mathcal{D}_{f^r}|q).
\end{aligned}$$

Therefore

$$\begin{aligned} b(p|\mathcal{D}_{f^1}\otimes\cdots\otimes\mathcal{D}_{f^r}|q) \leq & \frac{1}{4}\sum_{B\in\{\text{blocks of }(p|\mathcal{D}_{f^1}\otimes\cdots\otimes\mathcal{D}_{f^r}|q)\}} |B| \\ \leq & \frac{2\sum l_i}{4} \leq \frac{b(p)+b(q)}{2}. \end{aligned}$$

□

5.3.2 Convergence and asymptotic law for $\{X_f(n)\}_{f\in\mathbb{F}_2}$

As in Section 5.2, the proof of Theorem 5.27 is done in two parts. The first part gives a combinatorial expression of the moments and the second part deduces the law of $\{X_f(n)\}_{f\in\mathbb{F}_2}$ from this expression.

Moments formula for $\{X_f\}$ Theorem 5.8 can be adapted in the present situation to get the following result:

Proposition 5.33. *Let G_n be either U_n or U_n^+. Let $f^1,\dots,f^r$ be elements of $\mathbb{F}_2$ such that $\sum l_i \leq n$. We assume that $f^1,\dots,f^r$ are cyclically reduced if $G = U_n$. Then*

$$\int_{G_n} X_{f^1}(n)\dots X_{f^r}(n) = Card\{p\in\mathcal{C}|b(p|D_{f^1}\otimes\cdots\otimes D_{f^r}|p) = b(p)\}, \tag{5.3.3}$$

with $\mathcal{C}=\mathcal{U}$ if $G_n = U_n^+$ and $\mathcal{C}=\mathcal{U}_{class}$ if $G_n = U_n$.

Proof. The proof of the proposition follows the proof of Theorem 5.8, with the help of the previous combinatorial results.
Let $1\leq m\leq r$. The definition of $X_{f^m}(n)$ yields:

$$X_{f^m}(n) = \sum_{j=1}^{n}\left(\overrightarrow{\prod_{1\leq t\leq l_m}} u^{x^{\varepsilon(t)}_{\eta(t)}}\right)_{jj} = \sum_{j_1,\dots,j_{l_i}=0}^{n}\overrightarrow{\prod_{1\leq t\leq l_m}}\left(u^{x^{\varepsilon(i)}_{\eta(i)}}\right)_{j_tj_{t+1}},$$

with $j_{l_m+1}=j_1$, and η,ε the words associated to f^m in Section 5.3.1. From the definition of u^x for $x\in\{a,b,a^{-1},b^{-1}\}$, the values of $u_{ij}^{x^{\varepsilon(t)}_{\eta(t)}}$ depend on $\{\varepsilon(i),\eta(i)\}$ following the present rule:

$\eta\backslash\varepsilon$	1	-1
1	(u_{ij})	$(\overline{u_{ij}})$
-1	(u_{ji})	$(\overline{u_{ji}})$

If we write $u_{ij}^{-1}=\overline{u_{ij}}$ and $u_{(ij)^{-1}}=u_{ji}$, the expression of $X_{f^m}(n)$ becomes :

$$X_{f^m}(n) = \sum_{j_1,\dots,j_{l_m}=0}^{n}\overrightarrow{\prod_{1\leq t\leq l_m}} u^{\varepsilon(i)}_{(j_ij_{i+1})^{\eta(i)}} \quad (\text{with } j_{l_m+1}=j_1)\,.$$

This expression can be translated in terms of diagrams. Indeed, the right term is exactly :

$$X_{f^m}(n) = \sum_{\vec{i},\vec{j}\in[1,n]^{l_m},\ker(\vec{i},\vec{j})\geq\mathcal{D}_f} u^{\varepsilon(1)}_{i_1j_1}\dots u^{\varepsilon(l_m)}_{i_{l_m}j_{l_m}}.$$

A product of different X_{f^m} is just a concatenation of these expressions. With $L = \sum_{t=1}^r l_t$ and ε the concatenation of the words ε_{f^m} for $1 \leq m \leq r$, this yields:

$$X_{f^1}(n) \dots X_{f^r}(n) = \sum_{\substack{\vec{i},\vec{j} \in [1,n]^L \\ \ker(\vec{i},\vec{j}) \geq \mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^r}}} u_{i_1 j_1}^{\varepsilon(1)} \dots u_{i_L j_L}^{\varepsilon(L)}.$$

And this sum can be integrated with the Weingarten formula to get for $n \geq L$:

$$\begin{aligned}
\int X_{f^1}(n) \dots X_{f^r}(n) &= \int \sum_{\substack{\vec{i},\vec{j} \in [1,n]^L \\ \ker(\vec{i},\vec{j}) \geq \mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^r}}} u_{i_1 j_1}^{\varepsilon(1)} \dots u_{i_L j_L}^{\varepsilon(L)} \\
&= \sum_{\substack{\vec{i},\vec{j} \in [1,n]^L \\ \ker(\vec{i},\vec{j}) \geq \mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^r}}} \sum_{\substack{p \in \mathcal{C}(\varepsilon), p \leq \ker(\vec{i}) \\ q \in \mathcal{C}(\varepsilon), q \leq \ker(\vec{j})}} Wg_{G_n}(p,q) \\
&= \sum_{p,q \in \mathcal{C}(\varepsilon)} \sum_{\substack{\vec{i},\vec{j} \in [1,n]^L \\ \ker(\vec{i},\vec{j}) \geq (p|\mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^r}|q)}} Wg_{G_n}(p,q) \\
&= \sum_{p,q \in \mathcal{C}(\varepsilon)} n^{b(p|\mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^r}|q)} W_{G_n}(p,q).
\end{aligned}$$

Proposition 5.5 gives an asymptotic formula for Wg_{G_n}:

$$Wg_{G_n}(p,q) = (-1)^{\delta_{p \neq q}} n^{b(p \vee q) - b(p) - b(q)} (1 + O(\frac{1}{\sqrt{n}})).$$

Applying this to the previous computation yields:

$$\int X_{f^1}(n) \dots X_{f^r}(n) = \sum_{p,q \in \mathcal{C}(\varepsilon)} (-1)^{\delta_{p,q}} n^{b(p|\mathcal{D}|q) + b(p \vee q) - b(p) - b(q)} (1 + O(\frac{1}{\sqrt{n}})). \tag{5.3.4}$$

Since $\mathcal{C} \in \{\mathcal{U}, \mathcal{U}_{class}\}$, Proposition 5.31 yields the following inequalities:

$$\begin{cases} b(p|\mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^r}|q) \leq \frac{b(p)+b(q)}{2} \leq \max(b(p), b(q)) \\ b(p \vee q) \leq \min(b(p), b(q)) \end{cases}$$

In the large n limit, the non-vanishing terms in (5.3.4) are the ones such that $b(p|\mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^r}|q) + b(p \vee q) - b(p) - b(q) \geq 0$. By the previous inequalities, these terms must verify:

$$\begin{cases} b(p) = b(q) \\ b(p|\mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^r}|q) = b(p) \\ b(p \vee q) = b(q) \end{cases}$$

This implies that $p = q$ and $b(p|\mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^r}|p) = b(p)$. In this case $b(p|\mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^r}|q) + b(p \vee q) - b(p) - b(q) = 0$, which yields the final expression:

$$\lim_{n \to \infty} \int X_{f^1}(n) \dots X_{f^r}(n) = \#\{p \in \mathcal{C}(\varepsilon), b(p|\mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^r}|p) = b(p)\}.$$

□

Law of the $\{X_f\}$ It remains to describe the law of $\{X_f\}_{f\in\mathbb{F}_2}$. Note first that expanding f in the reduced word $f=\prod_{\rightarrow} x_{\eta(i)}^{\varepsilon(i)}$ yields

$$\begin{aligned}
\overline{X_f(n)} &= \overline{\sum_{j_1,\dots,j_r=0}^{n} \prod_{\rightarrow} u_{(j_ij_{i+1})^{\eta(i)}}^{\varepsilon(i)}} \quad (\text{with } j_{r+1}=j_1)\\
&= \sum_{j_1,\dots,j_r=0}^{n} \prod_{\rightarrow} \overline{u_{(j_{r+1-i}j_{r+2-i})^{\eta(i)}}^{\varepsilon(r+1-i)}} = \sum_{j_1,\dots,j_r=0}^{n} \prod_{\rightarrow} u_{(j_{r+1-i}j_{r+2-i})^{\eta(r+1-i)}}^{-\varepsilon(r+1-i)}\\
&= \sum_{j_1,\dots,j_r=0}^{n} \prod_{\rightarrow} u_{(j_ij_{i-1})^{\tilde{\eta}(i)}}^{\tilde{\varepsilon}(i)}
\end{aligned}$$

with $\tilde{\eta}(i)=-\eta(r+1-i)$, $\tilde{\varepsilon}(i)=-\tilde{\varepsilon}(r+1-i)$. Therefore for $f\in\mathbb{F}_2$,

$$\overline{X_f(n)} = X_{f^{-1}}(n).$$

Following a method already used in [14] we first compute the cumulants of the family $\{X_f\}_{f\in\mathbb{F}_2}$. Let us associate to each sequence of words $(f^1,\dots,f^r)$, with f^i having length l_i, the partition $p_{\vec{f}}$ of $P(\varepsilon)$, whose blocks are the sets $B_a=\{\sum_{i=0}^{a-1} l_i+1, \sum_{i=0}^{a-1} l_i + l_a\}$ for $1\le a\le r$.

Lemma 5.34. *Let $\mathcal{C}\in\{\mathcal{U},\mathcal{U}_{class}\}$. Let $r\geq 1$, $f^1,\dots,f^r\in\mathbb{F}_2$ such that f^i is cyclically reduced if $\mathcal{C}=\mathcal{U}_{class}$. If $\mathcal{C}=\mathcal{U}$ (resp. $\mathcal{C}=\mathcal{U}_{class}$) the free cumulant (resp classical cumulant) of $(f^1,\dots,f^r)$ is*

$$c_r(f^1,\dots,f^r)=\#\{p\in\mathcal{C}(\varepsilon), p\vee p_{\vec{f}}=\mathbf{1}, b(p|\mathcal{D}_{f^1}\dots\mathcal{D}_{f^r}|p)=b(p)\}.$$

Proof. The proof is essentially an adaptation of the proof of [14] in the context of bidiagrams. We write the proof in the free case, since the proof in the classical one is exactly the same. For each tuple $\vec{f}=(f^1,\dots,f^k)$, let $\int(f^1,\dots,f^k)=\int X_{f^1}\dots X_{f^k}$, and for σ a partition of $\{1,\dots,r\}$, let

$$\int_\sigma (f^1,\dots,f^k)=\prod_{\{i_1<\dots<i_s\}\in\sigma}\int (f^{i_1},\dots,f^{i_s}).$$

For σ partition of $\{1,\dots,r\}$, let $\sigma^{p_{\vec{f}}}$ be the the partition of $P(\varepsilon)$ obtained by linking the r blocks of $p_{\vec{f}}$ according to σ. Since $\mathcal{U}$ is block stable, Proposition 5.33 yields

$$\int_\sigma (f^1,\dots,f^r)=\#\{p\le\sigma^{p_{\vec{f}}}, b(p|\mathcal{D}_{f^1}\dots\mathcal{D}_{f^r}|p)=b(p)\}.$$

Therefore

$$\begin{aligned}
c_r(f^1,\dots,f^r) &= \sum_\sigma \mu(\sigma,\mathbf{1}_r)\int_\sigma (f^1,\dots,f^r)\\
&= \sum_\sigma \mu(\sigma,\mathbf{1}_r)\#\{p\le\sigma^{p_{\vec{f}}}, b(p|\mathcal{D}_{f^1}\dots\mathcal{D}_{f^r}|p)=b(p)\}\\
&= \sum_{p_{\vec{f}}\le\tau} \mu(\tau,\mathbf{1}_L)\#\{p\le\tau, b(p|\mathcal{D}_{f^1}\dots\mathcal{D}_{f^r}|p)=b(p)\},
\end{aligned}$$

where we use on the third equality the invariance property of the Moebius function μ. Writing $\#\{p \leq \tau, b(p|\mathcal{D}_{f^1} \dots \mathcal{D}_{f^r}|p) = b(p)\}$ as $\sum_{\substack{p\leq\tau,\\ b(p|\mathcal{D}_{f^1}\dots\mathcal{D}_{f^r}|p)=b(p)}} 1$ and interverting the sums yields

$$\begin{aligned} c_r(f^1, \dots, f^r) &= \sum_{b(p|\mathcal{D}_{f^1}\dots\mathcal{D}_{f^r}|p)=b(p)} \sum_{\tau \geq p\vee p_{\vec{f}}} \mu(\tau, \mathbf{1}_L) \\ &= \sum_{b(p|\mathcal{D}_{f^1}\dots\mathcal{D}_{f^r}|p)=b(p)} \delta_{p\vee p_{\vec{f}}, \mathbf{1}_\varepsilon} \\ &= \#\{p \vee p_{\vec{f}} = \mathbf{1}_\varepsilon, b(p|\mathcal{D}_{f^1} \dots \mathcal{D}_{f^r}|p) = b(p)\}, \end{aligned}$$

where the second equality uses again the properties of the Moebius function. $\square$

The last step is the computation of $|\{p \vee p_{\vec{f}} = \mathbf{1}, b(p|\mathcal{D}_{f^1} \dots \mathcal{D}_{f^r}|p) = b(p)\}|$ for arbitrary $f^1, \dots, f^r$. From now on, $c_r(f^1, \dots, f^r)$ denotes the classical or the free cumulant, depending on the situation. $f^1, \dots, f^r$ are assumed cyclically reduced if we are in the classical case. As in Section 2, the condition $b(p|\mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^r}|p) = b(p)$ yields restrictive properties on p. For $t \in \mathbb{Z}/l_a\mathbb{Z}$, let us denote by x_t^a the element $\sum_{i=0}^{a-1} l_i + t$ of B_a. We have the following result:

Lemma 5.35. *Let $r \geq 2$. Suppose that p is a pairing such that $b(p|\mathcal{D}_{f^1} \otimes \dots \otimes \mathcal{D}_{f^r}|p) = b(p)$ and $p \vee p_{\vec{f}} = \mathbf{1}_\varepsilon$. Let $x_t^a \in B_a$ and $x_{t'}^b \in B_b$ such that $a \neq b$ and $x_t^a \sim_p x_{t'}^b$. Then $l_a = l_b$, and there exists $\tau \in \{1, -1\}$ such that for all $1 \leq h \leq l_a$, $x_{t+h}^a \sim x_{t'+\tau h}^b$. In particular such p exists only if $r = 2$ and $c_r(f_1, \dots, f_r) = 0$ if $r \geq 3$.*

To simplify the proof, we use the same notation to denote elements of the partitions in $P(\varepsilon)$ and elements of the upper row of partitions in $P(\varepsilon, \varepsilon)$.

Proof. Let $r \geq 2$, $f^1, \dots, f^r \in \mathbb{F}_2$. Let $p \in \mathcal{C}$ be such that $b(p|\mathcal{D}_{f^1} \dots \mathcal{D}_{f^r}|p) = b(p)$ and $p \vee p_{\vec{f}} = \mathbf{1}_\varepsilon$. In the classical case, the proof of Proposition 5.31 yields that $(p|\mathcal{D}|p)$ has no block with only two elements. In the free case, this could happen only if these two elements are extremal in a particular block B of $p_{\vec{f}}$; in particular since p is non-crossing, any element of B is also paired to an element of B through p. Since $r \geq 2$, this contradicts the fact $p \vee p_{\vec{f}} = \mathbf{1}$.
Therefore any block of $(p|\mathcal{D}|p)$ has at least 4 elements. Since $b(p|\mathcal{D}_{f^1} \dots \mathcal{D}_{f^r}|p) = b(p) = \frac{L}{2}$, each block of $(p|\mathcal{D}|p)$ has exactly 4 elements. Let $t \in \mathbb{Z}/l_a\mathbb{Z}$ and $t' \in \mathbb{Z}/l_b\mathbb{Z}$ such that $a \neq b$ and $x_t^a \sim_p x_{t'}^b$. Thus $x_t^a \sim_{pR_h(p)} x_{t'}^b$ and x_t^a and $x_{t'}^b$ are in a same block B of $(p|\mathcal{D}|p)$. By $\mathcal{D}$, x_t^a is linked to exactly one element u among $\{x_{t-1}^a, x_{t+1}^a, \overline{x_{t-1}^a}, \overline{x_{t-1}^a}\}$. The same is true for $x_{t'}^b$ and one element v among $\{x_{t'-1}^b, x_{t'+1}^b, \overline{x_{t'-1}^b}, \overline{x_{t'-1}^b}\}$. Thus $B = \{u, v, x_t^a, x_{t'}^b\}$ forms a block of $(p|\mathcal{D}|p)$ and u is linked to an element of B through the pairing $pR_h(p)$. Since we already have $x_t^a \sim_p x_{t'}^b$, the only possibility is $u \sim_{pR_h(p)} v$. Doing the same for $\overline{x_t^a}$ and $\overline{x_{t'}^b}$ yields that either $x_{t+1}^a \sim_p x_{t'+1}^b$ or $x_{t+1}^a \sim_p x_{t-1}^a$.
Let us assume without loss of generality that $x_{t+1}^a \sim_p x_{t+1}^b$. Doing the same reasoning with $\{x_{t+1}^a, x_{t+1}^b\}$ yields that either $x_{t+2}^a \sim_p x_{t'+2}^b$ or $x_{t+2}^a \sim_p x_{t'}^b$. But the latter is impossible (except if $l_a = 2$) since $x_{t'}^b$ is already linked to x_t^a through p. Thus $x_{t+2}^a \sim_p x_{t'+2}^b$. By recursion, for all $1 \leq h \leq l_a$, $x_{t+h}^a \sim_p x_{t'+h}^b$ and $l_a \leq l_b$; by symmetry $l_a = l_b$.
Therefore any element of B_a is paired to an element of B_b through p, and $B_a \cup B_b$ is a block of $p \vee p_{\vec{l}}$. In particular if $r > 2$, this contradicts the assumption $p \vee p_{\vec{l}} = \mathbf{1}_\varepsilon$. $\square$

If $r \leq 2$, the situation is more evolved. First notice that $X_{ab} = X_{b^{-1}a^{-1}}$, $X_{ba} = X_{a^{-1}b^{-1}}$, $X_{ab^{-1}} = X_{b^{-1}a}$, $X_{a^{-1}b} = X_{ba^{-1}}$, $X_a = X_{b^{-1}}$ and $X_{a^{-1}} = X_b$; moreover $X_f = X_{f'}$ with $f \neq f'$ only in the previous cases. Let us write $f \sim f'$ if $X_f = X_{f'}$.

Proposition 5.36. *If $\mathcal{C} = \mathcal{U}$ (resp. $\mathcal{C} = \mathcal{U}_{class}$), $\{X_f\}_{f\in\mathbb{F}^2}$ is a family of free (resp. independent) (semi-)circular (resp. gaussian) variables. Let $f_1, f_2 \in \mathbb{F}_2$. In the free case, $c_2(f_1, f_2) = 0$ unless $f_2 \sim f_1^{-1}$, and in this case $k(f_1, f_1^{-1}) = 1$. $k(f) = 0$ unless $f \sim ab$ or $f \sim a^{-1}b^{-1}$, and in the latter case $k(f) = 1$.*
In the classical case,

$$c_2(f_1, f_2) = \sum_{1\leq k\leq r, \tau\in\{1,-1\}} \delta_{c^{k,\tau}(f_1)f_2=1},$$

where $c^{k,\tau}$ acts on a reduced word $x_1 \dots x_t \in \mathbb{F}_2$ as

$$c(x_1 \dots x_t) = x_{k+\tau}x_{k+2\tau} \dots x_{k+t\tau}.$$

$k_f = 0$ unless $f \sim ab$ or $f \sim a^{-1}b^{-1}$, where $k(f) = 1$.

Proof. The first part of the proposition is a straightforward application of Lemma 5.35.
Let $f_1, f_2 \in \mathbb{F}_2$ with $f_1 = x_{\eta_1}^{\varepsilon_1} \dots x_{\eta_r}^{\varepsilon_r}$ and $f_2 = x_{\eta'_1}^{\varepsilon'_1} \dots x_{\eta'_{r'}}^{\varepsilon'_{r'}}$. Let p be such that $b(p|\mathcal{D}(f_1) \otimes \mathcal{D}(f_2)|p) = b(p)$ and $p_{\vec{f}} \vee p = \mathbf{1}_{\varepsilon\varepsilon'}$.
The condition $p_{\vec{f}} \vee p = \mathbf{1}_{\varepsilon\varepsilon'}$ means that there exist $a \in B_{f_1}, b \in B_{f_2}$ with $a \sim_p b$. By the previous Lemma, this implies that $r = r'$, and there exists a bijection $\varphi : \{1, \dots, r\} \to \{1, \dots r\}$ such that $i \sim_p r + \varphi(i)$. By the same Lemma, there exist $r_0 \in [\![0, r-1]\!]$ and $\tau \in \{-1, 1\}$ such that $\varphi(x) = \tau x + r_0$ for all $1 \leq x \leq r$. Since p is in $\mathcal{U}$ or $\mathcal{U}_{class}$, i and $r + \varphi(i)$ have opposite colors and thus $\varepsilon'(\varphi(i)) = -\varepsilon(i)$. Having each block of $(p|\mathcal{D}_{f_1} \otimes \mathcal{D}_{f_2}|p)$ with exactly 4 elements requires that $\eta'(\varphi(i)) = -\eta(i)$ (except if $r = 2$, in which case the condition is $\eta'(1)\eta'(2) = \eta(1)\eta(2)$).

We must now distinguish the classical and the free case.

- In the free case there is only one way to achieve a non-crossing pairing in this way, namely $\varphi_0(x) = r + 1 - x$. This proves that $c_2(f_1, f_2) \leq 1$. From the first part of the proof this partition occurs if and only if $\eta'(r + 1 - i) = -\eta(i)$ and $\varepsilon'(r + 1 - i) = -\varepsilon'(i)$, which is equivalent to $f_1 \sim f_2^{-1}$.

- In the classical case, any map φ could arise. It remains to find which φ yields a pairing with the condition that each block of $(p|\mathcal{D}_{f_1} \otimes \mathcal{D}_{f_2}|p)$ has 4 elements. Suppose that φ is given by $r_0 \in [\![0, r-1]\!]$ and $\tau \in \{-1, 1\}$. By the first part of the proof the condition is satisfied if and only if $c^{r-r_0+1,-\tau}(f_1) = f_2^{-1}$. Summing on all r_0, τ yields the result.

The same proof yields that $c_1(f)$ is non-zero only if the length of f is at most 2 and $f \sim f^{-1}$. $\square$

Theorem 5.27 is the combination of Proposition 5.33 and Proposition 5.36.

Part III

Free wreath product

Chapter 6

Free wreath product with the free symmetric group

Introduction

In this chapter, we will consider the case of the free wreath product quantum groups defined by Bichon in [21]. This product was introduced as the most natural way to build the quantum automorphism group of the $n-$times disjoint union of a finite connected graph. The free wreath product $\wr_*$ associates to a compact quantum group $\mathbb{G}$ and a compact subgroup $\mathbb{F}$ of S_N^+ a new compact quantum group $\mathbb{G} \wr_* \mathbb{F}$. It is constructed as an analogue of the wreath products of classical groups. An example of this construction was studied by Banica and Vergnioux in [16], and then by Banica, Belinschi, Capitaine and Collins in [9]: they focused on the free wreath product of the dual of the cyclic group $\mathbb{Z}/s\mathbb{Z}$ with S_N^+. Banica and Vergnioux obtained the fusion rules and Banica, Belinschi, Capitaine and Collins obtained interesting probability results involving free compound Poisson variables.

François Lemeux generalized these results in [55] to the case of a free wreath product between the dual $\hat{\Gamma}$ of a discrete groupe Γ and S_N^+. Once again he was able to find the fusion rules of the quantum group as well as some operator algebraic properties by using certain results of Brannan on S_N^+ (see [25]). We investigate here the general problem of the free wreath product of any compact quantum group of Kac type $\mathbb{G}$ with S_N^+. In particular we construct the intertwiner spaces of $\mathbb{G} \wr_* S_N^+$ from the knowledge of the intertwiner spaces of $\mathbb{G}$, (see Theorem 6.16). We give also an expression of the Haar state of $\mathbb{G} \wr_* S_N^+$ from the Haar state of S_N^+. This yields the equality in law

$$\chi_{u \wr_* v} \sim \chi_u \boxtimes \chi_v,$$

with u being a representation of $\mathbb{G}$, v the fundamental representation of S_N^+ (with χ denoting in each case the associated character) and $\boxtimes$ is the free multiplicative convolution of two non-commutative variables . This is a positive answer to a special case of a conjecture raised by Banica and Bichon in [10] (See Subsection 6.4.1).

Using the description of the intertwiner spaces, François Lemeux and Jonas Wahl also obtained in [94],[56] further interesting results on the von Neumann algebra and C^*-algebras associated to $\mathbb{G} \wr_* S_N^+$. The chapter is organised as follows : the first section is dedicated to some preliminaries and notations. The second section gives some classical results and proofs that provide some insight into the final general description of the intertwiner spaces for the free wreath products. The description of the intertwiner spaces for the free wreath products $\mathbb{G} \wr_* S_N^+$ is the main result of the third section. In the fourth section, we give the probabilistic applications that one can deduce from the latter description.

6.1 Preliminaries

In this section, we recall a few facts and results about compact quantum groups and about free wreath products by the quantum permutation groups S_N^+, and we set the notations.
A compact quantum group (see [100]), or Woronowicz-C^*-algebra, is a pair $\mathbb{G} = (C(\mathbb{G}), \Delta)$ where $C(\mathbb{G})$ is a unital separable C^*-algebra and $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes_{min} C(\mathbb{G})$ is a unital $*$-homomorphism (i.e. it satisfies the coassociativity relation $(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta$), and such that the cancellation property holds (i.e $span\{\Delta(a)(b \otimes 1) : a, b \in C(\mathbb{G})\}$ and $span\{\Delta(a)(1 \otimes b) : a, b \in C(\mathbb{G})\}$ are norm dense in $C(\mathbb{G}) \otimes C(\mathbb{G})$). These assumptions allow to prove the existence and uniqueness of a Haar state $h : C(\mathbb{G}) \to \mathbb{C}$ satisfying the bi-invariance relations $(h \otimes \mathrm{id}) \circ \Delta(\cdot) = h(\cdot)1 = (\mathrm{id} \otimes h) \circ \Delta(\cdot)$. In this chapter we will deal with compact quantum groups of Kac type, which means that their Haar state h is a trace. Let $\lambda_h : C(\mathbb{G}) \to \mathcal{B}(L^2(\mathbb{G}, h))$ be the GNS representation associated to the Haar state h of $\mathbb{G} = (C(\mathbb{G}), \Delta)$ (also called the left regular representation). The reduced C^*-algebra associated to $\mathbb{G}$ is then defined by $C_r(\mathbb{G}) = \lambda_h(C(\mathbb{G})) \simeq C(\mathbb{G})/Ker(\lambda_h)$ and the von Neumann algebra of $\mathbb{G}$ by $L^\infty(\mathbb{G}) = C_r(\mathbb{G})''$. One can prove that $C_r(\mathbb{G})$ is again a Woronowicz-C^*-algebra whose Haar state extends to $L^\infty(\mathbb{G})$. We will denote simply by Δ and h the coproduct and Haar state on $C_r(\mathbb{G})$.

An N-dimensional (unitary) correpresentation $u = (u_{ij})_{ij}$ of $\mathbb{G}$ is a (unitary) matrix $u \in M_N(C(\mathbb{G})) \simeq C(\mathbb{G}) \otimes \mathcal{B}(\mathbb{C}^N)$ such that for all $i, j \in \{1, \dots, N\}$, one has

$$\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}.$$

The matrix $\overline{u} = (u_{ij}^*)$ is called the conjugate of $u \in M_N(C(\mathbb{G}))$ and in general it is not necessarily unitary (even if u is). However all the compact quantum groups we will deal with are of Kac type and in this case the conjugate of a unitary correpresentation is also unitary (see [65]) .

An intertwiner between two correpresentations

$$u \in M_{N_u}(C(\mathbb{G})) \text{ and } v \in M_{N_v}(C(\mathbb{G}))$$

is a matrix $T \in M_{N_u,N_v}(\mathbb{C})$ such that $v(1 \otimes T) = (1 \otimes T)u$. We say that u is equivalent to v, and we note $u \sim v$, if there exists an invertible intertwiner between u and v. We denote by $\mathrm{Hom}_{\mathbb{G}}(u, v)$ the space of intertwiners between u and v. A correpresentation u is said to be irreducible if $\mathrm{Hom}_{\mathbb{G}}(u, u) = \mathbb{C}\,\mathrm{id}$. We denote by $\mathrm{Irr}(\mathbb{G})$ the set of equivalence classes of irreducible correpresentations of $\mathbb{G}$.

As a Woronowicz-C^*-algebra, $C(\mathbb{G})$ contains a dense $*$-subalgebra denoted by $\mathrm{Pol}(\mathbb{G})$ and linearly generated by the coefficients of the irreducible correpresentations of $\mathbb{G}$ (see [65] for details on the subject). The coefficients of a $\mathbb{G}$-representation r acting on a Hilbert space H_r are given by $(\mathrm{id} \otimes \varphi)(r)$ for some $\varphi \in \mathcal{B}(H_r)^*$. This algebra has a Hopf-$*$-algebra structure and in particular there is a $*$-antiautomorphism $\kappa : \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})$ which acts on the coefficients of an irreducible correpresentation $r = (r_{ij})$ as $\kappa(r_{ij}) = r_{ji}^*$. This algebra is also dense in $L^2(\mathbb{G}, h)$. Since h is faithful on the $*$-algebra $\mathrm{Pol}(\mathbb{G})$, one can identify $\mathrm{Pol}(\mathbb{G})$ with its image in the GNS-representation $\lambda_h(C(\mathbb{G}))$. We will denote by χ_r the character of the irreducible correpresentation $r \in \mathrm{Irr}(\mathbb{G})$, that is $\chi_r = (\mathrm{id} \otimes \mathrm{Tr})(r)$.

A fundamental and basic family of examples of compact quantum groups is recalled in the following definition:

Definition 6.1. *([96]) Let $N \geq 2$. S_N^+ is the compact quantum group $(C(S_N^+), \Delta)$, where $C(S_N^+)$ is the universal C^*-algebra generated by N^2 elements u_{ij} such that the matrix $u = (u_{ij})$ is unitary and $u_{ij} = u_{ij}^* = u_{ij}^2, \forall i, j$ (i.e. u is a magic unitary) and such that the coproduct Δ is* *given by the usual relations making u a finite dimensional correpresentation of $C(S_N^+)$, that is $\Delta(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj}$, $\forall i,j \leq N$.*

In the cases $N = 2, 3$, one obtains the usual algebras $C(\mathbb{Z}_2), C(S_3)$. If $N \geq 4$, one can find an infinite dimensional quotient of $C(S_N^+)$ so that $C(S_N^+)$ is not isomorphic to $C(S_N)$, see e.g. [96], [7].

In [95], Wang defined the free product $\mathbb{G} = \mathbb{G}_1 * \mathbb{G}_2$ of compact quantum groups, showed that $\mathbb{G}$ is still a compact quantum group and gave a description of the irreducible correpresentations of $\mathbb{G}$ as alternating tensor products of nontrivial irreducible correpresentations of $\mathbb{G}_1$ and $\mathbb{G}_2$. For a correpresentation v of $\mathbb{G}$, denote by $\bar{v}$ the contragredient correpresentation.

Theorem 6.2. *([95]) Let $\mathbb{G}_1$ and $\mathbb{G}_2$ be compact quantum groups. Then the set* $\mathrm{Irr}(\mathbb{G})$ *of irreducible correpresentations of the free product of quantum groups $\mathbb{G} = \mathbb{G}_1 * \mathbb{G}_2$ can be identified with the set of alternating words in* $\mathrm{Irr}(\mathbb{G}_1) * \mathrm{Irr}(\mathbb{G}_2)$ *and the fusion rules can be recursively described as follows:*

- *If the words $x, y \in \mathrm{Irr}(\mathbb{G})$ end and start in $\mathrm{Irr}(\mathbb{G}_i)$ and $\mathrm{Irr}(\mathbb{G}_j)$ respectively with $j \neq i$ then $x \otimes y$ is an irreducible correpresentation of $\mathbb{G}$ corresponding to the concatenation $xy \in \mathrm{Irr}(\mathbb{G})$.*
- *If $x = vz$ and $y = z'w$ with $z, z' \in \mathrm{Irr}(\mathbb{G}_i)$, then there is a recurrence formula*

$$x \otimes y = \bigoplus_{1 \neq t \subseteq z \otimes z'} vtw \oplus \delta_{\bar{z}, z'} (v \otimes w),$$

  where the sum runs over all non-trivial irreducible correpresentations $t \in \mathrm{Irr}(\mathbb{G}_i)$ contained in $z \otimes z'$, with multiplicity.

In this chapter, we are interested in the free wreath product of quantum groups:

Definition 6.3. *([21, Definition 2.2]) Let A be a Woronowicz-C^*-algebra, $N \geq 2$ and $\nu_i : A \to A^{*N}$ be the canonical inclusion of the i-th copy of A in the free product A^{*N}, $i = 1, \ldots, N$.*

The free wreath product of A by $C(S_N^+)$ is the quotient of the C^*-algebra $A^{*N} * C(S_N^+)$ by the two-sided ideal generated by the elements

$$\nu_k(a) u_{ki} - u_{ki} \nu_k(a), \quad 1 \leq i, k \leq N, \quad a \in A.$$

It is denoted by $A *_w C(S_N^+)$.

In the next result, we use the Sweedler notation $\Delta_A(a) = \sum a_{(1)} \otimes a_{(2)} \in A \otimes A$.

Theorem 6.4. *([21, Theorem 2.3]) Let A be a Woronowicz-C^*-algebra, then the free wreath product $A *_w C(S_N^+)$ admits a Woronowicz-C^*-algebra structure: if $a \in A$, then*

$$\Delta(u_{ij}) = \sum_{k=1}^N u_{ik} \otimes u_{kj}, \forall i, j \in \{1, \ldots, N\},$$

$$\Delta(\nu_i(a)) = \sum_{k=1}^N \nu_i(a_{(1)}) u_{ik} \otimes \nu_k(a_{(2)}),$$

$$\varepsilon(u_{ij}) = \delta_{ij},\ \varepsilon(\nu_i(a)) = \varepsilon_A(a),\ S(u_{ij}) = u_{ji},\ S(\nu_i(a)) = \sum_{k=1}^N \nu_k(S_A(a)) u_{ki},$$

$$u_{ij}^* = u_{ij},\ \nu_i(a)^* = \nu_i(a^*).$$

Moreover, if $\mathbb{G}$ is a full compact quantum group, then $\mathbb{G} \wr_* S_N^+ = (A *_w C(S_N^+), \Delta)$ is also a full compact quantum group.

Remark 6.5. *The homomorphisms $\nu_i : A \to A^{*N} \subseteq A *_w C(S_N^+)$ are injective and we have $\nu_i = \pi \circ \bar{\nu}_i$, where*

$$\bar{\nu}_i = q \circ \nu_i : A \to A *_w C(S_N^+),$$

$q : A^{*N} * C(S_N^+) \to A *_w C(S_N^+)$ is the quotient map and $\pi : A *_w C(S_N^+) = id * \varepsilon$. Hence the morphisms $\bar{\nu}_i : A \to A *_w C(S_N^+)$ are injective.

Recall that the case of the dual of a discrete group $\mathbb{G} = \widehat{\Gamma}$ is investigated in [55]. In particular, a description of the irreducible representations is given and several operator algebraic properties are obtained from this description.

6.2 Classical wreath products by permutation groups.

In this section we provide a probabilistic formula for the moments of the character coming from certain wreath products of classical groups. This is in particular a hint for the formula in the free case. Recall that we denote by $\mathcal{P}(k)$ the set of all partitions of the set $\{1, \dots, k\}$.

Let G be a classical group, $n \geq 1$. Then S_n acts on G^n by the automorphisms

$$s : \sigma \in S_n \mapsto s(\sigma).(g_1, \dots, g_n) = (g_{\sigma^{-1}(1)}, \dots, g_{\sigma^{-1}(n)}). \qquad (*)$$

Definition 6.6. *The wreath product between G and S_n, denoted by $G \wr S_n$, is the semi-direct product of G^n and S_n, where S_n acts on G^n by $(*)$. In other words,*

$$G \wr S_n = \{((g_1, \dots, g_n), \sigma), g_i \in G, \sigma \in S_n\},$$

with the product

$$((g_1, \dots, g_n), \sigma) \cdot ((g_1', \dots, g_n'), \mu) = ((g_1 g'_{\sigma^{-1}(1)}, \dots, g_n g'_{\sigma^{-1}(n)}), \sigma\mu).$$

If G is a compact group, $G \wr S_n$ is compact as well and thus there exists a Haar measure on $G \wr S_n$. It is easy to see that $G \wr S_n$ is isomorphic to $G \times \cdots \times G \times S_n$ as a measure space and that the Haar measure on $G \wr S_n$ is given by $d\lambda_{G \wr S_n} = \bigotimes_i dg_i \otimes d\sigma$, where d_g denotes the Haar measure on G and $d\sigma$ the normalized counting measure on S_n.
If $\alpha : G \to U(V)$ is a unitary representation of G, then $G \wr S_n$ acts on $V^{\otimes n}$ via

$$\alpha^n((g_1, \dots, g_n), \sigma)(v_1 \otimes \cdots \otimes v_n) := \alpha(g_1)(v_{\sigma^{-1}(1)}) \otimes \cdots \otimes \alpha(g_n)(v_{\sigma^{-1}(n)}).$$

We will use the following notation in the sequel:

Notation 6.7. *Let $\beta : G \to U(H)$ be a unitary representation of a compact group G; then:*

- *χ_β denotes the character of β,*
- *F_β is the exponential generating serie of the moments of χ_β with respect to the Haar measure*

The purpose of this section is to describe the distribution of χ_{α^n} under $d\lambda_{G \wr S_n}$, when α is a represention of G. We will assume that $\alpha(G) \subseteq GL_p(\mathbb{R})$ for some $p \geq 1$. In particular χ_{α^n} is real. The computations are similar in the complex setting; we just have to deal separately with the real and imaginary part of χ_α.

Notation 6.8. *For each partition $\nu \in \mathcal{P}(k)$ with blocks $B_1, \dots B_r$ and sequence of numbers $(c_1, \dots, c_n, \dots)$ of length greater than k we write*

$$c_\nu = c_{|B_1|} c_{|B_2|} \dots c_{|B_r|},$$

with $|B_i|$ being the cardinal of the block B_i.

Proposition 6.9. *The exponential serie of the moments of χ_{α^n} is given by*

$$F_{\alpha^n}(x) = \sum m_{\alpha^n}(k) \frac{x^k}{k!},$$

with

$$m_{\alpha^n}(k) = \sum_{\nu \in \mathcal{P}(k), l(\nu) \leq n} m_\alpha(\nu),$$

where $l(\pi)$ is the length of a partition π (namely the number of blocks of π).

Proof. Let $t = ((g_1, \dots, g_n), \sigma) \in G \wr S_n$, we have for $x > 0$ small enough. Writing the action of t through α^n in block matrices yields the following result

$$\begin{aligned}
F_{\alpha^n}(x) =& \mathbb{E}_{G \wr S_n}\left(\exp(x \mathrm{Tr}(\alpha^n(t)))\right) = \int_{G \wr S_n} \exp\left(x \sum_{\substack{1 \leq i \leq n \\ \sigma(i) = i}} \mathrm{Tr}(\alpha(g_i)) \right) \prod dg_i d\sigma \\
=& \int_{S_n} \prod_{i \text{ fixed point of } \sigma} \left(\int_{G_i} \exp(x \times \mathrm{Tr}(\alpha(g_i))) dg_i \right) d\sigma \\
=& \int_{S_n} \prod_{i \text{ fixed point of } \sigma} F_\alpha(x) d\sigma \\
=& \int_{S_n} F_\alpha(x)^{\# \text{ fixed points of } \sigma} d\sigma \\
=& \int_{S_n} \exp\left(\log(F_\alpha(x)) \# \text{ fixed points of } \sigma\right) d\sigma.
\end{aligned}$$

Considering $\log(F_\alpha(x))$ as fixed in the last integral yields the equality

$$F_{\alpha^n}(x) = F_{S_n}(\log(F_\alpha(x))), \tag{6.2.1}$$

where F_{S_n} denotes the exponential generating serie of the moments of the fundamental representation $S_n \hookrightarrow M_n(\mathbb{C})$. Now, we can exploit the general facts that

$$F_\beta(x) = \sum m_\beta(k) \frac{x^k}{k!} \tag{6.2.2}$$

and

$$\log F_\beta(x) = \sum c_\beta(k) \frac{x^k}{k!}, \tag{6.2.3}$$

where $(m_\beta(k))_{k \geq 1}$ are the moments of the law of χ_β and $(c_\beta(k))_{k \geq 1}$ are the classical cumulants of this law. The latter is the only sequence of real numbers satisfying

$$m_\beta(k) = \sum_{\pi \in \mathcal{P}(k)} c_\beta(\pi) \tag{6.2.4}$$

for all $k \geq 1$. From the left-hand side of (6.2.1) and (6.2.2) we get

$$F_{\alpha^n}(x) = \sum_k m_{\alpha^n}(k) \frac{x^k}{k!},$$

and from the right-hand of (6.2.1) with (6.2.3) we compute

$$\begin{aligned} F_{\alpha^n}(x) &= \sum_r m_{S_n}(r) \frac{\left(\sum c_\alpha(u) \frac{x^u}{u!}\right)^r}{r!} \\ &= \sum_k \frac{x^k}{k!} \sum_r \frac{m_{S_n}(r)}{r!} \sum_{\substack{u_1 \times 1 + \cdots + u_k \times k = k \\ \sum u_i = r}} k! \frac{r!}{\prod u_i!} \left(\frac{c_\alpha(1)}{1!}\right)^{u_1} \cdots \left(\frac{c_\alpha(k)}{k!}\right)^{u_k}. \end{aligned}$$

The last equality above being due to the multinomial expansion. Hence, after identifying coefficients we obtain:

$$m_{\alpha^n}(k) = \sum_r m_{S_n}(r) \sum_{\substack{u_1 \times 1 + \cdots + u_k \times k = k \\ \sum u_i = r}} k! \frac{1}{u_1! \ldots u_r!} \left(\frac{c_\alpha(1)}{1!}\right)^{u_1} \cdots \left(\frac{c_\alpha(k)}{k!}\right)^{u_k}. \tag{6.2.5}$$

We say that a partition $p \in \mathcal{P}(k)$ is of type $(1^{u_1}, \ldots, k^{u_r})$, if it is a partition having u_1 blocks of cardinal 1, u_2 of cardinal 2 and so on. The number of partitions of $\{1, \ldots, k\}$ of type $(1^{u_1}, \ldots, k^{u_r})$ is exactly

$$\frac{k!}{u_1! \ldots u_k!} \frac{1}{1!^{u_1} \ldots k!^{u_k}},$$

(see e.g. page 22 in [60]). Thus summing over every types of partitions in (6.2.5) yields:

$$m_{\alpha^n}(k) = \sum_r m_{S_n}(r) \sum_{\pi \in \mathcal{P}(k), l(\pi) = r} c_\alpha(\pi). \tag{6.2.6}$$

Using the fact that (see [15], [72])

$$m_{S_n}(r) = \#\{\text{partitions of } \{1, \ldots, r\} \text{ having at most } n \text{ blocks}\}, \tag{6.2.7}$$

we can transform (6.2.6) into

$$\begin{aligned} m_{\alpha^n}(k) &= \sum_r \sum_{\substack{\nu \leq \mathbf{1}_r \\ l(\nu) \leq n}} \sum_{\substack{\pi \leq \mathbf{1}_k \\ l(\pi) = r}} c_\alpha(\pi) = \sum_r \sum_{\substack{\pi \leq \mathbf{1}_k \\ l(\pi) = r}} \sum_{\substack{\pi \leq \nu \leq \mathbf{1}_k \\ l(\nu) \leq n}} c_\alpha(\pi) \\ &= \sum_{\pi \leq \mathbf{1}_k} \sum_{\substack{\pi \leq \nu \leq \mathbf{1}_k \\ l(\nu) \leq n}} c_\alpha(\pi) = \sum_{\substack{\nu \leq \mathbf{1}_k \\ l(\nu) \leq n}} \sum_{\pi \leq \nu} c_\alpha(\pi) \\ &= \sum_{\substack{\nu \leq \mathbf{1}_k \\ l(\nu) \leq n}} m_\alpha(\nu). \end{aligned}$$

□

We can deduce from Proposition6.9 the aymptotic law of χ_{α^n} when n goes to infinity :

Corollary 6.10. *The following convergence in moments holds:*

$$\chi_{\alpha^n} \underset{n\to\infty}{\longrightarrow} \mathcal{P}(\chi_\alpha),$$

where $\mathcal{P}(\chi_\alpha)$ is the compound Poisson law with respect to the parameter 1 and the law α.

Proof. We have

$$\begin{aligned} m_{\alpha^n}(k) &= \sum_{\nu \leq \mathbf{1}_k, l(\nu)\leq n} m_\alpha(\nu) \\ &\underset{n\to\infty}{\longrightarrow} \sum_{\nu\leq \mathbf{1}_k} m_\alpha(\nu) = m_{\mathcal{P}(\chi_\alpha)}. \end{aligned}$$

□

Remark 6.11. *In the next section we will describe the intertwiner spaces for a free wreath product $\mathbb{G} \wr_* S_N^+$. The result and proofs can be easily adapted to get the same result in the classical case; one only needs to use all partitions instead of non-crossing ones.*

6.3 Intertwiner spaces in $\mathbb{G} \wr_* S_N^+$

Let $\mathbb{G} = (C(\mathbb{G}), v)$ be a compact matrix quantum group of Kac type, generated by a unitary v acting on H. In this section, the C^*-algebras associated with compact quantum groups are considered in their maximal versions. A generating magic unitary u of the free quantum permutation group S_N^+ acting on $\mathbb{C}^N$ is a matrix $(u_{ij})_{1\leq i,j\leq N}$ of orthogonal projections of $C(S_N^+)$, such that the algebra generated by $\{u_{ij}\}$ is dense in $C(S_N^+)$ and such that

$$\sum_i u_{ij} = \sum_j u_{ij} = 1, u_{ij}u_{ik} = \delta_{jk}u_{ij}, u_{ij}u_{lj} = \delta_{il}u_{ij}.$$

We recall that the correpresentation

$$\omega := (\omega_{ijkl})_{1\leq i,j\leq N}^{1\leq k,l\leq d_{\mathbb{G}}} = (u_{ij}v_{kl}^{(i)})_{i,j,k,l}$$

acting on $W := \mathbb{C}^N \otimes H$, is the generating matrix of the free wreath product quantum groups $\mathbb{G} \wr_* S_N^+$, see [21].

Let $\mathrm{Rep}(\mathbb{G})$ be the set of equivalence classes of unitary finite dimensional (not necessarily irreducible) correpresentations of $\mathbb{G}$ and we denote by $H^\alpha = \langle Y_1^\alpha, \dots, Y_{d_\alpha}^\alpha \rangle$ the representation space of α, for $\alpha \in \mathrm{Rep}(\mathbb{G})$. We have a natural family of $\mathbb{G} \wr_* S_N^+$-representations (see the proof of Theorem 2.3 in [21]) given by

$$\{r(\alpha) := \left(u_{ij}\alpha_{kl}^{(i)}\right) : \alpha \in I\}. \tag{6.3.1}$$

Notice that $r(\alpha)$ acts on the vector space $\mathbb{C}^N \otimes H^\alpha$. These correpresentations will be called basic correpresentations for $\mathbb{G} \wr_* S_N^+$.

Let $\{T_p\}_{p\in NC(k,l)}$ be the basis of $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$ for $k, l \in \mathbb{N}$. For each partition $p = \{B_1, \dots, B_r\}$, the blocks of p are ordered by the lexicographical order.

We want to describe the intertwiner spaces between tensor products of basic correpresentations of $\mathbb{G} \wr_* S_N^+$. These spaces will be described by linear maps associated with certain non-crossing partitions and with $\mathbb{G}$-morphisms. Indeed, let $[\alpha] := (\alpha_1, \dots, \alpha_k)$ and $[\beta] := (\beta_1, \dots, \beta_l)$

be tuples of $\mathbb{G}$-representations such that the points of p are decorated by these correpresentations. This means that in each block B_i, certain correpresentations $\alpha_1^i, \ldots, \beta_1^i, \ldots,$ are attached to the upper and lower points respectively. We make the convention that if $k = 0$, then the trivial correpresentation decorates the upper part of $p \in NC(0, l)$ and an similar convention if $l = 0$. The non-crossing partitions describing intertwiners in $\mathbb{G} \wr_* S_N^+$ will also be such that their blocks are decorated by $\mathbb{G}$-morphisms. To be more precise, let us introduced some notation.

Notation 6.12. *Let $p \in NC(k, l)$; its blocks are denoted by B_i for $1 \leq i \leq r$. We will simplify the notation B_i into B when the context is clear. We denote by:*

- *$B = U_B \cup L_B$ the upper and lower parts of each block B.*
- *$H^{U_B} = \bigotimes_{i \in U_B} H^{\alpha_i}$ the tensor product of spaces H^{α_i}, and similarly we write $H^{L_B} = \bigotimes_{j \in L_B} H^{\beta_j}$.*
- *$\alpha(U_B) = \bigotimes_{i \in U_B} \alpha_i$ the tensor product of correpresentations α_i and similarly we write $\beta(L_B) = \bigotimes_{j \in L_B} \beta_j$.*

Furthermore, we assume that "attached" to each block B there is a $\mathbb{G}$-morphism

$$S_B = \alpha(U_B) \to \beta(L_B) \in \mathcal{B}(H^{U_B}, H^{L_B}) \tag{6.3.2}$$

and we put

$$S = \bigotimes_B S_B : \bigotimes_B \alpha(U_B) \to \bigotimes_B \beta(L_B) \tag{6.3.3}$$

with the order on the blocks we gave above. We say that the blocks of p are *decorated* by $[S] = (S_1, \ldots, S_r)$ where r is the number of blocks in p.

Definition 6.13. *We say that the partition p decorated by representations $[\alpha], [\beta]$ is admissible if $\forall B \in p, \mathrm{Hom}_{\mathbb{G}}(\alpha(U_B); \alpha(L_B)) \neq 0$.*

Therefore, we can consider

$$T_p \otimes S \in \mathcal{B}\left((\mathbb{C}^N)^{\otimes k} \otimes \bigotimes_B H^{U_B}; (\mathbb{C}^N)^{\otimes l} \otimes \bigotimes_B^r H^{L_B}\right).$$

Remark 6.14. *Notice that if the $\mathbb{G}$-morphisms in $\mathcal{B}(H^{U_B}, H^{L_B})$ run over a basis of intertwiners $\alpha(U_B) \to \beta(L_B)$ then the family $(T_p \otimes S)_{p,S}$ is free.*

We shall twist this linear map to obtain a morphism

$$\widetilde{T_p \otimes S} \in \mathrm{Hom}_{\mathbb{G} \wr_* S_N^+}(r(\alpha_1) \otimes \cdots \otimes r(\alpha_k), r(\beta_1) \otimes \cdots \otimes r(\beta_l)).$$

Notation 6.15. *Let $p \in NC(k, l)$ be decorated by $\mathbb{G}$-representations $[\alpha], [\beta]$ and morphisms $[S]$ as in the above notation. One can consider a unitary t_p^U acting on vectors $x_i \in \mathbb{C}^N$, $y_i \in H^{\alpha_i}$, $i = 1, \ldots, k$*

$$t_p^U : (\mathbb{C}^N \otimes H^{\alpha_1}) \otimes \cdots \otimes (\mathbb{C}^N \otimes H^{\alpha_k}) \to (\mathbb{C}^N)^{\otimes k} \otimes \bigotimes_B H^{U_B},$$

$$\bigotimes_{i=1}^k (x_i \otimes y_i) \mapsto \bigotimes_{i=1}^k x_i \otimes \bigotimes_B \bigotimes_{i' \in U_B} y_{i'}$$

and a unitary t_p^L *acting on vectors* $x_j \in \mathbb{C}^N$, $y_j \in H^{\beta_j}$, $j = 1, \ldots, l$

$$t_p^L : (\mathbb{C}^N \otimes H^{\beta_1}) \otimes \cdots \otimes (\mathbb{C}^N \otimes H^{\beta_l}) \to (\mathbb{C}^N)^{\otimes l} \otimes \bigotimes_B H^{L_B},$$
$$\bigotimes_{j=1}^{l} (x_j \otimes y_j) \mapsto \bigotimes_{j=1}^{l} x_j \otimes \bigotimes_B \bigotimes_{j' \in L_B} y_{j'}.$$

We set

$$\begin{aligned} U^{p,S} &:= (t_L^p)^* \circ (T_p \otimes S) \circ t_U^p \\ &\in \mathcal{B}\Big((\mathbb{C}^N \otimes H^{\alpha_1}) \otimes \cdots \otimes (\mathbb{C}^N \otimes H^{\alpha_k}), (\mathbb{C}^N \otimes H^{\beta_1}) \otimes \cdots \otimes (\mathbb{C}^N \otimes H^{\beta_l})\Big). \end{aligned}$$

We can now prove the following result:

Theorem 6.16. *Let* $\mathbb{G} = (C(\mathbb{G}), \Delta)$ *be a compact quantum group of Kac type. Let* $\alpha_1, \ldots, \alpha_k$ *and* $\beta_1, \ldots, \beta_l$ *be finite dimensional correpresentations in* $\mathrm{Rep}(\mathbb{G})$. *We set* $[\alpha] = (\alpha_1, \ldots, \alpha_k)$ *and* $[\beta] = (\beta_1, \ldots, \beta_l)$. *Then*

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{G} \wr_* S_N^+}(r(\alpha_1) \otimes \ldots \otimes r(\alpha_k); r(\beta_1) \otimes \cdots \otimes r(\beta_l)) && (6.3.4) \\ &\qquad = \mathrm{span}\{U^{p,S} : p \in NC_{\mathbb{G}}([\alpha], [\beta]), S \text{ as below}\} && (6.3.5) \end{aligned}$$

where $U^{p,S} = (t_L^p)^* \circ (T_p \otimes S) \circ t_U^p$ *with*

- *the isomorphisms* t_U^p, t_L^p *defined in Notation 6.15,*
- $NC_{\mathbb{G}}([\alpha], [\beta])$ *consists of non-crossing partitions in* $NC(k, l)$ *decorated with correpresentations* $[\alpha], [\beta]$ *on the upper and lower points respectively,*
- $S = \bigotimes_B S_B : \bigotimes_B \alpha(U_B) \to \bigotimes_B \beta(L_B)$ *as in (6.3.3), where the* $\mathbb{G}$*-morphisms in* $\mathcal{B}(U_B, L_B)$ *which decorate the blocks* $B \in p$ *run over intertwiners in* $\mathrm{Hom}_{\mathbb{G}}(\alpha(U_B), \beta(L_B))$.

Proof. We first prove that

$$U^{p,S} \in \mathrm{Hom}_{\mathbb{G} \wr_* S_N^+}(r(\alpha_1) \otimes \cdots \otimes r(\alpha_k); r(\beta_1) \otimes \cdots \otimes r(\beta_l)),$$

which is the inclusion of the right hand space (6.3.5) in the left hand space (6.3.4).

The Frobenius reciprocity for C^*-tensor categories with conjugates provide the following isomorphisms:

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{G} \wr_* S_N^+}(r(\alpha_1) \otimes \cdots \otimes r(\alpha_k); r(\beta_1) \otimes \cdots \otimes r(\beta_l)) \\ &\qquad \simeq \mathrm{Hom}_{\mathbb{G} \wr_* S_N^+}\left(1; \overline{r(\alpha_1) \otimes \cdots \otimes r(\alpha_k)} \otimes r(\beta_1) \otimes \cdots \otimes r(\beta_l)\right) \\ &\qquad \simeq \mathrm{Hom}_{\mathbb{G} \wr_* S_N^+}\left(1; r(\bar{\alpha}_k) \otimes \cdots \otimes r(\bar{\alpha}_1) \otimes r(\beta_1) \otimes \cdots \otimes r(\beta_l)\right), \end{aligned}$$

$$\mathrm{Hom}_{\mathbb{G}}(\alpha_1 \otimes \cdots \otimes \alpha_k; \beta_1 \otimes \cdots \otimes \beta_l) \simeq \mathrm{Hom}_{\mathbb{G}}(1; \bar{\alpha}_k \otimes \cdots \otimes \bar{\alpha}_1 \otimes \beta_1 \otimes \cdots \otimes \beta_l).$$

Hence, it is enough to prove that

$$t_L^p(T_p \otimes \xi) \in \mathrm{Hom}_{\mathbb{G} \wr_* S_N^+}(1; r(\alpha_1) \otimes \cdots \otimes r(\alpha_k)) \tag{6.3.6}$$

for all $k \in \mathbb{N}$, $p \in NC(k)$ and all fixed vectors

$$\xi = \bigotimes_B \xi(L_B) : \mathbb{C} \to \bigotimes_B H^{L_B}.$$

Moreover it is enough to prove (6.3.6) for the one block partition 1_k since one can recover any $p \in NC$ by tensor products and compositions of partitions $1_k, k \geq 1$ and id.

We now fix $(e_i)_{i=1}^N$ a basis of $\mathbb{C}^N$ and $(Y_j^\alpha)_{j=1}^{d_\alpha}$ a basis of H^α, for any $\alpha \in \mathrm{Rep}(\mathbb{G})$. Let us prove taht $t_L^{1_k}(T_p \otimes \xi) \in \mathrm{Hom}_{\mathbb{G}\wr_* S_N^+}(1; r(\alpha_1) \otimes \cdots \otimes r(\alpha_k))$ for any

$$\xi = \sum_{[j]} \lambda_{[j]}^k Y_{j_1}^{\alpha_1} \otimes \cdots \otimes Y_{j_k}^{\alpha_k} \in \mathrm{Hom}_{\mathbb{G}}(1; \alpha_1 \otimes \cdots \otimes \alpha_k).$$

Setting $T_\xi^{1_k} := t_L^{1_k}(T_p \otimes \xi)$ yields

$$T_\xi^{1_k} \equiv \sum_{i,[j]} \lambda_{[j]}^k (e_i \otimes Y_{j_1}^{\alpha_1}) \otimes \cdots \otimes (e_i \otimes Y_{j_k}^{\alpha_k}),$$

so that

$$\begin{aligned} r_{\alpha_1} \otimes \cdots \otimes r_{\alpha_k}(T_\xi^{1_k} \otimes 1) = \sum_{i,[j]} \lambda_{[j]}^k \sum_{[r],[s]} (e_{s_1} \otimes Y_{r_1}^{\alpha_1}) \otimes \cdots \otimes (e_{s_k} \otimes Y_{r_k}^{\alpha_k}) \\ \otimes \left(u_{s_1 i}(\alpha_1)_{r_1 j_1}^{(s_1)} \ldots u_{s_k i}(\alpha_k)_{r_k j_k}^{(s_k)} \right). \end{aligned}$$

But the magic unitary u satisfies for all s, t, $u_{si}u_{ti} = \delta_{st}u_{si}, \sum_i u_{si} = 1$ and then combining this with the commuting relations in the free wreath product $C(\mathbb{G}) *_w C(S_N^+)$, we get

$$\begin{aligned} r_{\alpha_1} \otimes \cdots \otimes r_{\alpha_k}(T_\xi^{1_k} \otimes 1) &= \sum_{[j]} \lambda_{[j]}^k \sum_{[r],s_1} (e_{s_1} \otimes Y_{r_1}^{\alpha_1}) \otimes \cdots \otimes (e_{s_1} \otimes Y_{r_k}^{\alpha_k}) && (6.3.7)\\ &\quad \otimes \left((\alpha_1)_{r_1 j_1}^{(s_1)} \ldots (\alpha_k)_{r_k j_k}^{(s_1)} 1 \right) \\ &= \sum_{s_1} \sum_{[j]} \lambda_{[j]}^k \sum_{[r]} (e_{s_1} \otimes Y_{r_1}^{\alpha_1}) \otimes \cdots \otimes (e_{s_1} \otimes Y_{r_k}^{\alpha_k}) && (6.3.8)\\ &\quad \otimes \left((\alpha_1)_{r_1 j_1}^{(s_1)} \ldots (\alpha_k)_{r_k j_k}^{(s_1)} \right). && (6.3.9)\\ & && (6.3.10) \end{aligned}$$

Now, since

$$\xi = \sum_{[j]} \lambda_{[j]}^k Y_{j_1}^{\alpha_1} \otimes \cdots \otimes Y_{j_k}^{\alpha_k} \in \mathrm{Hom}_{\mathbb{G}}(1; \alpha_1 \otimes \cdots \otimes \alpha_k), \tag{6.3.11}$$

applying $(t_L^{1_k})^{-1}$ in (6.3.7), using (6.3.11) and then applying $t_L^{1_k}$ yield

$$\begin{aligned} r_{\alpha_1} \otimes \cdots \otimes r_{\alpha_k}(T_{\xi_{1_k}}^{1_k} \otimes 1) &= \sum_{[r],s_1} \lambda_{[r]}^k (e_{s_1} \otimes Y_{r_1}^{\alpha_1}) \otimes \cdots \otimes (e_{s_1} \otimes Y_{r_k}^{\alpha_k}) \otimes 1 \\ &= T_\xi^{1_k} \otimes 1. \end{aligned}$$

This proves that $T_\xi^{1_k} \in \mathrm{Hom}_{\mathbb{G}\wr_* S_N^+}(1; r(\alpha_1) \otimes \cdots \otimes r(\alpha_k))$.

A straightforward computation shows that the collection of $\mathbb{G} \wr_* S_N^+$-intertwiners spaces $\operatorname{span}\left\{U^{p,S} : p, S \text{ as in (6.3.5)}\right\}$ is stable by composition, tensor product and duality. Therefore this collection of vector spaces defines a rigid monoidal C^*-tensor category $\mathcal{T}$, with objects indexed by families $[\alpha]$ of $\mathbb{G}$-representations.
If one applies Woronowicz's Tannaka-Krein duality to this category $\mathcal{T}$, we get a compact matrix quantum group $(\mathbb{H}, \Omega)$ generated by a unitary Ω corresponding to $r(v) \in \mathcal{B}(\mathbb{C}^N \otimes H) \otimes C(\mathbb{H})$ and a family of correpresentations $(R_{\alpha_i})_{i \in I}$ such that

$$\begin{aligned}\operatorname{Hom}_{\mathbb{H}}(R_{\alpha_1} \otimes \ldots \otimes R_{\alpha_k}; R_{\beta_1} \otimes \cdots \otimes R_{\beta_l}) \\ = \operatorname{span}\left\{U^{p,S} : p, S \text{ as in (6.3.5)}\right\},\end{aligned}$$

with $p \in NC(k,l)$, $S : \bigotimes_B \alpha(U_B) \to \bigotimes_B \beta(L_B)$, $[\alpha] = (\alpha_1, \ldots, \alpha_k)$, $[\beta] = (\beta_1, \ldots, \beta_l)$.

We proved above that

$$U^{p,S} \in \operatorname{Hom}_{\mathbb{G} \wr_* S_N^+}(r(\alpha_1) \otimes \cdots \otimes r(\alpha_k); r(\beta_1) \otimes \cdots \otimes r(\beta_l)).$$

In particular, there is by universality a (surjective) morphism

$$\pi_1 : C(\mathbb{H}) \to C(\mathbb{G} \wr_* S_N^+), \quad \Omega_{ijkl} \mapsto \omega_{ijkl}.$$

To prove the theorem we shall construct a surjective morphism $\pi_2 : C(\mathbb{G} \wr_* S_N^+) \to C(\mathbb{H})$ such that

$$\pi_1 \circ \pi_2 = \mathrm{id} = \pi_2 \circ \pi_1.$$

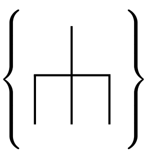
We define the following elements in $C(\mathbb{H})$

$$V_{kl}^{(i)} := \sum_j \Omega_{ijkl} \quad \text{and} \quad U_{ijk} = \sum_l \Omega_{ijkl} \Omega_{ijkl}^*. \tag{6.3.12}$$

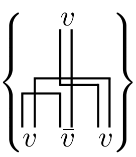
We shall prove that the generating relations in $C(\mathbb{G} \wr_* S_N^+)$ are also satisfied by the elements $V_{kl}^{(i)}$ and U_{ijk} in $C(\mathbb{H})$.

Since the generating matrix v of $\mathbb{G}$ is unitary, we get that $\xi = \sum_b Y_b \otimes \bar{Y}_b$ is a fixed vector of $v \otimes \bar{v}$ and thus $\xi \otimes \xi \in \operatorname{Hom}(1; v \otimes \bar{v} \otimes v \otimes \bar{v}) \simeq \operatorname{Hom}(v; v \otimes \bar{v} \otimes v)$. Via this isomorphism, we identify $\xi \otimes \xi$ with $Y \mapsto \sum_c Y_c \otimes \overline{Y_c} \otimes Y$.

We then have an intertwiner $T := t_L^p(T_p \otimes \xi \otimes \xi) \in \operatorname{Hom}(\Omega; \Omega \otimes \bar{\Omega} \otimes \Omega) \subseteq \mathcal{T}$ with

$p =$  ,

i.e. with Notation 6.12 and making plain the $\mathbb{G}$-morphisms on the block p,

$P =$ 

;

corresponding to the linear map

$$T(Y \otimes e_a) = \sum_c (e_a \otimes Y_c) \otimes \overline{(e_a \otimes Y_c)} \otimes (e_a \otimes Y).$$

We obtain for all $a = 1, \dots, N$ and $b = 1, \dots, d_{\mathbb{G}}$:

$$\begin{aligned}
&\sum_{[i],[k],c} (e_{i_1} \otimes Y_{k_1}) \otimes (e_{i_2} \otimes Y_{k_2}) \otimes (e_{i_3} \otimes Y_{k_3}) \otimes \Omega_{i_1 a k_1 c}\ \Omega^*_{i_2 a k_2 c}\ \Omega_{i_3 a k_3 b} \\
&= \sum_{i,k,r} (e_i \otimes Y_r) \otimes (e_i \otimes Y_r) \otimes (e_i \otimes Y_k) \otimes \Omega_{iakb}
\end{aligned}$$

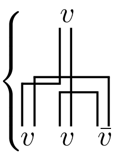
so that for all $[i] \in \{1, \dots, N\}^3$, $[k] \in \{1, \dots, d_{\mathbb{G}}\}^2$, $a \in \{1, \dots, N\}$ and $b \in \{1, \dots, d_{\mathbb{G}}\}$:

$$\left(\sum_c \Omega_{i_1 a k_1 c}\ \Omega^*_{i_2 a k_2 c} \right) \Omega_{i_3 a k_3 b} = \delta_{i_1, i_2, i_3} \delta_{k_1, k_2} \Omega_{i_3 a k_3 b}. \tag{6.3.13}$$

and taking adjoints:

$$\Omega^*_{i_3 a k_3 b} \left(\sum_c \Omega_{i_2 a k_2 c}\ \Omega^*_{i_1 a k_1 c} \right) = \delta_{i_1, i_2, i_3} \delta_{k_1, k_2} \Omega^*_{i_3 a k_3 b}. \tag{6.3.14}$$

Considering now

$P' = \Bigg\{$  $\Bigg\}$,

we can get the same way, for all $[i] \in \{1, \dots, N\}^3$, $[k] \in \{1, \dots, d_{\mathbb{G}}\}^2$, $a \in \{1, \dots, N\}$ and $b \in \{1, \dots, d_{\mathbb{G}}\}$, using $t^p_L(T_{p'} \otimes \xi \otimes \xi) \in \mathrm{Hom}(\Omega; \Omega \otimes \Omega \otimes \bar{\Omega}) \subseteq \mathcal{T}$,

$$\Omega_{i_3 a k_3 b} \left(\sum_c \Omega_{i_1 a k_1 c}\ \Omega^*_{i_2 a k_2 c} \right) = \delta_{i_1, i_2, i_3} \delta_{k_1, k_2} \Omega_{i_3 a k_3 b}, \tag{6.3.15}$$

and taking adjoints:

$$\left(\sum_c \Omega_{i_2 a k_2 c}\ \Omega^*_{i_1 a k_1 c} \right) \Omega^*_{i_3 a k_3 b} = \delta_{i_1, i_2, i_3} \delta_{k_1, k_2} \Omega^*_{i_3 a k_3 b}. \tag{6.3.16}$$

We shall obtain from (6.3.13), (6.3.14), (6.3.15), (6.3.16) all the necessary relations in $C(\mathbb{H})$ to reconstruct the free wreath product $\mathbb{G} \wr_* S_N^+$.

From these relations, we see in particular that the elements $U_{ijk} = \sum_c \Omega_{ijkc} \Omega^*_{ijkc}$ do not depend on k since

$$\begin{aligned}
U_{ijk} U_{ijk'} &= \sum_{c,d} \Omega_{ijkc} \Omega^*_{ijkc} \Omega_{ijk'd} \Omega^*_{ijk'd} \\
&= \sum_d \left(\sum_c \Omega_{ijkc} \Omega^*_{ijkc} \Omega_{ijk'd} \right) \Omega^*_{ijk'd} \\
&= \sum_d \Omega_{ijk'd} \Omega^*_{ijk'd} = U_{ijk'} && \text{(by (6.3.13))},
\end{aligned}$$

and similarly $U_{ijk} U_{ijk'} = U_{ijk}$, using (6.3.15). We then obtain $U_{ijk} = U_{ijk'}$. Let us simply write $U_{ij} := U_{ijk}$. Notice that the case $k = k'$ above shows that U_{ij} is an orthogonal projection (the relation $U_{ij}^* = U_{ij}$ is clear). In fact, the matrix (U_{ij}) is a magic unitary, since it is a unitary whose entries are orthogonal projections.

We now prove that for all $i = 1, \dots, N$ and all $\varepsilon_j, \varepsilon'_k \in \{1, *\}$,

$$\mathrm{Hom}_{\mathbb{G}}\left(v^{\varepsilon_1} \otimes \cdots \otimes v^{\varepsilon_k}; v^{\varepsilon'_1} \otimes \cdots \otimes v^{\varepsilon'_l}\right) \subseteq \mathrm{Hom}_{\mathbb{H}_i}(V^{(i)\varepsilon_1} \otimes \cdots \otimes V^{(i)\varepsilon_k}; V^{(i)\varepsilon'_1} \otimes \cdots \otimes V^{(i)\varepsilon'_l}\Big),$$

where $\mathbb{H}_i$ is the compact matrix quantum groups whose underlying Woronowicz-C^*-algebra is generated by the coefficients of $V^{(i)}$. By Frobenius reciprocity, it is enough to prove that any fixed vector in $\mathbb{G}$ is fixed in $\mathbb{H}_i$.

If $\xi_k = \sum_{[j]} \lambda_{[j]} Y_{j_1} \otimes \cdots \otimes Y_{j_k} \in \mathrm{Hom}(1; v^{\varepsilon_1} \otimes \cdots \otimes v^{\varepsilon_k})$, we have:

$$\sum_{[r][j]} \lambda_{[j]} Y_{r_1} \otimes \cdots \otimes Y_{r_k} \otimes v^{\varepsilon_1}_{r_1 j_1} \dots v^{\varepsilon_k}_{r_k j_k} = \sum_{[r]} \lambda_{[r]} Y_{r_1} \otimes \cdots \otimes Y_{r_k} \otimes 1,$$

i.e. $\forall [r] \in \{1, \dots, d_{\mathbb{G}}\}^k$, we have the following relations in $C(\mathbb{G})$:

$$\sum_{[j]} \lambda_{[j]} v^{\varepsilon_1}_{r_1 j_1} \dots v^{\varepsilon_k}_{r_k j_k} = \lambda_{[r]}. \tag{6.3.17}$$

Now, we use the morphism $(t^p_L)^* \circ (T_p \otimes \xi_k) \in \mathcal{T}$, with $p = 1_k \in NC(k)$ i.e.

$$\begin{aligned}(t^p_L)^* \circ (\xi_k \otimes T_p) &= \sum_{i[j]} \lambda_{[j]} (e_i \otimes Y_{j_1}) \otimes \cdots \otimes (e_i \otimes Y_{j_k}) \\ &\in \mathrm{Hom}(1; \Omega^{\varepsilon_1} \otimes \cdots \otimes \Omega^{\varepsilon_k}) \subseteq \mathcal{T}.\end{aligned}$$

We get

$$\sum_{[r][t]} (e_{r_1} \otimes Y_{t_1}) \otimes \cdots \otimes (e_{r_k} \otimes Y_{t_k}) \otimes \sum_{i[j]} \lambda_{[j]} \Omega^{\varepsilon_1}_{r_1 i t_1 j_1} \dots \Omega^{\varepsilon_k}_{r_k i t_k j_k} \tag{6.3.18}$$

$$= \sum_{r[t]} \lambda_{[t]} (e_r \otimes Y_{t_1}) \otimes \cdots \otimes (e_r \otimes Y_{t_k}) \otimes 1. \tag{6.3.19}$$

Notice that the relations (6.3.13), (6.3.14), (6.3.15), (6.3.16) yield for $\varepsilon = 1, *$ and all i, j, k, l:

$$U_{ij} \Omega^{\varepsilon}_{ijkl} = \Omega^{\varepsilon}_{ijkl} = \Omega^{\varepsilon}_{ijkl} U_{ij}. \tag{6.3.20}$$

Then using these commuting relations and the fact that (U_{ij}) is a magic unitary, we get from

(6.3.18):

$$\begin{aligned}
&\sum_{[r][t]}(e_{r_1}\otimes Y_{t_1})\otimes\cdots\otimes(e_{r_k}\otimes Y_{t_k})\otimes\sum_{i[j]}\lambda_{[j]}\Omega^{\varepsilon_1}_{r_1it_1j_1}\dots\Omega^{\varepsilon_k}_{r_kit_kj_k}\\
&\qquad=\sum_{[r][t]}(e_{r_1}\otimes Y_{t_1})\otimes\cdots\otimes(e_{r_k}\otimes Y_{t_k})\otimes\sum_{i[j]}\lambda_{[j]}(\Omega^{\varepsilon_1}_{r_1it_1j_1}U_{r_1i})\dots(\Omega^{\varepsilon_k}_{r_kit_kj_k}U_{r_ki})\\
&\qquad=\sum_{r_1[t]}(e_{r_1}\otimes Y_{t_1})\otimes\cdots\otimes(e_{r_1}\otimes Y_{t_k})\otimes\sum_{i[j]}\lambda_{[j]}(\Omega^{\varepsilon_1}_{r_1it_1j_1}\dots\Omega^{\varepsilon_k}_{r_1it_kj_k})(U_{r_1i}\dots U_{r_1i})\\
&\qquad=\sum_{r_1[t]}(e_{r_1}\otimes Y_{t_1})\otimes\cdots\otimes(e_{r_1}\otimes Y_{t_k})\otimes\sum_{[i][j]}\lambda_{[j]}(\Omega^{\varepsilon_1}_{r_1it_1j_1}\dots\Omega^{\varepsilon_k}_{r_1it_kj_k})(U_{r_1i_1}\dots U_{r_1i_k})\\
&\qquad=\sum_{r_1[t]}(e_{r_1}\otimes Y_{t_1})\otimes\cdots\otimes(e_{r_1}\otimes Y_{t_k})\otimes\sum_{[i][j]}\lambda_{[j]}(\Omega^{\varepsilon_1}_{r_1i_1t_1j_1}U_{r_1i_1})\dots(\Omega^{\varepsilon_k}_{r_1i_kt_kj_k}U_{r_1i_k})\\
&\qquad=\sum_{r_1[t]}(e_{r_1}\otimes Y_{t_1})\otimes\cdots\otimes(e_{r_1}\otimes Y_{t_k})\otimes\sum_{[j]}\lambda_{[j]}V^{(r_1)\varepsilon_1}_{t_1j_1}\dots V^{(r_1)\varepsilon_k}_{t_kj_k}.
\end{aligned}$$

Hence with (6.3.19), we obtain $\forall[t]\in\{1,\dots,d_{\mathbb{G}}\}^k$

$$\sum_{[j]}\lambda_{[j]}V^{(r_1)\varepsilon_1}_{t_1j_1}\dots V^{(r_1)\varepsilon_k}_{t_kj_k}=\lambda_{[t]},$$

so that $\xi_k=\sum_{[j]}\lambda_{[j]}Y_{j_1}\otimes\cdots\otimes Y_{j_k}\in\mathrm{Hom}_{\mathbb{H}_r}\left(1;V^{(r)\varepsilon_1}\otimes\cdots\otimes V^{(r)\varepsilon_k}\right)$ for all $r=1,\dots,N$.

Then, we obtain that $\mathrm{Rep}(\mathbb{G})\subseteq\mathrm{Rep}(\mathbb{H}_i)\subseteq\mathrm{Rep}(\mathbb{H})$ as full sub-categories. Woronowicz's Tannaka-Krein duality theorem then implies that for all $i=1,\dots,N$ there exists a morphism

$$\pi_i:C(\mathbb{G})\to C(\mathbb{H}_i)\subseteq C(\mathbb{H})$$

sending v to $V^{(i)}$.

Now, we prove that $V^{(i)}_{kl}U_{ij}=\Omega_{ijkl}=U_{ij}V^{(i)}_{kl}$. This follows from (6.3.20):

$$V^{(i)}_{kl}U_{ij}=\sum_J\Omega_{iJkl}U_{ij}=\Omega_{ijkl}U_{ij}=\Omega_{ijkl}$$

and similarly

$$U_{ij}V^{(i)}_{kl}=\Omega_{ijkl}.$$

It follows from what we have proved above that there exist morphisms

- $\pi_i:C(\mathbb{G})\to C(\mathbb{H}_i)$ such that $\pi_i\left(v^{(i)}_{kl}\right)=V^{(i)}_{kl}$, for all $i=1,\dots,N$,
- $\pi_{N+1}:C(S_N^+)\to C(\mathbb{H})$ such that $\pi_{N+1}(u_{ij})=U_{ij}$.

Thanks to the commuting relations we obtained above, these morphisms induce a morphism $\pi_2:C(\mathbb{G}\wr_* S_N^+)\to C(\mathbb{H})$, such that $\pi_2\left(v^{(i)}_{kl}u_{ij}\right)=V^{(i)}_{kl}U_{ij}$. By construction, we then get $\pi_1\circ\pi_2=\mathrm{id}=\pi_2\circ\pi_1$ and the proof is complete. □

Remark 6.17. *In the case where $\mathbb{G}$ is the dual of a discrete (classical) group $\mathbb{G}=\widehat{\Gamma}$, we recover the results of [16] and [55]. Indeed, in this case, the irreducible correpresentations of $\mathbb{G}=(C^*(\Gamma),\Delta)$ are the one-dimensional group like correpresentations $\Delta(g)=g\otimes g, g\in\Gamma$, the* *trivial one is the neutral element e and the tensor product of two irreducible correpresentations is their product in Γ. Any morphism*

$$S_{[g],[h]} : \mathbb{C} \simeq \mathbb{C}^{\otimes k} \to \mathbb{C}^{\otimes l} \simeq \mathbb{C}, g_1 \dots g_k \to h_1 \dots h_l$$

is determined by the image of $1 \in \mathbb{C}$ and the tensor products $S_{[g],[h]} \otimes T_p$ are scalar multiplication of the linear maps T_p. The space

$$Hom_{\widehat{\Gamma} \wr_* S_N^+}(r(g_1) \otimes \cdots \otimes r(g_k); r(h_1) \otimes \cdots \otimes r(h_l))$$

is generated by the maps T_p where $p \in NC(k,l)$ is an admissible diagram in $NC_{\widehat{\Gamma}}$ as in Definition 6.13. In this setting, p is admissible if $p \in NC(k,l)$ has the additional rules that if one decorates the points of p by the elements g_i, h_j then in each block, the product on top is equal to the product on bottom in Γ.

In the sequel, we denote by $1_{\mathbb{G}}$ the trivial $\mathbb{G}$-representation and simply by 1 the one of $\mathbb{G} \wr_* S_N^+$.

Corollary 6.18. *Let $N \geq 4$, then:*

1. *For all $\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_l \in Rep(\mathbb{G})$, we have*

$$\begin{aligned}\dim \mathrm{Hom}_{\mathbb{G} \wr_* S_N^+}&(r(\alpha_1) \otimes \cdots \otimes r(\alpha_k); r(\beta_1) \otimes \cdots \otimes r(\beta_l)) \\ &= \sum_{p \in NC_{\mathbb{G}}([\alpha],[\beta])} \prod_{B \in p} \dim \mathrm{Hom}_{\mathbb{G}}(\alpha(U_B), \beta(L_B)).\end{aligned}$$

2. *If $\alpha \in \mathrm{Irr}(\mathbb{G})$ is non-equivalent to $1_{\mathbb{G}}$ then $r(\alpha)$ is an irreducible $\mathbb{G} \wr_* S_N^+$-representation.*
3. *$r(1_{\mathbb{G}}) = (u_{ij}) = 1 \oplus \omega(1_{\mathbb{G}})$ for some $\omega(1_{\mathbb{G}}) \in \mathrm{Irr}(\mathbb{G} \wr_* S_N^+)$.*
4. *Denoting $\omega(\alpha) := r(\alpha) \ominus \delta_{\alpha,1_{\mathbb{G}}} 1$ then $(\omega(\alpha))_{\alpha \in \mathrm{Irr}(\mathbb{G})}$ is a family of pairwise non-equivalent $\mathbb{G} \wr_* S_N^+$-irreducible correpresentations.*

Proof. We use Theorem 6.16 and the independence of the linear maps

$$T_p \in \mathcal{B}((\mathbb{C}^N)^{\otimes k}, (\mathbb{C}^N)^{\otimes l}),\ p \in NC(k,l)$$

for all $N \geq 4$. The first assertion follows from this linear independence of the maps T_p. Indeed, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbb{G} \wr_* S_N^+}&(r(\alpha_1) \otimes \cdots \otimes r(\alpha_k); r(\beta_1) \otimes \cdots \otimes r(\beta_l)) \\ &= \bigoplus_{p \in NC_{\mathbb{G}}([\alpha],[\beta])} \mathrm{span}\left\{U^{p,S} : \forall B, S_B \in \mathrm{Hom}_{\mathbb{G}}(\alpha(U_B), \beta(L_B))\right\}\end{aligned}$$

and the first assertion follows by computing the dimension of the spaces on each side.

Now we prove simultaneously the last three relations. For $\alpha, \beta \in \mathrm{Irr}(\mathbb{G})$, the intertwiner space

$$\mathrm{Hom}_{\mathbb{G} \wr_* S_N^+}(r(\alpha), r(\beta))$$

is encoded by the following candidate diagrams:

$$p_1 = \left\{ \begin{array}{c} \alpha \\ | \\ \beta \end{array} \right\} \text{ and } p_2 = \left\{ \begin{array}{c} \alpha \\ \vdots \\ \beta \end{array} \right\}.$$

Since α and β are irreducible, we see that p_1 is an admissible diagram if and only if $\alpha \simeq \beta$ and p_2 is admissible if and only if $\alpha \simeq \beta \simeq 1_{\mathbb{G}}$.

Therefore, if α is not equivalent to β:

$$\dim\ \mathrm{Hom}_{\mathbb{G} \wr_* S_N^+}(r(\alpha), r(\beta)) = 0.$$

If $\alpha \simeq \beta$ are not the trivial correpresentation $1_{\mathbb{G}}$ then the only intertwiner $r(\alpha) \to r(\beta)$ arises from p_1:

$$\dim\ \mathrm{Hom}_{\mathbb{G} \wr_* S_N^+}(r(\alpha), r(\alpha)) = 1.$$

If $\alpha \simeq \beta \simeq 1_{\mathbb{G}}$, then the diagram p_2 also gives rise to an intertwiner $U^{p_2, S}_{(1_{\mathbb{G}}),(1_{\mathbb{G}})}$ with $S : 1_{\mathbb{G}} \to 1_{\mathbb{G}}$ the trivial inclusion. The independence of $T_{\{|\}} = \mathrm{id}_{\mathbb{C}^N}$ and $T_{\{\,\vdots\,\}}$ allows to conclude

$$\dim\ \mathrm{Hom}_{\mathbb{G} \wr_* S_N^+}(r(1_{\mathbb{G}}), r(1_{\mathbb{G}})) = 2.$$

$\square$

6.4 Probabilistic aspects of the free wreath product

We provide here some probabilistic consequences of the description of the intertwiner spaces of $\mathbb{G} \wr_* S_N^+$. In this section we are mainly interested in the non-commutative probability space arising from the Haar state on $C(\mathbb{G} \wr_* S_N^+)$ and the behavior of the coefficients of a correpresentation as random variables in this setting. Since most of the results involve the law of free compound Poisson laws, we shall recall its definition. We refer to [66] for an introductory course on non-commutative variables.

6.4.1 Laws of characters

Notation 6.19. *In the sequel $\varepsilon = \varepsilon(1) \dots \varepsilon(r)$ denotes a word in $\{1, *\}$ and $NC(\varepsilon)$ is the set of noncrossing partitions with each endpoint i colored with $\varepsilon(i)$. For $p \in NC(\varepsilon)$ and B a block of p, $\varepsilon(B)$ denotes the subword of ε coming from the points in the block B (with the same order as in p).*

Let (A, φ) be a noncommutative probability space, X an element of A with $*-$distribution μ_X depicted by all of its moments

$$m_X(\varepsilon) = \varphi(X^{\varepsilon(1)} \dots X^{\varepsilon(r)}).$$

Similarly as in (6.2.4), the free cumulants of X, $\{k_X(\varepsilon)\}_\varepsilon$ is the unique collection of complex numbers such that the following moment-cumulant formula holds for all ε :

$$m_X(\varepsilon) = \sum_{p \in NC(\varepsilon)} \prod_B k_X(\varepsilon(B)).$$

The existence and uniqueness of such a collection is easily proved by induction on the length of ε [66].

Definition 6.20. *The free compound Poisson distribution $\mathcal{P}_\lambda(\mu_X)$ with law μ_X and parameter $\lambda > 0$ is the $\star-$distribution defined by its free cumulants*

$$k_{\mathcal{P}_\lambda(\mu_X)}(\varepsilon) = \lambda m_X(\varepsilon). \tag{6.4.1}$$

In particular, if Y is a random variable following a free compound Poisson distribution with law μ_X and parameter 1, then we have the following moment formula :

$$m_Y(\varepsilon) = \sum_{p\in NC(\varepsilon)} \prod_B m_X(\varepsilon(B)).$$

We refer to [66] for the proof that there exists actually a propability space and a random variable on it with such a distribution.
The first result is a direct application of the Corollary 6.18. We refer to Definition 6.3.1 for the definition of the correpresentation $r(\alpha)$.

Proposition 6.21. *Let $\mathbb{G}$ be a compact quantum group of Kac type, $\alpha \in \mathrm{Rep}(\mathbb{G})$, $n \geq 4$. Then the law of the character $\chi(r(\alpha))$, with respect to the Haar state h, is a free compound Poisson with law $\chi(\alpha)$ and parameter* 1.

Proof. Let ε be a word in $\{1, \star\}$. Then the law of a free compound Poisson with law $\chi(\alpha)$ and parameter 1, $\mathcal{P}(\chi(\alpha))$ is described by its free cumulants, with the formula (6.4.1):

$$k_{\mathcal{P}(\chi(\alpha))}(\varepsilon(1)\dots\varepsilon(r)) = m_{\chi(\alpha)}(\varepsilon(1)\dots\varepsilon(r).$$

With the moment-cumulant formula, this is equivalent to the following expression for the moments of $\mathcal{P}(\chi(\alpha))$:

$$m_{\mathcal{P}(\chi(\alpha))} = \sum_{p\in NC_\varepsilon} \prod_B m_{\chi(\alpha)}(\varepsilon(B)).$$

By the Corollary 6.18 we have

$$\begin{aligned} h\left(\chi(r(\alpha)^{\varepsilon(1)}\dots\chi_n(r(\alpha))^{\varepsilon(r)}\right) &= \dim \mathrm{Hom}_{\mathbb{G}\wr_* S_N^+}(1; r(\alpha)^{\varepsilon(1)} \otimes \dots \otimes r(\alpha)^{\varepsilon(r)}) \\ &= \sum_{p\in NC_\varepsilon} \prod_B \dim \mathrm{Hom}_{\mathbb{G}}(1, \alpha(L_B)) \\ &= \sum_{p\in NC_\varepsilon} \prod_B m_{\chi(\alpha)}(\varepsilon(B)). \end{aligned}$$

The second equality is given by Corollay 6.18, and the third one by the definition of $\alpha(L_B)$ and the tensor product structure. ☐

A consequence of this result is a partial answer to the free product conjecture given by Banica and Bichon (see [10]) : for each compact matrix quantum group (A, v) we denote by $\mu(A, v)$ the law of the character of the fundamental representation with respect to the Haar measure. A quantum permutation group is a quantum subgroup of S_N^+ for some $N \geq 0$, in the following sense : it is a compact matrix quantum group (A, v) such that there exists a surjective C^*-morphism $\Phi : C(S_N^+) \to A$ sending the elements u_{ij} of $C(S_N^+)$ to v_{ij} (see [82] for a survey on the subject).

Corollary 6.22. *Let (A, v) be a quantum permutation group, and $S_N^+ = (C(S_N^+, u)$, $n \geq 4$. Then*

$$\mu(A \wr_* B, w) = \mu(A, v) \boxtimes \mu(C(S_N^+, u).$$

Proof. It is a direct consequence of the last proposition and the fact that in the law of a free compound poisson with law μ is the same as the free multiplicative convolution of μ with the free Poisson distribution. ☐

The conjecture asserts that this formula still holds when replacing S_N^+ with certain quantum subgroups of S_N^+. See [10] for more details.

6.4.2 Weingarten calculus

Let us construct a Weingarten calculus for a free wreath product. Weingarten calculus has been mainly developped in the framework of compact quantum groups and permutation quantum groups by Banica and Collins (see [12],[13]). This tool has mainly two advantages : on one hand it allows us sometimes to get some interesting formulae for the Haar state on the matrix entries of a correpresentation, and on the other hand it yields some asymptotic results on the joint law of a finite set of elements when the dimension of the quantum group goes to infinity.

Let us first sum up the pattern of this method coming from [12]: let $\mathbb{G} = (A, (u_{ij})_{1\leq i,j\leq n})$ be a matrix compact quantum group acting on $V^{\otimes k} = \langle X_i \rangle_{1\leq i\leq n}^{\otimes k}$ with the correpresentation α_k, and h the associated Haar measure. We will assume that $\mathbb{G}$ is orthogonal to simplify the notations, although it could be easily generalized to the general Kac type case : that means that the elements u_{ij} are all self-adjoint in A (see [99]). By the property of the Haar state,

$$(Id \otimes h) \circ \alpha_k(X_{i_1} \otimes \cdots \otimes X_{i_k}) = P(X_{i_1} \otimes \cdots \otimes X_{i_k}),$$

with P the orthogonal projection of $V^{\otimes k}$ on the invariant subspace of α_k. On the other hand,

$$(Id \otimes h) \circ \alpha_k(X_{i_1} \otimes \cdots \otimes X_{i_k}) = \sum h(u_{j_1 i_1} \dots u_{j_k i_k})(X_{j_1} \otimes \cdots \otimes X_{j_k}).$$

We get thus the following expression for the Haar state on $u_{j_1 i_1} \dots u_{j_k i_k}$:

$$h(u_{j_1 i_1} \dots u_{j_k i_k}) = \langle P(X_{i_1} \otimes \cdots \otimes X_{i_k}), X_{j_1} \otimes \cdots \otimes X_{j_k} \rangle.$$

The right-hand side may be hard to compute. Hopefully the Gram-Schmidt orthogonalisation process yields a nicer expression if we already know a basis of the invariant subspace S_k of α_k. Let $\{S_k(i)\}$ be a basis of this subspace, G_k being the Gram-Schmidt matrix of $\{S_k(i)\}$ defined by $G_k(i,j) = \langle S_k(i), S_k(j) \rangle$ and $W_k = G_k^{-1}$. A standard computation yields:

$$h(u_{j_1 i_1} \dots u_{j_k i_k}) = \sum_{i,j} \langle X_{i_1} \otimes \cdots \otimes X_{i_k}, S_k(i) \rangle W_k(i,j) \langle S_k(j), X_{j_1} \otimes \cdots \otimes X_{j_k} \rangle.$$

Of course the matrix $W_k(i,j)$ is hard to compute.

Let us see nonetheless what it gives in the case of a free wreath product $(\mathbb{G} \wr_* S_N^+, (w_{ij,kl}))$, with $\mathbb{G}$ an orthogonal matrix quantum group. A basis of S_k is given by the vectors $U^{p,S}, p \in NC(k)$, as defined in (6.3.5). The first task is to compute the matrix $W_k(i,j)$. Consider the following map

$$\begin{aligned} t_k : (\mathbb{C}^N \otimes V) \otimes \cdots \otimes (\mathbb{C}^N \otimes V) &\to (\mathbb{C}^N)^{\otimes k} \otimes V \otimes \cdots \otimes V \\ \bigotimes_{i=1}^{k} (x_i \otimes y_i) &\mapsto \bigotimes_{i=1}^{k} x_i \otimes \bigotimes_{i=1}^{k} y_i. \end{aligned}$$

t_k is unitary and and by definition of $U^{p,S}$,

$$t_k(U^{p,S}) = T_p \otimes S.$$

Recall that S depends implicitly on p through the definition (6.3.5): the latter is an invariant vector of the $k-$tensor product representation of $\mathbb{G}$ having the block structure of p. Nevertheless S is independent of N and in particular we have the expression

$$\begin{aligned} \langle U^{p,S}, U^{q,S'} \rangle =& \langle t_k(U^{p,S}), t_k(U^{q,S'}) \rangle \\ =& \langle T_p, T_q \rangle \langle S, S' \rangle = N^{b(p\vee q)} \langle S, S' \rangle. \end{aligned}$$

Remark 6.23. *Easy quantum groups form a particular family of compact quantum groups whose associated intertwiners spaces can be combinatorially described. Namely if $\mathbb{G}$ is an easy quantum group, the invariant subspace of the $k-$tensor-product representation is spanned by the vectors T_p, as defined in Section 1.1.3, with p belonging to a subcategory of $\mathcal{P}(k)$. See [15], [71] for more informations on the subject, and [50], [40] and [25] for some applications. In this case, the scalar product matrix has a simpler form. Indeed if $\mathbb{G}$ is an easy quantum group of dimension s and with category of partition $\mathcal{C}$, then a direct computation yields for $\alpha \leq p, \beta \leq q$ two partitions in $\mathcal{C}$:*

$$\langle U^{p,\alpha}, U^{q,\beta} \rangle = N^{b(p\vee q)} s^{b(\alpha\vee\beta)}.$$

The Weingarten formula has also a more combinatorial form since we can write:

$$h(w_{i_1 j_1, k_1 l_1} \dots w_{i_r j_r, k_r l_r}) = \sum_{\substack{\alpha \leq \ker(\vec{i}), \beta \leq \ker(\vec{j}) \\ \alpha \leq p \leq \ker(\vec{k}), \beta \leq q \leq \ker(\vec{l})}} G_k^{-1}((p,\alpha),(q,\beta)),$$

where $\ker(\vec{i})$ *is the partition whose blocks are the set of indices on which i has the same value.*

The scalar product matrix $G_k = (\langle U^{p,S}, U^{q,S'} \rangle)_{(p,S),(q,S')}$ is a block matrix, the blocks G_k^{pq} being indexed by $p, q \in NC(k)$. Note that as in [12], one can factorize this matrix as follows:

$$G_k = \Delta_{nk}^{1/2} \tilde{G} \Delta_{nk}^{1/2},$$

where Δ_{nk} is the diagonal matrix with diagonal coefficients

$$\Delta_{nk}((p,S),(p,S)) = N^{b(p)}$$

and

$$\tilde{G}_k((p,S),(q,S') = N^{b(p\vee q) - \frac{b(p)+b(q)}{2}} \langle S, S' \rangle.$$

Asymptotically with n going to infitiny, $\tilde{G}_k = D_k(1 + o(\frac{1}{\sqrt{n}}))$, D_k being the block diagonal matrix

$$D_k((p,S),(q,S')) = \delta_{p,q} \langle S, S' \rangle.$$

Finally we can remark that restricted on the subspace $V_{p_0} = Vect((U_{p_0,S})_S)$, the matrix $(\langle S, S' \rangle)_{S,S'}$ is the tensor product of the Gram-Schmidt matrices of $\mathbb{G}$ $G_{\mathbb{G},|B_i|}$, for each block $|B_i|$ of p_0. If we put all these considerations together, we get that

$$W_n((p,S),(q,S')) = \delta_{p,q} N^{-b(p)} \left(\bigotimes_{B \in p} W_{\mathbb{G}}^{-1} \right) (S,S')(1 + o(\frac{1}{\sqrt{n}})).$$

This formula allows to generalize the results in [12] to the free wreath product case. Define the following partial trace:

Definition 6.24. *Let $0 \leq s \leq n$ the partial trace of order s of the matrix $w = (w_{ij,kl})_{1\leq i,j\leq r, 1\leq k,l\leq n}$ is*

$$\chi^w(s) = \sum_{i=1}^{r} \sum_{k=1}^{s} w_{ii,kk}.$$

Let $t \in (0,1]$ and let $\mathbb{G}$ be a matrix compact quantum group of Kac type and dimension r. Denote by $\chi_{\mathbb{G}}$ the law of the character of its fundamental representation. Let $(\mathbb{G} \wr_* S_n^+, (w_{ij,kl})_{1\leq i,j\leq r, 1\leq k,l\leq n})$ be the matrix quantum group $\mathbb{G} \wr_* S_n^+$ with its fundamental representation w.

Theorem 6.25. *With respect to the Haar measure, if $s \sim tn$ for n going to infinity,*

$$\chi_w(s) \to \mathcal{P}_t(\chi_{\mathbb{G}}),$$

where $\mathcal{P}_t(\chi_{\mathbb{G}})$ is the free compound Poisson with parameter t and original law $\chi_{\mathbb{G}}$.

Proof. A similar computation as in [12], Theorem 5.1 gives

$$\int (\chi^w(s))^k = Tr(G_{k,n}^{-1} G_{k,s})$$

and with the asymptotic form of $G_{k,n}$ this gives us:

$$G_{k,n}^{-1} G_{k,s} = \Delta_{nk}^{-1/2} D_k^{-1} \Delta_{nk}^{-1/2} (Id + o(\frac{1}{\sqrt{n}})) \Delta_{sk}^{1/2} D_k \Delta_{sk}^{1/2} (Id + o(\frac{1}{\sqrt{n}})).$$

Since D_k is block diagonal and Δ_{nk}, Δ_{sk} are diagonal, and equal to the identity on each block, these three matrices commute, and

$$Tr(G_{k,n}^{-1} G_{k,s}) = Tr(\Delta_{s/n,k}(Id + o(\frac{1}{\sqrt{n}}))) \to Tr(\Delta_{t,k}).$$

Since

$$Tr(\Delta_{t,k}) = \sum_{p \in NC(k)} t^{b(p)} \dim V_p = \sum_{p \in NC(k)} t^{b(p)} \prod_{B \in p} m_{|B|}(\chi_{\mathbb{G}}),$$

$Tr(\Delta_{t,k})$ is exactly the $k-$th moment of the law $\mathcal{P}_t(\chi_{\mathbb{G}})$. $\square$

Remark 6.26. *All the results of this section can be transposed to the classical case. One just has to substitute classical compound Poisson laws for free compound Poisson laws, and use crossing partitions instead of non-crossing ones.*

6.4.3 Non-commutative symmetric functions as a probability space

We expose in this subsection a relation between the ring of non-commutative symmetric functions (as defined in Definition 3.38) and the representation theory of the free wreath product $\mathbb{U} \wr_* S_n^+$ for $n \geq 4$. This free wreath product is also the easy quantum group $H_n^{+,\infty}$ of Chapter 4 and 5. This quantum group has been deeply studied by Banica and Verginoux in [16]. In particular, they found the fusion rules of the irreducible representations:

Theorem 6.27 ([16])**.** *Let $n \geq 4$. The irreducible representations of $(C(H_n^{+,\infty}), (u_{ij})_{1 \leq i,j \leq n})$ are indexed by finite sequences of integers, with the fusion rules given by the recursive formula*

$$\begin{aligned}(j_1, \dots, j_r) \otimes (i_1, \dots, i_{r'}) = (j_1, \dots, j_r, i_1, \dots, i_{r'}) + (j_1, \dots, j_{r-1}, j_r + i_1, i_2, \dots, i_{r'}) \\ + \delta_{j_r + i_1 = 0} (j_1, \dots, j_{r-1}) \otimes (i_2, \dots, i_{r'}),\end{aligned}$$

such that (0) is the trivial representation and for k non zero, (k) is the irreducible representation given by the matrix $(u_{ij}^k)_{1 \leq ij \leq n}$.

This yields the following embedding of **NSym** in $Cl(H_n^{+,\infty})$:

Proposition 6.28. *The map Φ : **NSym** $\longrightarrow Cl(H_n^{+,\infty})$ defined by $\Phi(S_k) = \sum_{1 \leq i \leq k} u_{ii}^k$ is an embedding of **NSym** in $Cl(H_n^{+,\infty})$ such that for any composition I of n,*

$$\Phi(R_I) = \chi_I,$$

where χ_I is the character of the irreducible representation of $H_n^{+,\infty}$ indexed by I in Theorem 6.27.

This result can be seen as a non-commutative version of Theorem 2.23. The ring of non-commutative symmetric functions has already been related with the representations of a specialization at $q = 0$ of the quantum linear group and the Hecke algebra [52]. Proposition 6.28 gives a semisimple version of this result.

Proof. Since **NSym** is the free ring generated by the variables S_k, $k \geq 1$, there is a morphism of algebra $\Phi : \mathbf{NSym} \longrightarrow Cl(H_n^{+,\infty})$ sending S_i to $\chi_{(k)} = \sum_{1 \leq i \leq k} u_{ii}^k$. We prove by recursion of the length l of a composition I that $\Phi(R_I) = \chi_{(I)}$. If $l = 1$, this is true by the definition of I and the fact that $R_{(k)} = S_k$ for $k \geq 1$. Suppose the result true for $l-1$ and let $I = (i_1, \dots, i_l)$ be a composition of length l. On one hand, the product formula on the ribbon Schur basis yields

$$R_{(i_1,\dots,i_{l-1})} R_{(i_l)} = R_{(i_1,\dots,i_l)} + R_{(i_1,\dots,i_{l-2},i_{l-1}+i_l)}.$$

Thus applying the map Φ on both sides and using the induction hypothesis yields

$$\chi_{(i_1,\dots,i_{l-1})}\chi_{(i_l)} = \Phi(R_{(i_1,\dots,i_l)}) + \chi_{(i_1,\dots,i_{l-2},i_{l-1}+i_l)}.$$

Since each i_j is positive, the fusion rules of Theorem 6.27 give on the left hand side of the latter equation

$$\chi_{(i_1,\dots,i_{l-1})}\chi_{(i_l)} = \chi_{(i_1,\dots,i_{l-1}+i_l)} + \chi_{(i_1,\dots,i_l)}.$$

Therefore, $\Phi(R_{(i_1,\dots,i_l)}) = \chi_{(i_1,\dots,i_l)}$. Since the set $\{\chi_{(I)}\}_{I \text{ compositions}}$ is a set of characters of distinct irreducible representations (and thus linearly independent), Φ is injective. □

In particular, the Haar state on $Cl(H_n^{+,\infty})$ yields a scalar product on **NSym** by the formula

$$\langle F, G \rangle = h_{H_n^{+,\infty}}(\Phi(F)\Phi(G)^*),$$

and the basis of ribbon Schur functions is an orthonormal basis with respect to this scalar product.
From now on, **NSym** is identified with its image in $Cl(H_n^{+,\infty})$. By this identification, each variable S_i is a random variable with respect to the Haar measure on $C(H_n^{+,\infty})$. The law of the random vector $(S_i)_{i \geq 1}$ can be computed thanks to the description of the intertwiners of $\mathbb{U} \wr_* S_n^+$ in Section 6.3 (we could also directly use a result of [16], which already gives the description of the intertwiners of $\hat{\mathbb{Z}} \wr_* S_n^+$).

Proposition 6.29. *The family $\{S_i\}_{i \geq 1}$ is distributed as $(sz^i s)_{i \geq 1}$, where s is a semi-circular element and z is a uniform variable on the unit circle free from s.*

Proof. Let $i_1, \dots, i_r$ be integers distinct from zero, and write $S_{-i} = S_i^*$ for i positive. Then by the Tannaka-Krein duality,

$$h(S_{i_1} \dots S_{i_r}) = \dim \operatorname{Mor}_{H_n^{+,\infty}}(\mathbf{1}, \alpha(i_1) \otimes \dots \otimes \alpha(i_r)),$$

where $\alpha(k)$ is the irreducible representation $(u_{ij}^k)_{1 \leq i,j \leq n}$.
By Theorem 6.16 and the fact that the tensor product of the representations $z^{j_1}, \dots, z^{j_s}$ of $\mathbb{U}$ is trivial if and only if $\sum_{1 \leq m \leq s} j_m = 0$,

$$h(S_{i_1} \dots S_{i_r}) = \#\{\pi \in NC(i_1, \dots, i_r) | \forall B \in \pi, B \text{ is balanced}\},$$

where as in Chapter 5, $NC(i_1, \dots, i_r)$ denotes the set of non-crossing partitions with the element $1 \leq m \leq r$ colored with i_m, and where a block $\{j_1, \dots, j_s\}$ of a partition in $NC(i_1, \dots, i_r)$ is balanced if and only if $\sum_{1 \leq m \leq s} j_{i_m} = 0$. In the proof of Proposition 5.22, it has been proven that

$$\#\{\pi \in NC(i_1, \dots, i_r) | \forall B \in \pi, B \text{ is balanced}\} = m(sz^{i_1}s, sz^{i_2}s, \dots, sz^{i_r}s),$$

with s a semi-circular variable and z a uniform variable on the unit circle free from s. □

Chapter 7

Planar algebra of a free wreath product

This chapter is devoted to the general description of the intertwiner spaces for the free wreath product of two free permutation groups. This description leads to the proof of the following result, which was conjectured by Banica and Bichon in[10]:

Theorem 7.1 ([10], Conj 3.1)**.** *Let F and G be two non-commutative permutation groups such that* $\dim_F \mathrm{Mor}(0,1) = \dim_G \mathrm{Mor}(0,1) = 1$*. Then*

$$\mu(F \wr_* G) = \mu(F) \boxtimes \mu(G),$$

where $\mu(\mathbb{G})$ denotes the law (with respect to the Haar measure) of the character of the fundamental representation of a matrix compact quantum group $\mathbb{G}$.

The main ingredient in the proof of Theorem7.1 is the notion of free product of planar algebras, which has been introduced by Bisch and Jones in [23]. Section 1 is an introduction to the concept of planar algebra; several basic combinatorial results are given on planar tangles, the main objects in the construction of planar algebras. Section 2 gives an isomorphism between the intertwiner spaces of a free wreath product and the free product of certain planar algberas. Section 3 is devoted to the combinatorial proof of a dimension formula which has been found by Bisch and Jones in an unpublished paper [23]: this result yields the proof of Theorem 7.1 thanks to the isomorphism given in Section 2.

7.1 Planar algebras

As already said in Chapter 3, a planar algebra is a collection $(V_k)_{k\geq 1}$ of vector spaces with an action of planar tangles on thess vector spaces.

7.1.1 Definition of planar tangles

By a diffeomorphism of $\mathbb{R}^2$ we mean an orientation preserving diffeormophism defined on a domain D of $\mathbb{R}^2$. In this subsection we define the notion of irreducible planar tangle and we associate a non-crossing partition to each planar tangle.

Disk and intervals By a disk we mean an open subset of $\mathbb{R}^2$ whose boundary is a smooth Jordan curve.
Let $\mathbb{U}$ be the unit circle on $\mathbb{R}^2$. For $\omega \in \mathbb{U}$, we denote by $[1, \omega[$ (resp. $[1, \omega]$) the subset of $\mathbb{U}$

whose argument is less (resp. less or equal) to the one of ω (or to 2π if $\omega = 1$). If $\omega' \in [1, \omega[$, $[\omega', \omega]$ (resp. $]\omega', \omega[$) denotes the set $[1, \omega] \setminus [1, \omega'[$ (resp. $[1, \omega[\setminus [1, \omega']$).
Let D be a disk and let ∂D denote its boundary. For each finite subset $S \subseteq \partial D$ of cardinal k with a distinguished element $i_* \in S$, there is a canonical bijection from S to $[\![1, k]\!]$ obtained by numbering the elements of S counterclockwise, starting at i_*. The element of S which is numbered i is denoted by i_D.
Since ∂D is a Jordan curve, there is an orientation preserving diffeomorphism φ_D mapping ∂D to the unit circle $\mathbb{U}$. For each couple (i, j) of elements of S with $i < j$, the set $\varphi_D^{-1}(]\varphi_D(i), \varphi_D(j)[)$ is denoted by (i, j) and called the interval component of ∂D between i and j; the set $\varphi_D^{-1}(\mathbb{U} \setminus [\varphi_D(i), \varphi_D(j)])$ is denoted by (j, i) and called the interval component of ∂D between j and i.
When there is no possible confusion, $[\![1, k]\!]$ is always identified with $\mathbb{Z}/(k\mathbb{Z})$ and thus $k + 1 = 1$.
When the bounds of an interval component is not specified, this interval is always assume to be of type $(i, i + 1)$ for some $i \in [\![1, k]\!]$.
By a curve γ, we mean either a Jordan curve or a injective smooth map $\gamma : [0, 1] \to \mathbb{R}^2$. If γ is a Jordan curve, γ is called a closed curve.

Planar tangle A planar tangle is a particular collection of subsets of $\mathbb{R}^2$ which has been introduced by Jones in [46]. We can see it as a generalization of the two-level noncrossing partitions of Section 1.1.2 to a kind of multilevel noncrossing partition.

Definition 7.2. *A planar tangle P of degree $k \geq 0$ consists of the following objects:*

- *A disk D_0 of $\mathbb{R}^2$, called the outer disk.*
- *Some disjoint disks $D_1, \dots, D_n$ in the interior of D_0 which are called the inner disks.*
- *For each $0 \leq i \leq n$, a finite subset $S_i \in \partial D_i$ of cardinal $2k_i$ (such that $k_0 = k$) with a particular element $i_* \in S_i$. The elements of S_i are called the distinguished points of D_i and numbered counterclockwise starting from i_*. k_i is called the degree of the inner disk D_i.*
- *A finite set of disjoint smooth curves $\{\gamma_j\}_{1 \leq j \leq r}$ such that each $\mathring{\gamma}_j$ lies in the interior of $D_0 \setminus \bigcup_{i \geq 1} D_i$ and such that $\bigcup_{1 \leq j \leq r} \partial\gamma_j = \bigcup_{0 \leq i \leq n} S_i$; it is also required that each curve meets a disk boundary orthogonally, and that its endpoints have opposite (resp. same) parity if they both belong to inner disks or both belong to the outer disk (resp. one belongs to an inner disk and the other one to the outer disk).*
- *A region of P is a connected component of $D_0 \setminus (\bigcup_{i \geq 1} D_i \cup (\bigcup \gamma_j))$. Give a chessboard shading on the regions of P in such a way that the interval components of type $(2i + 1, 2i)$ are boundaries of shaded regions.*

The skeleton of P, denoted by ΓP, is the set $(\bigcup \partial D_i) \cup (\bigcup \gamma_j)$.

In the sequel, D_0 denotes always the outer disk of a planar tangle. If the degree of P is 0, then P is of degree 0_+ (resp. 0_-) if the boundary of the outer disk is the boundary of an unshaded region (resp. shaded). An example of planar tangle with its associated shading is given in Figure 7.1.

An isotopy of P is the image of P (i.e the family of images of all the given sets in the definition, such that a region and its image have same shading) by a diffeomorphism, and two planar tangles P and P' are said equivalent if there exists an isotopy φ such that $\varphi(P) = P'$. For any disk D of $\mathbb{R}^2$ with a set S of $2k$ distinguished points on ∂D (numbered counterclockwise from 1

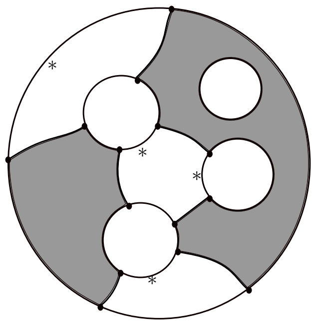


Figure 7.1: : Planar tangle of degree 4 with 4 inner disks.

to $2k$), there is an isotopy $\varphi_{D,S}$ of P such that $\varphi_{D,S}(\partial D_0) = \partial D$ and $\varphi_{D,S}(i_{D_0}) = i_D$ for each distinguished point of D.
A connected planar tangle is a planar tangle whose regions are simply connected; this implies that for any inner disk D and any element $x \in \partial D$, there is a path from x to the boundary of the outer disk which is contained in $(\bigcup \partial D_i) \cup (\bigcup \gamma_j)$. An irreducible planar tangle is a connected planar tangle such that each curve has an endpoint being a distinguished point of D_0 and the other one being on an inner disk. An example of connected planar tangle and of an irreducible planar tangle is given in 7.2.

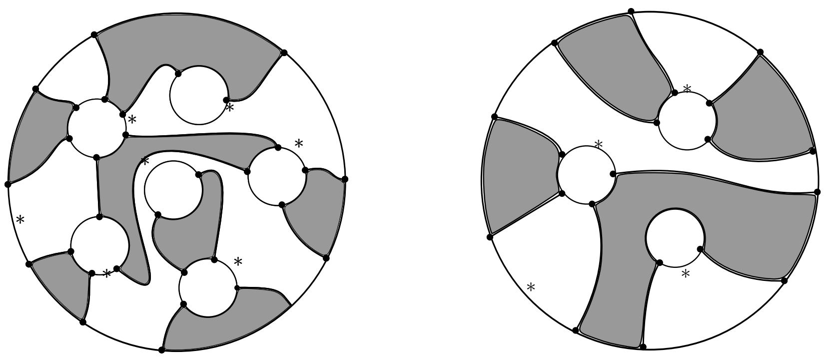


Figure 7.2: : A connected and an irreducible planar tangle.

Composition of planar tangles Let P and P' be two planar tangles of respective degree k and k', and let D be an inner disk of P. We assume that the degree of D is also k'. Let S be the set of distinguished points of D and let $\varphi_{D,S}$ be an isotopy of P' to ∂D respecting the distinguished points. For each distinguished point i of D, the union of the curve of P and the one of $\varphi_{D,S}(P')$ ending at i yields a new smooth curve. Thus, the union of P and $\varphi_{D,S}(P')$ with the disk D removed is a new planar tangle which is denoted by $P \circ_D P'$ and called the composition of P and P' with respect to D. An example of composition the planar tangle of Figure 7.1 with the second planar tangle of Figure 7.2 is given in Figure 7.3 (only an equivalent planar tangle is displayed in orther to get a clear picture).

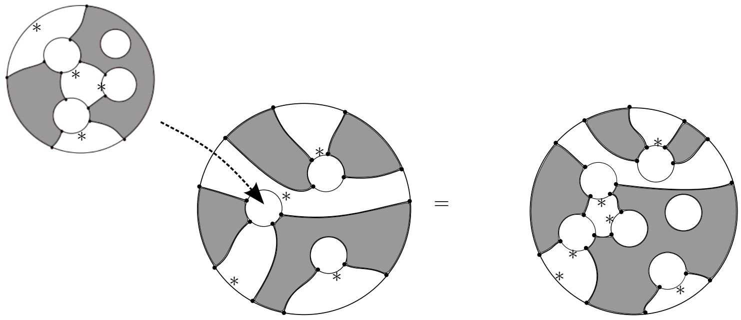


Figure 7.3: : Composition of two planar tangles.

Note that if D_1 and D_2 are two distinct inner disks of P and P_1 and P_2 are two planar tangles having respectively the same degree as D_1 and D_2, then $(P \circ_{D_1} P_1) \circ_{D_2} P_2 = (P \circ_{D_2} P_2) \circ_{D_1} P_1$. If $P_1, \dots, P_s$ are distinct planar tangles and $D_{i_1}, \dots, D_{i_n}$ are distinct inner disks of P such that $\deg P_j = \deg D_{i_j}$ for all $1 \leq i \leq s$, we denote by $P \circ_{(D_{i_1},\dots,D_{i_s})} (P_1, \dots, P_s)$ the planar tangle obtained by iterating the composition with respect to the different inner disks.

7.1.2 Non-crossing partition and irreducible planar tangles

Let P be a planar tangle of degree k. $\Gamma P \setminus \partial D_0$ is the union of a finite number of connected components $C_1, \dots, C_r$. For $1 \leq i \leq 2k$, the distinguished point i_{D_0} of the outer boundary belongs to the closure of a unique connected component $C_{f(i)}$. We define an equivalence relation $\sim_P$ on $[\![1, 2k]\!]$ by saying that $i \sim_P j$ if and only if $f_i = f_j$. We denote by π_P the partition associated to $\sim_P$ through the correspondence between partitions and equivalence relations which has been established in Section 1.1.1.

Lemma 7.3. *π_P is a non-crossing partition of $2k$ with even blocks.*

Recall that a block is called even if its cardinal is even.

Proof. Suppose that $1 \leq i < j < k < l \leq 2k$ with $i \sim_P k$ and $j \sim_P l$. Thus there exist a path γ_1 in $\Gamma P \setminus D_0$ between i and k and a path γ_2 in $\Gamma P \setminus D_0$ between j and l. Since $j \in (i, k)$ and $l \in (k, i)$, γ_1 and γ_2 intersect. Therefore, the four points are in the same connected component of $\Gamma P \setminus D_0$ and $i \sim_P j \sim_P k \sim_P l$. π_P is thus non-crossing.
Since each inner disk has an even number of distinguished points and each curve connects two distinguished points, a counting argument yields the parity of the size of the blocks. $\square$

Reciprocally, a noncrossing partition π of $2k$ with even blocks yields an irreducible planar tangle P_π such that $\pi_{P_\pi} = \pi$. The construction is done recursively on the number of blocks as follows:

1. If π is the one block partition, P_π is the planar tangle with one outer disk D_0 of degree $2k$, one inner disk D_1 of degree $2k$, and a curve between the point i of D_0 and the point i of D_1.

2. Suppose that P_π is constructed for all partitions having less than r blocks. Let $B = [\![i_1, i_2]\!]$ be an interval block of π, and let π' be the non-crossing partition obtained by removing this block (and relabelling increasingly the integers). π' has also even blocks.
Let $P_{\pi,B}$ be the planar tangle consisting of an outer disk D_0 of degree $2k$ and two inner disks D_1 and D_2 of respective degree $i_2 - i_1 + 1$ and $2k - (i_2 - i_1 + 1)$, and curves connecting

- i_{D_0} to i_{D_2} for $i < i_1$.
- i_{D_0} to $(i - i_1 + 1)_{D_1}$ for $i_1 \le i \le i_2$ if i_1 is odd, and i_{D_0} to $(i - i_1)_{D_1}$ for $i_1 \le i \le i_2$ if i_1 is even.
- i_{D_0} to $(i - (i_2 - i_1 + 1))$ for $i > i_2$.

Set $P_\pi = P_{\pi,B} \circ_{D_2} P_{\pi'}$. Note that the resulting planar tangle doesn't depend on the choice of the interval block B.

By construction, P_π is irreducible and $\pi_{P_\pi} = \pi$. The inner disk of P_π corresponding to the block B of π is denoted by D_B.
These particular irreducible planar tangles yield a decomposition of connected planar tangles. Let P be a connected planar tangle. Let $B_1, \ldots, B_r$ be the blocks of π_P ordered lexicographically and let $C_1, \ldots, C_r$ be the corresponding connected components of P. For $1 \le i \le r$, P_i is defined as the planar tangle $P \setminus (\bigcup_{j \neq i} C_j)$, where the distinguished points of the outer boundary of P which are in B_i have been counterclockwise relabelled in such a way that the first odd point is labelled 1. The planar tangle P_1 of the connected planar tangle of Figure 7.2 is depicted in Figure 7.4.

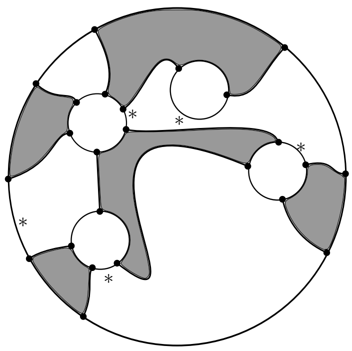


Figure 7.4: : First connected component of the first planar tangle of Figure 7.2.

Proposition 7.4. *Let P be a connected planar tangle, and set $\pi = \pi_P$. Then*

$$P = P_\pi \circ_{D_{B_1},\ldots,D_{B_r}} (P_1, \ldots, P_r).$$

Proof. It is possible to draw r disjoint Jordan curves $\{\gamma_i\}_{1 \le i \le r}$ such that γ_i intersects P $|B_i|$ times, once at each curve of C_i connected to a distinguished point of the outer boundary (or two times at a curve joining two distinguished points of the outer boundary). The intersection points are labelled counterclockwise around γ_i, in such a way that the intersection point with the curve coming from the first odd point of B_i is labelled 1.
Let Γ_i be the closed region delimited by γ_i and set $\tilde{P}_i = (P \cap \Gamma_i) \cup \gamma_i$, with the labelling of the distinguished points of γ_i given above. Then $\tilde{P}_i$ is a planar tangle which is an isotopy of P_i. Figure 7.5 shows a possible choice of Jordan curves for the connected planar tangle of Figure 7.2.

Let $\tilde{P}$ be the planar tangle whose inner disks are $\{\Gamma_i\}_{1 \le i \le r}$, with the distinguished points being the ones of γ_i, and whose skeleton is $\Gamma P \setminus (\bigcup \mathring{\Gamma}_i)$. Then $\tilde{P}$ is equivalent to P_π and

$$\tilde{P} \circ_{\Gamma_1,\ldots,\Gamma_r} (\tilde{P}_1, \ldots, \tilde{P}_r) = P.$$

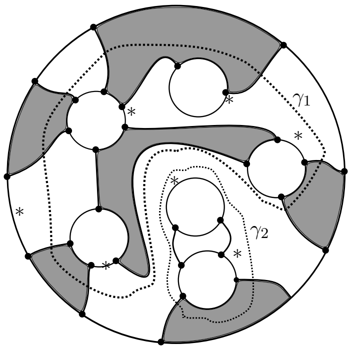


Figure 7.5: : Jordan curves surrounding the connected components of a planar tangle.

□

Shaded regions and Kreweras complement We refer to Section 5.2 for the notations on the Kreweras complement related to partial set partitions. If S is a set of cardinal k, $f : [\![1,k]\!] \longrightarrow$ is a bijective function and $\pi \in P(k)$, then $f(\pi)$ is the partition of S defined by $f(i) \sim_{f(\pi)} f(j)$ if and only if $i \sim_\pi j$. Let $k \geq 1$, and for $i \geq 1$, set $\delta(i) = 1$ if i is odd and 0 else. A partial partition $(\tilde{\pi}, S)$ of $4k$ is associated to each $\pi \in NC(2k)$ as follows:

- $S = \{2i - \delta(i)\}_{1 \leq i \leq 2k}$.
- $\tilde{\pi} = f(\pi)$ where $f : [\![1,2k]\!] \longrightarrow [\![1,4k]\!]$ given by $f(i) = 2i - \delta(i)$.

S is the set $\{1, 4, 5, 8, \ldots, 4k-3, 4k\}$. Let $\tilde{f}$ be the map from $[\![1,2k]\!]$ to $[\![1,4k]\!] \setminus S$ defined by $\tilde{f}(i) = 2i - (1 - \delta(i))$.

Definition 7.5. *Let $\pi \in NC(2k)$. The nested Kreweras complement of π, denoted by $kr'(\pi)$, is the partition of $2k$ such that $\tilde{f}(kr'(\pi)) = kr(\tilde{\pi}, S)$.*

The nested Kreweras complement of the partition $\{\{1,3,4\},\{2\},\{5,6\}\}$ is the partition $\{\{1,2\},\{3,4\},\{5,6\}\}$, as shown in Figure 7.6.

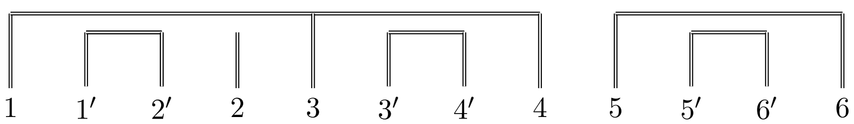


Figure 7.6: The partition $\{\{1,3,4\},\{2\},\{5,6\}\}$ and its nested Kreweras complement.

Contratry to the usual Kreweras complement, the nested Kreweras complement is not bijective. Let π_0 (resp π_1) be the partition of $2k$ with block $(2i, 2i+1)$ (resp. $(2i+1, 2i+2)$).

Lemma 7.6. $kr'(\pi) = kr'(\pi \vee \pi_0)$

Proof. Since $\pi \leq (\pi \vee \pi_0)$, $kr'(\pi \vee \pi_0) \leq kr'(\pi)$.
Suppose that $i \sim_{kr'(\pi)} j$. Then $2i - (1-\delta(i)) \sim_{kr(\tilde{\pi}, S^c)} 2j - (1-\delta(j))$. By Lemma 5.16, this implies that for all $k \in [\![2i-(1-\delta(i)), 2j-(1-\delta(j))]\!] \cap S$, $l \in S \setminus [\![2i-(1-\delta(i)), 2j-(1-\delta(j))]\!]$, $k \not\sim_\pi l$. If $k \in [\![2i-(1-\delta(i)), 2j-(1-\delta(j))]\!] \cap S$ and $l \notin [\![2i-(1-\delta(i)), 2j-(1-\delta(j))]\!]$,

then $l \neq k+1$ and thus $k \not\sim_{\pi_0} l$; therefore, for all $k \in [\![2i-(1-\delta(i)), 2j-(1-\delta(j))]\!] \cap S$, $l \in S \setminus [\![2i-(1-\delta(i)), 2j-(1-\delta(j))]\!]$, $k \not\sim_{\pi \vee \pi_0} l$. By Lemma 5.16, this implies that $i \sim_{kr'(\pi \vee \pi_0)} j$: $kr'(\pi) \leq kr'(\pi \vee \pi_0)$. □

The nested Kreweras complement is involved in the description of planar tangles in the following way:

Proposition 7.7. *Let P be a planar tangle of degree k. i and j are in the same block of $kr'(\pi_P)$ if and only if i_{D_0} and j_{D_0} are boundary points of the same shaded region.*

Proof. Let P be a planar tangle. Relabel i with $2-\delta(i)$: with this relabelling each interval of type $(4i+1, 4i+4)$ is the boundary interval of a shaded region, and $\pi(P)$ becomes a partial non-crossing partition of $4k$ with support S. Add two points $4i+2$ and $4i+3$ in $(4i+1, 4i+4)$ in such a way that $4i+2 \in (4i+1, 4i+3)$. Let $\sim$ be the relation on S^c defined by $i \sim j$ if and only if i and j are boundary points of a same shaded region. Let (π', S^c) be the partial partition associated to $\sim$: π' is non-crossing since two regions that intersect are the same.
Let $\pi = ((\pi_P, S) \vee (\pi', S^c))$. Let $1 \leq i < j < r < s \leq 4k$ with $i \sim_\pi r$ and $j \sim_\pi s$. Since π_P is non-crossing, if i, j, r, s are all in S then $i \sim_\pi j \sim_\pi r \sim_\pi s$. Assume from now on that they are not all in S, and suppose without loss of generality that $i \in S^c$. $i \sim_\pi r$, thus r is also in S^c and i and r are boundary points of a same shaded region σ. Since $j \in (i, r)$ and $s \in (r, i)$, any path on ΓP between j and s would cut σ in two distinct regions: thus if $j, s \in S$, then $j \not\sim_\pi s$. Therefore, the hypothesis $j \sim_\pi s$ yields that $j, s \in S^c$. π' being non-crossing, i, j, r, s are in the same block of π. Finally, $\pi_P \vee \pi'$ is non-crossing and thus $\pi' \leq kr'(\pi_P)$.
Let π_2 be a partial partition with support S^c, such that $\pi_P \vee \pi_2$ is non-crossing. Suppose that $i \sim_{\pi_2} j$, with $i, j \in S^c$. Let σ_i (resp. σ_j) be the shaded region having i (resp. j) as boundary point. $\pi_P \vee \pi_2$ is non-crossing, thus for all $r, s \in S$ such that $i \leq r \leq j$ and $j \leq s \leq i$, $r \not\sim_{\pi_P} s$. Thus, there is no path in ΓP between (i, j) and (j, i), and $\sigma_i = \sigma_j$: this yields $i \sim_{\pi'} j$. Therefore, $(\pi_2, S^c) \leq (\pi', S^c)$ and $(\pi', S^c) = kr(\pi_P, S)$.
Let $1 \leq i, j \leq 2k$. By the two previous paragraphs, i and j are in the same block of $kr'(\pi_P)$ and only if $2i-(1-\delta(i))$ and $2j-(1-\delta(j))$ are boundary points of a same shaded region. Since the points $2i-(1-\delta(i))$ and $2i-\delta(i)$ both belong to the interval $(2(i+\delta(i))-3, 2(i+\delta(i)))$ (which is part of the boundary of a shaded region), $2i-(1-\delta(i))$ and $2j-(1-\delta(j))$ are boundary points of a same shaded region if and only if $2i-\delta(i)$ and $2j-\delta(j)$ are boundary points of the same shaded region. Since we had relabelled i by $2-\delta(i)$, this yields the result. □

7.1.3 Planar algebra

Let us recall the definition of a planar algebra as given in Section 3.3.2. Planar algebras have been introduced by Jones in [46] in order to study the structure of subfactors.

Definition 7.8. *A planar algebra $\mathcal{P}$ is a collection of finite dimensional vector spaces $\{\mathcal{P}_n\}_{n \in \mathbb{N}^* \cup \{-,+\}}$ with $\dim \mathcal{P}_+ = \dim \mathcal{P}_- = 1$, such that each planar tangle P of order k with n inner disks of respective order $k_1, \ldots, k_n$ yields a multilinear map:*

$$T_P : \bigotimes_{1 \leq i \leq n} \mathcal{P}_{k_i} \longrightarrow \mathcal{P}_k$$

which is compatible with the composition of planar tangles and invariant under isotopy. Namely, if Q is another tangle of order k_{i_0} for some $1 \leq i_0 \leq n$ and φ is a diffeomorphism, then

$$T_P \circ (\bigotimes_{i \neq i_0} Id_{\mathcal{P}_{k_i}} \otimes T_Q) = T_{P \circ_{D_{i_0}} Q} \text{ and } T_{\varphi(P)} = T_P,$$

where $P \circ_{i_0} Q$ is the composition of tangles defined in Section 7.1.1 and $\varphi(P)$ is the isotopy of P with respect to φ.

In particular the compatibility with the composition of tangles yields that if P contains a close curve γ which delimits a simply connected region σ, $T_P = \delta_\sigma T_{\tilde{P}}$, where $\tilde{P}$ is the planar tangle P with the curve γ removed, and δ_σ is a scalar that depends only on the shading of σ. In the sequel δ_1 (resp. δ_2) will denote the value of δ_σ for a shaded (resp. unshaded) region.

Several operations are defined by the action of particular planar tangles :
Each $\mathcal{P}_n$ is an algebra with multiplication given by the planar tangle of Figure 7.7.

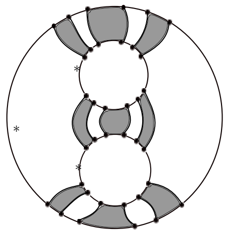


Figure 7.7: : Multiplication tangle of degree $\mathcal{P}_6$.

The is an inclusion of algebras $\mathcal{P}_n \subseteq \mathcal{P}_{n+1}$ with the planar tangle of Figure 7.8.

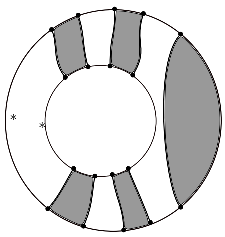


Figure 7.8: : Inclusion tangle from $\mathcal{P}_4$ to $\mathcal{P}_6$.

There exist two linear functionals Tr_R and Tr_L respectively defined by the first and second planar tangles of Figure 7.9.

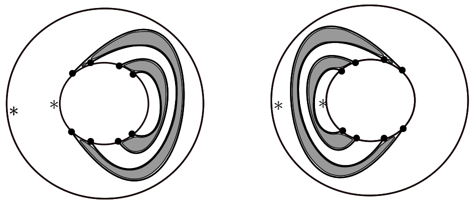


Figure 7.9: : Right trace tangle and Left trace tangle on $\mathcal{P}_4$.

Note that $\mathrm{Tr}_L(Id_{\mathcal{P}_1}) = \delta_2$ and $\mathrm{Tr}_R(Id_{\mathcal{P}_1}) = \delta_1$.

A planar subalgebra of $\mathcal{P}$ is a collection of vector subspaces $W_n \subseteq \mathcal{P}_n$ which is stable under the action of the planar tangles. If $\mathcal{P}$ is a planar algebra, there is a minimal planar subalgebra

of $\mathcal{P}$ denoted by $TL(\mathcal{P})$ and given by the image of all planar tangles without inner disk.
A morphism Φ between two planar algebras $\mathcal{P} : \{\mathcal{P}_n\}$ and $\mathcal{Q} = \{\mathcal{Q}_n\}$ is a collection of linear maps $\Phi_n : \mathcal{P}_n \longrightarrow \mathcal{Q}_n$ that commute with the action of the planar tangles. Namely $T_P \circ (\bigotimes_{D_i \text{ interior disk of } P} \Phi_i) = \Phi_k \circ T_P$ for any planar tangle of order k.

Subfactor planar algebra Two planar tangles P and Q of degree 0_+ or 0_- are related by a spherical symmetry if P can be obtained from Q by the composition of a Moebius transformation and a diffeomorphism. An example of two planar tangles related by a spherical symmetry is shown in Figure 7.10.

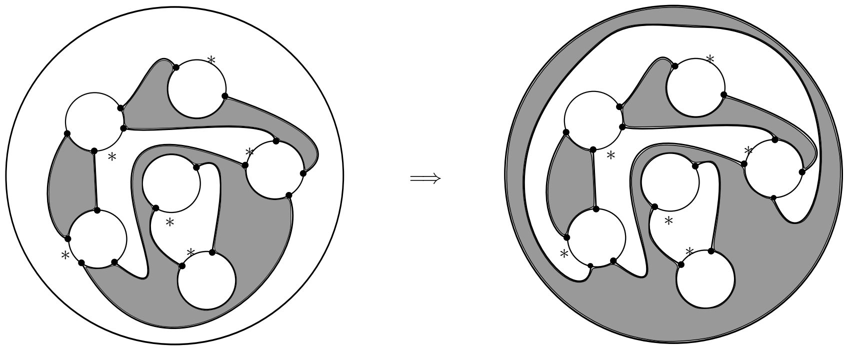


Figure 7.10: : Two planar tangles related by a spherical symmetry.

A planar algebra is called spherical if the action of a planar tangle P of degree 0_+ or 0_- is invariant under spherical symmetries. In such planar algebra, the linear functionals Tr_L and Tr_R are equal and simply denoted by Tr; in particular $\delta_1 = \delta_2$ and Tr is a trace on each algebra $\mathcal{P}_n$.

The conjugate P^* of a planar tangle P is defined as the image of P by any axial symmetry s, with the rule that the first distinguished point of a disk $s(D)$ is the last distinguished point of the inital disk D. The conjugate of the planar tangle of Figure 7.1 is drawn in Figure 7.11.

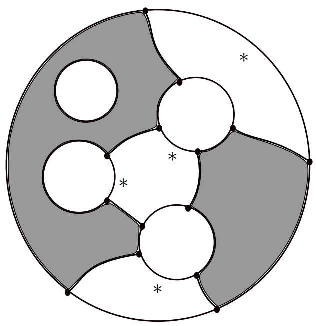


Figure 7.11: : Axial symmetry of the planar tangle of Figure 7.1.

Suppose that there exists an involutive antilinear map $*$ on each vector space $\mathcal{P}_n$. The planar algebra is called a $*-$planar algebra if $T_{P^*}(v_1^* \otimes \cdots \otimes v_n^*) = (T_P(v_1, \ldots, v_n))^*$ for any $v_1, \ldots, v_n$

with $v_i \in \mathcal{P}_{k_i}$.

Definition 7.9. *A planar algebra $\mathcal{P}$ is called a subfactor planar algebra if $\mathcal{P}$ is a spherical $*-$planar algebra such that the bilinear product $\langle .,. \rangle$ defined on each $\mathcal{P}_n$ by the formula $\langle x,y \rangle = \mathrm{Tr}(y^* x)$ is an hermitian form.*

Subfactor planar algebras have a very rich structure: a complete review of their properties can be found in [46]. The main result that will be needed in the present chapter is the following:

Theorem 7.10. *[Jones,[46]] Let $\mathcal{P}$ be a subfactor planar algebra. There exists a bipartite graph $G_{\mathcal{P}}$ with root vertex $*$ such that:*

$$\dim \mathcal{P}_k = \#\{ \text{ walk of length } 2k \text{ on } G_{\mathcal{P}} \text{ starting and ending at } *\}.$$

7.2 Intertwiner spaces of a free wreath product

A non-commutative permutation group f is called irreducible if $\dim(\mathrm{Mor}_F(0,1)) = 1$. This section is devoted to the description of the intertwiner spaces of the free wreath product of two irreducible non-commutative permutations groups F and G;

7.2.1 Intertwiner spaces of non-commutative permutation groups

Spin planar algebra As planar tangles are generalization of noncrossing partitions to higher dimensions, the spin planar algebra is a way to generalize the maps T_P's (as defined in Section 1.1.2) in order to build multilinear maps. This forms a planar algebra which has been introduced by Jones in [46]. Some cares are needed in order to define properly a planar algebra that possesses all the properties of a subfactor planar algebra.
Let V be a $d-$dimensional Hilbert space with a distinguished orthonormal basis $(e_i)_{1\leq i\leq d}$. We denote by V_n the vector space $V^{\otimes n}$ and we set $V_+ = \mathbb{C}, V_- = V$. For each $n \geq 1$, a basis of $V^{\otimes n}$ is given by $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}_{1\leq i_1,\dots,i_n\leq d}$. The action of planar tangle will be described on these bases. Namely let P be a planar tangle of degree k with r inner disks D_i, $1 \leq i \leq r$ of respective degree k_i. P defines a map T_P from $\bigotimes V_{k_i}$ to V_k whose expression on the respective bases is the following:

1. For each $1 \leq i \leq r$, let $e_{\vec{j}^i} = e_{j_1^i} \otimes \cdots \otimes e_{j_{k_i}^i}$ be a basis element of V_{k_i}.
2. Each inner disk D_i has k_i boundary components which are also the boundary of a shaded region (each interval $(2l-1, 2l)$ for $1 \leq l \leq k_i$): label the interval component between the points $2l-1$ and $2l$ of D_i with the value j_l^i.
3. A function f : {shaded regions of P} $\to [\![1;d]\!]$ is called compatible if the label of the boundary of an inner disk which is also the boundary of a shaded region σ is equal to $f(\sigma)$.
4. A compatible function f yields a labelling of the outer boundary by setting $j_l = f(\sigma)$ if the interval component $(2l-1, 2l)$ of D_0 is the boundary of the shaded region σ. The resulting vector $e_{j_1} \otimes \cdots \otimes e_{j_l}$ is denoted by e_f.
5. Set
$$T_P(e_{\vec{j}_1} \otimes \cdots \otimes e_{\vec{j}_r}) = \sum_{f \text{ compatible}} e_f.$$

Pictorially, this means that we impose the indices of the tensor products to be the same on the boundaries of a shaded regions. An example of such condition is given in Figure 7.12.

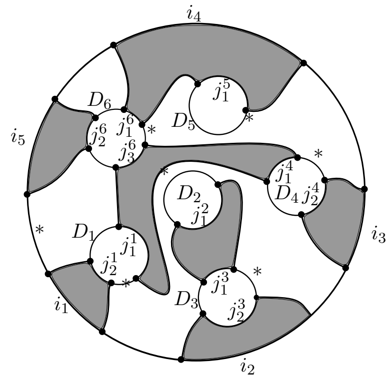


Figure 7.12: : Spin action a planar tangle.

The planar tangle P of Figure 7.12 yields the map $T_P : V^{\otimes 2} \otimes V \otimes V^{\otimes 2} \otimes V^{\otimes 2} \otimes V \otimes V^{\otimes 3} \longrightarrow V^{\otimes 5}$ defined by

$$\begin{aligned}
&T_P\left((e_{j_2^1} \otimes e_{j_2^1}) \otimes e_{j_1^2} \otimes (e_{j_1^3} \otimes e_{j_2^3}) \otimes (e_{j_1^4} \otimes e_{j_2^4}) \otimes e_{j_1^5} \otimes (e_{j_1^6} \otimes e_{j_2^6} \otimes e_{j_3^6})\right)\\
=&\delta_{j_1^1 j_1^4 j_3^6} \delta_{j_1^2 j_1^3} \delta_{j_1^5 j_1^6} e_{j_2^1} \otimes e_{j_2^3} \otimes e_{j_2^4} \otimes e_{j_1^5} \otimes e_{j_2^6}.
\end{aligned}$$

This action of planar tangles is clearly invariant under isotopy and is compatible with the composition of tangles. The spin planar algebra is not spherical since $\delta_1 = d$ and $\delta_2 = 1$. However it is possible to multiply each T_P by a scalar $\mu(P)$ in such a way that the resulting action still yields a planar algebra structure with $\delta_1 = \delta_2 = \sqrt{d}$.

Definition 7.11 ([46]). *The collection of vector spaces given by $V_n = V^{\otimes n}, V_- = V, V_+ = \mathbb{C}$, with the action of a planar tangle P given by $\mu(P)T_P$ as defined above, is called the spin planar algebra and denoted by $\mathcal{P}(V)$.*

Note that for $d > 1$, the spin planar algebra is not a planar algebra since $\dim V_- = d > 1$. However the following result of [46] holds:

Lemma 7.12 (Jones, [46]). *A planar algebra contained in the spin planar algebra is a spherical planar algebra.*

A planar algebra contained in spin planar algebra is called a spin planar subalgebra.
In particular the spin planar subalgebra $TL(V) \subseteq \mathcal{P}(V)$ which is given by the image of all planar tangles without inner disk is a spherical planar algebra. Constructing T_P for each of these planar tangles gives that

$$TL(V)_n = span(T_p, p \in NC(n)).$$

This planar algebra of noncrossing partitions has been introduced and studied by Sunder and Kodiyalam in [49]. It also yields the first connection between spin planar algebras and intertwiner spaces of non-commutative permutation groups, since by a result of [5]

$$TL(V) = \{\mathrm{Mor}_{S_n^+}(0, n)\}_{n \geq 1}.$$

Let $n \geq 1$. As explained in Definition 3.30, a non-commutative permutation group of size n is a quantum subgroup of the free symmetric group S_n^+. Therefore by the Tannaka-Krein duality of 3.15,

$$\text{For any } k, k' \geq 1,\ \mathrm{Mor}_{S_n^+}(k, k') \subseteq \mathrm{Mor}_F(k, k'). \tag{7.2.1}$$

We have seen in last section that the intertwiner spaces of S_n^+ form a particular planar subalgebra of the spin planar algebra. The relation (7.2.1) extends to the following result due to Banica in [6]:

Proposition 7.13 ([6],Sec.5). *Let F be a quantum permutation group of size n. If F is irreducible, then $\{\mathrm{Mor}_F(k, 0)\}_{k\geq 1}$ is a sub-planar algebra of the spin planar algebra $\mathcal{P}(V)$, V being the vector space of the fundamental representation of F.*
Moreover any sub-planar algebra of $\mathcal{P}(V)$ is of the form $\{\mathrm{Mor}_F(k, 0)\}_{k\geq 1}$ for an irreducible quantum permutation group F of size n, and the correspondence is bijective.

We denote by $\mathcal{P}(F)$ the planar algebra associated to the quantum permutation F by the previous Proposition. Recall that given a compact quantum subgroup G of O_n^+, the knowledge of $\{\mathrm{Mor}_G(k, 0)\}_{k\geq 1}$ is enough to get all the intertwiner spaces $\mathrm{Mor}_G(k, l)$ for $1 \leq k, l$, because of the isomorphism $\mathrm{Mor}_G(k, l) \simeq \mathrm{Mor}_G(k + l, 0)$. Therefore, $\mathcal{P}(F)$ describes all the representations of F.

7.2.2 Case of a free wreath product

If G and F are two irreducible quantum permutation groups of respective size d and n, then the free wreath product $G \wr_* F$ is again a quantum permutation group by the inclusion $G \wr_* F \subseteq S_{dn}^+$. The problem is to relate the intertwiner spaces of $G \wr_* F$ with the ones of G and F. As it will be proven in this subsection, $G \wr_* F$ is again irreducible; therefore by Proposition 7.13, the problem is equivalent to relating $\mathcal{P}(G \wr_* F)$ to $\mathcal{P}(G)$ and $\mathcal{P}(F)$. This relation involves the free product of planar algebras, a construction which has been done by Bisch and Jones in [23].

Tensor and free products of planar algebras

Definition 7.14. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar algebras. The tensor product planar algebra $\mathcal{P} \otimes \mathcal{Q}$ is the collection of vector spaces $(\mathcal{P} \otimes \mathcal{Q})_i = \mathcal{P}_i \otimes \mathcal{Q}_i$, with the action of any planar tangle given by the tensor product of the action on each component: Namely for a planar tangle P,*

$$T_P(\bigotimes_{D_i}(p_i \otimes q_i)) = T_P(\bigotimes_{D_i} p_i) \otimes T_P(\bigotimes_{D_i} q_i),$$

where $p_i \in \mathcal{P}_{k_i}$ and $q_i \in \mathcal{Q}_{k_i}$.

If $\mathcal{P}(V)$ and $\mathcal{P}(W)$ are two spin planar algebras respectively associated to the vector spaces V and W, then $\mathcal{P}(V) \otimes \mathcal{P}(W)$ is isomorphic to the spin planar algebra $\mathcal{P}(V \otimes W)$, with the isomorphism $V^{\otimes k} \otimes W^{\otimes k} \simeq (V \otimes W)^{\otimes k}$. The free product of two planar algebras $\mathcal{P}$ and $\mathcal{Q}$ is a subplanar algebra of $\mathcal{P} \otimes \mathcal{Q}$ defined by the image of certains planar tangles.
A pair (P, Q) of planar tangles of degree k is called free if there exists a planar tangle R of degree $2k$ and two isotopies φ_1 and φ_2, respectively of P and Q, such that:

- $\Gamma R = \Gamma\varphi_1(P) \cup \Gamma\varphi_2(Q)$, and the set of distinguished points of R is the image through φ_1 and φ_2 of the set of distinguished points of P and Q.
- $\varphi_1(P \setminus D_0(P)) \cap \varphi_2(Q \setminus D_0(Q)) = \emptyset$. This means that a connected component of R is the image of a connected component of either P or Q.

- The distinguished point numbered i of $\partial D_0(P)$ is sent by φ_1 to the distinguished point numbered $2i - \delta(i)$ of $\partial D_0(R)$.
- The distinguished point numbered i of $\partial D_0(Q)$ is sent by φ_2 to the distinguished point numbered $2i - (1 - \delta(i))$.
- A distinduished point of an inner disk coming from P is labelled as in P; a distinguished point i of an inner disk coming from Q is labelled $i - 1$.

The last condition ensure that curves of R have endpoints with correct parities. If P and Q are connected planar tangles and R exists, then R is unique up to orientation preserving diffeomorphism: this planar tangle is called the free composition of P and Q and denoted by $P * Q$. An example of a free pair of planar tangles, with the resulting free composition, is drawn in Figure 7.13.

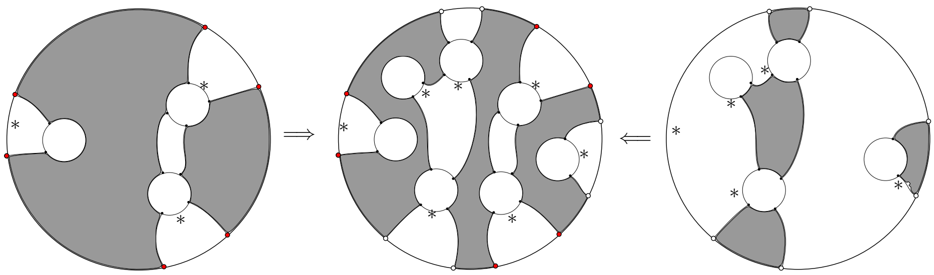


Figure 7.13: : Free composition of two planar tangles.

Definition 7.15. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar algebras. The free product planar algebra $\mathcal{P} * \mathcal{Q}$ is the collection of vector subspaces $(\mathcal{P} * \mathcal{Q})_k$ of $(\mathcal{P} \otimes \mathcal{Q})_k$ spanned by the image of the maps $T_P \otimes T_Q$ for all free pairs of planar tangles of degree k.*

$\{\mathcal{P} * \mathcal{Q})_k\}_{k \geq 1}$ is stable under the action of planar tangles:

Lemma 7.16. *$\mathcal{P} * \mathcal{Q}$ is a planar sub-algebra of $\mathcal{P} \otimes \mathcal{Q}$.*

Proof. It suffices to check the stability on the generating sets of the vector spaces $(\mathcal{P} * \mathcal{Q})_k$ given in Definition 7.15. Let P be a planar tangle, and for each inner disk D_i of P, let v_i be an element of $(\mathcal{P} * \mathcal{Q})_{k_i}$ of the form $T_{P_i} \otimes T_{Q_i}((\bigotimes_{D_j(P_i)} v^i_j) \otimes (\bigotimes_{D_j(Q_i)} w^i_j)$ (with (P_i, Q_i) free pair). By composition of actions of planar tangles,

$$\begin{aligned} T_P(\bigotimes_{D_i(P)} v_i) =& T_{P \circ_{(D_1, \dots D_n)}}(P_1, \dots, P_n)(\bigotimes_{D_i(P)} \bigotimes_{D_j(P_i)} v^i_j) \\ & \otimes T_{P \circ_{(D_1, \dots D_n)}}(Q_1, \dots, Q_n)(\bigotimes_{D_i(P)} \bigotimes_{D_j(Q_i)} w^i_j). \end{aligned}$$

Thus, it is enough to prove that $S = P \circ_{(D_1, \dots D_n)} (P_1, \dots, P_n)$ and $T = P \circ_{(D_1, \dots D_n)} (Q_1, \dots, Q_n)$ form a free pair. Let $\tilde{P}$ be the planar tangle of order $2k$ obtained from P by doubling all the curves of P and all the distinguished points (in such a way that the tangle still remains planar).

By this construction, a curve joining the point j of D_i to the point j' of $D_{i'}$ in P yields in $\tilde{P}$ two curves: one joining the point $2j-1$ of D_i to the point $2j'-1$ of $D_{i'}$ and the other one joining the point $2j$ of D_i to the point $2j'$ of $D_{i'}$. Since P is a planar tangle, the conditions on the parities of j and j' yield that in $\tilde{P}$, the curves join points labelled 0 or 1 modulo 4 (resp. 2 or 3 mod 4) to points labelled 0 or 1 modulo 4 (resp. 2 or 3 mod 4). Reciprocally, by removing from $\tilde{P}$ all the distinguished points labelled 0 and 1 modulo 4 and the curves joining them, we recover the planar tangle P. The same holds for the distinguished points labelled 2 and 3. Therefore, if we compose the tangle $P_i * Q_i$ inside each disk D_i, the resulting tangle is exactly $S * T$ (up to a relabelling). Thus $\tilde{P} \circ_{D_1,\dots,D_n} (P_1 * Q_1, \dots, P_n * Q_n) = S * T$ and $\mathcal{P} * \mathcal{Q}$ is stable by the action of planar tangles.

□

There is also a combinatorial way to characterize free pairs of planar tangles.

Lemma 7.17. *If (P, Q) is a free pair, then $\pi(P * Q) = (\pi_P, S) \vee (\pi_Q, S^c)$.*
In particular (P, Q) is a free pair if and only if (P_{π_P}, P_{π_Q}) is a free pair.

Proof. The first statement of the lemma is a direct consequence of the definition of S and the fact that $i \sim_P j$ (resp. $i \sim_Q j$ if and only if $2i - \delta(i) \sim_{P*Q} 2j - \delta(j)$ (resp. $2i - (1 - \delta(i)) \sim_{P*Q} 2j - (1 - \delta(j))$.
Thus, if (P, Q) is free pair, then $P_{\pi(P*Q)}$ is exactly the free composition of P_{π_P} with P_{π_Q} and (P_{π_P}, P_{π_Q}) is also a free pair.
Suppose that (P_{π_P}, P_{π_Q}) is a free pair. There exist $P_1, \dots, P_r, Q_1, \dots, Q_{r'}$ such that $P = P_{\pi_P} \circ_{D_1,\dots,D_R} (P_1, \dots, P_r)$ and $Q = P_{\pi_Q} \circ_{D'_1,\dots,D'_{r'}} (Q_1, \dots, Q_{r'})$. Therefore, $(P_{\pi_P} * P_{\pi_Q}) \circ_{D_1,\dots,D_r,D'_1,\dots,D'_{r'}} (P_1, \dots, P_r, Q_1, \dots, Q_{r'})$ gives the free composition of P and Q. □

Proposition 7.18. *Let P and Q be two connected planar tangles. Then (P, Q) is a free pair if and only if $\pi_Q \leq kr'(\pi_P)$. In particular if P, P' and Q, Q' satisfy $\pi_P = \pi_{P'}$ and $\pi_Q = \pi_{Q'}$, then (P, Q) is a free pair if and only if (P', Q') is a free pair.*

Proof. If (P, Q) is a free pair, then by the previous Lemma $(\pi_P, S) \vee (\pi_Q, S^c)$ is non-crossing. Therefore $\pi_Q \leq kr'(\pi_P)$.
If $\pi_Q \leq kr'(\pi_P)$, $\pi_0 = (\pi_P, S) \vee (\pi_Q, S^c)$ is noncrossing with even blocks. Therefore, P_{π_0} is a well-defined planar tangle. Let $\{C_i\}$ be the connected components of π_0 coming from blocks of (π_P, S) and $\{D_i\}$ the ones coming from blocks of (π_Q, S^c). Then $P_{\pi_0} \setminus \bigcup C_i$ is an irreducible planar tangle and $\pi(P_{\pi_0} \setminus \bigcup C_i) = \pi_Q$. Thus $P_{\pi_0} \setminus \bigcup C_i = P_{\pi_Q}$ up to a relabelling of the distinguished points, and similarly $P_{\pi_0} \setminus \bigcup D_i = P_{\pi_P}$ up to a relabelling. Therefore, P_{π_0} is, up to a relabelling, the free composition of P_{π_P} with P_{π_Q}. Thus, (P_{π_P}, P_{π_Q}) is a free pair and by the previous lemma, (P, Q) is also a free pair. □

Reduced free pair

Definition 7.19. *A free pair (P, Q) of planar tangles is called reduced if P and Q are irreducible, and any region of $P * Q$ is bounded by at most one connected component from P and one from Q.*

By a connected component from P (resp. Q) in $P * Q$, we mean the image by φ_1 (resp. φ_2) of a connected component of $\Gamma P \setminus \partial D_0(P)$ (resp. $\Gamma Q \setminus \partial D_0(Q)$), where φ_1 and φ_2 are the maps involved in the construction of $P * Q$. An example of reduced free pair is given in Figure 7.14.

In the sequel, a distinguished point is called an outer boundary point of a region σ if it is both a point of the boundary of σ and a distinguished point of the outer disk.

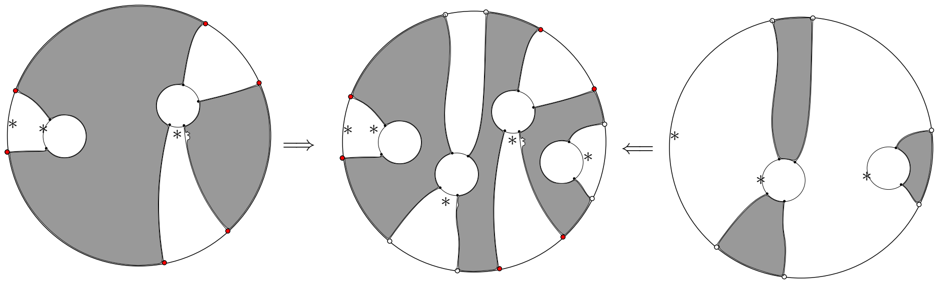


Figure 7.14: : Example of reduced free pair.

Lemma 7.20. *A free pair (P,Q) of degree k is reduced if and only if P and Q are irreducible and*

- *For $0 \leq i \leq k-1$, $\{(4i+2),(4i+3)\}$ is the set of outer boundary points of a region of $P*Q$. The same holds true for $\{4i,(4i+1)\}$.*
- *The set of outer boundary points of any other region has the form $\{4i-1,4i,4j+1,4j+2\}$ for $1 \leq i,j \leq k$.*

Proof. Suppose that (P,Q) is reduced. Let $0 \leq i \leq k-1$. $4i+2$ and $4i+3$ are two consecutive points of the outer boundary, thus they are boundary points of a same unshaded region of $P*Q$. They are both coming from Q and the region having $(4i+2,4i+3)$ as boundary component has at most one boundary component coming from Q, thus $(4i+2)$ and $(4i+3)$ are in the same connected component. Since Q is irreducible, they are both connected to the same inner disk: therefore, $\{(4i+2),(4i+3)\}$ is the set of outer boundary points of a region of $P*Q$. By symmetry between P and Q, the same holds true for $4i$ and $(4i+1)$.
Let σ be a region which has no outer boundary interval of the form $(4i,4i+1)$ or $(4i+2,4i+3)$. Since $P*Q$ is irreducible, σ has an outer boundary interval, which is by hypothesis of the form $(4i-1,4i)$ or $(4i+1,4i+2)$ (in particular σ is shaded). Let us assume without loss of generality that this boundary interval is of the form $(4i-1,4i)$, and order counterclockwise the distinguished points on the boundary of σ. Since $4i-1$ and $4i$ are not coming from the same tangle, they are connected to different inner disks. Moreover, the next outer boundary point of σ after $4i$ is in the same connected component as $4i$, and thus it is of the form $4j+1$ or $4j$. But consecutive boundary points of a connected component have to be of opposite parity (for example as a consequence to the fact that π_P has even blocks); thus the next outer boundary point of sigma is of the form $4j+1$ for some $0 \leq j \leq k$. Since σ is not bounded by other connected component coming from P, the only outer boundary points of σ coming from P are $4i$ and $4j+1$. Since σ is shaded and $(4j+1)$ is a boundary point of σ, $(4j+1,4j+2)$ is a boundary interval of σ and thus $4j+2$ is also a boundary point of σ. Let x be the next outer boundary point of σ after $4j+2$; by the same arguments, $x=4i'-1$ for some $1 \leq i' \leq k$. This implies also that $4i'$ is an outer boundary point of σ. Since $4i'$ comes from P, $4i'=4i$ and $x=4i-1$. The set of outer boundary points of σ is thus exactly $\{4i-1,4i,4j+1,4j+2\}$.
Reciprocally, suppose that $P*Q$ satisfies the two conditions of the lemma. By the first condition, any unshaded region is bounded by only one connected component. If σ is unshaded, then $\{4i-1,4i,4j+1,4j+2\}$ is the set of outer boundary points of σ: thus, σ has exactly one connected component coming from each tangle and $(P*Q)$ is reduced. ☐

Recall that π_0 (resp. π_1) is the pair partition of $2k$ with blocks $\{(2i,2i+1)\}$ (resp. $\{2i+1,,2I)\}$). The latter Lemma can be rephrased as follows:

Lemma 7.21. *A free pair (P, Q) is reduced if and only if P and Q are irreducible and $\pi_P \geq \pi_0$, $\pi_Q = kr'(\pi_P)$.*

Proof. Let P, Q be two irreducible tangles. It suffices to show that the two conditions of Lemma 7.20 is equivalent to the two conditions $\pi_P \geq \pi_0$ and $\pi_Q = kr'(\pi_P)$. The first condition is equivalent $\pi_P \geq \pi_0$, $\pi_Q \geq \pi_1$. The second condition of Lemma 7.20 is equivalent to the condition : if $4i \sim_{P*Q} 4j+1$ and for all $4i < k < 4j+1, 4j+1 < l < 4i$, $k \not\sim_{P*Q} l$, then $4j+2 \sim_{P*Q} 4i-1$. By Lemma 5.16 this is equivalent to $(\pi_Q, S^c) = kr(\pi_P, S)$ (which implies also that $\pi_Q \geq \pi_1$). □

Despite this rigid structure, only considering reduced free pairs is nonetheless enough to describe free product of planar algebras.

Proposition 7.22. *The free planar algebra $\mathcal{P} \otimes \mathcal{Q}$ is spanned by the images of $T_P \otimes T_Q$ for all reduced free pairs (P, Q).*

Proof. It suffices to prove that the image of $T_P \otimes T_Q$ with (P, Q) a free pair is contained in the image of (P', Q') with (P', Q') a reduced free pair.
The image of $T_P \otimes T_Q$ is contained in the image of $T_{P_{\pi_P}} \otimes T_{P_{\pi_Q}}$ by Proposition 7.4 and by Lemma 7.17 (P_{π_P}, P_{π_Q}) is again a free pair. We can thus assume that $P = P_\pi$ and $Q = P_{\pi'}$, with the condition $\pi' \leq kr'(\pi)$.
Suppose that $\mu \leq \nu$ are two noncrossing partitions of k. Let $B_1, \ldots, B_r$ be the blocks of ν in the lexicographical order. Since $\mu \leq \nu$, $\mu = \bigvee(\mu_{|B_i}, B_i)$. Therefore,

$$P_\mu = P_\nu \circ_{D_1,\ldots,D_r} (P_{\mu_{|B_1}}, \ldots, P_{\mu_{|B_r}}),$$

and the image of T_{P_μ} is contained in the one of T_{P_ν}.
Since $kr'(\pi) = kr'(\pi_0 \vee \pi)$, $\pi' \leq kr'(\pi_0 \vee \pi)$. $\pi \leq \pi \vee \pi_0$ and $\pi' \leq kr'(\pi \vee \pi_0)$, thus the image of T_{P_π} is included in the image of $T_{P_{\pi\vee\pi_0}}$ and the image of $T_{P_{\pi'}}$ is included in the image of $T_{P_{kr'(\pi\vee\pi_0)}}$. By Lemma 7.21, $(P_{\pi\vee\pi_0}, P_{kr'(\pi_0\vee\pi)})$ is reduced. □

Generating subset of a free product of planar algebras Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar algebras. We denote by $S_{\mathcal{P}}(k)$ (resp. $U_{\mathcal{P}}(k)$) the image in $\mathcal{P}_k$ of S_k (resp. U_k), the tangle without inner disk where $2i-1$ is linked to $2i$ (resp. $2i$ is linked to $2i+1$) for all $1 \leq i \leq k$. The picture of both tangles is drawn in Figure 7.15 for $k = 4$.

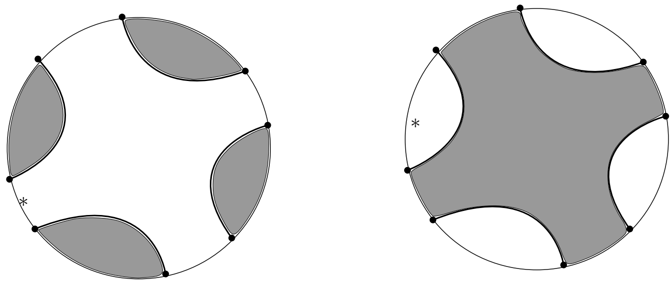


Figure 7.15: : Tangles S_4 and U_4.

Proposition 7.23. *Let $\mathcal{P}$ and $\mathcal{Q}$ be two planar algebras. Then $\mathcal{P} * \mathcal{Q}$ is the subplanar algebra of $\mathcal{P} \otimes \mathcal{Q}$ generated by $\{U_{\mathcal{P}}(k) \otimes \mathcal{Q}_k\}_{k\geq 1}$ and $\{\mathcal{P}_k \otimes S_{\mathcal{Q}}(k)\}_{k\geq 1}$.*

Proof. (U_k, Id_k) and (Id_k, S_k) are two free pairs of planar tangles, thus for all $k \geq 1$, $U_{\mathcal{P}}(k) \otimes \mathcal{Q}_k$ and $\mathcal{P}_k \otimes S_{\mathcal{Q}}(k)$ are subspaces of $\mathcal{P} * \mathcal{Q}$. In particular the subplanar algebra they generate is also a subplanar algebra of $\mathcal{P} * \mathcal{Q}$.
Reciprocally let (P, Q) be a reduced free pair of degree k. Let us show that there exists a planar tangle R of degree k with r inner disks D_i of respective degree k_i, such that $P = R_{D_1,\ldots,D_i}(X_1, \ldots, X_r)$ and $Q = R_{D_1,\ldots,D_i}(\tilde{X}_1, \ldots, X_r)$, where for each $1 \leq i \leq r$, $(X_i, \tilde{X}_i)$ is either (U_{k_i}, Id_{k_i}) or (Id_{k_i}, S_{k_i}).
Since (P, Q) is a free pair, the free composition $P * Q$ exists; since this pair is reduced, each inner disk of $P * Q$ is only connected to the outer boundary, and, by Lemma 7.20, both elements of $\{4i, 4i+1\}$ (resp. $\{4i-2, 4i-1\}$) are connected to a same inner disk. Color an inner disk D_i of $P * Q$ with 1 if it comes from P and with 2 if it comes from Q. We denote by γ_i the curve arriving on the distinguished point i of the outer boundary and by $\bar{i}$ the distinguished point of an inner disk which is connected to i.
We operate the following operation on $P * Q$: for each interval $(4i-1, 4i)$, let σ be the region for which $(4i-1, 4i)$ is a boundary interval; add a curve $\tilde{\gamma}$ in this region connecting $\overline{4i-1}$ to $\overline{4i}$ and erase $\gamma_{4i-1}, \gamma_{4i}$ and the distinguished points $4i-1$ and $4i$.
The degrees of the inner disks don't change and this yields a planar tangle R with $2k$ boundary points and r inner disks (where r is the sum of the number of inner disks in P and in Q). In the resulting planar tangle R, an odd point i of the outer boundary is still connected to the point $\bar{i}$ on a disk colored 1, and an even point i of the outer boundary is still connected to the point $\bar{i}$ on a disk colored 2. The construction of the tangle R is shown in Figure 7.16.

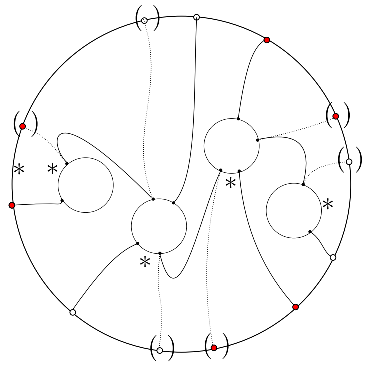


Figure 7.16: : Construction of the planar tangle R for the reduced free pair of Figure 7.14.

Set $X_i = U_{k_i}, \tilde{X}_i = Id_{k_i}$ if D_i is colored 2, and $X_i = Id_{k_i}, \tilde{X}_i = S_{k_i}$ if D_i is colored 1. Consider $R_1 = R_{D_1,\ldots,D_r}(X_1, \ldots, X_r)$. Each disk of R colored 2 is replaced by a planar tangle without inner disk, and thus disappears in R_1. A disk of R colored 1 is composed with the identity, and thus remains the same in R_1. An odd point $4i+1$ is already connected to $\overline{4i+1}$. An even point $4i+2$ is connected to an odd point $\overline{4i+2}$ of a disk D colored 2. Therefore since D is composed with U_{k_i}, $\overline{4i+2}$ is connected in R_1 by a curve to the following point of D in the counterclockwise order: since $4i+2$ and $4i+3$ are in the same connected component, the following point is exactly $\overline{4i+3}$. By the modification we made on $P * Q$, $\overline{4i+3}$ is connected by a curve to the point $\overline{4i+4}$. Therefore in R_1, $4i+2$ is connected to $\overline{4i}$. Thus relabelling the distinguished point $4i+2$ by $4i+4$ yields exactly the image of P in $P * Q$. This reconstruction of P is shown in Figure 7.17.

Likewise $R_2 = R_{D_1,\ldots,D_r}(\tilde{X}_1, \ldots, \tilde{X}_r)$ is equal to the image of Q in $P * Q$. $\square$

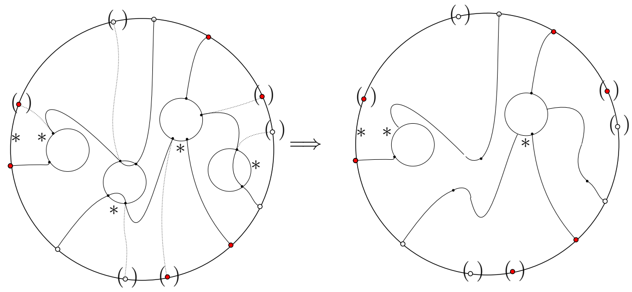


Figure 7.17: : Reconstruction of P from R.

Remark 7.24. *In [54], the free product of two planar algebras $\mathcal{P} * \mathcal{Q}$ is directly defined as the subplanar algebra of $\mathcal{P} \otimes \mathcal{Q}$ generated by $\{U_{\mathcal{P}}(k) \otimes \mathcal{Q}_k\}_{k\geq 1}$ and $\{\mathcal{P}_k \otimes S_{\mathcal{Q}}(k)\}_{k\geq 1}$.*

Free product of spin planar algebras

Let us now consider the free product of two spin planar algebras $\mathcal{P}(V)$ and $\mathcal{P}(W)$; note that $\mathcal{P}(V) * \mathcal{P}(W) \subseteq \mathcal{P}(V) \otimes \mathcal{P}(W) \simeq \mathcal{P}(V \otimes W)$. $\Phi = \{t_k\}$ denotes the isomorphism of planar algebras $\mathcal{P}(V) \otimes \mathcal{P}(W) \longrightarrow \mathcal{P}(V \otimes W)$ given by

$$t_k((e_{i_1} \otimes \cdots \otimes e_{i_k}) \otimes (f_{j_1} \otimes \cdots \otimes f_{j_k})) = (e_{i_1} \otimes f_{j_1}) \otimes \cdots \otimes (e_{i_k} \otimes f_{j_k})),$$

where $(e_i)_{1\leq i\leq n}$ is an orthonormal basis of V, $(f_j)_{1\leq j\leq m}$ is an orthonormal basis of W and $1 \leq i_1, \dots, i_k \leq n, 1 \leq j_1, \dots, j_k \leq m$.

We will give a more precise description of a free product of spin planar algebras.

Let S_0 be the set $\{2i+1\}_{0\leq i\leq k-1}$ and let $f : [\![1,k]\!] \longrightarrow S_0$ be the function $f(i) = 2i+1$. Let $\tilde{f} : [\![1,k]\!] \longrightarrow S_0^c$ be the function $\tilde{f}(i) = 2i$. For π a noncrossing partition of k, the Kreweras complement $Kr(\pi)$ of π (see [66]) is the non-crossing partition of $NC(k)$ such that $(\tilde{f}(Kr(\pi)), S_0^c) = kr(f(\pi), S_0)$. The Kreweras complement of the non-crossing partition $\{\{1,3,4\},\{2\},\{5,6\}\}$ is the partition $\{\{1,2\},\{3\},\{4,6\},\{5\}\}$, as shown in Figure 7.18.

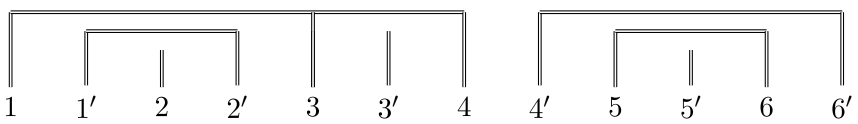


Figure 7.18: The partition $\{\{1,3,4\},\{2\},\{5,6\}\}$ and its Kreweras complement.

Let V be an $n-$dimensional vector space with basis $(e_i)_{1\leq i\leq n}$.

Definition 7.25. *Let π be a non-crossing partition of k. For each block B of π, let $\lambda^B \in V^{\otimes|B|}$. The composition of π with $\{\lambda^B\}_{B\in\pi}$ is the vector $T(\pi, \{\lambda^B\}_{B\in\pi})$ of $V^{\otimes k}$ defined by*

$$T(\pi, \{\lambda^B\}_{B\in\pi}) = \sum_{1\leq i(1)\dots,i(k)\leq n} (\prod_{\substack{B\in\pi \\ B=\{r_1<\cdots<r_{|B|}\}}} \lambda^B_{i(r_1),\dots,i(r_{|B|})}) e_{i(1)} \otimes \cdots \otimes e_{i(k)}.$$

The dual composition of π with $\{\lambda^B\}_{B\in\pi}$ is the vector $\tilde{T}(\pi, \{\lambda^B\}_{B\in\pi})$ of $V^{\otimes k}$ defined by

$$\tilde{T}(\pi, \{\lambda^B\}_{B\in\pi}) = \sum_{\substack{1\leq i(1),\dots,i(k)\leq n \\ \ker(i)\leq Kr'(\pi)}} (\prod_{\substack{B\in\pi \\ B=\{r_1<\cdots<r_{|B|}\}}} \lambda^B_{i(r_1),\dots,i(r_{|B|})}) e_{i(1)} \otimes \cdots \otimes e_{i(k)},$$

where $\ker(\vec{i})$ is as defined in 1.1.3.

Example 7.26. *Let $\pi = \{\{1,4,7\},\{2,3\},\{5,6\}\}$. Thus $Kr(\pi) = \{\{1,3\},\{2\},\{4,6\},\{5\},\{7\}\}$. In this case,*

$$T(\pi, \{\lambda^{1,4,7}, \lambda^{2,3}, \lambda^{5,6}\}) = \sum \lambda^{1,4,7}_{i_1,i_2,i_3} \lambda^{2,3}_{j_1,j_2} \lambda^{5,6}_{k_1,k_2} e_{i_1} \otimes e_{j_1} \otimes e_{j_2} \otimes e_{i_2} \otimes e_{k_1} \otimes e_{k_2} \otimes e_{i_3}.$$

and

$$\tilde{T}(\pi, \{\lambda^{1,4,7}, \lambda^{2,3}, \lambda^{5,6}\}) = \sum \lambda^{1,4,7}_{i_1,i_3,i_5} \lambda^{2,3}_{i_2,i_1} \lambda^{5,6}_{i_4,i_3} e_{i_1} \otimes e_{i_2} \otimes e_{i_1} \otimes e_{i_3} \otimes e_{i_4} \otimes e_{i_3} \otimes e_{i_5}.$$

We denote by T_π the linear map from $\bigotimes_{B\in\pi} V^{\otimes |B|}$ to $V^{\otimes k}$ defined by $T_\pi(\bigotimes_{B\in\pi} \lambda_B) = T(\pi, \{\lambda_B\}_{B\in\pi})$ and by $\tilde{T}_\pi$ the linear map from $\bigotimes_{B\in\pi} V^{\otimes|B|}$ to $V^{\otimes k}$ defined by $\tilde{T}_\pi(\bigotimes_{B\in\pi} \lambda_B) = \tilde{T}(\pi, \{\lambda_B\}_{B\in\pi})$. Compositions and dual compositions appear in the action of planar tangles on spin planar algebras. A partition π of $2k$ such that $\pi \geq \pi_1$ yields a partition of k denoted by $\pi/2$ and defined by $i \sim_{\pi/2} j$ if and only of $2i \sim_\pi 2j$. A partition π of $2k$ such that $\pi \geq \pi_0$ yields a partition of k also denoted by $\pi/2$ and defined by $i \sim_{\pi/2} j$ if and only of $2i-1 \sim_\pi 2j-1$. If $\pi \geq \pi_1$ (resp. $\pi \geq \pi_0$) and B is a block of $\pi/2$, $\tilde{B}$ denotes the block of π containing $2i-1$ and $2i$ (resp. $2i-2$ and $2i-1$) for each $i \in B$.

Lemma 7.27. *If $\pi \geq \pi_0$, $Kr(\pi/2) = kr'(\pi)/2$.*

Proof. Suppose that $i \sim_{Kr(\pi/2)} j$. For all $i < k \leq j, j < l \leq i$, $k \not\sim_{\pi/2} l$, thus for all $i < k \leq j, j < l \leq i$, $2k-1 \not\sim_\pi 2l-1$. Since $\pi \geq \pi_0$, for all $2i-1 < k \leq 2j-1, 2j-1 < l \leq 2i-1$, $k \not\sim_\pi l$.
Thus for all $4(i-1)+1 < k \leq 4(j-1)+1$, $4(j-1)+1 < l \leq 4(i-1)$ with $k,l \in S$, $k \not\sim_{(f(\pi),S)} l$, where f is the map $f(i) = 2i - \delta(i)$ defined in Section 7.1.2: this implies that $4(i-1)+2 \sim_{kr(f(\pi),S)} 4(j-1)+2$ and thus $2(i-1)+1 \sim_{kr'(\pi)} 2(j-1)+1$. Since $kr'(\pi) \geq \pi_1$, $2i \sim_{kr'(\pi)} 2j$ and $i \sim_{kr'(\pi)/2} j$. Thus $Kr(\pi/2) \leq kr'(\pi)/2$. The same proof yields the converse inequality. ☐

Proposition 7.28. *Let π be a partition of $2k$ such that $\pi \geq \pi_0$. The action of the irreducible planar tangle P_π on $\mathcal{P}(V)$ is $\tilde{T}_{\pi/2}$, and the action of $P_{kr'(\pi)}$ is $T_{Kr(\pi/2)}$.*

Proof. By Proposition 7.7, $(2i-1, 2i)$ and $(2j-1, 2j)$ are outer boundary components of the same shaded region of P_π if and only if $2i \sim_{kr'(\pi)} 2j$. By Lemma 7.27, this is equivalent to $i \sim_{Kr(\pi/2)} j$. For each block B of P_π, let D_B denote the inner disk of P_π corresponding to B.
Let us compute T_{P_π}. By the construction of the planar tangle P_π in Section 7.1.2, if $2i-1$ is the $(2s-1)-$th or $2s-$th element of a block B of π then $2i-1$ is linked by a curve to the point $2s-1$ of D_B. If i is the $s-$th element in the block B of $\pi/2$, then $2i-1$ is the $2s$-th element of the corresponding block $\tilde{B}$ in π if $1 \notin B$ and the $(2s-1)-$element of $\tilde{B}$ if $1 \in B$. In any case, $(2i-1, 2i)$ is a boundary interval of the same shaded region as the interval $(2s-1, 2s)$ of $D_{\tilde{B}}$.
Let f be a function from $[\![1,k]\!]$ to $[\![1,n]\!]$. For each block $B = \{i^B_1, \dots, i^B_{|B|}\}$ of $\pi/2$, let $e_B = e_{f(i^B_1)} \otimes \cdots \otimes e_{f(i^B_{|B|})}$ be the corresponding element of $V^{\otimes|B|}$. Then by the expression of T_{P_π} in the spin planar algebra and the above remarks,

$$T_{P_\pi}(\bigotimes_{B\in\pi/2} e_B) = \delta_{\ker(f(1),\dots,f(k)) \leq Kr(\pi/2)} e_{f(1)} \otimes \cdots \otimes e_{f(k)}.$$

This is exactly the expression of $\tilde{T}_{\pi/2}(\bigotimes_{B\in\pi/2} e_B)$.
Let us compute $T_{P_{kr'(\pi)}}$. Let B be a block of $kr'(\pi)$. It corresponds to an inner disk D_B of $P_{kr'(\pi)}$. Label i the interval $(2i-1, 2i)$ of the boundary of D_B, and do the same for the intervals of D_0.

$kr'(\pi) \geq \pi_1$, thus for all $1 \leq i \leq k$, the outer distinguished points $2i-1$ and $2i$ are in the same connected component of $P_{kr'(\pi)}$. Since $P_{kr'(\pi)}$ is also irreducible, each shaded region σ has exactly one boundary component which is an outer interval i_σ, and one boundary component which is an interval j_σ of an inner disk $D_{B(\sigma)}$. Therefore, there are as many shaded region as outer intervals of type $(2i-1, 2i)$, and we can identify i_σ with i.
Let f be a function from $[\![1,k]\!]$ to $[\![1,n]\!]$. For each block $B = \{i_1^B, \dots, i_{|B|}^B\}$ of $kr'(\pi)/2$, let $e_B = e_{f(i_1^B)} \otimes \cdots \otimes e_{f(i_{|B|}^B)}$ be the corresponding element of $V^{\otimes |B|}$. Then by the spin action of $T_{P_{kr'(\pi)}}$,

$$T_{P_{kr'(\pi)}}(\bigotimes_{B \in kr'(\pi)} e_B) = e_{f_1} \otimes \cdots \otimes e_{f_k},$$

and $T_{P_{kr'(\pi)}} = T_{kr'(\pi)/2}$. Since $kr'(\pi)/2 = Kr(\pi/2)$, this gives the second result of the Proposition.

□

This yields a simpler description of a free product of spin planar subalgebras:

Corollary 7.29. *Let $\mathcal{P}, \mathcal{Q}$ be two spin planar subalgebras. Then $(\mathcal{P} * \mathcal{Q})_k$ is spanned by the union of $\tilde{T}_\pi(\bigotimes_{B\in\pi} \mathcal{P}_{|B|}) \otimes T_{Kr(\pi)}(\bigotimes_{B \in Kr(\pi)} \mathcal{Q}_{|B|})$ for all $\pi \in NC(k)$.*

Proof. By Proposition 7.22 (and its proof), $(\mathcal{P} * \mathcal{Q})_k$ is spanned by the images of $(T_{P_\pi}, T_{P_{kr'(\pi)}})$ for all $\pi \in NC(2k)$ with $\pi \geq \pi_0$. Proposition 7.28 yields the result. □

In particular, for $\mathcal{P} = TL(V)$, $(\mathcal{P} * \mathcal{Q})_k$ is spanned by $\{T_p \otimes T_p(\mathcal{Q}_k)\}_{p \in NC(k)}$, where T_p is the vector defined from p in Chapter 1. Therefore if $\mathcal{Q} = \mathcal{P}(G)$ for an irreducible noncommutative permutation group G, then by Theorem 6.16 $\mathcal{P}(G \wr_* S_n^+) \simeq TL(V) * \mathcal{P}(G)$, the isomorphism being given by the collection of maps $\Phi = \{t_k\}_{k \geq 1}$ of Section 7.2.2.

Intertwiner spaces of a free wreath product

The following Theorem is the main result of the section: this was originally conjectured by Banica and Bichon in [10].

Theorem 7.30. *Let F and G be two irreducible free permutation groups. Then*

$$\mathcal{P}(G \wr_* F) \simeq \mathcal{P}(F) * \mathcal{P}(G),$$

with the isomorphism Φ given in Section 7.2.2.

Proof. The proof follows the same pattern as the proof of Theorem 6.16. Suppose that F acts on V and G on W: let $(u_{ij})_{1 \leq i \leq n}$ be the fundamental representation of F and $(v_{kl})_{1 \leq k,l \leq m}$ the one of G. Recall that the free wreath product $G \wr_* F$ is defined by the fundamental representation $(u_{ij}v_{kl}^i)_{\substack{1\leq i,j \leq n \\ 1 \leq k,l \leq m}}$ with the commutation relations $u_{ij}v_{kl}^i = v_{kl}^i u_{ij}$ (see 3.3.1).
Step 1: $\mathcal{P}(F) * \mathcal{P}(G)$ is isomorphic through Φ to a spin planar subalgebra of $\mathcal{P}(V \otimes W)$, thus by Theorem 7.13 there exists an irreducible permutation group H acting on $V \otimes W$, such that $(\mathcal{P}(F) * \mathcal{P}(G))_k \simeq \mathrm{Mor}_H(0,k)$ for all $k \geq 1$, each isomorphism map being given by t_k.
Step 2: Let us prove that $\mathcal{P}(F) * \mathcal{P}(G) \subseteq \mathcal{P}(G \wr_* F)$. Since both are planar algebras, it is enough to prove that a set of generating elements of $\mathcal{P}(F) * \mathcal{P}(G)$ is in $\mathcal{P}(G \wr_* F)$. By Proposition 7.23, a generating set is given by elements of two kinds:

- $(T_{\mathbf{1}_k} \otimes w_k)$ with $w_k \in \mathrm{Mor}_G(0,k)$. It has been proven in Theorem 6.16 that $t_k(T_{\mathbf{1}_k} \otimes w_k) \in \mathrm{Mor}_{G \wr_* S_n^+}(0,k)$, where t_k is the map defined in Section 7.2.2. Since $F \subseteq S_n^+$, $G \wr_* F \subseteq G \wr_* S_n^+$ and thus $t_k(T_{\mathbf{1}_k} \otimes w_k) \in \mathrm{Mor}_{G \wr_* F}(0,k)$.

- $t_k(w_k \otimes T_{\mathbf{0}_k})$ with $w_k \in \mathrm{Mor}_F(0,k)$. Let us expand w_k in the basis of $V^{\otimes k}$ as $\sum_{1\leq i_1,\dots,i_k\leq n} w_{\vec{i}} e_{i_1} \otimes \cdots \otimes e_{i_k}$. Then

$$t_k(w_k \otimes T_{\mathbf{0}_k}) = \sum_{\substack{1\leq i_1,\dots,i_k\leq n\\ 1\leq r_1,\dots,r_k\leq m}} w_{\vec{i}} (e_{i_1} \otimes f_{r_1}) \otimes \cdots \otimes (e_{i_k} \otimes f_{r_k}).$$

  Since $w_k \in \mathrm{Mor}_F(0,k)$, $\sum w_{\vec{i}} u_{j_1 i_1} \dots u_{j_k i_k} = w_{\vec{j}}$. Applying the action of $G \wr_* F$ yields (we use the abbreviated notations $e_{\vec{i}} = e_{i_1} \otimes \cdots \otimes e_{i_k}$)

$$\begin{aligned}
\alpha_{G\wr_* F}(t_k(w_k \otimes T_{\mathbf{0}_k})) &= \sum_{\substack{\vec{i},\vec{j}\\ \vec{r},\vec{s}}} w_{\vec{i}} t_k(e_{\vec{j}} \otimes f_{\vec{s}}) \otimes v^{j_1}_{s_1 r_1} u_{j_1 i_1} \dots v^{j_k}_{s_k r_k} u_{j_k i_k}\\
&= \sum_{\substack{\vec{i},\vec{j}\\ \vec{s}}} w_{\vec{i}} t_k(e_{\vec{j}} \otimes f_{\vec{s}}) \otimes (\sum_{r_1} v^{j_1}_{s_1 r_1}) u_{j_1 i_1} \dots (\sum_{r_k} v^{j_k}_{s_k r_k}) u_{j_k i_k}\\
&= \sum_{\substack{\vec{i},\vec{j}\\ \vec{s}}} w_{\vec{i}} t_k(e_{\vec{j}} \otimes f_{\vec{s}}) \otimes u_{j_1 i_1} \dots u_{j_k i_k}\\
&= \sum_{\vec{j},\vec{s}} w_{\vec{j}} t_k(e_{\vec{j}} \otimes f_{\vec{s}}) \otimes \mathbf{1} = t_k(w_k \otimes T_{\mathbf{0}_k}) \otimes \mathbf{1},
\end{aligned}$$

  where the third equality is due to the fact that G is a free permutation group (yielding the equality $\sum_r v^j_{sr} = 1$ for all $1 \leq j \leq n, 1 \leq s \leq m$).

Therefore $\mathcal{P}(F) * \mathcal{P}(G) \subseteq \mathcal{P}(G \wr_* F)$ and by the Tannaka-Krein duality, $G \wr_* F$ is a quantum subgroup of H.
Step 3: Let $(w_{ijkl})_{\substack{1\leq i,j\leq n\\ 1\leq k,\leq m}}$ be the fundamental representation of H. Let us show that $H \subseteq G \wr_* F$ by showing that all the relations satisfied by the fundamental representation of $G \wr_* F$ are also satisfied by the fundamental representation of H. Since $\mathcal{P}(S_n^+) * \mathcal{G} \subseteq \mathcal{P}(F) * \mathcal{P}(G)$ and from the end of Section 7.2.2 $\mathcal{P}(S_n^+) * \mathcal{G}) = \mathcal{P}(G \wr_* S_n^+)$, $H \subseteq G \wr_* S_n^+$.
Therefore, all the relations satisfied by the fundamental representation of $G \wr_* S_n^+$ are also satisfied by the fundamental representation of H. In particular from the proof of Theorem 6.16, $w_{ijkl} = \tilde{u}_{ij} \tilde{v}^j_{kl}$ for $\tilde{u}_{ij} = \sum_l w_{ijkl}$ and $\tilde{v}^i_{kl} = \sum_j w_{ijkl}$. Moreover from the same proof, $\tilde{u}_{ij} \tilde{v}^j_{kl} = \tilde{v}^j_{kl} \tilde{u}_{ij}$ and there exist a C^*-morphism $\pi_{n+1} : C(S_n^+) \to C(H)$ defined by $\pi_{n+1}(s_{ij}) = \tilde{u}_{ij}$ and n C^*-morphisms $\pi_i : C(G) \to C(H)$ defined by $\pi_i(v_{kl}) = \tilde{v}^i_{kl}$.
Moreover,

$$\begin{aligned}
\Delta \tilde{u}_{ij} = \Delta(\sum_l w_{ijkl}) &= \sum_{l,r,s} w_{irks} \otimes w_{rjsl}\\
&= \sum_{r,s} \tilde{u}_{ir} \tilde{v}^j_{ks} \otimes \tilde{u}_{rj} (\sum_l \tilde{v}^j_{sl})\\
&= \sum_r \tilde{u}_{ir} \sum_s \tilde{v}^j_{ks} \otimes \tilde{u}_{rj} = \sum_r \tilde{u}_{ir} \otimes \tilde{u}_{rj},
\end{aligned}$$

where the second and third equalities are due to the fact that $\sum_l v^j_{sl} = \sum_s v^j_{sl} = \mathbf{1}$. Thus $(\tilde{u}_{ij})_{1\leq i,j\leq n}$ is a representation of the compact quantum group H: let $\tilde{F}$ the C^*-algebra generated by the representation $(\tilde{u}_{ij})_{1\leq i,j\leq n}$. Let us prove that there is a surjective morphism from $C(F)$ to $C(\tilde{F})$ sending u_{ij} to $\tilde{u}_{ij}$. It suffices to show that $\mathrm{Mor}_F(0,k) \subseteq \mathrm{Mor}_{\tilde{F}}(0,k)$. Let $v_k = \sum_{\vec{i}} v_{\vec{i}} e_{\vec{i}}$ be in $\mathrm{Mor}_F(0,k)$. Then $t_k(v_k \otimes \mathbf{0}_k) \in \mathrm{Mor}_H(0,k)$. On one hand the action of H gives:

$$\begin{aligned}
\sum_{\vec{i},\vec{j},\vec{r},\vec{s}} & v_{\vec{i}} t_k(e_{\vec{j}} \otimes f_{\vec{s}}) \otimes \tilde{u}_{j_1 i_1} \tilde{v}^{j_1}_{s_1 r_1} \dots \tilde{u}_{j_k i_k} \tilde{v}^{s_k}_{s_k r_k} \\
&= \sum_{\vec{i},\vec{j},\vec{s}} v_{\vec{i}} t_k(e_{\vec{j}} \otimes f_{\vec{s}}) \otimes \tilde{u}_{j_1 i_1} (\sum_{r_1} \tilde{v}^{j_1}_{s_1 r_1}) \dots \tilde{u}_{j_k i_k} (\sum_{r_k} \tilde{v}^{s_k}_{s_k r_k}) \\
&= \sum_{\vec{i},\vec{j},\vec{s}} v_{\vec{i}} t_k(e_{\vec{j}} \otimes f_{\vec{s}}) \otimes \tilde{u}_{j_1 i_1} \dots \tilde{u}_{j_k i_k}.
\end{aligned}$$

Since $t_k(v_k \otimes \mathbf{0}_k) \in \mathrm{Mor}_H(0,k)$, this implies the equality:

$$\sum_{\vec{i},\vec{s}} v_{\vec{i}} t_k(e_{\vec{i}} \otimes f_{\vec{s}}) \otimes \mathbf{1} = \sum_{\vec{i},\vec{j},\vec{s}} v_{\vec{i}} t_k(e_{\vec{j}} \otimes f_{\vec{s}}) \otimes \tilde{u}_{j_1 i_1} \dots \tilde{u}_{j_k i_k},$$

which yields the equality $\sum_{\vec{i}} v_{\vec{i}} \tilde{u}_{j_1 i_1} \dots \tilde{u}_{j_k i_k} = v_{\vec{j}} \mathbf{1}$. In particular $v_k \in \mathrm{Mor}_{\tilde{F}}(0,k)$.
Thus, there is a $*-$morphism $\Phi_0 : C_0(F) \longrightarrow C(H)$ sending u_{ij} to $\tilde{u}_{ij}$ and n $*-$morphisms $\Phi_i : C_0(G) \longrightarrow C(H)$ sending v_{kl} to $\tilde{v}^i_{kl}$. Since for all $1 \leq i \leq n$, $\tilde{u}_{ij} \tilde{v}^i_{kl} = \tilde{v}^i_{kl} \tilde{u}_{ij}$, there exists a $*-$morphism

$$\Phi : C_0(G)^{*n} * C_0(F) \big/ \langle u_{ij} v^i_{kl} = v^i_{kl} u_{ij} \rangle \longrightarrow C(H)$$

sending $u_{ij} v^i_{kl}$ to $\tilde{u}_{ij} \tilde{v}^i_{kl}$. Since $C_0(G)^{*n} * C_0(F) \big/ \langle u_{ij} v^i_{kl} = v^i_{kl} u_{ij} \rangle = C_0(G \wr_* F)$ and Φ sends the fundamental representation of $G \wr_* F$ to the one of H, this yields that $H \subseteq G \wr_* F$.
Step 4: Since $G \wr_* F \subseteq H$ from Step 2 and $H \subseteq G \wr_* F$ from Step 3, $H = G \wr_* F$ and by the Tannaka-Krein duality ,

$$\mathcal{P}(G \wr_* F) = \mathcal{P}(H) = \Phi(\mathcal{P}(F) * \mathcal{P}(G)).$$

□

Theorem 7.30 with Corollary 7.29 yields a description of the intertwiner spaces of a free wreath product $G \wr_* F$. Namely,

$$\mathrm{Mor}_{G \wr_* F}(0,k) = \langle t_k(\tilde{T}_\pi(\bigotimes_{B \in \pi} \mathrm{Mor}_F(0,|B|) \otimes T_{Kr(\pi)}(\bigotimes_{B \in Kr(\pi)} \mathrm{Mor}_G(0,|B|)\rangle.$$

The main problem to deduce the law of $\chi_{G \wr_* F}$ is that it is difficult to extract a basis from these sets. In the next section, we will explain a result of Bisch and Jones that gives the dimension of each vector space $(\mathcal{P} * \mathcal{Q})_k$ thanks to the ones of $\mathcal{P}$ and $\mathcal{Q}$.

7.3 Free product formula of Bisch and Jones

Each planar algebra $\mathcal{P}$ yields a probability measure $\mu(\mathcal{P})$ by saying that the $k-$th moment $\mu(\mathcal{P})_k$ of $\mu(\mathcal{P})$ is $\dim(\mathcal{P}_k)$. It will be later clear that this sequence of moments actually defines a measure. In this section we review the following result of Bisch and Jones ([23]), which yields the proof of Theorem 7.1:

Theorem 7.31 (Bisch and Jones, [23]). *Let $\mathcal{P}, \mathcal{Q}$ be two planar algebras with $\mu(\mathcal{P}) = \mu(\mathcal{Q}) = 1$. Then*

$$\mu(\mathcal{P} * \mathcal{Q}) = \mu(\mathcal{P}) \boxtimes \mu(\mathcal{Q}).$$

Since the proof the this result still doesn't exist in the literature, we give a combinatorial proof of Theorem 7.31 which is based on the computation of the principal graph of a free product by Landau (see [54]).

Principal graph of a free product

One of the major consequence of the axioms of a subfactor planar algebra is the Theorem 7.10 of Section 7.1.3:

Theorem (Jones, [46]). *Let $\mathcal{P}$ be a subfactor planar algebra. There exists a bipartite graph $G_{\mathcal{P}}$ with root vertex $*$ such that:*

$$\dim \mathcal{P}_k = \#\{ \textit{ walk of length } 2k \textit{ on } G_{\mathcal{P}} \textit{ starting and ending at } *\}.$$

The principal graph of a free product $\mathcal{P} * \mathcal{Q}$ can be obtained from the principal graph of $\mathcal{P}$ and the one of $\mathcal{Q}$. This has been done by Landau in [54]. We will only give the result when $\dim \mathcal{P}_1 = \dim \mathcal{Q}_1 = 1$, since this is the only interesting case in this chapter.
If $\mathcal{P}$ is a planar algebra such that $\dim \mathcal{P}_1$, the root vertex $*$ of $G_{\mathcal{P}}$ is linked to only one other vertex $\tilde{*}$ by a unique edge e_* called the root edge. Let $G_{\mathcal{P}}$ and $G_{\mathcal{Q}}$ be respectively the principal graph of P and Q. Let $\tilde{G}_{\mathcal{P}}$ (resp. $\tilde{G}_{\mathcal{Q}}$) be the graph $G_{\mathcal{P}}$ with the root vertex and the root edge removed, and with the new root being $\tilde{*}$. We build recursively a graph $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ and a height function on edges of $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ as follows:

1. Let G_1 be a copy of $G_{\mathcal{P}}$. The root of $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ is the root $*$ of G_1. For all edges e of G_1, set $h(e) = 0$.

2. On each vertex at odd distance to $*$, add a copy of $\tilde{G}_{\mathcal{Q}}$ (that means that we identify the root of $\tilde{G}_{\mathcal{Q}}$ with the given vertex). For each edge e newly added, set $h(e) = 1$.

3. On each vertex added during the previous step which is at even distance to $*$, add a copy of $\tilde{G}_{\mathcal{P}}$. For each edge e belonging to a new copy G of $\tilde{G}_{\mathcal{P}}$, set $h(e) = h(e') + 1$, where e' is any edge not belonging to G and having the root of G as endpoint.

4. On each vertex added during the previous step which is at odd distance to $*$, add a copy of $\tilde{G}_{\mathcal{Q}}$. For each edge e belonging to a new copy G of $\tilde{G}_{\mathcal{Q}}$, set $h(e) = h(e') + 1$, where e' is any edge not belonging to G and having the root of G as endpoint.

5. Return to step 3.

By a copy G of a graph X in $G_{\mathcal{P}} \star G_{\mathcal{Q}}$, we mean the set of edges and vertices of $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ which were added by pasting one graph of type X at one particular vertex: X is called the initial graph of G. The height of a copy of graph is the height of any edge belonging to this copy of graph.
Note that each copy of graph is connected to exactly one copy of graph of lower height. Let $\mathcal{G}$ denotes the graph whose set of vertices is the set of copies of graph $\{G\}$, and such that there is an edge between G and G' if and only both share a vertex. By the previous remark $\mathcal{G}$ is tree; let the only copy of $G_{\mathcal{P}}$ be the root of this tree. For G a vertex of $\mathcal{G}$, the subtree T_G of $\mathcal{G}$ rooted at G is the subtree of $\mathcal{G}$ consisting of vertices whose unique path to the root goes through G.
There exists a map φ from the set of edges of $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ to the set of vertices of $\mathcal{G}$ such that $\varphi(e)$ is the copy of graph G such that $e \in G$.
Landau proved in [54] the following result:

Proposition 7.32 (Landau,[54]). *$G_{\mathcal{P}} \star G_{\mathcal{Q}}$ is the principal graph of $\mathcal{P} * \mathcal{Q}$.*

We will describe combinatorially the set of loops γ of length $2k$ on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ which start at the root. Such a loop is called a rooted loop on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$.
Since $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ is bipartite, the set of rooted loops of length $2k$ is exactly the set of words $\gamma = \gamma_1 \dots \gamma_{2k}$ in the edges of $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ of length $2k$, such that γ_1 and γ_{2k} has the root vertex as endpoint, and such that for all $1 \leq i \leq 2k-1$, γ_i and γ_{i+1} share a common vertex; in particular $|h(\gamma_{i+1}) - h(\gamma_i)| \leq 1$. Moreover each walk γ yields a lazy walk $\varphi(\gamma)$ of length $2k-1$ on $\mathcal{G}$ defined by $\varphi(\gamma)(i) = \varphi(\gamma_i)$ (a lazy walk is a walk that can be stationary).

Lemma 7.33. *Let γ be a rooted loop of length $2k$ on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ and let $1 \leq i \leq j \leq 2k$. If $h(\gamma_i) = h(\gamma_j)$ and for all $i \leq k \leq j$, $h(\gamma_k) \geq h(\gamma_i)$, then i and j are in the same copy of graph.*

Proof. Let G be the copy of graph containing γ_i. Since for all $i \leq k \leq j$, $h(\gamma_k) \geq h(\gamma_i)$, the lazy walk $\varphi(\gamma)$ restricted to $[\![i, j]\!]$ is a lazy walk on T_G. Since the only vertex of height $h(\gamma_i)$ of this subtree is G itself and $h(\gamma_j) = h(\gamma_i)$, γ_j belongs to G. □

Each rooted loop γ defines a symmetric relation $\sim_\gamma$ on $2k$ as follows: for $i \leq j$, $i \sim_\gamma j$ if and only if γ_i and γ_j belong to the same copy of graph, and for all $i \leq k \leq j$, $h(\gamma_k) \geq h(\gamma_j)$.

Lemma 7.34. *$\sim_\gamma$ is an equivalence relation and the associated partition π_γ is a non-crossing partition of $2k$.*

Proof. The reflexivity is assumed by definition and $i \sim_\gamma i$ for all $1 \leq i \leq 2k$. Let $1 \leq i, j, k \leq 2k$ be such that $i \sim_\gamma j$ and $j \sim_\gamma k$. γ_i and γ_j are in the same copy of graph and γ_j and γ_k are in the same copy of graph, thus γ_i and γ_k are in the same copy of graph and $h(\gamma_i) = h(\gamma_j) = h(\gamma_k)$. $h(\gamma_s) \geq h(\gamma_j)$ for $s \in [\![i, j]\!]$ and $h(\gamma_s) \geq h(\gamma_j)$ for $s \in [\![j, k]\!]$, thus $h(\gamma_s) \geq h(\gamma_i)$ for $s \in [\![i, k]\!]$. Therefore, the relation $\sim_\gamma$ is transitive and thus an equivalence relation. Let π_γ denote the associated partition of $2k$.
Let $1 \leq i < j < k < l \leq 2k$ be such that $i \sim_\gamma k$ and $j \sim_\gamma l$. Since $i \sim_\gamma k$ and $i \leq j \leq k$, $h(\gamma_j) \geq h(\gamma_k)$. Since $j \sim_\gamma l$ and $j \leq k \leq l$, $h(\gamma_k) \geq h(\gamma_l)$. Therefore $h(\gamma_k) = h(\gamma_j)$ and for all $j \leq r \leq k$, $h(\gamma_r) \geq h(\gamma_k)$. By Lemma 7.33, j and k are in the same copy of graph. π_γ is thus non-crossing. □

We define the value $h(B)$ of a block of π_γ as the height of the corresponding copy of graph. Since two neighbouring blocks correspond to adjacent copies of graph, they must have consecutive value (and thus opposite parity).
Let π be a non-crossing partition. An interval of a block B of π is a maximal interval subset $[\![i, j]\!] \subseteq B$. In particular $i - 1 \notin B$ and $j + 1 \notin B$. The set of intervals of a block B is ordered by the lexicographical order; if B has a unique interval, this interval is called a block interval. Otherwise, the first interval is called the initial interval of B and the last interval is called the final interval of B. The other ones are called intermediate intervals.

Lemma 7.35. *Let γ be a rooted loop of length $2k$ and let $B = \{i_1 < \dots < i_r\}$ be a block of π_γ. Then $i_1 = 1$ if and only if $i_r = 2k$, and if $i_1 \neq 1$ then $h(i_1 - 1) = h(i_2 + 1) = h(i_1) - 1$ and $i_1 - 1 \sim_\gamma i_2 + 1$.*

Proof. $h(\gamma_1) = h(\gamma_{2k}) = 0$ and for all $1 \leq i \leq 2k$, $h(\gamma_i) \geq 0$, thus 1 and $2k$ are in the same block.
Let $i_0 = \sup\{i < i_1, h(i) = h(i_1)\}$. $i_0 \neq i_1 - 1$ because $i_0 \not\sim_\gamma i_1$. On $[\![i_0 + 1, i_1 - 1]\!]$, either $h > h(i_1)$ or $h < h(i_1)$. Since $i_0 \not\sim_\gamma i_1$, $h < h(i_1)$ on $[\![i_0 + 1, i_1 - 1]\!]$ and thus $h(i_1 - 1) = h(i_1) - 1$. Likewise, $h(i_2 + 1) = h(i_2) - 1 = h(i_1) - 1$.
Moreover on $[\![i_1 - 1, i_2 + 1]\!]$, $h \geq h(i_1 - 1)$, thus by Lemma 7.33 $i_1 - 1$ and $i_2 + 1$ are in the same copy of graph and $i_1 - 1 \sim_\gamma i_2 + 1$. □

A partition π is called irreducible if $1 \sim_\pi 2k$ (see [2]). A subpartition of π is a subset $\mathcal{A} \subseteq \pi$ of blocks of π such that $\bigcup_{B\in\mathcal{A}} B$ is an interval.

Lemma 7.36. *Let γ be a rooted loop of length $2k$. Then π_γ and every subpartitions of π_γ are irreducible. The first and last elements of a block having even value are respectively odd and even. The first and last elements of a block having odd value are respectively even and odd.*

Proof. By Lemma 7.35, 1 and $2k$ are in the same block, thus π is irreducible.
Let $\tilde{\pi} = \{B_i\}_{i\in I_0}$ be a strict subpartition of π and set $[\![i_1, i_2]\!] = \bigcup_{i\in I_0} B_i$. Since $\tilde{\pi} \neq \pi$, $i_1 \neq 1$ and $i_2 \neq 2k$. Let $x = \inf\{i_1 - 1 < i, h(i) = h(i_1 - 1)\}$. On $[\![i_1 - 1, x]\!]$, either $h > h(i_1 - 1)$ or $h < h(i_1 - 1)$. By Lemma 7.35, $h(i_1) > h(i_1 - 1)$, thus $h > h(i_1 - 1)$ on $[\![i_1 - 1, x]\!]$ and $i_1 - 1 \sim x$. Since $x \notin [\![i_1, i_2]\!]$, $h(i_2) > h(i_1 - 1)$ and thus $h(i_2) \geq h(i_1)$. By symmetry, $h(i_1) \geq h(i_2)$ and finally $h(i_1) = h(i_2)$. Since $h > h(i_1) - 1$ on $[\![i_1, i_2]\!]$, $h \geq h(i_1)$ on $[\![i_1, i_2]\!]$: Lemma 7.33 yields that $i_1 \sim i_2$.
1 is the first element of a block with even value and is odd. Let $i > 1$ be the first element of a block B with even value. This means that $h(\gamma_i)$ is odd, and thus γ_i belong to a copy G of $\tilde{G}_{\mathcal{P}}$. Since i is the first element of B, $h(i-1) = h(i) - 1$, and thus γ_{i-1} belong to the only copy of $\tilde{G}_{\mathcal{Q}}$ with lower height than G: thus the vertex between γ_{i-1} and γ_i is the root vertex v of G. Since G is a copy of $\tilde{G}_{\mathcal{P}}$, v is at even distance to the root. $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ being bipartite, $i - 1$ is also even and thus i is odd. The same proof holds for the three other cases. □

Definition 7.37. *A non-crossing partition π is of type $\star$ if π is irreducible, all its subpartitions are irreducible, all its blocks have an even number of elements, and the first elements of two neighbouring blocks have opposite parities.*
The value $h(B)$ of a block B in π is 0 if B contains 1, 1 if the first element of B is even and 2 if the first element of B is odd and distinct from 1.

Let $P_\star(k)$ the set of partitions of type $\star$.

Lemma 7.38. $|P_\star(2k)| = NC(k)$.

Proof. Let $P_k = |P_\star(2k)|$. It suffices to show that $P_k = \sum_{i=1}^{k} P_{i-1}P_{k-i}$ with $P_0 = P_1 = 1$. We set a partition of $P_\star = \amalg P_x$ depending on the position of the second element x of the block containing 1. Since the blocks are even, $x \in \{2i | 1 \leq i \leq k\}$. Let $x_0 = 2i_0$ and $\pi \in P_{x_0}$. If $x_0 \geq 4$, $\pi_{[\![2,x_0-1]\!]}$ is again completely irreducible with blocks of even numbers, and the first elements of two neighbouring blocks of $\pi_{[\![2,x_0-1]\!]}$ have opposite parities: thus $\pi_{[\![2,x_0-1]\!]} \in P_\star(i_0 - 1)$. On the other a hand since the first point of the first block after x_0 is even, $x_0 + 1$ belongs again to the block containing 1 (and thus $2k$): thus likewise if $x_0 \leq 2k - 2$, $\pi_{[\![x_0+1,2k]\!]} \in P_\star(k - i_0)$. Therefore, there exists a map $\varphi : P_x \longrightarrow P_\star(k - x/2) \times P_\star(x/2 - 1)$; this map is clearly bijective and thus $|P_x| = |P_\star(k - x/2)| \times |P_\star(x/2 - 1)|$. Summing on $2 \leq 2x \leq 2k$ yields the result. □

Let γ be a rooted loop on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$. For $B = \{i_1 < \cdots < i_r\}$ a block of π_γ, we denote γ_B the word $\gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_r}$.

Lemma 7.39. *For γ a rooted loop on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ and B a block of π_γ, γ_B is a rooted loop on G_B, where G_B is the copy of graph containing $\{\gamma_i\}_{i\in B}$.*

Note that G_B is well-defined, since all edges of B are in the same copy of graph.

Proof. We have to show that γ_B defines indeed a walk, and that the first and last vertices of γ_B are the root vertex of G_B. Let $B = [\![r_1, s_1]\!] \cup [\![r_2, s_2]\!] \cup \cdots \cup [\![r_t, s_t]\!]$ with $r_i \leq s_i, s_i < r_{i+1} - 1$. For $r_i \leq j < s_i$, γ_j and γ_{j+1} are consecutive edges of γ_B that are also consecutive edges of γ: therefore, they share a vertex. Let $j = s_i$ with $i < t$. Since any subpartition of π_γ is

also irreducible, $x = s_i + 1$ and $y = r_{i+1} - 1$ are in a same block B' of π. By Lemma 7.35, $h(B') = h(B) + 1$, and thus the vertex between γ_{s_i} and γ_x is the root vertex between $G_{B'}$ and G_B. Likewise, the vertex between γ_y and $\gamma_{r_{i+1}}$ is the root vertex between $G_{B'}$ and G_B. Therefore, the final vertex of γ_{s_i} and the first vertex of $\gamma_{r_{i+1}}$ is the same. γ_B is thus a walk on G_B.
Since $h(r_1 - 1) = h(r_1) - 1$, the root vertex between γ_{r_1-1} and γ_{r_1} is the root vertex of G_B. For the same reason, the root vertex between γ_{s_t} and γ_{s_t+1} is the root vertex of G_B. Thus, γ_B is a rooted loop on G_B. $\square$

This implies in particular that the blocks of $\pi(\gamma)$ are even (this could have been proven directly from Lemma 7.36). The latter result yields also a combinatorial description of the rooted loops on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$.
Let ψ_G be the canonical bijection sending an edge of the copy of graph G to the same edge in the initial graph (which is either $G_{\mathcal{P}}, \tilde{G}_{\mathcal{P}}$ or $\tilde{G}_{\mathcal{Q}}$, depending on the value of B). ψ_G^{-1} maps an edge of the initial graph of G to the same edge in G. For each rooted loop γ and block B of π_γ, let $\tilde{\gamma}_B$ denote the image of γ_B by ψ_{G_B}.

Proposition 7.40. *There is a bijection between rooted loops of length $2k$ on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ and pairs $(\pi, \{\zeta_B\}_{B\in\pi})$, where*

- *$\pi \in P_\star(2k)$.*
- *ζ_B is a reduced loop of length $|B|$ on $\tilde{G}_{\mathcal{P}}$ (resp. $\tilde{G}_{\mathcal{Q}}$, resp. $G_{\mathcal{P}}$) if $h(B) = 2$ (resp. $h(B) = 1$, resp. $h(B) = 0$).*

Proof. Let A be the set of rooted loops of length $2k$ on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ and let B be the set of pairs $(\pi, \{\zeta_B\}_{B\in\pi})$ as in the statement of the proposition.
Let $\Phi : A \longrightarrow B$ be the map defined by $\Phi(\gamma) = (\pi_\gamma, \{\tilde{\gamma}_B\}_{B\in\pi})$. By Lemmas 7.36 and 7.39, this map is well defined.
Let $\gamma \neq \gamma'$. If $\pi_\gamma \neq \pi_{\gamma'}$ then $\Phi(\gamma) \neq \Phi(\gamma')$. Suppose that $\pi_\gamma = \pi_{\gamma'}$ and let $1 \leq i \leq 2k$ be the first element such that $\gamma_i \neq \gamma'_i$. Let B be the block containing i. For all $t < i$, $\gamma_t = \gamma'_t$, thus $G_B(\gamma) = G_B(\gamma')$. Thus γ_B and γ'_B are both walks on G_B. Since $\gamma_i \neq \gamma'_i$, these walks are distinct and thus $\tilde{\gamma}_B \neq \tilde{\gamma}'_B$: therefore, $\Phi(\gamma) \neq \Phi(\gamma')$.
Let $(\pi, \{\zeta_B\}_{B\in\pi})$ be a pair as in the statement of the proposition. We define recursively on each block a copy of graph G_B and a rooted loop γ as follows:

1. Let $B_0 = \{i_1 = 1 < \cdots < i_r = 2k\}$ be the block containing 1. We set $G_{B_0} = G_{\mathcal{P}}$. For $i_s \in B_0$, we set $\gamma_{i_s} = \psi_{G_{B_0}}(\zeta_{B_0}(s))$. Since ζ_{B_0} is a reduced loop on $G_{\mathcal{P}}$, γ_{B_0} is a reduced loop on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$ (which is contained in the only copy of $G_{\mathcal{P}}$). Set $t = 0$.

2. Let $B = \{i_1 < \cdots < i_r\}$ be the first block neighbouring B_t: note that $[\![1, i_1 - 1]\!] \subseteq B_t$. Suppose that $h(B) = 1$. Then i_1 is even and thus the length of $\gamma_{B_t \cap [\![1,i_1-1]\!]}$ is odd. Therefore, the walk $\gamma_{B_t \cap [\![1,i_1-1]\!]}$ ends at a vertex v which is at odd distance from the root, and thus there exists a copy G of $\tilde{G}_{\mathcal{Q}}$ whose root is exactly v. This implies that the height of $h(G)$ is $h(i_1 - 1) + 1$. Let $G_B = G$. For $i_s \in B$, set $\gamma_{i_s} = \psi_{G_B}(\zeta_B(s))$, $h(i) = h(G)$. Since by irreducibility $i_1 - 1$ and $i_r + 1$ belongs to a same block of π (which is thus included in B_t) and since γ_{B_t} is a rooted loop on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$, the final vertex of γ_{i_1-1} and the first vertex of $i_r + 1$ is the same vertex v which is the root of G_B. $\psi_{G_B}(\zeta_B)$ is a rooted loop on G_B, thus the final vertex of γ_{i_r} coincides also with the root v. Therefore $\gamma_{B_t \cup B}$ is again a rooted loop on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$. Let $B_{t+1} := B_t \cup B$.
   Do the same construction (with a copy of $\tilde{G}_{\mathcal{P}}$) if $h(B) = 2$.

3. Increase t by one and return to Step 2, until t is the number of blocks of π.

By construction γ is a rooted loop on $G_{\mathcal{P}} \star G_{\mathcal{Q}}$. Let $B = \{i_1 < \cdots < i_r\}$ and $B' = \{j_1 < \cdots < j_{r'}\}$ be two blocks of π such that $i_1 < j_1 < j_{r'} < i_r$ (written $B' \leq B$): in this case the construction yields that $h(G_{B'}) > h(G_B)$.
If $i \sim_\pi j$, then i and j are in the same copy of graph G_B and for all $i \leq k \leq j$ $h(\gamma_k) \geq h(\gamma_i)$: this implies $i \sim_\gamma j$ and $\pi \leq \pi_\gamma$. Suppose that $1 \leq i < j \leq 2k$ are such that $i \not\sim_\pi j$, and let $B_i = \{i_1 < \cdots < i_r\}$ (resp. $B_j = \{j_1 < \cdots < j_r\}$) be the block of i (resp. j) in π. If γ_i is in a different copy of graph than γ_j, $i \not\sim_\gamma j$. Let us assume that they are in the same copy of graph: since $h(G_{B_i}) = h(G_{B_j})$, $i_r < j_1$. Since π is completely irreducible and $i_1 \not\sim_\pi j_{r'}$, $\pi_{[\![i_1, j_{r'}]\!]}$ is not a subpartition of π and thus there exist $i_r < k < j_1$ and $l \in [\![1, 2k]\!] \setminus [\![i_1, j_{r'}]\!]$ such that $k \sim_\pi l$. Assume without loss of generality that $l \leq i_1$ and let B be the block of l and k. Since $B_i \leq B$, $h(G_B) > h(G_{B_i})$. Thus $h(\gamma_l) < h(\gamma_i)$ and $i \not\sim_{\pi_\gamma} j$. Therefore, $\pi = \pi_\gamma$. Since $\pi_\gamma = \pi$ and by construction $\gamma_B = \zeta_B$ for each block B of π, $\Phi(\gamma) = (\pi, \{\zeta_B\}_{B \in \pi})$. Therefore, Φ is surjective and bijective. □

Free product formula

The proof of Theorem 7.31 relies on the combinatorics of the free multiplicative convolution. We will first construct a bijection between $P_\star$ and $NC(k)$ that respect the cardinal of the blocks.
Let $p \in P_\star(2k)$ and let $l(p)$ denote the number of blocks of p. Two blocks B and B' are said neighbors if there exists $i \in B$ such that $i+1$ or $i-1$ belongs to B'.

Lemma 7.41. *Let $p \in P_\star(2k)$. If B and B' are two neighbouring blocks, then either $B \leq B'$ or $B' \leq B$.*

Proof. Suppose that $B \not\leq B'$ and $B' \not\leq B$. Let i_B and $i_{B'}$ (resp. f_B and $f_{B'}$) be the first (resp. last) elements of B and B'. Since $B \not\leq B'$, $B' \not\leq B$ and p is non crossing, either $f_B < i_{B'}$ or $f_{B'} < i_B$. Suppose without loss of generality that $f_B < i_{B'}$. Then $f_B \neq i_{B'} - 1$, since otherwise $p_{[\![i_B, f_{B'}]\!]}$ would be a subpartition of p with $i_B \not\sim f_{B'}$. Therefore B and B' are not neighbors. □

Let $p \in P_\star$. For B a block of p and $i \in B$, we set $h(i) = 0$ if $h(B) \equiv 0[2]$ and $h(i) = 1$ if $h(B) \equiv 1[2]$. Let $f_p : [\![1, 2k]\!] \longrightarrow [\![1, 4k]\!]$ be the function defined by $f_p(2i) = 4i - h(i)$ and $f_p(2i-1) = 4i-3+h(i)$. Since $f_p(2i) \in \{4i-1, 4i\}$ and $f_p(2i-1) \in \{4i-3, 4i-2\}$, f_p is strictly increasing and thus $(f(p), f([\![1, 2k]\!]))$ defines a partial partiton of $4k$. Complete this partition by saying that the elements of $[\![1, 4k]\!] \setminus f([\![1, 2k]\!])$ are singletons: this yields a non-crossing partition of $4k$ denoted by $\tilde{p}$.
We define also an involutive map $\bar{\cdot}$ on $[\![1, 4k]\!]$ by saying that $\overline{4i+1} = 4i$, $\overline{4i+2} = 4i+3$, $\overline{4i+3} = 4i+2$ and $\overline{4i} = 4i+1$ for $0 \leq i \leq k-1$.

Lemma 7.42. *Let $1 \leq i \leq k$ and suppose that $4i-2 \not\sim_{\tilde{p}} 4i-1$. Then at least one of them is a singleton. The same holds for $4i$ and $4i+1$.*

Proof. Let $1 \leq i \leq k$. Suppose that $4i-2$ and $4i-1$ are not singletons. Thus $4i-2$ and $4i-1$ are in $f_p([\![1, 2k]\!])$. This means that $2i-1$ and $2i$ are both in a block of p of odd value. Since two neighbouring blocks have values of opposite parities, $2i-1$ and $2i$ are in the same block, and $4i-2 \sim_{\tilde{p}} 4i-1$. The same holds for $4i$ and $4i+1$ and $1 \leq i \leq k-1$. Since $1 \sim_{\tilde{p}} 4k$, the implication holds also for $i = k$. □

Lemma 7.43. *$4i-2$ (resp. $4i+1$) is a singleton of $\tilde{p}$ and $4i-1$ (resp. $4i$) is not a singleton of $\tilde{p}$ if and only if $2i$ (resp. $2i+1$) is the first element of a block of p with value 1 (resp. 2).*

$4i-1$ (resp. $4i$) is a singleton of $\tilde{p}$ and $4i-2$ (resp. $4i+1$) is not a singleton of $\tilde{p}$ if and only if $2i-1$ (resp. $2i$) is the last element of a block of p with value 1 (resp. 2).

Proof. p has no singleton, thus $f(p)$ has no singleton. Therefore, $4i-2$ is a singleton of $\tilde{p}$ and $4i-1$ is not if and only if $f(2i) = 4i-1$ and $f(2i-1) = 4i-3$. By construction of f, the latter means that $h(2i) \equiv 1$, $h(2i-1) \equiv 0$, and the elements $2i-1$ and $2i$ belong to distinct neighbouring blocks B_{2i-1} and B_{2i} of p. The two blocks are neighbors and p is non-crossing, thus B_{2i-1} is distinct from B_{2i} if and only if either $2i-1$ is the last element of B_{2i-1} or $2i$ is the first of B_{2i}. Since $p \in P_\star(2k)$ the last element of a block of even value is even. Therefore, $2i-1$ is never the last element of B_{2i-1}, and thus B_{2i} and B_{2i-1} are distinct blocks if and only if $2i$ is the first element of B_{2i}.
The proof is the same in the three other cases (note that $4k$ and 1 are never singleton). $\square$

By the definition of f, elements of a block of p of even value are mapped to S and elements of a block of p of odd value are mapped to S^c. Thus $\tilde{p}$ has the form $((\pi(p), S) \vee (\pi'(p), S^c))$ for some partitions $\pi(p), \pi'(p)$ of $2k$. We define $\Theta(p)$ as the partition $((\pi(p) \vee \pi_0, S) \vee (\pi'(p) \vee \pi_1, S^c))$. By construction $\Theta(p)$ is a non-crossing partition of $4k$ of the form $((\pi, S) \vee (\pi', S^c))$ for some partition $\pi \geq \pi_0, \pi' \geq \pi_1$. Let B be a block of p. There exists a unique block $\vartheta_p(B)$ of $\Theta(p)$ such that $f(B) \subseteq \vartheta_p(B)$.

Lemma 7.44. *ϑ_p is an injective map. Moreover $|\vartheta_p(B)| = |B| + 2$ if $1 \neq B$, and $|\vartheta_p(B)| = |B|$ if $1 \in B$. For all block B not being in the image of ϑ_p, $|B| = 2$.*

Proof. Let $\tilde{B}$ be a block of $\tilde{p}$ which is not a singleton. And let us assume that $\tilde{B} = f(B)$ for a block B of p. By Lemma 7.42, if $x \in \tilde{B}$ then either $\bar{x}$ is a singleton or $\bar{x} \in \tilde{B}$. Therefore, each block of $\Theta(p)$ contains at most one block of $\tilde{p}$ which is not a singleton, and $\vartheta(p)$ is injective.
By Lemma 7.43, $x \in \tilde{B}$ and $\bar{x}$ is a singleton if and only if x is the first or last point of $\tilde{B}$ and $\tilde{B}$ doesn't contain 1. Therefore $\vartheta_p(B) = \tilde{B}$ if $1 \in B$ and $\vartheta_p(B)$ is the union of $\tilde{B}$ and two singletons if $1 \notin B$: this yields $|\vartheta_p(B)| = |B| + 2$ if $1 \notin B$, and $|\vartheta_p(B)| = |B|$ if $1 \in B$.
Let B be a block not being in the image of ϑ_p. Then B is a union of singletons of $\tilde{p}$. Since the blocks of (π_0, S) and (π_1, S^c) have size 2, $|B| = 2$. $\square$

Lemma 7.45. *Let $p \in P_\star$. Then $\Theta(p)$ is of the form $((\pi, S) \vee (kr(\pi), S^c))$ for some partition $\pi \geq \pi_0$ in $NC(2k)$.*

Proof. By construction, $\Theta(p)$ is of the form $((\pi, S) \vee (\pi', S^c))$ for some partitions $\pi \geq \pi_0, \pi' \geq \pi_1$ in $NC(2k)$. Thus $\pi' \leq kr'(\pi)$. Since $\pi \geq \pi_0$, Lemma7.27 yields $kr'(\pi)/2 = Kr(\pi/2)$. The Kreweras complement $Kr(\pi)$ of a partition π of k satisfies the relation $l(Kr(\pi)) + l(\pi) = k+1$ (see [66], Ch. 9). Therefore, $l(kr'(\pi)/2) + l(\pi/2) = k+1$. Since $l(kr'(\pi/2)) = l(kr'(\pi))$ and $l(\pi/2) = l(\pi)$, $l(\pi) + l(kr'(\pi)) = k+1$.
On the other hand, $|B_1| + \sum_{\substack{B \in p \\ 1 \notin B}} |B| = 2k$, thus by Lemma 7.44 $\sum_{B \in p} |\vartheta_p(B)| = 2k + 2(l(p)-1)$. Since all other blocks of $\Theta(p)$ have cardinal 2,

$$l(\Theta(p)) = l(p) + \frac{4k - (2k + 2(l(p)-1))}{2} = k+1.$$

Therefore, $l(\pi) + l(\pi') = k+1 = l(\pi) + l(kr'(\pi))$ and $l(\pi') = l(kr'(\pi))$. Thus, $\pi' = kr'(\pi)$. $\square$

Recall that a partition is formally defined as a set of subsets of $\{1, \ldots, k\}$. The former results give the desired bijection:

Proposition 7.46. *There exists a bijection $\Lambda : P_\star \to NC(k)$, and for each $p \in P_\star$ an injective map $\lambda_p :\longrightarrow \Lambda(p) \cup Kr(\Lambda(p))$ such that $|\lambda_p(B)| = |B|/2+1$ if $1 \notin B$ and $|\lambda_p(B)| = |B|/2$ if $1 \in B$; moreover if B is not in the image of λ_p, then B is a singleton.*

Proof. Let p and p' be two distinct partitions of $P_\star(2k)$. Let $1 \leq i, j \leq 2k$ such that i and j are in the same block B of p but in distinct blocks B_i and B_j of p'. Therefore, $f_p(i)$ and $f_p(j)$ are in the same block of $\Theta(p)$. But $f_{p'}(i)$ and $f_{p'}(j)$ are not in the same block of $\Theta(p')$: this would contradict the injectivity of $\vartheta_{p'}$. Thus $\Theta(p) \neq \Theta(p')$. Θ is injective and by Lemma 7.38, $|P_\star(2k)| = |NC(k)| = |\{\pi \in NC(2k), \pi \geq \pi_0\}|$, thus Θ is bijective.
If $\Theta(p) = ((\pi, S), (kr'(\pi), S^c))$, set $\Lambda(p) = \pi/2$. For each $B \in p$, set $\lambda_p(B) = \vartheta_p(B)/2$, where $B/2$ is the image of $B \in \pi$ through the map $\pi \longrightarrow \pi/2$. Λ and λ_p have all the desired properties. □

Denote by $M_k(G)$ the number of rooted loops of length $2k$ on a rooted graph G. If p is a partition, $B_1(p)$ denotes the block of p containing 1. Proposition 7.46 yields the following formula for the dimension of $\mathcal{P} * \mathcal{Q}$.

Lemma 7.47. *For all $k \geq 1$.*

$$\mathcal{P} * \mathcal{Q}_k = \sum_{p \in NC(k)} M_{|B_1|}(G_{\mathcal{P}}) \prod_{B \in p, 1 \notin B} M_{|B|-1}(\tilde{G}_{\mathcal{P}}) \prod_{B \in Kr(p)} M_{|B|-1}(\tilde{G}_{\mathcal{Q}}). \tag{7.3.1}$$

Proof. By Proposition 7.40,

$$\mathcal{P} * \mathcal{Q}_k = \sum_{p \in P_\star(k)} M_{|B_1|/2}(G_{\mathcal{P}}) \prod_{B \in p, h(B)=2} M_{|B|/2}(\tilde{G}_{\mathcal{P}}) \prod_{B \in p h(B)=1} M_{|B|/2}(\tilde{G}_{\mathcal{Q}}).$$

Applying the map λ_p on each block of p and using the results of Proposition 7.46 yield

$$\mathcal{P} * \mathcal{Q}_k = \sum_{p \in P_\star(k)} M_{|\lambda_p(B_1)|}(G_{\mathcal{P}}) \prod_{B \in p, h(B)=2} M_{|\lambda_p(B)|-1}(\tilde{G}_{\mathcal{P}}) \prod_{B \in p, h(p)=1} M_{|\lambda_p(B)|-1}(\tilde{G}_{\mathcal{Q}}).$$

By Proposition 7.46, any other block B of $\Lambda(p)$ is a singleton, therefore for such a B, $M_{|B|-1}(G) = M_0(G) = 1$. Thus, the above formula can be completed as

$$\mathcal{P} * \mathcal{Q}_k = \sum_{p \in P_\star(k)} M_{|\lambda_p(B_1)|}(G_{\mathcal{P}}) \prod_{B \in \Lambda(p), 1 \notin p} M_{|B|-1}(\tilde{G}_{\mathcal{P}}) \prod_{B \in Kr(p)} M_{|B|-1}(\tilde{G}_{\mathcal{Q}}).$$

Summing on $\Lambda(p)$ instead of p yields the result. □

Since the graph $\tilde{G}(\mathcal{P})$ is the graph $G_{\mathcal{P}}$ without the root vertex, the number of rooted loops of length $2k$ on $\tilde{G}(\mathcal{P})$ is equal to the number of rooted loops of length $2k+2$ on $G_{\mathcal{P}}$ that never pass through the vertex root (except at the first and last vertex).
If γ is a rooted loop of length $2k$ on a rooted bipartite graph G, we define an interval partition I_γ of $2k$ by saying that $i \sim_{I_\gamma} j$ if and only if the walk between i and j (i and j excluded) doesn't pass through the root: note that the blocks of I are necessarily even. For I an interval partition of $2k$ with even blocks, the number of rooted loops γ with $I_\gamma = I$ is exactly $\prod_{B \in I} \tilde{M}_{|I|/2}(G)$, where $\tilde{M}_k(G)$ is the number of rooted loops of length $2k$ that only pass through the root vertex at the first and last point. Thus,

$$M_k(G) = \sum_{I \in \mathcal{I}(k)} \prod_{B \in I} \tilde{M}_{|B|}(G),$$

where $\mathcal{I}(k)$ denotes the set of interval partitions of k. Therefore, $(\tilde{M}_k(G))_{k\geq 1}$ is the sequence of Boolean cumulants associated to $(M_k(G)) \geq k \geq 1$.

Denote by $b_k(\mu)$ the $k-$th boolean cumulant of a distribution μ. The formula (7.3.1) is thus

$$m_k(\mu(\mathcal{P} * \mathcal{Q})) = \sum_{p\in NC(k)} m_{|B_1|}(\mu_{\mathcal{P}}) \prod_{B\in p, 1\notin B} b_{|B|}(\mu_{\mathcal{P}}) \prod_{B\in Kr(p)} b_{|B|}(\mu_{\mathcal{Q}}). \tag{7.3.2}$$

Let $I = (I_1, \ldots, I_r)$ be an interval partition with $I_1 = [\![1, i_1]\!], \ldots, I_r = [\![i_{r-1}+1, k]\!]$. The descent set $D(I)$ of I is the set $\{i_1+1, \ldots, i_{r-1}+1\}$.

Lemma 7.48. *Let $I = (I_1, \ldots, I_r)$ be an interval partition of k. There is a bijection φ from the set $\{p \in NC(k), D(I) \subseteq B_1(p)\}$ to $\prod_j NC(|I_j|)$. such that φ preserves the size of the block not containing 1 and the Kreweras complement in the following sense: if $\varphi(p) = (\pi_1, \ldots, \pi_r)$, any block of π_j not containing $i_{j-1}+1$ is a block of π, and*

$$kr(p) = (kr(\pi_1), I_1) \vee (kr(\pi_2), I_2) \cdots \vee (kr(\pi_r), I_r).$$

Proof. Let $\varphi(p) = (p_{|I_1}, \ldots, p_{|I_r})$. φ preserves the size of the blocks not containing 1 in the sense of the statement of the lemma. Let π_I be the partition $(\mathbf{1}_{|D(I)|+1}, D(I)\cup\{1\})\vee(\mathbf{0}_{k-|D(I)|-1}, [\![1,k]\!]\setminus (D(I)\cup\{1\}))$: namely, π_I is the partition with only singletons, except one block $D(I)\cup\{1\}$. We define φ^{-1} by the formula

$$\varphi^{-1}(\pi_1, \ldots, \pi_r) = ((\pi_1, I_1) \vee (\pi_2, I_2) \cdots \vee (\pi_r, I_r)) \vee \pi_I.$$

Note that $\varphi^{-1}((\pi_1, \ldots, pi_r))$ is a non-crossing partition such that $D(I) \subseteq B_1$. Since each element of $D(I)\cup\{1\}$ is in a different interval of I, $\varphi \circ \varphi^{-1} = Id$.
Let i and j be in the same block I_s of I. Then $i \sim_{\varphi^{-1}\circ\varphi(p)} j$ if and only if $i \sim_{p_{I_s}} j$ (which is equivalent to $i \sim_p j$). If i and j are in different blocks, then $i \sim_{\varphi^{-1}\circ\varphi(p)} j$ if and only if $i, j \in D(I)\cup\{1\} \subseteq B_1$ (which is again equivalent to $i \sim_p j$).
Since $D(I) \subseteq B_1(p)$, $kr(\pi) \leq I$, which gives the result. □

The proof of Theorem 7.31 is based on the rewriting of Equation (7.3.2) with Lemma 7.48:

Proof of Theorem 7.31. Applying the Boolean moment-cumulant formula to $m_{|B_1|}(\mu_{\mathcal{P}})$ in (7.3.2) yields

$$m_k(\mu(\mathcal{P} * \mathcal{Q})) = \sum_{p\in NC(k)} \sum_{I\in\mathcal{I}(|B_1|)} b_I(\mu_{\mathcal{P}}) \prod_{B\in p, 1\notin B} b_{|B|}(\mu_{\mathcal{P}}) \prod_{B\in Kr(p)} b_{|B|}(\mu_{\mathcal{Q}}).$$

Let p be a partition and B_1 the block containing 1. Restricting an interval partition to B_1 yields a bijection ψ between interval partitions of k such that $D(I) \subseteq B_1$ and interval partitions of B_1. Moreover if $I = (I_1, \ldots, I_r)$ is an interval partition such that $D(I) \subseteq B_1$, the blocks of $\psi(I)$ are exactly the set $B_1 \cap I_j$ for $1 \leq j \leq r$. Thus

$$m_k(\mu(\mathcal{P} * \mathcal{Q})) = \sum_{p\in NC(k)} \sum_{I\in\mathcal{I}(k), D(I)\subseteq B_1} \left(\prod_{1\leq j\leq r} b_{|B_1\cap I_j|}(\mu_{\mathcal{P}}) \prod_{B\in p, B\neq B_1} b_{|B|}(\mu_{\mathcal{P}}) \right) b_{Kr(p)}(\mu_{\mathcal{Q}}),$$

where we use the notation $b_\pi = \prod_{B\in\pi} b_{|B|}$ for $\pi = Kr(p)$. Inverting the sums and using Lemma

7.48 yields

$$
\begin{aligned}
m_k(\mu(\mathcal{P} * \mathcal{Q}) &= \sum_{\substack{I \in \mathcal{I}(k) \\ I=(I_1,\dots,I_r)}} \sum_{\substack{p \in NC(k) \\ D(I) \subseteq B_1(p)}} \left(\prod_{1 \le j \le r} b_{|B_1 \cap I_j|}(\mu_{\mathcal{P}}) \prod_{B \in p, B \neq B_1} b_{|B \cap I_j|}(\mu_{\mathcal{P}}) \right) b_{kr(p)}(\mu_{\mathcal{Q}}) \\
&= \sum_{\substack{I \in \mathcal{I}(k) \\ I=(I_1,\dots,I_r)}} \sum_{(p_1,\dots,p_r) \in \prod NC(|I_j|)} \prod_{1 \le j \le r} b_{p_j}(\mu_{\mathcal{P}}) b_{kr(p_j)}(\mu_{\mathcal{Q}}). \\
&= \sum_{\substack{I \in \mathcal{I}(k) \\ I=(I_1,\dots,I_r)}} \prod_{1 \le j \le r} \left(\sum_{p \in NC(|I_j|)} (b_p(\mu_{\mathcal{P}}) b_{kr(p)}(\mu_{\mathcal{Q}} \right).
\end{aligned}
$$

Therefore, the Boolean cumulants of $\mu(\mathcal{P} * \mathcal{Q})$ are:

$$
b_k = \sum_{p \in NC(k)} (b_p(\mu_{\mathcal{P}}) b_{kr(p)}(\mu_{\mathcal{Q}})), \tag{7.3.3}
$$

for $k \geq 1$.

Let $\mathbb{B}$ be the Boolean Bercovici-Pata bijection (see [19]) from Boolean infinite divisible distributions to free infinite divisible distributions. $\mathbb{B}$ maps in particular a law μ having Boolean cumulants $(c_k)_{k \geq 1}$ to a law ν having free cumulants $(c_k)_{k \geq 1}$. Equation (7.3.3) together with the cumulant formula of the free multiplicative convolution (see [66]) yields

$$
\mathbb{B}(\mu(\mathcal{P} * \mathcal{Q})) = \mathbb{B}(\mu_{\mathcal{P}}) \boxtimes \mathbb{B}(\mu_{\mathcal{Q}}).
$$

But from [17], $\mathbb{B}$ is a semi-group homomorphism with respect to the free multiplicative convolution. Thus, $\mathbb{B}(\mu_{\mathcal{P}}) \boxtimes \mathbb{B}(\mu_{\mathcal{Q}}) = \mathbb{B}(\mu_{\mathcal{P}} \boxtimes \mu_{\mathcal{Q}})$, and applying $\mathbb{B}^{-1}$ yields

$$
\mu(\mathcal{P} * \mathcal{Q}) = \mu_{\mathcal{P}} \boxtimes \mu_{\mathcal{Q}}.
$$

□

Part IV

The multiplicative graph $\mathcal{Z}$ of quasisymmetric functions

Chapter 8

Combinatorics of large compositions

8.1 Introduction

A descent of a permutation σ of $n \in \mathbb{N}^*$ is an integer i such that $\sigma(i) > \sigma(i+1)$. For each permutation σ, the corresponding descent set $D(\sigma)$ is the set of all the descents of σ. Since descents can be located everywhere except on n, a descent set is just a subset of $\{1, \ldots, n-1\}$. Let us call a composition of n the data of n and a subset of $\{1, \ldots, n-1\}$. A composition D is represented by a ribbon Young diagram λ_D of n cells labelled 1 to n by the following rule : cells i and $i+1$ are neighbors and the cell $i+1$ is right to i if $i \notin D$, below i otherwise. Therefore, the descent set of a permutation σ is D if and only if inserting $\sigma(i)$ in each cell i of λ_D yields a standard ribbon Young tableau. For example, the composition $D = \{10, (3, 5, 9)\}$ gives the following ribbon Young diagram:

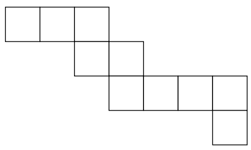


Figure 8.1: Ribbon Young diagram λ_D of to the composition $D = \{10, (3, 5, 9)\}$

The permutation $\sigma = (3, 5, 8, 4, 7, 1, 6, 9, 10, 2)$ has the descent set D since the associated filling of λ_D yields a ribbon Young tableau, as shown in figure 8.2.

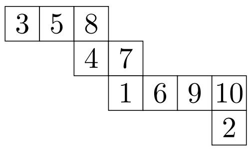


Figure 8.2: Standard filling of the composition $(3, 2, 4, 1)$

The descent statistic of a composition D is the number of standard fillings of the associated ribbon Young tableau λ_D (or, equivalently, the number of permutations having D as descent set). This latter number, denoted by $\beta(D)$, has been intensively studied in the last decades (see Viennot [87] and [88] , Niven [67], de Bruijn [31] , ...). Two main questions arose in this study: the first one is to find the compositions of n having a maximum descent statistic, and the second one is to find exact or asymptotic formulae for the descent statistic of large compositions having a given shape. For example, Niven and de Bruijn proved in [67] and [31] that

the two compositions of n maximizing the descent statistic are $D_1(n) = \{1, 3, 5, \dots\} \cap [1, n]$ and $D_2(n) = \{2, 4, 6, \dots, \} \cap [1, n]$: permutations having such descent sets are called alternating permutations. Désiré André already gave in [1] an asymptotic formula for the number of alternating permutations by showing that $\beta(D_1)(n) \sim 2(2/\pi)^n n!$ as n goes to infinity.
In order to evaluate the descent statistic of a broad class of compositions, Ehrenborg, Levin and Readdy formalized in [37] a probabilistic approach to the counting problem, by relating each permutation of n to a particular simplex of $[0, 1]^n$. Since the Lebesgue measure yields a probability measure on $[0, 1]^n$, it is possible to use probabilistic tools to get interesting results on descent statistics. Ehrenborg obtained in [36] an asymptotic formula for the descent statistics of the so-called nearly periodic permutations: the latter consist in permutations having the same descent pattern repeated several times, with some local perturbations. As for alternating permutations, the asympotic formula has the shape $K\lambda^n n!$, with K and λ being some constants depending on the situation. Using the approach of [37] with functional analysis tools, Bender, Helton and Richmond extended in [18] the previous results to a broader class of descent sets, and they found asymptotic formulae of the same shape as before.
The factorial term of the asymptotic formula is easy to understand, since it comes from the cardinality of the set of permutations of n elements. However, the term λ^n seems more mysterious. In [18], the authors identified in their examples the phenomenon that makes the term λ^n appear: namely, if we consider a large uniform random permutation with a fixed descent set, then the value of $\sigma(1)$ and $\sigma(n)$ are nearly independent, which causes a factorization in the asymptotic counting. Thus, the natural question is to know which compositions induce this phenomenon; it has been conjectured in [18] that every composition have this property as they become large.
In the present chapter we construct a family of probabilistic models, called sawtooth models, which extend the probabilistic approach of Ehrenborg, Readdy and Levin. These models are more general than the ones used in [18], but the combinatorial properties of the large descent sets appear more clearly in this broader case; thus, we first study these models in their full generality, before deducing some specific results on descent sets. A main consequence of the latter work is an affirmative answer to Conjecture 1 on asymptotic independence from Bender, Helton and Richmond ([18]). We are also able to give by the following intuitive result on compositions: *In the random filling of a composition, the contents of two distant cells are almost independent.*
In the next chapter, we will use the results of this chapter to study an analog of the Young lattice that was introduced by Gnedin and Olshanski in [42].

8.2 Preliminaries and results

8.2.1 Compositions

This paragraph gives definitions and notations concerning compositions.

Definition 8.1. *Let $n \in \mathbb{N}$. A composition λ of n is a sequence of positive integers $(\lambda_1, \dots, \lambda_r)$ such that $\sum \lambda_j = n$.*

A unique ribbon Young diagram with n cells is associated to each composition: each row j has λ_j cells, and the first cell of the row $j + 1$ is just below the last cell of the row j. For example the composition of 10, $(3, 2, 4, 1)$ is represented as in figure 8.1. This picture shows directly the link between Definition 8.1 and the definition we stated in the introduction : a composition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n yields a subset D_λ of $\{1, \dots, n-1\}$, namely the subset $\{\lambda_1, \lambda_1 + \lambda_2, \dots, \lambda_1 + \dots + \lambda_{r-1}\}$. The latter correspondence is clearly bijective.
The size $|\lambda|$ of a composition is the sum of the λ_j. When nothing is specified, λ will always be assumed to have the size n, and n will always denote the size of the composition λ.

A standard filling of a composition λ of size n is a standard filling of the associated ribbon Young diagram: this is an assignement of a number between 1 and n for each cell of the composition, such that every cells have different entries, and the entries are increasing to the right along the rows and decreasing to the bottom along the columns. An example for the composition of figure 8.1 is shown in figure 8.2.
In particular, reading the tableau from left to right and from top to bottom associates a permutation σ to each standard filling; moreover, the descent set of such a permutation σ, namely the set of indices i such that $\sigma(i+1) < \sigma(i)$, is exactly the set

$$D_\lambda = \{\lambda_1, \lambda_1 + \lambda_2, \ldots, \sum_1^{r-1} \lambda_i\}.$$

There is a bijection between the standard fillings of λ and the permutations of $|\lambda|$ with descent set D_λ. For example the filling in figure 8.2 yields the permutation $(3, 5, 8, 4, 7, 1, 6, 9, 10, 2)$.

8.2.2 Result on asymptotic independence

We present here the main results that are proven in the present chapter.

Notation 8.2. *Let λ be a composition. Let Σ_λ denote the set of all permutations with descent set D_λ. With the uniform counting measure $\mathbb{P}_\lambda$, it becomes a probability space, and σ_λ denotes the random permutation coming from this probability space. As usual $|\Sigma_\lambda|$ is the cardinality of the set Σ_λ.*

$|\Sigma_\lambda|$ is thus the descent statistic associated to the composition λ.
Denote for each random variable X by $\mu(X)$ its law and by d_X its density, and write $\mu \otimes \nu$ for the independent product of two laws. The goal of the chapter is to prove that distant cells in a composition have independent entries, namely:

Theorem 8.3. *Let $\varepsilon, r \in \mathbb{N}$. Then there exists $k \geq 0$ such that if λ is a composition of n and $0 < i_1 < \cdots < i_r \leq n$ are indices with $i_{j+1} - i_j \geq k$,*

$$\pi\left(\mu(\frac{\sigma_\lambda(i_1)}{n}, \ldots, \frac{\sigma_\lambda(i_r)}{n}), \mu(\frac{\sigma(i_1)}{n}) \otimes \cdots \otimes \mu(\frac{\sigma(i_r)}{n})\right) \leq \varepsilon,$$

with π denoting the Levy-Prokhorov metric on the set of measures of $[0,1]^r$.

If the variance of $\frac{\sigma_\lambda(i_1)}{n}$ and $\frac{\sigma_\lambda(i_n)}{n}$ remain bounded from below by a positive constant, then the approximate independence of $\frac{\sigma_\lambda(i_1)}{n}$ and $\frac{\sigma_\lambda(i_n)}{n}$ can be given with a stronger metric than the Levy-Prokhorov metric. This is the content of Conjecture 1 of [18], which is proven in this chapter and formulated in Theorem 8.36.

8.2.3 Runs of a composition

Let λ be a composition. We number the cells as we read them, from left to right and from top to bottom . The cells are identified with integers from 1 to n through this numbering. For example in the standard filling of figure (8.2), the number 7 is in the cell 5.
We call run any set consisting in all the cells of a given column or row. The set of runs is ordered with the lexicographical order. In the same example as before the runs are

$$s_1 = (1, 2, 3), s_2 = (3, 4), s_3 = (4, 5), s_4 = (5, 6), s_5 = (6, 7, 8, 9), s_6 = (9, 10),$$

where we put in the parenthesis the cells of each run.
Note that inside each run the cells are ordered by the natural order on integers. We call extreme cell a cell that is an extremum in a run with respect to this order, and denote by $\mathcal{E}_\lambda$ the set of extreme cells of λ. Apart from the first and last cells of the composition, each extreme cell belongs to two consecutive runs. Let P_λ be the set of extreme cells followed by a column, or preceeded by a row and V_λ the set of extreme cells followed by a row or preceeded by a column. The elements of P_λ are called peaks and the ones of V_λ valleys. The sets V_λ and P_λ are also ordered with the natural order:

$$P_\lambda = \{p_1 < \cdots < p_k\}, V_\lambda = \{q_1 < \cdots < q_{k'}\},$$

with $k - 1 \leq k' \leq k + 1$.
The first and last cells are always extreme points. A composition is said being of type ++ (resp. +-,-+,- -) if the first cell is a peak and the last cell is a peak (resp peak-valley, valley-peak, valley-valley).
Finally, let $l(s)$, the length of a run s, be the cardinality of s, and $L(\lambda)$, the amplitude of λ, be the supremum of all lengths $l(s)$.

8.2.4 The coupling method

In this paragraph we introduce a probabilistic tool called the coupling method, and set the relative notations for the sequel. We refer to [58] for a review on the subject. We will present the notions in the framework of random variables but we could have done the same with probability laws as well.

Definition 8.4. *Let $(E, \mathcal{E})$ be a probability space and X, Y two random variables on E. A coupling of (X, Y) is a random variable (Z_1, Z_2) on $(E \times E, \mathcal{E} \otimes, \mathcal{E})$ such that*

$$Z_1 \sim_{law} X, Z_2 \sim_{law} Y.$$

Such a coupling always exists : it suffices to consider two independent random variables Z_1 and Z_2 with respective law μ_X and μ_Y. However, a coupling is often useful precisely when the resulting random variables Z_1 and Z_2 are far from being independent. In particular, in this chapter we are mainly interested in the case where Z_1 and Z_2 respect a certain order on the set E. From now on E is a Polish space considered with its Borel $\sigma-$algebra $\mathcal{E}$, and $\lhd$ is a partial order on E such that the graph $\mathcal{G} = \{(x, y), x \lhd y\}$ is $\mathcal{E}-$measurable.

Definition 8.5. *Let X, Y be two random variables on E. Y stochastically dominates X (denoted $Y \succeq X$) if and only if*

$$\mathbb{P}(X \in A) \leq \mathbb{P}(Y \in A)$$

for any Borel set A such that

$$x \in A \Rightarrow \{y \in E, x \lhd y\} \subseteq A.$$

For example if $E = \mathbb{R}$ with the canonical order $\leq$ and $\sigma-$algebra $\mathcal{B}(\mathbb{R})$, then Y stochastically dominates X if and only if for all $x \in \mathbb{R}$,

$$\mathbb{P}(X \in [x, +\infty[) \leq \mathbb{P}(Y \in [x, +\infty[)$$

or equivalently, if we denote their respective cumulative distribution function by $F_X(t)$ and $F_Y(t)$:

$$F_Y(t) \leq F_X(t) \quad \text{for all } t \in \mathbb{R}.$$

There are several ways to characterize the stochastic dominance:

Proposition 8.6. *The three following statements are equivalent :*

- *Y stochastically dominates X*
- *there exists a coupling (Z_1, Z_2) of X, Y such that $Z_1 \lhd Z_2$ almost surely.*
- *for any positive measurable bounded function f that is non-decreasing with respect to $\lhd$,*

$$\mathbb{E}(f(X)) \leq \mathbb{E}(f(Y))$$

The proof is straightforward and can be found in [58]. This yields the following intuitive Lemma :

Lemma 8.7. *Let (X_1, X_2, Y_1, Y_2) be a random variable on E^4 such that :*

- *$X_1 \preceq Y_1$ and $Y_2 \preceq X_2$,*
- *(X_1, Y_1) is independent from (X_2, Y_2).*

Then

$$\mathbb{P}(X_1 \lhd X_2) \geq \mathbb{P}(Y_1 \lhd Y_2).$$

Proof. Let $\ll$ be the partial order on $E \times E$ defined by

$$(x, y) \ll (x', y') \leftrightarrow x \lhd x' \text{ and } y' \lhd y.$$

Since $Y_1 \succeq X_1$ and $X_2 \succeq Y_2$, there exists a coupling $(\hat{X}_1, \hat{Y}_1)$ (resp. $(\hat{X}_2, \hat{Y}_2)$) of X_1, Y_1 (resp. X_2, Y_2) such that almost surely $\hat{X}_1 \lhd \hat{Y}_1$ (resp $\hat{X}_2 \rhd \hat{Y}_2$). The random variables $(\hat{X}_1, \hat{Y}_1)$ and $(\hat{X}_2, \hat{Y}_2)$ can be chosen independent one from each other. Since (X_1, Y_1) and (X_2, Y_2) are also independent, this implies that $((\hat{X}_1, \hat{X}_2), (\hat{Y}_1, \hat{Y}_2))$ is a coupling of $((X_1, X_2), (Y_1, Y_2))$ with almost surely

$$(\hat{X}_1, \hat{X}_2) \ll (\hat{Y}_1, \hat{Y}_2).$$

But if $\hat{Y}_1 \lhd \hat{Y}_2$, then $\hat{X}_1 \lhd \hat{Y}_1 \lhd \hat{Y}_2 \lhd \hat{X}_1$ and thus

$$\mathbb{P}(Y_1 \lhd Y_2) = \mathbb{P}(\hat{Y}_1 \lhd \hat{Y}_2) \leq \mathbb{P}(\hat{X}_1 \lhd \hat{X}_2) = \mathbb{P}(X_1 \lhd X_2).$$

□

These results will be concretely applied on $\mathbb{R}^p, p \geq 1$, and thus we need to define a family of partial orders on those sets.

Definition 8.8. *Let $p \geq 1$. The partial order $\leq$ on $\mathbb{R}^p$ is the natural order on $\mathbb{R}$ for $p = 1$, and for $p \geq 2$ if $(x_i)_{1 \leq i \leq p}, (y_i)_{1 \leq i \leq p} \in \mathbb{R}^p$,*

$$(x_i)_{1 \leq i \leq p} \leq (y_i)_{1 \leq i \leq p} \Leftrightarrow \forall i \in [1; p], x_i \leq y_i.$$

For any word ε of length p in $\{+1, -1\}$ (or simply in $\{+, -\}$), the modified partial order $\leq_\varepsilon$ is defined as

$$(x_i)_{1 \leq i \leq p} \leq_\varepsilon (y_i)_{1 \leq i \leq p} \Leftrightarrow \forall i \in [1; p], \varepsilon_i x_i \leq \varepsilon_i y_i.$$

The easiest way to check the stochastic dominance is to look at the cumulative distribution function. The proof of the following Lemma is a direct application of Proposition 8.6.

Lemma 8.9. *Let $(X_i)_{1\leq i\leq p}$ and $(Y_i)_{1\leq i\leq p}$ be two random variables on $(\mathbb{R}^p, \leq_\varepsilon)$. Then $(Y_i)_{1\leq i\leq p}$ stochastically dominates $(X_i)_{1\leq i\leq p}$ if and only if for all $(t_i)_{1\leq i\leq p} \in \mathbb{R}^p$,*

$$F^{\varepsilon}_{(X_i)}(t_1, \dots, t_p) \geq F^{\varepsilon}_{(Y_i)}(t_1, \dots, t_p),$$

with $F^{\varepsilon}_{(X_i)}$ being the modified cumulative distribution function defined by

$$F^{\varepsilon}_{(X_i)}(t_1, \dots, t_p) = \mathbb{P}((X_i) \leq_\varepsilon (t_i)).$$

The stochastic dominance in the case $(\mathbb{R}^p, \leq_\varepsilon)$ is denoted as $(X_1, \dots, X_p) \preceq_\varepsilon (Y_1, \dots, Y_p)$. A consequence of the previous result is that if $(Y_1, \dots, Y_p)$ stochastically dominates $(X_1, \dots, X_p)$, then for all subsets $I = (i_1, \dots, i_r)$ of $\{1, \dots, p\}$, $(Y_{i_1}, \dots, Y_{i_r})$ also stochastically dominates $(X_{i_1}, \dots, X_{i_r})$.
Applying Lemma 8.9 to the case $p = 2$ yields the following Lemma:

Lemma 8.10. *Let $(U_1, V_1), (U_2, V_2)$ be two random variables on $[0, 1]$ such that U_2 and V_2 are independent. Suppose that for all $0 \leq t \leq 1$,*

$$F_{V_1}(t) \leq F_{V_2}(t)$$

and for all $v \in [0, 1]$,

$$F_{U_1|V_1=v}(t) \leq F_{U_2}(t).$$

There exists a coupling $((Z_1, \tilde{Z}_1), (Z_2, \tilde{Z}_2))$ of (U_1, V_1) and (U_2, V_2) such that almost surely

$$(Z_1, \tilde{Z}_1) \geq (Z_2, \tilde{Z}_2).$$

8.3 Sawtooth model

8.3.1 Definition of the model

In this section we introduce a statistical model of particles in a tube, which is a generalization of the probabilistic approach of Ehrenborg, Levin and Readdy in [37]. The model consists in a sequence of particles, each of them moving vertically in an horizontal two-dimensional tube. Each particle has a repulsive action on the two neighbouring particles, and moreover, the set of particles splits into two groups: the upper particles and the lower particles. The upper particles are always above the lower ones. The model is depicted in Figure 8.3.

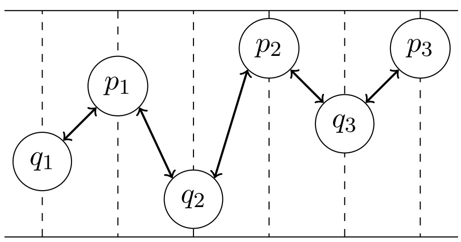


Figure 8.3: Upper particles $\{p_1, p_2, p_3\}$ and lower particles $\{q_1, q_2, q_3\}$ in a tube.

Such a system is called a Sawtooth model in the sequel.

Remark 8.11. *If there are k upper-particles, there must be k' lower particles with $k' \in \{k-1, k, k+1\}$, depending on the type of the first and the last particles. We define therefore the type $\varepsilon(\mathcal{S})$ of the model $\mathcal{S}$ as the word $\varepsilon_I\varepsilon_F$, with $\varepsilon_I = +$ (resp. $\varepsilon_F = +$) if the first (resp. last) particle is an upper one, and $\varepsilon_I = -$ (resp. $\varepsilon_F = -$) otherwise.*

Unless specified otherwise, the first particle is a lower particle (as in the picture). The particles are ordered from the left, and following this order the upper particles are written $\{p_1 < p_2 < \cdots < p_k\}$ and the lower particles $\{q_1 < \cdots < q_{k'}\}$. Since the nature of our results won't depend on the type of the model, we will also assume that there are $k+1$ lower particles, yielding that the last particle is a lower one too.
Denote by x_i the position of q_i and by y_i the position of p_i: by a rescaling, we can assume that $x_i, y_i \in [0,1]$. These positions are considered as random, and each configuration of positions is weighted according to repulsive interactions between neighbouring particles. This yields the following definition:

Definition 8.12. *A Sawtooth model $\mathcal{S}$ is the union of two families of random variables $\{X_i\}_{1\leq i\leq k+1}$ and $\{Y_j\}_{1\leq j\leq k}$ on $[0,1]$ with the multivariate density*

$$\mathbb{P}(\{X_i = x_i, Y_j = y_j\}) = \frac{1}{\mathcal{V}} \prod \mathbf{1}_{x_i \leq y_i \geq x_{i+1}} f_i(y_i - x_i) g_i(y_i - x_{i+1}) \prod dx_i \prod dy_j, \tag{8.3.1}$$

where $\{f_i, g_i\}_{1\leq i\leq k}$ is a family of increasing positive C^1 functions on $[0,1]$.
The quantity $\mathcal{V}$ is called the volume of $\mathcal{S}$ and is sometimes denoted by $\mathcal{V}(\mathcal{S})$ to avoid confusion. $\mathcal{S}$ is said normalized if $\int f_i = \int g_i = 1$ for $1 \leq i \leq k$.

The volume has the following expression:

$$\mathcal{V}(\mathcal{S}) = \int_{[0,1]^{2k+1}} \prod \mathbf{1}_{x_i \leq y_i \geq x_{i+1}} f_i(y_i - x_i) g_i(y_i - x_{i+1}) \prod dx_i dy_i. \tag{8.3.2}$$

In particular, an appropriate rescaling of the functions f_i, g_i can transform any Sawtooth model into a normalized one, without changing the probability space. Thus, from now on and unless stated otherwise, the model is assumed normalized. In case we are considering non-normalized models, we will use the notation f_i, g_i, etc. for the normalized quantities, and $\tilde{f}_i, \tilde{g}_i, etc.$ for the non-normalized ones.
Aiming the results we stated on compositions, we should answer these questions :

1. *As the number of particles goes to infinity, is there some independence between X_1 and X_{k+1} ?*
2. *It is possible to estimate the behavior of a particle X_r by only considering its neighbouring particles ?*

For each subset of particles $\Omega = (q_{i_1}, \ldots, q_{i_r}, p_{j_1}, \ldots, p_{j_{r'}})$ and measurable event $\mathcal{X}$, denote by

$$d_{\Omega|\mathcal{X}}(x_{i_1}, \ldots, x_{i_r}, y_{j_1}, \ldots, y_{j_{r'}})$$

the marginal density of Ω conditioned on $\mathcal{X}$. The subscripts will be dropped when there is no confusion, and we denote by X_I the first variable X_1 and X_F the last variable X_{k+1}. Finally, since the system is fully described by the functions $\{f_i, g_j\}$, we will refer sometimes to a particular system just by mentioning this set of functions.
The definition of a Sawtooth model yields directly two first results which are given in Lemma 8.13 and Lemma 8.14. The first one stresses the Markovian aspect of a Sawtooth model :

Lemma 8.13. *Let $\mathcal{S}$ be a Sawtooth model of size k, and $1 \leq i \leq k$. Let Z be the position of a particle right to X_i (namely $Z = X_j$ for $j > i$ or $Z = Y_j$ for $j \geq i$) and $\mathcal{X}$ be an event depending on the positions of particles right to Z. Then for $0 \leq z \leq 1$,*

$$d_{X_i|Z=z,\mathcal{X}} = d_{X_i|Z=z}.$$

Proof. It suffices to prove that the particles left to Z are independent of the particles right to Z conditionally on the value of Z. This is implied by the form of the density of the model, since the latter splits between the density of the particles left to Z and the ones right to Z. $\square$

The second one is a generalization of Lemma 3-(a) in [18]. :

Lemma 8.14. *Let $1 \leq r \leq k+1$, and let $\mathcal{X}$ be an event depending on the position of all particles except X_r. Then $d_{X_r|\mathcal{X}}(x_r)$ is decreasing in x_r.*

Proof. Let a be in $[0,1]$. By Lemma 8.13,

$$\begin{aligned} d_{X_r|\mathcal{X}}(a) &= \int_{[0,1]^2} d_{(X_r|\mathcal{X})|Y_{r-1}=z, Y_{r+1}=z'}(a) d_{Y_{r-1},Y_{r+1}|\mathcal{X}}(z,z')dzdz' \\ &= \int_{[0,1]^2} d_{X_r|Y_{r-1}=z, Y_{r+1}=z'}(a) d_{Y_{r-1},Y_{r+1}|\mathcal{X}}(z,z')dzdz'. \end{aligned}$$

Thus, it is enough to prove the monotonicity in the case of a conditioning on $Y_{r-1} = z, Y_{r+1} = z'$. In this case

$$d_{X_r|Y_{r-1}=z,Y_{r+1}=z'}(a) = \mathbf{1}_{z \geq a, z' \geq a} \frac{1}{R}(g_{r-1}(z-a) f_r(z'-a)),$$

with R a normalizing constant. Since g_{r-1} and f_r are increasing, this concludes the proof. $\square$

The same result holds for upper particles, but in this case the density is increasing.

8.3.2 The processes $\mathcal{S}_\lambda$ and Σ_λ

Let us see how these definitions fit into the framework of compositions. The main idea from [37] is to consider the set of all permutations with a given descent set D_λ as a probability space. $|\Sigma_\lambda|$ can indeed be related to the volume of a polytope in $[0,1]^n$ (see for example the survey of Stanley on alternating permutations, [78]) . For each sequence of distinct elements $\vec{z} = (z_1, \dots, z_n)$ in $[0,1]$, the ranking permutation of $\vec{z}$ is the permutation $\sigma(\vec{z})$ that assigns to each j the position of z_j in the ordered sequence $(z_{i_1} < \cdots < z_{i_n})$: namely, $\sigma(\vec{z})(j) = k$ if and only if $\#\{1 \leq i \leq n | z_i \leq z_j\} = k$.

Proposition 8.15 ([37])**.** *The law of σ_λ is the law of the ranking permutation for a sequence of independent uniform variables $Z_1, \dots, Z_n$ in $[0,1]$ conditioned on the event*

$$\{Z_i > Z_{i+1} \text{ if and only if } i \in D_\lambda\}.$$

In particular, the following expression of the number of permutations with descent set D_λ holds :

$$|\Sigma_\lambda| = n! \int_{[0,1]^n} \prod_{i \in D_\lambda} \mathbf{1}_{z_i \geq z_{i+1}} \prod_{i \notin D_\lambda} \mathbf{1}_{z_i \leq z_{i+1}} \prod dz_i,$$

with $z_{n+1} = 1$.

The proof of the latter proposition is straightforward as soon as we remark that the volume of the polytope $\{0 \leq z_1, \dots, z_n \leq 1\}$ is exactly $\frac{1}{n!}$. The processus $\{Z_i\}_{1 \leq i \leq n}$ in the previous proposition is denoted by $\tilde{\mathcal{S}}_\lambda$. Since the indicator function in the integrand depends on conditions between neighbouring points, this result can be rephrased in terms of Sawtooth model. Regrouping the inequalities between elements of the same run of λ yields:

$$|\Sigma_\lambda| = n! \int_{[0,1]^n} \mathbf{1}_{z_1 \leq z_2 \leq \cdots \leq z_{i_1}} \mathbf{1}_{z_{i_1} \geq z_{i_1+1} \geq \cdots \geq z_{i_1+i_2}} \dots \mathbf{1}_{z_{n-i_{2r}} \leq \cdots \leq z_n} \prod dz_i, \tag{8.3.3}$$

and by integrating over all the coordinates that do not correspond to extreme cells, we get

$$\begin{aligned}|\Sigma_\lambda| =& n! \int_{[0,1]^n} \mathbf{1}_{x_1^- \le x_1^+ \ge x_2^- \le \dots} \frac{1}{(l(s_1)-2)!} |x_1^+ - x_1^-|^{l(s_1)-2} \\ & \frac{1}{(l(s_2)-2)!} |x_1^+ - x_2^-|^{l(s_2)-2} \dots \frac{1}{(l(s_{2r})-2)!} |x_k^+ - x_{k+1}^-|^{l(s_k)-2} \prod_{i=1}^{k} dx_i^+ \prod_{i=1}^{k+1} dx_i^- .\end{aligned}$$

Let $\mathcal{S}_\lambda$ be the non-normalized Sawtooth model with the non-normalized density functions $\{\tilde{f}_j, \tilde{g}_j\}_{1 \le i \le r}$ such that

$$\tilde{f}_j(t) = \frac{1}{(l(s_{2j-1})-2)!} t^{l(s_{2j-1})-2}, \tilde{g}_j(t) = \frac{1}{(l(s_{2j})-2)!} t^{l(s_{2j})-2}.$$

A comparison between the latter expression of $|\Sigma_\lambda|$ and the expression (8.3.2) of the volume of a Sawtooth model gives

$$|\Sigma_\lambda| = |\lambda|! \mathcal{V}(\mathcal{S}_\lambda)$$

To sum up, three processes are constructed from λ. The first one, σ_λ comes from the uniform random standard filling of the ribbon Young tableau λ, the second one, $\tilde{\mathcal{S}}_\lambda$, comes from the probabilistic approach of [37], and the third one, $\mathcal{S}_\lambda$, is obtained from $\tilde{\mathcal{S}}_\lambda$ by considering only the extreme particles. They are of course intimately related, even if the first one is discrete and the second and third ones are continuous. σ_λ can be recovered from $\tilde{\mathcal{S}}_\lambda$ by the associated ranking permutation, and when $|\lambda|$ goes to infinity $\frac{\sigma_\lambda(i)}{n}$ and Z_i are approximately the same :

Lemma 8.16. *The following inequality always holds for $0 < A, n \in \mathbb{N}$:*

$$\mathbb{P}(\max(|\frac{\sigma_\lambda(i)}{n+1} - Z_i| > \frac{A}{\sqrt{n+2}}) \le \frac{1}{A^2}$$

In particular, if the densities of Z_i remains bounded by a constant B,

$$\|F_{Z_i} - F_{\frac{\sigma(i)}{n}}\|_\infty \to_{|\lambda| \to +\infty} 0.$$

Proof. Let us evaluate $\mathbb{P}(|\frac{\sigma_\lambda(i)}{n+1} - Z_i| > \frac{A}{n+2})$. Condition the event $\{|\frac{\sigma_\lambda(i)}{n+1} - Z_i| > \frac{A}{n+2}\}$ on a particular realization σ of σ_λ, and suppose that $\sigma(i) = k$. In this case, the conditional density of Z_i is :

$$\begin{aligned} d_{Z_i|\sigma_\lambda = \sigma}(z_i) =& n! \left(\int_{0 \le z_{\sigma^{-1}(1)} \le \dots \le z_{\sigma^{-1}(k-1)} \le z_i} \prod_{1 \le \sigma(j) \le k-1} dz_j \right) \\ & \left(\int_{z_i \le z_{\sigma^{-1}(k+1)} \le \dots \le z_{\sigma^{-1}(n)} \le 1} \prod_{k+1 \le \sigma(j) \le 1} dz_j \right) \\ =& \frac{n!}{(k-1)!(n-k)!} z_i^{k-1} (1-z_i)^{n-k}. \end{aligned}$$

Computing the conditional expectation yields $\mathbb{E}(Z_i|\sigma_\lambda = \sigma) = \frac{k}{n+1}$ and

$$Var(Z_i|\sigma_\lambda = \sigma) = \left(\frac{k}{n+1} \frac{n+1-k}{n+1} \right) \frac{1}{n+2} \le \frac{1}{n+2}.$$

Thus, by the Chebyshev's inequality,

$$\mathbb{P}_{Z_i|\sigma_\lambda=\sigma}\left(|Z_i - \frac{\sigma(1)}{n+1}| > \frac{A}{\sqrt{n+2}}\right) \leq \frac{1}{A^2}.$$

Integrating this inequality on all the disjoint events σ on which Z_i can be conditioned yields the first part of the Lemma. $\square$

From now on, let $\tilde{\gamma}_r$ denote for $r \geq 2$ the function $\tilde{\gamma}_r(t) = \frac{1}{(r-2)!}t^{r-2}$, and $\gamma_r(t) = (r-1)t^{r-2}$ its normalized density function.

8.4 Convex Sawtooth Model

In this section, we study the behavior of the extreme particles for a Sawtooth model respecting a particular convexity property. The results of this section are much easier to get in the particular case of the Sawtooth models $\mathcal{S}_\lambda$ of the last section, since the density functions $\{f_i, g_i\}$ are explicitly given. We will use this particular Sawtooth models as examples for our more general computations.

8.4.1 Log-concave densities

To be able to get some results on the behavior of the particles, it is necessary to impose some conditions on the density functions $\{f_i, g_i\}$. Actually the condition we need is quite natural from a physical point of view, since we will require that the repulsive forces in the definition of the Sawtooth model come from a convex potential : the consequence is that the density functions should be log-concave. This motivates the following definition :

Definition 8.17. *A Sawtooth model is called convex if all the functions $(f_i, g_i)_{1\leq i\leq k}$ are log-concave. This means that for all $1 \leq i \leq k$, $\frac{f_i'(t)}{f_i(t)}$ and $\frac{g_i'(t)}{g_i(t)}$ are decreasing.*

The main advantage of the log-concavity is that the behavior of the particles becomes monotone in a certain sense.
For $1 \leq s \leq k+1$, let $\mathcal{S}_{\to X_s}$ (resp. $\mathcal{S}_{X_s\leftarrow}$) denote the Sawtooth model obtained by keeping only the particles between X_I and X_s (resp. between X_s and X_F) and the functions $\{f_i, g_i\}_{i\leq s}$ (resp. $\{f_i, g_i\}_{i\geq s+1}$). Likewise, let $\mathcal{S}_{\to Y_s}$ (resp. $\mathcal{S}_{Y_s\leftarrow}$) denote the Sawtooth model obtained by keeping only the particles between X_I and Y_s (resp. between Y_s and X_F) and the functions $\{f_i, g_j\}_{\substack{i\leq s\\ j\leq s-1}}$ (resp. $\{f_i, g_j\}_{\substack{i\geq s+1\\ j\geq s}}$).
In order to emphasize a specific Sawtooth model $\mathcal{S}$, we write $X_i^{\mathcal{S}}$ to denote the particle X_i in $\mathcal{S}$, and $F_{X_i,\mathcal{S}}$ to denote the cumulative distribution function of X_i in $\mathcal{S}$ (and the same for Y_i).

Proposition 8.18. *Let $\{f_i, g_i\}$ be a convex Sawtooth model. Then for $1 \leq s \leq k$, $(X_s|Y_s = y)$ is increasing with y (in terms of stochastic dominance) and $(Y_s|X_{s+1} = x)$ is increasing with x. Moreover,*

$$X_s^{\mathcal{S}_{\to X_s}} \succeq (X_s|Y_s = y), Y_s^{\mathcal{S}_{\to Y_s}} \succeq (Y_s|X_{s+1} = x).$$

Proof. Let $1 \leq s \leq k$. To prove the first part of the proposition, it is enough to show that for $0 \leq t \leq 1$, $F_{X_s|Y_s=y}(t)$ is decreasing in y and $F_{Y_s|X_{s+1}=x}(t)$ is decreasing in x.
Let $d(x)$ be the density of X_s in $\mathcal{S}_{\to X_s}$. Then by the definition of the probability density of $\mathcal{S}$,

the density of X_s in $\mathcal{S}$ conditioned on the value of Y_s is $\mathbf{1}_{x\leq y}\frac{d(x)f_s(y-x)}{A}$, with A a normalizing constant. Thus, the cumulative distribution function $F_y(.)$ of X_s conditioned on $Y_s = y$ is

$$F_y(t) = \frac{\int_0^{t\wedge y} d(x)f_s(y-x)dx}{\int_0^y d(x)f_s(y-x)dx}.$$

For $t > y$ it is clear that $\frac{\partial}{\partial y}F_y(t) = 0$, and from now on we only consider $t \leq y$. Since the logarithm function is increasing, it is enough to show that $\frac{\partial}{\partial y}\log(F_y(t)) \leq 0$. This derivative is equal to

$$\frac{\partial}{\partial y}\log(F_y(t)) = \frac{\int_0^t d(x)f_s'(y-x)dx}{\int_0^t d(x)f_s(y-x)dx} - \frac{\int_0^y d(x)f_s'(y-x)dx}{\int_0^y d(x)f_s(y-x)dx} - \frac{d(y)f_s(0)}{\int_0^y d(x)f_s(y-x)dx}.$$

Since $(-\frac{d(y)f_s(0)}{\int_0^y d(x)f_s(y-x)dx}) \leq 0$, the non-positivity of the remaining part of the sum suffices. Denote

$$\Delta = \int_0^t d(x)f_s'(y-x)dx \int_0^y d(x)f_s(y-x)dx - \int_0^y d(x)f_s'(y-x)dx \int_0^t d(x)f_s(y-x)dx.$$

Thus, we have to show that $\Delta \leq 0$. For $t \leq y$,

$$\begin{aligned}\Delta &= \int_0^t d(x)f_s'(y-x)dx \left(\int_0^t d(x)f_s(y-x)dx + \int_t^y d(x)f_s(y-x)dx\right) \\ &\quad - \left(\int_0^t d(x)f_s'(y-x)dx + \int_t^y d(x)f_s'(y-x)dx\right)\int_0^t d(x)f_s(y-x)dx \\ &= \int_0^t d(x)f_s'(y-x)dx \int_t^y d(x)f_s(y-x)dx \\ &\quad - \int_t^y d(x)f_s'(y-x)dx \int_0^t d(x)f_s(y-x)dx.\end{aligned}$$

Expressing products of integrals as double integrals yields

$$\begin{aligned}\Delta &= \int_{\substack{0\leq z_1\leq t\\ t\leq z_2\leq y}} d(z_1)d(z_2)f_s'(y-z_1)f_s(y-z_2)dz_1dz_2 \\ &\qquad - \int_{\substack{0\leq z_1\leq t\\ t\leq z_2\leq y}} d(z_1)d(z_2)f_s(y-z_1)f_s'(y-z_2)dz_1dz_2 \\ &= \int_{\substack{0\leq z_1\leq t\\ t\leq z_2\leq y}} d(z_1)d(z_2)(f_s'(y-z_1)f_s(y-z_2) - f_s(y-z_1)f_s'(y-z_2))dz_1dz_2.\end{aligned}$$

Since $d(z_1)d(z_2)$ is positive and $\frac{f_s'(t)}{f_s(t)}$ is decreasing, $\Delta \leq 0$ and the first part of the proposition is proven.
The second part of the proposition is equivalent to the inequalities

$$F_{X_s|Y_s=y}(t) \geq F_{X_s,\mathcal{S}_{\to X_s}}(t)$$

and

$$F_{Y_s|X_{s+1}=x}(t) \leq F_{Y_s,\mathcal{S}_{\to Y_s}}(t)$$

for all $0 \le t \le 1$.
From the first part of the Proposition, it suffices to prove the first inequality only for $y = 1$. Since f_s is increasing, there exists a measure μ on $[0,1]$ such that $f_s(x) = \int_0^x d\mu(u)$. Thus,

$$F_1(t) = \frac{\int_0^t d(x)\left(\int_0^{1-x} d\mu(u)\right)dx}{\int_0^1 d(x)\left(\int_0^{1-x} d\mu(u)\right)dx} = \frac{\int_{[0,1]^2} \mathbf{1}_{x\le t, u\le 1-x} d(x)d\mu(u)dx}{\int_{[0,1]^2} \mathbf{1}_{u\le 1-x} d(x)d\mu(u)dx}.$$

The main point is to express the latter quantity as the expectation of a random variable almost surely greater than $\int_0^t d(x)dx$. Changing the order of the integrals yields

$$F_1(t) = \frac{\int_0^1 \left(\int_0^{t\wedge(1-u)} d(x)dx\right)d\mu(u)}{\int_0^1 \left(\int_0^{1-u} d(x)dx\right)d\mu(u)} = \frac{\int_0^1 \left(\int_0^{t\wedge(1-u)} \frac{d(x)}{\int_0^{1-u} d(x)dx}dx\right)\left(\int_0^{1-u} d(x)dx\right)d\mu(u)}{\int_0^1 \left(\int_0^{1-u} d(x)dx\right)d\mu(u)}.$$

Let $\tilde{U}$ be a random variable absolutely continuous with respect to μ and having the density

$$d_{\tilde{U}}(u) = \frac{\left(\int_0^{1-u} d(x)dx\right)d\mu(u)}{\int_0^1 \left(\int_0^{1-u} d(x)dx\right)d\mu(u)}.$$

Then

$$F_1(t) = \mathbb{E}_{\tilde{U}}\left(\frac{\int_0^{t\wedge(1-\tilde{U})} d(x)dx}{\int_0^{1-\tilde{U}} d(x)dx}\right).$$

Since for each $u \ge 0$

$$\frac{\int_0^{t\wedge 1-u} d(x)dx}{\int_0^{1-u} d(x)dx} \ge \int_0^t d(x)dx,$$

this concludes the proof.
It is exactly the same for $F_{Y_s|X_{s+1}=x}(t)$. $\square$

Remark 8.19. *In the case of a Sawtooth model $\mathcal{S}_\lambda$, a simpler proof of the monotonicity result of Proposition 8.18 can be done by induction on the length of the run of λ between x_s^- and x_s^+. Namely, if the run has length 2,*

$$F_{X_s|Y_s=y}(t) = \frac{\int_0^{t\wedge y} d_{X_s,\mathcal{S}_\lambda\to X_s}(x)dx}{\int_0^y d_{X_s,\mathcal{S}_\lambda\to X_s}(x)dx},$$

which is decreasing in y. If the run has length $r > 2$, the expression of the density in the integral of (8.3.3) yields

$$F_{X_s|Y_s=y}(t) = \frac{\int_0^y F_{\tilde{X}_s|\tilde{Y}_s=y'}(t)d_{\tilde{Y}_s,\mathcal{S}_{\tilde{\lambda}}\to\tilde{Y}_s}(y')dy'}{\int_0^y d_{\tilde{Y}_s,\mathcal{S}_{\tilde{\lambda}}\to\tilde{Y}_s}(y')dy'},$$

where $\tilde{\lambda}$ is the composition λ with the run between x_s^- and x_s^+ reduced to $r-1$, and $\tilde{X}_s$ and $\tilde{Y}_s$ correspond to the variables x_s^- and x_s^+ in $\mathcal{S}_{\tilde{\lambda}}$. By recurrence hypothesis, $F_{\tilde{X}_s|\tilde{Y}_s=y'}(t)$ is decreasing in y', and thus, $\frac{\int_0^y F_{\tilde{X}_s|\tilde{Y}_s=y'}(t)d_{\tilde{Y}_s,\mathcal{S}_{\tilde{\lambda}}\to\tilde{Y}_s}(y')dy'}{\int_0^y d_{\tilde{Y}_s,\mathcal{S}_{\tilde{\lambda}}\to\tilde{Y}_s}(y')dy'}$ is decreasing in y.

8.4.2 Alternating pattern of a convex sawtooth model

Proposition 8.18 yields two main features for the model. The first one is an extension of the previous result.

Proposition 8.20. *Let $1 \leq s$, $0 \leq t \leq 1$. Then for $r \geq s$, $F_{X_s|X_r=x}(t)$ is decreasing in x and $F_{X_s|Y_r=y}(t)$ is decreasing in y. Likewise, $F_{X_s|X_r=0}(t)$ is decreasing in r and $F_{X_s|Y_r=1}(t)$ is increasing in r.*
Moreover,

$$F_{X_s,\mathcal{S}_{\to X_r}}(t) \leq F_{X_s|Y_r=y}(t)$$

and

$$F_{X_s,\mathcal{S}_{\to Y_r}}(t) \geq F_{X_s|X_{r+1}=x}(t).$$

Proof. Let $s \geq 1$ and let us prove the monotonicty on x and y by recurrence on r, starting at $s = r$. $F_{X_s|X_s=x}(t)$ is clearly decreasing in x and from Proposition 8.18, $F_{X_s|Y_s=y}(t)$ is decreasing in y. Thus, the initialization is done.
Suppose the result proved until X_r. Then

$$F_{X_s|X_{r+1}=x}(t) = \int_0^1 F_{X_s|Y_r=y,X_{r+1}=x}(t) d_{Y_r|X_{r+1}=x}(y)dy,$$

and by an integration by part, since from Lemma 8.13 $F_{X_s|Y_r=y,X_{r+1}=x}(t) = F_{X_s|Y_r=y}(t)$,

$$F_{X_s|X_{r+1}=x}(t) = F_{X_s|Y_r=1}(t) - \int_0^1 \frac{\partial}{\partial y} F_{X_s|Y_r=y}(t) F_{Y_r|X_{r+1}=x}(y)dy.$$

Thus,

$$\frac{\partial}{\partial x} F_{X_s|X_{r+1}=x}(t) = -\int_0^1 \frac{\partial}{\partial y} F_{X_s|Y_r=y}(t) \frac{\partial}{\partial x} F_{Y_r|X_{r+1}=x}(y)dy.$$

By recurrence $\frac{\partial}{\partial y} F_{X_s|Y_r=y}(t)$ is negative and by Proposition 8.18 $\frac{\partial}{\partial x} F_{Y_r|X_{r+1}=x}(y)$ is negative, thus $\frac{\partial}{\partial x} F_{X_s|X_{r+1}=x}(t)$ is also negative. It is exactly the same for $F_{X_s|Y_{r+1}=y}(t)$.
Let $r \geq s$. $F_{X_s|X_{r+1}=0}(t) = \int_0^1 F_{X_s|X_r=x}(t) d_{X_r|X_{r+1}=0}(x)dx$, thus by Proposition 8.18

$$F_{X_s|X_{r+1}=0}(t) \leq \int_0^1 F_{X_s|X_r=0}(t) d_{X_r|X_{r+1}=0}(x)dx \leq F_{X_s|X_r=0}(t).$$

The same proof holds to show that $F_{X_s|Y_r=1}(t)$ is increasing in r.
Let us prove the second part of the proposition and let $y \in [0,1]$. Conditioning X_s on X_r in $\mathcal{S}_{\to X_r}$ yields

$$F_{X_s,\mathcal{S}_{\to X_r}}(t) = \mathbb{E}\left(F_{X_s|X_r=\tilde{X}_r}(t)\right),$$

with $\tilde{X}_r$ following the law of q_r in $\mathcal{S}_{\to X_r}$.
On one hand from the first part of the proposition, $F_{X_s|X_r=x}(t)$ is decreasing in x. On the other hand from Proposition 8.18, $\tilde{X}_r$ stochastically dominates $(X_r|Y_r = y)$. Thus, from Proposition 8.6,

$$F_{X_s,\mathcal{S}_{\to X_r}}(t) = \mathbb{E}\left(F_{X_s|X_r=\tilde{X}_r}(t)\right) \leq F_{X_s|Y_r=y}(t).$$

The same pattern proves the second inequality. □

There is an immediate consequence of this Proposition on the behavior of $F_{X_s,\mathcal{S}_{\to X_u}}(t)$ with $u \geq s$.

Corollary 8.21. *The following inequalities hold for $k \geq s$:*

$$F_{X_s,\mathcal{S}_{\to X_s}}(t) \leq \cdots \leq F_{X_s,\mathcal{S}_{\to X_u}}(t) \leq \cdots \leq F_{X_s,\mathcal{S}_{\to Y_u}}(t) \cdots \leq F_{X_s,\mathcal{S}_{\to Y_s}}(t).$$

Proof. The previous Proposition yields directly the following inequalities :

$$F_{X_s,\mathcal{S}_{\to Y_r}}(t) \geq F_{X_s|Y_r=1}(t) \geq F_{X_s,\mathcal{S}_{\to X_r}}(t).$$

Moreover,

$$\begin{aligned} F_{X_s,\mathcal{S}_{\to X_{u+1}}}(t) &= \int_{[0,1]} F_{X_s|Y_u=y}(t) d_{Y_n,\mathcal{S}_{\to X_{u+1}}}(y)dy \\ &\geq \int_{[0,1]} F_{X_s,\mathcal{S}_{\to X_u}}(t) d_{Y_u,\mathcal{S}_{\to X_{u+1}}}(y)dy \\ &\geq F_{X_s,\mathcal{S}_{\to X_u}}(t), \end{aligned}$$

the first inequality being due to Proposition 8.18. By symmetry between X_u and Y_u the general result holds. □

8.4.3 Estimates on the behavior of extreme particles

As a second consequence of Proposition 8.18 we can get a more accurate estimate on the behavior of the first and last particles of $\mathcal{S}$. In particular, we can achieve a coupling of (X_I, X_F) with two couples of random variables, which only depend on f_1 and g_n and give some bounds on (X_I, X_F) in the sense of the stochastic domination.
In this paragraph we will not assume that the first and last particles are lower ones, and deal with model of any type (refer to Remark 8.11 for the definition of the type of a model). Moreover, to describe the bounding random variables, we introduce two particular transforms Γ^+ and Γ^-:

Definition 8.22. *Let f be a positive function on $[0,1]$. Then $\Gamma^+(f)$ and $\Gamma^-(f)$ are the functions defined on $[0,1]$ as :*

$$\Gamma^-(f)(t) = \frac{\int_{1-t}^1 f(u)du}{\int_0^1 f(u)du},$$

and

$$\Gamma^+(f)(t) = \frac{\int_0^t f(u)du}{\int_0^1 f(u)du}.$$

Remark that $\Gamma^-(f)(t)$ (resp. $\Gamma^+(f)(t)$) is the cumulative distribution function of the random variable $1 - Z$ (resp. Z), Z being the random variable with density $\frac{f(x)}{\int_0^1 f(x)dx}$.

Proposition 8.23. *Let $\mathcal{S}$ be a convex Sawtooth model of type ε with density functions $\{f_i, g_i\}_{1\leq i\leq k}$ and at least four particles. There exists a probability space and two couples of random variables $(X_+, Y_+), (X_-, Y_-)$ on it, such that :*

- $(X_-, Y_-) \preceq_\varepsilon (X_I, X_F) \preceq_\varepsilon (X_+, Y_+)$.
- *X_+ and Y_+ are independent with distribution function*

$$F_{X_+,Y_+}(s,t) = \Gamma^{\varepsilon_1}(f_1)(s)\Gamma^{\varepsilon_2}(g_n)(t).$$

- *X_- and Y_- are independent with distribution function*

$$F_{X_-,Y_-}(s,t) = \left(\Gamma^{\varepsilon_1} \circ \Gamma^{\varepsilon_1^*}(f_1)\right)(s) \left(\Gamma^{\varepsilon_2} \circ \Gamma^{\varepsilon_2^*}(g_n)\right)(t).$$

with $-^* = +$ *and* $+^* = -$.

Proof. We assume without loss of generality that each f_i, g_i is normalized and, since the type of the Sawtooth model doesn't change the pattern of the proof, we assume that $\mathcal{S}$ is of type $--$. On one hand the conditional law of (X_I, X_F) given the value of $Y_1 = y_1, Y_k = y_k$ has for cumulative distribution function :

$$\begin{aligned}
F_{X_I,X_F|Y_1=y_1,Y_k=y_k}(t_1,t_2) =& \frac{\left(\int_0^{t_1\wedge y_1} f_1(y_1-x)dx\right)\left(\int_0^{t_2\wedge y_k} g_k(y_k-y)dy\right)}{(\int_0^{y_1} f_1(x)dx)(\int_0^{y_k} g_k(x)dx)} \\
=& F_{X_I|Y_1=y_1}(t_1)F_{X_F|Y_k=y_k}(t_2).
\end{aligned}$$

This together with Proposition 8.18 gives the bound

$$\begin{aligned}
F_{X_I,X_F|Y_1=y_1,Y_k=y_k}(t_1,t_2) =& F_{X_I|Y_1=y_1}(t_1)F_{X_F|Y_k=y_k}(t_2) \\
\geq& F_{X_I|Y_1=1}(t_1)F_{X_F|Y_k=1}(t_2).
\end{aligned}$$

Since

$$F_{X_I|Y_1=1}(t_1)F_{X_F|Y_k=1}(t_2) = (1-F_{f_1}(1-t_1))(1-F_{g_k}(1-t_2)) = \Gamma^-(f_1)(t_1)\Gamma^-(g_k)(t_2),$$

this gives the upper part of the stochastic bound.
On the other hand, the density of (Y_1, Y_k) conditioned on the value of (X_2, X_k) is

$$\begin{aligned}
& d_{Y_1,Y_k|X_2=x_2,X_k=x_k}(y_1,y_k) \\
=& \mathbf{1}_{y_1\geq x_2, y_k\geq x_k} \frac{\left(\int_0^{y_1} f_1(y_1-x)dx\right) g_1(y_1-x_2)}{\int_{x_2}^1 \left(\int_0^z f_1(z-x)dx\right) g_1(z-x_2)dz} \frac{\left(\int_0^{y_k} g_k(y_k-x)dx\right) f_k(y_k-x_k)}{\int_{x_k}^1 \left(\int_0^z g_k(z-x)dx\right) f_k(z-x_k)dz} \\
=& \mathbf{1}_{y_1\geq x_2, y_k\geq x_k} \frac{F_{f_1}(y_1)g_1(y_1-x_2)}{\int_{x_2}^1 F_{f_1}(z)g_1(z-x_2)dz} \frac{F_{g_k}(y_k)f_k(y_k-x_k)}{\int_{x_k}^1 F_{g_k}(z)f_k(z-x_k)dz}.
\end{aligned}$$

Factorizing the latter density yields

$$d_{Y_1,Y_k|X_2=x_2,X_k=x_k}(y_1,y_k) = d_{Y_1|X_2=x_2}(y_1)d_{Y_k|X_k=x_k}(y_k).$$

Let us first consider Y_1. Recall that g_1 is an increasing $\mathcal{C}^1$ function. This means in particular that

$$g_1(x) = \frac{1}{K}\int_0^x d\lambda(u),$$

with λ a probability measure on $[0,1]$ having eventually a dirac mass at 0 and then a continuous density function on $]0,1]$. Thus, the density of Y_1 conditioned on the value of X_2 is

$$d_{Y_1|X_2=x_2}(y_1) = \frac{1}{A}\mathbf{1}_{y_1\geq x_2}F_{f_1}(y_1)\int_{x_2}^{y_1} d\lambda(u-x_2),$$

with A a normalizing constant. Let d_u be the density function defined for $0 \leq u \leq 1$ by

$$d_u(y) = \frac{1}{A_u}\mathbf{1}_{y\geq u}F_{f_1}(y),$$

with A_u a normalizing constant depending on u and let $F_u(t)$ be the associated cumulative distribution function. On one hand

$$\begin{aligned}F_{Y_1|X_2=x_2}(t) =&\frac{\int_0^t \mathbf{1}_{y_1\geq x_2} F_{f_1}(y_1)\int_{x_2}^{y_1} d\lambda(u-x_2)dy_1}{\int_0^1 \mathbf{1}_{y_1\geq x_2} F_{f_1}(y_1)\int_{x_2}^{y_1} d\lambda(u-x_2)dy_1}\\ =&\frac{\int_0^t \int_{x_2}^1 \mathbf{1}_{y_1\geq u} F_{f_1}(y_1) d\lambda(u-x_2)dy_1}{\int_0^1 \int_{x_2}^1 \mathbf{1}_{y_1\geq u} F_{f_1}(y_1) d\lambda(u-x_2)dy_1},\end{aligned}$$

and after changing the order of the integrals, since $F_u(1) = 1$,

$$\begin{aligned}F_{Y_1|X_2=x_2}(t) =&\frac{\int_{x_2}^1 \left(\int_0^t \mathbf{1}_{y_1\geq u} F_{f_1}(y_1)dy_1\right) d\lambda(u-x_2)}{\int_{x_2}^1 \left(\int_0^1 \mathbf{1}_{y_1\geq u} F_{f_1}(y_1)dy_1\right) d\lambda(u-x_2)}\\ =&\frac{\int_{x_2}^1 A_u F_u(t) d\lambda(u-x_2)}{\int_{x_2}^1 A_u d\lambda(u-x_2)}\\ =&\mathbb{E}_{\tilde{U}}(F_{\tilde{U}}(t)),\end{aligned}$$

with $\tilde{U}$ a random variable with law $d\tilde{U}(u) = \mathbf{1}_{u\geq x_2}\frac{A_u d\lambda(u-x_2)}{\int_{x_2}^1 A_u d\lambda(u-x_2)}$.

On the other hand

$$F_u(t) = \mathbf{1}_{t\geq u}\frac{\int_u^t F_{f_1}(u)du}{\int_u^1 F_{f_1}(u)du} = \mathbf{1}_{t\geq u}\frac{\mathcal{F}_{f_1}(t) - \mathcal{F}_{f_1}(u)}{\mathcal{F}_{f_1}(1) - \mathcal{F}_{f_1}(u)},$$

with $\mathcal{F}_{f_1}$ being the primitive of F_{f_1} taking the value 0 at 0. This yields

$$\begin{aligned}\frac{\partial}{\partial u}F_u(t) =&\frac{\partial}{\partial u}\left(\mathbf{1}_{u\leq t}\frac{\mathcal{F}_{f_1}(t) - \mathcal{F}_{f_1}(u)}{\mathcal{F}_{f_1}(1) - \mathcal{F}_{f_1}(u)}\right)\\ =&\mathbf{1}_{u\leq t}\frac{\partial}{\partial u}\left(\left(\mathcal{F}_{f_1}(t) - \mathcal{F}_{f_1}(1)\right)\frac{1}{\mathcal{F}_{f_1}(1) - \mathcal{F}_{f_1}(u)} + 1\right)\\ =&\mathbf{1}_{u\leq t}\left(\mathcal{F}_{f_1}(t) - \mathcal{F}_{f_1}(1)\right)\frac{\partial}{\partial u}\left(\frac{1}{\mathcal{F}_{f_1}(1) - \mathcal{F}_{f_1}(u)}\right)\\ =&\mathbf{1}_{u\leq t}\left(\mathcal{F}_{f_1}(t) - \mathcal{F}_{f_1}(1)\right)\frac{F_{f_1}(u)}{\left(\mathcal{F}_{f_1}(1) - \mathcal{F}_{f_1}(u)\right)^2} \leq 0,\end{aligned}$$

and thus

$$F_u(t) \leq F_0(t) = \frac{\mathcal{F}_{f_1}(t)}{\mathcal{F}_{f_1}(1)}.$$

Integrating with respect to $\tilde{U}$ yields

$$F_{Y_1|X_2=x_2}(t) = \mathbb{E}_{\tilde{U}}\left(F_{\tilde{U}}(t)\right) \leq \mathbb{E}_{\tilde{U}}\left(F_0(t)\right),$$

and finally, $F_{Y_1|X_2=x_2}(t) \leq \frac{\mathcal{F}_{f_1}(t)}{\mathcal{F}_{f_1}(1)}$. We can now integrate this inequality to get a bound on the cumulative distribution function of X_I conditioned on X_2 :

$$\begin{aligned}
F_{X_I|X_2=x_2}(t) = & \int_0^1 F_{X_I|Y_1=y}(t) d_{Y_1|X_2=x_2}(y) dy \\
= & F_{X_I|Y_1=1}(t) - \int_0^1 \frac{\partial}{\partial y} F_{X_I|Y_1=y}(t) F_{Y_1|X_2=x_2}(y) dy \\
\leq & F_{X_I|Y_1=1}(t) - \int_0^1 \frac{\partial}{\partial y} F_{X_I|Y_1=y}(t) \frac{\mathcal{F}_{f_1}(y)}{\mathcal{F}_{f_1(1)}} dy \\
\leq & \int_0^1 F_{X_I|Y_1=y}(t) \frac{F_{f_1}(y)}{\mathcal{F}_{f_1(1)}} dy.
\end{aligned}$$

Note that the direction of the inequality on the third line is due to the negative sign of $\frac{\partial}{\partial y} F_{X_I|Y_1=y}(t)$. Since

$$\begin{aligned}
\int_0^1 F_{X_I|Y_1=y}(t) \frac{F_{f_1}(y)}{\mathcal{F}_{f_1(1)}} dy = & \int_0^1 \frac{\int_0^{t\wedge y} f_1(y-u) du}{F_{f_1}(y)} \frac{F_{f_1}(y)}{\mathcal{F}_{f_1}(1)} dy \\
= & \int_0^t \int_u^1 \frac{f_1(y-u)}{\mathcal{F}_{f_1}(1)} dy du \\
= & \frac{\int_0^t F_{f_1}(1-u) du}{\mathcal{F}_{f_1}(1)} = \Gamma^-(F_{f_1})(t),
\end{aligned}$$

this yields the inequality

$$F_{X_I|X_2=x_2}(t) \leq \Gamma^- \circ \Gamma^+(f_1)(t).$$

Note that the latter inequality is valid even if the model has only three particles (see the next Corollary). Finally, since in our case there are at least four particles, $X_F \neq X_2$, and thus $F_{X_I|X_2=x_2, X_F=y}(t) = F_{X_1|X_2=x_2}(t)$. Therefore

$$F_{X_I|X_F=y}(t) \leq \Gamma^- \circ \Gamma^+(f_1)(t),$$

and by averaging on y,

$$F_{X_I}(t) \leq \Gamma^- \circ \Gamma^+(f_1)(t).$$

Doing the same with X_F gives the bound :

$$F_{X_F}(t) \leq \Gamma^- \circ \Gamma^+(g_k)(t).$$

The result follows from Lemma 8.10. □

Remark 8.24. *The case of a Sawtooth model $\mathcal{S}_\lambda$ illustrates the pattern of the proof in the general case. Namely, suppose that λ has a first run of length r which is increasing. Then, conditioning the law of x_1 on the value of the first particle after the first peak (which is x_{r+1} in this case) yields the formula:*

$$F_{x_1|x_{r+1}=z}(t) = \frac{\int_0^t (\int_{x\wedge z}^1 (y-x)^r dy) dx}{\int_0^1 (\int_{x\wedge z}^1 (y-x)^r dy) dx}.$$

Computing the integral in the numerator and in the denominator yields

$$F_{x_1|x_{r+1}=z}(t) = \frac{[1-(1-t)^r] - [z^r - ((t \vee z) - t)^r]}{1-z^r}. \tag{8.4.1}$$

By Proposition 8.18, $F_{x_1|x_{r+1}=1}(t) \leq F_{x_1|x_{r+1}=z}(t) \leq F_{x_1|x_{r+1}=0}(t)$*: therefore, the bounds are given by the cases* $z=1$ *and* $z=0$*. By Equation* (8.4.1)*,* $F_{x_1|x_{r+1}=0}(t) = 1-(1-t)^r = \Gamma^-\Gamma^+(\tilde{\gamma}_r)(t)$*. Suppose that* $z \geq t$*: rewriting the right hand side of* (8.4.1) *as* $\frac{h(1)-h(z^r)}{1-z^r}$ *with* $h(x) = x-(x^{1/r}-t)$ *yields*

$$F_{x_1|x_{r+1}=1}(t) = h'(1) = 1-(1-t)^{r-1} = \Gamma^-(\gamma_r)(t).$$

The proof of Proposition 8.23 is actually a generalization of the proof in the case $\mathcal{S}_\lambda$*.*

In particular, as a corollary of Proposition 8.23 (and as a corollary of the proof in the case $k=2$), the following result holds :

Corollary 8.25. *Let* $\mathcal{S}$ *be a convex Sawtooth model of type* ε *with density functions* $\{f_i, g_i\}_{1\leq i\leq k}$ *. There exists a couple of random variables* $(Z^{(1)}, Z^{(2)})$ *such that for* $y \in [0,1]$*,*

- $Z^{(1)} \preceq_{\varepsilon(1)} (X_I|X_F = y) \preceq_{\varepsilon(1)} Z^{(2)}$*,*
- *The cumulative distribution function of* $Z^{(2)}$ *is :*

$$F_{Z^{(2)}}(t) = \Gamma^{\varepsilon(1)}(f_1)(t).$$

- *The cumulative distribution function of* $Z^{(1)}$ *is*

$$F_{Z^{(1)}}(t) = \Gamma^{\varepsilon(1)} \circ \Gamma^{\varepsilon(1)^*}(f_1)(t).$$

Proof. For $k \geq 3$, the result is deduced from the latter Proposition. In the case $k=2$, the proof is exactly the same as in the latter Proposition, except that we only deal with the left case, and thus we don't need anymore the fact that $X_2 \neq X_F$. □

In the case of a composition λ with first run of length $R+1$, the latter corollary yields that for $\mathcal{S}_\lambda$:

$$1-(1-t)^R \leq F_{X_I}(t) \leq 1-(1-t)^{R+1},$$

if the first run is increasing, and

$$t^{R+1} \leq F_{X_I}(t) \leq t^R,$$

if the first run is decreasing.

8.5 The independence theorem in a bounded Sawtooth Model

This section is devoted to the proof of the approximate independence of X_I and X_F when the number of particles grows whereas the repulsion forces remain bounded. In this section the Sawtooth model is assumed normalized.

8.5.1 Decorrelation principle and bounding Lemmas

Definition 8.26. *Let* $A > 0$*. A Sawtooth model* $\mathcal{S}$ *with density functions* $\{f_i, g_i\}$ *is bounded by* A *if*

$$\max(\|f_i\|_{[0,1]}, \|g_i\|_{[0,1]}) \leq A.$$

The purpose is to prove the following Theorem :

Theorem 8.27. *Let $A > 0$. For all $\varepsilon > 0$ there exists $N_A \geq 0$ such that for any Sawtooth model $\mathcal{S}$ bounded by A and with $2k \geq N_A$ particles we have :*

$$\|d_{X_I,X_F}(x,y) - d_{X_I}(x)d_{X_F}(y)\|_\infty \leq \varepsilon$$

The pattern of the proof is the following : conditioned on the fact that a particle P - from now on called a splitting particle - is close to the boundary of the domain, the left part $\mathcal{S}_{\to P}$ and the right part $\mathcal{S}_{\leftarrow P}$ of the system are almost not correlated anymore (see Figure 8.4).

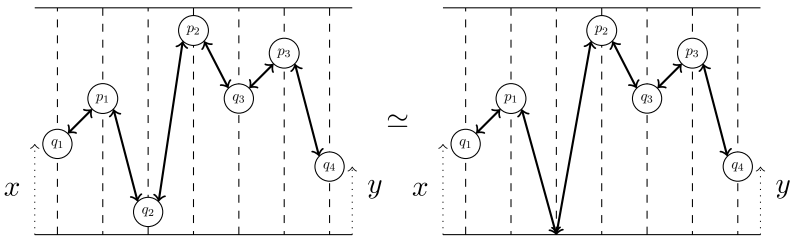


Figure 8.4: Decorrelation of the process

However, we may still not have independence if the law of X_I and X_F depends on which particle splits the system. Thus, we have to find a set of particles that is large enough, so that with probability close to one an element of this set is close to the boundary, and such that nonetheless conditioning on having any particle from this set close to the boundary yields the same law on (X_I, X_F).
Let us first begin by bounding the density of (X_I, X_F).

Lemma 8.28. *Suppose that $\|f_1\|_\infty \leq A$ and let $\mathcal{S}$ be a Sawtooth model larger than 2. Then there exists K_A only depending on A such that for any event $\mathcal{X}$ depending on $\{X_i, Y_i\}_{i\geq 2}$,*

$$\|d_{X_I|\mathcal{X}}\|_\infty \leq K_A.$$

More precisely $K_A = 4A^2$ fits.

This Lemma was already mentioned in the specific context of compositions in [18]. We provide here a different proof.

Proof. By Lemma 8.13, it suffices to prove it for a conditioning on $\{X_2 = x_2\}$. From Lemma 8.14, $d_{X_I|X_2=x_2}(x)$ is decreasing in x and thus it is enough to bound $d_{X_I|X_2=x_2}(0)$. We have

$$d_{X_I|X_2=x_2}(0) = \frac{\int_{x_2}^1 f_1(z)g_1(z-x_2)dz}{\int_{x_2}^1 F_{f_1}(z)g_1(z-x_2)dz} \leq A\frac{\int_{x_2}^1 g_1(z-x_2)dz}{\int_{x_2}^1 F_{f_1}(z)g_1(z-x_2)dz}.$$

Remark that

$$\frac{\int_{x_2}^1 g_1(z-x_2)dz}{\int_{x_2}^1 F_{f_1}(z)g_1(z-x_2)dz} = \frac{1}{\mathbb{E}_{\tilde{Z}}(F_{f_1}(\tilde{Z}))},$$

with $\tilde{Z}$ being a random variable with density $\mathbf{1}_{z\geq x_2}g_1(z-x_2)$. Since $\|F'_{f_1}\| \leq A$ and $F_{f_1}(1) = 1$, $F_{f_1}(t) \geq 1/2$ on $[1-1/(2A)]$; moreover, $z \mapsto g_1(z-x_2)$ is increasing, thus $\mathbb{P}(\tilde{Z} \in [1-1/(2A), 1]) \geq \frac{1}{2A}$ and by Markov's inequality $\mathbb{E}_{\tilde{Z}}(F_{f_1}(\tilde{Z})) \geq 1/4A$. Finally,

$$d_{X_I|X_2=x_2}(0) \leq 4A^2.$$

□

The next step is to get a bound on the first derivative of d_{X_I}. This is possible only if g_1 is also bounded by A and the model is large enough.

Lemma 8.29. *Suppose that* $\max(\|f_1\|_\infty, \|g_1\|_\infty) \leq A$ *and that* $\mathcal{S}$ *is a Sawtooth model with at least four particles. Then there exists a constant* R_A *only depending on* A *such that for any event* $\mathcal{X}$ *depending on* $\{X_{i+1}, Y_i\}_{i\geq 2}$,

$$\|(d_{X_I|\mathcal{X}})'\|_\infty \leq R_A.$$

Proof. For exactly the same reasons as in the previous proof, it suffices to bound the derivative of the density conditioned on $\mathcal{X} = \{Y_2 = y_2\}$. The expression of the density probability yields

$$d_{X_I|Y_2=y_2}(x) = \frac{\int_x^1 f_1(y_1 - x) d_{Y_1|Y_2=y_2}(y_1) dy_1}{\int_0^1 \left(\int_x^1 f_1(y_1 - x) d_{Y_1|Y_2=y_2}(y_1) dy_1\right) dx}.$$

Let $\Delta = \int_0^1 \left(\int_x^1 f_1(y_1 - x) d_{Y_1|Y_2=y_2}(y_1) dy_1\right) dx$, which is independent of x. Then

$$\begin{aligned}
|\frac{\partial}{\partial x} d_{X_I|Y_2=y_2}(x)| =& \frac{1}{\Delta} |\frac{\partial}{\partial x} \int_x^1 f_1(y_1 - x) d_{Y_1|Y_2=y_2}(y_1) dy_1| \\
=& \frac{1}{\Delta} |\int_x^1 (\frac{\partial}{\partial x} f_1(y_1 - x)) d_{Y_1|Y_2=y_2}(y_1) dy_1 - f_1(0) d_{Y_1, \mathcal{S}_{Y_2\leftarrow}|Y_2=y_2}(x)| \\
\leq& \frac{1}{\Delta} \left(|\int_x^1 -(\frac{\partial}{\partial x} f_1)(y_1 - x) d_{Y_1|Y_2=y_2}(y_1) dy_1| + |f_1(0) d_{Y_1|Y_2=y_2}(x)| \right),
\end{aligned}$$

Let us first bound the numerator. By the expression of the density of Y_1 conditioned on $Y_2 = y_2$,

$$d_{Y_1|Y_2=y_2}(y_1) = \frac{F_{f_1}(y_1) d_{Y_1, \mathcal{S}_{Y_1\leftarrow}|Y_2=y_2}(y_1)}{\mathbb{E}_{\tilde{Y}_1}(F_{f_1}(\tilde{Y}_1))},$$

with $\tilde{Y}_1$ having the density $d_{Y_1, \mathcal{S}_{Y_1\leftarrow}|Y_2=y_2}$. Since g_1 is bounded by A, from Lemma 8.28, $|d_{Y_1, \mathcal{S}_{Y_1\leftarrow}|Y_2=y_2}| \leq K_A$. From Lemma 8.14, $d_{Y_1, \mathcal{S}_{Y_1\leftarrow}|Y_2=y_2}(y)$ is increasing in y, and $|F'_{f_1}| \leq A$, thus $\mathbb{E}_{\tilde{Y}_1}(F_{f_1}(\tilde{Y}_1)) \geq \frac{1}{4A^2}$ and

$$|f_1(0) d_{Y_1, \mathcal{S}_{Y_2\leftarrow}|Y_2=y_2}(x)| \leq 4A^2 K_A^2.$$

Let us bound also the first term of the sum: f_1 being increasing, $\frac{\partial}{\partial x} f_1(y_1 - x) \leq 0$ and we can thus remove the absolute value in this first term. An other application of Lemma 8.28 yields:

$$\begin{aligned}
\int_x^1 -(\frac{\partial}{\partial x} f_1)(y_2 - x) d_{Y_1|Y_2=y_2}(y_1) dy_1 &\leq K_A(\int_x^1 (\frac{\partial}{\partial x} f_1)(y_2 - x) dy_2) \\
&\leq K_A((f_1(1 - x) - f_1(0)) \leq A \times K_A.
\end{aligned}$$

The numerator is thus bounded by $AK_A + 4A^2 K_A^2$.
Changing the order of the integrals in Δ yields :

$$\Delta = \int_0^1 F_{f_1}(y_1) d_{Y_1|Y_2=y_2}(y_1) dy_1.$$

Since F'_{f_1} is bounded by A and $F_{f_1}(1) = 1$, we can conclude as in the previous proof that $F_{f_1}(t) \geq \frac{1}{2A}$ on $[1 - 1/(2A), 1]$. Moreover, Y_1 is an upper particle, and thus by Lemma 8.14, $d_{Y_1|Y_2=y_2}(y_1)$ is increasing in y_2. Since $\int_{[0,1]} d_{Y_1|Y_2=y_2} = 1$, this implies that

$$\int_{1-1/(2A)}^{1} d_{Y_1|Y_2=y_2}(y_1)dy_1 \geq \frac{1}{2A},$$

and yields $\Delta \geq \frac{1}{4A^2}$. The bounds on the numerator and on Δ yield :

$$|\frac{\partial}{\partial x} d_{X_I|Y_2=y_2}(x)| \leq 4A^3(K_A + 4AK_A^2).$$

□

As an application of Lemma 8.29, we can also prove that $y \mapsto F_{X_I|X_F=y}(t)$ is Lipchitz :

Proposition 8.30. *Let $\mathcal{S}$ be a Sawtooth model with $k \geq 3$ lower particles. Suppose that $\{f_1, g_1, f_k, g_k\}$ are bounded by $A > 0$. Let R_A be the constant of Lemma 8.29 (with $R_A \geq 1$). Then on a neighbourhood $[0, 1/R_A]$ of 0,*

$$\mathcal{F} : \begin{cases} [0, 1/R_A] & \rightarrow & (\mathcal{C}([0,1], \mathbb{R}), \|.\|) \\ y & \mapsto & F_{X_I|X_F=y} \end{cases}$$

is Lipschitz with a Lipschitz constant B_A only depending on A.

Proof. It suffices to prove that for $x \in [0,1]$, $y \mapsto d_{X_I|X_F=y}(x)$ is Lipschitz on $[0, 1/R_A]$ with a Lipschitz constant independent of x.
From Lemma 8.14, d_{X_F} is decreasing and thus on $[1/R_A, 1]$, $d_{X_F} \leq d_{X_F}(1/R_A)$. From Lemma 8.29, $|\frac{\partial}{\partial y} d_{X_F}(y)| \leq R_A$ and thus on $[0, 1/R_A]$, $d_{X_F}(y) \leq d_{X_F}(1/R_A) + R_A(1/R_A - y)$. This implies that

$$\begin{aligned}\int_{[0,1]} d_{X_F}(y)dy \leq & \int_0^{1/R_A} d_{X_F}(1/R_A) + R_A(1/R_A - y)dy + \int_{1/R_A}^{1} d_{X_F}(1/R_A) \\ \leq & d_{X_F}(1/R_A) + \frac{1}{2R_A}.\end{aligned}$$

Since $\int_{[0,1]} d_{X_F} = 1$, this implies that $d_{X_F}(1/R_A) \geq 1 - \frac{1}{2R_A}$, and thus that $d_{X_F} \geq 1 - \frac{1}{2R_A}$ on $[0, 1/R_A]$.
From Lemma 8.29, $\|\frac{\partial}{\partial y} d_{X_F|X_I=x}\| \leq R_A$. Thus, since $\|f_1\| \leq A$, this yields by applying Lemma 8.28 on $d_{X_I,X_F}(x,y) = d_{X_F|X_I=x}(y)d_{X_I}(x)$:

$$|\frac{\partial}{\partial y} d_{X_I,X_F}(x,y)| \leq K_A R_A.$$

Thus, on $[0, 1/R_A]$,

$$\begin{aligned}|\frac{\partial}{\partial y} d_{X_I|X_F=y}(x)| = & \frac{1}{d_{X_F}(y)} |\frac{\partial}{\partial y} d_{X_I,X_F}(x,y) - \frac{d_{X_I,X_F}(x,y)\frac{\partial}{\partial y} d_{X_F}(y)}{d_{X_F}(y)}| \\ \leq & \frac{1}{1 - 1/(2R_A)}(K_A R_A + \frac{R_A K_A^2}{1 - 1/(2R_A)}).\end{aligned}$$

Set $B_A = \frac{1}{1-1/(2R_A)}(K_A R_A + \frac{R_A K_A^2}{1-1/(2R_A)})$. Then $\mathcal{F}$ is B_A−Lipschitz on $[0, 1/R_A]$. □

8.5.2 Behavior of $\{X_i\}$ for large models

The purpose of this subsection is to find for a model $\mathcal{S}$ a large set of intermediate particles $\{X_r\}$ for which one of these particles is close to 0 with high probability and such that $F_{X_I|X_r=0}$ is essentially the same for all particles of this set.
The first part is a essentially probability computation :

Proposition 8.31. *Let $\eta > 0, \varepsilon > 0$. There exists N_0 such that for any model $\mathcal{S}$ of size N larger than $N_0 + 4$ and for any $2 \leq r \leq N - N_0$, $y_{r+N_0} \in [0,1]$,*

$$\mathbb{P}(\bigcup_{r\leq i\leq r+N_0} \{X_i < \eta\}|Y_{r+N_0} = y_{r+N_0}) \geq 1 - \varepsilon.$$

Proof. Let N_0 be an integer to specify later and let $\mathcal{S}, r$ be as in the statement of the Proposition.
Let $\tilde{P} = \mathbb{P}(\bigcap_{r\leq i\leq r+N_0}\{X_i \geq \eta\}|Y_{r+N_0} = y_{r+N_0})$.
Let $0 \leq y_{r-1}, \ldots, y_{r+N_0} \leq 1$ and condition $(\bigcap_{r\leq i\leq r+N_0}\{X_i \geq \eta\}|Y_{r+N_0} = y_{r+N_0})$ on the event $\bigcap_{r-1\leq i\leq r+N_0-1}\{Y_i = y_i\}$. We denote by $P_{\vec{y}}$ the probability of this conditioned event. By Lemma 8.13, the random variables $\{X_i\}_{r\leq i\leq r+N_0}$ are conditionally independent given the value of $\{Y_i\}_{r-1\leq i\leq N_0}$; therefore,

$$P_{\vec{y}} = \prod_{i=r}^{r+N_0} \mathbb{P}(X_i \geq \eta|Y_{i-1} = y_{i-1}, Y_i = y_i).$$

Moreover, Lemma 8.14 yields that $d_{X_i|Y_{i-1}=y_{i-1},Y_i=y_i}$ is decreasing: thus, $\mathbb{P}(X_i \geq \eta|Y_{i-1} = y_{i-1}, Y_i = y_i) \leq (1-\eta)$. This yields

$$P_{\vec{y}} \leq (1-\eta)^{N_0+1}.$$

Integrating $P_{\vec{y}}$ with respect to $y_{r-1}, \ldots, y_{N_0-1}$ gives $\tilde{P} \leq (1-\eta)^{N_0+1}$. Let N_0 be such that $(1-\eta)^{N_0+1} \leq \varepsilon$. For $N \geq N_0$,

$$\mathbb{P}(\bigcup_{r\leq i\leq r+N_0} \{X_i < \eta\}|Y_{r+N_0} = y_{r+N_0}) \geq 1 - \varepsilon.$$

□

As said before, it is also necessary that $F_{X_I|X_r=0}$ remains almost constant among this subset of particles. This is possible for large Sawtooth models, thank to the monotonicity results of Proposition 8.20 :

Proposition 8.32. *Let $A, \varepsilon > 0$, $M \in \mathbb{N}^*$. There exists $N_{\varepsilon,A,M}$ such that for any Sawtooth model bounded by A and of size $N \geq N_{\varepsilon,A,M}$, there exists $1 \leq r \leq N - M$ such that for $r \leq i, j \leq r + M$,*

$$\|F_{X_I|X_i=0} - F_{X_I|X_j=0}\|_\infty \leq \varepsilon.$$

Proof. Let $\mathcal{S}$ be a Sawtooth model bounded by A and of size N.
Denote by F_i the function $t \mapsto F_{X_I|X_i=0}(t)$ for $2 \leq i \leq N$. By Lemma 8.28, all the F_i are K_A−Lipschitz. Let $K = \lfloor \frac{2K_A}{\varepsilon} \rfloor$. It suffices to find $r \geq 2$ such that for all $r \leq i, j \leq r + M$, and all $0 \leq k \leq K$,

$$|F_i(\frac{k}{K}) - F_j(\frac{k}{K})| \leq \frac{\varepsilon}{3}.$$

Denote by $v_i \in [0,1]^{K+1}$ the vector $(F_i(\frac{k}{K}))_{0\leq k\leq K}$ and let $N_{\varepsilon,A,M} = (M+1)(\lfloor \frac{3}{\varepsilon} \rfloor + 1)^{K+1}$. Suppose that $N \geq N_{\varepsilon,A,M}$. For $\vec{m} \in [\![0, \lfloor \frac{3}{\varepsilon} \rfloor]\!]^{K+1}$, denote by $C_{\vec{m}}$ the hypercube $\{\vec{x} \in [0,1]^{K+1}|\forall 1 \leq$

$i \leq K+1, m_i\frac{\varepsilon}{3} \leq x_i < (m_i+1)\frac{\varepsilon}{3}\}$. $\{C_{\vec{m}}\}_{\vec{m}\in[\![0,\lfloor\frac{3}{\varepsilon}\rfloor]\!]^{K+1}}$ is a partition of $[0,1]^{K+1}$ in $(\lfloor\frac{3}{\varepsilon}\rfloor+1)^{K+1}$ subsets. If v_i and v_j are both in a same $C_{\vec{m}}$, then for all $0 \leq k \leq K$, $|v_i(k)-v_j(k)| \leq \frac{\varepsilon}{3}$.
Since $N \geq (M+1)(\lfloor\frac{3}{\varepsilon}\rfloor+1)^{K+1}$, Dirichlet's principle yields the existence of $\vec{m}_0 \in [\![0,\lfloor\frac{3}{\varepsilon}\rfloor]\!]^{K+1}$ such that $\#(\{v_i\}_{1\leq i\leq N} \cap C_{\vec{m}_0}) \geq M+1$. Let $i_0 < \cdots < i_M$ be such that for all $0 \leq j \leq M$, $v_{i_j} \in C_{\vec{m}_0}$; in particular, $i_M \geq i_0 + M$. From the previous paragraph, for all $0 \leq k \leq K$, $|v_{i_M}(k) - v_{i_0}(k)| \leq \frac{\varepsilon}{3}$. By Proposition 8.20, $F_i(\frac{k}{K})$ is decreasing in i; thus, since $v_i(k) = F_i(\frac{k}{K})$, for all $i_0 \leq j \leq i_M$ and all $0 \leq k \leq K$

$$v_{i_0}(k) \geq v_i(k) \geq v_{i_M}(k).$$

Since $i_0 + M \leq i_M$, this yields $\|v_i - v_j\|_\infty \leq \frac{\varepsilon}{3}$ for $i_0 \leq i, j \leq i_0 + M$. □

8.5.3 Proof of Theorem 8.27

Theorem 8.27 is a consequence of the following proposition :

Proposition 8.33. *Let $A > 0$. For all $\varepsilon > 0$, there exists a number $N_{A,\varepsilon} \geq 0$ such that for any Sawtooth model $\mathcal{S}$ bounded by A and with $2k \geq N_{A,\varepsilon}$ particles, the following inequality holds:*

$$|F_{X_I|X_F=y}(t) - F_{X_I}(t)| \leq \varepsilon.$$

for all $t, y \in [0,1]$.

Proof. Set $\eta = \inf(\frac{1}{R_A}, \frac{\varepsilon}{B_A})$ with R_A, B_A the constants given respectively by Lemma 8.29 and Proposition 8.30. Let N_0 be the constant given for η and ε by Proposition 8.31. Finally, set $N_{A,\varepsilon} = N_{\varepsilon/4,A,N_0} + 4$ given by Proposition 8.32.
Let $\mathcal{S}$ be a Sawtooth model bounded by A of size larger than $N_{A,\varepsilon}$. Then by Proposition 8.32, there exists $2 \leq r \leq N_{A,\varepsilon} - 2 - N_0$ such that for all $r \leq i, j \leq r + N_0$,

$$\|F_{X_I|X_i=0} - F_{X_I|X_j=0}\|_\infty \leq \varepsilon.$$

Denote $t = r + N_0$ and let $y_t \in [0,1]$. For $r \leq i \leq r + N_0$, set $L_i = \{X_i \leq \eta \cap \{\forall s > i, X_s > \eta\}\}$. Note that $L_i \cap L_j = \emptyset$ for all $i \neq j$ and $\bigcup L_i = L$ with $L = \bigcup_{r\leq i\leq r+N_0}\{X_i \leq \eta\}$. Moreover, since L_i is $(X_s, Y_s)_{s\geq i}$−measurable, by Lemma 8.13, conditioning X_I on $\{X_i = u, Y_t = y_t\} \cap L_i$ is the same as conditioning X_I on $\{X_i = u\}$. Thus,

$$\begin{aligned}\|F_{X_I|L_i,Y_t=y_t} - F_{X_I|X_r=0}\|_\infty =& \|\int_0^\eta (F_{X_I|X_i=u} - F_{X_I|X_r=0}) d_{X_i|L_i,Y_t=y_t}(u)du\|_\infty \\ \leq& \int_0^\eta \|F_{X_I|X_i=u} - F_{X_I|X_r=0}\|_\infty d_{X_i|L_i,Y_t=y_t}(u)du \\ \leq& 2\varepsilon,\end{aligned}$$

by the choice of η. Recall that if $A = \bigcup A_i$, with A_i disjoint events, then for any event C,

$$\mathbb{P}(C|A) = \sum \mathbb{P}(C|A_i)\mathbb{P}(A_i|A)$$

In particular, for $L = \bigcup_i L_i$ this yields

$$\begin{aligned}\|F_{X_I|L,Y_t=y_t} - F_{X_I|X_r=0}\| =& \|\sum_i (F_{X_I|L_i,Y_t=y_t} - F_{X_I|X_r=0})\mathbb{P}(L_i|L, Y_t = y_t)\|_\infty \\ \leq& \sum_i \|(F_{X_I|L_i,Y_t=y_t} - F_{X_I|X_r=0}\|_\infty \mathbb{P}(L_i|L, Y_t = y_t) \\ \leq& 2\varepsilon.\end{aligned}$$

By Proposition 8.31 and the choice of N_0, $\mathbb{P}(L|Y_t = y_t) \geq 1-\varepsilon$, and thus

$$\|F_{X_I|Y_t=y_t} - F_{X_I|X_r=0}\|_\infty \leq 3\varepsilon.$$

By averaging on y_t with the density $d_{Y_t|X_F=y}$ we get

$$\|F_{X_I|X_F=y} - F_{X_I}\|_\infty \leq 4\varepsilon.$$

□

Let us end the proof of the Theorem 8.27, which consists essentially in a rewriting in terms of densities of the latter Proposition.

Proof. Let $A > 0, \varepsilon > 0$. Set $\varepsilon_1 = \frac{(\varepsilon/K_{A)}^2}{4R_A}$ and let $\mathcal{S}$ be a Sawtooth model bounded by A of size larger than N_{A,ε_1} (N_{A,ε_1} being given by Proposition 8.33). Then from Proposition 8.33, for $y \in [0,1]$,

$$\|F_{X_I|X_F=y} - F_{X_I}\|_\infty \leq \frac{(\varepsilon/K_A)^2}{4R_A}. \tag{8.5.1}$$

Moreover, the following result holds for C^1-functions on $[0,1]$:

Lemma 8.34. *Let $f, g : [0,1] \to [0,1]$ be two C^1- functions, such that $\|f'\|_\infty, \|g'\|_\infty \leq M$. Then for $\varepsilon > 0$ small enough, if F, G are two primitives of f, g and*

$$\|F - G\|_\infty \leq \frac{\varepsilon^2}{4M},$$

then $\|f - g\|_\infty \leq \varepsilon$.

Proof. This is implied by proving that if $f : [0,1] \longrightarrow \mathbb{R}$ verifies $\|f\|_\infty \leq \frac{\varepsilon^2}{4M}$ and $\|f''\|_\infty \leq M$, then $\|f'\|_\infty \leq \varepsilon$. But the majoration on f'' yields that if $|f'(x)| \geq \varepsilon$,

$$\max(|\int_x^{x+\varepsilon/M} f'(x)dx|, |\int_{x-\varepsilon/M}^x f'(x)dx|) \geq \frac{\varepsilon^2}{2M}.$$

Thus,

$$\max(|f(x+\varepsilon/M)|, |f(x)|, |f(x-\varepsilon/m)|) \geq \frac{\varepsilon^2}{4M}.$$

□

Applying this Lemma to (8.5.1) yields for $y \in [0,1]$,

$$\|d_{X_I|X_F=y} - d_{X_i}\|_\infty \leq \varepsilon/K_A.$$

Finally,

$$|d_{X_I,X_F}(x,y) - d_{X_I}(x)d_{X_F}(y)| = |d_{X_F}(y)||d_{X_I|X_F=y}(x) - d_{X_I}(x))| \leq K_A\frac{\varepsilon}{K_A} \leq \varepsilon.$$

□

8.6 Application to compositions

Theorem 8.27 can be applied to the framework of compositions :

Corollary 8.35. *Let $A \geq 0, \varepsilon > 0$. There exists $n \geq 0$ such that for any composition λ of size larger than n with every runs bounded by A,*

$$\|d_{\mathcal{S}_\lambda}(x,y) - d_{\mathcal{S}_\lambda}(x)d_{\mathcal{S}_\lambda}(y)\| < \varepsilon$$

Proof. Each run of λ of length l yields a density function γ_l in $\mathcal{S}_\lambda$, and $\|\gamma_l\|_\infty = l-1$. Thus, if any run of λ is bounded by A, then all the density functions $\{f_i, g_i\}$ in $\mathcal{S}_\lambda$ are bounded by $A-1$. It suffices then to apply Theorem 8.27. □

The purpose of this section is to strengthen Corollary 8.35 and to prove the following Theorem :

Theorem 8.36. *Let $\varepsilon > 0$, $A \geq 0$. There exists $n \geq 0$ such that for any composition λ of size larger than n with first and last run bounded by A,*

$$\|d_{\mathcal{S}_\lambda}(x,y) - d_{\mathcal{S}_\lambda}(x)d_{\mathcal{S}_\lambda}(y)\| < \varepsilon. \tag{8.6.1}$$

This Theorem was Conjecture 1 in [18]. The proof of Theorem 8.36 is followed by some applications.

8.6.1 Effect of a large run on the law of (X_I, X_F)

From Corollary 8.35, it is enough to prove that the presence of a large run inside the composition disconnects the behaviors of X_I and X_F. The main reason for this is the Lemma below: for each composition λ, denote by λ^+ the composition λ with a cell added on the last run, and by λ^- the composition λ with a cell removed on the last run.

Lemma 8.37. *Let $A > 0$ and let λ be a composition with more than three runs and with the first run smaller than A. If the last run of λ is of size R,*

$$\|d_{X_I,\mathcal{S}_\lambda} - d_{X_I,\mathcal{S}_{\lambda^+}}\|_\infty \leq \frac{K_A}{R-1},$$

where K_A is the bound on the density of X_I as defined in Lemma 8.28.

Proof. Let us prove it in the case where the first run of λ is increasing and the last run decreasing, the other cases having the same proof. The expression (8.3.3) yields

$$d_{(X_I,X_F),\mathcal{S}_{\lambda^+}}(x,y) = \frac{\int_y^1 d_{(X_I,X_F),\mathcal{S}_\lambda}(x,z)dz}{\int_{[0,1]^2}\left(\int_y^1 d_{(X_I,X_F),\mathcal{S}_\lambda}(x,z)\right)dxdy}.$$

Thus, by integrating with respect to y and then changing the order of the integrals, this yields

$$\begin{aligned} d_{X_I,\mathcal{S}_{\lambda^+}}(x) =& \frac{\int_0^1\left(\int_0^1 d_{(X_I,X_F),\mathcal{S}_\lambda}(x,z)\mathbf{1}_{y\leq z}dy\right)dz}{\int_{[0,1]^2}\left(\int_0^1 d_{(X_I,X_F),\mathcal{S}_\lambda}(x,z)\mathbf{1}_{y\leq z}dy\right)dxdz} \\ =& \frac{\int_0^1 d_{(X_I,X_F),\mathcal{S}_\lambda}(x,z)zdz}{\int_{[0,1]^2} d_{(X_I,X_F),\mathcal{S}_\lambda}(x,z)zdzdx}. \end{aligned}$$

Factorizing by $d_{X_I,\mathcal{S}_\lambda}(x)$ makes a conditional expectation appear and thus

$$d_{X_I,\mathcal{S}_{\lambda+}}(x) = d_{X_I,\mathcal{S}_\lambda}(x)\frac{\mathbb{E}_{\mathcal{S}_\lambda}(X_F|X_I = x)}{\mathbb{E}_{\mathcal{S}_\lambda}(X_F)}.$$

Moreover, Proposition 8.23 yields

$$F_{Z_1} \leq F_{X_F|X_I=x} \leq F_{Z_2},$$

with $F_{Z_1} = \Gamma^-(F_{\gamma_R})$ and $F_{Z_2} = \Gamma^-(\gamma_R)$. Since $\Gamma^-(F_{\gamma_R})(t) = 1-(1-t)^R$ and $\Gamma^-(\gamma_R)(t) = 1-(1-t)^{R-1}$, by stochastic dominance, applying Proposition 8.6 gives

$$\frac{1}{R} \leq \mathbb{E}_{\mathcal{S}_\lambda}(X_F|X_I = x) \leq \frac{1}{R-1}.$$

Integrating the latter result on x yields $\frac{1}{R} \leq \mathbb{E}_{\mathcal{S}_\lambda}(X_F) \leq \frac{1}{R-1}$, and thus

$$\frac{R-1}{R} \leq \frac{\mathbb{E}_{\mathcal{S}_\lambda}(X_F|X_I = x)}{\mathbb{E}_{\mathcal{S}_\lambda}(X_F)} \leq \frac{R}{R-1}.$$

This yields

$$|d_{X_I,\mathcal{S}_{\lambda+}}(x) - d_{X_I,\mathcal{S}_\lambda}(x)| \leq |d_{\mathcal{S}_\lambda}(x)|\frac{1}{R-1} \leq \frac{K_A}{R-1}.$$

□

In particular, the previous Lemma can be used to bound the conditional law of the first particle with respect to the last one. For each composition λ, and any cells $i, j \in \lambda$, denote by $\lambda_{\to i}$ (resp $\lambda_{i\to}$, resp $\lambda_{i\to j}$) the composition consisting of the cells of λ from 1 to i (resp. from i to n, resp. from i to j). Moreover, denote by $R_{int}(\lambda)$ the set of all runs of λ except the first and last ones.

Proposition 8.38. *Let $A \geq 0$ and λ a composition with first run bounded by A. Then*

$$\|F_{X_I|X_F=x} - F_{X_I}\|_\infty \leq \frac{K_A}{\max_{s\in R_{int}(\lambda)} l(s) - 2}.$$

Proof. Let $t \in [0,1]$. Let s_0 be the run with maximal length R in R_{int} and let i_0 be the rightest cell of this run. This cell corresponds to a particle X_i or Y_i in $\mathcal{S}_\lambda$. Let us assume without loss of generality that this particle is a lower one. From Proposition 8.18, $F_{X_1|X_r=x}(t)$ is decreasing in x and thus

$$\begin{aligned}
|F_{X_I|X_F=x}(t) - F_{X_I}(t)| =& |F_{X_I|X_F=x}(t) - \int_{X_F} F_{X_I|X_F=x}(t) d_{X_F}(x)dx| \\
\leq& |F_{X_I|X_F=0}(t) - F_{X_I|X_F=1}(t)| \\
\leq& F_{X_I|X_F=0}(t) - F_{X_I|Y_k=1}(t).
\end{aligned}$$

Moreover, from Proposition 8.18 and Proposition 8.20,

$$F_{X_I|X_F=0}(t) \leq F_{X_I,\mathcal{S}_\lambda\to Y_k}(t) \leq F_{X_I,\mathcal{S}_\lambda\to Y_i}(t) \leq F_{X_I|X_i=0},$$

and

$$F_{X_I|Y_k=1}(t) \geq F_{X_I,\mathcal{S}_\lambda\to X_k}(t) \geq F_{X_I,\mathcal{S}_\lambda\to X_i}(t).$$

These inequalities imply

$$|F_{X_I|X_F=x}(t) - F_{X_I}(t)| \leq F_{X_I|X_i=0}(t) - F_{X_I,\mathcal{S}_\lambda \to X_i}(t).$$

From the expression (8.3.3), $F_{X_I,\mathcal{S}_\lambda \to X_i}(t) = F_{X_I,\mathcal{S}_{\lambda_{\to i_0}}}(t)$ and $F_{X_I|X_i=0}(t) = F_{X_1,\mathcal{S}_{\lambda^-_{\to i_0}}}(t)$. Thus, with the previous Lemma, since the last run of $\lambda^-_{\to i_0}$ is of size $R-1$,

$$|F_{X_I|X_F=x}(t) - F_{X_I}(t)| \leq |F_{X_I,\mathcal{S}_{\lambda_{\to i_0}}}(t) - F_{X_I,\mathcal{S}_{\lambda^-_{\to i_0}}}(t)| \leq \frac{K_A}{R-2}.$$

□

8.6.2 Proof of Theorem 8.36

The latter Proposition together with Lemma 8.34 yields Theorem 8.36 in case d'_{X_I} remains bounded. However, the bound of the derivative in Lemma 8.29 requires also a bound on the second run, and the latter is not assumed in our case. We should thus deal with this case before getting the general proof. Let us first consider a particular case.

Lemma 8.39. *Let λ_b be the composition with three runs of respective length 2, b and 2, and $d_b(x,y) = d_{X_I,|Y_2=y}(x)$. Then the following convergence holds:*

$$\lim_{b\to\infty} \sup_{[0,1]^2} (d_b(x,y) - (1-x^b)) = 0.$$

In particular, the asymptotic independence :

$$\lim_{b\to\infty} \sup_{x,y,y'} (d_b(x,y) - d_b(x,y')) = 0. \tag{8.6.2}$$

is valid.

Proof. After integrating in (8.3.3) the coordinates of the particles inside the composition :

$$d_b(x,y) = \frac{1 - x^b - (1-y)^b + ((x-y)\wedge 0)^b}{(1-1/(b+1))(1-(1-y)^b) + y/(b+1)(1-y)^b}. \tag{8.6.3}$$

Let us show that $\lim_{b\to\infty} d_b(x,y) - (1-x^{b+1}) = 0$ uniformly in x and y. In the denominator of (8.6.3), letting b go to $+\infty$ yields

$$(1 - \frac{1}{b+1})(1-(1-y)^b) + y/(b+1)(1-y)^b \sim_{b\to\infty} 1-(1-y)^b,$$

with the equivalent being uniform in x and y. Indeed

$$\frac{y/(b+1)(1-y)^b}{1-(1-y)^b} = \frac{1}{b+1}\frac{(1-y)^b}{\sum_{k=0}^{b-1}(1-y)^k} \leq \frac{1}{b+1}.$$

Since for $x \in [0,1/2], y \in [1/2,1]$, $d_b(x,y)$ converges uniformly to 1, it suffices to consider in the sequel that $x \in [1/2,1]$ and $y \in [0,1/2]$. Let Δ be defined as

$$\begin{aligned}\Delta(x,y) =& \frac{1 - x^b - (1-y)^b + (x-y)^b}{1-(1-y)^b} - (1-x^b) \\ =& (1 - \frac{x^b - (x-y)^b}{1-(1-y)^b}) - (1-x^b) = \frac{(x-y)^b - (1-y)^b x^b}{1-(1-y)^b}.\end{aligned}$$

A derivative computation shows that $\Delta(x,y) \leq \frac{1}{b}$, which proves the uniform convergence. Since $\lim_{b\to\infty} \|d_b(x,y) - (1-x^{b+1})\|_{\infty,[0,1]^2} = 0$,

$$\lim_{b\to\infty} \sup_{y,y',x} (d_b(x,y) - d_b(x,y')) = 0.$$

□

From the latter result can be deduced the asymptotic independence with a large second run :

Lemma 8.40. *Let $A, \varepsilon > 0$. There exist $B_A \in \mathbb{N}$ such that if λ is a composition with at least three runs, the extreme runs bounded by A and the second run larger than B_A, then*

$$\|d_{X_I,X_F} - d_{X_I} d_{X_F}\|_\infty \leq \varepsilon$$

Proof. Let λ be a composition with first run of length a and second run of length b. From the definition of the density d_{X_I,X_F} in (8.3.3), conditioning the law of X_I on the position x_P of the particle $P = a+b$ yields

$$d_{X_I|x_p=y}(x) = \frac{\int_x^1 \left(\int_0^{z_1\wedge y} (z_1-x)^{a-2}(z_1-z_2)^{b-2} dz_2\right) dz_1}{\mathcal{Z}}.$$

Let $2 \leq a \leq A$. Then

$$d_{X_I|x_p=y}(x) = \frac{\int_x^1 (u-x)^{a-3} d_b(u,y) du}{\frac{1}{a-2}\int_0^1 u^{a-2} d_b(u,y) du}$$

From the first part of Lemma 8.39, $|d_b(u,y) - (1-u^b)| \to_{b\to\infty} 0$ uniformly in u and y, and thus

$$\frac{1}{a-2}\int_0^1 u^{a-2} d_b(u,y) du \to_{b\to\infty} \frac{1}{(a-2)(a-1)},$$

uniformly in y. Since a is bounded by A, and from the second part of Lemma 8.39,

$$\|d_{X_I|x_p=y} - d_{X_I|x_p=y'}\|_\infty \leq A^2 \sup_{y,y',x} (d_b(x,y) - d_b(x,y')) \to 0$$

uniformly in y. Thus, for b large enough, $\|d_{X_I|x_p=y} - d_{X_I|x_p=y'}\| < \varepsilon/A$ for all y, y'; then averaging on the law of x_p conditioned on $X_F = y$ yields $|d_{X_I|X_F=y} - d_{X_I|X_F=y'}| < \varepsilon/A$ for all y, y'. Finally, this implies that

$$\|d_{X_I,X_F} - d_{X_I} d_{X_F}\|_\infty \leq \varepsilon.$$

□

The proof of Theorem 8.36 is just a gathering of all the previous results :

Proof. Let $A, \varepsilon > 0$. Since the first and last runs are bounded by A, any composition large enough has at least three runs. Let B_A be given by Lemma 8.40, R be the associate constante given by Lemma 8.29 for B_A, and set $C = \frac{4K_A R}{(\varepsilon/A)^2}$. Finally, let n be the integer given by Corollary 8.35 for compositions of runs bounded by C. Suppose that λ is a composition larger than n. By Lemma 8.40, if the second run is larger than B_A, (8.6.1) is verified. Thus, we can suppose that

the second run is bounded by B_A.
If λ has a run larger than C, then from Proposition 8.38,

$$\|F_{X_I|X_F=x} - F_{X_I}\|_\infty \leq \frac{K_A}{C-1} \leq \frac{(\varepsilon/A)^2}{4R}.$$

But from Lemma 8.29, d'_{X_I} is bounded by R, thus the latter inequality yields with Lemma 8.34 :

$$\|d_{X_I|X_F=y} - d_{X_I}\| \leq \varepsilon/A.$$

And d_{X_I} being bounded by A, this yields (8.6.1).
Thus, we can assume that all the runs of λ are bounded by C. Once again by the choice of n and Corollary 8.35, (8.6.1) is verified. □

Note that we actually proved something stronger than Theorem 8.36, namely :

Corollary 8.41. *Let $A, \varepsilon > 0$. There exists n_0 such that for every composition λ of size larger than n_0 and first run bounded by A, and for all $y, y' \in [0,1]$,*

$$\|d_{X_I|X_F=y} - d_{X_I|X_F=y'}\| \leq \varepsilon.$$

8.6.3 Consequences and proof of Theorem 8.3

Here are some interesting consequences of Theorem 8.36. Let us first remove the constraints on the extreme runs.

Lemma 8.42. *Let $\varepsilon > 0$. There exists $n \geq 0$ such that for all compositions larger than n with at least two runs,*

$$\sup_{(y,y')\in[0,1]^2} (\|F_{X_I|X_F=y} - F_{X_I|X_F=y'}\|_\infty) \leq \varepsilon.$$

Proof. Let R be the length of the first run of a composition λ. From Proposition 8.23 applied to $\mathcal{S}_\lambda$,

$$1 - (1-t)^R \leq F_{X_I|X_F=y}(t) \leq 1 - (1-t)^{R-1}.$$

Since $\sup_{[0,1]}(u^{R-1} - u^R) \to_{R\to\infty} 0$, there exists A such that for any composition with first run larger than A,

$$\sup_{[0,1]^2} \|F_{X_I|X_F=y} - F_{X_I|X_F=y'}\|_\infty \leq \varepsilon.$$

Applying Corollary 8.41 to A, ε yields that there exists n such that for any composition larger than n,

$$\sup_{[0,1]^2} \|F_{X_I|X_F=y} - F_{X_I|X_F=y'}\|_\infty \leq \varepsilon.$$

□

This result can be adapted to show that the law of the first particle depends only on the neighbouring particles : for any composition λ of size N, and $n \leq N$, denote by $\lambda(n)$ the composition λ containing only the n first cells.

Proposition 8.43. *Let $\varepsilon > 0$. There exists $n_0 \geq 1$ such that for any $n \geq n_0$ and any composition λ of size larger than n with first run smaller than n,*

$$\|F^{\mathcal{S}_\lambda}_{X_I} - F^{\mathcal{S}_{\lambda(n)}}_{X_I}\|_\infty \leq \varepsilon.$$

The proof consists only in an averaging of the inequality of the previous Lemma.
We will close this chapter by proving Theorem 8.3.
Let λ be a composition and let $s = [\![i_1, i_2]\!]$ be a run of λ. For a cell i in s, the position of i in s, denoted by a_i, is the ratio $a_i = \frac{i-i_1}{i_2-i_1}$ (resp. $\frac{i_2-i}{i_2-i_1}$) if the run is increasing (resp. decreasing). When a run is large, the behavior of a cell in this run is approximately frozen:

Lemma 8.44. *Let $\varepsilon > 0$. There exists $R_\varepsilon > 0$ such that for any composition λ of n and $1 \leq i \leq n$ such that i is in a run s of size larger than R_ε,*

$$\mathbb{P}(|\frac{\sigma_\lambda(i)}{n} - a_i| \geq \varepsilon) \leq \varepsilon,$$

where a_i is the position of i in s as previously defined.

Proof. Let λ be a composition of n, and let $1 \leq i \leq n$ be a cell of λ in a run s of length R. Let $i_1 \leq i_2$ be the extreme cells of the run s and suppose without loss of generality that s is increasing. We use the probabilistic model $\tilde{\mathcal{S}}_\lambda$ of Section 8.3.2. By Lemma 8.16, it suffices to prove that for R large enough,

$$\mathbb{P}(|Z_i - a_i| \geq \varepsilon) \leq \varepsilon.$$

Conditioning Z_{i_1} on the value of Z_{i_1-1} and Z_{i_2} gives the conditional expectation:

$$\mathbb{E}(Z_{i_1}|Z_{i_1-1} = z, Z_{i_2} = z') = \frac{\int_0^{z\wedge z'} x(z'-x)^{R-2}dx}{\int_0^{z\wedge z'}(z'-x)^{R-2}dx} \leq \frac{1}{R},$$

where the last bound is given by a computation of the integral. Since the bound is independent of z and z', for R large enough $\mathbb{P}(Z_{i_1} \geq \varepsilon) \leq \varepsilon$. Likewise, for R large enough, $\mathbb{P}(Z_{i_2} \leq 1-\varepsilon) \leq \varepsilon$. This gives the result if $i = i_1$ or $i = i_2$. Suppose that $i \neq i_1$ and $i \neq i_2$.
Conditioned on the value of Z_{i_1} and Z_{i_2}, the law of Z_i is

$$d_{Z_i|Z_{i_1}=z, Z_{i_2}=z'}(x) = \frac{\mathbf{1}_{z\leq x\leq z'}(z'-x)^{i_2-i-1}(x-z)^{i-i_1-1}}{\int_z^{z'}(z'-x)^{i_2-i-1}(x-z)^{i-i_1-1}dx}.$$

Thus, by a computation, the conditional expectation of $Z_i - z$ is

$$\mathbb{E}\left(Z_i - z|Z_{i_1} = z, Z_{i_2} = z'\right) = (z'-z)\frac{i-i_1}{i_2-i_1},$$

and the conditional variance of $Z_i - z$ is

$$Var\left(Z_i - z|Z_{i_1} = z, Z_{i_2} = z'\right) = (z'-z)^2\frac{i-i_1}{i_2-i_1}\left(\frac{i-i_1+1}{i_2-i_1+1} - \frac{i-i_1}{i_2-i_1}\right) \leq (z'-z)^2\frac{1}{R}.$$

Thus, for R large enough, $\mathbb{P}(|Z_i - (Z_{i_1} + a_i(Z_{i_2} - Z_{i_1}))| \geq \varepsilon) \leq \varepsilon$.
By the first part of the proof, for R large enough $\mathbb{P}(Z_{i_1} \geq \varepsilon) \leq \varepsilon$ and $\mathbb{P}(Z_{i_2} \leq 1-\varepsilon) \leq \varepsilon$; thus, for R large enough,

$$\mathbb{P}(|Z_i - a_i| \geq \varepsilon) \leq \varepsilon.$$

□

We can improve the result of Corollary 8.42 by considering the case of a cell in the middle of a composition.

Lemma 8.45. *Let $\varepsilon > 0, R > 0$. There exists $k_R \geq 1$ such that for any composition λ and $1 \leq j_1 < i < j_2 \leq n$ such that i is in a run bounded by R and $|i-j_1|, |j_2-i| \geq k_R$, then*

$$\|d_{Z_i|Z_{j_1}=z_1, Z_{j_2}=z_2} - d_{Z_i|Z_{j_1}=z_1', Z_{j_2}=z_2'}\|_\infty \leq \varepsilon$$

for all $0 \leq z_1, z_2, z_1', z_2' \leq 1$, where Z_i is the random variable corresponding to the particle i in $\tilde{\mathcal{S}}_\lambda$. Likewise,

$$\|d_{Z_i|Z_{j_1}=z_1} - d_{Z_i|Z_{j_1}=z_1'}\|_\infty \leq \varepsilon$$

and

$$\|d_{Z_i|Z_{j_2}=z_2} - d_{Z_i|Z_{j_2}=z_2'}\|_\infty \leq \varepsilon$$

for all $0 \leq z_1, z_2, z_1', z_2' \leq 1$.

Proof. We will only prove the first part of the Lemma, since the proof of the second part is a simpler version of the one of the first part.
Let λ be a composition and let $1 \leq j_1 < i < j_2 \leq n$ be three cells of λ. By the expression of the density in (8.3.3),

$$d_{Z_i|Z_{j_1}=z_1, Z_{j_2}=z_2}(x) = \frac{d_{X_F|X_I=z_1, \mathcal{S}_{\nu_1}}(x) d_{X_I|X_F=z_2, \mathcal{S}\nu_2}(x)}{\int_0^1 d_{X_F|X_I=z_1, \mathcal{S}_{\nu_1}}(x) d_{X_I|X_F=z_2, \mathcal{S}\nu_2}(x)dx},$$

where $\nu_1 = \lambda_{j_1 \to i}$ and $\nu_2 = \lambda_{i \to j_2}$. Since i is in a run bounded by R in λ, i is in a run bounded by R in ν_1 and in ν_2. Therefore by Corollary 8.41, there exists n_ε such that if $|\nu_1| \geq n_\varepsilon$ and $|\nu_2| \geq n_\varepsilon$, then

$$\|d_{X_F|X_I=z_1, \nu_1} - d_{X_F|X_I=z_1', \nu_1}\|_\infty \leq \varepsilon$$

and

$$\|d_{X_I|X_F=z_2, \nu_2} - d_{X_I|X_F=z_2', \nu_2}\|_\infty \leq \varepsilon,$$

for all $0 \leq z_1, z_2, z_1', z_2' \leq 1$. Moreover, by Lemma 8.28, $d_{X_F|X_I=z_1,\nu_1}$ is bounded by some constant K only depending on R, and the same holds for $d_{X_I|X_F=z_2,\nu_2}$. Therefore

$$\|d_{X_F|X_I=z_1,\nu_1}(x) d_{X_I|X_F=z_2,\nu_2}(x) - d_{X_F|X_I=z_1',\nu_1}(x) d_{X_I|X_F=z_2',\nu_2}(x)\|_\infty \leq 2A\varepsilon$$

for $0 \leq z_1, z_1', z_2, z_2' \leq 1$. In particular,

$$|\int_0^1 d_{X_F|X_I=z_1,\nu_1}(x) d_{X_I|X_F=z_2,\nu_2}(x) - d_{X_F|X_I=z_1',\nu_1}(x) d_{X_I|X_F=z_2',\nu_2}(x)dx| \leq 2A\varepsilon.$$

Set

$$A_{z_1,z_2} = \int_0^1 d_{X_F|X_I=z_1,\nu_1}(x) d_{X_I|X_F=z_2,\nu_2}(x)dx, B_{z_1,z_2} = d_{X_F|X_I=z_1,\nu_1}(x) d_{X_I|X_F=z_2,\nu_2}(x).$$

By the above computations,

$$\begin{aligned} |\frac{B_{z_1,z_2}}{A_{z_1,z_2}} - \frac{B_{z_1',z_2'}}{A_{z_1',z_2'}}| \leq & |\frac{B_{z_1,z_2}}{A_{z_1,z_2}} - \frac{B_{z_1',z_2'}}{A_{z_1,z_2}}| + |\frac{B_{z_1',z_2'}}{A_{z_1,z_2}} - \frac{B_{z_1',z_2'}}{A_{z_1',z_2'}}| \\ \leq & \frac{1}{A_{z_1,z_2}}(2R\varepsilon) + \frac{B_{z_1',z_2'}}{A_{z_1,z_2} A_{z_1',z_2'}}(2R\varepsilon). \end{aligned}$$

It remains to show that $\frac{1}{A_{z_1,z_2}}$ and $\frac{B_{z_1',z_2'}}{A_{z_1,z_2} A_{z_1',z_2'}}$ are bounded by a constant only depending on R. Since i is in a run bounded by R in ν_1 and ν_2, $|B_{z_1,z_2}|$ is bounded by K^2, where K is the

constant given Lemma 8.28 for a run of size R.
Let us show that A_{z_1,z_2} admits a lower bound only depending on R; suppose without loss of generality that the run of λ containing i is increasing and that i is not an extreme cell. Let R_1 be the length of the run containing i in ν_1 and let R_2 be the length of the run containing i in ν_2; since these both runs are part of the run of i in λ, they are both increasing and $R_1+R_2=R+1$. By Corollary 8.25, $t^{R_1} \leq F_{X_F|X_I=z_1,\nu_1}(t) \leq t^{R_1-1}$ and $1-(1-t)^{R_2-1} \leq F_{X_I|X_F=z_2,\nu_2}(t) \leq 1-(1-t)^{R_2}$ for $0 \leq t \leq 1$. By Lemma 8.14, $d_{X_F|X_I=z_1,\nu_1}$ is increasing and $d_{X_I|X_F=z_2,\nu_2}$ is decreasing, thus $F_{X_F|X_I=z_1,\nu_1}$ is convex and $F_{X_I|X_F=z_2,\nu_2}$ is concave. The convexity of $F_{X_F|X_I=z_1,\nu_1}$ yields that

$$F'_{X_F|X_I=z_1,\nu_1}(t) \geq \frac{F_{X_F|X_I=z_1,\nu_1}(t)-F_{X_F|X_I=z_1,\nu_1}(0)}{t-0} \geq t^{R_1-1}.$$

Likewise, the concavity of $F_{X_I|X_F=z_2,\nu_2}$ yields that

$$F'_{X_I|X_F=z_2,\nu_2}(t) \geq \frac{F_{X_I|X_F=z_2,\nu_2}(1)-F_{X_I|X_F=z_2,\nu_2}(t)}{1-t} \geq (1-t)^{R_2-1}.$$

Therefore,

$$A_{z_1,z_2} \geq \int_0^1 x^{R_1-1}(1-x)^{R_2-1}dx = \frac{(R_1-1)!(R_2-1)!}{(R_1+R_2-1)!} \geq \frac{1}{(R_1+R_2-1)!}.$$

Since $R_1+R_2-1=R$, $A_{z_1,z_2} \geq \frac{1}{R!}$. This yields

$$|\frac{B_{z_1,z_2}}{A_{z_1,z_2}} - \frac{B_{z'_1,z'_2}}{A_{z'_1,z'_2}}| \leq (2R\varepsilon)(R!+K^2(R!)^2).$$

Thus, if $\min(|\nu_1|,|\nu_2|) \geq n_\varepsilon$, then

$$\|d_{Z_i|Z_{j_1}=z_1,Z_{j_2}=z_2} - d_{Z_i|Z_{j_1}=z'_1,Z_{j_2}=z'_2}\|_\infty \leq (2R\varepsilon)(R!+K^2(R!)^2),$$

for all $0 \leq z_1,z_2,z'_1,z'_2 \leq 1$. Setting $k_R = n_{\varepsilon/(2R(R!+K^2(R!)^2)}$ gives the appropriate constant for the statement of the Lemma. □

We can now prove Theorem 8.3.

Proof of Theorem 8.3. The proof is done by recurrence on r.
Let $r=2$. Let $\varepsilon>0$ and R_ε be the constant from Lemma 8.44. Let λ be a composition of n and let $1 \leq i < j \leq n$ be two cells of λ. If i and j are both in runs larger than R_ε, then by Lemma 8.44, $\mathbb{P}(|\frac{\sigma_\lambda(i)}{n}-a_i| \geq \varepsilon) \leq \varepsilon$ and $\mathbb{P}(|\frac{\sigma_\lambda(j)}{n}-a_j| \geq \varepsilon) \leq \varepsilon$. Therefore,

$$\begin{aligned}\pi\left(\mu\left(\frac{\sigma_\lambda(i)}{n},\frac{\sigma_\lambda(j)}{n}\right),\mu(\frac{\sigma_\lambda(i)}{n})\otimes\mu(\frac{\sigma_\lambda(j)}{n})\right) &\leq \pi\left(\mu(\frac{\sigma_\lambda(i)}{n},\frac{\sigma_\lambda(j)}{n}),\delta_{a_i}\otimes\delta_{a_j}\right) \\ &+\pi\left(\delta_{a_i}\otimes\delta_{a_j},\mu(\frac{\sigma_\lambda(i)}{n})\otimes\mu(\frac{\sigma_\lambda(j)}{n})\right) \leq 2\varepsilon.\end{aligned}$$

Suppose without loss of generality that i is in a run smaller than R_ε. On the one hand, for $0 \leq t_1,t_2 \leq 1$,

$$F_{Z_i,Z_j}(t_1,t_2)-F_{Z_i}(t_1)F_{Z_j}(t_2) = \int_0^{t_2}\left(\int_0^{t_1} d_{Z_i|Z_j=y}(x)-d_{Z_i}(x)dx\right)d_{Z_j}(y)dy.$$

On the other end, by Lemma 8.45, there exists k such that if $|j-i| \geq k$,

$$\|d_{Z_i|Z_j=z}-d_{Z_i|Z_j=z'}\|_\infty \leq \varepsilon$$

for any $0 \leq z, z' \leq 1$. Therefore, for $|j-i| \geq k$, $\|d_{Z_i|Z_j=y} - d_{Z_i}\|_\infty \leq \varepsilon$ for $0 \leq y \leq 1$. This yields

$$|F_{Z_i,Z_j}(t_1,t_2) - F_{Z_i}(t_1)F_{Z_j}(t_2)| \leq \int_0^{t_2} t_1 \varepsilon d_{Z_j}(y)dy \leq \varepsilon.$$

In particular,

$$\pi\left(\mu(Z_i,Z_j), \mu(Z_i) \otimes \mu(Z_j)\right) \leq \varepsilon.$$

Lemma 8.16 concludes the case $r=2$.

Suppose that $r > 2$. Let λ be a composition and let $1 \leq i_1, \ldots, i_r \leq n$ be distinct cells of λ. If $i_1, \ldots, i_r$ are all in runs larger than R_ε, by the same reason as before,

$$\pi\left(\mu\left(\frac{\sigma_\lambda(i_1)}{n}, \ldots, \frac{\sigma_\lambda(i_r)}{n}\right), \mu(\frac{\sigma_\lambda(i_1)}{n}) \otimes \cdots \otimes \mu(\frac{\sigma_\lambda(i_r)}{n})\right) \leq 2\varepsilon.$$

Suppose without loss of generality that i_r is in a run bounded by R_ε, and let k be the constant associated to R_ε in Lemma 8.45. By recurrence hypothesis, there exists k_1 such that if $i_j - i_{j-1} \geq k_1$ for $2 \leq j \leq r-1$, then

$$\pi\left(\mu(Z_{i_1}, \ldots, Z_{i_{r-1}}), \mu(Z_{i_1} \otimes \cdots \otimes \mu(Z_{i_{r-1}})\right) \leq \varepsilon.$$

On the one hand for $\vec{t} \in [0,1]^r$,

$$\begin{aligned} &F_{(Z_i)_{1\leq i \leq r}}(\vec{t}) - F_{Z_{i_r}}(t_r)F_{(Z_{i_s})_{s<r}}((t_s)_{s<r}) \\ &\qquad = \int_{x_s \in [0,t_s]} \left(d_{Z_{i_r}|Z_{i_s}=x_s, s<r}(x_r) - d_{Z_{i_r}}(x_r)\right) d_{(Z_{i_s})_{s<r}}((x_s)_{s<r}) \prod_{s=1}^{r} dx_s. \end{aligned}$$

By Formula (8.3.3), $d_{Z_{i_r}|Z_{i_1}=x_1, \ldots Z_{i_{r-1}}=x_{r-1}}(x_r) = d_{Z_{i_r}|Z_{i_a}=x_a, Z_{i_b}=x_b}(x_r)$, where a and b are such that i_a is the cell of $\{i_1, \ldots, i_{r-1}\}$ directly before i_r and i_b is the cell of $\{i_1, \ldots, i_{r-1}\}$ directly after i_r. By Lemma 8.45, if $i_r - i_a \geq k$ and $i_b - i_r \geq k$, then

$$\|d_{Z_{i_r}|Z_{i_a}=x_a, Z_{i_b}=x_b} - d_{Z_{i_r}}\|_\infty \leq \varepsilon.$$

Thus,

$$|F_{(Z_{i_s})_{1\leq s\leq r}}(\vec{t}) - F_{Z_{i_r}}(t_r)F_{(Z_{i_s})_{s<r}}((t_i)_{i<r})| \leq \int_{x_s \in [0,t_s], s<r} \varepsilon d_{(Z_{i_s})_{s<r}}((x_s)_{s<r}) \prod_{s<r} dx_s \leq \varepsilon,$$

which yields

$$\pi(\mu((Z_{i_1}, \ldots, Z_{i_r}), \mu(Z_{i_r}) \otimes \mu((Z_{i_s})_{s<r})) \leq \varepsilon.$$

Finally,

$$\begin{aligned} \pi\left(\mu\left(Z_{i_1}, \ldots, Z_{i_r}\right), \mu(Z_{i_1}) \otimes \cdots \otimes \mu(Z_{i_r})\right) &\leq \pi\left(\mu\left(Z_{i_1}, \ldots, Z_{i_r}\right), \mu(Z_{i_r}) \otimes \mu((Z_{i_s})_{s<r}\right) \\ &+ \pi\left(\mu(Z_{i_r}) \otimes \mu((Z_{i_s})_{s<r}), \mu(Z_{i_1}) \otimes \cdots \otimes \mu(Z_{i_r})\right) \leq \varepsilon + \varepsilon \leq 2\varepsilon. \end{aligned}$$

□

Chapter 9

Martin bounday of $\mathcal{Z}$

9.1 Introduction

The lattice $\mathcal{Z}$ of zigzag diagrams is a graded graph whose vertices of degree n are labelled by compositions of n (which can be seen as ribbon Young diagrams). The study of this kind of lattices drew increasing interests these last decades, due to their interactions with representations of semi-simple algebras and with discrete random walks. In particular an other example of graded graph, the Young lattice $\mathcal{Y}$, has been deeply studied by Vershik, Kerov and other authors (see [47] for a review on the subject), yielding major breakthroughs on the representation theory of S_∞ and on the asymptotic study of certain particle systems. As explained in [42], the lattices $\mathcal{Z}$ and $\mathcal{Y}$ are somehow related, since the latter can be seen as a projection of the former.
The connection between the lattice structure and its probabilistic applications is made through the study of harmonic functions on the associated graph. One of the first tasks is therefore the characterization of harmonic functions on the lattice ; it is then possible to single out particular harmonic functions and study the random variables they generate. A general framework for the representation of harmonic functions on a graph E has been initiated by Martin in [61], with the concept of Martin boundary $\partial_M E$ and minimal boundary $\partial E_{\min}$: the Martin boundary is a topological space coming from the graph and allowing to establish a bijection between positive harmonic functions and measures whose support is included in a particular subset of $\partial_M E$. The latter subset is precisely the minimal boundary $\partial E_{\min}$. It is therefore important to know both $\partial_M E$ and $\partial E_{\min}$ to provide a topological and measure theoretic approach to the kernel representation of harmonic functions (see [34] for an exhaustive review on the subject).
In general $\partial E_{\min}$ is strictly included in $\partial_M E$. However in many cases the two coincide, as it happens for example for the lattice $\mathcal{Y}$. In this chapter we prove that the two boundaries also coincide for the lattice $\mathcal{Z}$. The minimal boundary of $\mathcal{Z}$ has already been described by Gnedin and Olshanski in[42], through the so-called oriented paintbox construction, and thus it remains to prove that any element of the Martin boundary fits in this construction. As an application we provide a precise link between harmonic measures on $\mathcal{Y}$ and harmonic measures on $\mathcal{Z}$: this link was already exposed in [42], and in the present chapter we explain this relation by mapping directly paths on $\mathcal{Z}$ to paths on $\mathcal{Y}$. Finally we study the behavior of a random path with respect to the Plancherel measure by providing a Central Limit Theorem.
Section 2 and 3 are devoted to preliminaries : the first gives necessary backgounds on Martin boundary, and the second describes the graph $\mathcal{Z}$ together with its link with compositions. The results of Gnedin and Olshanski on this graph are given in Section 4. In this section we provide also the pattern of the proof for the identification of the Martin boundary.
The proof heavily relies on combinatorics of compositions. In particular the Martin kernel of $\mathcal{Z}$, a two parameters function that characterizes the Martin boundary, is related to standard

fillings of compositions. Two constructions are needed in order to identify the Martin boundary: Section 5 deals with the first one, which is the construction of a sequence of random variables that relates the Martin kernel to the oriented Paintbox construction of Gnedin and Olshanski. The second one has been done in Chapter 8 and is a general framework that gives combinatorial estimates on compositions. Some results of Chapter 8 are recalled in Section 6. Section 7 and 8 show the identification of the Martin boundary. Finally Section 9 gives the map between paths on $\mathcal{Z}$ and paths on $\mathcal{Y}$ and exposes probabilistic results associated to a particular point of the Martin boundary, called the Plancherel measure (due to its relations with the Plancherel measure on the graph $\mathcal{Y}$).
We should stress that, as it has been explained to us by Jean-Yves Thibon, the map between the paths on the two graphs appears clearly by using the algebra **FQSym** of Free Quasi-Symmetric functions; although this algebra won't be described in this chapter, the interested reader should refer to the Chapter 3 of [35] for an introduction to **FQSym** and an explanation from a Hopf algebraic point of view to the construction we are doing in Section 9.2 of the present chapter.

9.2 Graded graphs and Martin boundary

This section is a discussion that introduces the concept of Martin boundary and motivates its role in the framework of graded graphs. All these results and proofs can be found in [34].

9.2.1 Graded graphs and random walks

The notations used here are from [79]. A rooted graded graph $\mathcal{G}$ is the data of a triple (V, ρ, E) where :

- V is a denumerable set of vertices with a distinguished element μ_0.
- $\rho : V \to \mathbb{N}$ is a rank function with $\rho^{-1}(\{0\}) = \{\mu_0\}$.
- The adjacency matrix E is a $V \times V$-matrix with entries in $\{0, 1\}$, such that $E(\mu, \nu)$ is zero if $\rho(\nu) \neq \rho(\mu) + 1$.

We write $\mu \nearrow \nu$ if $E(\mu, \nu) = 1$. A path on $\mathcal{G}$ is sequence of vertices $(\mu_1, \dots, \mu_n, \dots)$ of increasing degree such that for all $i \geq 1$, $\mu_i \nearrow \mu_{i+1}$. For a given graded graph the paths counting function $d : \mathcal{V} \to \mathbb{N}^*$ is the function that gives for each vertex $\mu \in \mathcal{G}$ the number of paths between μ_0 and μ. When the endpoints of a path are not specified, the path is considered as an infinite path starting at the root.
There is a natural way of constructing random walks that respect the structure of the graph $\mathcal{G}$: such a random walk starts at μ_0, and at each step the successor is chosen according to a transition matrix P, with the condition that $P(\mu, \nu) = 0$ if $E(\mu, \nu) = 0$ and $\sum_\nu P(\mu, \nu) = 1$. Thus each transition matrix P associates to any path $\lambda = (\mu_0 \nearrow \mu_1 \cdots \nearrow \mu_n)$ a weight p_λ which is the probability of the realization of λ, namely

$$p_\lambda = \mathbb{P}(X_0 = \mu_0, X_1 = \mu_1, \dots, X_n = \mu_n) = \frac{1}{Z} P(\mu_0, \mu_1) \dots P(\mu_{n-1}, \mu_n).$$

For some transition matrices P on $\mathcal{G}$, the weight $p(\lambda)$ only depends on the final vertex of the path (in this case we write $p(\lambda) = p(\mu)$ for any path λ between μ_0 and μ); such a transition matrix is called a harmonic matrix. In this case, a staightforward computation shows that p, the associated weight function, must verify

$$p(\mu) = \sum_{\mu \nearrow \nu} p(\nu), \tag{9.2.1}$$

and conversely, any positive solution p of (9.2.1) such that $p(\mu_0) = 1$ yields a harmonic matrix. This can be interpreted in terms of potential theory.
Let X be a denumerable states space with transition matrix P. Let $H(X,P)^+$ (resp. $M(X,P)$) denote the set of positive harmonic functions (resp. positive harmonic measures), which is the set of functions $f : X \to \mathbb{R}^+$ satisfying $\sum_y P(x,y)f(y) = f(x)$ (resp. $\sum_x f(x)P(x,y) = f(y)$). For each $\alpha \in M(X,P)$ let the dual transition matrix P^t_α be defined by the expression

$$P^t_\alpha(x,y) = \mathbf{1}_{\alpha(x)\neq 0}\frac{\alpha(y)}{\alpha(x)}P(y,x),$$

if $x \neq y$, and $P^t(x,x) = 1 - \sum_{x\neq y} P^t(x,y)$. Then P^t_α is indeed a transition matrix on X and the following maps are well-defined:

$$H_\alpha : \begin{cases} H(X,P)^+ & \to & M(X,P^t_\alpha) \\ h & \mapsto & (x \mapsto \mathbf{1}_{\alpha(x)>0}\frac{1}{\alpha(x)}h(x)) \end{cases},$$

and

$$M_\alpha : \begin{cases} M(X,P) & \to & H(X,P^t_\alpha)^+ \\ m & \mapsto & (x \mapsto \alpha(x)m(x)) \end{cases}.$$

The two maps are bijective if $\alpha > 0$ on X.
Let P be a transition matrix on a graded graph $\mathcal{G}$; by a recursive computation there exists a unique invariant measure α_P with respect to P such that $\alpha_P(\mu_0) = 1$. If P_p is a harmonic matrix, with p the associated weight function, then $P(\mu,\nu) = \mathbf{1}_{\mu\nearrow\nu}\frac{p(\nu)}{p(\mu)}$ and $\alpha_p = d(\mu)p(\mu)$. Thus the dual transition matrix is

$$P^t_{\alpha_p}(\nu,\mu) = \mathbf{1}_{\mu\nearrow\nu}\frac{d(\mu)p(\mu)}{d(\nu)p(\nu)}\frac{p(\nu)}{p(\mu)} = \mathbf{1}_{\mu\nearrow\nu}\frac{d(\mu)}{d(\nu)}.$$

In particular $P^t_{\alpha_p}$ is independent of p and, by H_{α_p}, any harmonic function of P comes from an invariant measure of P^t. Conversely let α be an invariant measure of P^t. Then the dual matrix $(P^t)^t_\alpha$ is exactly $P_{\alpha/d}$, the harmonic matrix associated to the weight function $p = \alpha/d$. We can check that the duality yields indeed a bijection between harmonic matrices of $\mathcal{G}$ and elements of $M(\mathcal{G},P^t)$ taking the value 1 on μ_0.
Thus the problem of finding the harmonic matrices on $\mathcal{G}$ is equivalent to the dual problem of finding harmonic measures with respect to P^t. Moreover an answer to the latter problem gives also by duality all the harmonic functions with respect to a harmonic matrix.
Fortunately a general framework, the Martin entrance boundary, describes exactly the harmonic measures associated to a transition matrix.

9.2.2 Martin entrance boundary

Let us take a closer look at the Markov chain $(\mathcal{G},P^t)$. Let $n_0 \geq 1$ and ν a vertex of degree n_0. The random walk $X = (X_n)_{n\geq 0}$ with transition matrix P^t and initial distribution δ_ν goes backward from ν to μ_0 and stops at μ_0 at the times n_0. Let λ be a path between μ_0 and ν; from the definition of the kernel P^t, the probability that X follows the path λ is independent of λ and is therefore $\frac{1}{d(\nu)}$.
For μ of degree $m \leq n_0$, denote by $d(\mu,\nu)$ the number of paths between μ and ν (and by extension $d(\mu,\nu) = 0$ if the degree of μ is larger than the one of ν). By counting the paths going from μ_0 to ν and passing through μ, the probability that $X_{n_0-m} = \mu$ is thus

$$\mathbb{P}(X_{n_0-m} = \mu) = \frac{d(\mu)d(\mu,\nu)}{d(\nu)}.$$

In particular setting $\alpha_\nu(\mu) = \frac{d(\mu)d(\mu,\nu)}{d(\nu)}$ yields a measure α_ν that is harmonic with respect to P^t, except on the vertex ν. To construct actual harmonic measures, it seems thus natural to look at the behavior of α_ν when $\nu \to \infty$. Making the latter rigorous requires to specify a convergence mode for sequences of vertices of increasing degree. Let $K_\mu(\nu) = \frac{d(\mu,\nu)}{d(\nu)}$ be the Martin kernel of $\mathcal{G}$, and define on $\mathcal{G}$ the metric :

$$D(\nu_1, \nu_2) = \sum_\mu \frac{1}{2^{\Gamma(\mu)}} |K_\mu(\nu_1) - K_\mu(\nu_2)|,$$

Γ being any injective function $V \to \mathbb{N}$. Identifying $\nu \in V$ with $K_\cdot(\nu)$, V is seen through this metric as a subset of the space of functions from V to $[0,1]$ with the pointwise convergence topology. Thus by Tychonoff's Theorem the completion $\tilde{V}$ of V with respect to D is a compact space, and by construction K_μ extends continuously on this completion. Actually the completion is exactly the set of sequences $(\nu_n)_{n\geq 1}$ such that for each μ, $K_\mu(\nu_n)$ converges, with two sequences $(\nu_n^1)_{n\geq 1}$, $(\nu_n^2)_{n\geq 1}$ being identified whenever for each μ, $K_\mu(\nu_n^1)$ and $K_\mu(\nu_n^2)$ have the same limit. Denote by $\partial_M \mathcal{G}$ the set $\tilde{V} \setminus V$. The latter is called the Martin entrance boundary of the graded graph $\mathcal{G}$ and is a compact subset of $\tilde{V}$. Each element $\omega = \lim\limits_{n\to\infty} \nu_n$ in $\partial_M \mathcal{G}$ defines a function on V by the formula

$$\omega(\mu) = \lim K_\mu(\nu_n).$$

The following Theorem is a special case of a Theorem from Doob ([34]).

Theorem 9.1. *With the notations above, the two following results hold:*

- *There exists a Borel subset $\partial_{\min} \mathcal{G} \subseteq \partial_M \mathcal{G}$ (called minimal boundary) such that for any measure α harmonic with respect to P^t, there exists a unique measure λ_α on $\partial_{\min} \mathcal{G}$ giving the kernel representation*
$$\alpha(\mu) = \int_{\partial_{\min} \mathcal{G}} K_\mu(x) d\lambda_\alpha(x).$$
- *For any reverse random walk $(X_n)_{n\leq 0}$ that respects P^t, the path $(X_0, X_{-1}, \dots)$ converges almost surely to a $\partial_{\min} \mathcal{G}-$valued random variable $X_{-\infty}$. Moreover the probability that $(X_n)_{n\leq 0}$ goes through μ is exactly $d(\mu)\mathbb{E}(K_\mu(X_{-\infty}))$.*

There exists a more general construction of the Martin boundary from Kunita and Watanabe in [53], which encompasses the case of discrete random walks as well as more general Markov processes (including the Brownian motion on a domain). However our situation is much simpler and the previous Theorem is enough.
To summarize, the Martin entrance boundary gives a topological framework to represent harmonic measures, whereas the minimal entrance boundary gives the corresponding measure theoretic framework. The situation is simpler when the two boundaries coincide. In the case of the graph $\mathcal{Z}$ that we are studying, the minimal entrance boundary was already described by Gnedin and Olshanski in [42]. The purpose of the present chapter is to extend this desciption to the Martin entrance boundary by proving that the two boundary coincide.

9.3 The graph $\mathcal{Z}$

This section is devoted to an introduction to the graph $\mathcal{Z}$ and its relation with sequences of permutations. All the results from this section can de found in [42].

9.3.1 Compositions

Let us first recall the definition of a composition:

Definition 9.2. *Let $n \in \mathbb{N}$. A composition λ of n, written $\lambda \vdash n$, is a sequence of positive integers $(\lambda_1, \ldots, \lambda_r)$ such that $\sum \lambda_j = n$.*

Let D_λ be the subset of $[1; n]$ defined by $D_\lambda = \{\lambda_1, \lambda_1 + \lambda_2, \ldots, \sum_1^{r-1} \lambda_i\}$. Since there is a bijection between subsets of $[1; n-1]$ and compositions of n, D_λ is often simply denoted by λ. To a composition is also associated a unique ribbon Young diagram with n cells: each row j has λ_j cells, and the first cell of the row $j+1$ is just below the last cell of the row j. For example the composition $(3, 2, 4, 1)$ of 10 is represented in Figure 9.1.

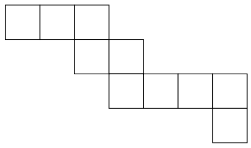


Figure 9.1: Skew Young tableau associated to the composition $\lambda = (3, 2, 4, 1)$.

The size n is included in the definition of composition itself, since n is equal to the sum of all λ_j. If we want to emphasize the size of a compostion λ, we denote it as $|\lambda|$. When nothing is specified, λ is always assumed to have the size n, and n always denotes the size of the composition λ.
A standard filling of a composition λ of size n is a standard filling of the associated ribbon Young diagram: it is the assignement of an integer from 1 to n to each cell of the composition, such that every cells have different entries, and the entries are increasing to the right along the rows and decreasing to the bottom along the columns. An example for the composition of Figure 9.1 is shown in Figure 9.2.

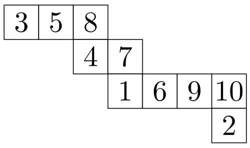


Figure 9.2: Standard filling of the composition $(3, 2, 4, 1)$.

In particular, reading the tableau from left to right and from top to bottom gives for each standard filling a permutation σ; moreover the descent set $des(\sigma)$ of σ, namely the set of indices i such that $\sigma(i+1) < \sigma(i)$, is exactly the set D_λ. There is a bijection between the standard fillings of λ and the permutations of $|\lambda|$ with descent set D_λ. For example the filling in Figure 9.2 yields the permutation $(3, 5, 8, 4, 7, 1, 6, 9, 10, 2)$.

9.3.2 The graph $\mathcal{Z}$

The graded graph $\mathcal{Z}$, which was introduced by Viennot in [89], is defined as follows:

1. The set $\mathcal{Z}_n$ of vertices of degree n of $\mathcal{Z}$ is the set of compositions of n. The vertex of degree 0 is denoted $\emptyset$.

2. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ be two compositions. There is an edge between μ and λ if and only if $|\lambda| = |\mu| + 1$ and

- either $r = s$ and for each i except one $\mu_i = \lambda_i$ (thus exactly one μ_{i_0} is increased by one)
- either $r = s+1$, and there exists j such that: for $k < j$, $\lambda_k = \mu_k$, $\lambda_j + \lambda_{j+1} - 1 = \mu_j$, and for $k > j$, $\lambda_{k+1} = \mu_k$ (namely one μ_i is split, and one cell is added at the end of the first piece).

The first four levels of $\mathcal{Z}$ are displayed in Figure 9.3.

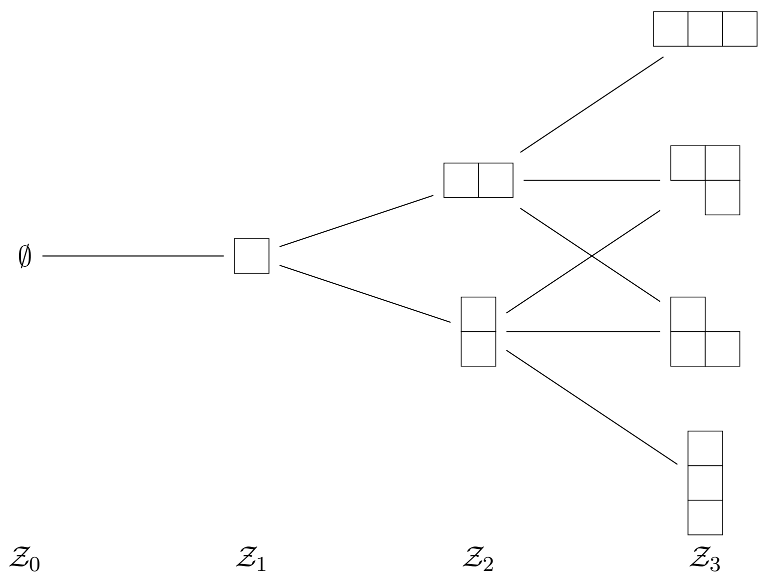


Figure 9.3: Vertices of $\mathcal{Z}$ of degree 0 to 3.

For a composition λ, let Ω_λ be the set of paths between $\emptyset$ and λ. It has been shown in [89] that $\Omega_\lambda \simeq \{\sigma \in S_{|\lambda|}, des(\sigma) = D_\lambda\}$. One way to see this is to remark that Ω_λ is the set of all standard fillings of the ribbon diagram associated to λ. Thus these sets have same cardinality and

$$d(\lambda) = |\Omega_\lambda| = \#\{\sigma \in S_{|\lambda|}, des(\sigma) = D_\lambda\}.$$

Let $\mathbb{P}_\lambda$ denote the uniform distribution on Ω_λ; from Section 2, this is equivalent to considering the random walk starting at λ with transition matrix P^t. This random walk gives n random variables σ_k^λ, $1 \leq k \leq n$, each of them being the random path restricted to the vertices of degree smaller than k.
Since there is a bijection between paths on $\mathcal{Z}$ from $\emptyset$ to μ and permutations of $|\mu|$ with descent set D_μ, each σ_k^λ is a random permutation in S_k, and the law of $\sigma_\lambda = \sigma_n^\lambda$ is the uniform distribution on the set of permutations with descent set D_λ. Moreover a counting argument yields that for $\sigma \in \mathfrak{S}_k$ with $des(\sigma) = D_\mu$, under the probability $\mathbb{P}_\lambda$,

$$\mathbb{P}_\lambda(\sigma_\lambda^k = \sigma) = \frac{d(\mu, \lambda)}{d(\lambda)}. \tag{9.3.1}$$

By abuse of notation a finite path starting at $\emptyset$ on $\mathcal{Z}$ and the corresponding permutation are both usually denoted by σ. In particular if $\sigma \in \Omega_\lambda$, σ_k denotes the path after k steps, whereas $\sigma(i)$ will denote the image of i by the permutation associated to σ (the same for $\sigma(A)$ with A a subset of $\{1, \dots, n\}$).

9.3.3 Arrangement on $\mathbb{N}$

In this paragraph a permutation $\sigma \in \mathfrak{S}_k$ is written as a word in the alphabet $\{1, \dots, k\}$, where $i_j = \sigma(j)$. For $k \geq 2$ and $\sigma = (i_1 \dots i_k)$, $\sigma_\downarrow \in \mathfrak{S}_{k-1}$ is defined as the permutation $(i_1 \dots \cancel{k} \dots i_k)$. If $\sigma \in \mathfrak{S}_n$, $\sigma_{\downarrow k}$ denotes the $(n-k)-$iteration of the $\downarrow$-operation : namely all the indices between $k+1$ and n have been erased.
An arrangement of $\mathbb{N}$ is a sequence $(\sigma_1, \dots, \sigma_k, \dots)$ such that for all $k \geq 1$, $\sigma_k \in \mathfrak{S}_k$, and such that the following compatibility condition holds :

$$(\sigma_k)_\downarrow = \sigma_{k-1}.$$

For example the following sequence is the first part of an arrangement :

$$((1), (21), (231), (2341), (52341), \dots).$$

The set of all arrangements is denoted $\mathcal{A}$. For $k \geq 1$, let $\pi_k : \mathcal{A} \to \mathfrak{S}_k$ be the map which consists in the projection of the sequence $(\sigma_1, \sigma_2, \dots)$ on the $k-$th element σ_k. $\mathcal{A}$ is considered with the initial topology with respect to the maps π_k, and with the corresponding borelian $\sigma-$algebra. Thus by the Kolmogorov's extension Theorem, any random variable Π on $\mathcal{A}$ is uniquely determined by the law of its finite-dimensional projections $(\pi_1(\Pi), \dots, \pi_k(\Pi))$.
The result of the previous subsection yields that there is a bijection between infinite paths on $\mathcal{Z}$ and arrangements of $\mathbb{N}$, and from Section 2 this bijection extends to a bijection between harmonic measures α with respect to P^t and random arrangements Π such that

$$\mathbb{P}(\pi_1(\Pi) = \sigma_1, \dots, \pi_k(\Pi) = \sigma_k) = p(des(\sigma_k)),$$

with p a positive function on $\mathcal{Z}$ given by $p = \frac{\alpha}{d}$. This correspondance is convenient since it allows to describe the solutions of the problem (9.2.1) in terms of random arrangements.

9.4 Paintbox construction and Minimal boundary

Thanks to the latter correspondance, Gnedin and Olshanski described the minimal entrance boundary of $\mathcal{Z}$ in terms of random arrangements. This description is the purpose of the following paragraph.

9.4.1 Paintbox construction

The description is based on a topological space consisting in pairs of disjoint open sets of $[0, 1]$:

Definition 9.3. *The topological space $\mathcal{U}^{(2)}$ is the space*

$$(\{(U_\uparrow, U_\downarrow) | U_\uparrow \text{ and } U_\downarrow \text{ disjoint open sets of }]0, 1[\}, d),$$

with the distance d between $(U_\uparrow, U_\downarrow)$ and $(V_\uparrow, V_\downarrow)$ given by

$$d((U_\uparrow, U_\downarrow), (V_\uparrow, V_\downarrow)) = \sup(d_{Haus}(U_\uparrow^c, V_\uparrow^c), d_{Haus}(U_\downarrow^c, V_\downarrow^c)).$$

Let $M_1(\mathcal{U}^{(2)})$ denote the set of probability measures with respect to the $\sigma-$algebra coming from the above topology.
From the definition of the metric, $(U_\uparrow(j), U_\downarrow(j)))_{j\geq 1}$ converges to $(V_\uparrow, V_\downarrow))$ if and only if, for each $\varepsilon > 0$, all of the following phenomena occur:

- for j large enough, the number of connected components of size larger than ε in $U_\uparrow$ and $V_\uparrow$ are the same,
- the boundaries of the connected components of size larger than ε in $U_\uparrow$ converge to the ones of $V_\uparrow$,
- the same holds by switching $\uparrow$ and $\downarrow$.

In particular $(U_\uparrow(j), U_\downarrow(j))$ converges to $(\emptyset, \emptyset)$ if and only if the size of the largest components in $U_\uparrow(j)$ and $U_\downarrow(j)$ tends to 0. The following important result holds for $\mathcal{U}^{(2)}$:

Proposition 9.4 ([42]). *$\mathcal{U}^{(2)}$ is compact space.*

The minimal entrance boundary of $\mathcal{Z}$ is described by random arrangements constructed from elements of $\mathcal{U}^{(2)}$.

Definition 9.5. *Let $U = (U_\uparrow, U_\downarrow)$ be fixed, $(X_1, \dots, X_k, \dots)$ a sequence of $[0, 1]$. For each $k \geq 1$, $\sigma_U(X_1, \dots, X_k) \in \mathfrak{S}_k$ is defined by the following rule:*
$(\sigma_U(X_1, \dots, X_k))^{-1}(i)$ is less than $(\sigma_U(X_1, \dots, X_k))^{-1}(j)$ if and only if one of the three following situations arises :

- *X_i and X_j are not in the same connected component of $U_\uparrow$ or $U_\downarrow$ and $X_i < X_j$*
- *X_i and X_j are in the same connected component of $U_\uparrow$ and $i < j$*
- *X_i and X_j are in the same connected component of $U_\downarrow$ and $j < i$.*

The random variable $\sigma_U(X_1, \dots, X_k)$ defined for an infinite family $(X_1, \dots, X_k, \dots)$ of independent uniform variables on $[0, 1]$ is denoted $\sigma_U(k)$. The sequence $(\sigma_U(1), \sigma_U(2), \dots)$ is denoted σ_U.

The construction of $\sigma_U(X_1, \dots, X_k)$ from $(X_1, \dots, X_k)$ and $U \in \mathcal{U}^{(2)}$ is well-defined and unique. If $U = (\emptyset, \emptyset)$, $\sigma_{(\emptyset,\emptyset)}(X_1, \dots, X_k)$ is just the permutation associated to the reordering $(X_{i_1} < X_{i_2} < \dots X_{i_k})$. This permutation is denoted by $Std^{-1}(X_1, \dots, X_k)$. For each k, the random variable $\sigma_{(\emptyset,\emptyset)}(k)$ has a uniform distribution on $\mathfrak{S}_k$.
The next Theorem is due to Gnedin and Olshanski in [42] (based on an important work of Jacka and Warren in [45]) and identifies $\mathcal{U}^{(2)}$ with the minimal entrance boundary of the graded graph $\mathcal{Z}$:

Theorem 9.6. *Each random variable σ_U defines a random arrangement $\mathcal{A}$ that comes from a harmonic probability measure on $(\mathcal{Z}, P^t)$, and there is an isomorphism :*

$$\Phi : M_1(\mathcal{U}^{(2)}) \longrightarrow M_1(\partial_{\min}\mathcal{Z})$$

which restricts to a bijective map $p : \mathcal{U}^{(2)} \longrightarrow \partial_{\min}\mathcal{Z}$ mapping $\delta_{(U_\uparrow,U_\downarrow)}$ to $\sigma_{(U_\uparrow,U_\downarrow)}$.

In particular for each $k \geq 1$ and $\sigma \in \mathfrak{S}_k$, $\mathbb{P}(\sigma_U(k) = \sigma)$ only depends on the descent set μ of σ and is thus denoted by $p_U(\mu)$.

9.4.2 Martin entrance boundary of $\mathcal{Z}$

The question is to know if $\partial_{\min}\mathcal{Z} = \partial_M\mathcal{Z}$. The problem is summed up in Conjecture 45 of [42]. To each composition λ of n is associated an element $U_\lambda = (U_\uparrow(\lambda), U_\downarrow(\lambda))$ of $\mathcal{U}^{(2)}$ as follows : for each $s \leq n-1$ set $I_s = [\frac{s-1}{n-1}, \frac{s}{n-1}]$, and define

$$U_\uparrow(\lambda) = int(\bigcup_{i \notin des(\lambda)} I_s), U_\downarrow(\lambda) = int(\bigcup_{i \in des(\lambda)} I_s),$$

with int denoting the interior of a set. Then the conjecture states the following :

Conjecture 9.7. a) *A sequence* $(\lambda_n)_{n\geq 1}$ *is in* $\partial_M\mathcal{Z}$ *if and only if* U_{λ_n} *converges in* $\mathcal{U}^{(2)}$.

b) $U_{\lambda_n} \to_{\mathcal{U}^{(2)}} (U_\uparrow, U_\downarrow)$ *is equivalent to* $K_\mu(\lambda_n) \to p_{(U_\uparrow,U_\downarrow)}(\mu)$ *for all* $\mu \in \mathcal{Z}$.

c) *The Martin boundary of the graph* $\mathcal{Z}$ *actually coincides with its minimal boundary :* $\partial_M\mathcal{Z} = \mathcal{U}^{(2)}$

Actually, the only difficult part is to prove the first implication of *b*):

$$(U_{\lambda_n} \to_{\mathcal{U}^{(2)}} (U_\uparrow, U_\downarrow)) \Longrightarrow (\forall \mu \in \mathcal{Z}, K_\mu(\lambda_n) \to p_{(U_\uparrow,U_\downarrow)}(\mu)). \tag{9.4.1}$$

Indeed suppose that the latter is true :

Proof. *a)* Let $\omega = (\lambda_n)_{n\geq 1}$ be in $\partial_M\mathcal{Z}$. Since $\mathcal{U}^{(2)}$ is compact, proving the convergence of U_{λ_n} in $\mathcal{U}^{(2)}$ is the same as proving that every convergent subsequences of U_{λ_n} have the same limit. Let $(\lambda_{\varphi(n)})_{n\geq 1}$ and $(\lambda_{\varphi'(n)})_{n\geq 1}$ be such that

$$U_{\lambda_{\varphi(n)}} \to (U^1_\uparrow, U^2_\downarrow), U_{\lambda_{\varphi'(n)}} \to (U^2_\uparrow, U^2_\downarrow)$$

Then by (9.4.1), for all $\mu \in \mathcal{Z}$, $\omega(\mu) = p_{U^1_\uparrow,U^1_\downarrow}(\mu)$ and $\omega(\mu) = p_{U^2_\uparrow,U^2_\downarrow}(\mu)$. Since $p : \mathcal{U}^{(2)} \to \partial_{\min}\mathcal{Z}$ is injective, necessarily $(U^1_\uparrow, U^1_\downarrow) = (U^2_\uparrow, U^2_\downarrow)$. This shows that U_{λ_n} converges.
Conversely if U_{λ_n} converges in $\mathcal{U}^{(2)}$, the assumption (9.4.1) implies directly that $(\lambda_n) \in \partial_M\mathcal{Z}$.

b) The direct implication is exactly (9.4.1); for the converse implication, the convergence of $K_\mu(\lambda_n)$ for all $\mu \in \mathcal{Z}$ implies that $(\lambda_n)_{n\geq 1} \in \partial_M\mathcal{Z}$. Thus from *a*), U_{λ_n} converges in $\mathcal{U}^{(2)}$. By injectivity of p, U_{λ_n} converges to $(U_\uparrow, U_\downarrow)$.

c) This is the summary of 1) and 2).

□

The following sections are devoted to the proof of the implication (9.4.1), which implies Conjecture 9.7 :

Theorem 9.8. *Let* λ_n *be a sequence of compositions such that* $\lambda_n \vdash n$. *If* U_{λ_n} *converges to* $(U_\uparrow, U_\downarrow)$, *then for all* $\mu \in \mathcal{Z}$,

$$K_\mu(\lambda_n) \to p_{(U_\uparrow,U_\downarrow)}(\mu).$$

The result of Theorem 9.8 roughly means that, for k fixed and λ a large composition such that U_λ is close to $(U_\uparrow, U_\downarrow)$, the restriction to $\{1, \ldots, k\}$ of the uniform random filling λ yields a random variable on $\mathfrak{S}_k$ close to the Paintbox construction $\sigma_{(U_\uparrow,U_\downarrow)}(k)$. Since the Paintbox construction involves for each integer $1 \leq i \leq k$ an independent uniform random variable on $[0, 1]$, we will also create in Section 5 a random variable ξ^λ_i which mimicks the position of i in the uniform random filling of λ. The proof of Theorem 9.8 consists then essentially in proving that the family $(\xi^\lambda_i)_{1\leq i\leq k}$ becomes a family of independent uniform random variables on $[0, 1]$ when λ becomes large. The latter convergence implies that the permutation σ_λ is close to σ_{U_λ}. The fact that U_λ is approximately $(U_\uparrow, U_\downarrow)$ will conclude the proof.
Note that there are two kinds of limit involved in the proof: the limit of the law of $(\xi^{\lambda_n}_i)_{1\leq i\leq k}$ and the topological limit of U_{λ_n}. In order to finalize the proof, we need a final result showing that the order of the limits does not matter. Since the proof of the latter fact is straightforward but lengthy, it is postponed to the Appendix.
The convergence to the family of independent uniform random variables is not clear and explained in Sections 7 and 8. The proof uses the results of Chapter 8, which deal with combinatorics of large compositions and which are summarized in Section 6.

9.5 The familiy $(\xi_i^\lambda)_{i\geq 1}$

Some definitions on compositions are needed before defining the family $(\xi_i^\lambda)_{i\geq 1}$.

9.5.1 Combinatorics of compositions

Let λ be a composition of n with its associated descent set $D_\lambda = (\lambda_1, \lambda_1+\lambda_2, \dots, \lambda_1+\cdots+\lambda_{r-1})$. An integer $i \in [1;n]$ is a peak of λ if $i \in D_\lambda \cup \{n\}$ and $i-1 \notin D_\lambda$, and $i \in [1;n]$ is a valley if $i \notin D_\lambda$ and $i-1 \in D_\lambda \cup \{0\}$. This definition makes sense if we consider any standard filling σ of λ : $\sigma(i)$ is a local maximum (resp. minimum) of $\sigma = \sigma(1)\dots\sigma(n)$ if and only if i is a peak (resp. valley) of λ. Let V denote the set of valleys, P the set of peaks, and $\mathcal{E} = V \cup P$ the set of extreme cells.

A run s of λ is an interval $[a;b]$ of $[1;n]$ such that a, b are consecutive integers of $\mathcal{E}$. A run $[a;b]$ is called descending if $a \in P$ and ascending if $a \in V$. The runs are ordered by the lower endpoint of the corresponding interval, and this yields a total ordered set $S = \{s_i\}_{1\leq i\leq t}$. Each element s_i of S corresponds to an interval $[a_i; a_{i+1}]$, with $a_1 = 1$ and $a_{t+1} = n$. In particular two consecutive runs s_i and s_{i+1} overlap on a_{i+1}. The length of a run s_i is defined as the value $l_i = a_{i+1} - a_i$. For example if $\lambda = (3,2,4,1)$, $V = \{1,4,6,10\}$, $P = \{3,5,9\}$ and $S = \{[1;3],[3;4],[4;5],[5,6],[6;9],[9,10]\}$.

For any cell i of λ, the slope of i, $\mathfrak{s}(i) = [x(i); y(i)]$, is defined as the maximum subinterval of $[1;n]$ that contains i and no other peak or valley. In the previous example, $\mathfrak{s}(7) = [7;8]$ and $\mathfrak{s}(6) = [6;8]$.

9.5.2 Definition of $(\xi_i^\lambda)_{i\geq 1}$

Let $(X^i(p,q))_{\substack{i\geq 1\\ \{p,q\}\subseteq\mathbb{Q}}}$ be a family of independent variables such that $X^i(p,q)$ follows the uniform law on $[p,q]$.

Definition 9.9. *Let λ be a composition of n, $\sigma \in \Omega_\lambda$. The averaged coordinate of i with respect to σ is the random variable defined by $\xi_i(\sigma) = 0$ if $i > n$, and*

$$\xi_i(\sigma) = X^i\left(\frac{x(\sigma^{-1}(i)) - 1}{n}, \frac{y(\sigma^{-1}(i))}{n}\right),$$

for $1 \leq i \leq n$.
For σ_λ chosen uniformly among Ω_λ, $\xi_i(\sigma_\lambda)$ is denoted ξ_i^λ. $\xi^\lambda(k)$ denotes the vector $(\xi_1^\lambda, \dots, \xi_k^\lambda)$ and $\xi^\lambda(n)$ is simply written ξ^λ.

Basically constructing ξ_i^λ means that we sample a uniformly random standard filling σ_λ of λ, we look at the cell containing i with respect to this filling, and then sample a random variable uniformly distributed on the rescaled slope of this cell. The advantage is that the knowledge of $\xi^\lambda(k)$ is enough to reconstruct σ_k^λ. This reconstruction needs a slightly modified version of U_λ:

Definition 9.10. *The run paintbox $\tilde{U}_\lambda$ associated to λ is an element of $\mathcal{U}^{(2)}$ consisting in the following open subsets:*

- $\tilde{U}_\uparrow(\lambda) = \bigcup_{a_i\in V}]\frac{a_i-1}{n}, \frac{a_{i+1}-1}{n}[$
- $\tilde{U}_\downarrow(\lambda) = \bigcup_{a_i\in P}]\frac{a_i-1}{n}, \frac{a_{i+1}-1}{n}[$

with $a_{i+1} = n+1$ if $a_i = n$.

The run paintbox $\tilde{U}_\lambda$ becomes close to U_λ when n goes to infinity:

Lemma 9.11. *Let λ be a composition of n. With respect to the distance on $\mathcal{U}^{(2)}$,*

$$d(U_\lambda, \tilde{U}_\lambda) \leq \frac{1}{n}.$$

Proof. The definition of U_λ yields the following open sets:

$$U_\uparrow(\lambda) = \bigcup_{a_i \in V, a_i \neq n}]\frac{a_i - 1}{n-1}, \frac{a_{i+1} - 1}{n-1}[,$$

and

$$U_\downarrow(\lambda) = \bigcup_{a_i \in P, a_i \neq n}]\frac{a_i - 1}{n-1}, \frac{a_{i+1} - 1}{n-1}[.$$

Let us show that $U_\uparrow(\lambda)^c$ is included in the $\frac{1}{n}$−inflation of $\tilde{U}_\uparrow(\lambda)^c$ and conversely (the proof for $U_\downarrow(\lambda)^c$ and $\tilde{U}_\downarrow(\lambda)^c$ is the same). The $\frac{1}{n}$−inflation of $U_\uparrow(\lambda)^c$ is

$$U_\uparrow(\lambda)^{c,1/n} = \left(\bigcup_{a_i \in P, a_i \neq n} [(\frac{a_i - 1}{n-1} - \frac{1}{n}) \vee 0, (\frac{a_{i+1} - 1}{n-1} + \frac{1}{n}) \wedge 1] \right) \cup [0, \frac{1}{n}] \cup [1 - \frac{1}{n}, 1].$$

On the other hand

$$\tilde{U}_\uparrow(\lambda)^c = \left(\bigcup_{a_i \in P} [\frac{a_i - 1}{n}, \frac{a_{i+1} - 1}{n}] \right) \cup \{0\} \cup \{1\}.$$

Suppose that $a_i \neq n$. Then for all $1 \leq k \leq n-1$, $\frac{k}{n-1} - \frac{1}{n} \leq \frac{k}{n} \leq \frac{k}{n-1} + \frac{1}{n}$, thus

$$[\frac{a_i - 1}{n}, \frac{a_{i+1} - 1}{n}] \subseteq [(\frac{a_i - 1}{n-1} - \frac{1}{n}) \vee 0, (\frac{a_{i+1} - 1}{n-1} + \frac{1}{n}) \wedge 1] \subseteq U_\uparrow(\lambda)^{c,1/n}.$$

If $a_i = n$, $[\frac{a_i - 1}{n}, \frac{a_{i+1} - 1}{n}] = [1 - 1/n, 1] \subseteq U_\uparrow(\lambda)^{c,1/n}$. Finally $\tilde{U}_\uparrow(\lambda)^c \subseteq U_\uparrow(\lambda)^{c,1/n}$.
For the converse inclusion the $\frac{1}{n}$−inflation of $\tilde{U}_\uparrow(\lambda)^c$ is

$$\tilde{U}_\uparrow(\lambda)^{c,1/n} = \left(\bigcup_{a_i \in P} [(\frac{a_i - 1}{n} - \frac{1}{n}) \vee 0, (\frac{a_{i+1} - 1}{n} + \frac{1}{n}) \wedge 1] \right) \cup [0, \frac{1}{n}] \cup [1 - \frac{1}{n}, 1],$$

and

$$U_\uparrow(\lambda)^c = \left(\bigcup_{a_i \in P, a_i \neq n} [\frac{a_i - 1}{n-1}, \frac{a_{i+1} - 1}{n-1}] \right) \cup \{0\} \cup \{1\}.$$

Since for $1 \leq k \leq n-1$, $\frac{k}{n} - \frac{1}{n} \leq \frac{k}{n-1} \leq \frac{k}{n} + \frac{1}{n}$, for each $a_i \neq n$,

$$[\frac{a_i - 1}{n-1}, \frac{a_{i+1} - 1}{n-1}] \subseteq [(\frac{a_i - 1}{n} - \frac{1}{n}) \vee 0, (\frac{a_{i+1} - 1}{n} + \frac{1}{n}) \wedge 1],$$

and therefore $U_\uparrow(\lambda)^c \subseteq \tilde{U}_\uparrow(\lambda)^{c,1/n}$.
Doing the same for $U_\downarrow(\lambda)$ and $\tilde{U}_\downarrow(\lambda)$ concludes the proof. □

The previous Lemma implies that for any sequence $(\lambda_n)_{n \geq 1}$ with $|\lambda_n| \to \infty$, the convergence of U_{λ_n} is equivalent to the convergence of $\tilde{U}_{\lambda_n}$, and both have the same limit. The advantage is that the knowledge of ξ_k^λ and $\tilde{U}_{\lambda_n}$ is enough to recover σ_k^λ. Recall that $\sigma_{\tilde{U}_\lambda}((\xi_i(\sigma))_{1 \leq i \leq k})$ is the paintbox construction relative to the tuple $(\xi_i(\sigma))_{1 \leq i \leq k}$ and the paintbox $\tilde{U}_\lambda$ as defined in subsection 9.4.1.

Proposition 9.12. *For each $1 \leq k \leq n$, $\sigma \in \Omega_\lambda$,*

$$\sigma_{\tilde{U}_\lambda}((\xi_i(\sigma))_{1\leq i \leq k}) = \sigma_{\downarrow k}.$$

In particular the random variables σ_k^λ and $\sigma_{\tilde{U}_\lambda}(\xi^\lambda(k))$ have the same law.

Proof. It is enough to prove it for $k = n$. Denote $\xi_i(\sigma) = \xi_i$ and $\xi = (\xi_i(\sigma))_{1\leq i\leq n}$. It is equivalent to prove that for $1 \leq i, j \leq n$,

$$\left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(i) < \left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(j) \Leftrightarrow \sigma_\lambda^{-1}(i) < \sigma_\lambda^{-1}(j).$$

Let $1 \leq i < j \leq n$. Then $\sigma_\lambda^{-1}(i) < \sigma_\lambda^{-1}(j)$ implies that i is left to j in the associated filling of λ. This is possible in one of the two following situations :

1. $\mathfrak{s}(i)$ and $\mathfrak{s}(j)$ are disjoint and $\mathfrak{s}(i)$ is left to $\mathfrak{s}(j)$. In this case ξ_i and ξ_j are not in the same interval component of $\tilde{U}_\lambda$ and ξ_i is in an interval component left to the one of ξ_j. By the run Paintbox construction,
$$\left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(i) < \left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(j).$$
2. $\mathfrak{s}(i)$ and $\mathfrak{s}(j)$ overlap. This implies that i and j are in a same run $s = [a_i; a_{i+1}]$ of λ. Let $I_s =]\frac{a_i-1}{n}, \frac{a_{i+1}-1}{n}[$. Since $i < j$ and $\sigma_\lambda^{-1}(i) < \sigma_\lambda^{-1}(j)$, the run s has to be an ascending one and thus $a_i \in V$ and $a_{i+1} \in P$. In particular $\sigma_\lambda^{-1}(i)$ cannot be a peak, and $\sigma_\lambda^{-1}(j)$ cannot be a valley. Thus ξ_i is either in an interval component left to I_s, either in I_s. For similar reasons, ξ_j is either in an interval component right to I_s, either in $_s$. This implies that if ξ_i or ξ_j is not in I_s, $\left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(i) < \left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(j)$. But if ξ_i and ξ_j are both in I_s, since the latter is in $\tilde{U}_\uparrow(\lambda)$, the same inequality holds.

Finally, in any case,

$$\sigma_\lambda^{-1}(i) < \sigma_\lambda^{-1}(j) \Longrightarrow \left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(i) < \left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(j).$$

The pattern is exactly the same to prove that

$$\sigma_\lambda^{-1}(i) > \sigma_\lambda^{-1}(j) \Longrightarrow \left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(i) > \left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(j),$$

yielding the first part of the Proposition. This first part implies clearly the second one. □

It is also possible to recover exactly the position of $\{1, \ldots, k\}$ in the filling σ of λ from $(\xi_i(\sigma))_{1\leq i\leq k}$:

Lemma 9.13. *Let σ, σ' be two permutations of Ω_λ. If $(\sigma^{-1}(1), \ldots, \sigma^{-1}(k))$ is not equal to $(\sigma'^{-1}(1), \ldots, \sigma'^{-1}(k))$, then $(\xi_1(\sigma), \ldots, \xi_k(\sigma))$ and $(\xi_1(\sigma'), \ldots, \xi_k(\sigma'))$ have disjoint supports.*

Proof. The proof is done by recurrence on $k \geq 1$. Let $k = 1$. 1 has to be located in a valley of λ. If $\sigma^{-1}(1) \neq \sigma'^{-1}(1)$, 1 is located in a different valley in σ and σ'. Thus the slopes of $\sigma^{-1}(1)$ and $\sigma'^{-1}(1)$ are disjoint, and $\xi_1(\sigma)$ and $\xi_1(\sigma)$ have disjoint supports.
Let $k > 1$. Suppose that $(\sigma^{-1}(1), \ldots, \sigma^{-1}(k)) \neq (\sigma'^{-1}(1), \ldots, \sigma'^{-1}(k))$. By recurrence hypothesis, if $(\sigma^{-1}(1), \ldots, \sigma^{-1}(k-1))$ is not equal to $(\sigma'^{-1}(1), \ldots, \sigma'^{-1}(k-1))$, $(\xi_1(\sigma), \ldots, \xi_{k-1}(\sigma))$ and $(\xi_1(\sigma'), \ldots, \xi_{k-1}(\sigma'))$ have disjoint supports. This yields also that $(\xi_1(\sigma), \ldots, \xi_k(\sigma))$ and $(\xi_1(\sigma'), \ldots, \xi_k(\sigma'))$ have disjoint supports.
Thus let us assume that $(\sigma^{-1}(1), \ldots, \sigma^{-1}(k-1)) = (\sigma'^{-1}(1), \ldots, \sigma'^{-1}(k-1))$. This implies that $\sigma^{-1}(k) \neq \sigma'^{-1}(k)$; since the position of $\{1, \ldots, k-1\}$ is the same in the fillings σ and σ' of λ, the cell containing k in σ is in a different run than the cell containing k in σ'. Therefore their slopes are disjoint, and $(\xi_1(\sigma), \ldots, \xi_k(\sigma))$ and $(\xi_1(\sigma'), \ldots, \xi_k(\sigma'))$ have disjoint supports. □

This section ends by a convergence result. Although this result is crucial for the proof of Theorem 9.4.1, its proof is rather technical and has been postponed to the Appendix.

Proposition 9.14. *Let U_n be a sequence of $\mathcal{U}^{(2)}$ and $((X^n(i))_{i\geq 1})_{n\geq 1}$ a sequence of random infinite vectors on $[0,1]$. Let $(X^0(1),\dots,X^0(n),\dots)$ be a random infinite vector on $[0,1]$. Suppose that each finite dimensional marginal law of any of these random vectors admits a density with respect to the Lebesgue measure. If $U_n \to U \in \mathcal{U}^{(2)}$ and for each $k\geq 1$, $X^n_k = (X^n(1),\dots,X^n(k))$ converges in law to $X^0_k = (X^0(1),\dots,X^0(k))$, then for each $k \geq 1$,*

$$\sigma_{U_n}(X^n_k) \longrightarrow_{law} \sigma_U(X^0_k).$$

9.6 Combinatorics of large compositions

The purpose of this section is to introduce the background material to prove that the family $(\xi^\lambda_u)_{1\leq u\leq k}$ converges in law to a family of independent uniform random variables on $[0,1]$. Since ξ^λ_u depends uniquely on the runs in which u is located in a random filling σ_λ of λ, it is necessary to evaluate the probability for u to be located in a particular run s of λ. For a composition λ and $i\in\lambda$ a fixed cell, denote by $\lambda_{\leq i}$ (resp. $\lambda_{\geq i}$, $\lambda_{<i}$, $\lambda_{>i}$) the composition λ restricted to cells left (resp. right, res. strictly left, resp. strictly right) to i. Recall that $d(\lambda)$ denotes the number of standard fillings of the ribbon Young diagram associated to λ.
Let us focus here on the location of 1 in σ_λ. Since 1 is necessary a local minimum in any filling of λ, it has to be located in a valley $v\in V$. For a fixed valley v of λ, the cardinal of standard fillings of λ such that 1 is located in v is exactly the number of possibilities to fill in the part of λ left of v, with any subset S of cardinal $|\lambda_{<v}|$ of $[2,n]$, and to fill in independently the part of λ right to v with the complementary subset of S in $[2,n]$. Thus

$$\mathbb{P}_{\sigma_\lambda}(1\in v) = \frac{(|\lambda|-1)!}{|\lambda_{<v}|!|\lambda_{>v}|!}\frac{d(\lambda_{>v})d(\lambda_{<v})}{d(\lambda)}. \tag{9.6.1}$$

The problem is therefore essentially to relate $d(\lambda_{>v})d(\lambda_{<v})$ to $d(\lambda)$.

9.6.1 Probabilistic approach to descent combinatorics

Ehrenborg, Levin and Readdy (see [37]) formalized in the context of descent sets an old relation between permutations of n and polytopes in $[0,1]^n$. Namely since the volume of the set $R_\sigma = \{x_{\sigma(1)} < \dots < x_{\sigma(n)}\}$ is exactly $\frac{1}{n!}$, it is possible to determine probabilistic quantities on $\mathfrak{S}_n$ by integrating certain functions that are constant on each region R_σ. The appropriate functions for descent sets were found in [37], yielding some new estimates as in [36] and [18]. The model of Ehrenborg, Levin and Readdy is exposed in this paragraph, but in a modified way to focus only on the set of extreme cells $\mathcal{E}$ (as defined in the paragraph 5.1). This yields the following framework : let λ be a composition of $n\geq 2$ with set of extreme cells $\mathcal{E} = \{a_1 = 1, a_2,\dots,a_r = n\}$. Suppose for example that the first cell is a valley (namely $a_1\in V$) and denote by s_j the run between a_j and a_{j+1}, with l_j its length. To each λ is associated the couple of random variable (X_λ, Y_λ) on $[0,1]^2$ with density

$$d_{X_\lambda,Y_\lambda}(x_1,x_r) = \frac{1}{\mathcal{V}_\lambda}\int_{[0,1]^{r-2}} \prod \mathbf{1}_{x_{2i-1}<x_{2i}>x_{2i+1}} \prod_{1\leq i\leq r-1} \frac{|x_i - x_{i+1}|^{l_i-1}}{(l_i-1)!} \prod_{2\leq i\leq r-1} dx_i. \tag{9.6.2}$$

If the first cell is a peak (i.e $a_1\in P$), the inequalities in the expression of the density are reversed. If $\lambda = \square$, the expression for the distribution of $(X_\square, Y_\square)$ (in the distributional sense) is simply:

$$d_{X_\square,Y_\square}(u,v) = \delta_{u=v}.$$

The latter probabilistic model is related to the combinatorics of descent sets through the equality

$$d(\lambda) = |\lambda|!\mathcal{V}_\lambda, \tag{9.6.3}$$

whose proof can be found in [37].
The first advantage of this model is that it behaves simply with respect to concatenation of compositions.

Definition 9.15. *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ be two compositions of m and n. The concatenated composision $\lambda + \mu$ is the composition of $n + m$*

$$\lambda + \mu = (\lambda_1, \dots, \lambda_r + \mu_1, \mu_2, \dots, \mu_s),$$

and the concatenated composition $\lambda - \mu$ is the composition of $n + m$

$$\lambda - \mu = (\lambda_1, \dots, \lambda_r, \mu_1, \mu_2, \dots, \mu_s).$$

This definition has a simple meaning in terms of associated ribbon Young diagrams: namely the diagram of $\lambda + \mu$ (resp. $\lambda - \mu$) is the juxtaposition of the one of λ and the one of μ such that the last cell of λ is left to (resp. above) the first cell of μ. An application of the section 2 of [37] (see also Lemma 2 in [18]) implies the following expression of the concatenation in the probabilistic framework :

Proposition 9.16. *Let λ, μ be two compositions, $\varepsilon \in \{-, +\}$. Then*

$$\mathcal{V}_{\lambda\varepsilon\mu} = \mathcal{V}_\lambda \mathcal{V}_\mu \mathbb{E}(Y_\lambda \leq_\varepsilon X_\mu)$$

and

$$d_{X_{\lambda\varepsilon\mu}, Y_{\lambda\varepsilon\mu}}(x, y) = \frac{1}{\mathbb{E}(Y_\lambda \leq_\varepsilon X_\mu)} \int_{[0,1]^2} d_{X_\lambda, Y_\lambda}(x, u) \mathbf{1}_{u \leq_\varepsilon v} d_{X_\mu, Y_\mu}(v, y) dudv,$$

where $\leq_- = \geq$ and $\leq_+ = \leq$, and the couples (X_λ, Y_λ) and (X_μ, Y_μ) are considered as independent.

The previous Proposition yields a particular case that helps to compute the law of ξ_1^λ. Denote by F_X the distributive cumulative function of a random variable X.

Corollary 9.17. *Let λ be a composition of n and v a valley of λ. Then*

$$\mathbb{P}_\lambda(1 \in v) = \frac{1}{n} \frac{1}{\int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t))(1 - F_{X_{\lambda_{>v}}}(t))dt},$$

with the convention $X_{\lambda_{>n}} = \delta_1$ and $Y_{\lambda_{<1}} = \delta_1$.

Proof. Since v is a valley, λ can be written $\lambda_{<v} - \square + \lambda_{>v}$. Thus the previous Proposition yields

$$\mathcal{V}_{\lambda_{<v} - \square + \lambda_{>v}} = \mathcal{V}_{\lambda_{<v}} \mathcal{V}_{\lambda_{\geq v}} \mathbb{E}(Y_{\lambda_{<v}} \geq X_{\lambda_{\geq v}}).$$

Conditioning the expectation on the value of $X_{\lambda_{\geq v}}$ gives by independence,

$$\mathbb{E}(Y_{\lambda_{<v}} \geq X_{\lambda_{\geq v}}) = \int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t)) d_{X_{\lambda_{\geq v}}}(t) dt.$$

On the other hand from the previous Proposition, since $X_{\lambda_{\geq v}} = X_{\square + \lambda_{>v}}$,

$$\begin{aligned} d_{X_{\lambda_{\geq v}}}(t) =& \frac{1}{\mathbb{E}(X_{\lambda_{>v}} \geq Y_\square)} \int_{[0,1]^3} \delta(t, u) \mathbf{1}_{u \leq v} d_{X_{\lambda_{>v}}, Y_{\lambda_{>v}}}(v, y) dudvdy \\ =& \frac{\mathcal{V}_{\lambda_{>v}} \mathcal{V}_\square}{\mathcal{V}_{\lambda_{\geq v}}} (1 - F_{X_{\lambda_{>v}}}(t)), \end{aligned}$$

and finally

$$\mathcal{V}_{\lambda_{<v}-\square+\lambda_{>v}} = \mathcal{V}_{\lambda_{<v}}\mathcal{V}_{\lambda_{>v}} \int_0^1 (1-F_{Y_{\lambda_{<v}}}(t))(1-F_{X_{\lambda_{>v}}}(t))dt.$$

Using the latter result in the equalities (9.6.1) and (9.6.3) yields

$$\begin{aligned}
\mathbb{P}_{\sigma_\lambda}(1\in v) =& \frac{(|\lambda|-1)!}{|\lambda_{<v}|!|\lambda_{>v}|!}\frac{d(\lambda_{>v})d(\lambda_{<v})}{d(\lambda)}\\
=& \frac{(|\lambda|-1)!}{|\lambda_{<v}|!|\lambda_{>v}|!}\frac{|\lambda_{<v}|!|\lambda_{>v}|!}{|\lambda|!}\frac{\mathcal{V}_{\lambda_{>v}}\mathcal{V}_{\lambda_{<v}}}{\mathcal{V}_\lambda}\\
=& \frac{1}{|\lambda|}\frac{\mathcal{V}_{\lambda_{>v}}\mathcal{V}_{\lambda_{<v}}}{\mathcal{V}_{\lambda_{<v}}\mathcal{V}_{\lambda_{>v}}\int_0^1(1-F_{Y_{\lambda_{<v}}}(t))(1-F_{X_{\lambda_{>v}}}(t))dt}\\
=& \frac{1}{|\lambda|}\frac{1}{\int_0^1(1-F_{Y_{\lambda_{<v}}}(t))(1-F_{X_{\lambda_{>v}}}(t))dt}.
\end{aligned}$$

□

9.6.2 Estimates on (X_λ, Y_λ)

The latter corollary shows that the knowledge of F_{X_μ} and F_{Y_μ} for any subcomposition μ of λ yields estimates on the location of 1 in σ_λ. The results on the behavior of F_{X_μ}, F_{Y_μ} obtained in Chapter 8 are summarized in this paragraph, and the reader should refer to this chapter for the corresponding proofs. The first result is a bound of F_{X_λ} depending on the length of the first run of λ and corresponds to Corollary 2 in Chapter 8 (and the following paragraph):

Proposition 9.18 (Cor.2 Chapter 8)**.** *Let λ be a composition with at least two runs, and with first run of length R. If the first run is increasing, the following inequality holds :*

$$1-(1-t)^R \le F_{X_\lambda}(t) \le 1-(1-t)^{R+1}.$$

If the first run is decreasing, the inequality is

$$t^{R+1} \le F_{X_\lambda}(t) \le t^R.$$

The same result holds for Y_λ after exchanging increasing run and decreasing run. The latter inequalities are very accurate when R is large, but when the runs remain bounded, the result is not so useful. It is still possible to show that the distribution of X_λ only depends on the first cells of the composition. This corresponds to Proposition 11 in Chapter 8.

Proposition 9.19 (Prop. 11,Ch. 8)**.** *Let $\varepsilon > 0$. There exists $n_0 \ge 1$ such that for any $n \ge n_0$ and any composition λ of size larger than n with first run smaller than n,*

$$\|F_{X_{\lambda_{\le n}}} - F_{X_\lambda}\|_\infty \le \varepsilon.$$

In the latter result, n_0 depends only on ε, and not on the shape of λ.

9.7 Asymptotic law of ξ_1^λ

This section is devoted to the asymptotic law of ξ_1^λ.

9.7.1 Preliminary results

Propositions 9.18 and 9.19 imply that $\mathbb{P}_\lambda(1 \in v)$ only depends on the shape of λ around v.

Lemma 9.20. *Let $\varepsilon > 0$. There exists $n_\varepsilon \in \mathbb{N}$ such that for $n_0 \geq n_\varepsilon$ and two compositions $\lambda \vdash n$ and $\mu \vdash m$ with the first run of μ smaller than n_0,*

$$1 - \varepsilon \leq \frac{\int_0^1 (1 - F_{Y_\lambda}(t))(1 - F_{X_\mu}(t))dt}{\int_0^1 (1 - F_{Y_\lambda}(t))(1 - F_{X_{\mu_{\leq n_0}}}(t))dt} \leq 1 + \varepsilon.$$

Proof. Let λ, μ be two compositions, with L the size of the last run of λ and R the size of the first run of μ. Set $\varepsilon_1 = +$ if the last run of λ is increasing, $\varepsilon_1 = -$ else, and the same with ε_2 and the first run of μ. Let $\Delta = \int_0^1 (1 - F_{Y_\lambda}(t))(1 - F_{X_\mu}(t))dt$. From Proposition 9.18, integrating the inequalities yields the following bounds on Δ:

- If $\varepsilon_1 = +, \varepsilon_2 = +$,

$$\frac{1}{R+2}\left(1 - \frac{1}{(R+3)\dots(R+L+2)}\right) \leq \Delta \leq \frac{1}{R+1}\left(1 - \frac{1}{(R+2)\dots(R+L+2)}\right),$$

- if $\varepsilon_1 = -, \varepsilon_2 = +$,

$$\frac{1}{L+R+3} \leq \Delta \leq \frac{1}{L+R+1},$$

- if $\varepsilon_1 = +, \varepsilon_2 = -$,

$$1 - \frac{1}{R+1} - \frac{1}{L+1} + \frac{1}{R+L+1} \leq \Delta \leq 1 - \frac{1}{R+2} - \frac{1}{L+2} + \frac{1}{L+R+3},$$

- if $\varepsilon_1 = -, \varepsilon_2 = -$,

$$\frac{1}{L+2}\left(1 - \frac{1}{(L+3)\dots(L+R+2)}\right) \leq \Delta \leq \frac{1}{L+1}\left(1 - \frac{1}{(L+2)\dots(L+R+2)}\right).$$

The latter bounds are independent of the shape of λ, μ apart from the lengths of the last run of λ and the first run of μ. Denote by $A_{L,R}^{\varepsilon_1,\varepsilon_2}$ each upper bound in the previous list, and $B_{L,R}^{\varepsilon_1,\varepsilon_2}$ each lower bound. Then as $\min(L, R) \to \infty$,

$$\frac{B_{L,R}^{\varepsilon_1,\varepsilon_2}}{A_{L,R}^{\varepsilon_1,\varepsilon_2}} \to 1.$$

Thus there exists K such that if $\min(L, R) \geq K$, whatever is the shape of λ, μ outside these runs and $n_0 \geq R$,

$$1 - \varepsilon \leq \frac{\int_0^1 (1 - F_{Y_\lambda}(t))(1 - F_{X_\mu}(t))dt}{\int_0^1 (1 - F_{Y_\lambda}(t))(1 - F_{X_{\mu_{\leq n_0}}}(t))dt} \leq 1 + \varepsilon.$$

From now on, let us assume that the last run of λ and the first run of μ are bounded by K. Set $\eta = \inf_{\substack{\varepsilon_1,\varepsilon_2 \\ L,R \leq K}} B_{L,R}^{\varepsilon_1,\varepsilon_2}$. Since $L, R \geq 1$, each $B_{L,R}^{\varepsilon_1,\varepsilon_2}$ is strictly positive. The family $\{B_{L,R}^{\varepsilon_1,\varepsilon_2}\}_{\substack{\varepsilon_1,\varepsilon_2 \\ L,R \leq K}}$ being finite, this yields $\eta > 0$.
By Proposition 9.19, there exists $n_\varepsilon \geq 1$ such that for $n_0 \geq n_\varepsilon$ and any composition ν of size $n \geq n_0$ and first run smaller than n_0, $F_{X_{\nu_{\leq n_0}}} = F_{X_\nu} + g$ with $\|g\|_\infty \leq \varepsilon\eta$. Let $n_0 \geq n_\varepsilon$ and

suppose that $\lambda \vdash n, \mu \vdash m$ with the first run of μ smaller than n_0. Then there exists g such that $\|g\|_\infty \leq \varepsilon\eta$, and $F_{X_{\mu_{\leq n_0}}} = F_{X_\mu} + g$. This implies

$$\begin{aligned}\int_0^1 (1 - F_{Y_\lambda}(t))(1 - F_{X_{\mu_{\leq n_0}}}(t))dt &= \int_0^1 (1 - F_{Y_\lambda}(t))(1 - F_{X_\mu}(t) - g(t))dt \\ &= \Delta - \int_0^1 g(t)(1 - F_{Y_\lambda}(t))dt.\end{aligned}$$

Since $|\int_0^1 g(t)(1 - F_{Y_\lambda}(t))dt| \leq \varepsilon\eta$ and $\Delta \geq \eta$,

$$1 - \varepsilon \leq \frac{\int_0^1 (1 - F_{Y_\lambda}(t))(1 - F_{X_{\mu_{\leq n_0}}}(t))dt}{\Delta} \leq 1 + \varepsilon.$$

□

A corollary of the previous lemma yields a precise estimate of the probability that 1 is located in a particular valley v when the length of the slope of v goes to $+\infty$.

Corollary 9.21. *Let $\varepsilon > 0$. There exists $n_0 \geq 1$ such that if λ is a composition, and $v \in \lambda$ is a valley with slope $\mathfrak{s}(v) = [a; b]$ of size $b - a \geq n_0$, then*

$$(1 - \varepsilon)\frac{b - a}{n} \leq \mathbb{P}_\lambda(1 \in v) \leq (1 + \varepsilon)\frac{b - a}{n}.$$

Proof. Since v is a valley, v belongs to two runs s_i, s_{i+1}. If $b - a \geq 1$, then at least one run containing v is of size larger than 2. Assume without loss of generality that $l(s_{i+1}) \geq 2$. This means that the first run of $\lambda_{>v}$ is increasing. Let $\Delta = \int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t))(1 - F_{X_{\lambda_{>v}}}(t))dt$. Let L denote the length of the last run of $\lambda_{<v}$, and R the length of the first run of $\lambda_{>v}$.
If $l(s_i) = 1$, the last run of $\lambda_{<v}$ is increasing. Moreover in this case $b - a = R$. Thus the bounds on Δ from the proof of the last Lemma yield

$$\frac{1}{b - a + 2}(1 - \frac{1}{\prod_{i=3}^{L+2}(b - a + i)}) \leq \Delta \leq \frac{1}{b - a + 1}(1 - \frac{1}{\prod_{i=2}^{L+2}(b - a + i)}).$$

Thus independently from L, there exists n_1 such that if $l(s_i) = 1$ and $b - a \geq n_1$, then $(1 - \varepsilon)(b - a) \leq \Delta^{-1} \leq (1 + \varepsilon)(b - a)$.
If $l(s_i) > 1$, the last run of $\lambda_{<v}$ is decreasing. Then $b - a = L + R - 1$, and the bounds on Δ from the proof of the previous Lemma yield

$$\frac{1}{b - a + 2} \leq \Delta \leq \frac{1}{b - a}.$$

There exists n_2 such that if $l(s_i) > 1$ and $b - a \geq n_2$, then $(1 - \varepsilon)(b - a) \leq \Delta^{-1} \leq (1 + \varepsilon)(b - a)$.
Set $n_0 = \max(n_1, n_2)$, and let $n \geq n_0$. From Corollary9.17 $\mathbb{P}_\lambda(1 \in v) = \frac{1}{n\Delta}$, and thus

$$(1 - \varepsilon)\frac{b - a}{n} \leq \mathbb{P}_\lambda(1 \in v) \leq (1 + \varepsilon)\frac{b - a}{n}.$$

□

From the bounds $\{A_{R,L}^{\varepsilon_1,\varepsilon_2}, B_{R,L}^{\varepsilon_1,\varepsilon_2}\}$ on Δ that were found in the proof of Lemma 9.20, it is also possible to deduce a bound on the location probability of 1 in σ_λ:

Lemma 9.22. *Let λ be a composition of n, and $a < b$ be two peaks of λ. Then*

$$\mathbb{P}_\lambda(1 \in \lambda_{>a,<b}) \leq 3\frac{b-a}{n}.$$

Proof. Since 1 has to be located in a valley of λ,

$$\mathbb{P}(1 \in \lambda_{>a,<b}) = \sum_{\substack{v \in V \\ a<v<b}} \mathbb{P}(1 \in v).$$

From Corollary 9.17, for each $v \in V$,

$$\mathbb{P}(1 \in v) = \frac{1}{|\lambda|} \frac{1}{\int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t))(1 - F_{X_{\lambda_{>v}}}(t))dt},$$

Suppose that $v \in s_i \cap s_{i+1}$. By the bounds on Δ from the proof of Lemma 9.20,

$$\frac{1}{\int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t))(1 - F_{X_{\lambda_{>v}}}(t))dt} \leq l(s_i) + l(s_{i+1}) + 3 \leq 3(l(s_i) + l(s_{i+1})).$$

The latter inequality yields

$$\mathbb{P}(1 \in \lambda_{>a,<b}) \leq \frac{3}{|\lambda|} \sum_{\substack{v \in V \\ a<v<b}} l(s_i) + l(s_{i+1}) \leq \frac{3(b-a)}{n}.$$

□

9.7.2 Convergence to a uniform distribution

Let us show the convergence in law of ξ_1^λ. Let π denote the Levy-Prokhorov metric on the set $\mathcal{M}_1[0,1]$ of probability measures on $[0,1]$.

Proposition 9.23. *Let $\varepsilon > 0$. There exists n_0 such that for $n \geq n_0$, $\lambda \vdash n$,*

$$\pi(\xi_1^\lambda, \mathcal{U}([0,1])) \leq \varepsilon.$$

Proof. Let $\varepsilon > 0$. Since $F_{\mathcal{U}([0,1])} = Id_{[0,1]}$ is continuous, it is enough to prove that for $s \in [0,1]$ and for λ large enough,

$$|\mathbb{P}(\xi_1^\lambda \in [0,s]) - s| \leq \varepsilon.$$

Let $0 < s < 1$ and n_ε be the constant given by Lemma 9.20 for ε. Let $\lambda \vdash n$ and let v_n denote the last valley such that the associated slope intersects $[0, ns]$, namely $\mathfrak{s}(v^n) \cap [0, ns] \neq \emptyset$: since $0 < s$, such v_n always exists for n large enough. Let $[a_n; b_n]$ denote the slope of v_n. Thus $a_n \leq ns < b_n + 1$.
If $1 \in \lambda_{<a_n}$, $\xi_1^\lambda \in [0, \frac{a_n}{n}[\subseteq [0,s]$. Moreover

$$\begin{aligned}
\mathbb{P}_\lambda(1 \in \lambda_{<a_n}) &= \sum_{\substack{v \in V \\ v<a_n}} \mathbb{P}_\lambda(1 \in v) \\
&= \sum_{\substack{v \in V \\ v<a_n}} \frac{1}{|\lambda|} \frac{1}{\int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t))(1 - F_{X_{\lambda_{>v}}}(t))dt}.
\end{aligned}$$

If $v < a_n$ is a valley, there is necessarily a peak between v and v_n, and thus the first run of $\lambda_{>v}$ is of size smaller than $v_n - v$. Therefore from Lemma 9.20,

$$\begin{aligned}\mathbb{P}_\lambda(1 \in \lambda_{<a_n}) \leq & \sum_{\substack{v\in V\\ v<a_n}} \frac{1}{n} \frac{1+\varepsilon}{\int_0^1 (1-F_{Y_{\lambda_{<v}}}(t))(1-F_{X_{\lambda_{>v,<v_n+n_\varepsilon}}}(t))dt} \\ \leq & (1+\varepsilon)\frac{v_n+n_\varepsilon}{n}\mathbb{P}_{\lambda_{<v_n+n_\varepsilon}}(1 \in \lambda_{<a_n}),\end{aligned}$$

and for the same reasons,

$$\mathbb{P}_\lambda(1 \in \lambda_{<a_n}) \geq (1-\varepsilon)\frac{v_n+n_\varepsilon}{n}\mathbb{P}_{\lambda_{<v_n+n_\varepsilon}}(1 \in \lambda_{<a_n}).$$

The proof is now divided into two complementary cases :

- λ is such that $\frac{a_n}{v_n+n_\varepsilon} > 1-\varepsilon$: in this case, $\frac{v_n+n_\varepsilon-a_n}{v_n+n_\varepsilon} < \varepsilon$. Thus from Lemma 9.22, $\mathbb{P}_{\lambda_{<v_n+n_\varepsilon}}(1 \in \lambda_{\geq a_n}) \leq 3\varepsilon$. This yields

  $$1-3\varepsilon \leq \mathbb{P}_{\lambda_{<v_n+n_\varepsilon}}(1 \in \lambda_{<a_n}) \leq 1,$$

  and thus

  $$(1-3\varepsilon)^2\frac{v_n+n_\varepsilon}{n} \leq \mathbb{P}_\lambda(1 \in \lambda_{<a_n}) \leq (1+\varepsilon)\frac{v_n+n_\varepsilon}{n}.$$

  The hypothesis $\frac{a_n}{v_n+n_\varepsilon} > 1-\varepsilon$ yields also

  $$(1-3\varepsilon)^2\frac{a_n}{n} \leq \mathbb{P}_\lambda(1 \in \lambda_{<a_n}) \leq \frac{1+\varepsilon}{1-\varepsilon}\frac{a_n}{n}.$$

  On the other hand by independence between σ_λ and the family $(X^1_{(p,q)})_{p,q\in\mathbb{Q}}$,

  $$\begin{aligned}\mathbb{P}(1 \in v_n \cap \xi_1^\lambda \in [0,s]) =& \mathbb{P}_\lambda(1 \in v_n)\frac{Leb([0,s]\cap[(a_n-1)/n, b_n/n])}{(b_n-a_n+1)/n} \\ =& \mathbb{P}_\lambda(1 \in v_n)(\frac{s-a_n+1}{b_n-a_n+1} \wedge 1).\end{aligned}$$

  From the latter computation and from the bounding Lemma 9.22, $\mathbb{P}(1 \in v \cap \xi_1^\lambda \in [0,s]) \leq \frac{3(b_n-a_n)}{n}$. Thus the latter quantity doesn't become negligible for n large only if at least one of the two runs s_i or s_{i+1} surrounding v_n tends to infinity when n grows. But in this case from Corollary 9.21,

  $$\mathbb{P}(1 \in v_n) \sim_{b_n-a_n\to\infty} (b_n-a_n)/n.$$

  Therefore in any case:

  $$\begin{aligned}\mathbb{P}(1 \in v_n \cap \xi_1^\lambda \in [0,s]) =& _{n\to\infty}(b_n-a_n)/n\frac{s-a_n/n}{b_n/n-a_n/n} + o(1) \\ =& _{n\to+\infty}Leb([a_n/n,s]) + o(1),\end{aligned}$$

  with $o(1)$ being a quantity converging to zero with n, independently of the shape of λ. Summing the probabilities yields for n large enough

  $$\begin{aligned}\mathbb{P}(\xi_1^\lambda \in [0,s]) \leq & \frac{1+\varepsilon}{1-\varepsilon}Leb([0,\frac{a_n}{n}]) + Leb([\frac{a_n}{n},s]) + o(1) \\ \leq & \frac{1+\varepsilon}{1-\varepsilon}Leb([0,s]) + o(1),\end{aligned}$$

and for the same reasons

$$(1-3\varepsilon)^2 Leb([0,s]) + o(1) \leq \mathbb{P}(\xi_1^\lambda \in [0,s]).$$

There exists thus n_1 such that for $n \geq n_1$, if $\frac{a_n}{v_n+n_\varepsilon} > 1-\varepsilon$,

$$(1-3\varepsilon)^2 Leb([0,s]) - \varepsilon \leq \mathbb{P}(\xi_1^\lambda \in [0,s]) \leq \frac{1+\varepsilon}{1-\varepsilon} Leb([0,s]) + \varepsilon.$$

- λ is such that $\frac{a_n}{v_n+n_\varepsilon} \leq 1-\varepsilon$: this implies that either v_n remains bounded or $v_n - a_n$ goes to $+\infty$ as n grows.
  Suppose that v_n remains bounded by K as n grows. In this case by Lemma 9.22, $\mathbb{P}(1 \in \lambda_{<v_n}) \to 0$. Thus
  $$\mathbb{P}(\xi_1^\lambda \in [0,s]) = \mathbb{P}(1 \in v \cap \xi_1^\lambda \in [0,s]) + o(1).$$
  Since v_n remains bounded by K and $b_n + 1 > ns$, the slope of v_n tends to $+\infty$, and therefore from Corollary 9.21,
  $$\mathbb{P}(\xi_1^\lambda \in [0,s]) =_{n\to+\infty} (\frac{b_n - a_n}{n} + o(1))\frac{ns-a_n}{b_n-a_n} = s + o(1).$$
  Suppose that $v_n - a_n$ goes to $+\infty$. Since n_ε is a fixed integer and $v_n - a_n$ goes to $+\infty$, the size of the slope of v_n in $\lambda_{<v_n+\varepsilon}$ is equivalent to $v_n + n_\varepsilon - a_n$ as n goes to $+\infty$. Thus from Corollary 9.21,
  $$\mathbb{P}_{\lambda_{<v_n+n_\varepsilon}}(1 \in v_n) =_{n\to\infty} \frac{v_n + n_\varepsilon - a_n}{v_n + n_\varepsilon} + o(1).$$
  The same Corollary yields moreover
  $$\mathbb{P}_\lambda(1 \in v_n) =_{n\to\infty} \frac{b_n - a_n}{n} + o(1).$$
  From Lemma 9.22, $\mathbb{P}_{\lambda_{<v_n+n_\varepsilon}}(1 \in \lambda_{>v_n,<v_n+n_\varepsilon}) \leq \frac{3n_\varepsilon}{v_n+n_\varepsilon} = o(1)$ and thus
  $$\begin{aligned}\mathbb{P}_{\lambda_{<v_n+n_\varepsilon}}(1 \in \lambda_{<a_n}) =& 1 - \mathbb{P}_{\lambda_{<v_n+n_\varepsilon}}(1 \in v_n) - \mathbb{P}_{\lambda_{<v_n+n_\varepsilon}}(1 \in \lambda_{>v_n,<v_n+n_\varepsilon})\\ =& \frac{a_n}{v_n+n_\varepsilon} + o(1).\end{aligned}$$
  Thus
  $$\mathbb{P}(\xi_1^\lambda \in [0,s]) \leq (1+\varepsilon)\frac{a_n}{n} + \frac{b_n - a_n}{n}\frac{ns - a_n}{b_n - a_n} + o(1) \leq (1+\varepsilon)s + o(1),$$
  and for the same reasons $(1-\varepsilon)s + o(1) \leq \mathbb{P}(\xi_1^\lambda \in [0,s])$. This yields the existence of n_2 such that if $n \geq n_2$ and $\frac{a_n}{v_n+n_\varepsilon} \leq 1-\varepsilon$,
  $$(1-\varepsilon)Leb([0,s]) - \varepsilon \leq \mathbb{P}(\xi_1^\lambda \in [0,s]) \leq (1+\varepsilon)Leb([0,s]) + \varepsilon.$$

By the results from both cases, there exists n_0 such that for $n \geq n_0, \lambda \vdash n$,

$$|\mathbb{P}(\xi_1^\lambda \in [0,s]) - s| \leq \varepsilon.$$

□

9.8 Martin boundary of $\mathcal{Z}$

This section is devoted to the proof of Conjecture 9.7, yielding the identification of the Martin boundary of $\mathcal{Z}$ with its minimal boundary.

9.8.1 Generalization of Proposition 9.23

The result of the previous section can be generalized for $k \geq 2$:

Proposition 9.24. *Let λ_n be a sequence of compositions of size tending to infinity. Then for $k \geq 1$,*

$$(\xi_i^{\lambda_n})_{1 \leq i \leq k} \rightarrow_{law} (X_1, \dots, X_k),$$

with $(X_1, \dots, X_k)$ a vector of k independent uniform random variables on $[0, 1]$.

Proof. Let us prove by recurrence on $k \geq 1$ that for $\varepsilon > 0$, there exists $n_k \in \mathbb{N}$ such that for $n \geq n_k$, $\lambda \vdash n$,

$$\pi((\xi_i^{\lambda})_{1 \leq i \leq k}, (X_1, \dots, X_k)) \leq \varepsilon,$$

π denoting the Levy-Prokhorov metric on $[0, 1]^k$.
The initialization of the recurrence is done by Proposition 9.23. Let $k \geq 2$. It suffices to show that the law ξ_k^{λ} conditioned on $(\xi_i^{\lambda})_{1 \leq i \leq k-1}$ is close to the uniform law on $[0, 1]$ when n becomes large.
Let $s \in [0, 1] \setminus \mathbb{Q}, \varepsilon > 0$. Let

$$\Upsilon_\eta = \cap_{1 \leq i \leq k-1} \{(x_1, \dots, x_{k-1}) \in [0, 1]^{k-1} | x_i \notin [s - \eta, s + \eta]\}.$$

For all η, $Leb(\partial \Upsilon_\eta) = 0$ and $Leb(\lim_{\eta \to 0} \Upsilon_\eta) = 1$, thus by the recurrence hypothesis and the portemanteau theorem, there exists $\eta > 0$ such that for λ large enough,

$$\mathbb{P}((\xi_i^{\lambda})_{1 \leq i \leq k-1} \in \Upsilon_\eta) \geq 1 - \varepsilon, \tag{9.8.1}$$

and

$$\pi(((\xi_i^{\lambda})_{1 \leq i \leq k-1} | A_\eta), ((X_i)_{1 \leq i \leq k-1} | B_\eta)) \leq \varepsilon, \tag{9.8.2}$$

with $A_\eta = \{(\xi_i^{\lambda})_{1 \leq i \leq k-1} \in \Upsilon_\eta\}$ and $B_\eta = \{(X_i)_{1 \leq i \leq k-1} \in \Upsilon_\eta\}$.
Let $\lambda \vdash n$ and $\vec{i} = (i_1, \dots, i_{k-1})$ such that $\mathbb{P}(\sigma_\lambda^{-1}(1) = i_1, \dots, \sigma_\lambda^{-1}(k-1) = i_{k-1}) \neq 0$. Let us further assume that $\vec{i}$ satisfies the following condition :

$$\forall 1 \leq j \leq k - 1, \mathfrak{s}(i_j) \not\subseteq [n(s - \eta), n(s + \eta)], \tag{$*$}$$

where $\mathfrak{s}(i_j)$ denotes the slope of i_j as defined in Section 9.5.1. Then λ can be decomposed as

$$\lambda = \lambda_1 - \mu_1 + \lambda_2 - \cdots - \mu_r + \lambda_{r+1},$$

with μ_i consisting only in cells included in $\vec{i}$. From the latter construction, each run of λ intersects at most one λ_i.
Conditioned on $\mathcal{X}_{\vec{i}} = \{\sigma_\lambda^{-1}(1) = i_1, \dots, \sigma_\lambda^{-1}(k-1) = i_{k-1}\}$, the random filling of λ consists in sampling a uniformly random multiset $\vec{R} = (R_1, \dots, R_{r+1})$ of cardinal $(|\lambda_1|, \dots, |\lambda_{r+1}|)$ among $[k; n]$, and then independently filling each subcomposition $\lambda_1, \dots, \lambda_{r+1}$ respectively with $R_1, \dots, R_{r+1}$. Since k is the lowest element of $[k; n]$, for $v \in \lambda_i$, $\mathbb{P}(k \in v | \mathcal{X}_{\vec{i}}) \neq 0$ if and only if v is a valley of λ_i, and if this is the case,

$$\mathbb{P}(k \in v | \mathcal{X}_{\vec{i}}) = \mathbb{P}_{(R_1, \dots, R_{r+1})}(k \in R_i) \mathbb{P}_{\lambda_i}(1 \in v).$$

Let $s_p = [a_p; a_{p+1}]$ be the run of λ such that $s \in [\frac{a_p-1}{n}; \frac{a_{p+1}}{n}]$. If i_j is a peak, necessarily the two runs overlapping on i_j contain only elements lower than j, and thus the slope of i_j is smaller than j. Thus since for all $1 \leq j \leq k-1$, $\mathfrak{s}(i_j) \not\subseteq [n(s-\eta), n(s+\eta)]$, for n large enough, the peak of s_p cannot be in any μ_i for $1 \leq i \leq r$. This yields the existence of a unique i_0 such that $s_p \cap \lambda_{i_0} \neq \emptyset$.
Set $\lambda_{i_0} = \lambda_{\geq a, <b}$. Since the cells $a-1$ and b (if they exist) are in compositions of type μ, their content is smaller than $k-1$ and therefore smaller than the content of the cells of λ_{i_0}. Thus a valley v of λ_{i_0} such that the slope $\mathfrak{s}_\lambda(v)$ of v in λ is not included in λ_{i_0} has to be a or $b-1$. In particular if v is a valley of λ_{i_0} different from a and $b-1$, a rescaling of $\xi_1^{\lambda_{i_0}}$ yields that

$$(\xi_k^\lambda | k \in v) =_{law} (\frac{a}{n} + \frac{b-a}{n}\xi_1^{\lambda_{i_0}} | 1 \in v).$$

Thus

$$\mathbb{P}_\lambda(\{k \in v\} \cap \{\xi_k^\lambda \leq s\} | \mathcal{X}_{\vec{i}}) = \mathbb{P}_{\vec{R}}(k \in R_{i_0}) \mathbb{P}_{\lambda_{i_0}}(\{1 \in v\} \cap \{\xi_1^{\lambda_{i_0}} \in [0, \frac{ns-a}{b-a}]\}).$$

Suppose that a is a valley of λ_{i_0}. This implies that in λ, a is neither a peak nor a valley. Therefore the slope of a in λ_{i_0} is $[1, x]$ for some integer x, and the slope of a in λ is $[a-r, a+x]$ for the same x and some integer $1 \leq r \leq k-1$. Thus as n goes to infinity, this implies that

$$\pi((\xi_k^\lambda | k \in a), (\frac{a}{n} + \frac{b-a}{n}\xi_1^{\lambda_{i_0}} | 1 \in a)) \to_{n \to +\infty} 0,$$

and the rate of convergence only depends on n and k (and is therefore independent of the shape of λ and the choice of $\vec{i}$).
The same holds for $b-1$. Thus if $v = a$ or b, since $|\lambda_{i_0}| = b-a$,

$$\mathbb{P}_\lambda(\{k \in v\} \cap \{\xi_k^\lambda \leq s\} | \mathcal{X}_{\vec{i}}) = \mathbb{P}_{\vec{R}}(k \in R_{i_0})(\mathbb{P}_{\lambda_{i_0}}(\{1 \in v\} \cap \{\xi_1^{\lambda_{i_0}} \in [0, \frac{ns-a}{b-a}]\}) + o(1)).$$

Summing the probabilities yields

$$\mathbb{P}(\{\xi_k^\lambda \in [0, s]\} \cap \{k \in \lambda_{i_0}\} | \mathcal{X}_{\vec{i}}) = \mathbb{P}_{\vec{R}}(k \in R_{i_0}) \mathbb{P}_{\lambda_{i_0}}(\xi_1^{\lambda_{i_0}} \leq \frac{ns-a}{b-a}) + o(1).$$

If $i < i_0$, $k \in \lambda_i$ implies that $\xi_k^\lambda \in [0, s]$. A standard counting argument shows that

$$\mathbb{P}_{\vec{R}}(k \in R_i) = \frac{R_i}{\sum_j R_j},$$

and thus,

$$\begin{aligned}
\mathbb{P}(\xi_k^\lambda \in [0, s] | \mathcal{X}_{\vec{i}}) &= \left(\sum_{i < i_0} \mathbb{P}_{\vec{R}}(k \in R_i) + \mathbb{P}_{\vec{R}}(k \in R_{i_0}) \mathbb{P}_{\lambda_{i_0}}(\xi_1^{\lambda_{i_0}} \in [0, \frac{ns-a}{R_{i_0}}]) \right) + o(1) \\
&= \left(\sum_{i < i_0} \frac{R_i}{n} + \frac{R_{i_0}}{n} \mathbb{P}_{\lambda_{i_0}}(\xi_1^{\lambda_{i_0}} \in [0, \frac{ns-a}{R_{i_0}}]) \right) + o(1) \\
&= \frac{a}{n} + \frac{R_{i_0}}{n} \mathbb{P}_{\lambda_{i_0}}(\xi_1^{\lambda_{i_0}} \in [0, \frac{ns-a}{R_{i_0}}]) + o(1).
\end{aligned}$$

Either R_{i_0} remains bounded as $n \to \infty$ and $\frac{R_{i_0}}{n} \mathbb{P}_{\lambda_{i_0}}(\xi_1^{\lambda_{i_0}} \in [0, \frac{ns-a}{b-a}]) \to 0$, either R_{i_0} goes to infinity, and by Proposition 9.23,

$$\mathbb{P}_{\lambda_{i_0}}(\xi_1^{\lambda_{i_0}} \in [0, \frac{ns-a}{b-a}]) = \frac{ns-a}{R_{i_0}} + o(1).$$

Thus in any case,

$$\mathbb{P}(\xi_k^\lambda \in [0,s] | \mathcal{X}_{\vec{i}}) = \frac{a}{n} + \frac{R_{i_0}}{n}\frac{ns-a}{R_{i_0}} + o(1) \to s,$$

and the convergence is uniform in $\lambda, \vec{i}$.
Let $(x_i)_{1\le i\le k-1} \in \Upsilon_\eta$. If $(\xi_i^\lambda)_{1\le i\le k-1} = (x_i)_{1\le i\le k-1}$, then $(\sigma_\lambda^{-1}(1), \dots, \sigma_\lambda^{-1}(k-1))$ verifies the condition $(*)$. Moreover from Lemma 9.13, $(\xi_1^\lambda, \dots, \xi_{k-1}^\lambda) \mapsto (\sigma_\lambda^{-1}(1), \dots, \sigma_\lambda^{-1}(k-1))$ is well-defined and

$$(\xi_k^\lambda | (\xi_i^\lambda)_{1\le i\le k-1}) = (\xi_k^\lambda | \sigma_\lambda^{-1}(\{1, \dots, k-1\}).$$

Thus for n going to $+\infty$,

$$\mathbb{P}(\xi_k^\lambda \in [0,s] | (\xi_i^\lambda)_{1\le i\le k-1} = (x_i)_{1\le i\le k-1}) \to s,$$

and the convergence is uniform in $(x_i)_{1\le i\le k-1} \in \Upsilon_\eta$.
From the latter convergence and from (9.8.2), for n large enough,

$$\pi((\xi_i^\lambda)_{1\le i\le k}|A_\eta), ((X_i)_{1\le i\le k}|B_\eta)) \le \varepsilon.$$

If ε is small enough, then $\mathbb{P}(A_\eta) \ge 1-\varepsilon$ and $\mathbb{P}(B_\eta) \ge 1-\varepsilon$ imply that

$$\pi(((\xi_i^\lambda)_{1\le i\le k}|A_\eta), (\xi_i^\lambda)_{1\le i\le k}) \le 2\varepsilon,$$

and

$$\pi(((X_i)_{1\le i\le k}|B_\eta), (X_i)_{1\le i\le k}) \le 2\varepsilon.$$

Thus for n large enough,

$$\pi((\xi_i^\lambda)_{1\le i\le k}, (X_i)_{1\le i\le k}) \le 5\varepsilon.$$

This conclude the proof of the proposition. □

9.8.2 Proof of Theorem 9.8

Proof. Let $(\lambda_n)_{n\ge1}$ be a sequence of compositions and $U = (U_\uparrow, U_\downarrow) \in \mathcal{U}^{(2)}$ such that $\lambda_n \vdash n$ and $U_{\lambda_n} \to U$ in $\mathcal{U}^{(2)}$. By Lemma 9.11, $\tilde{U}_{\lambda_n} \to U$, with $\tilde{U}_\lambda$ the run paintbox defined for λ in Section 5.
Let $\mu \in \mathcal{Z}$, $\mu \vdash k$. Since $K_\mu(\lambda_n) = \frac{d(\mu,\lambda_n)}{d(\lambda_n)}$, by equality (9.3.1),

$$K_\mu(\lambda_n) = \mathbb{P}(\sigma_k^{\lambda_n} = \sigma),$$

σ being any permutation such that $des(\sigma) = \mu$. By Proposition 9.12,

$$\mathbb{P}(\sigma_k^{\lambda_n} = \sigma) = \mathbb{P}(\sigma_{\tilde{U}_{\lambda_n}}((\xi_i^{\lambda_n})_{1\le i\le k}) = \sigma).$$

By Proposition 9.24, as n goes to $+\infty$, $(\xi_i^{\lambda_n})_{1\le i\le k}$ converges in law to a sequence $(X_1, \dots, X_k)$ of uniform independent random variables on $[0,1]$.
Thus since $\tilde{U}_{\lambda_n} \to U$, by Proposition 9.14

$$\sigma_{\tilde{U}_{\lambda_n}}((\xi_i^\lambda)_{1\le i\le k}) \to_{law} \sigma_U((X_1, \dots, X_k)) = \sigma_U.$$

Therefore

$$K_\mu(\lambda_n) = \mathbb{P}(\sigma_{\lambda_n} = \sigma) \to p_{(U_\uparrow, U_\downarrow)}(\mu).$$

□

As explained in Section 4, the latter Theorem implies Conjecture 9.7.

Corollary 9.25. *Conjecture 9.7 is true and for the graded graph $\mathcal{Z}$,*

$$\partial_{\min}\mathcal{Z} = \partial_M \mathcal{Z}.$$

We end this section by showing that the topology on $\hat{\mathcal{Z}} = \mathcal{Z} \cup \partial_M \mathcal{Z}$, abstractly constructed in Section 2.1, can be concretely described in terms of oriented Paintbox construction. From the work of Gnedin and Olshanski in [42] (Proposition 36), $\partial_{\min}\mathcal{Z}$ with the induced topology of Section 2.1 is homeomorphic to $\mathcal{U}^{(2)}$. Since from the latter Corollary, $\partial_{\min}\mathcal{Z} = \partial_M \mathcal{Z}$, as topological spaces

$$\partial_M \mathcal{Z} = \mathcal{U}^{(2)}.$$

It remains to describe the topology of $\hat{\mathcal{Z}} = \mathcal{Z} \cup \partial_M \mathcal{Z}$. Let $\mathcal{U}_n \subseteq \mathcal{U}^{(2)}$ be the set of $(U_\uparrow, U_\downarrow)$ such that $[0,1] \setminus U_\uparrow \cup U_\downarrow \subseteq \{\frac{k}{n-1}, 0 \leq k \leq n-1\}$. Then $\hat{\mathcal{Z}}$ is characterized as follows:

Corollary 9.26. *Let $\mathcal{T} = [0,1] \times \mathcal{U}^{(2)}$ with the product topology. As topological spaces,*

$$\hat{\mathcal{Z}} \simeq (\{0\} \times \mathcal{U}^{(2)}) \cup \bigcup_{n \geq 1} (\{\frac{1}{n}\} \times \mathcal{U}_n) \subseteq \mathcal{T},$$

the space on the right being considered with the induced topology from $\mathcal{T}$.

Proof. The bijection Φ is achieved by sending $\lambda \vdash n$ to $\frac{1}{n} \times U_\lambda$ and $\omega = \lim \lambda_n \in \partial_M \mathcal{Z}$ to $0 \times \lim U_{\lambda_n}$. Since $\hat{\mathcal{Z}}$ is compact, the only thing to prove is the continuity of the map.
Let $x_n \to \omega \in \hat{\mathcal{Z}}$. If $\omega \in \mathcal{Z}$, the sequence is stationary and the convergence is straightforward. Suppose that $\omega \in \partial_M \mathcal{Z}$, and divide x_n into two complementary subsequences $(x_{\varphi(n)})$ and $(x_{\varphi^c(n)})$ such that $x_{\varphi(n)} \in \mathcal{Z}$ and $x_{\varphi^c(n)} \in \partial_M \mathcal{Z}$.
By Proposition 36 of [42],

$$\Phi(x_{\varphi^c(n)}) \to \Phi(\omega).$$

By Corollary 9.25, since $x_{\varphi(n)} \to_{\hat{\mathcal{Z}}} w$,

$$\Phi(x_{\varphi(n)}) = U_{x_{\varphi(n)}} \to \Phi(\omega),$$

which concludes the proof. □

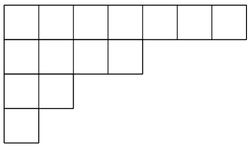
9.9 The Plancherel measure

The purpose of this section is to investigate the Plancherel measure on the graph $\mathcal{Z}$, which is the point $(\emptyset, \emptyset)$ of $\partial_M \mathcal{Z}$. We first recall the link between $\mathcal{Z}$ and the Young graph $\mathcal{Y}$ to justify the name of Plancherel measure. This link was already explained in [42] in terms of associated algebras of functions, but it seems to us that no direct link on the level of paths was clearly defined. It is the purpose of the second paragraph to clearly establish this link on the level of paths.

9.9.1 The graph $\mathcal{Y}$

A partition ρ of n is the data of a decreasing sequence of positive integers $(\rho_1 \geq \cdots \geq \rho_r)$ such that $\sum \rho_i = n$. n is called the degree of ρ and is denoted $\rho \vdash n$. Let us denote by $l(\rho)$ the length of the sequence. The set of partitions of n is denoted $\mathcal{Y}_n$, and the set of all partitions $\mathcal{Y}$. $\mathcal{Y}$ is ordered by saying that $\rho \preceq \tau$ if and only if $l(\rho) \leq l(\tau)$ and for all $1 \leq i \leq l(\rho)$, $\rho_i \leq \tau_i$.

As for compositions, a Young diagram is associated to each partition by drawing ρ_1 cells on the first row, ρ_2 cells on the second row and so on, such that the first cell of the row $i+1$ is just below the first cell of the row i. A standard filling of ρ is a filling of ρ with elements of $\{1, \dots, n\}$, such that the filling is increasing to the right and to the bottom. We denote by T_ρ the set of standard fillings of ρ (also called stantard tableau of shape ρ). Here is an example of a partition $\rho = (7, 4, 2, 1)$ and a standard filling of the associated diagram.



1	3	4	6	7	10	11
2	5	9	14			
8	12					
13						

Figure 9.4: Young diagram of $(7, 4, 2, 1)$ and an example of standard filling

We say that $\rho \nearrow \tau$ if and only if $\deg \tau = \deg \rho + 1$ and $\rho \preceq \tau$. When $T \in T_\tau$ is a standard tableau of shape $\tau \vdash n$, $T_\downarrow$ is defined as the standard tableau obtained by deleting the cell containing n. In particular $T_\downarrow$ has a shape ρ such that $\rho \nearrow \tau$. Adding an edge from ρ to τ if and only if $\rho \nearrow \tau$ transforms $\mathcal{Y}$ into a graded graph. The latter graph is a major construction for the representation theory of the symmetric groups $(\mathfrak{S}_n)_{n \geq 1}$, since the irreducible representations V_τ of $\mathfrak{S}_n$ are indexed by elements τ of $\mathcal{Y}_n$, and there is a decomposition

$$Res(V_\tau)_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} = \sum_{\rho \nearrow \tau} V_\rho.$$

As for the graph $\mathcal{Z}$, the set of paths on $\mathcal{Y}$ between the root $\emptyset$ and a partition ρ is in bijection with the set of standard tableaux of shape ρ, and each element of $\partial_{\min}\mathcal{Y}$ yields a random path on the graph $\mathcal{Y}$ (namely an infinite standard tableau). The minimal and Martin boundaries of $\mathcal{Y}$ have been intensively studied (see [76],[48],[85]) and fully described. In particuler the equality $\partial_{\min}\mathcal{Y} = \partial_M \mathcal{Y}$ holds also in this setting, and

$$\partial_M \mathcal{Y} = \{(a_1 \geq a_2 \geq \dots \geq 0), (b_1 \geq b_2 \geq \dots \geq 0), \sum a_j + b_j \leq 1\}.$$

For each $\omega \in \partial_M \mathcal{Y}$, ρ_ω denotes the random path on $\mathcal{Y}$ according to the harmonic measure ω. The next paragraph establishes a link between $\mathcal{Y}$ and $\mathcal{Z}$ based on the algorithm RSK of Robinson, Schensted and Knuth. The relation between both graphs has been already established through the ring of symmetric functions and the one of quasisymmetric functions. The reader should refer to [42] for a complete review of the subject.

9.9.2 RSK algorithm and the projection $\mathcal{Z} \to \mathcal{Y}$

Let us first recall the RSK algorithm in the special case of permutations. This algorithm, initiated by Robinson in [30] and created by Schensted in [75], establishes a bijection between $\mathfrak{S}_n$ and pairs of standard tableaux of n of the same shape. Let $\sigma = (\sigma(1), \dots, \sigma(n)) \in \mathfrak{S}_n$. The algorithm constructs a pair of standard tableaux from σ as follows :

1. Start with an infinite array $A^0 = (a^0_{k,l})_{k,l \geq 1}$ such that each cell is filled with the entry $n+1$ (namely $a^0_{k,l} = n+1$), and an infinite array $B = (b_{k,l})_{k,l \geq 1}$ such that each cell is empty (B is called the recording tableau).

2. At each step i, $1 \leq i \leq n$, the following insertion is done on the array A^{i-1}:

   - Let $(1, l_1)$ be the first cell (starting from the left) on the first row of A^{i-1} such that $\sigma(i) \leq a^{i-1}_{1,l_1}$. Set $a^i_{1,l_1} = \sigma(i)$.

- Let $(2, l_2)$ be the first cell on the second row of A^{i-1} such that $a^{i-1}_{1,l_1} \leq a^{i-1}_{2,l_2}$. Set $a^i_{2,l_2} = a^{i-1}_{1,l_1}$.
- Continue the process until the step k_0 where $a^{i-1}_{k_0,l_{k_0}} > n$. For $k > k_0$ or $k \leq k_0, l \neq l(k)$, define $a^i_{k,l} = a^{i-1}_{k,l}$. Return $A^i = (A^i_{k,l})_{k,l\geq 1}$. Set $b_{k_0,l_{k_0}} = i$.

3. Let $P(\sigma)$ be the part of the array A^n containing entries lower or equal to n, and $Q(\sigma)$ the part of the array B consisting in non empty cells.

Then the following Theorem holds ([75], [30]):

Theorem 9.27. *The map $S : \sigma \mapsto (P(\sigma), Q(\sigma))$ is a bijection between $\mathfrak{S}_n$ and pairs of standard tableaux of n of the same shape. Moreover*

$$(P(\sigma^{-1}), Q(\sigma^{-1})) = (Q(\sigma), P(\sigma)).$$

From now on $\rho(\sigma)$ denotes the shape of $P(\sigma)$ (or $Q(\sigma)$).
The link between $\mathcal{Z}$ and $\mathcal{Y}$ resides in the following proposition, mapping paths on $\mathcal{Z}$ to paths on $\mathcal{Y}$.

Proposition 9.28. *Let $(\sigma_k)_{k\geq 1}$ be a path on $\mathcal{Z}$. Then $(\rho(\sigma_k))_{k\geq 1}$ is a path on $\mathcal{Y}$. Moreover if $\sigma = (\sigma_k)_{k\geq 1}$ is a random path on $\mathcal{Z}$, then $\rho(\sigma) = (\rho(\sigma_k))_{k\geq 1}$ is a random path on $\mathcal{Y}$ and for $P \in \mathcal{T}_\tau$ a path on $\mathcal{Y}$ between $\emptyset$ and $\tau \vdash k_0$,*

$$\mathbb{P}((\rho(\sigma_1), \dots, \rho(\sigma_{k_0})) = P) = \sum_{\substack{\sigma \in \mathfrak{S}_{k_0} \\ P(\sigma) = P}} \mathbb{P}(\sigma_{k_0} = \sigma).$$

Proof. Let $\sigma = (i_1, \dots, i_{k-1}, n, i_{k+1}, \dots) \in \mathfrak{S}_n$. If suffices to prove that

$$P(\sigma_\downarrow) = P(\sigma)_\downarrow.$$

Although the latter equality appears clearly in the algorithm, the proof is easier to write by using σ^{-1} : indeed write $\sigma^{-1} = (j_1, \dots, j_{n-1}, k)$. Since $\sigma_\downarrow = (i_1, \dots, i_{k-1}, i_{k+1}, \dots)$, $(\sigma_\downarrow)^{-1} = (j_1^*, \dots, j_{n-1}^*)$, with $j_l^* = j_l$ if $j_l < k$ and $j_l^* = j_l - 1$ if $j_l > k$. All the j_l^* (resp j_l) are distinct and thus

$$std((j_1^*, \dots, j_{n-1}^*)) = std((j_1, \dots, j_{n-1})).$$

Since the Schensted algorithm only depends on the relative values of the entries, the recording tableaux B of the algorithm for $(\sigma_\downarrow)^{-1}$ and σ^{-1} after $n-1$ steps are the same. Therefore

$$Q((\sigma_\downarrow)^{-1}) = Q(\sigma^{-1})_\downarrow.$$

Thus from Theorem 9.27,

$$P(\sigma_\downarrow) = Q(\sigma_\downarrow^{-1}) = Q(\sigma^{-1})_\downarrow = P(\sigma)_\downarrow.$$

This yields that $\rho(\sigma_\downarrow) \nearrow \rho(\sigma)$ and for any arrangement $(\sigma_k)_{k\geq 1}$, the sequence $(\rho(\sigma_k))_{k\geq 1}$ is a well-defined path on $\mathcal{Y}$.
In particular if $(\sigma_k)_{k\geq 1}$ is a random path on $\mathcal{Z}$ and $P \in T_\tau$, $\tau \vdash k_0$, summing the probabilities of each path yields

$$\mathbb{P}((\rho(\sigma_1), \dots, \rho(\sigma_{k_0})) = P) = \sum_{\substack{\alpha \in \mathfrak{S}_{k_0} \\ P(\alpha) = P}} \mathbb{P}_{\sigma_k}(\sigma_{k_0} = \alpha).$$

□

The important fact is that harmonic measures on $\mathcal{Z}$ yield harmonic measures on the graph $\mathcal{Y}$.

Corollary 9.29. *Let $\sigma = (\sigma_k)_{k\geq 1}$ be a random arrangement such that $\mathbb{P}(\sigma_k = \alpha)$ depends only on $Q(\alpha)$. Then $\rho(\sigma)$ yields a harmonic measure on $\mathcal{Y}$.*
In particular harmonic measures on $\mathcal{Z}$ yield harmonic measures on the graph $\mathcal{Y}$.

Proof. From Section 2, a random path $\rho = (\rho_k)_{k\geq 1}$ on $\mathcal{Y}$ comes from a harmonic measure if and only if for any partition $\tau \vdash n$, and $P_1, P_2 \in T_\tau$,

$$\mathbb{P}((\rho_1, \dots, \rho_n) = P_1) = \mathbb{P}((\rho_1, \dots, \rho_n) = P_2).$$

Let $\sigma = (\sigma_k)_{k\geq 1}$ be a random arrangement such that $\mathbb{P}(\sigma_k = \alpha) = p(Q(\alpha))$, with p a positive function on standard Young tableaux. From Proposition 9.28, for $k_0 \geq 1, \tau \vdash k_0$ and $P \in T_\tau$,

$$\begin{aligned}\mathbb{P}((\rho_1, \dots, \rho_{k_0}) = P) &= \sum_{\substack{\alpha \in \mathfrak{S}_{k_0} \\ P(\alpha) = P}} \mathbb{P}(\sigma_{k_0} = \alpha) \\ &= \sum_{\substack{\alpha \in \mathfrak{S}_{k_0} \\ P(\alpha) = P}} p(Q(\alpha)) = \sum_{Q \in T_\tau} p(Q),\end{aligned}$$

the last equality being due to Theorem 9.27. Thus $\mathbb{P}((\rho_1, \dots, \rho_{k_0}) = P)$ is independent of $P \in T_\tau$.
Let φ be a harmonic measure on $\mathcal{Z}$. From Section 3, φ yields a random arrangement $\sigma = (\sigma_k)_{k\geq 1}$ such that $\mathbb{P}(\sigma_k = \alpha) = p(des(\alpha))$, for a particular function $p : \mathcal{Z} \to \mathbb{R}^+$. By a standard combinatorial result (see [79]), i is a descent of α if and only if $i+1$ is in a strictly lower row than i in $Q(\alpha)$. Thus if $Q(\alpha) = Q(\alpha')$, then $des(\alpha) = des(\alpha')$ and $\mathbb{P}(\sigma_k = \alpha) = \mathbb{P}(\sigma_k = \alpha')$. From the first part of the Corollary, $\rho(\sigma)$ yields a harmonic measure on $\mathcal{Y}$. □

In general, for Q a standard tableau, $des(Q)$ denotes the set of indices i such that $i+1$ is in a strictly lower row than i. This yields in particular the following equality for $\lambda \in \mathcal{Z}$:

$$d_{\mathcal{Z}}(\emptyset, \lambda) = \sum_{\tau \in \mathcal{Y}} d_{\mathcal{Y}}(\emptyset, \tau) \#\{Q \in T_\tau, des(Q) = D_\lambda). \tag{9.9.1}$$

The latter equation yields the law of $\rho(\sigma_\lambda)$, when σ_λ is chosen uniformly on the set of paths on $\mathcal{Z}$ between $\emptyset$ and λ. Let $K^{\mathcal{Y}}_\tau(\rho) = \frac{d_{\mathcal{Y}}(\tau,\rho)}{d_{\mathcal{Y}}(\emptyset,\rho)}$ denote the Martin kernel on $\mathcal{Y}$.

Lemma 9.30. *Let $\lambda \vdash n$ be a composition and σ_λ be a uniform random path between $\emptyset$ and λ. Then $\rho(\sigma_\lambda)$ is a random path on $\mathcal{Y}$ with law*

$$\mathbb{P}(\rho(\sigma_\lambda)(k) = \tau) = d_{\mathcal{Y}}(\emptyset, \tau) K^{\mathcal{Y}}_\tau(\rho_\lambda),$$

for $\tau \in \mathcal{Y}_k$ and ρ_λ a random element of $\mathcal{Y}_n$ with law

$$\mathbb{P}(\rho_\lambda = \rho) = \mathbb{P}(Q(\sigma_\lambda) \in T_\rho).$$

Proof. Let us apply the Schensted algorithm to σ_λ. If $P \in T_\tau$, with τ a Young diagram of k cells ($k \leq n$),

$$\mathbb{P}(\rho(\sigma_\lambda)(k) = \tau) = \frac{1}{d_{\mathcal{Z}}(\lambda)} \#\{\sigma \in \mathfrak{S}_n, des(\sigma) = \lambda, P(\sigma_{\downarrow k}) = \tau\}.$$

Equation (9.9.1) transforms the latter expression into

$$\begin{aligned}\mathbb{P}(\rho(\sigma_\lambda)(k)=\tau)=&\frac{\sum_{\rho\in\mathcal{Y}}\#\{Q\in T_\rho, des(Q)=\lambda)\}\#\{\sigma, Q(\sigma)=Q, P(\sigma_{\downarrow k})=\tau)\}}{\sum_{\rho\in\mathcal{Y}} d_{\mathcal{Y}}(\emptyset,\rho)\#\{Q\in T_\rho, des(Q)=\lambda\}}\\ =&\sum_{\rho\in\mathcal{Y}}\frac{d_{\mathcal{Y}}(\emptyset,\rho)\#\{Q\in T_\rho, des(Q)=\lambda\}}{\sum_{\rho\in\mathcal{Y}} d_{\mathcal{Y}}(\emptyset,\rho)\#\{Q\in T_\rho, des(Q)=\lambda\}}\frac{d_{\mathcal{Y}}(\emptyset,\tau)d_{\mathcal{Y}}(\tau,\rho)}{d_{\mathcal{Y}}(\emptyset,\rho)}\\ =&d_{\mathcal{Y}}(\emptyset,\tau)K_\tau(\rho_\lambda),\end{aligned}$$

with $\rho_\lambda=\rho(\sigma_\lambda)$ a random variable on $\mathcal{Y}_n$ with law

$$\mathbb{P}(\rho_\lambda=\rho)=\frac{d_{\mathcal{Y}}(\emptyset,\rho)\#\{Q\in T_\rho, des(Q)=\lambda\}}{\sum_{\rho\in\mathcal{Y}} d_{\mathcal{Y}}(\emptyset,\rho)\#\{Q\in T_\rho, des(Q)=\lambda\}}=\mathbb{P}(Q(\sigma_\lambda)\in T_\rho).$$

□

We finally prove that on $\partial_M\mathcal{Z}$, ρ restricts to a surjective map $p:\partial_M\mathcal{Z}\to\partial_M\mathcal{Y}$.

Proposition 9.31. *Let $U=(U_\uparrow,U_\downarrow)\in\partial_M\mathcal{Z}$. Let $(a_1\geq a_2\geq\cdots\geq 0)$ (resp. $(b_1\geq b_2\geq\cdots\geq 0)$) be the lengths of the interval components of $U_\uparrow$ (resp. $U_\downarrow$) in decreasing order. Then $\omega(U)=((a_1\geq a_2\geq\cdots\geq 0),((b_1\geq b_2\geq\cdots\geq 0))\in\partial_M\mathcal{Y}$ and $\rho(\sigma_U)=\rho_{\omega(U)}$. Moreover the induced map*

$$p:\partial_M\mathcal{Z}\to\partial_M\mathcal{Y}$$

is surjective.

Proof. Since the Schensted algorithm relates a finite number of permutations to each Young diagram, the map ρ sending random paths of $\mathcal{Z}$ to random paths of $\mathcal{Y}$ is clearly continuous with respect to the topology of convergence in law. Thus it is enough to prove the result on a dense subset of $\partial_M\mathcal{Z}$. Let $U=(U_\uparrow,U_\downarrow)\in\partial_M\mathcal{Z}$ be such that $\bar{U}=[0,1]$ and U has a finite number of interval components. Denote by $(a_1\geq a_2\geq\cdots\geq a_n)$ (resp. $(b_1\geq b_2\geq\cdots\geq b_m)$) the lengths of the interval components of $U_\uparrow$ (resp. $U_\downarrow$) in decreasing order, and $\omega_U=((a_1\geq a_2\geq\cdots\geq a_n),(b_1\geq b_2\geq\cdots\geq b_m))$. Then σ_U can be approximated by a sequence σ_{λ_n} with $\lambda_n\vdash n$, $U(\lambda_n)\to U$. By Greene's Theorem (see [43]) and Lemma 2 of the paper [86] of Kerov and Vershik, almost surely $\rho(\sigma_{\lambda_n})$ converges in $\mathcal{Y}\cup\partial_M\mathcal{Y}$ to ω_U. Thus by identification of the Martin boundary on $\mathcal{Y}$ and Lemma 9.30,

$$\mathbb{P}(\rho(\sigma_{\lambda_n})(k)=\tau)\to d(\emptyset,\tau)K_{\omega_U}(\tau),$$

for $\tau\vdash k$.
In particular $\rho(\sigma_U)=\rho_{\omega(U)}$. Since the subset

$$\{U\in\mathcal{U}^{(2)},\bar{U}=[0,1],U\text{ has a finite number of components}\}$$

is dense in $\mathcal{U}^{(2)}$, the latter equality holds on $\mathcal{U}^{(2)}$.
For any element $\omega=((a_1\geq a_2\geq\cdots\geq 0),(b_1\geq b_2\geq\cdots\geq 0))\in\partial_M\mathcal{Y}$, it is possible to find $U\in\mathcal{U}^{(2)}$ such that $\omega_U=\omega$, thus the map

$$p:\begin{cases}\partial_M\mathcal{Z} & \longrightarrow & \partial_M\mathcal{Y}\\ U & \mapsto & \omega_U\end{cases}$$

is surjective. □

9.9.3 Asymptotic of λ_n under the Plancherel measure

The purpose of this subsection is to explore further the behavior of $\sigma_{(\emptyset,\emptyset)}$. $\sigma_{(\emptyset,\emptyset)}$ is called the Plancherel measure on $\mathcal{Z}$ since it is the only element of $\partial_M\mathcal{Z}$ that yields the Plancherel measure on $\mathcal{Y}$ through the map p of the last paragraph.
In order to describe the descent set of a permutation $\sigma \in \mathfrak{S}_{n+1}$ we introduce the following notations. Let f_σ be the piecewise linear function on $[0, n-1]$ such that $f_\sigma(0) = 0$, and

$$f_\sigma(i) - f_\sigma(i-1) = \begin{cases} -1 & \text{if } i \text{ is a descent of } \sigma \\ +1 & \text{othewise} \end{cases}.$$

To describe the asymptotic value of f_σ for σ following the probability measure $p_{(U_\uparrow,U_\downarrow)}$ ($(U_\uparrow, U_\downarrow) \in \mathcal{U}^{(2)}$), we define also the following function $f_{(U_\uparrow,U_\downarrow)}$: it is the unique a.e differentiable function on $[0, 1]$ such that

$$\begin{cases} f_{(U_\uparrow,U_\downarrow)}(0) = 0 & \\ f'_{(U_\uparrow,U_\downarrow)}(t) = 1 & \text{if } t \in U_\uparrow \\ f'_{(U_\uparrow,U_\downarrow)}(t) = -1 & \text{if } t \in U_\downarrow \\ f'_{(U_\uparrow,U_\downarrow)}(t) = 0 & \text{if } t \in [0,1] \setminus U \end{cases}.$$

The map $(U_\uparrow, U_\downarrow) \mapsto f_{(U_\uparrow,U_\downarrow)}$ is continuous from $\mathcal{U}^{(2)}$ to $\mathcal{C}([0, 1], \mathbb{R})$, and the following result holds :

Proposition 9.32. *Let $U \in \mathcal{U}^{(2)}$. Then*

$$(t \mapsto \frac{1}{n} f_{\sigma_U(n)}(nt)) \to_{p.s,\|.\|_\infty} f_{(U_\uparrow,U_\downarrow)}.$$

The proof is a deduction from Theorem 9.1, since $U(\sigma_U(n)) \to_{\mathcal{U}^{(2)}} U$.
The next step is to get the fluctuations of f_{σ_U}. Only the case $U = (\emptyset, \emptyset)$ is done here. The result consists mainly in a mathematical formalization of the results obtained by Oshanin and Voituriez from a physical point of view in [68]. The reader should refer to the latter paper for interesting additional informations on the process $f_{\sigma_{\emptyset,\emptyset}}$.

Theorem 9.33. *For σ_n being uniformly sampled among $\mathfrak{S}_n$,*

$$(t \mapsto \frac{1}{\sqrt{n}} f_{\sigma_n}(nt)) \to \frac{1}{\sqrt{3}}\mathcal{B},$$

$\mathcal{B}$ denoting the Brownian motion on $[0, 1]$.

Proof. Recall from Section 4 that σ_n can be sampled from a family of independent uniform random variables $(x_i)_{i\geq 1}$ on $[0, 1]$ by applying the map std^{-1} on the sequence $(x_i)_{1\leq i\leq n}$. Since $\sigma \mapsto \sigma^{-1}$ is a measure preserving map (uniquely for the uniform measure), $f_{\sigma_n} \sim f_{\sigma_n^{-1}}$. The property noticed by Oshanin and Voituriez is that $((f_{\sigma_n^{-1}}, x_n))_{n\geq 1}$ is a Markov chain : indeed i is a descent of σ_n^{-1} if and only if $x_i > x_{i+1}$. Therefore, $Des(\sigma_{n+1}^{-1}) \cap \{1, \dots, n-1\} = Des(\sigma_n^{-1})$, and $n \in Des(\sigma_{n+1}^{-1})$ if and only if $x_n > x_{n+1}$. In the sequel $f_{\sigma_n^{-1}}(i)$ is denoted by Y_i (the subscript n is dropped, since this depends only on $(\sigma_n)_{\downarrow i}$).
This yields that for $R = [r_1; r_1 + r_2]$ and $S = [s_1; s_1 + s_2]$, with $s_1 \geq r_1 + r_2 + 2$, $n \geq s_1 + s_2$, we have

$$(\#Des(\sigma_n) \cap R, \#Des(\sigma_n) \cap S)) \sim X_1 \otimes X_2,$$

with $X_1 \sim \#Des(\sigma_{r_2+1})$ and $X_2 \sim \#Des(\sigma_{s_2+1})$. Moreover the number of permutations of n with k descents is given by the Eulerian number A_k^n, and its asymptotic value (see [79]) gives:

Lemma 9.34. *For n going to infinity, and σ_n uniform on $\mathfrak{S}_n$,*

$$\frac{1}{\sqrt{n}}(\#Des - \frac{n}{2}) \to_{law} \mathcal{N}(0, 1/12).$$

The latter Lemma together with the strong Markov property shows that if we write $\tilde{f}_{\sigma_n}(t) = \frac{1}{\sqrt{n}} f_{\sigma_n}(nt)$, the marginal distributions of $\tilde{f}_{\sigma_n}$ converge towards the ones of $\frac{1}{\sqrt{3}}\mathcal{B}$. An adequate bound for $\|f_{\sigma_n}\|$ is needed to be able to conclude by stantard tightness arguments. We follow Theorem 8.4 of the book [22] of Billingsley :

Theorem 9.35. *Let $(Y_i)_{i\geq 0}$ be a real random process. Let $f_n : [0,1] \to \mathbb{R}$ define the linear interpolation between the points*

$$f_n(\frac{i}{n}) = \frac{1}{\sqrt{n}} Y_i.$$

Suppose that for all $\varepsilon > 0$, there exists $\lambda > 0, n_0 \geq 0$ such that for all $k \in \mathbb{N}, n \geq n_0$,

$$\mathbb{P}(\max_{i \leq n} |Y_{k+i} - Y_k| \geq \lambda\sigma\sqrt{n}) \leq \varepsilon/\lambda^2.$$

Then the sequence f_n is tight.

The hypothesis of the Theorem is verified through the following Lemma, that mimicks the situation coming from a usual random walk.

Lemma 9.36. *Let $S_n = \sup_{[0,n]} Y_n, a > 0$ and $b \leq a - 2$. Then*

$$\mathbb{P}(S_n \geq a, Y_n \leq b) \leq \mathbb{P}(Y_n \geq 2a - b - 2),$$

and $F_{S_n}(t) \geq F_{|Y_n|}(t)$ for all $t \in \mathbb{R}$.

Proof. In the Markov chain (Y_n, x_n), $T = \inf(u \in \mathbb{N}, Y_u = a)$ is a stopping time. Since $\{S_n \geq a\} = \{T \leq n\}$, $\{S_n \geq a\} \in \mathcal{F}_T$ and by the strong Markov property,

$$\begin{aligned}
\mathbb{P}(S_n \geq a, Y_n \leq b) =& \mathbb{P}((T \leq n) \cap (Y_n - Y_T \leq b - a)) \\
=& \mathbb{E}(\mathbf{1}_{T \leq n} \mathbb{P}_{(Y_T, x_T)}(\tilde{Y}_{n-T} - \tilde{Y}_0 \leq b - a)) \\
\leq& \mathbb{E}(\mathbf{1}_{T \leq n} \mathbb{P}_{(Y_T, x_T)}(\tilde{Y}_{n-T} - \tilde{Y}_1 \leq b - a + 1)),
\end{aligned}$$

with $(\tilde{Y}, \tilde{x}_i)$ being an independent random walk starting at (Y_T, x_T). Since $\tilde{Y}_{n-T} - \tilde{Y}_1$ is independent of the value $\tilde{Y}_0 = Y_T$,

$$\mathbb{E}(\mathbf{1}_{T \leq n} \mathbb{P}_{(Y_T, x_T)}(\tilde{Y}_{n-T} - \tilde{Y}_1 \leq b - a + 1)) = \mathbb{E}(\mathbf{1}_{T \leq n} \mathbb{P}_{(0, x_T)}(\tilde{Y}_{n-T} - \tilde{Y}_1 \leq b - a + 1)).$$

Moreover $\tilde{Y}_{n-T} - \tilde{Y}_1 \sim -(\tilde{Y}_{n-T} - \tilde{Y}_1)$, thus

$$\begin{aligned}
\mathbb{P}(S_n \geq a, Y_n \leq b) \leq& \mathbb{E}(\mathbf{1}_{T \leq n} \mathbb{P}_{(0, x_T)}(-(\tilde{Y}_{n-T} - \tilde{Y}_1) \leq b - a + 1)) \\
=& \mathbb{E}(\mathbf{1}_{T \leq n} \mathbb{P}_{(0, x_T)}(\tilde{Y}_{n-T} \geq a - (b+1) + \tilde{Y}_1)) \\
\leq& \mathbb{E}(\mathbf{1}_{T \leq n} \mathbb{P}_{(0, x_T)}(\tilde{Y}_{n-T} \geq a - (b+1) - 1)) \\
\leq& \mathbb{P}((T \leq n) \cap (Y_n \geq 2a - b - 2)) \leq \mathbb{P}(Y_n \geq 2a - b - 2),
\end{aligned}$$

the last equality being due to the fact that $(Y_n \geq 2a - b - 2) \subseteq (T \leq n)$. This yields

$$\begin{aligned}
\mathbb{P}(S_n \geq a) \leq& \mathbb{P}((S_n \geq a) \cap (Y_n \leq a - 2)) + \mathbb{P}((S_n \geq a) \cap (Y_n \geq a)) \\
\leq& \mathbb{P}(Y_n \geq a) + \mathbb{P}(Y_n \geq a) \leq \mathbb{P}(|Y_n| \geq a),
\end{aligned}$$

the last equality being due to the fact that the law of Y_n is symmetric. This yields

$$F_{S_n}(u) \geq F_{|Y_n|}(u).$$

$\square$

In particular from the latter Lemma, for $\varepsilon > 0$ and λ such that $\mathbb{P}(\mathcal{N}(0, 1/3) \geq \lambda) \leq \frac{\varepsilon}{4\lambda^2}$, and for n large enough, $k \geq 0$,

$$\begin{aligned}
\mathbb{P}(\max_{i \leq n} |Y_{k+i} - Y_k| \geq \lambda\sqrt{n}) \leq & 2\mathbb{P}(|Y_{k+n} - Y_k| \geq \lambda\sqrt{n}) \\
\leq & 2\mathbb{P}(\mathcal{N}(0, 1/3) \geq \lambda) + \frac{\varepsilon}{2\lambda^2} \\
\leq & \frac{\varepsilon}{\lambda^2}.
\end{aligned}$$

And this concludes the proof of the Proposition. $\square$

Appendix: convergence result for the Paintbox construction

This appendix is dedicated to the proof of Proposition 9.14. Some notations and two preliminary results are first given.

9.9.4 Cluster sets

Let $k \geq 1$ and A a given set. We define an $A-$cluster of k as a map $f : A \to \mathcal{P}([1; k])$ such that $f(a_1) \cap f(a_2) = \emptyset$ for $a_1 \neq a_2$. The residue of f is the set $R_f = [1; k] \setminus \bigcup f(a)$ and the support of f is the set S_f of $a \in A$ such that $f(a) \neq \emptyset$. The degree of f is the minimum of the cardinals of non-empty sets $f(a)$.The set of $A-$clusters (resp. $A-$clusters of degree larger than s) is denoted $C^k(A)$ (resp $C^k_s(A)$). For $1 \leq s \leq k$, the $s-$level of the $A-$cluster f, denoted f^s, is the $A-$cluster of $C^k_s(A)$ defined by :

$$f^s(a) = \begin{cases} f(a) & \text{if } |f(a)| \geq s \\ \emptyset & \text{else} \end{cases}.$$

Let $\vec{x} = (x_1, \ldots, x_k)$ be a sequence on a space $(\Omega^k, \mathcal{A}^{\otimes k})$. Then any set J and any collection of disjoint subsets $\mathcal{A} = (\mathcal{A}_j)_{j \in J}$ of Ω yields a $J-$cluster map of k

$$f_{\vec{x},\mathcal{A}}(j) = \{i | x_i \in \mathcal{A}_j\}.$$

For $U \in \mathcal{U}^{(2)}$, denote by $\mathcal{U} = \{U_\alpha\}_{\alpha \in A_U}$ the collection of interval components of $U_\uparrow \cup U_\downarrow$. We say that $\alpha \in A^+_U$ (resp. A^-_U) if $U_\alpha \subseteq U_\uparrow$ (resp. $U_\alpha \subseteq U_\downarrow$).

Lemma 9.37. *Let $U = (U_\uparrow, U_\downarrow) \in \mathcal{U}^{(2)}$. For $\sigma \in \mathfrak{S}_k$ and $f \in C^k_2(A_U)$ define the sets*

$$X_{\sigma,f}(U) = \{\vec{x} \in [0, 1]^k | std^{-1}(\vec{x}) = \sigma\} \cap \{(f_{\vec{x},\mathcal{U}})^2 = f\}.$$

Then the sets $X_{\sigma,f}(U)$ are disjoint open sets and σ_U is constant on each of these sets. In particular if $X_k = (X(i))_{1 \leq i \leq k}$ is a random variable with density on $[0, 1]^k$, $\sigma_U(X_1, \ldots, X_k)$ is $((X_{\sigma,f}(U))_{f,\sigma}$ measurable.

Proof. Let us write simply $X_{\sigma,f}$ instead of $X_{\sigma,f}(U)$. Suppose that $X_k \in X_{\sigma,f} \cap X_{\sigma',f'}$. Then $std^{-1}(X_k) = \sigma = \sigma'$. Moreover $f = (f_{X_k,\mathcal{U}})^2 = f'$ and $X_{\sigma,f} = X_{\sigma',f'}$. The events $X_{\sigma,f}$ are thus disjoint for distinct pairs (σ, f). They are open from their definition and the fact that $U_\uparrow, U_\downarrow$ are open sets.
Each $\tau \in \mathfrak{S}_k$ is entirely defined by the set $S_\tau = \{(i,j)|i < j, \tau^{-1}(i) \leq \tau^{-1}(j)\}$. Let $\sigma \in \mathfrak{S}_k$. Then from the Paintbox construction, $\sigma_U^{-1}(\{\tau\}) \cap \{X_k|std^{-1}(X_k) = \sigma\}$ is precisely the set of X_k such that by writing $f = f_{X_k,\mathcal{U}}$:

- $(i,j) \in S_\sigma \cap S_\tau \Rightarrow \forall a \in A_U^-, \{i,j\} \not\subseteq f(a)$,
- $(i,j) \in S_\sigma \setminus S_\tau \Rightarrow \exists a \in A_U^-, \{i,j\} \subseteq f(a)$,
- $(i,j) \in S_\tau \setminus S_\sigma \Rightarrow \exists a \in A_U^+, \{i,j\} \subseteq f(a)$,
- $(i,j) \notin S_\sigma \cup S_\tau \Rightarrow \forall a \in A_U^+, \{i,j\} \not\subseteq f(a)$.

Define by $D(\sigma, \tau)$ the set of $A(U)-$clusters that respect the above four conditions. Thus

$$\sigma_U^{-1}(\{\tau\}) \cap \{X_k|std^{-1}(X_k) = \sigma\} = \bigcup_{f \in D(\sigma,\tau)} X_{\sigma,f}.$$

Since X_k admits a density function, $[0,1]^k \setminus \bigcup\{X_k|std^{-1}(X_k) = \sigma\}$ is a null-set and thus :

$$\sigma_U^{-1}(\{\tau\}) = \bigcup_\sigma \bigcup_{f \in D(\sigma,\tau)} X_{\sigma,f},$$

which proves the Lemma. □

9.9.5 Convergence in law with conditioning

The pattern of the proof of Proposition 9.14 implies the following question: suppose that A, B are metric spaces, considered as measure spaces with a given measure on each associated borelian $\sigma-$algebra. Let $f_n : A \to B$ be a sequence of measurable functions that converges pointwise almost surely to a continuous function $f : A \to B$. Let $(X_n)_{n\geq 1}$ be a sequence of random variables on A that converges in law to a random variable X. Do we have the convergence in law $f_n(X_n) \to f(X)$? The answer is negative in general, but in a very particular case the result holds.

Lemma 9.38. *Let $(\mathcal{X}_m)_{m\geq 1}$ be a family of measurable spaces of A with the following conditions :*

- $\lim_{m\to\infty} \mathbb{P}(X \in \mathcal{X}_m) = 1$.
- $\forall m \geq 1, \mathbb{P}(X \in \partial\mathcal{X}_m) = 0$.
- *For all $m \geq 1$, $f_{n|\mathcal{X}_m} \to f_{|\mathcal{X}_m}$ uniformly.*

Then $f_n(X_n)$ converges in law to $f(X)$.

Proof. Let $g : B \to \mathbb{R}$ be a $1-$Lipschitz function bounded by 1. It suffices to show that $\mathbb{E}(g \circ f_n(X_n)) - \mathbb{E}(g \circ f(X)) \longrightarrow 0$. For each $m \geq 1$, the difference can be bounded by

$$\begin{aligned}
&|\mathbb{E}(g \circ f_n(X_n)) - \mathbb{E}(g \circ f(X))| \leq \\
&\quad |\mathbb{E}(g \circ f_n(X_n)) - \mathbb{E}(g \circ f_n(X_n)|\mathcal{X}_m)| + |\mathbb{E}(g \circ f_n(X_n)|\mathcal{X}_m) - \mathbb{E}(g \circ f(X_n)|\mathcal{X}_m)| \\
&+|\mathbb{E}(g \circ f(X_n)|\mathcal{X}_m) - \mathbb{E}(g \circ f(X)|\mathcal{X}_m)| + |\mathbb{E}(g \circ f(X)|\mathcal{X}_m) - \mathbb{E}(g \circ f(X))|.
\end{aligned}$$

Let m be such that $\mathbb{P}(X \in \mathcal{X}_m) \geq 1 - \varepsilon$. Since $\mathbb{P}(X \in \partial\mathcal{X}_m) = 0$, by the convergence in law there exists n_0 such that for $n \geq n_0$, $\mathbb{P}(X_n \in \mathcal{X}_m) \geq 1 - 2\varepsilon$. For $n \geq n_0$,

$$\mathbb{E}(g \circ f_n(X_n)) = \mathbb{P}(X_n \in \mathcal{X}_m)\mathbb{E}(g \circ f_n(X_n)|\mathcal{X}_m) + \mathbb{P}(X_n \notin \mathcal{X}_m)\mathbb{E}(g \circ f_n(X_n)|\mathcal{X}_m^c).$$

Since g is bounded by 1 and $\mathbb{P}(X_n \notin \mathcal{X}_m) \leq 2\varepsilon$,

$$\begin{aligned}|\mathbb{E}(g \circ f_n(X_n)) - \mathbb{E}(g \circ f_n(X_n)|\mathcal{X}_m)| \leq & 2\varepsilon + |1 - \mathbb{P}(X_n \in \mathcal{X}_m)| \\ \leq & 4\varepsilon.\end{aligned}$$

For the same reasons,

$$|\mathbb{E}(g \circ f(X)|\mathcal{X}_m) - \mathbb{E}(g \circ f(X))| \leq 2\varepsilon.$$

Let $n_1 \geq n_0$ such that for $n \geq n_1$, $\|f_{n|\mathcal{X}_m} - f_{|\mathcal{X}_m}\| \leq \varepsilon$. Since g is $1-$Lipschitz, for $n \geq n_1$,

$$|\mathbb{E}(g \circ f_n(X_n)|\mathcal{X}_m) - \mathbb{E}(g \circ f(X_n)|\mathcal{X}_m)| \leq \varepsilon.$$

Since $\mathbb{P}(X \in \partial\mathcal{X}_m) = 0$, $(X_n|\mathcal{X}_m)$ converges in law to $(X|\mathcal{X}_m)$ and thus there exists $n_2 \geq n_1$ such that for $n \geq n_2$,

$$|\mathbb{E}(g \circ f(X_n)|\mathcal{X}_m) - \mathbb{E}(g \circ f(X)|\mathcal{X}_m)| \leq \varepsilon.$$

Therefore for $n \geq n_2$,

$$|\mathbb{E}(g \circ f_n(X_n)) - \mathbb{E}(g \circ f(X))| \leq 5\varepsilon,$$

which implies the Lemma. □

9.9.6 Proof of Proposisition 9.14

Let us recall here the statement of Proposition 9.14:

Proposition. *Let U_n be a sequence of $\mathcal{U}^{(2)}$ and $((X^n(i))_{i\geq 1})_{n\geq 1}$ a sequence of random infinite vectors on $[0,1]$. Let $(X^0(1), \dots, X^0(n), \dots)$ be a random infinite vector on $[0,1]$. Suppose that each finite dimensional marginal law of any of these random vectors admits a density with respect to the Lebesgue measure. If $U_n \to U \in \mathcal{U}^{(2)}$ and for each $k \geq 1$, $X_k^n = (X^n(1), \dots, X^n(k))$ converges in law to $X_k^0 = (X^0(1), \dots, X^0(k))$, then for each $k \geq 1$,*

$$\sigma_{U_n}(X_k^n) \longrightarrow_{law} \sigma_U(X_k^0).$$

Proof. Let $k \geq 1$ and set $X = X_k^0$, $X_n = X_k^n$.
Let $A = \bigcup_{\substack{\sigma \in \mathfrak{S}_k \\ f \in C_2^k(A_U)}} X_{f,\sigma}$ (refer to Lemma 9.37 for the definition of $X_{f,\sigma}$) with the induced topology from $[0,1]^k$, and $B = \mathfrak{S}_k$ with the discrete topology. Then from Lemma 9.37, $\sigma_U : A \to B$ is constant on each connected component $X_{f,\sigma}$ of A, thus σ_U is continuous.
By the definition of the convergence on $\mathcal{U}^{(2)}$, for $\vec{X} = (X_1, \dots, X_k) \in A$, $\sigma_{U_n}(\vec{X})$ converges to $\sigma_U(\vec{X})$.
Since $[0,1]^k \setminus A$ is of Lebesgue measure 0, we can suppose that X_n, X are random variables on A. It remains to build a sequence of measurable sets $\mathcal{X}_m$ that respects the hypothesis of Lemma 9.38.
Let $m \geq 1$. For $\eta > 0$, define $\Delta_\eta = \bigcup_{1\leq i,j\leq k}\{(x_1, \dots, x_k) \in [0,1]^k, |x_i - x_j| \leq \eta\}$. Then $\partial_{[0,1]^k}\Delta_\eta \subseteq \bigcup_{1\leq i,j\leq k}\{(x_1, \dots, x_k) \in [0,1]^k, |x_i - x_j| = \eta\}$. Since the latter is of Lebesgue measure 0, $\mathbb{P}(X \in \partial_{[0,1]^k}\Delta_\eta) = 0$. Since Δ_η is decreasing in η and $Leb(\bigcap \Delta_\eta) = 0$, there exists $\eta_1^m > 0$ such that $\mathbb{P}(X \in \Delta_{\eta_1^m}) \leq \frac{1}{m}$.

Denote by $\mathcal{U} = \{U_\alpha =]r_\alpha, s_\alpha[\}_{\alpha \in \mathcal{A}}$ the finite ordered collection of interval components of $U_\downarrow \cup U_\uparrow$ of size larger than η_1^m. $\mathcal{A}$ is a subset of $A(U)$. For $\eta > 0$, let

$$B_\eta = \bigcup_{i,\alpha} \{(x_1, \dots, x_k) \in [0,1]^k, x_i \in]r_\alpha - \eta, r_\alpha + \eta[\cup]s_\alpha - \eta, s_\alpha + \eta[\}.$$

Once again $Leb(\partial_{[0,1]^k} B_\eta) = 0)$, and since $Leb(\bigcap_\eta B_\eta) = 0$, there exists η_2^m such that $\mathbb{P}(X \in B_{\eta_2^m}) \leq \frac{1}{m}$. Let $K_m = B_{\eta_2^m} \cup \Delta_{\eta_1^m}$, $\mathcal{X}_m$ be the set $\{\vec{x} \notin K\}$. Then $\mathbb{P}(X \in \partial \mathcal{X}_m) = 0$ and $\lim_{m \to +\infty} \mathbb{P}(X \in \mathcal{X}_m) = 1$.
Let $\mathcal{X}_m$ be fixed, with associated complementary set $K_m = B_{\eta_2^m} \cup \Delta_{\eta_1^m}$. Set $\eta_m = \inf(\eta_1^m, \eta_2^m)$, and let n_m be such that for $n \geq n_0$, $d_{\mathcal{U}^{(2)}}(U_n, U) \leq \eta_m$. Suppose from now on that $n \geq n_m$. Since $d_{\mathcal{U}^{(2)}}(U_n, U) \leq \eta_m \leq \eta_1^m$ the interval components of $U_\downarrow^n$ (resp. $U_\uparrow^n$) of size larger than η_1^m are in order respecting bijection with those of $U_\downarrow$ (resp. $U_\uparrow$). Denote these interval components of U^n by $\mathcal{U}_n = \{U_\alpha^n =]r_\alpha^n, s_\alpha^n[\}_{\alpha \in \mathcal{A}}$, with $\mathcal{A} \subseteq A(U_n)$. Moreover since $d_{\mathcal{U}^{(2)}}(U_n, U) \leq \eta_m \leq \eta_2^m$, $|r_\alpha^n - r_\alpha| < \eta_2^m$ and $|s_\alpha^n - s_\alpha| < \eta_2^m$.
Since on $\mathcal{X}_m$, $|x_i - x_j| \geq \eta_1$, if $f \in C_2^k(A(U))$ and $S(f) \not\subseteq \mathcal{A}$, then $X_{\sigma,f}(U) \cap \mathcal{X}_m = \emptyset$. Thus we can consider that $f \in C_2^k(\mathcal{A})$. The same is true for $f \in C_2^k(A(U_n)), S(f) \not\subseteq \mathcal{A}$ with $(X_{\sigma,f}(U_n) \cap A) \cap \mathcal{X}_m$.
Let $f \in C_2^k(\mathcal{A})$ and suppose that $\vec{x} \in X_{\sigma,f}(U) \cap \mathcal{X}_m$. Let $\alpha \in S_f$ and suppose that $x_i \in U_\alpha =]r_\alpha, s_\alpha[$; since $\vec{x} \in \mathcal{X}_m$, $x_i \in]r_\alpha + \eta_2^m, s_\alpha - \eta_2^m[$. But $|r_\alpha^n - r_\alpha| < \eta_2^m$ and $|s_\alpha^n - s_\alpha| < \eta_2^m$, thus $x_i \in U_\alpha^n$. Conversely is $\alpha \in S_f$ and $x_i \in U_\alpha^n$, for the same reasons $x_i \in U_\alpha$. This shows that $X_{\sigma,f}(U) \cap \mathcal{X}_m = X_{\sigma,f}(U_n) \cap \mathcal{X}_m$. This yields that $\sigma_{U|X_{\sigma,f}(U)} = \sigma_{U_n|X_{\sigma,f}(U_n)}$. Finally we have proven that for $n \geq n_m$, $\sigma_{U_n|\mathcal{X}_m} = \sigma_{U|\mathcal{X}_m}$, which implies obviously the uniform convergence $\sigma_{U_n|\mathcal{X}_m} \to_{n \to \infty} \sigma_{U|\mathcal{X}_m}$.
The application of Lemma 9.38 concludes the proposition. □

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