Tropical Covers, Moduli Spaces & Mirror Symmetry

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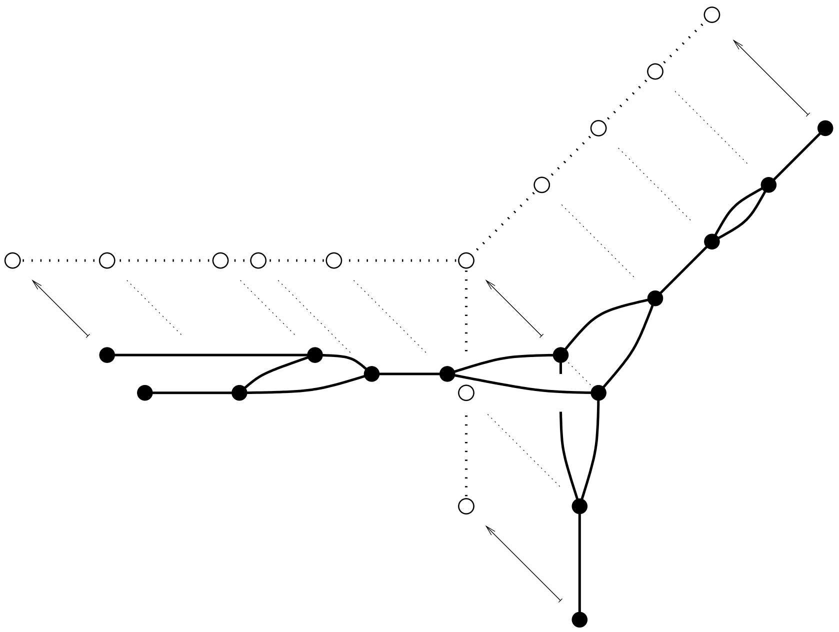
Die Gesamtheit der n-blättrigen Riemann'schen Flächen zu untersuchen, welche an w gegebenen Stellen in vorgeschriebener Weise verzweigt sind,[...].

Adolf Hurwitz,
Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten, 1891



Abstract

Classical Hurwitz theory studies ramified coverings of curves and provides many interesting connections between different fields a mathematical research. The idea of tropical geometry in general is to degenerate objects from algebraic geometry to piece-wise linear "tropical" objects and attack problems of the algebraic geometry world by understanding the combinatorics of the related objects in the tropical world. Tropical geometry appeared as an interesting tool for the study of Hurwitz theory. The purpose of this thesis is to lay clear foundations for tropical Hurwitz theory, bringing different aspects from literature together. We unify definitions from literature and provide moduli spaces for the important case of triple Hurwitz numbers. These can be seen as the building blocks of any higher Hurwitz number. Moreover we provide an important application of tropical Hurwitz theory concerning Mirror Symmetry, a currently very active field of mathematical research inspired by string theory.

Übersicht

Klassische Hurwitztheorie untersucht verzweigte Überlagerung von Kurven und liefert viele Verknüpfungen zwischen verschiedenen Forschungsgebieten der Mathematik. Das Konzept von Tropischer Geometrie im Allgemeinen ist es Objekte der Algebraischen Geometrie zu stückweise linearen „tropischen“ Objekten zu degenerieren und Probleme aus der Welt der Algebraischen Geometrie anzugehen, indem man die Kombinatorik der zugehörigen Objekte in der tropischen Welt versteht. Tropische Geometrie hat sich als ein vielversprechendes Werkzeug für Forschung in der Hurwitztheorie erwiesen. Das Ziel dieser Doktorarbeit ist es eine klare Grundlage für tropische Hurwitztheorie zu schaffen, indem sie verschiede Aspekte aus der Literatur zusammenführt. Wir vereinheitlichen die Definitionen der Literatur und stellen Modulräume für den wichtigen Fall der dreifachen Hurwitzzahlen bereit. Diese können als die Grundbausteine einer jeden höheren Hurwitzzahl angesehen werden. Des Weiteren stellen wir eine wichtige Anwendung der tropischen Hurwitztheorie in Bereich der Spiegelsymmetrie bereit, einem gegenwärtig äußerst aktiven, von der Stringtheorie inspirierten mathematischen Forschungsgebiet.

Contents

Preface

My thesis deals with tropical Hurwitz theory. Hurwitz theory is the study of ramified covers of curves and provides many interesting connections between geometry of covers, moduli spaces of curves, combinatorics, representation theory and matrix models. The idea of tropical geometry in general is to degenerate objects from algebraic geometry to piece-wise linear "tropical" objects and attack problems of the algebraic geometry world by understanding the combinatorics of the related objects in the tropical world. Tropical geometry appeared as an interesting tool for the study of Hurwitz theory. The purpose of this thesis is to lay clear foundations for tropical Hurwitz theory, bringing different aspects from literature together, and to provide an important application of tropical Hurwitz theory concerning Mirror Symmetry, a currently very active field of mathematical research inspired by string theory.

The classical Hurwitz Theory goes back to Adolf Hurwitz [**23**], who formulated and dealt with the problem of finding the number of d-sheeted covers of a fixed curve with fixed ramification profiles at fixed branch points. The ramification profile of a branch point is a partition of the degree d. If this partition equals $(2, 1, \ldots, 1)$ we call the ramification *simple*. The number of branch points, the lengths of their ramification profiles, the degree and the genera of source and target curve are related by the Riemann-Hurwitz formula. Hurwitz himself tackled the problem by relating these covers to tuples of permutations in the symmetric group $\mathbb{S}_d$ that fulfill certain properties. This makes Hurwitz numbers accessible combinatorially. The symmetric group approach provides for example a method to express arbitrary Hurwitz numbers in terms of static triple Hurwitz numbers via the so-called degeneration formula [**9, 29, 30**]. *Static triple Hurwitz numbers* count covers of $\mathbb{P}^1$ with exactly three branch points.

In modern research Hurwitz numbers play a role in connection with Gromov-Witten theory, i.e. the study of Chow rings of moduli spaces of stable curves and stable maps. Hurwitz numbers can be expressed as the degree of the branch map from the space of relative stable maps to the space of branch divisors [**16**]. Also they can be expressed in terms of Gromov-Witten invariants by the Gromov-Witten/Hurwitz correspondence by Okounkov and Pandharipande [**36**]. Furthermore the famous ELSV-formula [**13**] relates Hurwitz numbers to intersection products on the space of stable curves and has interesting applications including results about the structure of Hurwitz numbers and the proof of Witten's conjecture by Okounkov and Pandharipande [**37**].

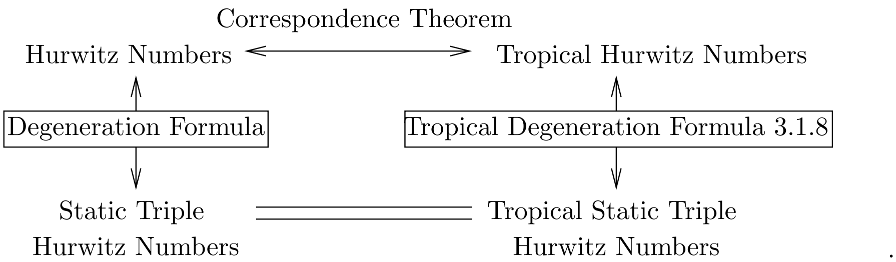
Tropical Hurwitz theory is a recent object of study. At the beginning of this thesis there were two articles concerning tropical Hurwitz numbers, [**10**] and [**4**].

The first one is an article by Cavalieri, Johnson and Markwig. This paper is concerned with tropical double Hurwitz numbers, i.e. weighted numbers of tropical covers with two (fixed) special ramification profiles and all other ramifications simple. They develope a natural definition of Hurwitz numbers as tropical intersection products, introducing an appropiate moduli space of tropical covers (which quite naturally comes with the structure of a *weighted polyhedral complex*) and a tropical branch map. This branch map is a *morphism of weighted polyhedral complexes* and it is quite easy to see that the degree of the branch map is constant, i.e. it does not depend on the position of the simple branch points, as long as they are in general position. The weights Cavalieri, Johnson and Markwig choose on the top-dimensional polyhedra of the moduli space are quite common to tropical geometers and in fact the degree — which can be seen as a weighted number of preimages — of the tropical branch map is as desired, meaning it equals the degree of the corresponding classical branch map, i.e. the Hurwitz number counting the isomorphism classes of covers in the moduli space considered. This equality between classical ant tropical Hurwitz numbers is referred to as *correspondence theorem*. The method of proof for the correspondence theorem is via the symmetric group.

In [**4**] on the other hand, Bertrand, Brugallé and Mikhalkin define tropical covers for any number of special ramifications and any (tropical) base curve. Their multiplicity for the covers match the multiplicities of [**10**] for tropical double covers. By proving a correspondence theorem, they show that their definition of tropical Hurwitz numbers as (weighted) count of tropical covers agree with their classical counterparts. The method of proof is via topological arguments. One problem is that their definition neither provides moduli spaces of covers nor branch maps.

The definition of tropical Hurwitz numbers given in section 2.3 is an approach to bring the definitions of both articles — [**4**] and [**10**] — together, allowing a maximum of generality **and** the possibilty to establish moduli spaces of covers in a very natural way. In this thesis we study more general moduli spaces of tropical covers and their branch maps, see chapter 4. The main result of this chapter is theorem 4.3.3, in which we establish the degree of the tropical branch map as a tropical intersection-theoretic invariant. Furthermore this degree equals (the tropical and thus) the classical Hurwitz number by theorem 4.3.6. With this chapter we enrich the correspondence theorem of [**4**] with a study of the involved tropical moduli spaces and intersection theory.

In chapter 3 we fully exploit the connection to the symmetric group. By proving a tropical version of the degeneration formula which boils the computation of arbitrary tropical Hurwitz numbers down to the computation of static classical triple Hurwitz numbers totally analogously to the classical degeneration formula we reprove the correspondence theorem of [**4**] by combinatorial methods using the symmetric group:

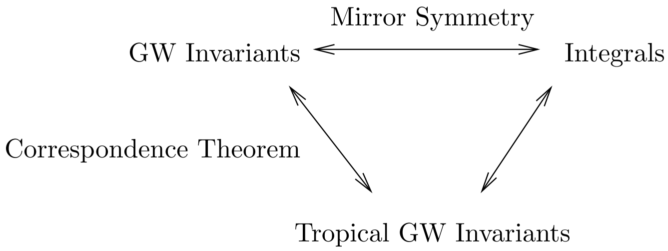


Furthermore we use symmetric group techniques to prove a correspondence theorem for covers of elliptic curves that will be important in chapter 5 in connection with Mirror Symmetry. The main results of chapter 3 are theorem 3.1.8 proving the tropical degeneration formula and the corresponce theorem for elliptic curves 3.2.3. This chapter sheds light on the deep connections between the symmetric group, degeneration techniques and tropical geometry.

Notice that theorem 4.3.3, theorem 4.3.6 and theorem 3.1.8 are formulated as local versions of the general results. We can understand the constructions involved in theorem 4.3.3 and theorem 4.3.6 as building blocks for general moduli spaces of tropical covers covering any rational, trivalent tropical curve. Our tropical degeneration formula describes the effect of cutting one bounded edge of the target tropical curve.

In the last chapter 5 finally we make use of tropical Hurwitz theory to study Mirror Symmetry for elliptic curves. Mirror Symmetry is a deep duality relation inspired by string theory. It associates to an algebraic variety X a mirror partner $X^\vee$ such that important invariants of X and $X^\vee$ get interchanged. In this thesis we consider Mirror Symmetry of elliptic curves together with Hurwitz numbers as invariants. On the mirror, which is also an elliptic curve, we consider Feynman integrals, which are certain integrals of a propagator function over a graph. In the general case certain Gromov-Witten invariants play the role of Hurwitz numbers and more general integrals the role of Feynman integrals.

Tropical geometry has proved to be an interesting new tool for Mirror Symmetry (see e.g. [**1, 18, 19, 20, 8**]). Our study can be viewed as a sequel and extension of Gross' paper [**18**], where he provides tropical methods for the study of Mirror Symmetry of $\mathbb{P}^2$. The main purpose of his paper is of a philosophical nature: he suggests tropical geometry as a new and worthwhile method for the study of Mirror Symmetry. More precisely, you can find (a version of) the following triangle in the introduction of [**18**]:



The relation between Gromov-Witten invariants and integrals (the top arrow) is a consequence of Mirror Symmetry. Tropical geometry comes in naturally, because there are many instances of correspondence theorems that relate Gromov-Witten invariants with their tropical analogues (the first of these is due to Mikhalkin [**32**]). The connection between tropical geometry and integrals is in general yet to be understood.

Gross studies the triangle in the situation of $\mathbb{P}^2$. He concentrates on proving the right arrow. For the left arrow only a partial Correspondence Theorem is known. Therefore his studies do not provide a complete proof for the Mirror Symmetry statement (the top arrow) yet.

In this thesis we present — for the case of elliptic curves — a complete proof using the detour via tropical geometry as depictured in figure . Beyond this application we provide a tropical Mirror Symmetry statement involving more refined invariants, which is interesting on its own.

For the precise statement of the tropical Mirror Symmetry theorem see theorem 5.2.6 — the main result of chapter 5.

The tropical Mirror Symmetry theorem has more consequences than the reproof of the classical Mirror Symmetry statement: First, the tropical approach provides computationally accessible algorithms for the involved invariants [**5**]. We can also use it to prove new results aboput the quasimodularity of certain generating functions of Hurwitz numbers, see [**6**]. Furthermore we can give a combinatorial

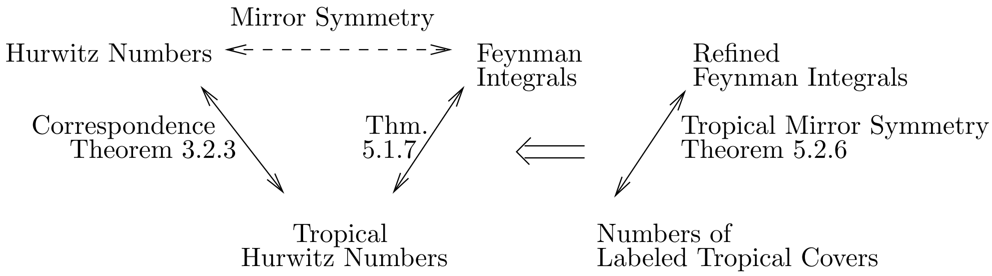


FIGURE 1. Mirror symmetry of the elliptic curve and the proof via a detour

characterisation for the vanishing of Feynman graphs, see corollary 5.4.10 of chapter 5.

Contents

The thesis is organized as follows:

In chapter 1 we introduce Hurwitz numbers and review known facts and historical roots.

In chapter 2 we introduce tropcial curves and covers. In this chapter we mainly follow ideas present in the literature. However, the definition of tropical Hurwitz numbers given in section 2.3 unifies the existing definitions of [**4**] and [**10**].

In chapter 3 we use symmetric group techniques to prove correspondence theorems. The tropical analogue of the classical degeneration formula as presented in section 3.1 plays an important role. This chapter is based on my own results. However, the correspondence theorem for elliptic curves is also published in a preprint joint with Janko Böhm, Kathrin Bringmann and Hannah Markwig [**6**].

In chapter 4 we construct general tropical moduli spaces of covers and connect the degree of the suitable branch maps to tropical Hurwitz numbers. The content of this chapter is based on the joint paper with Hannah Markwig [**7**] published in *"Communications in Contemporary Mathematics"* in 2013 (DOI: 10.1142/S0219199713500454).

In chapter 5 we study tropical Mirror Symmetry of elliptic curves. The content of this chapter is also based on the preprint with Janko Böhm, Kathrin Bringmann and Hannah Markwig [**6**].

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Die Arbeit an einer Dissertation ist — zumindest hier scheinen sich fast alle Wissenschaftler einig — mit vielen Höhen und Tiefen verbunden und kann bisweilen auch an den Nerven zehren. In meiner Zeit als Promotionsstudent habe ich in vielerlei Hinsicht und auf vielerlei Ebenen von Menschen in meiner Umgebung Unterstützung erfahren. Diesen Menschen möchte ich im Folgenden Dank sagen.

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CHAPTER 1

Ramified Covers

Before we define covers properly we will give a short historical review. For the moment it is enough to think of a ramified cover as a continuous finite mapping φ between complex curves that are somehow "smooth" (e.g. real Riemann surfaces or non-singular algebraic varieties) such that all but finitely many points have the same (finite) number of preimages. For the remaining points the number of preimages is less or equal and they are called the *branch points* of the mapping. We can think of the preimage curve having several sheets that come together in so-called *ramification points* over the branch points. The number of sheets coming together in a ramification points is called its *ramification index* and the tuple of all ramification indices of the preimages of a branch point is called its *ramification profile*.

1.1. Historical Roots

The problem of counting ramified covers dates back to the nineteenth century. The special case of degree 3 covers has already been considered in the year 1876 by Hermann Kasten in his dissertation in Göttingen. Moreover the question was touched by Felix Klein and Walther von Dyck and the former was the first one who formulated the task to determine "all Riemann surfaces with given branch points" in 1882 (cf. [**26**, p. 64]). Although other mathematicians — e.g. David Hilbert — also referred to the problem briefly, Adolf Hurwitz was the first one to try to discuss it in full generality in his articles [**23**] and [**22**]. At least for the case where all but one ramification are simple he succeeded in giving a closed formula in the latter paper. We will give a brief overview about how Hurwitz formulated the question and state some of his results.

Hurwitz talks about covers of degree n as so-called "n-sheeted Riemann surfaces" and defines them as topological objects. He can therefore easily see that the number N of such surfaces having a certain *ramification profile* over w fixed points of the complex plane is a topological invariant, that is it does not depend on the position of the points as long as they are different. Hurwitz' definition of such surfaces is naturally connected to systems of permutations in the symmetric group $\mathbb{S}_n$. We will therefore describe it briefly in modern terms and give the definitions required.

Definition 1.1.1. Let $\alpha = (m_1, \dots, m_k)$ be a partition of n into k positive integers, then a permutation $\sigma \in \mathbb{S}_n$ is of *cycle type* α if σ permits a decomposition $\sigma = \sigma_1 \circ \cdots \circ \sigma_k$ into k disjoint cycles such that the cycle σ_i has length m_i for all $i = 1, \dots, k$.

The set of all permutations in $\mathbb{S}_n$ of cycle type $\alpha = (m_1, \dots, m_k)$ is denoted by $\mathbb{S}_n^{(\alpha)}$ or $\mathbb{S}_n^{(m_1,\dots,m_k)}$.

The construction of a n-sheeted Riemann surface with w fixed ramification points and given ramification profiles works as described in the following.

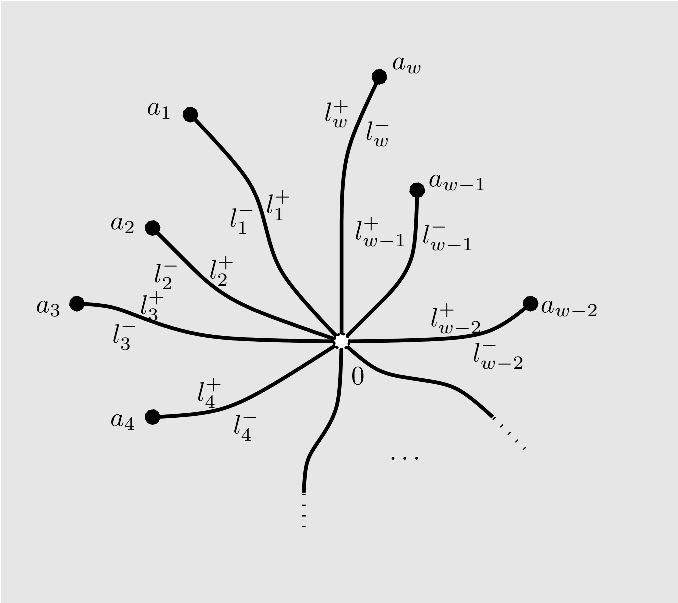


FIGURE 1. The Riemann surface E^*

CONSTRUCTION 1.1.2 (Hurwitz). Let E be the complex plane and $a_1, \ldots, a_w \in E \setminus \{0\}$ pairwise different points, each assigned with a ramification profile α_i. Assume there exists a tuple $(\sigma_1, \ldots, \sigma_w)$ of permutations in $\mathbb{S}_n$ such that

(i) for $i = 1, \ldots, w$ the permutation σ_i has cycle type α_i,
(ii) the subgroup $\langle \sigma_1, \ldots, \sigma_w \rangle \in \mathbb{S}_n$ generated by the σ_i acts transitively on $\{1, \ldots, n\}$ and
(iii) the product of all σ_i yields $\sigma_1 \circ \ldots \circ \sigma_w = \mathrm{id}_{\mathbb{S}_n}$.

Then we can construct a n-sheeted Riemann surface with ramification profile α_i over a_i, for $i = 1, \ldots, w$ in the following manner:

Choose w non-intersecting paths from 0 to each of the w points and denote by E^* the (simply connected) surface we get by cutting E along these paths. We may assume that the paths are ordered counter-clockwise and denote the $2w$ boundaries along the paths by l_i^+ and l_i^- as sketched in figure 1, such that if we travel counter-clockwise around the origin in a small neighbourhood we pass the boundaries in the order $l_1^+, l_1^-, l_2^+, l_2^-, \ldots, l_w^+, l_w^-$. Now take n copies of E^* and label them with $1, \ldots, n$, also relabel the boundaries l_i^+ and l_i^- of the j-th copy for all $i = 1, \ldots, w$ with $l_{j,i}^+$ and $l_{j,i}^-$ respectively. Finally for all $i = 1, \ldots, w$ and $j = 1, \ldots, n$ glue the boundary $l_{j,i}^+$ to the boundary $l_{\sigma_i(j),i}^-$. The resulting Riemann surface $S = S((a_i, \sigma_i)_{i=1,\ldots,w})$ carries the desired properties.

REMARK 1.1.3 (cf. [**23**]). The transitivity property in the above construction is equivalent to S being connected. The identity $\sigma_1 \circ \ldots \circ \sigma_w = \mathrm{id}_{\mathbb{S}_n}$ ensures that there is no ramification at the origin (see also section 1.3).

Hurwitz already realized that different choices of paths yield exactly the same covers. So it is sufficient to fix the branch points a_i and their ramification profiles α_i to count covers. Two covers are considered the same if they only differ in their labeling — or in terms of the symmetric group — two covers represented by the tuples $(\sigma_1, \ldots, \sigma_w)$ and $(\sigma'_1, \ldots, \sigma'_w)$ are identic if and only if there exists a permutation τ such that $\tau \circ \sigma_i \circ \tau^{-1} = \sigma_i'$ for all $i = 1, \ldots, w$. This has to be taken into account when counting ramified covers via the symmetric group.

1.2. Branched Covers

Hurwitz numbers count branched covers of smooth, complex, projective algebraic curves with given ramification profile over a finite number of points. If we use the term "algebraic curve" we always consider it to be smooth, complex and projective.

Since it is well-known that the categories of smooth, projective complex algebraic curves with algebraic morphisms and compact Riemann surfaces with holomorphic maps are equivalent, it is sufficient to consider branched covers in the latter category. Hurwitz himself, in fact, constructed Riemann surfaces S together with holomorphic maps $S \to \mathbb{C}$. Taking the Riemann sphere $\mathbb{C}_\infty$ instead of the plane $E = \mathbb{C}$ in his construction one creates holomorphic maps between compact Riemann surfaces which are in 1-on-1 correspondence to those constructed by Hurwitz. As we will see later every branched cover of the Riemann sphere has only finitely many branch points and their position is not important for counting covers, so — assuming without loss of generaility that the point ∞ is not a branch point — Hurwitz' considerations are sufficient to count branched covers of rational curves. We generalize his definitions by allowing base curves of arbitrary genus.

The following definitions and well-known facts about Riemann surfaces and holomorphic maps between them can be found in [**33**, II.4.]. Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a non-constant holomorphic map between compact Riemann surfaces for the remaining part of this subsection. The first thing to notice is that every non-constant holomorphic map between compact Riemann surfaces is surjective (see e.g. [**33**]). This leads to the following property of such maps.

Proposition 1.2.1. *Every non-constant holomorphic map $\varphi : \mathcal{C} \to \mathcal{D}$ between compact Riemann surfaces is finite and surjective, that is each point $p \in \mathcal{D}$ has non-empty finite fiber.*

Proof. Since φ is a non-constant, holomorphic map between compact Riemann surfaces it is surjective. Let p be a point in $\mathcal{D}$. We choose local charts on $\mathcal{D}$ and $\mathcal{C}$ which are centered at p and one of its preimages, respectively. Then φ restricted to these local charts is a holomorphic map, so its zero set (which is the set of preimages of p) is discrete and closed. Therefore the set of preimages of p on $\mathcal{C}$ under φ is discrete and closed and discrete, closed subsets of compact spaces are finite. □

A very nice feature of holomorphic maps between Riemann surfaces is the fact that locally they simply look like power maps if we choose the right coordinates. More precisely the following statement holds.

Proposition 1.2.2 (Local Normal Form, see e.g. [**33**]). *Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a non-constant holomorphic map between Riemann surfaces and p a fixed point on $\mathcal{C}$. Then there is a unique positive integer m such that for every chart $\phi_2 : U_2 \to V_2$ on $\mathcal{D}$ centered at $\varphi(p)$, there exists a chart $\phi_1 : U_1 \to V_1$ on $\mathcal{C}$ centered at p such that $\phi_2(\varphi(\phi^{-1}(z))) = z^m$.*

This fact allows us to define the multiplicity of a point on $\mathcal{C}$.

Definition 1.2.3. Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a non-constant holomorphic map between Riemann surfaces and p a fixed point on $\mathcal{C}$. Then the integer m from proposition 1.2.2 is called the *multiplicity* or *ramification index (of φ at p)*, denoted by $\mathrm{mult}_p(\varphi)$.

Remark 1.2.4. It is a well-known fact that the set of points having multiplicity more than 1 is a discrete set and it is finite if $\mathcal{C}$ is compact, see e.g. [**33**].

Definition 1.2.5. Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a non-constant holomorphic map between Riemann surfaces. A point $p \in \mathcal{C}$ is called *ramification point (of φ)* if $\mathrm{mult}_p(\varphi) \geq 2$. A point $q \in \mathcal{D}$ is a *branch point (of φ)* if it is the image of a ramification point. Moreover, if the surfaces are compact $\deg_q(\varphi) := \sum_{p\in\varphi^{-1}(q)} \mathrm{mult}_p(\varphi)$ is called the *(local) degree of φ at q* and the partition $(\mathrm{mult}_p(\varphi))_{p\in\varphi^{-1}(q)}$ of $\deg_q(\varphi)$ is called the *ramification profile* of φ over q. The map φ is *simply ramified* or equivalently has a *simple ramification* over q if it has ramification profile $(2, 1, 1, \ldots, 1)$.

We stick to the case where $\varphi : \mathcal{C} \to \mathcal{D}$ is a non-constant holomorphic map between *compact* Riemann surfaces. It is clear, that for a holomorphic function of type z^m the degree does not depend on the choice of image point. Moreover, if we fix any $q \in \mathcal{D}$ together with a chart, then due to the existience of local normal forms we have charts $\phi_i : U_i \to V_i$ for the preimages p_i of q. So in an appropriate neighbourhood of q the map φ looks like a disjoint union of power maps. In fact we only have to see that near q points have no preimages outside the V_i, but that is a consequence of the compactness of $\mathcal{C}$.

Proposition 1.2.6 ([**33**]). *Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a non-constant holomorphic map between compact Riemann surfaces. Then $\deg_q(\varphi)$ is the same for every $q \in \mathcal{D}$.*

Proof. The consideration before this proposition show that $\deg_q(\varphi)$ is locally constant (with respect to q). Since Riemann surfaces are connected by definition, it is constant. □

The proposition allows the following definition.

Definition 1.2.7. For a non-constant holomorphic map $\varphi : \mathcal{C} \to \mathcal{D}$ between compact Riemann surfaces we define the *degree* $\deg(\varphi)$ to be the degree $\deg_q(\varphi)$ of an arbitrary point $q \in \mathcal{D}$.

Remark 1.2.8. The definition of degree above agrees with the degree of φ as a (non-constant) morphism of smooth, complex, projective algebraic curves.

We see that apart from finitely many points, namely the branch points, a non-constant holomorphic map φ between compact Riemann surfaces is just a *covering map*, i.e. for any point p on the base surface we find a neighbourhood U, such that $p^{-1}(U)$ is a disjoint union of open sets each of which is mapped homeomorphically to U by φ. Moreover, in a neighbourhood of the branch points, it looks like a disjoint union of power maps. Of course this remains true if we consider such a map $\varphi : \mathcal{C} \to \mathcal{D}$ as a non-constant morphism of smooth, complex, projective algebraic curves (if we understand the term "neighbourhood" in the Euclidean, not the Zariski sense).

Definition 1.2.9. A *ramified* or *branched cover* is a non-constant holomorphic map between compact Riemann surfaces or, in other words, a non-constant morphism of smooth, complex, algebraic curves. Two branched covers $\varphi : \mathcal{C} \to \mathcal{D}$ and $\varphi' : \mathcal{C}' \to \mathcal{D}$ are *isomorphic* if there exists an isomorphism $\Phi : \mathcal{C} \to \mathcal{C}'$ such that $\varphi = \varphi' \circ \Phi$.

We conclude this section with the *Riemann-Hurwitz-Formula* (often related to as "Hurwitz' formula") as it is stated in [**33**, II.4.16.].

Theorem 1.2.10 (Riemann-Hurwitz-formula). *Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a non-constant holomorphic map between compact Riemann surfaces. Denote by $g(\mathcal{C})$ and $g(\mathcal{D})$ the* *genus of $\mathcal{C}$ and $\mathcal{D}$, respectively. Then*

$$2g(\mathcal{C}) - 2 = \deg(\varphi)(2g(\mathcal{D}) - 2) + \sum_{p \in \mathcal{C}} (\mathrm{mult}_p(\varphi) - 1), \tag{1}$$

Proof. We will just give a short sketch of the proof. For a more detailed version see [**33**].

Choose a triangulation T of $\mathcal{D}$ such that all branch points are vertices and lift this triangulation to a triangulation T' of $\mathcal{C}$. Then the negative Euler number of $\mathcal{C}$ is given by $2g(\mathcal{C}) - 2 = -v' + e' - t'$, where v', e', t' denote the number of vertices, edges and triangles in T', respectively. Clearly e' and t' coincide with $\deg(\varphi)$ times the number of egdes e and $\deg(\varphi)$ times the number of triangles t in T, respectively. Due to the ramification points v' is smaller than $\deg(\varphi) \cdot v$, namely $v' = \deg(\varphi) \cdot v - \sum_{p \in \mathcal{C}} (\mathrm{mult}_p(\varphi) - 1)$. This implies the desired formula. □

Note that in algebraic geometry, there is a more general version of the Riemann-Hurwitz-formula allowing ground fields to have arbitrary characteristic. Moreover it can be proved purely algebraically. Both can, for example, be found in [**21**, IV.2].

1.3. Monodromy

Assume we are given a branched cover $\varphi : \mathcal{C} \to \mathcal{D}$. Let B be the *branch locus* in $\mathcal{D}$, i.e. the set of branch points of φ and denote by $\mathcal{D}^\circ := \mathcal{D} \setminus B$ as well as $\mathcal{C}^\circ := \mathcal{C} \setminus \varphi^{-1}(B)$ the Riemann surfaces without branch locus and its preimage, respectively. Then restricting φ gives a covering map $\varphi^\circ : \mathcal{C}^\circ \to \mathcal{D}^\circ$. A nice feature of covering maps is that paths on $\mathcal{D}$ can be lifted to paths on $\mathcal{C}$. This leads us to the so-called *monodromy representation* of a branched cover. Let for this section $f : \mathcal{S} \to \mathcal{T}$ be a covering map between Riemann surfaces.

Lemma 1.3.1. *Let $f : \mathcal{S} \to \mathcal{T}$ be a covering map between Riemann surfaces and $\gamma : [0,1] \to \mathcal{T}$ a path in $\mathcal{T}$. Then, if we fix a point p in the fiber of $\gamma(0)$, there is a unique path $\tilde{\gamma} : [0,1] \to \mathcal{S}$ which starts at p and is a lifting of γ, i.e. $\tilde{\gamma}(0) = p$ and $f \circ \tilde{\gamma} = \gamma$.*

Proof. See e.g. [**34**, Lemma 54.1.] □

As a matter of fact not only paths but also homotopies of paths lift, i.e. if γ and δ are homotopic paths in $\mathcal{T}$ and $\tilde{\gamma}$ and $\tilde{\delta}$ are lifts with same starting point, then $\tilde{\gamma}$ and $\tilde{\delta}$ are homotopic as well. Therefore we can "lift elements from the fundamental group of $\mathcal{T}$": Fix a base point $q \in \mathcal{T}$ and take a point p in its fiber under f. For an element $[\gamma]$ of the fundamental group $\pi_1(\mathcal{T}, q)$ take a representative loop (e.g. γ) and its unique lift with p as starting point, then the lifted path has an end point over q which only depends on p and the homotopy class of γ. In fact this gives a group action of $\pi_1(\mathcal{T}, q)$ on the fiber of q.

Now let $f : \mathcal{S} \to \mathcal{T}$ be a *finite* covering map between (non-compact) Riemann surfaces, let us say of degree d. Again fix a point q in $\mathcal{T}$ and let $p_1, p_2, \ldots, p_d$ be the points in its fiber. Then the action of $[\gamma]$ on $\{p_1, p_2, \ldots, p_d\}$ provides a permutation in $\mathbb{S}_d$. This gives a group homomorphism $\rho : \pi_1(\mathcal{T}, q) \to \mathbb{S}_d$, which depends only on the labeling of $f^{-1}(q)$.

Definition 1.3.2. The map ρ defined above is called *monodromy representation* of f.

It is natural to extend this definition to branched covers: Given a branched cover $\varphi : \mathcal{C} \to \mathcal{D}$ then its *monodromy representation* is the monodromy representation of the corresponding covering map $\varphi^\circ : \mathcal{C}^\circ \to \mathcal{D}^\circ$, where branch points and its preimages are deleted.

It is clear that the fundamental group $\pi_1(\mathcal{D}^\circ, q)$ of $\mathcal{D}^\circ$ is generated by the generators of $\pi_1(\mathcal{D}, q)$ and elements represented by loops around branch points as sketched in figure 2. Now fix a branch point b with ramification profile $(m_1, \ldots, m_k)$, then the permutation associated to a loop around b has cycle type $(m_1, \ldots, m_k)$. Due to the connectedness of $\mathcal{C}^\circ$, the image $\rho(\pi_1(\mathcal{C}^\circ, q))$ acts transitively on the set $\{1, \ldots, d\}$. Moreover two branched covers φ and φ' are isomorphic if their monodromy representations ρ and ρ' are *conjugated*, i.e. there is a permutation $\sigma \in \mathbb{S}_d$ such that for all loops γ we have $\rho([\gamma]) = \sigma^{-1} \circ \rho'([\gamma]) \circ \sigma$. This conjugation obviously just amounts to a relabeling of the points in the fiber.

Now let $\mathcal{D}$ be a compact Riemann surface with a finite set B on it and a base point q. Denote by $\mathcal{D}^\circ$ the pointed Riemann surface $\mathcal{D} \setminus B$ and assume we are given group homomorphism $\rho : \pi(\mathcal{D}^\circ, q) \to \mathbb{S}_d$ whose image acts transitively. It is well-known that there exists a universal cover $\mathcal{C}_0 \to \mathcal{D}^\circ$, which is unique up to isomorphism. Let H be the preimage of $\mathrm{Stab}_{\{1\}}(\mathbb{S}_d)$ under ρ. Then there is a covering map $\varphi^\circ_\rho : \mathcal{C}^\circ := \mathcal{C}_0/H \to \mathcal{D}^\circ$ of degree d, whose monodromy representation is given by ρ. Locally around a point in B the cover looks like a power map between two punctured discs. Going back to the compactifaction $\mathcal{D}$ of $\mathcal{D}^\circ$ there is a unique compactification $\mathcal{C}$ of $\mathcal{C}^\circ$ together with an extension φ_ρ of the cover which locally extends the power maps to the origin. This extension is naturally a branched cover of degree d with monodromy representation ρ.

Altogether the former considerations lead to the following result, which can be used to give Hurwitz numbers a combinatorial representation, see the definitions 1.4.6 and 1.4.10 and the propositions following them, respectively.

Proposition 1.3.3. *Given a compact Riemann surface $\mathcal{D}$, finite subset B and a point $q \notin B$, there is a 1-to-1-correspondence between isomorphism classes of branched covers of $\mathcal{D}$ (with branch points in B) and conjugacy classes of group homomorphisms $\rho : \pi_1(\mathcal{D} \setminus B, q) \to \mathbb{S}_d$ whose image acts transitively.*

1.4. Classical Hurwitz Numbers

In this section let $\mathcal{D}$ be a compact Riemann surface of genus g', furthermore r and g non-negative integers and d a positive integer. Moreover fix r partitions $\mu_1, \ldots, \mu_r$ of d such that

$$s := s(g, g', d, (\mu_1, \ldots, \mu_r)) := 2g - 2 - d \cdot (2g' - 2) - rd + \sum_{i=1}^{r} |\mu_i| \geq 0, \quad (2)$$

where $|\mu_i|$ denotes the length of the partition μ_i. Now assume we are given a degree-d cover $\varphi : \mathcal{C} \to \mathcal{D}$ with source curve $\mathcal{C}$ of genus g and with r branch points of ramification profile $\mu_1, \ldots, \mu_r$. Assume moreover that all remaining ramifications are simple. Then due to the Riemann-Hurwitz formula (see theorem 1.2.10) the number of simple ramifications required equals s. We see that the non-negativity of the number s in equation (2) is a necessary condition for the existence of such a cover. Let us therefore assume for the remaining part of this section that all parameters are chosen in such a way that $s \geq 0$.

We will start with the standard definition of Hurwitz numbers and add a slight modification afterwards.

Definition 1.4.1 (Hurwitz Numbers). Let $\mathcal{D}$ be a compact Riemann surface of genus g'. Furthermore let d be a positive and g a non-negative integer and $\mu_1, \ldots, \mu_r$ be partitions of d. We fix a finite subset $B := \{b_1, \ldots, b_r, b_{r+1}, \ldots, b_{r+s}\}$ of pairwise different points in $\mathcal{D}$, where s is defined as in equation (2). The *Hurwitz number* $\check{H}_{d,g}(\mathcal{D}, (\mu_1, \ldots, \mu_r))$ is defined to be the *weighted* number of isomorphism classes of degree-d branched covers $\mathcal{C} \to \mathcal{D}$ such that

(i) the source curve $\mathcal{C}$ has genus g,
(ii) the cover has ramification profile μ_i over b_i for $i = 1 \dots, r$,
(iii) it has simple ramifications over $b_{r+1}, \dots, b_{r+s}$ and
(iv) it is unramified everywhere else.

The *weight* of each such isomorphism class of covers is $\frac{1}{|Aut(\varphi)|}$, for any representative φ.

REMARK 1.4.2. Note that the set B does not appear in the notation. In fact Hurwitz numbers do not depend on the position of the branch points as long as they are pairwise different.

In fact we would like to introduce Hurwitz numbers slightly different, labeling the preimages of the fixed ramification points.

DEFINITION 1.4.3 (labeled Hurwitz Numbers). Let $\mathcal{D}$ be a compact Riemann surface of genus g'. Furthermore let d be a positive and g a non-negative integer and $\mu_1, \dots, \mu_r$ be partitions of d. We fix a finite subset $B := \{b_1, \dots, b_r, b_{r+1}, \dots, b_{r+s}\}$ of pairwise different points in $\mathcal{D}$, where s is defined as in equation (2). The *(labeled) Hurwitz number* $H_{d,g}(\mathcal{D}, (\mu_1, \dots, \mu_r))$ is defined to be the *weighted* number of isomorphism classes of degree-d, *labeled* branched covers $\mathcal{C} \to \mathcal{D}$ such that

(i) the source curve $\mathcal{C}$ has genus g,
(ii) the cover has ramification profile μ_i over b_i for $i = 1 \dots, r$,
(iii) it has simple ramifications over $b_{r+1}, \dots, b_{r+s}$ and
(iv) it is unramified everywhere else.
(v) for $i = 1, \dots, r$ the preimages of b_i are labeled in order to be distinguishable.

The *weight* of each such isomorphism class $[\varphi]$ of covers is $\frac{1}{|Aut_{lab}(\varphi)|}$, where by $\mathrm{Aut}_{lab}(\varphi)$ we denote the group of automorphisms respecting the labels of φ.

In order to understand the difference between the two above definitions let us introduce the following notation.

REMARK 1.4.4. Let d be a positive integer and $\underline{p} = (p_1, \dots, p_k)$ a partition of d. Then $\mathbb{S}_k$ naturally acts on $\underline{p}$. We define the automorphism group of $\underline{p}$ as

$$\mathrm{Aut}(\underline{p}) := \{\sigma \in \mathbb{S}_k \mid \sigma(\underline{p}) = \underline{p}\}. \tag{3}$$

Obviously $|\mathrm{Aut}(\underline{p})| = \prod_{j=1}^{d} n_j!$, where n_j is the number of p_i that are equal to j.

PROPOSITION 1.4.5. *The difference between the two definitions for Hurwitz numbers is the number of automorphisms of a curve. In particular the second definition of Hurwitz numbers differs from the classical one in a factor of* $\prod_{i=1}^{r} |\mathrm{Aut}(\mu_i)|$*:*

$$H_{d,g}(\mathcal{D}, (\mu_1, \dots, \mu_r)) = \prod_{i=1}^{r} |\mathrm{Aut}(\mu_i)| \cdot \check{H}_{d,g}(\mathcal{D}, (\mu_1, \dots, \mu_r)).$$

PROOF. Assume we are given a unlabeled cover contributing to the Hurwitz number $\check{H}_{d,g}(\mathcal{D}, (\mu_1, \dots, \mu_r))$. Neglecting that different labels may produce the same (labeled) curve, there are $\prod_{i=1}^{r} |\mathrm{Aut}(\mu_i)|$ many ways to label the preimages of the branch points. Indeed preimages of the same point with different ramification index can already be distinguished, only those of same multiplicity have to get different labels.

Now assume that two labelings of a cover φ are non-trivially isomorphic, then the isomorphism is implied by an automorphism ψ of the unlabeled curve. But then clearly, since ψ cannot be the identity, it is not an automorphism for any labeling of the cover φ. Therefore

$$\frac{1}{|Aut_{lab}(\varphi_{lab})|} \cdot \prod_{i=1}^{r} |\mathrm{Aut}(\mu_i)| = \frac{1}{|Aut(\varphi)|} \cdot |L(\varphi)|,$$

where φ_{lab} denotes an abitrary labeling of φ and $L(\varphi)$ is the set of labeled covers based on the unlabeled cover φ. This implies that — when counting labeled covers instead of unlabeled — we "overcount" by a factor of $\prod_{i=1}^{r} |\operatorname{Aut}(\mu_i)|$. □

In focus of our attention will be Hurwitz numbers where the target curve $\mathcal{D}$ has genus 0 or 1. In the first case we will consider covers with $r = 3$ arbitrary ramifications. These numbers are called triple Hurwitz numbers. Clearly up to isomorphism $\mathcal{D}$ then equals $\mathbb{P}^1_{\mathbb{C}}$. In the second case, where $\mathcal{D}$ is an elliptic curve (by convention denoted by $\mathcal{E}$) we consider the case where $r = 0$ and we only have simple ramifications. We will use the correspondence in proposition 1.3.3 to establish alternative definitions for these numbers.

1.4.1. Hurwitz Numbers of $\mathbb{P}^1$. Let us fix as target curve $\mathbb{P}^1 =$ but let $r \geq 0$ be arbitrary. Assume we are given a degree $d \geq 1$ and r partitions $\mu_1, \ldots, \mu_r$ as well as a number $s \geq 0$ of simple ramification. Moreover fix a subset of points $B = \{b_1, \ldots, b_{r+s}\} \subset \mathbb{P}^1$. Assume that there exists a branched cover $\mathcal{C} \to \mathbb{P}^1$ with ramification profile μ_i over the first r branch points b_i, respectively, and simple ramifications over the remaining points in B. The genus of the source curve is then determined by the Riemann-Hurwitz formula. The 1-to-1-correspondence in proposition 1.3.3 relates the isomorphism class of this cover to a certain conjugacy class of group homomorphisms $\rho : \pi_1(\mathbb{P}^1 \setminus B, q) \to \mathbb{S}_d$, where $q \notin B$ is a fixed base point. A representative of this conjugacy class can be created by fixing an ordering on the d points in the fiber of q. So instead of counting covers we can count certain group homomorphisms — or in other words — certain tuples of images of the generators of $\pi_1(\mathbb{P}^1 \setminus B, q)$. This leads to an alternative definition of Hurwitz numbers for the case that the target curve is $\mathbb{P}^1$.

Definition 1.4.6 (Hurwitz numbers via the symmetric group). Let d be a positive integer and $\mu_1, \ldots, \mu_r$ be partitions of d. Moreover let g be a non-negative integer. The Hurwitz number $H_{d,g}(\mathbb{P}^1, (\mu_1, \ldots, \mu_r))$ is defined to be

$$\frac{1}{d!} \cdot \left| \left\{ (\sigma_1, \ldots, \sigma_r, \tau_1, \ldots, \tau_s) \in (\mathbb{S}_d)^{r+s} \mid \text{(i)-(iv) below is fulfilled} \right\} \right|, \tag{4}$$

where $s = \sum_{i=1}^{r} |\mu_i| + 2g - 2 - d \cdot (r-2)$ is defined as in inequality (2) and the set restricts to tuples of permutations that fulfill:

(i) For $1 = 1 \ldots, r$ the permutation σ_i has cycle type μ_i, denoted by $\sigma_i \in \mathbb{S}_d^{(\mu_i)}$,
(ii) all τ_j are transpositions,
(iii) the group $\langle \sigma_1, \ldots, \sigma_r, \tau_1, \ldots, \tau_s \rangle$ acts transitively on $\{1, \ldots, d\}$ and
(iv) the product of all σ_i, τ_j fulfills $\tau_s \circ \ldots \circ \tau_1 \circ \sigma_r \circ \ldots \circ \sigma_1 = \mathrm{id}_{\mathbb{S}_d}$.

Proposition 1.4.7. *The definition above coincides with definition 1.4.1 for the case where the base curve $\mathcal{D}$ is the complex projective line $\mathbb{P}^1$.*

Proof. The proposition follows from the correspondence in proposition 1.3.3: Instead of covers with branch locus B we count group homomorphisms $\pi_1(\mathbb{P}^1 \setminus B, q) \to \mathbb{S}_d$, where B is a set of $r+s$ points in $\mathbb{P}^1$ and q an arbitrary base point. In fact such a map is fully determined by the images of the generators. As generators we can choose for example $r+s$ loops $\gamma_1, \ldots, \gamma_{r+s}$ around the points in B as depicted in figure 2. Now the count of covers simplifies to the question how many possible tuples of images there are for our choice of generators of $\pi_1(\mathbb{P}^1 \setminus B, q)$.

From section 1.3 it is clear that a cover contributing to $H_{d,g}(\mathbb{P}^1, (\mu_1, \ldots, \mu_r))$ corresponds to a group homomorphisms where the loop around a branch point b_i is mapped to a permutation of cycle type μ_i for $i = 1, \ldots, r$ and to a transposition for $i = r+1, \ldots, r+s$. This explains the properties (i) and (ii) in the definition. Property (iii) follows from the fact that the source curves of our covers are

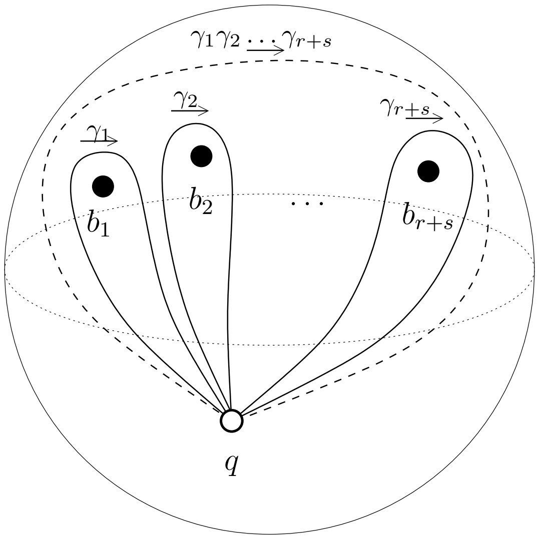


FIGURE 2. The generators of the fundamental group of the punctured sphere

connected. The last property ensures that our choice of images admits a group homomorphism: The loop that goes around all points in B (see the dashed path in figure 2) is homotopic to the 0-loop since we consider paths on the Riemann sphere. On the other hand it is homotopic to the composition of loops $\gamma_1, \dots, \gamma_{r+s}$. This implies $\gamma_1\gamma_2 \dots \gamma_{r+s} \cong 0$ and this is the only relation in $\pi_1(\mathbb{P}^1 \setminus B, q)$. So a choice of images σ_i and τ_j of the generators provides a group homomorphism $\rho : \pi_1(\mathbb{P}^1 \setminus B, q) \to \mathbb{S}_d$ if and only if $\rho(\gamma_{r+s}) \circ \dots \circ \rho(\gamma_1) = \rho(\gamma_1 \dots \gamma_{r+s})$ equals $\rho(0) = \mathrm{id}_{\mathbb{S}_d}$. That is exactly the property (iv) in the above definition.

Finally the factor $\frac{1}{d!}$ is due to the fact that we are overcounting and that we count *weighted* covers: In fact we have to count *conjugacy classes* of homomorphisms and not homomorphisms and weight each one with a certain factor. More precisely, assume we are given a valid tuple $(\sigma_1, \dots, \sigma_r, \tau_1, \dots, \tau_s)$ of images for the generating loops. If we conjugate all permutations of this tuple with the same permutation $\alpha \in \mathbb{S}_d$ we either get the same tuple again or we get a different but conjugated tuple. In the first case α corresponds to an automorphisms of the branched cover related to $(\sigma_1, \dots, \sigma_r, \tau_1, \dots, \tau_s)$ and in the second case the corresponding cover is isomorphic to the original one. Therefore we have to divide the number of valid tuples by the cardinality of $\mathbb{S}_d$. □

REMARK 1.4.8. Note that we just gave a definition that is equivalent to the classical definition 1.4.1 of Hurwitz numbers. Labelling the cycles in the cycle decomposition of each σ_i would provide a definition via the symmetric group that matches definition 1.4.3. We will omit the precise definition having in mind that labeled Hurwitz numbers differ from standard ones in a factor of $\prod_{i=1}^{r} |\operatorname{Aut}(\mu_i)|$.

In this thesis the case $r = 3$ plays a special role.

DEFINITION 1.4.9. Hurwitz numbers counting covers of $\mathbb{P}^1$ with $r = 3$ special ramifications are called *triple Hurwitz numbers*. If additionally the number s of simple ramifications is zero, they are called *static* triple Hurwitz numbers.

1.4.2. Simply Branched Covers of an Elliptic Curve. The second class of Hurwitz numbers we will consider are covers of elliptic curves that have only simple

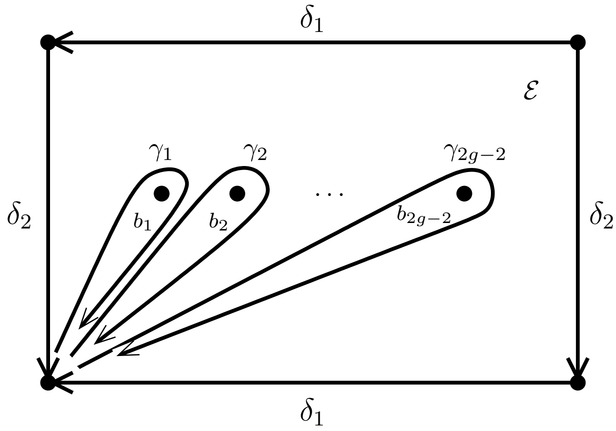


FIGURE 3. The generators of the fundamental group of $\mathcal{E} \setminus \{b_1, \ldots, b_{2g-2}\}$.

ramifications. Since Hurwitz numbers are topological invariants and all complex elliptic curves are homeomorphic to the real torus, those Hurwitz numbers do not depend on the choice of the base curve. So fix an arbitrary complex elliptic curve $\mathcal{E}$ and consider the Hurwitz numbers $H_{d,g}(\mathcal{E})$, counting isomorphism classes of degree d covers of $\mathcal{E}$ with genus g source curve. Due to the Riemann-Hurwitz-formula 1.2.10 such covers have $s = 2g - 2$ simple ramifications.

We establish a definition via tuples of permutations as in the case of rational target curves by using the correspondency in proposition 1.3.3 (cf.[**38**]):

DEFINITION 1.4.10 (Hurwitz numbers via the symmetric group II). Fix a degree $d > 0$ and a genus $g \geq 0$. Then the Hurwitz number $H_{d,g}(\mathcal{E})$ is defined to be

$$\frac{1}{d!} \cdot \left| \left\{ (\tau_1, \ldots, \tau_{2g-2}, \alpha, \sigma) \in (\mathbb{S}_d)^{s+2} \mid \text{ (i)-(iii) below is fulfilled} \right\} \right|, \tag{5}$$

where $s = 2g - 2$ is defined as in inequality (2) and the set only contains tuples of permutations that fulfill:

(i) All τ_i are transpositions,
(ii) the group $\langle \tau_1, \ldots, \tau_{2g-2}, \sigma, \alpha \rangle$ acts transitively on $\{1, \ldots, d\}$ and
(iii) the identity $\tau_{2g-2} \circ \ldots \circ \tau_1 = \alpha \circ \sigma \circ \alpha^{-1} \circ \sigma^{-1}$ holds.

PROPOSITION 1.4.11. *The definition above coincides with definition 1.4.3 for the case where the source curve $\mathcal{D}$ is elliptic and the covers counted do only have simple ramifications.*

PROOF. The proposition is proved analogously to proposition 1.4.7. Let B be a set of $2g-2$ distinct points on $\mathcal{E}$ and $q \notin B$ any base point, then the fundamental group $\pi_1(\mathcal{E} \setminus B, q)$ is generated by the loops around the $2g - 2$ points and two generators δ_1, δ_2 of $\pi_1(\mathcal{E}, q)$. Figure 3 depicts these generators on the torus cut along δ_1 and δ_2 for a nice choice of δ_1 and δ_2. Obviously going around all points in B is homotopic to walking along the path $(-\delta_2)(-\delta_1)\delta_2\delta_1$. Similar to the case of rational target curves this fact provides the condition (iii) in the definition above, where the images of δ_1 and δ_2 are denoted by α and σ, respectively. □

1.5. The Moduli Space of Relative Stable Maps and its Branch Map

In this section we will briefly explain the *space of relative stable maps*, which will be needed in section 4.4 to prove the main theorem of chapter 4. For an introduction to relative stable maps, see [**41**] and [**16, 29, 30**]. We will start with giving a brief introduction to moduli spaces in general. For an easy introduction on moduli spaces see [**27**].

The philosophy of moduli is to give a "nice" parameter space for geometric objects which have the same fixed properties up to some equivalence. Such problems a referred to as *moduli problems* in the following. The attribute "nice" above can mean several properties.

- The space should allow an appropiate definition of neighbourshoods, i.e. a moduli space should tell us how "close" two objects are to each other.
- We would like the moduli space to have a nice structure, e.g. a variety or a scheme.
- We would like our moduli spaces to be compact, in order to use intersection theory.

For the easiest cases of geometric objects moduli spaces are *fine*, as defined in the following.

Assume that we are given a moduli problem and find a natural parameter space M for our objects. A *family* X/B over such objects is just a morphism $X \xrightarrow{\pi} B$, where the base B parametrizes the objects, i.e. every fiber $\pi^{-1}(b)$ for $b \in B$ is one of our objects. Moreover, depending on the moduli problem, we demand some extra structure. For example, for n-marked points on curves, we add n sections. Furthermore, sometimes the morphism has to fulfill certain properties, such as flatness.

Definition 1.5.1 (Universal Family, Fine Moduli Space). Let M be a parameter space for a moduli problem, then U/M is called its *universal family* if for each family X/B of objects considered, there is a unique morphism $\gamma : B \to M$ such that the pullback $\gamma^* U$ along γ is equivalent to X (as family over B).

If M allows a universal family U/M, it is called *fine moduli space.*

Remark 1.5.2. If B is just a point p, then a family over B is just an object. $\{p\} \to M$ maps to a single point m in M and therefore the points in M are in bijection to (equivalence classes of) objects. The fiber U_m of m is equivalent to the object over b.

This holds even more generally: For any family X/B a point b is sent to the unique point m whose fiber U_m ist equivalent to the fiber X_b of b.

Example 1.5.3. Let us consider the example of the moduli problem of parametrizing rational, smooth, complex, projective curves with 4 (pairwise different) marked points $p_1, \ldots, p_4$ up to isomorphism. All these curves are isomorphic to $\mathbb{P}^1$ and the tuple of three points (p_1, p_2, p_3) can always be mapped to $(0, 1, \infty)$ by an appropiate automorphism of $\mathbb{P}^1$. So a natural parametrization of these objects can be given by the position of p_4 after these identifications, which is in fact the cross ratio of $p_1, \ldots, p_4$. Therefore the appropiate moduli space $M_{0,4}$ is just given by $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. It is a fine moduli space with universal family $M_{0,4} \times \mathbb{P}^1 \xrightarrow{\pi} M_{0,4}$ (together with four section $\sigma_1, \ldots, \sigma_4$ which stand for the markings) where π is the projection on the first component, σ_1, σ_2 and σ_3 map constantly to $0, 1$ and ∞, respectively, in the second component and σ_4 maps to the diagonal in $M_{0,4} \times \mathbb{P}^1$. The space is obviously not compact.

Unfortunately, for most interesting moduli problems a universal family does not exist. The problem often is the existence of non-trivial automorphisms of the objects considered. Often we encounter these problems if we try to find nice compactifications. In this case one has basically two options:
Firstly, one can weaken the universal family property and might establish at least a *coarse moduli space.* Secondly, we can drop the requirement of M being a scheme,

going over to a much more complicated category — the category of *stacks* — hoping that, in this case, a universal family for M exists.

We will introduce the *moduli stack of relative stable maps* in the following. On the way we will see Hurwitz spaces together with their branch maps.

DEFINITION 1.5.4. Let d be a positive integer, g, r integers and $\mu_1, \dots, \mu_r$ partitons of d. We fix the data of r pairwise different fixed points $q_1, \dots, q_r$ on $\mathbb{P}^1$. Then by $\mathcal{H}_{d,g}(\mathbb{P}^1, \mu_1, \dots, \mu_r)$ we denote the *Hurwitz space*, that is the space of isomorphism classes of covers $\varphi : \mathcal{C} \to \mathbb{P}^1$ together with s pairwise different marked points $p_1, \dots, p_s \in \mathbb{P}^1$ (s given by the Riemann-Hurwitz-formula 1.2.10), such that $\mathcal{C}$ is a irreducible, smooth, complex, projective curve of genus g and φ has ramification profile μ_i over q_i for $i = 1, \dots r$, is simply ramified at each p_j and unramified everywhere else.

The thus defined Hurwitz space is a smooth scheme in most of the cases. Problems only appear if the ramification profiles are chosen in a special way, so that they allow non-trivial automorphisms.

It is quite obvious that the spaces above are not compact: The space does not contain limits as two branch points run into each other. Since the special branch points q_i are fixed and pairwise different right from the beginning, this means that at least one of the branch points involved has to be simple.

Before we take a look at compactifications of $\mathcal{H}_{d,g}(\mathbb{P}^1, \mu_1, \dots, \mu_r)$ we will introduce the *branch map* for such spaces, which records the positions of the simple ramifications for each cover.

DEFINITION 1.5.5. For each Hurwitz space $\mathcal{H}_{d,g}(\mathbb{P}^1, \mu_1, \dots, \mu_r)$ we define a *branch map*

$$\mathrm{br} : \mathcal{H}_{d,g}(\mathbb{P}^1, \mu_1, \dots, \mu_r) \to \mathrm{Sym}^s(\mathbb{P}^1),$$

where is s is the number of simple ramifications as given by the Riemann-Hurwitz-formula 1.2.10, as follows:
If $\varphi : \mathcal{C} \to \mathbb{P}^1$ is a cover together with the data $p_1, \dots, p_s$ of simple branch points in $\mathbb{P}^1$, then we define $\mathrm{br}(\varphi) = (p_1, \dots, p_s)$.

REMARK 1.5.6. If we denote by $\Delta := \{(p_1, \dots, p_s) \mid \exists\, i < j \ :\ p_i = p_j\} \subset \mathrm{Sym}^s(\mathbb{P}^1)$ the union of diagonals, then br is an etalé cover of $\mathrm{Sym}^s(\mathbb{P}^1) \setminus \Delta$ (see e.g [**17**]. In particular its degree is the same at every point at $\mathrm{Sym}^s(\mathbb{P}^1) \setminus \Delta$ and tautologically equals the Hurwitz number $H_{d,g}(\mathbb{P}^1, \mu_1, \dots, \mu_r)$.

The suitable compactification for our later considerations the *space of relative stable maps*. For the case $r = 0$ Fantechi and Pandharipande show that the branch map extends to the space of stable maps and that the virtual degree still recovers the appropiate Hurwitz numbers, see [**16**]. In [**29, 30**] Li introduces relative stable maps to $\mathbb{P}^1$. For a nice introduction to relative stable maps to $\mathbb{P}^1$ with at most two fixed ramifications see also [**41**]. There Vakil also shows that for this compactification of the Hurwitz space the branch map can be extended and the virtual degree still gives back the right Hurwitz number. We will just introduce the space of relative stable maps relative to three points with **exactly one** simple ramification. In contrast to the general case the covers do not involve contracted components in this case. We will first define the space itself and have a look at its objects (especially on the boundary) in the following.

Assume that we are given a degree $d > 1$, a genus $g \geq 0$ and three partitions μ_0, μ_1, μ_∞ of d, such that the number s of simple ramifications as given by the Riemann-Hurwitz-formula 1.2.10 is equal to one.

DEFINITION 1.5.7. Fix the three points 0, 1 and ∞ in $\mathbb{P}^1$. We consider *relative stable maps to* $\mathbb{P}^1$, relative to these three points, with profiles — meaning partitions of d — μ_0, μ_1 and μ_∞, respectively. We denote by $\overline{M}_{g,|\mu|,\mu}(\mathbb{P}^1, d)$ the *space of* such *relative stable maps*, where $|\mu|$ stands for $\sum_{i=0,1,\infty} |\mu_i|$ (which is the number of marked points on the source curves). Its objects are described in remark 1.5.8.

REMARK 1.5.8. The space $\overline{M}_{g,|\mu|,\mu}(\mathbb{P}^1, d)$ is a one-dimensional moduli stack ([**29, 30**]). Points in $\overline{M}_{g,|\mu|,\mu}(\mathbb{P}^1, d)$ roughly correspond to maps of a nodal source curve C to a chain of $\mathbb{P}^1$'s such that the kissing condition is satisfied above each node: (i.e. the ramification profiles on both twigs agree), the three ramification profiles are satisfied above two points in the first copy of $\mathbb{P}^1$ of the chain and one point in the last copy and all preimages of the branch points are marked. Due to the choice of d, g, μ, the stability condition implies that there is at most one node in the target, i.e. at most two copies of $\mathbb{P}^1$. Points in the interior of the moduli space are ramified covers of $\mathbb{P}^1$ with the three special ramification profiles as above and one further simple ramification at a point $t \neq \{0, 1, \infty\}$. At the boundary, i.e. when t moves to one of these three points, the covers degenerate to covers of two copies of $\mathbb{P}^1$ as follows:

Consider the situation where t moves to 0. Then we have covers of a chain of two $\mathbb{P}^1$'s that satisfy the kissing condition above the node, say the ramification profile above the node is $\tilde{\mu}$. On one copy of $\mathbb{P}^1$, we then have three ramification points with profiles μ_u, a simple ramification and $\tilde{\mu}$. On the other, we have $\tilde{\mu}$, μ_v and μ_w. The possibilities for $\tilde{\mu}$ are restricted by the cut-and-join relations: to obtain $\tilde{\mu}$, we can either divide one entry of μ_u into two parts or sum two parts of μ_u. This follows from proposition 1.3.3: by matching a cover with a tuple of elements in the symmetric group, the simple ramification corresponds to a transposition τ while μ_u and $\tilde{\mu}$ correspond to permutations σ_u and $\tilde{\sigma}$ of appropriate cycle type satisfying $\sigma_u \circ \tau = \tilde{\sigma}$. A transposition can either cut a cycle or join two cycles in a permutation.

EXAMPLE 1.5.9. For some pictures of objects of the space $\overline{M}_{1,|\mu|,\mu}(\mathbb{P}^1, 5)$ for $\mu = ((3, 1, 1), (5), (3, 2))$ see example 4.4.3. Figure 7 sketches a cover corresponding to an interior point of $\overline{M}_{1,|\mu|,\mu}(\mathbb{P}^1, 5)$. A cover corresponding to a boundary point in the moduli space is sketched in figure 8.

We come to the main statement of this section, which is needed in section 4.4 to prove the main theorem of chapter 4.

PROPOSITION 1.5.10. *The branch map* $\mathrm{br} : \overline{M}_{g,|\mu|,\mu}(\mathbb{P}^1, d) \to \mathbb{P}^1$ *taking a cover to the image of its simple branch point is itself a cover of* $\mathbb{P}^1$ *of degree* $H_{d,g}(\mu)$, *branched above* 0, 1 *and* ∞. *In particular,* $\mathrm{br}^*(0) \sim \mathrm{br}^*(1) \sim \mathrm{br}^*(\infty)$, *and each consists of the boundary points desribed above.*

PROOF. The branch map is etalé and of correct degree apart from the boundary, see remark 1.5.6. By [**16**], br is a natural map of stacks (see also [**41**], Section 6.2). The statement about the degree and the branching is obvious. □

CHAPTER 2

Tropical Covers

At the beginning of this thesis there were two articles concerning tropical Hurwitz numbers, [**10**] and [**4**].

The first one is an article by Cavalieri, Johnson and Markwig. This paper is concerned with tropical double Hurwitz numbers, i.e. weighted numbers of tropical covers with two (fixed) special ramification profiles and all other ramifications simple. They develope a natural definition of Hurwitz numbers as tropical intersection products, introducing an appropiate moduli space of tropical covers (which quite naturally comes with the structure of a *weighted polyhedral complex*) and a tropical branch map. This branch map is a *morphism of weighted polyhedral complexes* and it is quite easy to see that the degree of the branch map is constant, i.e. it does not depend on the position of the simple branch points, as long as they are in general position. The weights Cavalieri, Johnson and Markwig choose on the top-dimensional polyhedrals of the moduli spaces are quite common to tropical geometers and in fact the degree — which can be seen as a weighted number of preimages — of the tropical branch map is as desired, meaning it equals the degree of the corresponding classical branch map, i.e. the Hurwitz number counting the isomorphism classes of covers in the moduli space considered.
They way they prove this is the following:

Hurwitz numbers can be expressed as cardinality of sets of tuples of permutations (see prop 1.4.7). Now given a tuple of permutations accounting to the Hurwitz number considered, one can look at its *cut-and-join-graph.* Even more, one can sort the tuples taken into account by the *combinatorial type* of their cut-and-join-graph and — weighting each graph with its cut-and-join-multiplicitiy — just count the weighted number of such graphs in order to acquire the right Hurwitz number.

In fact these graphs also come with a projection to a closed interval of $\mathbb{R}$. These maps already look like something similar to what one would expect to be a tropical cover: all we have to do is to add lengths data and assign positive integer weights to the edges (that actually come from the fact that each edge of a cut-and-join-graph belongs to a certain cycle permutation with certain length, which will be this weight). Now taking a generic point of the "base graph" it is easy to see that counting its preimages — weighted with the weight of the edge they lie on — we always get the same number: the degree of the corresponding cover.

Altogether these consideration give a correspondence between cut-""and-""join graphs and (top-dimensional) combinatorial types of tropical covers, where the two ramification profiles that we fixed appear as tuples of weights of the ends of the source curve having the same direction. This correspondence shows that the (weighted) count of cut-and-join-graphs agrees with the (weighted) count of tropical double covers. Therefore tropical double Hurwitz numbers in fact agree with their classical counterparts.

In [**4**] on the other hand, Bertrand, Brugallé and Mikhalkin define tropical covers (and also "open covers", i.e. covers of non-compact Riemann surfaces) for any number of special ramifications and any (tropical) base curve. Their multiplicity for the covers match the multiplicities of [**10**] for tropical double covers. By proving a correspondence theorem, they show that their definition of tropical (open) Hurwitz numbers as (weighted) count of tropical covers agree with their classical counterparts. One problem is that their definition neither provides moduli spaces of covers nor branch maps, since all ramifications live over the leaves of the base curve.

The definition of tropical Hurwitz numbers given in section 2.3 can be seen as an approach to bring the definitions of both articles — [**4**] and [**10**] — together, allowing a maximum of generality **and** the possibilty to establish moduli spaces of covers in a very natural way.

2.1. Tropical Curves

We will introduce a notion of graphs and some terms concerning its properties. For us a *graph* will be a finite, non-empty set of *vertices* V together with a finite multiset of *edges* E whose elements live in $\mathrm{Sym}(V)$, that are pairs of vertices (u,v) with an identification $(u,v)=(v,u)$. In particular one edge might appear more than once. Edges of the form (u,u) are called *loops*.

For an edge $e=(u,v)$ the vertices u and v are called its *endpoints* and e is *incident* to each u and v. The endpoints of an edge are called *adjacent* to each other. Adjacency is obviously a equivalence relation. We say Γ is *connected* if it has only one adjacency class. The *valency* of a vertex u is the number of incident edges, that is

$$\mathrm{val}(u)=|\{(u,v)\in E|v\in V\setminus\{u\}\}|+2\cdot|\{(u,u)\in E\}|.$$

A vertex of valency 1 is called a *leaf*, the *set of leaves* is denoted by $V_\infty(\Gamma)$. An edge incident to a leaf is called *end*, the set of ends is denoted by $E_\infty(\Gamma)$. Given a Graph Γ we denote by $V(\Gamma)$ and $E(\Gamma)$ the sets of vertices and edges, respectively.

A *metric graph* is a graph together with a length function $l:E\to\mathbb{R}_{>0}\cup\{\infty\}$. Identifying an edge of length l with the closed real interval $[0,l]\subset\mathbb{R}$, a metric graph can be considered as a finite set of intervals, whose endpoints are glued together at vertices. A *point* $p\in\Gamma$ is defined to be a point on one of these (glued) intervals.

Definition 2.1.1. A *tropical curve* (Γ,gen) is a finite, connected, metric graph Γ with a *genus function* $\mathrm{gen}:\Gamma\to\mathbb{Z}_{\geq 0}$ such that:

(i) $\mathrm{gen}(p)=0$ for all $p\in V_\infty$
(ii) $\mathrm{gen}(p)>0$ only for finitely many points $p\in\Gamma$,
(iii) an edge has lenghts ∞ if an only if it is an end.

The set of *essential vertices* of (Γ,gen), denoted by $V(\Gamma,\mathrm{gen})$, is the union of $V(\Gamma)$ with the set of points of positive genus. A finite set V of points in Γ is a *vertex set* of the tropical curve (Γ,gen) if it contains $V(\Gamma,\mathrm{gen})$. Elements of $V\setminus V^\infty(\Gamma)$ are called *inner vertices.*

The number $g(\Gamma,\mathrm{gen})=b^1(\Gamma)+\sum_p\mathrm{gen}(p)$, where b_1 denotes the first Betti number, is called the *genus of* (Γ,gen). The *combinatorial type of a tropical curve* is obtained by omitting the length data.

Remark 2.1.2. By abuse of notation we will just write Γ for a tropical curve defined as above. The genus map should be clear from the context. Any point in a vertex set of a curve that is not a vertex of the underlying graph will be considered as a two valent vertex splitting the edge it lies on into two edges. Tropical curves which only differ in their vertex sets are considered the same.

Now let v be a vertex of a tropical curve Γ. The metric on Γ naturally gives a topology on Γ, where a "small" neighbourhood of v is a set of $\mathrm{val}(v)$ many half-open intervals glued together. So locally around v the curve looks like a "star", i.e. finitely many rays emanating from on vertex. We would like to make this more precise and define the set of *tangent directions* $T_v(\Gamma)$ at v (see also [**2**]) to be $\varinjlim_{U_v} \pi_0(U_v\ v)$, where the limit is taken over small neighborhoods of v. Obviously $T_v(\Gamma)$ has val(v) elements, called the *directions at* v, which correspond to a unique edge each.

Finally we say when two tropcial curves are isomorphic. Such curves are considered the same in the following.

Definition 2.1.3. An isomorphism of tropical curves is an isometry of the underlying metric graphs respecting the genus functions.

2.2. Tropical Covers

Since we would like to define the tropical pendant to classical Hurwitz numbers, we have to introduce tropical covers.

In the following, assume that $\pi : \Gamma \to \Gamma'$ is a non-constant, continuous map between two tropical curves, such that the image $\pi(e)$ of each edge e of Γ is completely contained in one of the edges of Γ'.

Definition 2.2.1. We say that π is *integral affine-linear on each edge* if, for every edge e of Γ, there is a non-negative integer w_e such that $\pi_{|e}$ can be expressed as $w_e t + a$ for $t \in e = [0, l(e)]$, where a is a starting point on the edge e is mapped to. We then call the integer w_e the *weight* of e. Such a map π is called *finite* if w_e is a positive integer for all edges.

Remark 2.2.2. If π is finite, then every point in Γ' has only finitely many preimages.

As for embeddings of tropical curves (or more general tropical varieties) in higher dimensional tropical spaces, a tropical morphism, i.e. a function that maps one tropical curve to another, is supposed to fulfill some kind of balancing condition.

Let v be a vertex of the source curve Γ and assume that $v' = \pi(v)$ is a vertex of Γ'. Each direction of $T_v(\Gamma)$ corresponds to an edge e of Γ, which is mapped to a unique edge e' of Γ'. Since π is continuous, e' must be incident to v'. If π is finite the source curve does not have loops and thus each incident edge e' can uniquely be related to a direction in $T_{v'}(\Gamma')$. This gives a mapping

$$\pi_v : T_v(\Gamma) \to T_{v'}(\Gamma').$$

Moreover we assign a weight $w_{\vec{r}}$ to each direction $\vec{r}$ of $T_v(\Gamma)$, which is defined to be the weight of the corresponding edge.

Definition 2.2.3. Assume the map $\pi : \Gamma \to \Gamma'$ considered above is integer affine on each edge. Let v be a vertex of Γ whose image v' is a vertex in Γ'. We say that π is *balanced at* v or *fulfills the balancing condition at* v if

$$\sum_{\vec{r} \in T_v(\Gamma) : \pi(\vec{r}) = \vec{r}\,'} w_{\vec{r}} \qquad (6)$$

is the same for every direction $\vec{r}\,'$ in $T_{v'}(\Gamma')$. We define this number to be the *(local) degree of* π *at* v, denoted by $\deg_v(\pi)$.

Definition 2.2.4. Given a finite map $\pi : \Gamma \to \Gamma'$ between tropical curves with vertex sets V and V', respectively, we say that V and V' are π*-compatible*, if $V = \pi^{-1}(V')$.

Definition 2.2.5. Assume that $\pi : \Gamma \to \Gamma'$ is a non-constant, continuous map between tropical curves with π-compatible vertex sets V and V'. Then π is a *morphism of tropical curves* or a *tropical cover of* Γ' if it is

(i) integer affine-linear on each edge,
(ii) finite,
(iii) balanced at every vertex $v \in V$,
(iv) fulfills the *Riemann-Hurwitz condition*, that is

$$r_v := [\mathrm{val}(v) + 2 \cdot \mathrm{gen}(v) - 2] - \deg_v(\pi) \cdot [\mathrm{val}(v') + 2 \cdot \mathrm{gen}(v') - 2] \tag{7}$$

is non-negative for every vertex $v \in V$ with image v'. We will call this number the *Riemann-Hurwitz-number* or, for short, *RH-number* of v.

Remark 2.2.6. The Riemann Hurwitz condition for tropical cover comes from the classical Riemann-Hurwitz formula as described in e.g. [**4, 2**].

Convention 2.2.7. We can always adapt the vertex set of the source to the vertex set of the target curve in order to make them compatible. So whenever we consider a tropical morphisms we assume that the vertex sets are compatible.

Proposition 2.2.8. *Let $\pi : \Gamma \to \Gamma'$ be a tropical morphism. Then*
a) ends of Γ are mapped to ends of Γ',
b) leaves of Γ are mapped to leaves of Γ',
c) the number of (weighted) preimages is the same for every point in Γ'.

Proof. The first two properties follows from the fact that tropical morphisms dilate edges with a positive factor. So since ends have infinite lenghts their image can only live in edges with infinite lengths, i.e. ends. The remaining part is an immediate consequence from the balancing condition. □

Definition 2.2.9. The *degree* of a tropical morphism π, denoted by $\deg(\pi)$, is the number of weighted preimages of an arbitrary point.

Note that you can find different definitions for tropical morphisms in literature. For example weights are sometimes allowed to be zero (see e.g. [**4, 35**]). Anyway, it is sufficient for us to consider covers with positive weights, since we will count tropical cover with certain multiplicities which contain all edge weights as factors, so non-finite covers would have a multiplicity of zero anyway.

Definition 2.2.10. Let $\pi : \Gamma \to \Gamma'$ be a tropical cover. We define a *labeling* on the vertices of Γ by assigning one label to each leaf of Γ and r_v many labels to every inner vertex v. A tropical cover with labeling is called *labeled tropical cover*. The *combinatorial type* of such a cover is the cover without the length data.

Convention 2.2.11. By abuse of notation we will refer to labeled tropical covers as tropical covers in the following, leaving out the adjective.

As in the classical case we would like to count covers of a fixed target curve having certain ramification profile over fixed branch points. Therefore we have to define *branch points* and *ramification profiles* for tropical covers.

Definition 2.2.12. Let $\pi : \Gamma \to \Gamma'$ be a tropical cover. Denote by V and V' the vertex sets of Γ and Γ', respectively. Then $v' \in V'$ is called *branch point* of π if there is a vertex v in its preimage $\pi^{-1}(v')$ that has positive RH-number r_v. For a vertex v of the sourve curve we define $r_v + 1$ to be the *ramification index* at v. For any $v' \in V'$ the multiset ramification indices $(r_v + 1)_{\pi(v)=v'}$ is called the *ramification profile over* v'. If it is equal to $(2, 1, 1, \dots, 1)$ we say that π is *simply ramified* over v'.

Example 2.2.13. If v is a leaf of Γ then its ramification index is the weight of the incident end. So the ramification profile over a leaf v' of the target curve is given by the multiset of weights of the ends incident to leaves over v'.

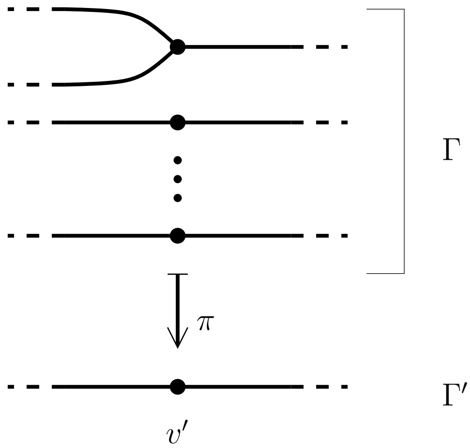


FIGURE 1. A simple ramification over a two-valent vertex

EXAMPLE 2.2.14. Let $\pi : \Gamma \to \Gamma'$ be a tropical cover and v' a 2-valent vertex of the base curve Γ'. Then, if π is simply ramified over v', exactly one preimage of v' is 3-valent and the remaining preimages have valency 2, as depicted in figure 1.

DEFINITION 2.2.15. Two covers $\pi : \Gamma \to \Gamma'$ and $\tilde{\pi} : \tilde{\Gamma} \to \Gamma'$ of Γ' are isomorphic (and will be considered the same), if there exist an isomorphism $\psi : \Gamma \to \tilde{\Gamma}$ of tropical curves respecting the weights and labelings such that $\tilde{\pi} \circ \psi = \pi$.

NOTATION 2.2.16. Let α be a fixed combinatorial type of cover, then all covers of that type have the same number of automorphisms, denoted by $|\operatorname{Aut}(\alpha)|$.

2.3. Tropical Hurwitz Numbers

As in the classical case, Hurwitz numbers in the tropical world count covers of curves with certain ramification profiles. We will first give a general definition and establish the two kind of Hurwitz numbers we are especially interested in — tropical triple Hurwitz numbers and Hurwitz numbers for elliptic curves — afterwards.

In order to introduce the weights tropical curves are counted with, we have to define *local Hurwitz numbers*.

DEFINITION 2.3.1. Let $\pi : \Gamma \to \Gamma'$ be a degree d tropical cover and v and v' vertices of Γ and Γ', respectively, such that $\pi(v) = v'$. Moreover let us label the edges emanating from v' by $e_1, \ldots, e_{\mathrm{val}(v)}$ and let $\mu_1, \ldots, \mu_{\mathrm{val}(v')}$ be partitions of the local degree $d' := \deg_v(\pi)$, where μ_i is given by the multiset of weights of all edges incident to v that are mapped e_i.

Then H_v, the *local Hurwitz numbers at* v, is defined to be the (classical) labeled Hurwitz number of genus $\mathrm{gen}(v)$ degree d' covers of $\mathbb{P}^1$ with ramification profiles $\mu_1, \ldots, \mu_{\mathrm{val}(v')}$, i.e. $H_{d',\mathrm{gen}(v)}(\mu_1, \ldots, \mu_{\mathrm{val}(v')})$ as defined in definition 1.4.3.

Our definition of *tropical Hurwitz numbers* will be as follows.

DEFINITION 2.3.2. Let Γ' be a tropical curve with r leaves, whose genus function is identically zero. Furthermore let d be a positive integer and $\mu_1, \ldots, \mu_r$ be partitions of d. We label the leaves $l_1, \ldots, l_r$ and fix s pairwise distinct 2-valent vertices $p_1, \ldots, p_s$ of Γ', where s is given by $s := 2g-2-d\cdot(2g'-2)-rd+\sum_{i=1}^{r}|\mu_i| \geq 0$ (without loss of generality we can assume that Γ' has that many 2-valent vertices in its vertex set). For $r \leq 2$ we require $s > 0$.

The tropical Hurwitz number $H^{trop}_{d,g}(\Gamma', (\mu_1, \ldots, \mu_r))$ is defined to be the weighted number of isomorphism classes of degree-d tropical covers $\Gamma \to \Gamma'$ such that

- the source curve Γ has genus g,
- the cover has ramification profile μ_i over l_i for $i = 1\dots, r$,
- is simply ramified over $p_1, \dots, p_s$ and
- it is unramified everywhere else.

The *multiplicity* of an isomorphism class is

$$\frac{1}{\mathrm{Aut}(\pi)} \cdot \prod_e w_e \cdot \prod_v H_v,$$

where π is an arbitrary representative of that class and the products run over all edges of the source curve that are not ends and all inner vertices, respectively.

Remark 2.3.3. The requirement that $s > 0$ if $r \leq 2$ ensures that for each cover considered the source curve has at least one inner vertex.

Notation 2.3.4. We will say that a tropical cover is *trivalent* or of *trivalent type* if the cover is taken into account for some tropical Hurwitz number as defined above. In particular over each 2-valent vertices there is at most one trivalent vertex and 2-valent vertices else. Moreover all these vertices have genus 0.

Remark 2.3.5. Tropical Hurwitz numbers are well-defined, i.e. they do not depend on the vertex set V' chosen for Γ'.

Proof. It remains to show that adding and deleting a two valent vertex to V' does not change the weighted count above. Assume that Γ' has vertex set V' and let $u' \in V'$ be one of its vertices. Then a cover $\pi : \Gamma \to \Gamma'$ accounting to the Hurwitz number of covers of Γ' with vertex set V' corresponds one-on-one to a cover of Γ' with vertex set $V' \setminus \{u'\}$ — in fact, we just delete all point set $W := \pi^{-1}(u')$ from the vertex set of Γ. Obviously each point in $u \in W$ splits an edge e of Γ into two edges e_1, e_2. Nevertheless both covers have the same multiplicity: Since $w_e = w_{e_1} = w_{e_2}$ and $H_u = \frac{1}{w_e}$ we have $w_e = w_{e_1} \cdot w_{e_2} \cdot H_u$. □

By definition, simple ramifications live over 2-valent vertices (denoted by p_i, $i = 1, \dots, s$). In order to provide ramification profile $(2, 1, \dots, 1)$, the preimages of each p_i have to be 2-valent genus-0-vertices except for one trivalent genus-0-vertex. In fact this is the only possibility over a fixed vertex to get exactly one vertex with RH-number equal to 1 and all others zero.

This implies that in the definition of the weight it is sufficient to let the second product run over all inner vertices living over at least trivalent points and flatten all two-valent vertices of the source curve, linking the two edges they are incident to, respectively.

In fact for any vertex v being mapped to 2-valent point (necessarily a simple branch point) we have $H_v = 1$:

Assume first that v itself is 2-valent. Then H_v counts covers with two special ramifications of type (d), where $d = \deg(v)$, and without any additional simple ramifications Performing, for example, a count of these numbers via tuples of permutations (as in section 1.4.1), we easily see that these numbers are $\frac{1}{d}$. But when flattening such a vertex we loose one inner edge of weight d, which contributed to the product of weights before.

We compute the local Hurwitz Numbers of a trivalent vertex mapping to a two-valent vertex. We just have to use a well-known fact about classical Hurwitz numbers and remember that our definition for Hurwitz numbers differs from the unlabeled one by the product of automorphisms of the special ramification profiles. Writing $d = \deg(v)$ we have for the "unlabeled" local Hurwitz number

$$\check{H}_v = \check{H}_{d,0}((d), (d_1, d_2)) = \begin{cases} 1 \text{ if } d_1 \neq d_2 \\ \frac{1}{2} \text{ if } d_1 = d_2 \end{cases}$$

(This can for example be seen using the definition via the symmetric group 1.4.6). But in the second case we also have $\mathrm{Aut}((d_1, d_2)) = 2$ eliminating the factor $\frac{1}{2}$. In any case we see $H_v = 1$.

From the discussion above we see that the following proposition follows.

PROPOSITION 2.3.6. *In the definition of weights of covers in definition 2.3.2 it is sufficient to consider the source curve with flattened* 2*-valent vertices and to let the second product run over all inner vertices that are mapped to vertices of valency at least* 3*.*

REMARK 2.3.7. A generic curve is trivalent, i.e. the valency of each vertex is at most three. Because of proposition 2.3.6, for a trivalent curve as target, in the weights defined in definition 2.3.2 the local Hurwitz numbers are all static triple Hurwitz numbers (see definition 1.4.9).

We will now briefly point out of difference between our definition of Hurwitz numbers and the definition given in [**4**]:

- In [**4**] tropical covers are not labeled. This effects the size of Automorphism classes of the covers taken into account (see corollary 2.3.10 below).
- We consider labeled curves, whereas Bertrand, Brugallé and Mikhalkin consider unlabeled.
- Bertrand, Brugallé and Mikhalkin only allow $r_v = 0$ for interior vertices v of the source curves. This would mean that only leaves are allowed to have labels in case that they labeled their covers.
- Furthermore, in [**4**], tropical *open* Hurwitz numbers are defined. In fact, on the classical side, curves (i.e. Riemann surfaces) with boundary are allowed as source curves. This amounts to the fact that on the tropical side ends of the source curve do not necessarily have to have length ∞. We will not consider "open covers" in this thesis.

NOTATION 2.3.8. Let $\mu_1, \ldots, \mu_r$ be partitions of d and $\mu = (\mu_1, \ldots, \mu_r)$. The we denote by $|\mathrm{Aut}(\mu)|$ the product of the number of automorphisms of the μ_i, i.e.

$$|\mathrm{Aut}(\mu)| = \prod_{i=1}^{r} |\mathrm{Aut}(\mu_i)|,$$

where $\mathrm{Aut}(\mu_i)$ is defined as in equation (3).

PROPOSITION 2.3.9. *There is a natural correspondence between the covers taken into account in our definitions and those allowed in* [**4**]*.*

PROOF. Let $\pi : \Gamma \to \Gamma'$ be a cover (as defined in definition 2.3.2) with ramification profile $\mu_1, \ldots, \mu_r$ over the ends. By definition we take only covers into account whose branch points $p_1, \ldots, p_s$ are two-valent vertices and their ramifications over the p_i are simple. Now forget the labelings and attach an end to the target curve to each p_i and simultanously attach $\deg_v(\pi) - r_v$ many ends to each $v \in \pi^{-1}(p_i)$. The just attached ends of the source curve are mapped to the just attached ends of the base curve in the obvious manner. The weights of the new edges are chosen as follows: For each v in one of the $\pi^{-1}(p_i)$ choose any partition of $\deg_v(\pi)$ of length $\deg_v(\pi) - 1$ and distribute its entries to the ends attached at v: that is $(2, 1, \ldots, 1)$ if $r_v = 1$ and $(1, 1, \ldots)$ for the unramified vertices over p_i.

One can easily convince oneself that all interior vertices still fulfill the balancing condition and that all interior vertices have RH-number equal to 0 now. This gives a correspondence between a class of covers (those which do only differ in their labeling) and a single of cover as in [**4**]. In fact, shrinking ends of the source curve (at fixed simple ramification points) of a cover that is defined as in [**4**] and labeling their vertices appropiately, we get back the cover taken into account in our definition of Hurwitz numbers. □

Corollary 2.3.10. *The Hurwitz numbers defined in definition 2.3.2 agrees with the definition of Hurwitz numbers in* [**4**] *up to a factor of* $|\operatorname{Aut}(\mu)|$ *due to the labeling.*

Proof. We will take the correspondence between covers from above and show that the multiplicities of curves corresponding to each other agree up to the factor of $|\operatorname{Aut}(\mu)|$:

Let us discuss the labeling: Firstly, deleting the labeling of interior vertices might admit more automorphisms. But since we demand interior branch points to be pairwise different, two differently labeled points of the source curve are mapped to different trivalent points after attaching the desired ends. Secondly, deleting labels from the ends increases the number of automorphisms by a certain factor. But analogously to the classical case this factor is just the number $|\operatorname{Aut}(\mu)|$.

Finally, attaching the ends at the source curve does not change the local Hurwitz numbers to be considered. In particular double and triple Hurwitz numbers are considered, respectively.

Conclusionally, each class of covers $[\pi]$ with our definition provides a cover $\tilde{\pi}$ matching the definition of [**4**], such that $\sum_{\pi' \in [\pi]} \operatorname{mult}(\pi') = |\operatorname{Aut}(\mu)| \cdot m(\tilde{\pi})$, where $m(\tilde{\pi})$ is the multiplicity in [**4**]. This yields the desired statement. □

Remark 2.3.11. We will consider this correspondence (slightly more explicit) for the case of triple tropical covers in the proof of lemma 4.3.5.

Theorem 2.3.12. *The tropical Hurwitz numbers defined in definition 2.3.2 agree with their classical pendants, as defined in definition 1.4.3, i.e.*

$$H_{d,g}^{trop}(\Gamma', (\mu_1, \dots, \mu_r)) = H_{d,g}(\mathbb{P}^1, (\mu_1, \dots, \mu_r)).$$

Proof. This follows immediately from the above corollary and the correspondence theorem [**4**, Theorem 2.11.]. □

Remark 2.3.13. If we consider static triple Hurwitz numbers, i.e. if $r = 3$ and $s = 0$ in the the above situation, the correspondence in theorem 2.3.12 holds by definition: There is only one tropical cover, where the source curve is star-shaped, i.e. it has one vertex and no interior edges. Therefore $\frac{1}{\operatorname{Aut}(\pi)} \cdot \prod_e w_e = 1$ and the weight the cover is counted with consists only of the local Hurwitz number, which is equal to $H_{d,g}(\mathbb{P}^1, (\mu_1, \mu_2, \mu_3))$.

CHAPTER 3

Tropical Correspondence Theorems via the Symmetric Group

In this chapter we fully exploit the connection of Hurwitz numbers to the symmetric group. By proving a tropical version of the degeneration formula which boils the computation of arbitrary tropical Hurwitz numbers down to the computation of static classical triple Hurwitz numbers totally analogously to the classical degeneration formula we reprove the correspondence theorem of [**4**] by combinatorial methods using the symmetric group. Furthermore we use symmetric group techniques to prove a corresponce theorem for covers of elliptic curves that will be important in chapter 5 in connection with mirror symmetry. The main results of chapter 3 are theorem 3.1.8 proving the tropical degeneration formula and the corresponce theorem for elliptic curves 3.2.3. This chapter sheds light on the deep connections between the symmetric group, degeneration techniques and tropical geometry.

3.1. The Tropical Degeneration Formula

In this section we will give a tropical version of the degeneration formula for a cover with **four** fixed complicated ramification profiles. In fact it is straightforward to extend this considerations to Hurwitz numbers over rational curves (i.e. over $\mathbb{P}^1$) with five or more fixed ramifications.

Let us first introduce some definitions needed to state the well-known degeneration formula for Hurwitz numbers over $\mathbb{P}^1$. First of all we would like to extend our definition of Hurwitz numbers to *disconnected* ones, i.e. we will allow our source curves to have more than one connected component (or in other words more than one irreducible component). As usual for a smooth complex curve $\mathcal{C}$ with connected components $\mathcal{C}_1, \dots, \mathcal{C}_m$ of genus $g(\mathcal{C}_1), \dots, g(\mathcal{C}_m)$, respectively, we define the *genus of* $\mathcal{C}$ to be $g(\mathcal{C}) := 1 - m + \sum_{i=1}^{m} g(\mathcal{C}_i)$. In fact curves can have negative genus now. Although the definitions of disconnected Hurwitz numbers is quite self-explaining, we will briefly introduce them.

Convention 3.1.1. For the whole section — if not stated else — let Hurwitz numbers count *unlabeled* covers, i.e. we are working with the classical definition for Hurwitz numbers, see definition 1.4.1.

3.1.1. Disconnected Hurwitz Numbers. We will briefly introduce Hurwitz numbers counting (unlabeled) disconnected covers as well as their tropical pendants.

Definition 3.1.2. A *disconnected (branched) cover* is a non-constant morphism of smooth, complex, projective algebraic curves, where the source curve is possibly disconnected. Two branched covers $\varphi : \mathcal{C} \to \mathcal{D}$ and $\varphi' : \mathcal{C}' \to \mathcal{D}$ are *isomorphic* if there exists an isomorphism $\Phi : \mathcal{C} \to \mathcal{C}'$ such that $\varphi = \varphi' \circ \Phi$.

Let $\varphi : \mathcal{C} \to \mathcal{D}$ be a disconnected cover and let $\mathcal{C}_1, \dots, \mathcal{C}_m$ be the irreducible components of $\mathcal{C}$. Then $\varphi_{|\mathcal{C}_i}$ is a cover in the sense of definition 1.2.9 for each

$i = 1, \dots, m$. In fact the other direction is also true: If $\varphi_{|\mathcal{C}_i}$ is a cover in the sense of definition 1.2.9 for each $i = 1, \dots, m$, then $\varphi : \mathcal{C} \to \mathcal{D}$ is a disconnected cover. In particular the degree of $\varphi_{|\mathcal{C}_i}$, the genus of $\mathcal{C}_i$ and ramifications on $\mathcal{C}_i$ have to be compatible in the sense the Riemann-Hurwitz formula (see theorem 1.2.10), for each $i = 1, \dots, m$.

The definitions of *degree*, *branch points* and *ramification profiles* are considered earlier.

Definition 3.1.3. Let $\mathcal{D}$ be a smooth, rational, complex curve. Furthermore let d be a positive integer and g an integer and let $\mu_1, \dots, \mu_r$ be partitions of d. We fix a finite subset $B := \{b_1, \dots, b_r, b_{r+1}, \dots, b_{r+s}\}$, where s is the desired number of simple ramifications as given by the Riemann-Hurwitz formula (see equation (2) in section 1.4). Then the *(unlabeled) disconnected Hurwitz number* $H^{\bullet}_{d,g}(\mathcal{D}, (\mu_1, \dots, \mu_r))$ is defined to be the *weighted* number of isomorphism classes of unlabeled, *possibly disconnected*, branched covers $\mathcal{C} \to \mathcal{D}$ of degree d, such that

(i) the source curve $\mathcal{C}$ is a (possibly disconnected) smooth complex curve of genus g,
(ii) the cover has ramification profile μ_i over b_i for $i = 1 \dots, r$,
(iii) it has simple ramifications over $b_{r+1}, \dots, b_{r+s}$ and
(iv) it is unramified everywhere else.

The *weight* of each such isomorphism class $[\varphi]$ of covers is $\frac{1}{|Aut(\varphi)|}$, where $\mathrm{Aut}(\varphi)$ is the group of automorphisms of φ.

We can define *tropical disconnected Hurwitz numbers* analogously: We allow tropical curves to be disconnected in in this section and extend the definition of covers to being allowed to have disconnected source curves. For a source curve Γ with connected components $\Gamma_1, \dots, \Gamma_m$ the genus of Γ is given by $g(\Gamma) = \sum_{i=1}^{m} g(\Gamma_i) + 1 - m$. As in the classical case the definitions for degree, branch points and ramification profiles stay the same as in the connected case. Moreover we will mainly consider *unlabeled* Hurwitz numbers, i.e. we count unlabeled covers. That means that we do **not** label any leaves of the source curve.

Definition 3.1.4. Let Γ' be a tropical curve with $r \geq 3$ leaves, whose genus function is identically zero. Furthermore let d be a positive integer, g an integer and $\mu_1, \dots, \mu_r$ be partitions of d, such that $s := 2g - 2 - d \cdot (2g' - 2) - rd + \sum_{i=1}^{r} |\mu_i| \geq 0$. We label the leaves $l_1, \dots, l_r$ and fix s pairwise distinct 2-valent vertices $p_1, \dots, p_s$ on Γ'.

The *(unlabeled) disconnected tropical Hurwitz number* $H^{\bullet, trop}_{d,g}(\Gamma', (\mu_1, \dots, \mu_r))$ is defined to be the *weighted* number of isomorphism classes of unlabeled disconnected tropical covers $\Gamma \to \Gamma'$ of degree d such that

- the (possibly disconnected) tropical source curve Γ has genus g,
- the cover has ramification profile μ_i over l_i for $i = 1 \dots, r$,
- is simply ramified over $p_1, \dots, p_s$ and
- it is unramified everywhere else.

The *weight* of an isomorphism class is

$$\frac{1}{\mathrm{Aut}(\pi)} \cdot \prod_e w_e \cdot \prod_v H_v,$$

where π is an arbitrary representative of that class and the products run over all edges of the source curve that are not ends and all inner vertices, respectively. H_v, as before, still denotes labeled Hurwitz numbers, see definition 2.3.1

Remark 3.1.5. Notice that the condition $r \geq 3$ above can be dropped. The definition of multiplicities then have to be adapted, see remark 3.2.8.

3.1.2. The Degeneration Formula for Quadruple Hurwitz Numbers. We will now consider quadruple Hurwitz numbers, counting covers of a rational curve with 4 fixed ramifications. For the whole chapter we assume that the degree d, the genus g and the 4 ramification profiles $\mu_1, \ldots, \mu_4$ are chosen in such a way, that the number s of simple ramifications is zero, i.e. $s = 2g-2d-2+\sum_{i=1}^{4} |\mu_i| = 0$. We state the degeneration formula for quadruple Hurwitz numbers, see also [**9, 29, 30**].

THEOREM 3.1.6. *Given a degree $d \geq 1$, a genus $g \in \mathbb{Z}$ and ramification profiles $\mu_1, \ldots, \mu_4$, such that $2g - 2d - 2 + \sum_{i=1}^{4} |\mu_i| = 0$, we can express the disconnected quadruple Hurwitz number $H^{\bullet}_{d,g}(\mathbb{P}^1, (\mu_1, \ldots, \mu_4))$ as follows, using only static triple Hurwitz numbers:*

$$H^{\bullet}_{d,g}(\mathbb{P}^1, (\mu_1, \ldots, \mu_4)) = \sum_{\eta, g_1, g_2} C_\eta \cdot H^{\bullet}_{d,g_1}(\mathbb{P}^1, (\mu_1, \mu_2, \eta)) H^{\bullet}_{d,g_2}(\mathbb{P}^1, (\eta, \mu_3, \mu_4)), \quad (8)$$

where the sum goes over all positive partitions η of d and over all integers $g_1, g_2 \in \mathbb{Z}$ with $g_1 + g_2 + |\eta| - 1 = g$ and C_η stands for the cardinality of the centralizer of a permutation of cycle type η.

SKETCH OF THE PROOF. The theorem is proved using the description of Hurwitz numbers via tuples of permutations, similar to the definition in section 1.4.1. In fact, for disconnected covers, we just have to drop the property that the subgroup generated by the permutations acts transitively. The factor C_η comes from simple combinatorial observations. For a full proof see [**9**]. □

Note, that the Hurwitz numbers on the right are only non-zero for a finite number of choices η, g_1, g_2, so in fact the sum is finite. For example, it is clear that both, g_1 and g_2, have to be at least $-d$.

REMARK 3.1.7. For a (positive) partition $\eta = (\eta_1, \ldots, \eta_k)$ the number C_η as above is given by $\prod_i \eta_i \cdot |\operatorname{Aut}(\eta)|$.

PROOF. The centralizer of a permutation of cycle type η is a semidirect product of the cartesian product of the centralizers of its cycles on the one hand and $\operatorname{Aut}(\eta)$ on the other hand. □

We would like to see that the degeneration formula also holds for tropical Hurwitz numbers. The following statement is the main statement of this section.

THEOREM 3.1.8. *Given a positive integer degree d, a genus $g \in \mathbb{Z}$, four partitions $\mu_1, \ldots, \mu_4$ of d and a (connected) tropical curve Λ with 4 ends, labeled by $l_1, \ldots, l_4$, such that $2g - 2d - 2 + \sum_{i=1}^{4} |\mu_i| = 0$, we can express the disconnected, tropical quadruple Hurwitz number $H^{\bullet,trop}_{d,g}(\Lambda, (\mu_1, \ldots, \mu_4))$ using only disconnected, tropical, static triple Hurwitz numbers:*

$$H^{\bullet,trop}_{d,g}(\Lambda, (\mu_1, \ldots, \mu_4)) = \sum_{\eta, g_1, g_2} C_\eta \cdot H^{\bullet,trop}_{d,g_1}(\mathcal{L}, (\mu_1, \mu_2, \eta)) H^{\bullet,trop}_{d,g_2}(\mathcal{L}, (\eta, \mu_3, \mu_4)), \quad (9)$$

where the sum goes over all positive partitions η of d and over all integers $g_1, g_2 \in \mathbb{Z}$ with $g_1+g_2+|\eta|-1 = g$ and $C_\eta = \prod_i \eta_i \cdot |\operatorname{Aut}(\eta)|$ is the cardinality of the centralizer of a permutation of cycle type η.

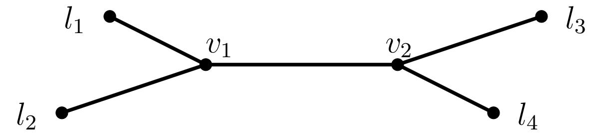
The proof of this theorem will follow later.

Of course, on the classical side as well as the tropical side, Hurwitz numbers over $\mathbb{P}^1$ are invariant under permutation of the ramification profiles. So in fact it is not important in the above theorems whether we split the quadruple $(\mu_1, \ldots, \mu_4)$ into the pairs (μ_1, μ_2) and (μ_3, μ_4) or e.g. (μ_1, μ_3) and (μ_2, μ_4). Another way to split would nevertheless give an alternate way to write down the degeneration formulas, e.g. in the classical case:

$$H^{\bullet}_{d,g}(\mathbb{P}^1,(\mu_1,\dots,\mu_4)) = \sum_{\eta,g_1,g_2} C_\eta \cdot H^{\bullet}_{d,g_1}(\mathbb{P}^1,(\mu_1,\mu_3,\eta))H^{\bullet}_{d,g_2}(\mathbb{P}^1,(\eta,\mu_2,\mu_4)).$$

In order to prove the tropical degeneration formula for quadruple Hurwitz numbers in the form stated in equation (9), we will fix the following labeled tropical curve as target.

NOTATION 3.1.9. In the following we fix Λ to the labeled tropical curve sketched below.



As before we will omit lengths data. Since Λ is supposed to be the target curve, both vertices v_1 and v_2 have genus equal to zero.

In the preceeding sections we always assumed tropical covers to be labeled tropical covers (see definition 5.2.1), meaning leaves and inner vertices of the source curves are labeled. In this section we worked with unlabeled covers. Before we prove theorem 3.1.8 we introduce a third category of Hurwitz numbers, so-called *semi-labeled* tropical Hurwitz numbers.

DEFINITION 3.1.10. Let Γ' be a tropical curve with r leaves, whose genus function is identically zero. Furthermore let d be a positive integer, g an integer and $\mu_1,\dots,\mu_r$ be partitions of d, such that $s := 2g-2-d\cdot(2g'-2)-rd+\sum_{i=1}^{r}|\mu_i| \geq 0$. We label the leaves $l_1,\dots,l_r$ and fix s pairwise distinct 2-valent vertices $p_1,\dots,p_s$ of Γ'.

For any $q = 1,\dots,r$ the *disconnected, semi-labeled, tropical Hurwitz number* $H^{\bullet,trop}_{d,g}(\Gamma',(\mu_1,\dots,\mu_q,\breve{\mu}_{q+1},\dots,\breve{\mu}_r))$ is defined to be the weighted number of isomorphism classes of disconnected *semi-labeled* tropical covers $\Gamma \to \Gamma'$ of degree d such that

- the (possibly disconnected) tropical source curve Γ has genus g,
- the cover has ramification profile μ_i over l_i for $i = 1\dots,r$, *where the leaves over $l_1,\dots,l_q$ are **labeled** and the leaves over $l_{q+1},\dots,l_r$ are **not labeled***,
- is simply ramified over $p_1,\dots,p_s$ and
- it is unramified everywhere else.

The *multiplicity* of an isomorphism class is

$$\frac{1}{\mathrm{Aut}(\pi)} \cdot \prod_e w_e \cdot \prod_v H_v,$$

where π is an arbitrary representative of that class and the products run over all edges of the source curve that are not ends and all inner vertices, respectively. The H_v are again labeled triple Hurwitz numbers.

With this definition labeled tropical Hurwitz numbers can be denoted by $H^{\bullet,trop}_{d,g}(\Gamma',(\mu_1,\dots,\mu_r))$ and unlabeled are denoted by $H^{\bullet,trop}_{d,g}(\Gamma',(\breve{\mu}_1,\dots,\breve{\mu}_r))$. We will use this kind of notation to make things clear, in the following.

REMARK 3.1.11. Obviously semi-labeled tropical Hurwitz numbers differ only in a factor $\prod_{i=1}^{q}|\mathrm{Aut}(\mu_i)|$ from unlabeled Hurwitz numbers, which comes from a different cardinality of $\mathrm{Aut}(\pi)$ due to the labeling. More precisely:

$$H^{\bullet,trop}_{d,g}(\Gamma',(\breve{\mu}_1,\dots,\breve{\mu}_r)) = \prod_{i=1}^{q}|\mathrm{Aut}(\mu_i)| \cdot H^{\bullet,trop}_{d,g}(\Gamma',(\mu_1,\dots,\mu_q,\breve{\mu}_{q+1},\dots,\breve{\mu}_r))$$

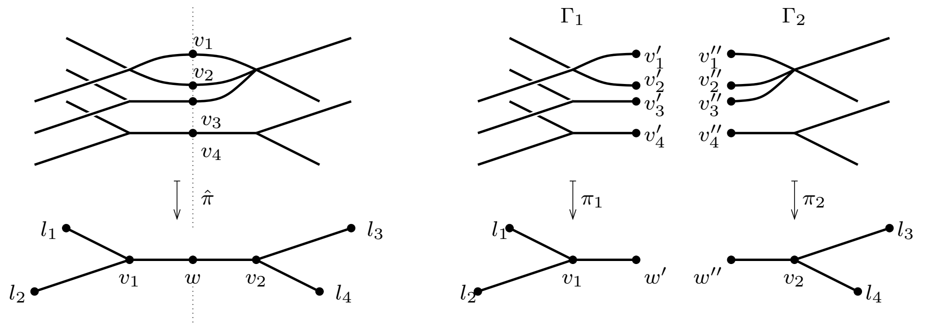


FIGURE 1. Cutting a quadruple cover into two triple covers

We are now ready to prove the tropical degeneration theorem for quadruple Hurwitz numbers.

PROOF OF THEOREM 3.1.8. The main point of the proof is the bijection between diconnected quadruple covers (with some extra labeling) and pairs of disconnected semi-labeled triple Hurwitz numbers.

Let us fix a vertex w in the interior of the unique inner edge of Λ and, for each cover $\pi : \Gamma \to \Lambda$ accounting to the left hand side of equation (9), consider the cover $\hat{\pi}$, where the preimages of w are labeled. For such π or $\hat{\pi}$, respectively, let us denote by $\eta_\pi = \eta_{\hat{\pi}}$ the partition of d given by the multiset of weights of the vertices over w (i.e. the weights of the edges they lie on). Then the number of labelings of such a cover π is obviously given by $\frac{|\operatorname{Aut}(\eta_\pi)|}{|\operatorname{Aut}_{int}(\pi)|}$, where $\operatorname{Aut}_{int}(\pi)$ stands for those automorphisms of π that map the ends identically, i.e. which permute only interior edges. If we denote by $\operatorname{Aut}_{ext}(\pi)$ those automorphisms that map interior edges identically and only permute ends, then clearly

$$|\operatorname{Aut}(\pi)| = |\operatorname{Aut}_{int}(\pi)||\operatorname{Aut}_{ext}(\pi)|. \tag{10}$$

Moreover, if we donte by $\operatorname{Aut}_{l_i}(\pi)$ those automorphisms which do only permute ends with leaves over l_i, we can also factorize:

$$|\operatorname{Aut}_{ext}(\pi)| = \prod_{i=1}^{4} |\operatorname{Aut}_{l_i}(\pi)|. \tag{11}$$

Let us furthermore denote by $g_1^{(\pi)}$ and $g_2^{(\pi)}$ the genera of the two (disconnected) curves Γ_1 and Γ_2 that we get when cutting Γ over w, see figure 1. Analogously we define $g_1^{(\hat{\pi})}$ and $g_2^{(\hat{\pi})}$.

By reordering the covers by their ramification profile η_π and genera $g_1^{(\pi)}$ and $g_2^{(\pi)}$, we can write the left hand side of equation (9) as follows:

$$\begin{aligned}
&H_{d,g}^{\bullet,trop}(\Lambda, (\mu_1, \ldots, \mu_4)) \\
&= \sum_{\eta, g_1, g_2} \sum_{\pi} \frac{1}{|\operatorname{Aut}(\pi)|} \prod_e w(e) \prod_{v \mapsto v_1} H_v \prod_{v \mapsto v_2} H_v \\
&= \sum_{\eta, g_1, g_2} \sum_{\hat{\pi}} \frac{1}{|\operatorname{Aut}(\eta)|} \frac{1}{|\operatorname{Aut}_{ext}(\pi)|} \prod_i \eta_i \prod_{v \mapsto v_1} H_v \prod_{v \mapsto v_2} H_v, \qquad (12) \\
&= \sum_{\eta, g_1, g_2} \frac{1}{|\operatorname{Aut}(\eta)|} \prod_i \eta_i \sum_{\hat{\pi}} \frac{1}{|\operatorname{Aut}_{ext}(\pi)|} \prod_{v \mapsto v_1} H_v \prod_{v \mapsto v_2} H_v, \qquad (13)
\end{aligned}$$

where the first sum in all rows runs over the same η, g_1, g_2 as in equation (9) and the second over all covers π or labeled covers $\hat{\pi}$ such that

$$\left(\eta_\pi = \eta \quad \wedge \quad g_1^{(\pi)} = g_1 \quad \wedge \quad g_2^{(\pi)} = g_2\right) \text{ or}$$
$$\left(\eta_{\hat{\pi}} = \eta \quad \wedge \quad g_1^{(\hat{\pi})} = g_1 \quad \wedge \quad g_2^{(\hat{\pi})} = g_2\right),$$

respectively. As usual, in the first row the products go over the weights of the interior edges and the local Hurwitz numbers at vertices over v_1 and v_2, respectively. Equation (12) holds since we have to multiply by a factor $\frac{|\operatorname{Aut}_{int}(\pi)|}{|\operatorname{Aut}(\eta)|}$ when counting extra labeled covers instead of normal ones. Moreover we use equation (10) and the fact that each interior edge of Γ contains exactly one preimage of w (in fact, each interior edge of Γ is mapped bijectively to the unique interior edge of Λ) and therefore $\prod_e w(e) = \prod_i \eta_i$. Equation (13) is obvious.

It remains to show that the second sum over the labeled covers $\hat{\pi}$ in equation (13) can be expressed as

$$|\operatorname{Aut}(\eta)|^2 H^{\bullet,trop}_{d,g_1}(\mathcal{L}, (\check{\mu}_1, \check{\mu}_2, \check{\eta})) H^{\bullet,trop}_{d,g_2}(\mathcal{L}, (\check{\eta}, \check{\mu}_3, \check{\mu}_4)).$$

In order to prove this, assume that η, g_1, g_2 are fixed and $\hat{\pi}$ is a labeled cover with the desired properties. We cut the cover $\hat{\pi}$ over w and at w (splitting w and its preimages each into two vertices and keeping the labels as depicted in figure 1) yielding two semi-labeled covers $\pi_1 : \Gamma_1 \to \mathcal{L}$ and $\pi_2 : \Gamma_2 \to \mathcal{L}$ accounting to the triple Hurwitz numbers $H^{\bullet,trop}_{d,g_1}(\mathcal{L}, (\check{\mu}_1, \check{\mu}_2, \eta))$ and $H^{\bullet,trop}_{d,g_2}(\mathcal{L}, (\eta, \check{\mu}_3, \check{\mu}_4))$. Due to the labeling of the preimages of w' and w'' there is a unique (canonical) way to glue both covers together, giving back the cover $\hat{\pi}$. On the other hand, given any pair (π_1, π_2) of covers accounting to $H^{\bullet,trop}_{d,g_1}(\mathcal{L}, (\check{\mu}_1, \check{\mu}_2, \eta))$ and $H^{\bullet,trop}_{d,g_2}(\mathcal{L}, (\eta, \check{\mu}_3, \check{\mu}_4))$, respectively, such that the labels are compatible — meaning the leaves corresponding to the ramification profile η in both covers have the same set of labels for the sets of leaves with same weight — we can uniquely glue these covers to a cover $\hat{\pi}$ (with extra labeling) having the desired properties and cutting this cover at w gives back the pair (π_1, π_2). To sum up:

There is a bijection between quadruple covers with extra labeling and pairs of semi-labeled triple covers accounting to $H^{\bullet,trop}_{d,g_1}(\mathcal{L}, (\check{\mu}_1, \check{\mu}_2, \eta))$ and $H^{\bullet,trop}_{d,g_2}(\mathcal{L}, (\eta, \check{\mu}_3, \check{\mu}_4))$, respectively.

So instead of summing over all labeled $\hat{\pi}$ we can sum over pairs (π_1, π_2) of triple covers where the leaves of the η-ramification are labeled:

$$\begin{aligned}
& \sum_{\hat{\pi}} \frac{1}{|\operatorname{Aut}_{ext}(\pi)|} \prod_{v \mapsto v_1} H_v \prod_{v \mapsto v_2} H_v \\
= & \sum_{(\pi_1,\pi_2)} \frac{1}{|\operatorname{Aut}_{ext}(\pi)|} \prod_{v \mapsto v_1} H_v \prod_{v \mapsto v_2} H_v \\
= & \sum_{(\pi_1,\pi_2)} \frac{1}{|\operatorname{Aut}_{l_1}(\pi)|} \frac{1}{|\operatorname{Aut}_{l_2}(\pi)|} \prod_{v \mapsto v_1} H_v \frac{1}{|\operatorname{Aut}_{l_3}(\pi)|} \frac{1}{|\operatorname{Aut}_{l_4}(\pi)|} \prod_{v \mapsto v_2} H_v \qquad (14) \\
= & \underbrace{\sum_{\pi_1} \frac{1}{|\operatorname{Aut}_{l_1}(\pi_1)|} \frac{1}{|\operatorname{Aut}_{l_2}(\pi_1)|} \prod_{v \mapsto v_1} H_v}_{=H^{\bullet,trop}_{d,g_1}(\mathcal{L},(\check{\mu}_1,\check{\mu}_2,\eta))} \cdot \underbrace{\sum_{\pi_2} \frac{1}{|\operatorname{Aut}_{l_3}(\pi_2)|} \frac{1}{|\operatorname{Aut}_{l_4}(\pi_2)|} \prod_{v \mapsto v_2} H_v}_{=H^{\bullet,trop}_{d,g_2}(\mathcal{L},(\eta,\check{\mu}_3,\check{\mu}_4))} \qquad (15) \\
= & |\operatorname{Aut}(\eta)| H^{\bullet,trop}_{d,g_1}(\mathcal{L}, (\check{\mu}_1, \check{\mu}_2, \check{\eta})) \cdot |\operatorname{Aut}(\eta)| H^{\bullet,trop}_{d,g_2}(\mathcal{L}, (\check{\eta}, \check{\mu}_3, \check{\mu}_4)).
\end{aligned}$$

where in equation (14) we use equality (11) and in equation (15) we use that the bijection created above maintains automorphisms of the ends attached to leaves over the l_i and that the local Hurwitz numbers stay the same after cutting our quadruple cover. Finally, the last equation follows from remark 3.1.11. □

THEOREM 3.1.12 (Correspondence Theorem for Static Quadruple Hurwitz Numbers). *Given a positive integer degree d, a genus $g \in \mathbb{Z}$, four partitions $\mu_1, \dots, \mu_4$ of d and a (connected) tropical curve Λ with 4 ends, labeled by $l_1, \dots, l_4$, such that $2g - 2d - 2 + \sum_{i=1}^{4} |\mu_i| = 0$, we have*

$$H_{d,g}^{\bullet,trop}(\Lambda, (\mu_1, \dots, \mu_4)) = H_{d,g}^{\bullet}(\mathbb{P}^1, (\mu_1, \dots, \mu_4)).$$

PROOF. Comparing the two degeneration formulas 3.1.8 and 3.1.6 it remains to show that tropical, static triple Hurwitz numbers agree with their tropical counterparts. But that is clear by remark 2.3.13, since corresponding labeled tropical and classical Hurwitz numbers differ from their unlabeled pendants by the same factor. □

3.2. The Correspondence Theorem for Elliptic Curves

In this section we will establish a correspondence theorem for Hurwitz numbers of elliptic curves with only simple ramifications. As is section 1.4.2 we fix an arbitrary complex elliptic curve $\mathcal{E}$ and consider the Hurwitz numbers $H_{d,g}(\mathcal{E})$, counting isomorphism classes of degree d covers of $\mathcal{E}$ with genus g source curve. Due to the Riemann-Hurwitz-formula 1.2.10 such covers have $s = 2g - 2$ simple ramifications.

On the tropical side, since we do not fix any complicated ramification profiles, the number of ends of the source curve has to be zero. More precisely, when counting genus-g-covers of an elliptic curve of any degree, we have to fix a tropical genus-1-curve without ends as source curve E, which necessarily is a 2-valent graph. We also require E to have $2g - 2$ vertices $p_1, \dots, p_{2g-2}$. We will consider the tropical Hurwitz numbers $H_{d,g}^{trop}(E)$ counting degree d covers of E by genus g curves with simple branching over $p_1, \dots, p_{2g-2}$.

EXAMPLE 3.2.1. See example 5.1.6 for a picture of a tropical cover of an elliptic curve.

NOTATION 3.2.2. Since the respective Hurwitz numbers do not depend on the elliptic target curves, we will just use the notations $H_{d,g}$ instead of $H_{d,g}(\mathcal{E})$ and $H_{d,g}^{trop}$ instead of $H_{d,g}^{trop}(E)$ in the following.

The aim of this section is to bring the above Hurwitz numbers into connection, yielding a correspondence theorem for Hurwitz numbers of elliptic curves (i.e. the left arrow of the triangle in figure in the introduction). The following is the precise formulation of this arrow:

THEOREM 3.2.3 (Correspondence Theorem). *The classical and tropical Hurwitz numbers of simply ramified covers of an elliptic curve coincide (see definition 1.4.3 and 2.3.2), more precisely for any integers $d, g \geq 1$ we have*

$$H_{d,g}^{trop} = H_{d,g}.$$

We prove theorem 3.2.3 by cutting covers of E at the preimages of a fixed base point p_0 (which is not a branch point), thus producing a (possibly reducible) cover of the tropical line $\mathbb{TP}^1 := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ as in [**10**]. We use the correspondence theorem of [**10**] relating the numbers of tropical covers to certain tuples of elements of the symmetric group that correspond to classical covers of $\mathbb{P}^1$, as described in section 1.3. We study "gluing factors" that relate the tropical multiplicity of a cover of E to the tropical multiplicity of the cut cover. These factors equal the number of ways to produce a tuple of elements of the symmetric group corresponding to a cover of an elliptic curve $\mathcal{E}$ from a tuple corresponding to a cover of $\mathbb{P}^1$.

Remember that by pairing a cover of $\mathcal{E}$ with a monodromy representation, the Hurwitz number $H_{d,g}$ of definition 1.4.3 equals the following count of tuples of permutations:

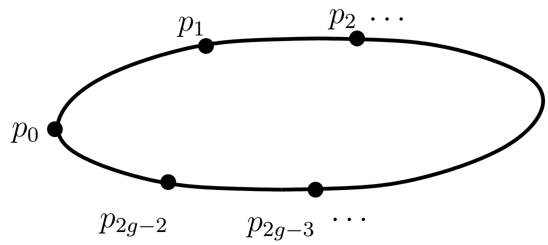
Remark 3.2.4 (see also definition 1.4.10).

$$\frac{1}{d!} \cdot \left|\left\{(\tau_1, \ldots, \tau_{2g-2}, \alpha, \sigma) \in (\mathbb{S}_d)^{s+2}\right\}\right|, \tag{16}$$

where the set only contains to tuples of permutations that fulfill:

(i) All τ_i are transpositions,
(ii) the group $\langle \tau_1, \ldots, \tau_{2g-2}, \sigma, \alpha \rangle$ acts transitively on $\{1, \ldots, d\}$ and
(iii) the identity $\tau_{2g-2} \circ \ldots \circ \tau_1 = \alpha \circ \sigma \circ \alpha^{-1} \circ \sigma^{-1}$ holds.

Given a tuple as in remark 3.2.4 we now construct an associated tropical cover. As a convention, we fix the base point p_0 and the $2g-2$ branch points $p_1, \ldots, p_{2g-2}$ (ordered clockwise) on E:



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Construction 3.2.5. Given a tuple as in remark 3.2.4 we construct a tropical cover of E with branch points $p_1, \ldots, p_{2g-2}$ as follows:

(1) For each cycle c of σ of length m draw an edge of weight m over p_0 and label it with the corresponding cycle.
(2) For $i = 1, \ldots, 2g-2$, successively cut or join edges over p_i according to the effect of τ_i on $\tau_{i-1} \circ \ldots \circ \tau_1 \circ \sigma$. Label the new edges as before.
(3) Glue the outcoming edges attached to points over p_{2g-2} with the edges over p_0 according to the action of α on the cycles of σ. More precisely: Glue the edge with the label $\alpha \circ c \circ \alpha^{-1}$ over p_{2g-2} to the edge with label c over p_0.
(4) Forget all the labels on the edges.

Note that for a cycle $c = (n_1 \ldots n_l)$ of length $l \geq 2$ we have $\alpha \circ c \circ \alpha^{-1} = (\alpha(n_1) \ldots \alpha(n_l))$. We use the same convention for cycles of length 1.

Example 3.2.6. Let $g = 2$ and $d = 4$ and consider the tuple of permutations

$$(\tau_1, \tau_2, \tau_3, \tau_4, \alpha, \sigma) = ((1\,3), (2\,4), (1\,2), (1\,3), (2\,3\,4), (2\,3))$$

in $\mathbb{S}_4$. We see that $\sigma = (2\,3) = (1)(2\,3)(4)$ has cycle type $(2, 1, 1)$. Moreover $\alpha \circ \sigma \circ \alpha^{-1} = (1)(2)(3\,4) = \tau_4 \circ \ldots \circ \tau_1 \circ \sigma$ is fulfilled, so the tuple contributes to the count of $H_{4,2}$. Figure 2 sketches the construction of remark 3.2.5 up to the gluing step (the object can be considered as a cover of the tropical line $\mathbb{TP}^1$ as described below).

In the gluing step the vertices p_0 and p_0' are going to be identified. Since we have

$$\begin{aligned} \alpha \circ (1) \circ \alpha^{-1} &= (1) \\ \wedge\ \alpha \circ (2\,3) \circ \alpha^{-1} &= (3\,4) \\ \wedge\ \alpha \circ (4) \circ \alpha^{-1} &= (2), \end{aligned}$$

the ends of the source curve are glued according to the red numbers in figure 2. The result is the cover of E depicted on the left in figure 3. Note that choosing $\alpha = (2\,4)$

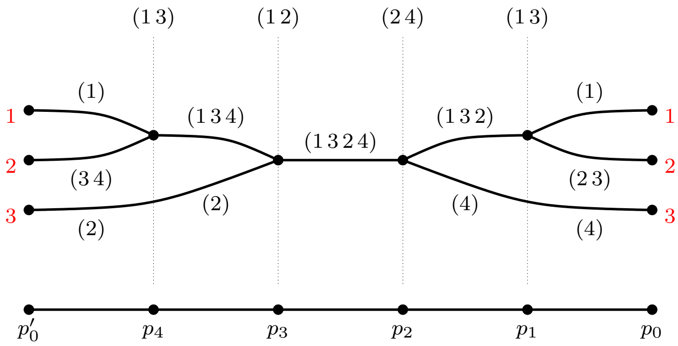


FIGURE 2. The cover of $\mathbb{TP}^1$ associated to a tuple of permutations.

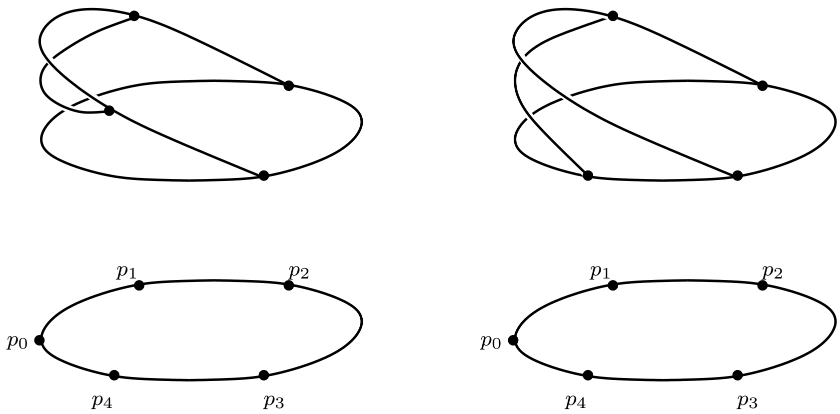


FIGURE 3. Two different gluings of the same cover of the line.

yields the same gluing, while $\alpha' = (1\,2\,4)$ also fulfills $\alpha' \circ \sigma \circ \alpha'^{-1} = \tau_4 \circ \ldots \circ \tau_1 \circ \sigma$, but since

$$\begin{aligned} \alpha' \circ (1) \circ \alpha'^{-1} &= (2) \\ \wedge\ \alpha' \circ (2\,3) \circ \alpha'^{-1} &= (3\,4) \\ \wedge\ \alpha' \circ (4) \circ \alpha'^{-1} &= (1), \end{aligned}$$

it provides a different gluing, sketched on the right side of figure 3. In particular the combinatorial types of the source curves are different.

Now we describe how to cut tropical covers of E in general, thus producing covers of a line. As usual, we neglect edge lengths — to be precise, they have to adapted accordingly.

CONSTRUCTION 3.2.7. To every cover $\pi : C \to E$ of degree d we associate a (possibly disconnected) tropical cover $\tilde{\pi} : \tilde{C} \to \mathbb{TP}^1$ of the line $\mathbb{TP}^1$ of the same degree, by cutting E at p_0 and the source curve C at every preimage of p_0.

REMARK 3.2.8. We have already defined disconnected Hurwitz numbers in definition 3.1.4 for the case of $r \geq 3$ fixed complicated ramification profiles. For the definition of connected tropical covers of $\mathbb{TP}^1$, see definition 2.3.2. This definition can easily be generalized by allowing the source curve to be disconnected. But for the case $r = 2$ considered we have to adapt the multiplicity differently than in the case $r \geq 3$:

Notice that the multiplicity of a single edge of weight m covering $\mathbb{TP}^1$ is $\frac{1}{m}$ (this case is not taken care of in definition 3.1.4 where $r \geq 3$ ensures that every source curve contains at least one vertex). The *multiplicity* of a disconnected cover $\tilde{\pi} : \tilde{C} \to \mathbb{TP}^1$ is

$$\operatorname{mult}(\tilde{\pi}) := \prod_K \frac{1}{w_K} \cdot |\operatorname{Aut}(\tilde{\pi})| \cdot \prod_e w_e,$$

where the first product goes over all connected components K of $\tilde{C}$ that just consist of one single edge mapping to $\mathbb{TP}^1$ with weight w_K and the second product goes over all bounded edges e of $\tilde{C}$, with w_e denoting their weight.

The factor $|\operatorname{Aut}(\tilde{\pi})|$ can be simplified to $\frac{1}{2}^{l_1+l_2}$, where l_1 denotes the number of balanced forks (i.e. adjacent ends of same weight) and l_2 denotes the number of wieners (i.e. pairs of bounded edges of same weight sharing both end vertices), see also [**10**]. Since we allow disconnected covers, we will have other contributions to the automorphism group: connected components consisting of single edges of the same weight as above can be permuted. So we get a contribution of $\frac{1}{r!}$ to $|\operatorname{Aut}(\tilde{\pi})|$ if for a certain weight m there are exactly r copys of connected components consisting of a single edge of weight m.

For a cover $\pi : C \to E$, we denote by Δ the partition of d given by the weights of the edges over p_0. For the cut cover $\tilde{\pi}$ (see construction 3.2.7) these are exactly the ramification profiles over $-\infty$ and ∞.

Example 3.2.9. The two covers of E depicted in figure 3 cut at p_0 both give a cover of the line as sketched in figure 2 (where we dropped the labels on the edges). The multiplicity of the cover of the line equals the product of the weight of the bounded edges, i.e. $3^2 \cdot 4$, since there are no automorphisms.

Example 3.2.10. Figure 4 shows a disconnected cover $\tilde{\pi}$ of the line of degree 17 with 6 (simple) ramifications and profiles $(4, 4, 4, 1, 1, 1, 1, 1)$ over the ends. Its multiplicity equals

$$\operatorname{mult}(\tilde{\pi}) = \underbrace{\left(\frac{1}{2}\right)^2 \cdot \frac{1}{2} \cdot \frac{1}{2!} \cdot \frac{1}{3!}}_{=|\operatorname{Aut}(\tilde{\pi})|} \cdot \underbrace{\frac{1}{4 \cdot 4 \cdot 1 \cdot 1 \cdot 1}}_{=\prod_K \frac{1}{w_K}} \cdot \underbrace{4 \cdot 2 \cdot 2 \cdot 2 \cdot 2}_{=\prod_e w_e} = \frac{1}{24},$$

where the first factor contributing to the automorphisms comes from the two balanced forks, the second from the wiener and the other two from two single-edge components of weight 4 and three single-edge components of weight 1 respectively.

The correspondence theorem in [**10**] matches tropical covers of $\mathbb{TP}^1$ as above with algebraic covers of $\mathbb{P}^1$ having two ramifications of profile Δ over 0 and ∞ respectively and only simple ramifications else.

Similar to remark 3.2.4, the associated Hurwitz numbers can be written in terms of tuples of elements of the symmetric group.

Remark 3.2.11. The double Hurwitz number $H_{d,g}(\mathbb{P}^1, \Delta, \Delta)$ counting the number of (isomorphism classes of) covers $\phi : \mathcal{C} \to \mathbb{P}^1$ of degree d (each weighted with $\frac{1}{|\operatorname{Aut}(\phi)|}$), where $\mathcal{C}$ is a possibly disconnected curve such that the sum of the genera of its connected components equals g, having ramification profile Δ over 0 and ∞ and only simple ramifications else, equals $\frac{1}{d!}$ times the number of tuples $(\tau_1, \ldots, \tau_{2g-2}, \sigma, \sigma')$ in $\mathbb{S}_d$ such that

- σ and σ' are permutations of cycle type Δ,
- the τ_i are transpositions for all $i = 1, \ldots, 2g - 2$,
- the equation $\sigma' \circ \tau_{2g-2} \circ \ldots \circ \tau_1 \circ \sigma = \operatorname{id}_{\mathbb{S}_d}$ holds.

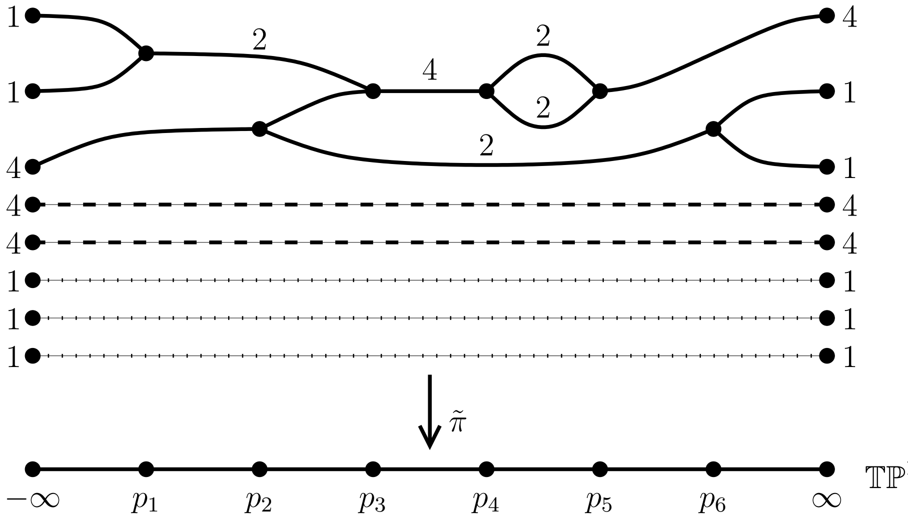


FIGURE 4. A (disconnected) tropical cover of the line.

Note that as in definition 1.4.3, it follows from the Riemann-Hurwitz formula that the number of simple ramifications is $2g-2$. The condition $\sigma' \circ \tau_{2g-2} \circ \ldots \circ \tau_1 \circ \sigma = \mathrm{id}_{\mathbb{S}_d}$ reflects the fact that the fundamental group $\pi_1(\mathbb{P}^1)$ is trivial. We do not include a condition about transitivity here, since we allow also disconnected covers.

As in construction 3.2.5, we can associate a tropical cover of the line to a tuple as in remark 3.2.11. The procedure is the same, we just drop the gluing step (3). The statement of the correspondence theorem 5.28 in [**10**] is that for a fixed tropical cover $\tilde{\pi} : \tilde{C} \to \mathbb{TP}^1$, the tropical multiplicity equals $\frac{1}{d!}$ times the number of tuples that yield $\tilde{\pi}$ under the above procedure.

We now relate the tuples in remarks 3.2.4 and 3.2.11. resp. the multiplicities of a tropical cover $\pi : C \to E$ and the cut cover $\tilde{\pi} : \tilde{C} \to \mathbb{TP}^1$ of construction 3.2.7.

DEFINITION 3.2.12. Given a cover $\pi : C \to E$ and the cut cover $\tilde{\pi} : \tilde{C} \to \mathbb{TP}^1$ of construction 3.2.7, we choose a tuple $(\tau_1, \ldots, \tau_{2g-2}, \sigma, \sigma')$ that yields $\tilde{\pi}$ when applying construction 3.2.5 (minus the gluing in step (3)). We define $n_{\tilde{\pi},\pi}$ to be the number of $\alpha \in \mathbb{S}_d$ satisfying $\alpha \circ \sigma \circ \alpha^{-1} = \sigma'$ and, when labeling $\tilde{\pi}$ with cycles according to our choice of tuple $(\tau_1, \ldots, \tau_{2g-2}, \sigma, \sigma')$ and performing step (3) and (4) of construction 3.2.5 (gluing and forgetting the cycle labels), we obtain π.

Note that $n_{\tilde{\pi},\pi}$ is well-defined (i.e. does not depend on the choice of the tuple $(\tau_1, \ldots, \tau_{2g-2}, \sigma, \sigma')$): This is true since any representative $(\bar{\tau}_1, \ldots, \bar{\tau}_{2g-2}, \bar{\sigma}, \bar{\sigma}')$ for the cover $\tilde{\pi}$ is a conjugate of $(\tau_1, \ldots, \tau_{2g-2}, \sigma, \sigma')$ and therefore the desired $\bar{\alpha}$ are in one-to-one correspondence to the desired α.

PROPOSITION 3.2.13. *For a cover $\pi : C \to E$ with partition $\Delta = (m_1, \ldots, m_r)$ over the base point, the number $n_{\tilde{\pi},\pi}$ of definition 3.2.12 is given by*

$$n_{\tilde{\pi},\pi} = m_1 \cdot \ldots \cdot m_r \cdot \frac{|\operatorname{Aut}(\tilde{\pi})|}{|\operatorname{Aut}(\pi)|}.$$

PROOF. As in the definition of $n_{\tilde{\pi},\pi}$ (see definition 3.2.12), fix a tuple of permutations $(\tau_1, \ldots, \tau_{2g-2}, \sigma, \sigma')$ that yields $\tilde{\pi}$ when applying construction 3.2.5 minus the gluing step (3).

The set of α such that $\alpha \circ \sigma \circ \alpha^{-1} = \sigma'$ is a coset of the stabilizer of σ with respect to the operation of $\mathbb{S}_d$ on itself via conjugation: $(\alpha, \sigma) \mapsto \alpha \circ \sigma \circ \alpha^{-1}$. Assume

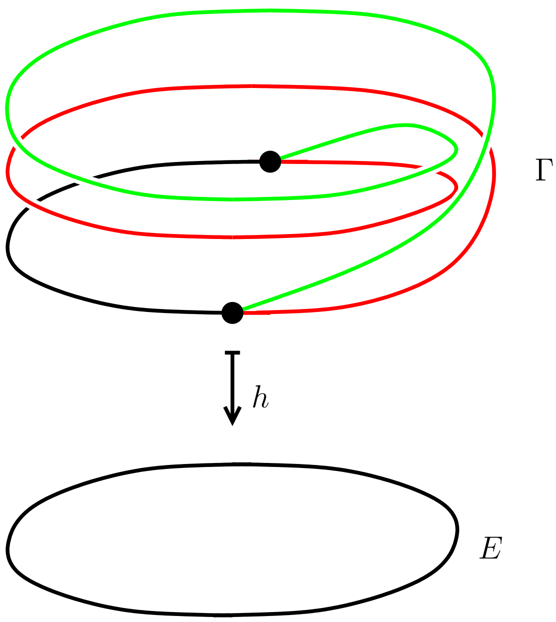


FIGURE 5. A tropical elliptic cover with a long wiener. The two wiener-edges (i.e. the red and the green edge) have the same weight.

that Δ consists of k_i weights w_i for $i = 1, \ldots, s$, then this stabilizer is isomorphic to the semidirect product $\prod_{i=1}^{s} C_{w_i}^{k_i} \rtimes \prod_{i=1}^{s} \mathbb{S}_{k_i}$ of cyclic groups C_{w_i} of length w_i and symmetric groups $\mathbb{S}_{k_i}$. This can be seen as follows: for each weight w_i (i.e. length of a cycle of σ) we can choose an element of $\mathbb{S}_{k_i}$ permuting the cycles of length w_i in σ. Assume the cycle c_1 of σ is mapped to the cycle c_2 by this permutation. Then we consider permutations α' in the group of bijections of the entries of c_2 to the entries of c_1 that satisfy $\alpha' \circ c_1 \circ \alpha'^{-1} = c_2$, there are w_i such α' (and they form a cyclic group). Since the cycles of σ are disjoint, the choices for α' for each pair of cycles (c_1, c_2) where c_1 is mapped to c_2 under the permutations in $\mathbb{S}_{k_i}$ that we choose for each i can be combined to a unique α in the stabilizer of σ.

We label the edges of $\tilde{C}$ with cycles as given by the choice of our tuple. Transferred to our situation, the argument above shows that when searching for α that satisfy both requirements of definition 3.2.12, we always get the contributions from the C_{w_i}, (leading to a factor of $\prod_{i=1}^{s} w_i^{k_i} = m_1 \cdot \ldots \cdot m_r$). To prove the lemma, it remains to see that $\frac{|\operatorname{Aut}(\tilde{\pi})|}{|\operatorname{Aut}(\pi)|}$ equals the number of ways to choose permutations of the cycles of the same length (resp. permutations of the ends of $\tilde{C}$) that correspond to a gluing of $\tilde{\pi}$ equal to π when applying construction 3.2.5, step (3).

So let us now analyze the automorphism groups and compare the quotient of their sizes to the possibilities to glue the cover $\tilde{\pi}$ (with labeled ends) to π.

The automorphism group of $\tilde{\pi}$ is, as mentioned above, a direct product of symmetric groups each corresponding to a wiener, a balanced fork or the set of connected components consisting of a single edge of fixed weight. The automorphism group of π is a direct product of symmetric groups of size two corresponding to wieners. Notice that we can have long wieners as in figure 5, where the two edges of the same weight are curled equally. Clearly automorphisms that come from wieners that are not cut cancel in the quotient and we can thus disregard them. Since therefore all contributions to the automorphism groups we have to

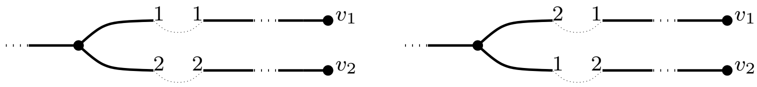


FIGURE 6. Two ways to glue a fork to two distinguishable ends.

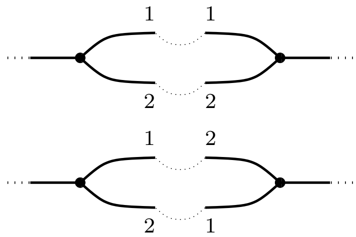


FIGURE 7. Gluing two balanced forks to a wiener.

consider come from ends of $\tilde{C}$, and the possibilities to glue the cover $\tilde{\pi}$ to π also only depend on the ends of $\tilde{C}$, we can analyze the situation locally on the level of the involved ends.

We say that an end of $\tilde{C}$ is distinguishable if it is not part of a balanced fork and not an end of a component consisting of a single edge. Distinguishable ends do not contribute to the automorphisms of $\tilde{\pi}$.

We have to consider several cases. We first consider cases not involving connected components consisting of a single edge.

(1) If we glue two distinguishable ends of $\tilde{C}$ to get back C, there are no choices for different gluings. Since distinguishable ends do not contribute to the automorphisms, the equality of contributions from these ends holds.
(2) Assume that an edge of C is cut in such a way that one of the ends is part of a balanced fork and the other is distinguishable. Then obviously there are 2 ways to glue, see figure 6. The balanced fork contributes with a factor 2 to $|\operatorname{Aut}(\tilde{\pi})|$. After gluing, the fork is not part of a wiener, so the contribution to $|\operatorname{Aut}(\pi)|$ is 1. Again, we see that the contributions coming from these ends to the quotient of the sizes of the automorphism groups on the one hand and to the possibilities of gluing on the other hand coincide.
(3) If two balanced forks are glued, we obtain a wiener. The contribution to $|\operatorname{Aut}(\tilde{\pi})|$ and $|\operatorname{Aut}(\pi)|$ is 4 and 2 respectively. The ways to glue the forks to a wiener is 2, as illustrated in figure 7.

Now we have to consider cases involving ends of connected components consisting of a single edge, say of weight m. Assume there are l components consisting of a single edge of weight m. These ends contribute a factor of $l!$ to $|\operatorname{Aut}(\tilde{\pi})|$.

(4) Assume that l_0 of the components are not part of a long wiener after gluing. They do not contribute to the automorphisms of π. Note that the components nevertheless might be attached to balanced forks. In this case the fork is either part of a pseudo-wiener in π (i.e. two edges sharing the same end vertices and having the same weight, but curled differently, see figure 8) or the two edges of the fork have different end vertices.

Let us now determine the number of ways to glue these ends of $\tilde{C}$ to get back C. We can choose l_0 of the l single-edge-components, and distribute them to l_0 distinguishable places. Also, we get a factor of 2 for

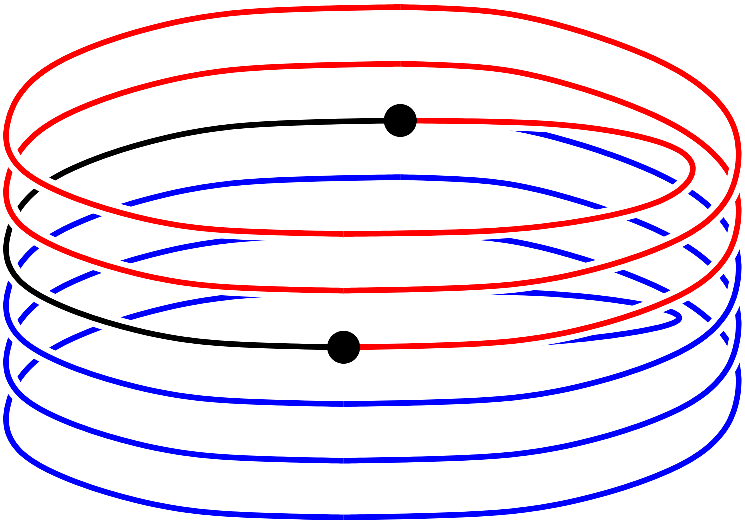


FIGURE 8. A cover with a pseudo-wiener.

each balanced fork involved. The result is $\binom{l}{l_0} \cdot l_0! \cdot 2^f$, where f is the number of balanced forks involved.

The remaining $l - l_0$ components must be part of long wieners in π. Let n be the number of long wieners in π, giving a contribution of 2^n to $|\operatorname{Aut}(\pi)|$, then $2n$ balanced forks from $\tilde{\pi}$ are involved in the gluing process contributing with a factor of 2^{2n} to $|\operatorname{Aut}(\tilde{\pi})|$. The number of ways to glue is the number of ways to distribute the $l - l_0$ components to $l - l_0$ gluing places and a factor of 2 for every wiener we get, just as in figure 7. Altogether the contribution to the number of gluings providing the desired cover equals

$$\binom{l}{l_0} \cdot l_0! \cdot 2^f \cdot (l - l_0)! \cdot 2^n = l! \cdot 2^f \cdot 2^n.$$

The contribution to the quotient of the sizes of the automorphism groups equals

$$\frac{l! \cdot 2^f \cdot 2^{2n}}{2^n}.$$

Obviously, the two expressions coincide and we are done.

□

We are now ready to prove the correspondence theorem 3.2.3, that is the equality of tropical and classical Hurwitz numbers of simply ramified covers of elliptic curves.

PROOF OF THEOREM 3.2.3. By remark 3.2.4

$$H_{d,g} = \frac{1}{d!} \cdot |\{(\tau_1, \ldots, \tau_{2g-2}, \alpha, \sigma)\}|,$$

where $\alpha, \sigma, \tau_i \in \mathbb{S}_d$, the τ_i are transpositions, the equality $\tau_{2g-2} \circ \ldots \circ \tau_1 \circ \sigma = \alpha \circ \sigma \circ \alpha^{-1}$ holds and $\langle \tau_1, \ldots, \tau_{2g-2}, \sigma \rangle$ acts transitively on the set $\{1, \ldots, d\}$. We can group the tuples in the set according to the tropical cover $\pi : C \to E$ they provide under construction 3.2.5 and write the sum above as

$$\frac{1}{d!} \cdot \sum_{\pi} |\{(\tau_1, \ldots, \tau_{2g-2}, \alpha, \sigma) \text{ yielding the cover } \pi\}|.$$

For a fixed cover π, instead of counting tuples yielding π, we can count tuples $(\tau_1, \ldots, \tau_{2g-2}, \sigma, \sigma')$ yielding the cut cover $\tilde{\pi}$ from construction 3.2.7 and then multiply with the number of appropriate α, i.e. with $n_{\tilde{\pi},\pi}$ (see definition 3.2.12):

$$\frac{1}{d!} \cdot \sum_{\pi} |\{(\tau_1, \ldots, \tau_{2g-2}, \sigma, \sigma') \text{ that provide the cover } \tilde{\pi}\}| \cdot n_{\tilde{\pi},\pi}.$$

By [**10**] (see also remark 3.2.11) the count of the tuples yielding a cover $\tilde{\pi}$ divided by $d!$ coincides with its tropical multiplicity $\operatorname{mult}(\tilde{\pi}) = \frac{1}{|\operatorname{Aut}(\tilde{\pi})|} \cdot \prod_K \frac{1}{w_K} \cdot \prod_{\tilde{e}} w_{\tilde{e}}$ where the first product goes over all components K consisting of a single edge of weight w_K and the second product goes over all bounded edges $\tilde{e}$ of $\tilde{C}$ and $w_{\tilde{e}}$ denotes their weight (see (3.2.8)). Using proposition 3.2.13, the number $n_{\tilde{\pi},\pi}$ can be substituted by $\prod_{e'} w_{e'}^{c_{e'}} \cdot \frac{|\operatorname{Aut}(\tilde{\pi})|}{|\operatorname{Aut}(\pi)|}$ where the product goes over all edges e' of C that contain a preimage of the base point p_0 of E and $c_{e'}$ denotes the number of preimages in e', $c_{e'} = |\pi^{-1}(p_0) \cap e'|$. We obtain

$$H_{d,g} = \sum_{\pi} \frac{1}{|\operatorname{Aut}(\tilde{\pi})|} \cdot \prod_{\tilde{e}} w_{\tilde{e}} \cdot \prod_{K} \frac{1}{w_K} \cdot \prod_{e'} w_{e'}^{c_{e'}} \cdot \frac{|\operatorname{Aut}(\tilde{\pi})|}{|\operatorname{Aut}(\pi)|}.$$

An edge e' of C of weight $w_{e'}$ having $c_{e'}$ preimages over the base point provides exactly $c_{e'} - 1$ single-edge-components of weight $w_{e'}$ in the cut cover $\tilde{\pi}$. Vice versa, each such component comes from an edge with multiple preimages over the base point. Therefore the expression $\prod_K \frac{1}{w_K} \cdot \prod_{e'} w_{e'}^{c_{e'}}$ simplifies to $\prod_{e'} w_{e'}$. We obtain

$$H_{d,g} = \sum_{\pi} \frac{1}{|\operatorname{Aut}(\pi)|} \prod_{e} w_e = H_{d,g}^{trop}$$

and the theorem is proved. $\square$

CHAPTER 4

Tropical Moduli Spaces of Covers

Remember that our definition of tropical Hurwitz numbers given in section 2.3 unifies the definitions of [**4**] and [**10**], allowing a maximum of generality **and** the possibilty to establish moduli spaces of covers in a very natural way. In this chapter we study more general moduli spaces of tropical covers and their branch maps. The main result is theorem 4.3.3, in which we establish the degree of the tropical branch map as a tropical intersection-theoretic invariant. Furthermore this degree equals (the tropical and thus) the classical Hurwitz number by theorem 4.3.6. With this chapter we enrich the correspondence theorem of [**4**] with a study of the involved tropical moduli spaces and intersection theory.

4.1. Tropical Triple Hurwitz Numbers

Triple Hurwitz numbers in classical algebraic geometry count covers of a smooth genus 0 curve, i.e. $\mathbb{P}^1_{\mathbb{C}}$, simply denoted by $\mathbb{P}^1$, with three special ramification profiles and all other ramifications simple. On the tropical side, each special ramification of a cover lies over a leaf. So in order to consider tropical triple Hurwitz numbers, we have to count covers of the *tripod*-model of the tropical projective line (see definition 4.1.1). Note that g' (the genus of the base curve) will be zero through the whole section.

The structure and the results of this section are already presented in article [**7**], which is joint work with Hannah Markwig. Some statements may nevertheless be formulated slightly different — mainly because of different notations due to a less general definition for tropical curves and covers in [**7**] than in this thesis.

Definition 4.1.1. Let $\mathcal{L}$ be the abstract curve that corresponds to a generic tropical line in the tropical projective plane, i.e. a curve with one vertex that we denote by c and three ends (of infinite lengths) adjacent to c, each attached to a leaf that we call u, v and w (see figure 1). The corresponding ends are referred to as u-, v- and w-ends, respectively.

In the case of covers of a base curve of genus 0, the automorphisms are easy to describe.

Remark 4.1.2. Automorphisms of a trivalent cover $\pi : \Gamma \to \mathcal{L}$ can only arise due to *wieners*, that are pairs of parallel edges of Γ being mapped with same weight (see figure 2 and also [**10**]). More precisely, $|\operatorname{Aut}(\pi)| = 2^k$, where k is the number of wieners of π, since each wiener allows a permutation of the two edges involved.

Let us see an example for a cover of $\mathcal{L}$.

Example 4.1.3. Figure 3 shows a tropical curve of genus 5. The red numbers denote the genus on vertices, the black numbers are the edge lengths, vertex labels as well as markings of the ends are left out. All vertices without a red number have genus zero.

Now figure 4 shows an example of a cover of $\mathcal{L}$ by the curve from figure 3. We only mark the edge weights in blue, the other values can be deduced from figure 3.

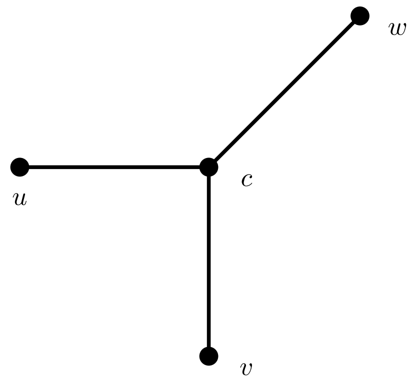


FIGURE 1. The tropical line $\mathcal{L}$.

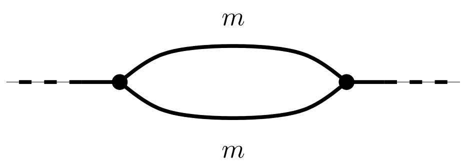


FIGURE 2. A wiener.

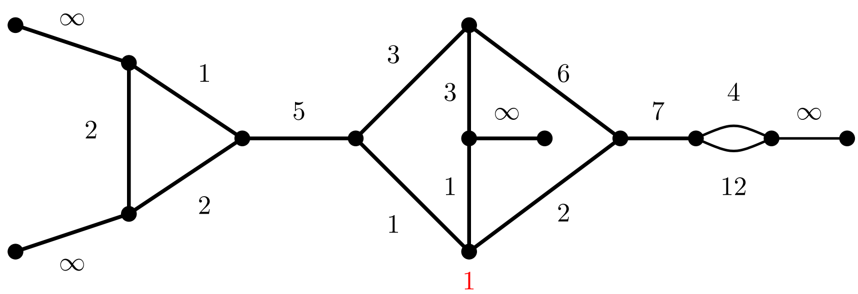


FIGURE 3. A tropical curve.

Note that the RH-numbers of the vertices mapping to c are zero. In particular the trivalent vertex of local degree 3 has genus 1. When drawing pictures of a cover of $\mathcal{L}$ in the following, we will leave out the target $\mathcal{L}$. The map to $\mathcal{L}$ should be self-explaining in the respective figure.

Obviously this cover is trivalent, i.e. it matches definition 2.3.2, where $r = 3$ (the number of leaves of the base curve), $(\mu_u, \mu_v, \mu_w) := (\mu_1, \mu_2, \mu_3) = ((3,1),(4),(4))$ (i.e. $d = 4$), $g = 5$ (remember that we have one vertex of genus 1 over c) and conclusionally $s = 2g - 2 - d(2g' - 2) - rd + \sum_{i=1}^{r} |\mu_i| = 10$ (the number of simple ramifications). In other words this cover accounts to the Hurwitz number $H_{4,4}^{trop}(\mathcal{L}, ((3,1),(4),(4)))$. Its multiplicity is $\frac{1}{1} \cdot (1^5 \cdot 2^2 \cdot 3^4 \cdot 4^2) \cdot (1 \cdot 2) = 10368$. Note that due to proposition 2.3.6 we only have to consider the local Hurwitz numbers of the two vertices over c. The number of Automorphisms is 1 due to remark 4.1.2.

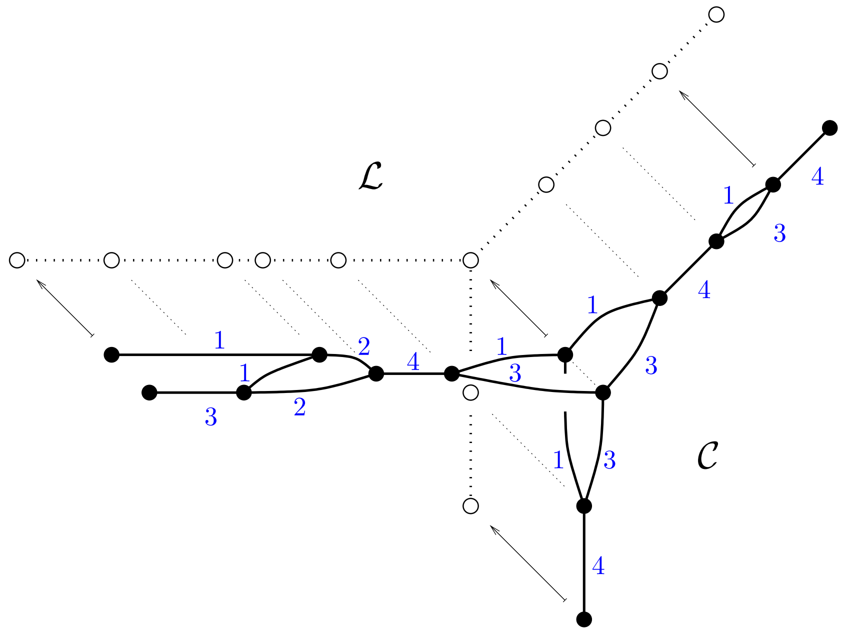


FIGURE 4. A tropical cover of $\mathcal{L}$.

4.2. The Moduli Space

The moduli space structure of tropical triple covers is as follows. We have to distinguish different covers by their *combinatorial type* (see definition 2.2.10) in order to get a fan structure on the set of all covers. Let $g \geq 0$ and $d > 0$ be fixed integers in the following. Moreover we fix a triple $\mu = (\mu_u, \mu_v, \mu_w)$ of partitions of d such that $s = 2g - 2 + (2-r)d + \sum_{i=1}^{r} |\mu_i| \geq 0$.

Let us think of 2-valent vertices of the source without genus to be smoothened, i.e. the adjacent edges are joined and the vertex is deleted. This is possible since these 2-valent vertices are never labeled due to definition 2.2.5. Moreover smoothening theese points does not have an effect on the multiplicities of the covers as shown in 2.3.6. Moreover, by convention, we assume that the only 2-valent vertices of $\mathcal{L}$ are the desired branch points. Basically we can move around in the moduli space that we will construct in the following by "moving branch points".

Let $\pi : \Gamma \to \mathcal{L}$ be a tropical cover of degree d with source curve of genus g and ramification profile μ_u, μ_v and μ_w over u, v and w, respectively. Denote by α its combinatorial type. Then the set $D_\Gamma := D_\alpha$ of covers of type α naturally forms an open polyhedral cone: We can vary the lengths of the bounded edges in order to move within D_α, but we cannot vary them independently since we need to cover $\mathcal{L}$. Inside the open positive orthant of $\mathbb{R}^b$, where b is the number of bounded edges, the conditions can be expressed as integral linear equations. We will see an example in 4.2.7.

The closures of these cones can naturally be glued together. To see this let us investigate their boundary.

DEFINITION 4.2.1. Points on the boundary of the cone $\overline{D}_\alpha$ would correspond to covers where some lengths of edges are shrunk to zero (remember that we actually do not allow edge lengths to be zero). We remove edges of zero lengths, identify their adjacent vertices and adjust the genus at vertices and their labels as follows: Denote by G' a connected subgraph of edges whose lengths go to zero. Let $v_1, \ldots, v_k$ be the vertices of G'. Replace G' by a vertex v of genus $\operatorname{gen}(v) = \sum_{i=1}^k \operatorname{gen}(v_i) + b^1(G')$ and with the union of all labels of the v_i as labels.

We call the new cover (resp. the new combinatorial type) obtained in this way a *contraction* of π (resp. of a contraction of α).

Lemma 4.2.2. *Definition 4.2.1 is well-defined, i.e. a contraction corresponding to a point on the boundary of the cone $\overline{D}_\alpha$ of a combinatorial type α is indeed a tropical cover in the sense of definition 2.2.5.*

Proof. We just have to check that the labeling of the inner vertices matches definition 2.2.5. Let v be a new vertex replacing the connected subgraph G' of edges going to zero. Assume G' has k vertices $v_1, \ldots, v_k$ and E edges. Then v has $\sum_i r_{v_i}$ labels and we have to see that it has RH-number $r_v = \sum_i r_{v_i} \geq 0$. Assume that for $l < k$ the vertices $v_1, \ldots, v_l$ are mapped to the center c of $\mathcal{L}$ while the v_i with $i > l$ are mapped to a ray. Then the RH-numbers of the v_i equal $r_{v_i} = \operatorname{val}(v_i) + 2\operatorname{gen}(v_i) - 2 - d_i$ if $i \leq l$ where d_i denotes the local degree at v_i, and $r_{v_i} = \operatorname{val}(v_i) + 2\operatorname{gen}(v_i) - 2$ else. If $l = 0$ all the v_i as well as the new vertex v must be mapped to the same ray and we have $r_v = \operatorname{val}(v) + 2\operatorname{gen}(v) - 2$. If $l > 0$, the new vertex v must be mapped to c and we have $r_v = \operatorname{val}(v) + 2\operatorname{gen}(v) - 2 - d$, where d denotes the local degree at v. Obviously $d = \sum d_i$ in this case. In any case we have

$$\begin{aligned}
\sum_i r_{v_i} &= \sum_{i=1}^k (\operatorname{val}(v_i) + 2\operatorname{gen}(v_i) - 2) - \sum_{i=1}^l d_i \\
&= \sum_{i=1}^k \operatorname{val}(v_i) + 2\sum_{i=1}^k \operatorname{gen}(v_i) - 2k - \sum_{i=1}^l d_i \\
&= \sum_{i=1}^k \operatorname{val}(v_i) - 2E + 2\sum_{i=1}^k \operatorname{gen}(v_i) + 2(E - k + 1) - 2 - \sum_{i=1}^l d_i \\
&= \operatorname{val}(v) + 2\operatorname{gen}(v) - 2 - \sum_{i=1}^l d_i = r_v,
\end{aligned}$$

where the third equality is obtained by adding zero and the last equality holds because the Euler-characteristic of G' yields $b^1(G') = E - k + 1$. ☐

We now take the set of all cones D_α such that the combinatorial type α is trivalent or a contraction of a trivalent type. We glue these cones by identifying points on the boundary of $\overline{D}_\alpha$ with the corresponding point in the cone of its contraction. With this identification, the set of cones becomes an abstract polyhedral complex in the sense of [**25**, Definition 3.4] that we call $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu_u, \mu_v, \mu_w)$, the *moduli space of tropical covers of $\mathcal{L}$ with source curve of genus g and ramification profiles μ_u, μ_v, μ_w (over u, v, w, respectively).*

By definition, the cones corresponding to trivalent types are the maximal cones of $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu)$. We will compute their dimension in the following.

Lemma 4.2.3. *Let α be a trivalent type of degree d genus g covers of $\mathcal{L}$ with ramification profiles μ_u, μ_v, μ_w. Then the dimension of the cone D_α of α is* $\dim(D_\alpha) = \sum_{i=u,v,w} |\mu_i| + 2g - 2 - d$, *where $|\mu_i|$ denotes the length of the partition μ_i.*

Proof. For a trivalent cover of type α, the dimension $\dim(D_\alpha)$ clearly equals the number of vertices which are not mapped to c: we can vary the lengths of the edges, staying within D_α, in such a way that the images of these vertices move on $\mathcal{L}$ (see also remark 4.2.6). Each such moving image yields one degree of freedom. It follows that for a trivalent graph, the dimension $\dim(D_\alpha)$ equals the total number of vertex labels. The star-shaped cover — i.e. the cover with one interior vertex adjacent to ends of weights μ which are mapped to u, v and w accordingly — is a contraction of every trivalent type. Since contraction by definition preserves the number of vertex labels, we can compute the number of vertex labels of the star-shaped cover in order to obtain the number of vertex labels of any trivalent cover. By the RH-condition, the star-shaped cover has $\sum_{i=u,v,w} |\mu_i| + 2g - 2 - d$ many labels. The claim follows. □

Below, we equip each maximal cone of $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu_u, \mu_v, \mu_w)$ with a weight, so that we can conclude the following result about the structure of $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu_u, \mu_v, \mu_w)$:

THEOREM 4.2.4. *The moduli space $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu_u, \mu_v, \mu_w)$ of tropical covers of $\mathcal{L}$ of genus g with ramification profile μ_u, μ_v, μ_w is an abstract weighted polyhedral complex of pure dimension $\sum_{i=u,v,w} |\mu_i| + 2g - 2 - d$.*

4.2.1. The Weights on the Maximal Cones. To introduce the weights of maximal cones, we need the following preparations.

Let $f : \mathbb{Z}^n \to \mathbb{Z}^m$ be a linear map. We define the *index of* f, denoted by I_f, to be the index of the sublattice $f(\mathbb{Z}^n)$ inside $\mathbb{Z}^m$.

DEFINITION 4.2.5. Let α be a combinatorial type of cover. In the underlying graph G, identify all vertices mapping to c to one vertex. We call the graph obtained in this way G'. Pick $b^1(G')$ independent cycles, i.e. generators of $H_1(G', \mathbb{Z})$. Each such generator is given as a chain of directed edges around the cycle. In order to obtain a cover of type α, we can choose lengths for the bounded edges, but we cannot choose them independently. The condition can be rephrased by stating that the images of the loops of G' have to close up. In this way, we obtain $b^1(G')$ independent integral linear equations that cut out D_α from $(\mathbb{R}_{>0})^B$, where B denotes the number of bounded edges. We use the integral equations as defined by the weights of the edges that appear and do not reduce common factors (for example for a wiener with edges e_1 and e_2, each of weight 2, the equation is $2l(e_1) - 2l(e_2) = 0$ and not $l(e_1) - l(e_2) = 0$).

REMARK 4.2.6. In the above situation an Euler-characteristic computation for G' minus its ends shows that $1 - b^1(G') = 1 + |\{v|\pi(v) \neq c\}| - B$, i.e. the number of equations in definition 4.2.5 equals $B - |\{v|\pi(v) \neq c\}|$ (here, B denotes again the number of bounded edges). It follows that the dimension of D_α equals $B - (B - |\{v|\pi(v) \neq c\}|) = |\{v|\pi(v) \neq c\}|$. Indeed, in D_α we can vary the images of the vertices not mapped to c, and we have used this fact in the proof of lemma 4.2.3.

EXAMPLE 4.2.7. Consider the cover of example 4.1.3. We enumerate the inner edges as indicated in figure 5 by the black numbers and red numbers. The black numbers represent a set of edges forming a spanning tree of G', i.e. a set of edges whose lengths we can vary independently. The red edges each close a loop in G', i.e. they depend on the lengths of the black edges. Denoting $x_i = l(e_i)$ we get the following six linear equations that cut out D_α from $\mathbb{R}^{13}_{>0}$:

$$\begin{aligned} x_1 - 3x_9 &= 0 \\ 2x_3 - 2x_4 + x_{10} &= 0 \\ 3x_5 - x_{11} &= 0 \\ 3x_7 - x_{12} &= 0 \\ x_8 - 3x_{13} &= 0. \end{aligned}$$

DEFINITION 4.2.8. For a combinatorial type α, we define I_α to be the index of the linear map A_α defined by the matrix that we get from the equations in definition 4.2.5.

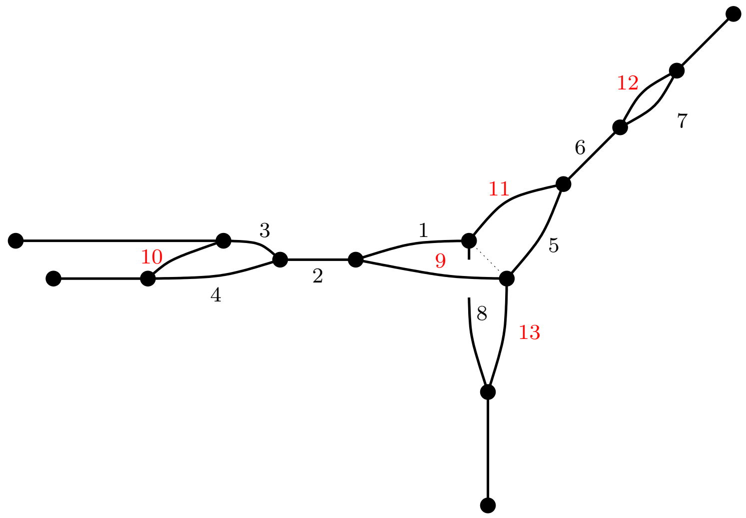


FIGURE 5. Equations for cutting out D_α.

Note that while the matrix A_α depends on the choice of generators of $H_1(G', \mathbb{Z})$, its minors and therefore the index I_α do not (see [**10**, chapter 5]).

DEFINITION 4.2.9. For a maximal cone of $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu_u, \mu_v, \mu_w)$, resp. for a trivalent type α, we define its weight $\omega(\alpha)$ to be

$$\omega(\alpha) := \frac{1}{|\operatorname{Aut}(\alpha)|} \cdot I_\alpha \cdot \prod_v H_v,$$

where $|\operatorname{Aut}(\alpha)|$ is defined as in notation 2.2.16 and the product runs over all vertices v mapping to c (see proposition 2.3.6).

Note that this definition is natural when compared to other definitions of weights in tropical moduli spaces, see e.g. [**25**, definition 3.5] or [**10**, definition 5.10]. Also, it is natural from the point of view of tropical intersection theory, since the cones D_α are cut out by the equations of index I_α.

4.3. The Tropical Branch Map

We will establish a tropical pendant $\mathrm{br}^{trop} : \mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu_u, \mu_v, \mu_w) \to \mathcal{L}^r$ to the branch map from classical Hurwitz theory (see section 1.5) and show that — as in the classical case — its degree is the corresponding Hurwitz number $H_{d,g}(\mathcal{D}, (\mu_0, \mu_1, \mu_\infty))$, where the μ_i are three partitions of d.

DEFINITION 4.3.1. The *(tropical) branch map* on the tropical moduli space $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu_u, \mu_v, \mu_w)$ is defined as

$$\begin{aligned} \mathrm{br}^{trop} : \mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu_u, \mu_v, \mu_w) &\to \mathcal{L}^r \\ (\pi : \Gamma \to \mathcal{L}) &\mapsto (\pi(v_1), \pi(v_2), \dots, \pi(v_r)), \end{aligned}$$

where $r = \sum_{i \in \{u,v,w\}} \mu_i + 2g - 2 - d$ is the total number of labels (and the dimension of $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu_u, \mu_v, \mu_w)$) and v_i is the vertex of Γ that is labeled with i.

It follows easily that br^{trop} is a *morphism of weighted polyhedral complexes* of the same dimension in the sense of [**25**, Definition 4.1]:

The *degree of a morphism f of weighted polyhedral complexes* of the same dimension is defined to be the sum of the weights of cones times the local multiplicities of cones (we denote the latter by $\text{mult}_D f$ for each cone D), where the sum goes over all inverse images of a point p *in general position* [**25**, Definition 4.1]. Written as formula this means

$$\deg(f) = \sum_{q \mid f(q)=p} \omega(D(q)) \, \text{mult}_{D(q)} f,$$

where, for each $q \in f^{-1}(p)$, the expression $D(q)$ denotes the unique maximal cone that contains q in its interior.

LEMMA 4.3.2. *Let α be the combinatorial type of a trivalent cover. We have*

$$I_\alpha \cdot \text{mult}_{D_\alpha} \text{br}^{trop} = \prod_e w_e,$$

where the product goes over all bounded edges of α (i.e. the edges of its underlying graph), w_e denotes the weight of the edge e, I_α the lattice index defined in definition 4.2.8 and $\text{mult}_{D_\alpha} \text{br}^{trop}$ *the local multiplicity of the branch map just as above.*

PROOF. This is a straight-forward generalization of Remark 5.19 and Lemma 5.26 of [**10**]. □

We now state our main result.

THEOREM 4.3.3. *The degree of* br^{trop} *is constant, i.e. it does not depend on the choice of the point in general position that we pull back.*

The proof will be made in section 4.5. We first consider the consequences of theorem 4.3.3.

REMARK 4.3.4. Obviously the degree of br^{trop} on a tropical moduli space of covers equals the tropical Hurwitz number corresponding to this moduli space.

LEMMA 4.3.5. *The degree of the tropical branch map agrees with the tropical Hurwitz numbers of* [**4**]*, up to a factor of* $|\text{Aut}(\mu)|$ *that arises because we mark the ends.*

PROOF. Remember: The definition of tropical Hurwitz number in [**4**] counts covers where all the ramification data is imposed at the ends, i.e. simple ramification in the interior appearing as trivalent vertices is not considered. To interpret our covers in this context, we need to add an extra end to $\mathcal{L}$ at the image of every trivalent vertex x not mapping c, and analogously add one (unmarked) end of weight 2 and $\deg_\pi x - 2$ (unmarked) ends of weight 1 to x as well as $\deg_\pi y$ (unmarked) ends of weight one to every $y \in \pi^{-1}(\pi(x)) \setminus \{x\}$. We call the new tropical curves obtained in this way Γ' and $\mathcal{L}'$ respectively. We extend π to a cover $\pi' : \Gamma' \to \mathcal{L}'$ such that the new ends of Γ' are mapped to the new ends of $\mathcal{L}'$ in the obvious manner.

By lemma 4.3.2 we can write the contribution of each combinatorial type of cover to our count as

$$\omega(\alpha) \cdot \text{mult}_{D_\alpha} \text{br}^{trop} = \frac{1}{2^k} \cdot \prod_e w_e \cdot \prod_x H_x, \tag{17}$$

where k denotes the number of wieners. Note that by remark 4.1.2 $|\text{Aut}(\pi)| = 2^k$. On the other hand, $\pi' : \Gamma' \to \mathcal{L}'$ is counted in [**4**] with multiplicity

$$\frac{1}{|\text{Aut}(\pi')|} \cdot \prod_{e'} w_{e'} \cdot \prod_{x'} H_{x'}. \tag{18}$$

(Note that in [**4**] the authors work with a definition of (classical) Hurwitz numbers where we do not mark the preimages of the three special ramification points, consequently they have to multiply their Hurwitz number with a factor reflecting the local automorphisms, i.e. the automorphisms of the three local partitions.) The automorphisms $\mathrm{Aut}(\pi')$ here consist of automorphisms of the unmarked ends, and the wieners as before. We now analyze the difference between the two expressions.

- Assume x is a trivalent vertex not mapping to c, then x does not contribute any Hurwitz number to (17). In (18) the corresponding vertex x' provides a factor
$$H_{x'} = (\deg_\pi(x) - 2)!$$
(this number reflects the number of ways to mark the preimages of the simple branch point). This factor is annihilated by the corresponding global automorphisms in the whole product.
- Let $y \in \pi^{-1}(\pi(x)) \setminus \{x\}$ for a vertex x as above. Similar to the former case y yields no contribution to (17) and in (18) we get
$$H_{y'} = \frac{1}{\deg_\pi y} \cdot (\deg_\pi y)!.$$
By adding the extra ends at y we subdivide an edge e of Γ into two edges providing an additional factor of $w_e = \deg_\pi y$ which together with the new global automorphisms cancels the contribution of $H_{y'}$.

Furthermore, vertices mapping to c yield the same contributions to both counts (17) and (18). We count covers with marked ends, so for each cover π' we have to multiply by a factor taking into account the possibilities to mark the ends. This factor times the contribution to $|\mathrm{Aut}(\pi')|$ arising from these ends equals $|\mathrm{Aut}(\mu)|$. The contribution to $|\mathrm{Aut}(\pi')|$ of the newly attached ends all cancel as discussed above. There remain only contributions from wieners, which we also have in (17). It follows that the two expressions agree up to a factor of $|\mathrm{Aut}(\mu)|$. □

As a consequence of lemma 4.3.5, we can conlude:

Theorem 4.3.6. *The degree of the tropical branch map on an appropriate moduli space equal the corresponding classical Hurwitz numbers $H_{d,g}(\mu_u, \mu_v, \mu_w)$ (see definition 1.4.3).*

Proof. This follows from lemma 4.3.5 and the correspondence theorem 2.11 in [**4**]. □

We will have to make some preparing considerations before we will be able to proof theorem 4.3.3. Actually, the main ingredient for the proof of theorem 4.3.3 is a duality between tropical resolutions of a codimension-one combinatorial type of cover (i.e. a combinatorial type that belongs to a cone of codimension one in the moduli space of tropical covers) and boundary points of a one-dimensional algebraic moduli space. We first explain the one-dimensional case in detail before deducing the consequences for the general situation.

4.4. The One-dimensional Case

Throughout this subsection, fix a degree $d > 0$, a triple of ramification profiles $\mu := (\mu_u, \mu_v, \mu_w)$ (i.e. partitions of d) and a genus $g \geq 0$ such that $\sum_{i=u,v,w} |\mu_i| + 2g - d - 2 = 1$, i.e. covers in $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu)$ have exactly one label. Then there is exactly one (combinatorial type of) cover that is not trivalent (in the sense of notation 2.3.4), namely the star-shaped cover with a vertex of genus g and 1 label over c. Obviously, $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu)$ as abstract polyhedral complex is just a star itself: a collection of one-dimensional rays adjacent to the star-shaped curve. Each ray corresponds to a possible resolution of the star-shaped curve, i.e. to a cover of $\mathcal{L}$ with one trivalent vertex mapping to one of the rays of $\mathcal{L}$. Topologically, there are three different types for such resolutions: we can either

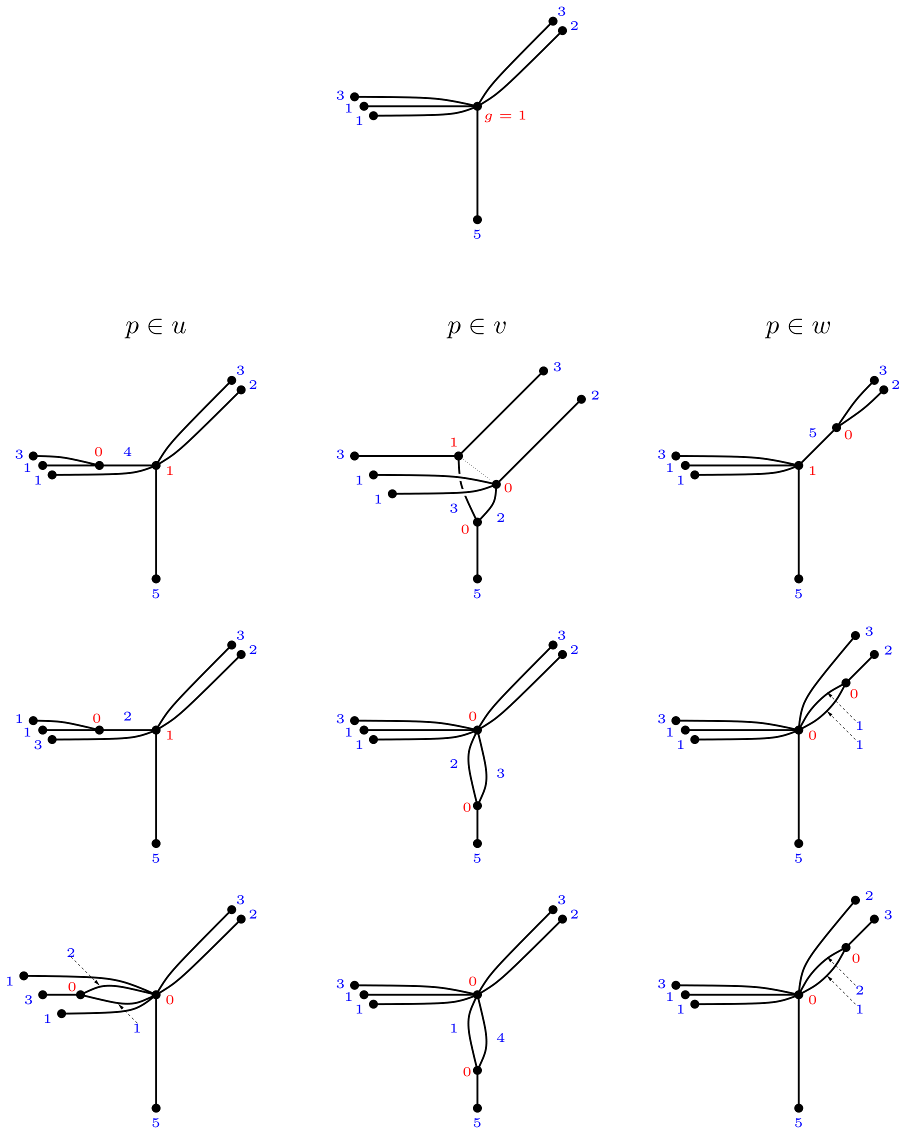


FIGURE 6. Resolutions of the star-shaped cover in $\mathrm{M}_1^{\mathrm{trop}}(\mathcal{L}, ((3,1,1),(5),(3,2)))$.

(i) join two edges (as e.g. in the top row on the left of figure 6),
(ii) split an edge while extracting genus from the vertex over c (as e.g. in the bottom row on the left of figure 6) or
(iii) split an edge and the interior vertex (as e.g. in the top row in the middle of figure 6).

EXAMPLE 4.4.1. Consider the moduli space $\mathrm{M}_1^{\mathrm{trop}}(\mathcal{L}, ((3,1,1),(5),(3,2)))$. The star-shaped combinatorial type in this space has an interior vertex of genus one. Its resolutions, i.e. the trivalent combinatorial types in this space — ordered by the position of the image p of their labeled point on the different ends of $\mathcal{L}$ — are depicted in figure 6. (As before blue numbers denote edge weights and red numbers are the genus on vertices.) In the picture, we neglect the marking of the ends as usual. This implies that e.g. the picture in the top row on the left actually combines two marked pictures, for the two possibilities to mark the two ends of weight one.

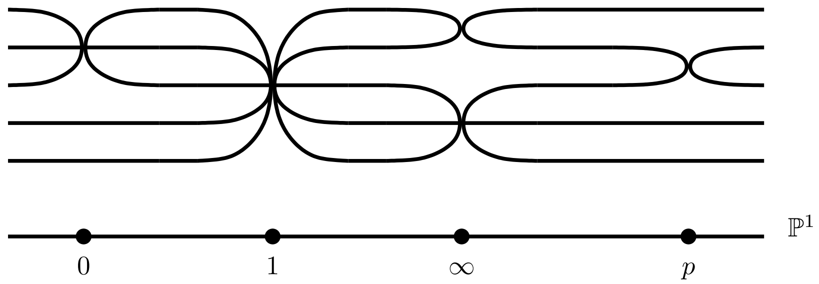


FIGURE 7. Sketch of an (classical) cover dual to the star-shaped tropical cover in figure 6.

We will consider $\overline{M}_{g,|\mu|,\mu}(\mathbb{P}^1, d)$, the space of relative stable maps (as defined in definition 1.5.7). To remember: In the situation considered in this section, the points in the interior $M_{g,|\mu|,\mu}(\mathbb{P}^1, d)$ correspond to branched covers of $\mathbb{P}^1$ with three fixed ramifications μ_u, μ_v, μ_w over 0, 1 and ∞, respectively, and one simple ramification over an arbitrary point $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. The boundary $\partial M_{g,|\mu|,\mu}(\mathbb{P}^1, d)$ correspond to the limits for t to 0, 1 or ∞. For example if t moves to 0, we consider covers of a chain of two $\mathbb{P}^1$'s satisfying a kissing condition above the node and having the μ_u-ramification and the simple ramification over the same copy and the μ_v- and μ_w-ramification over the other copy of $\mathbb{P}^1$.

The duality between boundary points of $\overline{M}_{g,|\mu|,\mu}(\mathbb{P}^1, d)$ and rays of $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu)$ goes by the dual graph construction:

DEFINITION 4.4.2. For an element of $\overline{M}_{g,|\mu|,\mu}(\mathbb{P}^1, d)$, we construct its *dual graph* as follows:

- For every component C_i of the source curve C, we draw a vertex with genus $g(C_i)$.
- For every node of component C_i and C_j we draw an edge between the vertices i and j — the weight of the edge equals the intersection multiplicity of the two components at the node.
- For every marked point on C_i we draw a marked end adjacent to the vertex i, the weight of the end equals the ramification index at the marked point.

We straighten two-valent vertices. We interpret the outcome as a combinatorial type of tropical covers of $\mathcal{L}$ by mapping the marked points over 0, 1 and ∞ to u, v and w, respectively.

Obviously, the dual graph of a cover in the interior of $\overline{M}_{g,|\mu|,\mu}(\mathbb{P}^1, d)$ is just the star-shaped cover in $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu)$.

EXAMPLE 4.4.3. Figure 7 sketches a cover corresponding to an interior point of $\overline{M}_{1,|\mu|,\mu}(\mathbb{P}^1, 5)$ where $\mu = ((3, 1, 1), (5), (3, 2))$. A cover corresponding to a boundary point in the moduli space is sketched in figure 8. We neglect markings. Figure 9 very roughly sketches all covers corresponding to boundary points of $\overline{M}_{1,|\mu|,\mu}(\mathbb{P}^1, 5)$ dual to the tropical covers in figure 6. The order is the same in both pictures. Also here, we neglect the markings of the preimages of the three special branch points, the picture on the top left actually combines two marked pictures. The top left picture represents the same cover as figure 8, the kissing condition is indicated by the broken line.

PROPOSITION 4.4.4. *The boundary points of* $\overline{M}_{g,|\mu|,\mu}(\mathbb{P}^1, d)$ *are in 1:1-correspondence with rays of* $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu)$ *via the dual graph construction. More precisely, boundary points*

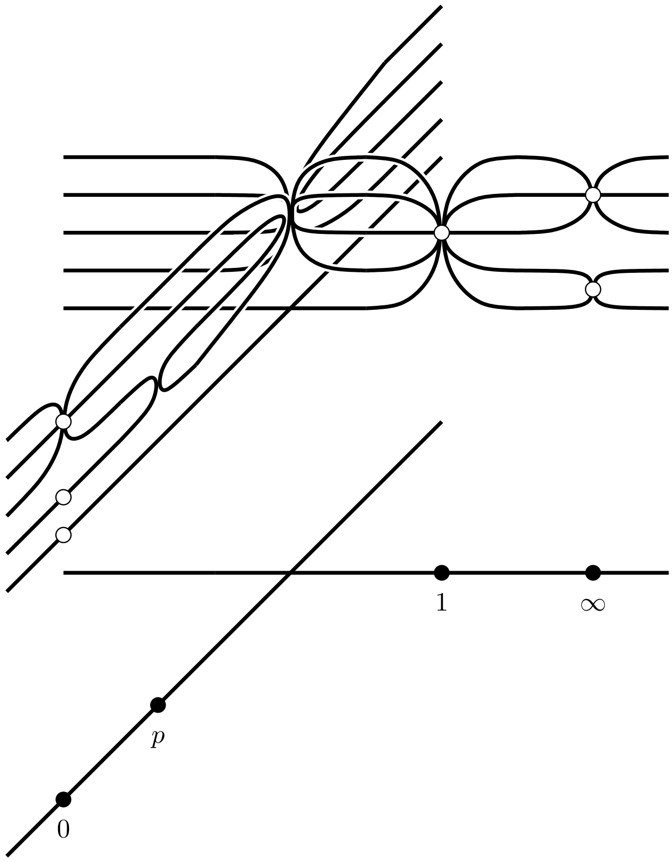


FIGURE 8. The boundary point of $\overline{M}_{1,|\mu|,\mu}(\mathbb{P}^1, 5)$ dual to the tropical cover on the left of the first row in figure 6

- *where t goes to 0 correspond to tropical covers with a trivalent vertex above the u-end,*
- *where t goes to 1 to covers with a trivalent vertex above the v-end and*
- *where t goes to ∞ to covers with a trivalent vertex above the w-end.*

PROOF. Take a point in the boundary of $\overline{M}_{g,|\mu|,\mu}(\mathbb{P}^1, d)$, say where t moved to 0. We claim that the dual graph Γ is a possible resolution of the star-shaped cover in $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu)$ with a trivalent vertex above u. As described in remark 1.5.8, such a boundary point is a cover of two copies of $\mathbb{P}^1$, one copy with ramification profiles μ_u, simple and $\tilde{\mu}$, the other with $\tilde{\mu}$, μ_v and μ_w. The possibilities for $\tilde{\mu}$ are given by the cut-and-join relations (see remark 1.5.8) . A cover with profiles μ_u, simple and $\tilde{\mu}$ contains one rational component C_1 with the simple ramification and two more ramification profiles, one totally ramified and the other in two parts. The dual vertex is a trivalent vertex of genus zero which is mapped to u. The remaining components are mapped trivially (and thus also rational), thus their dual vertex is two-valent, with one adjacent marked end and one bounded edge of the same weight connecting it to a vertex corresponding to a component cover the other copy of $\mathbb{P}^1$. We have the following possibilities:

(1) There is exactly one component cover the other copy of $\mathbb{P}^1$, and it meets C_1 in two nodes. The dual graph then is as e.g. in the bottom row on the left of figure 6.

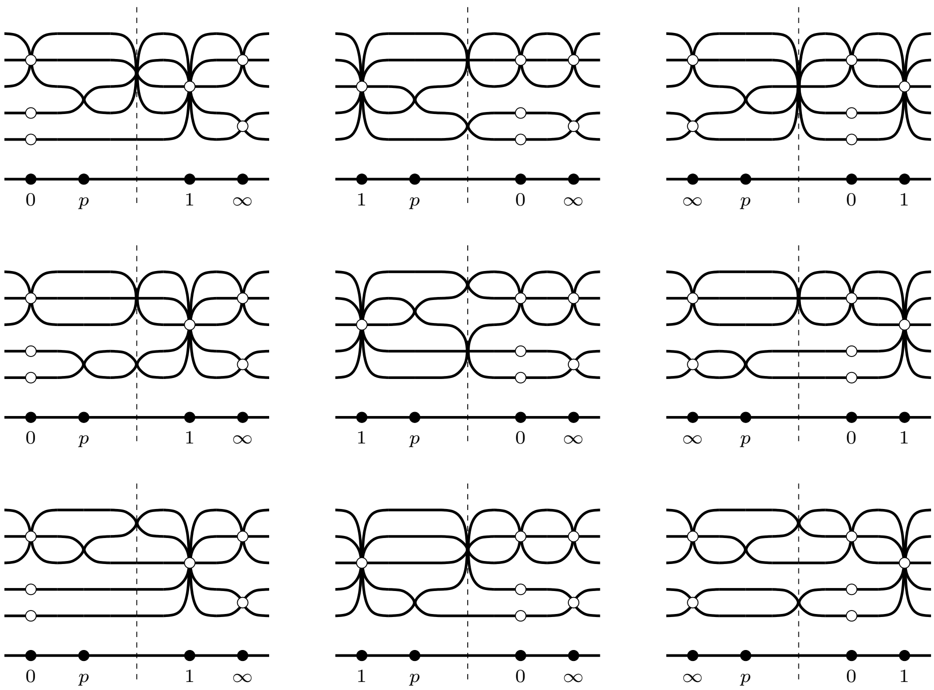


FIGURE 9. Boundary points of $\overline{M}_{1,|\mu|,\mu}(\mathbb{P}^1, 5)$ dual to the tropical resolutions in figure 6.

(2) There is exactly one component cover the other copy of $\mathbb{P}^1$, and it meets C_1 in one node. Then $\tilde{\mu}$ is obtained from μ_u by summing two parts, and consequently we have two marked points in C_1. The dual graph is as e.g. in the top row on the left of figure 6.

(3) There are two components cover the other copy of $\mathbb{P}^1$, each meeting C_1 in one node. The dual graph is as e.g. in the top row in the middle of figure 6.

Vice versa, we can obviously construct for each combinatorial type of tropical cover corresponding to a ray of $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu)$ a boundary point in $\overline{M}_{g,|\mu|,\mu}(\mathbb{P}^1, d)$ whose dual graph equals the combinatorial type. □

LEMMA 4.4.5. *The multiplicity of a boundary point of* $\overline{M}_{g,|\mu|,\mu}(\mathbb{P}^1, d)$ *in* $\mathrm{br}^*(p)$ *for* $p = 0$, 1 *or* ∞ *equals the tropical multiplicity of the combinatorial type of cover given by the dual graph.*

PROOF. We formulate the argument for $p = 0$ to keep notation simple. Since br is a branched cover, we can determine the multiplicity of a boundary point in $\mathrm{br}^*(0)$ by counting the number of covers in $\overline{M}_{g,|\mu|,\mu}(\mathbb{P}^1, d)$ with the simple ramification at t close to 0 that degenerate to the given boundary point.

We count these covers in terms of monodromy representations as in section 1.3. As every ramification point over 0, 1 and ∞ is marked, we can think of μ_u, μ_v and μ_w as marked partitions, where the marking is induced by the marks of the preimages of 0, 1 and ∞, respectively. We also consider permutations $\sigma \in \mathbb{S}_d$ together with a marking of their cycles and call this a *marked permutation*. By abuse of notation, we still denote a marked permutation by $\sigma \in \mathbb{S}_d$.

We say that a marked permutation σ is of *marked cycle type* μ_u and write $\sigma \in \mathbb{S}_d^{(\mu_u)}$ if the marked tuple of its cycle lengths agrees with the marked partition μ_u.

Following proposition 1.3.3, the Hurwitz number $H_{d,g}(\mu)$ equals

$$H_{d,g}(\mu) = \frac{1}{d!} \cdot |\{(\sigma_u, \sigma_v, \sigma_w, \tau)\}|,$$

where the tuples in the braces satisfy

- σ_u, σ_v and σ_w are marked permutations satisfying $\sigma_u \in \mathbb{S}_d^{(\mu_u)}$, $\sigma_v \in \mathbb{S}_d^{(\mu_v)}$ and $\sigma_w \in \mathbb{S}_d^{(\mu_w)}$ respectively,
- τ is an unmarked transposition in $\mathbb{S}_d$,
- $\tau \circ \sigma_u \circ \sigma_v \circ \sigma_w = \mathrm{id}_{\mathbb{S}_d}$ and
- $\langle \tau, \sigma_u, \sigma_v, \sigma_w \rangle_{\mathbb{S}_d}$ acts transitively on $\{1, \dots, d\}$.

Now consider a possible kissing condition $\tilde{\mu}$. As in remark 1.5.8, it is obtained from μ_u by either splitting one part into two or summing to parts to one. In the first case, we consider $\tilde{\mu}$ as a partially marked partition (where the two new parts are not marked). Analogously, we also consider partially marked permutations and say they are of partially marked cycle type $\tilde{\mu}$, if the partially marked partition of cycle lengths agrees with $\tilde{\mu}$. By abuse of notation, we also write $\tilde{\sigma} \in \mathbb{S}_d^{(\tilde{\mu})}$ if $\tilde{\sigma}$ is of partially marked cycle type $\tilde{\mu}$. In the following, it should always be clear from the context whether a permutation is marked, partially marked or unmarked.

Fix a boundary point in $\mathrm{br}^*(0)$ with kissing condition $\tilde{\mu}$. Remember from remark 1.5.8 that for a boundary point, the target consists of two copies of $\mathbb{P}^1$ meeting in a node. One copy is covered with ramification profiles μ_u, simple and $\tilde{\mu}$, the other by $\tilde{\mu}$, μ_v and μ_w. There is one component called C_1 above the first copy of $\mathbb{P}^1$ which contains the simple ramification.

Assume first the dual graph of the boundary point is as in case (1) of the proof of proposition 4.4.4, i.e. as e.g. in the bottom row on the left of figure 6. Then $\tilde{\mu}$ is obtained from μ_u by splitting the part m into positive intergers m_1 and m_2 with $m_1 + m_2 = m$. If we consider covers with simple ramification at t close to 0, we can count the ones which degenerate to this boundary point as follows:

$$\frac{1}{d!} \cdot \left| \left\{ (\sigma_u, \sigma_v, \sigma_w, \tau) \;\middle|\; \begin{array}{l} \bullet\ \sigma_u \in \mathbb{S}_d^{(\mu_u)}, \sigma_v \in \mathbb{S}_d^{(\mu_v)}, \sigma_w \in \mathbb{S}_d^{(\mu_w)} \\ \bullet\ \tau \text{ an unmarked transposition in } \mathbb{S}_d \\ \bullet\ \tau \circ \sigma_u \circ \sigma_v \circ \sigma_w = \mathrm{id}_{\mathbb{S}_d} \\ \bullet\ \langle \tau, \sigma_u, \sigma_v, \sigma_w \rangle_{\mathbb{S}_d} \text{ acts transitively on } \{1, \dots, d\} \\ \bullet\ \langle \tau \circ \sigma_u, \sigma_v, \sigma_w \rangle_{\mathbb{S}_d} \text{ acts transitively on } \{1, \dots, d\} \\ \bullet\ \tau \circ \sigma_u \in \mathbb{S}_d^{(\tilde{\mu})} \end{array} \right\} \right|. \tag{19}$$

The second transitivity condition reflects the fact that there is only one component above the other copy of $\mathbb{P}^1$ which meets C_1 in two nodes. Obviously the first transitivity condition is obsolete. We can order the set of tuples by the result of $\sigma_u \circ \tau$ and accordingly write the number as $\frac{1}{d!}$ times the sum over all $\tilde{\sigma} \in \mathbb{S}_d^{(\tilde{\mu})}$ of products of two factors:

$$\left| \left\{ (\sigma_v, \sigma_w) \;\middle|\; \begin{array}{l} \bullet\ \sigma_v \in \mathbb{S}_d^{(\mu_v)}, \sigma_w \in \mathbb{S}_d^{(\mu_w)} \\ \bullet\ \sigma_w \circ \sigma_v \circ \tilde{\sigma} = \mathrm{id}_{\mathbb{S}_d} \\ \bullet\ \langle \tilde{\sigma}, \sigma_v, \sigma_w \rangle_{\mathbb{S}_d} \text{ acts transitively on } \{1, \dots, d\} \end{array} \right\} \right|$$

and

$$\left|\left\{(\sigma_u,\tau)\;\middle|\;\begin{array}{l}\bullet\ \sigma_u\in\mathbb{S}_d^{(\mu_u)}\\ \bullet\ \tau \text{ an (unmarked) transposition in } \mathbb{S}_d\\ \bullet\ \tau\circ\sigma_u=\tilde\sigma\end{array}\right\}\right|.$$

For the second factor, it is easier to multiply with $\tau=\tau^{-1}$ and count the number of transpositions τ satisfying $\tau\circ\tilde\sigma\in\mathbb{S}_d^{(\mu_u)}$. The requirement is satisfied if and only if both entries of τ come from the two different cycles of $\tilde\sigma$ which are joined to one cycle. We can thus choose one entry of the m_1 entries of one cycle, and one of the m_2 entries of the other, leading to $m_1\cdot m_2$ choices. Since this holds true for any $\tilde\sigma$, we can pull this factor in front of the sum. Our number then equals

$$(m_1+m_2)\cdot\frac{1}{d!}\cdot\sum_{\tilde\sigma\in\mathbb{S}_d^{(\tilde\mu)}}\left|\left\{(\sigma_v,\sigma_w)\;\middle|\;\begin{array}{l}\bullet\ \sigma_v\in\mathbb{S}_d^{(\mu_v)},\sigma_w\in\mathbb{S}_d^{(\mu_w)}\\ \bullet\ \sigma_w\circ\sigma_v\circ\tilde\sigma=\mathrm{id}_{\mathbb{S}_d}\\ \bullet\ \langle\tilde\sigma,\sigma_v,\sigma_w\rangle_{\mathbb{S}_d}\text{ acts trans. on }\{1,\dots,d\}\end{array}\right\}\right|.$$

The sum times $\frac{1}{d!}$ equals $H_{d,g}(\tilde\mu,\mu_v,\mu_w)$ if $m_1\neq m_2$ and $\frac{1}{2}H_{d,g}(\tilde\mu,\mu_v,\mu_w)$ if $m_1=m_2$ (because if $m_1=m_2$ there are two ways to mark the two preimages with ramification index $m_1=m_2$ above the point with ramification profile $\tilde\mu$ which we count only once here since we have only a partially marked partition). Since the dual graph has a wiener if and only if $m_1=m_2$ (leading to a factor of $\frac{1}{2}$ in the tropical multiplicity), the product equals the tropical multiplicity.

Now assume that the dual graph of the boundary point is as in case (2) of the proof of proposition 4.4.4, i.e. as e.g. in the top row on the left of figure 6. There is one component cover the other copy of $\mathbb{P}^1$, and it meets C_1 in one node. Then $\tilde\mu$ is obtained from μ_u by summing two parts m_1 and m_2. Again, if we consider covers with simple ramification at t close to 0, we can count the ones which degenerate to this boundary point just as in equation (19).

We claim that if $\langle\tau,\sigma_u,\sigma_v,\sigma_w\rangle_{\mathbb{S}_d}$ acts transitively on $\{1,\dots,d\}$ then so does $\langle\tau\circ\sigma_u,\sigma_v,\sigma_w\rangle_{\mathbb{S}_d}$, hence we can drop the second transitivity condition. Assume $\langle\tau,\sigma_u,\sigma_v,\sigma_w\rangle_{\mathbb{S}_d}$ acts transitively. For arbitrary $k,l\in\{1,\dots,d\}$ we would like to have a word in $\tau\circ\sigma_u$, σ_v, σ_w and their inverses which as a permutation maps k to l. Let τ be (τ_1,τ_2). It joins two cycles c_1 and c_2 of σ_u (containing the elements τ_1 and τ_2 respectively) to a cycle c in $\tilde\sigma$ (obviously containing τ_1 and τ_2). The remaining cycles are the same in both permutations. Therefore there are $s,t\in\mathbb{N}$ such that $\tilde\sigma^s(\tau_1)=c^s(\tau_1)=\tau_2$ and $\tilde\sigma^t(\tau_2)=c^t(\tau_2)=\tau_1$. Since $\langle\tau,\sigma_u,\sigma_v,\sigma_w\rangle_{\mathbb{S}_d}$ acts transitively, we have a product $\delta_r\circ\ldots\circ\delta_1$ where each δ_i is one of the permutations τ,σ_u,σ_v and σ_w or their inverses, and which maps k to l. Let k_i be $\delta_i\circ\ldots\circ\delta_1(k)$ for $i=1,\dots,r$ and $k_0=k$. Assume $\delta_i=\tau$ and k_{i-1} is in the support of τ. If $k_{i-1}=\tau_1$ define $\delta_i'=\tilde\sigma^s$ and $\delta_i'=\tilde\sigma^t$ otherwise. Then clearly $\delta_r\circ\ldots\circ\delta_i'\circ\ldots\circ\delta_1(k)=\delta_r\circ\ldots\circ\delta_i\circ\ldots\circ\delta_1(k)$. Analogously if $\delta_i=\sigma_u$ (or σ_u^{-1}) with k_{i-1} in the support of c_1 or c_2, we can substitute δ_i with powers of $\tilde\sigma$ (or $\tilde\sigma^{-1}$). In this way we produce the desired word in the permutations in $\langle\tau\circ\sigma_u,\sigma_v,\sigma_w\rangle_{\mathbb{S}_d}$ mapping k to l.

After dropping the second transitivity condition in equation (19), we can as before write the number as a product of two factors

$$\frac{1}{d!}\cdot\sum_{\tilde\sigma\in\mathbb{S}_d^{(\tilde\mu)}}\left|\left\{(\sigma_v,\sigma_w)\;\middle|\;\begin{array}{l}\bullet\ \sigma_v\in\mathbb{S}_d^{(\mu_v)},\sigma_w\in\mathbb{S}_d^{(\mu_w)}\\ \bullet\ \tilde\sigma\circ\sigma_v\circ\sigma_w=\mathrm{id}_{\mathbb{S}_d}\\ \bullet\ \langle\tilde\sigma,\sigma_v,\sigma_w\rangle_{\mathbb{S}_d}\text{ acts transitively on }\{1,\dots,d\}\end{array}\right\}\right|$$

and

$$\left|\left\{(\sigma_u,\tau)\;\middle|\;\begin{array}{l}\bullet\ \sigma_u\in\mathbb{S}_d^{(\mu_u)}\\ \bullet\ \tau \text{ an (unmarked) transpositions in } \mathbb{S}_d\\ \bullet\ \tau\circ\sigma_u=\tilde\sigma\end{array}\right\}\right|,$$

where we can pull the second factor out of the sum because it is the same for each $\tilde\sigma$: we count transpositions τ satisfying $\tau\circ\tilde\sigma\in\mathbb{S}_d^{(\mu_u)}$. To obtain such a τ, we can pick any entry in the joined cycle in $\tilde\sigma$, and pick as second entry one which is m_1 numbers away. If $m_1\neq m_2$, we have m_1+m_2 different choices. If $m_1=m_2$, we have m_1 choices but then for each choice two options for the marking of $\tau\circ\tilde\sigma$, so altogether we get $2m_1=m_1+m_2$ also.

The first factor equals $H_{d,g}(\tilde\mu,\mu_v,\mu_w)$. The product equals the tropical multiplicity.

Finally, assume that the dual graph of the boundary point is as in case (3) of the proof of proposition 4.4.4, i.e. as e.g. in the top row in the middle of figure 6.

There are two components D_1 and D_2 of genus g_1 and g_2 cover the other copy of $\mathbb{P}^1$ with degree d_1 and d_2 respectively, each meeting C_1 in one node. Then $\tilde\mu$ is obtained from μ_u by splitting the part m into two parts m_1 and m_2. Moreover, $\tilde\mu$ is naturally divided into two partitions $\tilde\mu^{(1)}$ and $\tilde\mu^{(2)}$ of d_1 and d_2 respectively depending on whether the corresponding ramification point is in D_1 or D_2. In the same way, the partitions μ_v and μ_w are divided into $\mu_v^{(1)},\mu_v^{(2)}$ and $\mu_w^{(1)},\mu_w^{(2)}$ respectively.

If we consider covers with simple ramification at t close to 0, we can count the ones which degenerate to this boundary point as follows:

$$\frac{1}{d!}\cdot\sum_{S\subset\{1,\dots,d\},|S|=d_1}|\{(\sigma_u,\sigma_v,\sigma_w,\tau)\}|,$$

where the tuples in the braces satisfy

(C1) $\sigma_u\in\mathbb{S}_d^{(\mu_u)}$,
(d1) $\sigma_v\in\mathbb{S}_d^{(\mu_v)},\sigma_w\in\mathbb{S}_d^{(\mu_w)}$;
(C2) τ an (unmarked) transposition in $\mathbb{S}_d$;
(d2) $\tau\circ\sigma_u\circ\sigma_v\circ\sigma_w=\mathrm{id}_{\mathbb{S}_d}$;
(d3) $\langle\tau,\sigma_u,\sigma_v,\sigma_w\rangle_{\mathbb{S}_d}$ acts transitively on $\{1,\dots,d\}$;
(C3) $\tau\circ\sigma_u=\tilde\sigma^{(1)}\circ\tilde\sigma^{(2)}$ (where $\tilde\sigma^{(1)}$ and $\tilde\sigma^{(2)}$ are disjoint permutations acting on the subset $S\subset\{1,\dots,d\}$ resp. S^c satisfying $\tilde\sigma^{(1)}\in\mathbb{S}_S^{(\tilde\mu^{(1)})}$ resp. $\tilde\sigma^{(2)}\in\mathbb{S}_{S^c}^{(\tilde\mu^{(2)})}$);
(A1) there are permutations $\sigma_x^{(1)}\in\mathbb{S}_S^{(\mu_x^{(1)})}$ for $x=v,w$ and
(B1) $\sigma_x^{(2)}\in\mathbb{S}_{S^c}^{(\mu_x^{(2)})}$ for $x=v,w$ satisfying
(d4) $\sigma_x^{(1)}\circ\sigma_x^{(2)}=\sigma_x$ for $x=v,w$;
(A2) $\tilde\sigma^{(1)}\circ\sigma_v^{(1)}\circ\sigma_w^{(1)}=\mathrm{id}_{\mathbb{S}_S}$;
(B2) $\tilde\sigma^{(2)}\circ\sigma_v^{(2)}\circ\sigma_w^{(2)}=\mathrm{id}_{\mathbb{S}_{S^c}}$;
(A3) $\langle\tilde\sigma^{(1)},\sigma_v^{(1)},\sigma_w^{(1)}\rangle_{\mathbb{S}_S}$ acts transitively on S;
(B3) $\langle\tilde\sigma^{(2)},\sigma_v^{(2)},\sigma_w^{(2)}\rangle_{\mathbb{S}_{S^c}}$ acts transitively on S^c.

Due to (C3) and (C1), τ must have one entry in S and one in S^c, so the transitivity condition (d3) is implied by (A3) and (B3). Moreover, (B2) and (A2) imply (d2). Below, we count the possibilities for tuples $(\sigma_v^{(i)},\sigma_w^{(i)})$ for $i=1,2$. The permutation σ_x is then by (d4) given as the product of the two entries and we can thus neglect it and condition (d1) which is implied by (A1) and (B1). Finally, ordering the tuples by the different possibilities for $\tilde\sigma^{(1)}$ and $\tilde\sigma^{(2)}$ we can write the above number as

$$\frac{1}{d!}\cdot\sum_{S\subset\{1,\ldots,d\},|S|=d_1}\ \sum_{\tilde{\sigma}^{(1)}\in\mathbb{S}_S^{(\tilde{\mu}^{(1)})}}\ \sum_{\tilde{\sigma}^{(2)}\in\mathbb{S}_{S^c}^{(\tilde{\mu}^{(2)})}}\left|\left\{(\sigma_v^{(1)},\sigma_w^{(1)})\mid(\mathrm{A})\right\}\right|\cdot$$
$$\left|\left\{(\sigma_v^{(2)},\sigma_w^{(2)})\mid(\mathrm{B})\right\}\right|\cdot\left|\{(\sigma_u,\tau)\mid(\mathrm{C})\}\right|,$$

where a capital letter stands for the three conditions labeled accordingly.

The last factor in each summand equals $m_1\cdot m_2$ for all choices of the $\tilde{\sigma}^{(i)}$ by the same argument as in the first case. Instead of summing over all subsets of $\mathbb{S}_d$ of size d_1 we can fix without restriction $S=\{1,\ldots,d_1\}$ and multiply by $\binom{d}{d_1}$. Furthermore, the two factors in each summand each depend on only one summation index, so we can sort the sums accordingly. Notice that

$$\sum_{\tilde{\sigma}^{(1)}\in\mathbb{S}_S^{(\tilde{\mu}^{(1)})}}\left|\left\{(\sigma_v^{(1)},\sigma_w^{(1)})|(\mathrm{A})\right\}\right|=d_1!\cdot H_{d_1,g_1}(\tilde{\mu}^{(1)},\mu_v^{(1)},\mu_w^{(1)})$$

and

$$\sum_{\tilde{\sigma}^{(2)}\in\mathbb{S}_{S^c}^{(\tilde{\mu}^{(2)})}}\left|\left\{(\sigma_v^{(2)},\sigma_w^{(2)})\,|(\mathrm{B})\right\})\right|=d_2!\cdot H_{d_2,g_2}(\tilde{\mu}^{(2)},\mu_v^{(2)},\mu_w^{(2)}),$$

so we get

$$\begin{aligned}&m_1\cdot m_2\cdot\frac{1}{d!}\cdot\binom{d}{d_1}d_1!\cdot H_{d_1,g_1}(\tilde{\mu}^{(1)},\mu_v^{(1)},\mu_w^{(1)})\cdot d_2!\cdot H_{d_2,g_2}(\tilde{\mu}^{(2)},\mu_v^{(2)},\mu_w^{(2)})\\ =\;&m_1\cdot m_2\cdot H_{d_1,g_1}(\tilde{\mu}^{(1)},\mu_v^{(1)},\mu_w^{(1)})\cdot H_{d_2,g_2}(\tilde{\mu}^{(2)},\mu_v^{(2)},\mu_w^{(2)}).\end{aligned}$$

This equals the tropical multiplicity. □

Corollary 4.4.6. *The degree of the tropical branch map* $\mathrm{br}^{trop}:\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L},\mu)\to\mathcal{L}$ *from a* one-*dimensional space* $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L},\mu)$ *(i.e.* $2g-2-d+|\mu|=1$*) is constant. In particular, if we consider all resolutions of the star-shaped cover and group their multiplicities into three sums corresponding to the three ends of* $\mathcal{L}$ *to which the trivalent vertex can be mapped, the three sums agree.*

Proof. This follows from lemma 4.4.5 and proposition 1.5.10. □

Example 4.4.7. If we add the tropical multiplicities for each column in figure 6, we get

$$\begin{aligned}&2\cdot4\cdot H_{5,1}((4,1),(5),(3,2))+2\cdot H_{5,1}((2,3),(5),(2,3))\\&+2\cdot H_{5,0}((1,1,1,2),(5),(2,3))\\=&2\cdot4\cdot2+2\cdot1+2\cdot6=30\end{aligned}$$

for the left column (the first factor 2 comes from the fact that the upper left figure stands for two different types of cover due to the different possibilities to mark the weight-1-edges over u). This is the sum of tropical multiplicities of resolutions where the trivalent vertex is mapped to u. In the same way, we get

$$\begin{aligned}&2\cdot3\cdot H_{2,0}((1,1),(2),(2))\cdot H_{3,1}((3),(3),(3))\\&+2\cdot3\cdot H_{5,0}((3,1,1),(2,3),(2,3))+4\cdot H_{5,0}((3,1,1),(4,1),(2,3))\\=&(2\cdot3)\cdot(1\cdot\frac{1}{3})+(2\cdot3)\cdot2+4\cdot4=30\end{aligned}$$

for the middle column corresponding to resolutions where the trivalent vertex is mapped to v and

$$\begin{aligned}&5 \cdot H_{5,1}((3,1,1),(5),(5)) + \frac{1}{2} \cdot H_{5,0}((3,1,1),(5),(3,1,1))\\&+ 2 \cdot H_{5,0}((3,1,1),(5),(2,2,1))\\=&5 \cdot 4 + \frac{1}{2} \cdot 4 + 2 \cdot 4 = 30\end{aligned}$$

for the right column corresponding to resolutions where the trivalent vertex is mapped to w.

4.5. The Proof of Theorem 4.3.3

Using the duality in the one-dimensional case, we can now prove theorem 4.3.3. So assume we are given μ, d and g such that $2g - 2 - d + |\mu| \geq 0$.

PROOF OF THEOREM 4.3.3. We refine the fan $\mathcal{L}^r$ by adding the diagonals D_{ij} defined as $\{(p_1, \ldots, p_r) \mid p_k \neq c \ \forall k = 1, \ldots, r, \ p_i = p_j\}$ for $i \neq j$ as codimension-1-faces, where c as before denotes the center of the line $\mathcal{L}$. Let us call the new fan by abuse of notation $\mathcal{L}^r$ as well. The point configurations in the interior of top-dimensional faces of $\mathcal{L}^r$ are in general position. The degree of br^{trop} is constant on any top-dimensional face, since the preimages of two different point configurations in the same face contain the same combinatorial types.

As $\mathcal{L}^r$ is connected in codimension 1 it is sufficient to see that the degree of br^{trop} does not change if we cross a codimension-1-face in $\mathcal{L}^r$.

Let us first assume that we cross a diagonal, that is beginning from a point configuration P in general position two branch points on one of the ends of $\mathcal{L}$ change their positions. We call the new point configuration P'. One can see easily that we have exactly the same combinatorial types of curves in the preimages of P and P', they just differ by their vertex labelings (see also [**10**], Lemma 5.27). Thus the degree of br^{trop} is constant when crossing this diagonal.

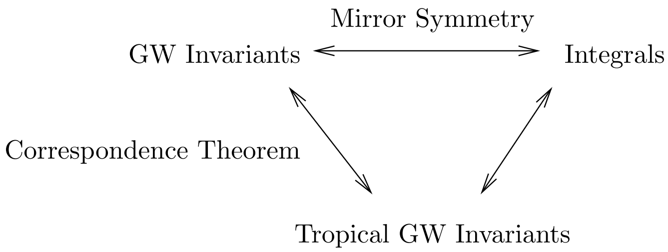
Now let us fix a point configuration P on a codimension-1-face in $\mathcal{L}^r$ which is not a diagonal, that is a point configuration where exactly one point is the center c of $\mathcal{L}$. The combinatorial types of the preimages with respect to br^{trop} have exactly one simple ramification over the center and all other simple ramifications over the ends. For a fixed type α the top-dimensional cones adjacent to D_α in $\mathrm{M}_{\mathrm{g}}^{\mathrm{trop}}(\mathcal{L}, \mu)$ correspond to the resolutions of the simple ramification over the center as described in section 4.4. We can thus interpret their contribution to the degree of br^{trop} as a product of a local factor corresponding to the one-dimensional resolution and factors from the remaining parts of the cover, which are the same in any case. Since by corollary 4.4.6 the local factors add to a contribution which does not depend on the end of $\mathcal{L}$ above which we pull the simple ramification, the degree of br^{trop} is constant locally around P. □

CHAPTER 5

Mirror Symmetry

Mirror symmetry is a deep symmetry relation motivated by dualities in string theory. Many results and conjectures of different flavors are related to mirror symmetry (see e.g. [**14, 15, 19, 24, 8, 39, 40**]). Here, we focus on statements relating Gromov-Witten invariants of a variety X to certain integrals on a mirror partner $X^\vee$.

Tropical geometry has proved to be an interesting new tool for Mirror Symmetry (see e.g. [**1, 18, 19, 20, 8**]). In this chapter we study the tropical Mirror Symmetry theorem for elliptic curves. This study can be viewed as a sequel and extension of Gross' paper [**18**], where he provides tropical methods for the study of Mirror Symmetry of $\mathbb{P}^2$. The main purpose of his paper is of a philosophical nature: he suggests tropical geometry as a new and worthwhile method for the study of mirror symmetry. More precisely, you can find (a version of) the following triangle in the introduction of [**18**]:



The relation between Gromov-Witten invariants and integrals (the top arrow) is a consequence of Mirror Symmetry. Tropical geometry comes in naturally, because there are many instances of correspondence theorems that relate Gromov-Witten invariants with their tropical analogues (the first of these is due to Mikhalkin [**32**]). The connection between tropical geometry and integrals is in general yet to be understood.

Gross studies the triangle in the situation of $\mathbb{P}^2$: here, the mirror is $(\mathbb{C}^*)^2$. For this case, statements relating Gromov-Witten invariants to integrals are known already [**3**]; but again, the purpose is to a lesser extent to give a new proof of this Mirror Symmetry relation, but to outline a new path for future progress in Mirror Symmetry. In the case of $\mathbb{P}^2$, the Mirror Symmetry relation involves descendant Gromov-Witten invariants of $\mathbb{P}^2$. Only a partial correspondence theorem is proved to relate some of these invariants to their tropical counterparts [**31**]. Gross concentrates on proving the right arrow in his situation, i.e., he provides a natural connection between integrals and tropical Gromov-Witten invariants. This connection very roughly relates monomials in a big generating function that yield a nonzero contribution to an integral with pieces of tropical curves that glue to one big tropical curve satisfying the requirements. The heart of the argument is thus a purely combinatorial hunt of monomials respectively pieces of tropical curves, and the fact that both sides can be boiled down to combinatorics that fits together very naturally strongly recommends the tropical approach for future experiments in Mirror Symmetry. However, since the existing correspondence theorems are not sufficient yet to cover the whole situation needed for Mirror Symmetry of $\mathbb{P}^2$, this exciting new approach does not give an alternative proof of the Mirror Symmetry statement for $\mathbb{P}^2$ (the top arrow) yet.

Here we demonstrate that tropical geometry can actually lead to a complete alternative proof of Mirror Symmetry, and that this approach is very natural and requires not much more than a careful analysis of the underlying combinatorics. We prove a tropical Mirror Symmetry theorem for tropical elliptic curves that in particular implies Mirror Symmetry for elliptic curves.

We study the triangle above for elliptic curves. As in the case of $\mathbb{P}^2$, Mirror Symmetry of elliptic curves is known and can to the best of our knowledge even be considered as folklore in Mirror Symmetry [**11**], i.e., in principle the top arrow is taken care of already (see theorem 5.1.5). The known proof is inspired by physics and uses quantum field theory.

We provide the alternative route via the left and right arrows. The Gromov-Witten invariants involved in the upper left vertex of the triangle are nothing but Hurwitz numbers — numbers of covers of an elliptic curve having simple ramification above some fixed branch points. The integrals in the upper right vertex are certain integrals over Feynman graphs.

As described in chapter 2 correspondences between Hurwitz numbers and their tropical counterparts have been studied. In chapter 3 we provide such a theorem concretely for the case of elliptic curves. As in the case of $\mathbb{P}^2$, most exciting is the right arrow. It turns out that a more general formulation of Mirror Symmetry is more natural on the tropical side. We prove a tropical Mirror Symmetry theorem (Theorem 5.2.6) relating numbers of labeled tropical covers with refined Feynman integrals. A careful analysis of the combinatorial principles underlying the count of labeled tropical covers on the one hand and nonzero contributions to refined integrals over Feynman graphs on the other hand reveals that they can be related very naturally. The right arrow then follows easily from our tropical mirror symmetry theorem, see theorem 5.1.7.

To sum up, for the case of elliptic curves, we complete the picture of the triangle in figure of the introduction of this thesis. We provide proofs for all solid arrows, in particular this implies the dashed arrow, the Mirror Symmetry statement for elliptic curves. We thus provide a complete and very natural combinatorial proof of Mirror Symmetry for elliptic curves by means of tropical geometry.

While our method of proof may at the first glance seem similar to the method used in [**18**] — a combinatorial hunt of monomials in a big generating function on one side, and tropical covers on the other side — the details are very different in the situation of an elliptic curve. Also, our tropical mirror symmetry statement provides more than the proof of the right arrow — we generalize the statement to labeled tropical covers and refined integrals. We were inspired by [**18**], but nevertheless our result is beyond a mere generalization of this paper, and hopefully will shed more light on other more adventurous situations of Mirror Symmetry. Note that our case is the first instance where tropical Mirror Symmetry is understood for a compact Calabi-Yau variety, and for arbitrary genus Gromov-Witten invariants.

As a side product we also give a combinatorial characterization of graphs whose corresponding Feynman integral is zero: we prove in corollary 5.4.10 that a graph yields a zero Feynman integral if and only if it contains a bridge.

5.1. Mirror Symmetry for Elliptic Curves

In this section, we define the relevant invariants (i.e. Hurwitz numbers and Feynman integrals) and present a precise statement of the top arrow of the triangle in the introduction. In particular we consider covers of elliptic curves. We first fix some notations.

Remember, that Hurwitz numbers are naturally topological invariants, in particular they do not depend on the position of the branch points as long as these are pairwise different. Moreover, since all complex elliptic curves are homeomorphic to the real torus, numbers of covers of an elliptic curve do not depend on the choice of the base curve. We thus fix an arbitrary complex elliptic curve $\mathcal{E}$ in this chapter. Let $\mathcal{C}$ be a non-singular curve of genus g and $\varphi : \mathcal{C} \to \mathcal{E}$ a cover of degree d. For our purpose, it is sufficient to consider covers which are *simply ramified*. It follows from the Riemann-Hurwitz formula (see 1.2.10) that a simply ramified cover of $\mathcal{E}$ has exactly $2g-2$ branch points.

Remember that for the sake of simplicity we will denote $H_{d,g} := H_{d,g}(\mathcal{E})$, see section 3.2

REMARK 5.1.1. Rebember that, by convention, we mark the branch points p_i in this definition, leading to a factor of $(2g-2)!$ when compared with other definitions.

DEFINITION 5.1.2. We package the Hurwitz numbers $H_{d,g}$ into a generating function as follows:

$$F_g(q) := \sum_{d=1}^{\infty} H_{d,g} q^{2d}.$$

The Mirror Symmetry statement relates the generating function $F_g(q)$ to certain integrals which we are going to define now. We start by defining the function which we are going to integrate.

DEFINITION 5.1.3 (The propagator). We define the *propagator*

$$P(z,q) := \frac{1}{4\pi^2}\wp(z,q) + \frac{1}{12}E_2(q^2)$$

in terms of the *Weierstraß-P-function* $\wp$ and the *Eisenstein series*

$$E_2(q) := 1 - 24\sum_{d=1}^{\infty} \sigma(d) q^d.$$

Here, $\sigma = \sigma_1$ denotes the sum-of-divisors function $\sigma(d) = \sigma_1(d) = \sum_{m|d} m$.

The variable q above should be considered as a coordinate of the moduli space of elliptic curves, the variable z is the complex coordinate of a fixed elliptic curve.

DEFINITION 5.1.4 (Feynman graphs and integrals). A *Feynman graph* Γ of genus g is a trivalent connected graph of genus g. For a Feynman graph, we throughout fix a reference labeling $x_1, \ldots, x_{2g-2}$ of the $2g-2$ trivalent vertices and a reference labeling $q_1, \ldots, q_{3g-3}$ of the edges of Γ.

For an edge q_k of Γ connecting the vertices x_i and x_j, we define a function

$$P_k := P(z_i - z_j, q),$$

where P denotes the propagator of definition 5.1.3 (the choice of sign i.e. $z_i - z_j$ or $z_j - z_i$ plays no role, more about this in section 5.3). Pick a total ordering Ω of the vertices and starting points of the form $iy_1, \ldots, iy_{2g-2}$ in the complex plane, where the y_j are pairwise different small numbers. We define integration paths $\gamma_1, \ldots, \gamma_{2g-2}$ by

$$\gamma_j : [0,1] \to \mathbb{C} : t \mapsto iy_j + t,$$

such that the order of the real coordinates y_j of the starting points of the paths equals Ω. We then define the integral

$$I_{\Gamma,\Omega}(q) := \int_{z_j \in \gamma_j} \prod_{k=1}^{3g-3} (-P_k). \tag{20}$$

Finally, we define

$$I_\Gamma(q) = \sum_\Omega I_{\Gamma,\Omega}(q),$$

where the sum runs over all $(2g-2)!$ orders of the vertices.

The following is the precise statement of the top arrow in the triangle in the introduction (see Theorem 3 of [**11**]):

THEOREM 5.1.5 (Mirror Symmetry for elliptic curves). *Let $g > 1$. For the definition of the invariants, see definitions 1.4.3 and 5.1.4. We have*

$$F_g(q) = \sum_{d=1}^{\infty} H_{d,g} q^{2d} = \sum_\Gamma I_\Gamma(q) \cdot \frac{1}{|\operatorname{Aut}(\Gamma)|},$$

where $\operatorname{Aut}(\Gamma)$ *denotes the automorphism group of* Γ *and the sum goes over all trivalent graphs* Γ *of genus* g.

5.1.1. Tropical Covers of Elliptic Curves and their Hurwitz Numbers. We give a proof of theorem 5.1.5 by a detour to tropical geometry and tropical Mirror Symmetry. Let us briefly review the objects we consider, that are simply ramified tropical covers of an elliptic curve.

Since we do not want to fix any complicated ramification profiles, the number of ends of the source curve has to be zero. More precisely, when counting genus-g-covers of an elliptic curve of any degree, we have to fix a tropical genus-1-curve without ends as source curve E, which necessarily is a 2-valent graph. We require E to have $2g-2$ vertices $p_1, \ldots, p_{2g-2}$. Moreover, to fix notation, in the following we denote by C a tropical curve of genus g and combinatorial type Γ.

EXAMPLE 5.1.6. Figure 1 shows a tropical cover of degree 4 with a genus 2 source curve. The red numbers close to the vertex P are the weights of the corresponding edges, the black numbers denote the lengths. The cover is balanced at P since there is an edge of weight 3 leaving in one direction and an edge of weight 2 plus an edge of weight 1 leaving in the opposite direction.

We can see that the length of an edge of C is determined by its weight and the length of its image. We will therefore in the following not specify edge lengths anymore.

Using the correspondence theorem 3.2.3, it is obviously sufficient to prove the following theorem in order to obtain a tropical proof of theorem 5.1.5. This theorem can be viewed as the right arrow in the triangle in the introduction.

THEOREM 5.1.7. *Let $g > 1$. For the definition of the invariants, see definitions 3.2.2 and 5.1.4. We have*

$$\sum_{d=1}^{\infty} H_{d,g}^{trop} q^{2d} = \sum_\Gamma I_\Gamma(q) \cdot \frac{1}{|\operatorname{Aut}(\Gamma)|},$$

where the sum goes over all trivalent graphs Γ *of genus* g.

5.2. Labeled Tropical Covers

To deduce theorem 5.1.7, we prove a more general statement that implies this theorem, namely our tropical Mirror Symmetry theorem for elliptic curves 5.2.6. To state the result, we need to introduce tropical covers with additional labeling (additional to to the labeling of the inner vertices as descirbed in defintion 2.2.10) and a refined version of the Feynman integrals from above. This more general tropical Mirror Symmetry theorem is more natural on the tropical side, since the combinatorics involved in the hunt of monomials contributing to the refined Feynman integrals resp. in counting labeled tropical covers can be related very naturally.

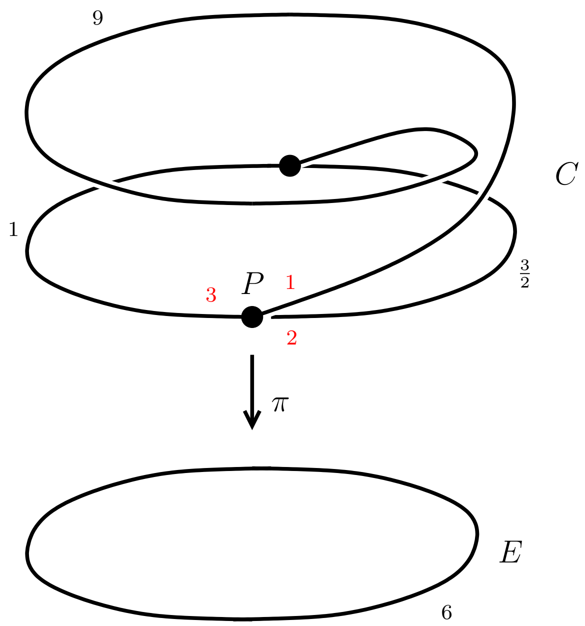


FIGURE 1. A tropical cover of E of degree 4 with genus 2 source curve.

DEFINITION 5.2.1 (Tropical Covers with additional Labeling). Let $\pi : C \to E$ be a tropical cover as in definition 2.2.5. Let Γ be the combinatorial type of the tropical curve C. We fix not only a labeling $x_1, \ldots, x_{2g-2}$ of the vertices, but also a reference labeling $q_1, \ldots, q_{3g-3}$ of the edges of Γ, as in definition 5.1.4 for Feynman graphs. We then consider the *labeled tropical cover* $\hat{\pi} : C \to E$, where the source curve C is in addition equipped with the labeling. The important difference between tropical covers and labeled tropical covers is the definition of isomorphism: for a labeled tropical cover, we require an isomorphism to respect the labels. As usual, we consider labeled tropical covers only up to isomorphism.

The combinatorial type of a labeled tropical cover is the combinatorial type of the source curve together with the labels, i.e. a Feynman graph.

EXAMPLE 5.2.2. Figure 2 shows a labeled cover of degree 4. The edges labeled q_2, q_3 and q_6 are supposed to have weight 1, edge q_1 and q_4 weight 2 and q_5 weight 3. The underlying Feynman graph is the one of figure 3.

The definition of the generating series $F_g(q)$ for Hurwitz numbers has to be refined accordingly:

DEFINITION 5.2.3. We fix once and for all a base point p_0 in E. For a tuple $\underline{a} = (a_1, \ldots, a_{3g-3})$, we define $H^{trop}_{\underline{a},g}$ to be the weighted number of labeled tropical covers $\hat{\pi} : C \to E$ of degree $\sum_{i=1}^{3g-3} a_i$, where C has genus g, having their branch points at the prescribed positions and satisfying

$$|\hat{\pi}^{-1}(p_0) \cap q_i| \cdot w_i = a_i$$

for all $i = 1, \ldots, 3g-3$. Each labeled tropical cover is counted with multiplicity $\prod_{i=1}^{3g-3} w_i$. Here, w_i denotes the weight of the edge q_i. We call $\underline{a}$ the *branch type* of the tropical cover at p_0.

We also define for a Feynman graph Γ the number $H^{trop}_{\underline{a},\Gamma}$ to be the weighted number of labeled tropical covers as above with source curve of type Γ.

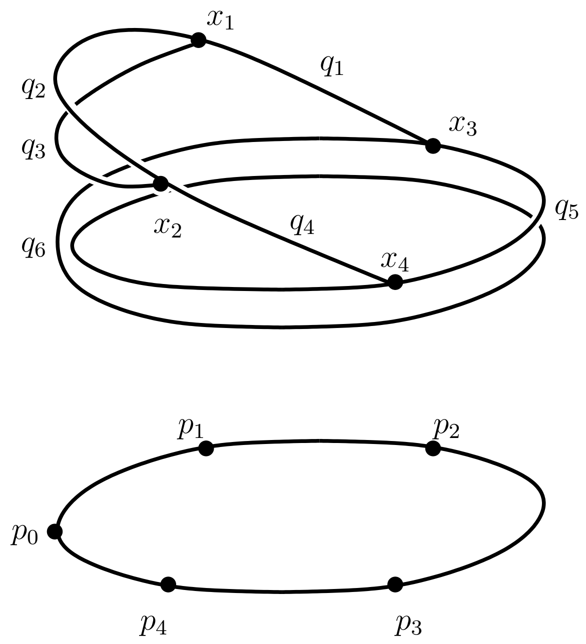


FIGURE 2. A labeled tropical cover of E.

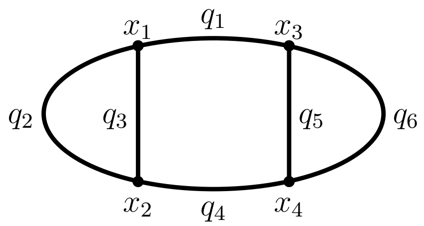


FIGURE 3. The Feynman graph Γ.

We set

$$F_g(q_1, \ldots, q_{3g-3}) = \sum_{\underline{a}} H^{trop}_{\underline{a},g} q^{2\cdot \underline{a}}.$$

Here, the sum goes over all $\underline{a} \in \mathbb{N}^{3g-3}$ and $q^{2\cdot\underline{a}}$ denotes the multi-index power $q^{2\cdot\underline{a}} = q_1^{2a_1} \cdot \ldots \cdot q_{3g-3}^{2a_{3g-3}}$.

EXAMPLE 5.2.4. Choose for example $g = 3$ and $\underline{a} = (0, 2, 1, 0, 0, 1)$. Let Γ be the Feynman graph depicted in figure 3. Then $H^{trop}_{\underline{a},\Gamma} = 256$. All labeled covers contributing to $H^{trop}_{\underline{a},\Gamma}$ together with their multiplicities are depicted in figure 4. The number next to each label q_i stands for the weight of the labeled edge. The white dots are the points in the fiber of p_0 under π.

Similarly, we refine the definition of the integrals of definition 5.1.4:

DEFINITION 5.2.5. Let Γ be a Feynman graph. As usual, the vertices are labeled with x_i and the edges with q_i. For the edge q_k of Γ connecting the vertices x_i and x_j, we change the definition of the integrand to

$$P_k := P(z_i - z_j, q_k),$$

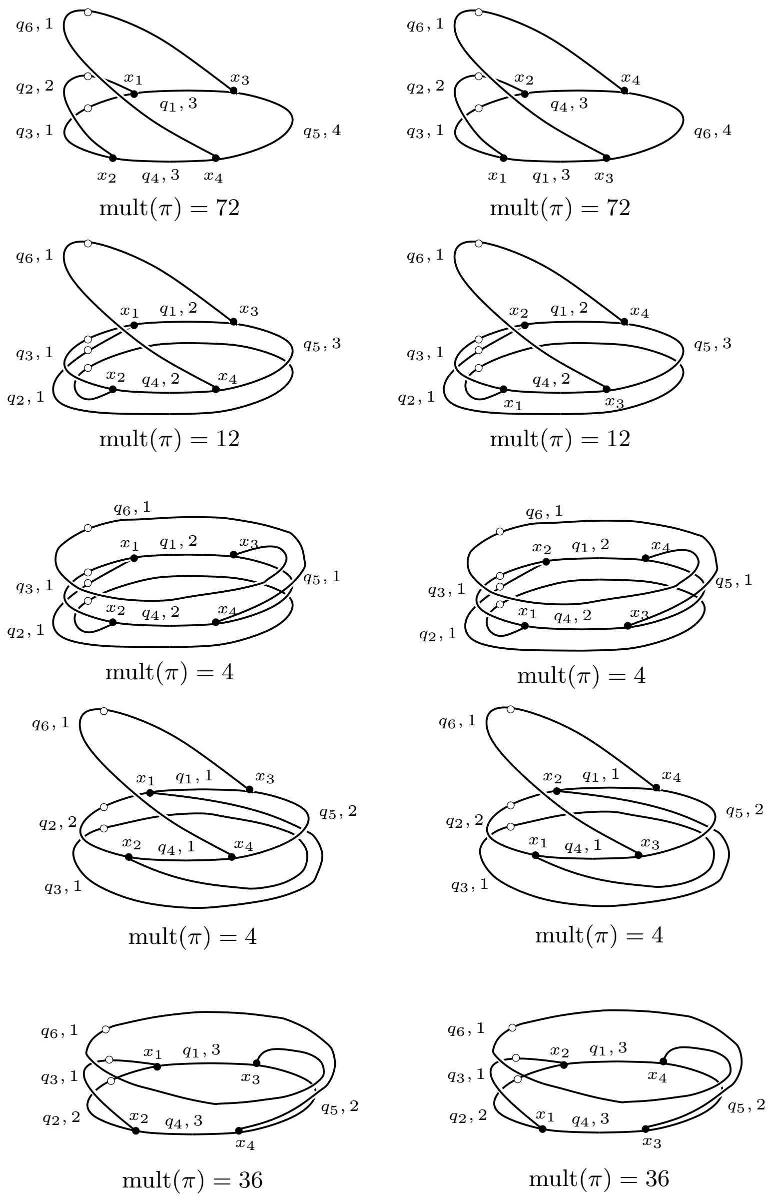


FIGURE 4. All labeled tropical covers contributing to the tropical Hurwitz number $H^{trop}_{(0,2,1,0,0,1),\Gamma} = 256$.

where P denotes the propagator of definition 5.1.3. For a total ordering Ω of the vertices we then define the integral

$$I_{\Gamma,\Omega}(q_1,\ldots,q_{3g-3}) := \int_{z_j\in\gamma_j} \prod_{1}^{3g-3} (-P_k)$$

just as in definition 5.1.4.

We also set

$$I_\Gamma(q_1,\ldots,q_{3g-3}) = \sum_{\Omega} I_{\Gamma,\Omega}(q_1,\ldots,q_{3g-3}),$$

where the sum goes over all $(2g-2)!$ orders of the vertices.

We can now present the main result of this chapter, the tropical Mirror Symmetry theorem for elliptic curves in its refined version:

THEOREM 5.2.6 (Tropical Mirror Symmetry for elliptic curves). *Let $g > 1$. For the definition of the invariants, see definitions 5.2.1 and 5.2.5. We have*

$$F_g(q_1, \ldots, q_{3g-3}) = \sum_{\underline{a}} H^{trop}_{\underline{a},g} q^{2\cdot\underline{a}} = \sum_{\Gamma} I_\Gamma(q_1, \ldots, q_{3g-3}).$$

More precisely, the coefficient of the monomial $q^{2\cdot\underline{a}}$ in $I_\Gamma(q_1, \ldots, q_{3g-3})$ equals $H^{trop}_{\underline{a},\Gamma}$.

The proof or theorem 5.2.6 follows immediatly from Theorem 5.4.5.

Note that the ropical Mirror Symmetry theorem naturally gives an interpretation of the Hurwitz number generating function $F_g(q_1, \ldots, q_{3g-3})$ in terms of a sum over Feynman graphs — we can write it as

$$F_g(q_1, \ldots, q_{3g-3}) = \sum_{\Gamma} \left(\sum_{\underline{a}} H^{trop}_{\underline{a},\Gamma} q^{2\cdot\underline{a}} \right),$$

and theorem 5.2.6 implies that the equality to the Feynman integral holds on the level of the summands for each graph. The same is true of course after setting $q_k = q$ for all k, thus going back to (unlabeled) tropical covers and (unrefined) Feynman integrals. This is particularly interesting since all statements that hold on the level of graphs (such as the quasimodularity shown in [**6**, section 3] now become meaningful on the A-model side (i.e. for the generating function of Hurwitz numbers.)

We now show how one can deduce theorem 5.1.7 from this more refined version:

PROOF OF THEOREM 5.1.7 USING THEOREM 5.2.6. For a fixed graph Γ, let $H^{trop}_{d,\Gamma}$ be the number of (unlabeled) tropical covers of degree d as in definition 3.2.2, where the combinatorial type of the source curve is Γ. As in definition 2.3.2, each cover $\pi : C \to E$ is counted with multiplicity $\frac{1}{|\operatorname{Aut}(\pi)|} \prod_e w_e$, where the product goes over all edges e of Γ and w_e denotes the weight of the edge e. As usual for Feynman graphs, we fix a reference labeling x_i of the vertices and q_i of the edges (see definition 5.1.4). There exists a forgetful map ft from the set of labeled tropical covers satisfying the ramification conditions to the set of unlabeled covers by just forgetting the labels. We would like to study the cardinality of the fibers of ft. Let $\pi : C \to E$ be an (unlabeled) cover such that the combinatorial type of C is Γ. The automorphism group of Γ, $\operatorname{Aut}(\Gamma)$, acts transitively on the fiber $\operatorname{ft}^{-1}(\pi)$ in the obvious way. So, to determine the cardinality of the set $\operatorname{ft}^{-1}(\pi)$, we think of it as the orbit under this action and obtain $|\operatorname{ft}^{-1}(\pi)| = \frac{|\operatorname{Aut}(\Gamma)|}{|\operatorname{Aut}(\pi)|}$, since the stabilizer of the action equals the set of automorphisms of π. Each labeled cover in the set $\operatorname{ft}^{-1}(\pi)$ is counted with the same multiplicity $\prod_e w_e$, where the product goes over all edges e in Γ.

Thus we obtain

$$\begin{aligned} \sum_{\underline{a} \mid \sum a_i = d} H^{trop}_{\underline{a},\Gamma} &= \sum_{\hat{\pi}: C \to E} \prod_e w_e = \sum_{\pi: C \to E} \quad \sum_{\hat{\pi}: C \to E \mid \operatorname{ft}(\hat{\pi}) = \pi} \prod_e w_e \\ &= \sum_{\pi: C \to E} \frac{|\operatorname{Aut}(\Gamma)|}{|\operatorname{Aut}(\pi)|} \prod_e w_e = |\operatorname{Aut}(\Gamma)| \cdot H^{trop}_{d,\Gamma}, \end{aligned}$$

where the second sum goes over all labeled covers $\hat{\pi} : C \to E$ of degree d and genus g satisfying the conditions and such that the combinatorial type of C is Γ, the third sum goes analogously over all (unlabeled) covers $\pi : C \to E$ and over the labeled covers in the fiber of the forgetful map, the third equality holds true because of the orbit argument we just gave and the last equality since an (unlabeled) cover is counted with multiplicity $\frac{1}{|\operatorname{Aut}(\pi)|}\prod_e w_e$.

We conclude

$$\sum_d H_{d,g}^{trop} q^{2d} = \sum_d \sum_\Gamma H_{d,\Gamma}^{trop} q^{2d} = \sum_d \sum_\Gamma \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_{\underline{a} | \sum a_i = d} H_{\underline{a},\Gamma}^{trop} q^{2d}$$
$$= \sum_\Gamma \frac{1}{|\operatorname{Aut}(\Gamma)|} \sum_d \sum_{\underline{a} | \sum a_i = d} H_{\underline{a},\Gamma}^{trop} q^{2d}.$$

Now we can replace $H_{\underline{a},\Gamma}^{trop}$ by the coefficient of $q^{2\cdot\underline{a}}$ in $I_\Gamma(q_1, \ldots, q_{3g-3})$ by theorem 5.2.6. If we insert $q_k = q$ for all $k = 1, \ldots, 3g-3$ in $I_\Gamma(q_1, \ldots, q_{3g-3})$ we can conclude that the coefficient of q^{2d} in $I_\Gamma(q)$ equals $\sum_{\underline{a} | \sum a_i = d} H_{\underline{a},\Gamma}^{trop}$. Thus the above expression equals

$$\sum_\Gamma \frac{1}{|\operatorname{Aut}(\Gamma)|} I_\Gamma(q),$$

and theorem 5.1.7 is proved. □

Remark 5.2.7. From a computational point of view, it makes sense to introduce a base point to the computations above. In terms of labeled tropical covers, we then count covers as above which satisfy in addition the requirement that a fixed vertex, say e.g. x_1, is mapped to a fixed base point p. In terms of integrals, we set the variable $z_1 = 0$. The analogous statement to theorem 5.2.6 relating numbers of labeled tropical covers sending x_1 to p to coefficients of integrals where we set $z_1 = 0$ holds true and can be proved along the same lines as the proof of theorem 5.2.6 presented here.

5.3. The Propagator

In this subsection, we study the combinatorics of Feynman integrals. We show that the computation of a Feynman integral can be boiled down to a combinatorial hunt of monomials in a big generating function. We also express the integrals in terms of a constant coefficient of a multi-variate series, an expression which is also important for proving the quasimodularity of Feynman integrals, see [**6**, section 3] In order to compute the integrals of definition 5.2.5, it is helpful to make a change of variables $x_j = e^{i\pi z_j}$ for each $j = 1, \ldots, 2g-2$. Under this change of variables, each integration path γ_j goes to (half) a circle around the origin. The integral is then nothing else but the computation of residues. We start by giving a nicer expression of the propagator after the change of variables:

Theorem 5.3.1 (The propagator). *The propagator $P(x,q)$ of definition 5.1.3 with $x = e^{i\pi z}$ equals*

$$P(x,q) = -\frac{x^2}{(x^2-1)^2} - \sum_{n=1}^{\infty} \left(\sum_{d|n} d \left(x^{2d} + x^{-2d}\right) \right) q^{2n}. \tag{21}$$

Proof. The claim of the theorem follows by comparing Taylor coefficients of both sides. To be more precise (see [**28**])

$$\wp(z,q) = z^{-2} + \sum_{k=2}^{\infty} (2k-1) G_{2k}(q) z^{2k-2},$$

where $G_{2k}(q)$ is the classical weight $2k$ Eisenstein series, normalized to have a Fourier expansion of the shape

$$G_{2k}(q) = 2\zeta(2k) + \frac{2(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^{2n},$$

where $\zeta(s)$ denotes the Riemann zeta function and $\sigma_\ell(n) := \sum_{d|n} d^\ell$ is the ℓth divisor sum. Note that at even integers the ζ-funtion may be written in terms of Bernoulli numbers, defined via its generating functions (see e.g. [**12**])

$$\frac{t}{e^t-1} = \sum_{m=0}^{\infty}\frac{B_m}{m!}t^m. \tag{22}$$

To be more precise, we have

$$\zeta(2k) = (-1)^{k+1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}.$$

We next determine the Taylor expansion of the right-hand side of (21). Differentiating (22) gives that

$$-\frac{e^{2\pi iz}}{(e^{2\pi iz}-1)^2} = \frac{1}{(2\pi z)^2} + \sum_{m=0}^{\infty}\frac{m-1}{m!}B_m(2\pi iz)^{m-2}.$$

Moreover, in the second term we use the series expansion of the exponential function to obtain

$$\begin{aligned}
-\sum_{n=1}^{\infty}\sum_{d|n} d\left(x^{2d}+x^{-2d}\right)q^{2n} &= -\sum_{n=1}^{\infty}\sum_{d|n} d\sum_{\ell=0}^{\infty}\left(1+(-1)^\ell\right)\frac{(2\pi idz)^\ell}{\ell!}q^{2n}\\
&= -2\sum_{n=1}^{\infty}\sum_{d|n} d\sum_{\ell=0}^{\infty}\frac{(2\pi idz)^{2\ell}}{(2\ell)!}q^{2n}\\
&= -2\sum_{n=1}^{\infty}\sum_{\ell=0}^{\infty}\frac{(2\pi idz)^{2\ell}}{(2\ell)!}\sum_{d|n} d^{2\ell+1}q^{2n}\\
&= -2\sum_{n=1}^{\infty}\left(\sum_{\ell=0}^{\infty}\frac{(2\pi idz)^{2\ell}}{(2\ell)!}\sigma_{2\ell+1}(n)\right)q^{2n}.
\end{aligned}$$

Now the claim follows by comparing coefficients of $z^{2\ell}$. □

Now let us go back to a fixed Feynman graph Γ and consider the function we have to integrate. Denote the vertices of the edge q_k of Γ by x_{k_1} and x_{k_2}. Since the derivative of $\ln(x_i)$ is $\frac{1}{x_i}$, after the coordinate change we integrate the function

$$P_\Gamma := \prod_{k=1}^{3g-3}\left(-P\Big(\frac{x_{k_1}}{x_{k_2}}, q_k\Big)\right)\cdot\frac{1}{i\pi x_1\cdot\ldots\cdot i\pi x_{2g-2}}.$$

After the coordinate change, the integration paths are half-circles around the origin. Since our function is symmetric (there are only even powers of x), we can compute this integral as $\frac{1}{2}$ times the integral of the same function along a whole circle. Since the function has only one pole at zero (within the range of integration), we can compute the integral along the whole circle as $2i\pi$ times the residue at zero by the Residue Theorem.

It follows that the integral equals the constant coefficient of

$$P'_\Gamma := \prod_{k=1}^{3g-3} \left(-P\Big(\frac{x_{k_1}}{x_{k_2}}, q_k\Big) \right). \tag{23}$$

Note that it follows from theorem 5.3.1 that $-P\big(\frac{x}{y}, q\big) = -P\big(\frac{y}{x}, q\big)$. This is obvious for the (Laurent-polynomial) coefficients of q^d with $d > 0$. For the constant coefficient, it follows since

$$\frac{\left(\frac{x}{y}\right)^2}{\left(\left(\frac{x}{y}\right)^2 - 1\right)^2} = \frac{x^2y^2}{(x^2-y^2)^2} = \frac{x^2y^2}{(y^2-x^2)^2} = \frac{\left(\frac{y}{x}\right)^2}{\left(\left(\frac{y}{x}\right)^2 - 1\right)^2}. \tag{24}$$

Therefore it is not important which vertex of q_k we call x_{k_1} and which x_{k_2} (this also explains the independence of the sign $z_i - z_j$ resp. $z_j - z_i$ in definition 5.1.4). To compute the (in the x_k) constant coefficient of P'_Γ, we have to express the (in q_k) constant coefficient of each factor,

$$\frac{\left(\frac{x_{k_1}}{x_{k_2}}\right)^2}{\left(\left(\frac{x_{k_1}}{x_{k_2}}\right)^2 - 1\right)^2},$$

as a series. Depending on whether $\left|\frac{x_{k_1}}{x_{k_2}}\right| < 1$ or $\left|\frac{x_{k_2}}{x_{k_1}}\right| < 1$, we can use the left or the right expression of equation (24) for the constant coefficient and expand the denominator as product of geometric series. The following lemma shows how to expand the constant coefficient as a series, depending on the absolute value of the ratio of the two involved variables. This explains why different orders Ω can produce different integrals $I_{\Gamma,\Omega}(q_1, \dots, q_{3g-3})$: the position of the integration paths determine the series expansion of the constant coefficients.

LEMMA 5.3.2. *Assume $|x| < 1$. Then*

$$\frac{x^2}{(x^2-1)^2} = \sum_{w=1}^{\infty} w \cdot x^{2w}.$$

The proof follows easily after expanding the factors as geometric series.

The discussion of this subsection can be summed up as follows:

LEMMA 5.3.3. *Fix a Feynman graph Γ and an order Ω as in definition 5.1.4, and a tuple $(a_1, \dots, a_{3g-3})$ as in definition 5.2.3. We express the coefficient of $q^{2\cdot \underline{a}}$ in $I_{\Gamma,\Omega}(q_1, \dots, q_{3g-3})$ of definition 5.2.5. Assume k is such that the entry $a_k = 0$, and assume the edge q_k connects the two vertices x_{k_1} and x_{k_2}. Choose the notation of the two vertices x_{k_1} and x_{k_2} such that the chosen order Ω implies $\left|\frac{x_{k_1}}{x_{k_2}}\right| < 1$ for the starting points on the integration paths. Then the coefficient of $q^{2\cdot\underline{a}}$ in $I_{\Gamma,\Omega}(q_1, \dots, q_{3g-3})$ equals the constant term of the series*

$$\prod_{k|a_k=0} \left(\sum_{w_k=1}^{\infty} w_k \cdot \left(\frac{x_{k_1}}{x_{k_2}}\right)^{2w_k} \right) \cdot \prod_{k|a_k\neq 0} \left(\sum_{w_k|a_k} w_k \left(\left(\frac{x_{k_1}}{x_{k_2}}\right)^{2w_k} + \left(\frac{x_{k_2}}{x_{k_1}}\right)^{2w_k} \right) \right). \tag{25}$$

5.4. The Bijection

For a fixed Feynman graph Γ and tuple $(a_1, \dots, a_{3g-3})$, we are now ready to directly relate nonzero contributions to the constant term of the series given in (25) for each order Ω to tropical covers contributing to $H^{trop}_{\underline{a},\Gamma}$, thus proving theorem 5.2.6.

We express the constant term as a sum over products containing one term of each factor of the series in (25):

DEFINITION 5.4.1. Fix Γ, Ω and $(a_1,\ldots,a_{3g-3})$ as in definition 5.1.4 resp. 5.2.3. Consider a tuple of powers a_k and terms of the series in (25)

$$\big((a_k,T_k)\big)_{k=1,\ldots,3g-3} = \left(\left(a_k, w_k\cdot\left(\frac{x_{k_i}}{x_{k_j}}\right)^{2w_k}\right)\right)_{k=1,\ldots,3g-3},$$

where $i=1$ and $j=2$ if $a_k=0$, and $\{i,j\}=\{1,2\}$ otherwise. We require the product of the terms, $\prod_{k=1}^{3g-3}T_k$, to be constant in each x_i, $i=1,\ldots,2g-2$.

We denote the set of all such tuples by $\mathcal{T}_{\underline{a},\Gamma,\Omega}$.

Obviously, each tuple yields a summand of the constant term of the series in (25) (and thus, by lemma 5.3.3, a contribution to the $q^{2\cdot\underline{a}}$-coefficient of the integral $I_{\Gamma,\Omega}(q_1,\ldots,q_{3g-3})$), and vice versa, each summand arises from a tuple in $\mathcal{T}_{\underline{a},\Gamma,\Omega}$. Note that the contribution of a tuple equals

$$\prod_{k=1}^{3g-3} T_k = \prod_{k=1}^{3g-3} w_k. \tag{26}$$

DEFINITION 5.4.2. Let $\hat{\pi}: C\to E$ be a labeled tropical cover as in definition 5.2.1 contributing to $H^{trop}_{\underline{a},\Gamma}$. We cut E at the base point p_0 and flatten it to an interval following clockwise orientation. We define an order of the vertices x_i of C given by the natural order of their image points on the interval. For a given order Ω as in definition 5.1.4, we let $H^{trop}_{\underline{a},\Gamma,\Omega}$ be the weighted number of labeled tropical covers as in definition 5.2.1 (i.e. of degree $\sum_{i=1}^{3g-3}a_i$, where the source curve has combinatorial type Γ, having their branch points at the prescribed positions and satisfying $|(\hat{\pi}^{-1}(p_0)\cap q_i|\cdot w_i = a_i$ for all $i=1,\ldots,3g-3$, where w_i denotes the weight of the edge q_i), and in addition satisfying that the above order equals Ω. As usual, each cover is counted with multiplicity $\prod_{k=1}^{3g-3}w_k$.

Note that obviously we have $\sum_\Omega H^{trop}_{\underline{a},\Gamma,\Omega} = H^{trop}_{\underline{a},\Gamma}$, where the sum goes over all $(2g-2)!$ orders Ω of the vertices.

EXAMPLE 5.4.3. The cover in figure 2 cut at p_0 and flattened to an interval is depicted in figure 5. Its vertex ordering Ω is given by $x_1<x_2<x_3<x_4$.

EXAMPLE 5.4.4. Go back to example 5.2.4 where we determined $H^{trop}_{\underline{a},\Gamma}$ for the Feynman graph Γ in figure 3 and $\underline{a}=(0,2,1,0,0,1)$. Note that there are two orders Ω that yield nonzero contributions, namely $x_1<x_3<x_4<x_2$ (call it Ω_1) and $x_2<x_4<x_3<x_1$ (call it Ω_2). In figure 4, the covers with order Ω_1 appear in the left column, the covers with Ω_2 in the right column. For both orders, we have $H^{trop}_{\underline{a},\Gamma,\Omega_i}=128$, $i=1,2$. Altogether, we have $H^{trop}_{\underline{a},\Gamma} = H^{trop}_{\underline{a},\Gamma,\Omega_1}+H^{trop}_{\underline{a},\Gamma,\Omega_2} = 128+128=256$.

THEOREM 5.4.5. *Fix a Feynman graph Γ, an order Ω and a tuple $(a_1,\ldots,a_{3g-3})$ as in definition 5.1.4 resp. 5.2.3.*

There is a bijection between the set of labeled tropical covers contributing to $H^{trop}_{\underline{a},\Gamma,\Omega}$ and the set $\mathcal{T}_{\underline{a},\Gamma,\Omega}$ of tuples contributing to the $q^{2\cdot\underline{a}}$-coefficient of the integral $I_{\Gamma,\Omega}(q_1,\ldots,q_{3g-3})$ (see definition 5.4.2 and 5.4.1).

The bijection identifies the coefficients w_k of the terms T_k of a tuple with the weights of the edges of the corresponding labeled tropical cover. In particular, the contribution of a tuple to the coefficient of $q^{2\cdot\underline{a}}$ in $I_{\Gamma,\Omega}(q_1,\ldots,q_{3g-3})$ equals the multiplicity of the corresponding labeled tropical cover, with which it contributes to $H^{trop}_{\underline{a},\Gamma,\Omega}$.

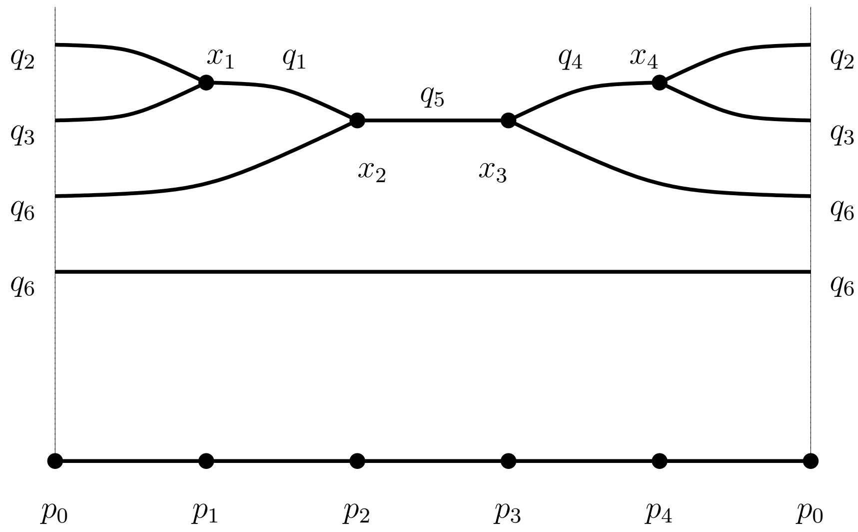


FIGURE 5. The cover of figure 2 cut at p_0.

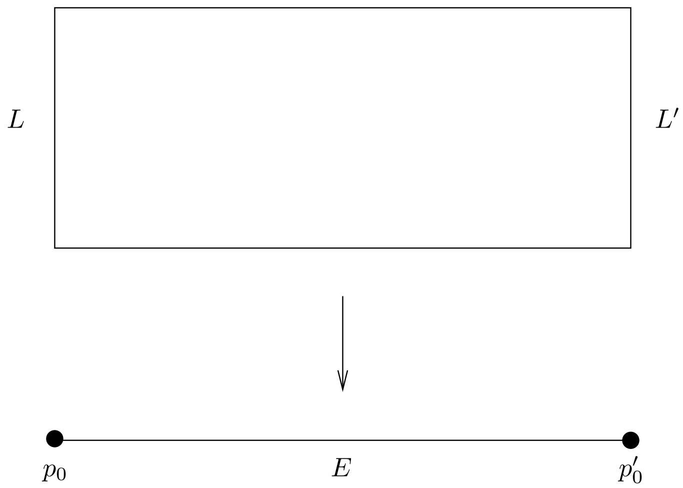


FIGURE 6. Preparation to construct a tropical cover from a tuple.

Note that it follows immediatly from theorem 5.4.5 that $H^{trop}_{\underline{a},\Gamma,\Omega}$ equals the coefficient of $q^{2\cdot\underline{a}}$ in $I_{\Gamma,\Omega}(q_1,\ldots,q_{3g-3})$, hence the coefficient of $q^{2\cdot\underline{a}}$ in $I_\Gamma(q_1,\ldots,q_{3g-3})$ equals $H^{trop}_{\underline{a},\Gamma}$ and theorem 5.2.6 is proved.

To prove theorem 5.4.5, we set up the map sending a tuple in $\mathcal{T}_{\underline{a},\Gamma,\Omega}$ to a tropical cover in construction 5.4.6. The theorem then follows from lemma 5.4.8 stating that this construction indeed yields a map as required and lemma 5.4.9 stating that it has a natural inverse and therefore is a bijection.

As usual, let Γ, Ω and $\underline{a}$ be fixed as in definition 5.1.4 resp. 5.2.3.

CONSTRUCTION 5.4.6. Draw an interval from p_0 to p'_0 that can later be glued to E by identifying p_0 and p'_0, and a rectangular box above in which we can step by step draw a cover of E following the construction (see figure 6). The vertical sides of the box are called L and L' and represent points which (if they belong to the cover after the construction) are pairwise identified and mapped to the base point.

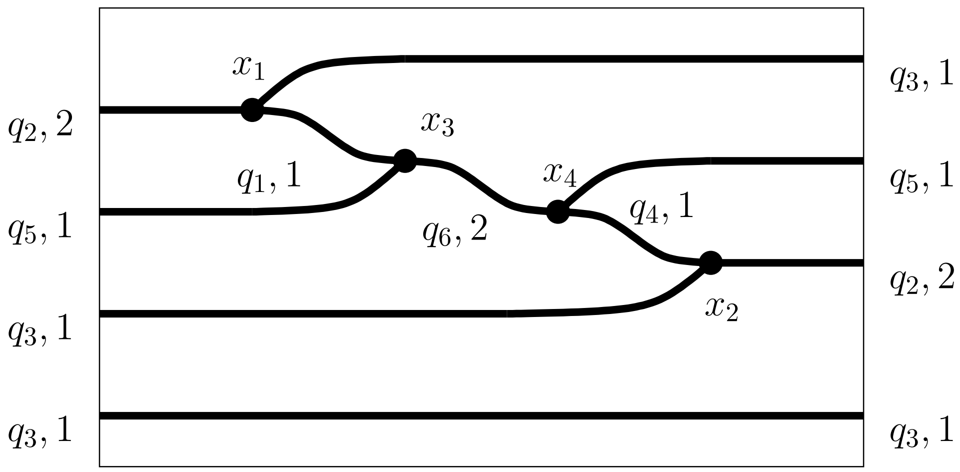


FIGURE 7. Applying Construction 5.4.6 in Example 5.4.7.

Given a tuple

$$\big((a_k, T_k)\big)_{k=1,\dots,3g-3} = \left(\left(a_k, w_k \cdot \left(\frac{x_{k_i}}{x_{k_j}}\right)^{2w_k}\right)\right)_{k=1,\dots,3g-3}$$

of $\mathcal{T}_{\underline{a},\Gamma,\Omega}$,

- draw dots labeled $x_1, \dots, x_{3g-3}$ into the box, from left to right (and slightly downwards, to keep some space to continue the picture), as determined by the order Ω, one dot above each point condition $p_i \in E$ where we fix the branch points;
- for the term $T_k = w_k \cdot \left(\frac{x_{k_i}}{x_{k_j}}\right)^{2w_k}$ draw an edge leaving vertex x_{k_i} to the right and entering vertex x_{k_j} from the left — if $a_k = 0$, let this edge be a straight line connecting these two vertices, if $a_k \neq 0$ let it first leave the box at L' and enter again at L, altogether $\frac{a_k}{w_k}$ times, before it enters x_{k_j};
- give the edges drawn in the item before weight w_k. As always, the lengths of the edges are then determined by the differences of the image points of the x_i and the weights.
- Glue the corresponding points on L and L' to obtain a cover of E.

EXAMPLE 5.4.7. Let Γ be the Feynman graph of figure 3 and let $\underline{a} = (0, 2, 2, 0, 1, 0)$. Moreover, choose the ordering $x_1 < x_3 < x_4 < x_2$ and pick the terms $T_1 = \left(\frac{x_1}{x_3}\right)^2$, $T_2 = 2 \cdot \left(\frac{x_2}{x_1}\right)^{2\cdot 2}$, $T_3 = \left(\frac{x_1}{x_2}\right)^2$, $T_4 = \left(\frac{x_4}{x_2}\right)^2$, $T_5 = \left(\frac{x_4}{x_3}\right)^2$ and $T_6 = 2 \cdot \left(\frac{x_3}{x_4}\right)^{2\cdot 2}$ from the series in (25). When applying construction 5.4.6 we obtain the picture shown in figure 7 before gluing.

LEMMA 5.4.8. *Construction 5.4.6 defines a map from $\mathcal{T}_{\underline{a},\Gamma,\Omega}$ to the set of tropical covers contributing to $N^{trop}_{\underline{a},\Gamma,\Omega}$ (see definition 5.4.2 and 5.4.1).*

PROOF. Since the integrand of definition 5.2.5 is set up such that we have a term containing a power of $\frac{x_i}{x_j}$ in the tuples in $\mathcal{T}_{\underline{a},\Gamma,\Omega}$ if and only if there is an edge q_k connecting x_i and x_j, it is clear that we produce a cover whose source curve has combinatorial type equal to the labeled Feynman graph Γ (see also equation (23)). It is also obvious that the vertices are mapped to the interval respecting the order Ω. To see that it is a tropical cover at all, we have to verify the balancing condition at each vertex x_i. This follows from the fact that we require the product of all terms to be constant in x_i: since Γ is trivalent, we have three edges adjacent to x_i, to fix notation call them (without restriction) q_1, q_2, and q_3. Assume (also without restriction) that the other vertex of q_j, $j = 1, \ldots, 3$ is x_j. The only three terms in the product $\prod_{k=1}^{3g-3} T_k$ involving x_i are then

$$w_1 \cdot \left(\frac{x_1}{x_i}\right)^{2w_1}, w_2 \cdot \left(\frac{x_i}{x_2}\right)^{2w_2}, \text{ and } w_3 \cdot \left(\frac{x_i}{x_3}\right)^{2w_3},$$

where we picked an arbitrary choice between a quotient such as $\frac{x_1}{x_i}$ and its inverse in each term for now. This choice is again made without restriction, just to fix the notation in the terms T_k of the given tuple. (Of course, in general, if some of the a_j,$j = 1, \ldots, 3$ are zero, the choice has to respect the order Ω.) In our fixed but arbitrary choice the requirement that the product is constant in x_i translates to the equation

$$-2w_1 + 2w_2 + 2w_3 = 0.$$

The construction implies that here, the edge q_1 enters x_i from the left with weight w_1 while q_2 and q_3 leave the vertex x_i to the right with weight w_2 and w_3, hence the balancing condition is fulfilled. Since the direction of the edges we draw in the construction reflects the fact that the corresponding two vertices show up in the numerator resp. denominator of the quotient, it is obvious that the instance for which we fixed a notation generalizes to any situation; and the balancing condition at x_i is always equivalent to the requirement that the product is constant in x_i. It is clear from the construction that the labeled tropical cover we have built has its branch points at the required positions. If $a_k = 0$, the construction implies that $\hat{\pi}^{-1}(p_0) \cap q_k = \emptyset$ and thus $|\hat{\pi}^{-1}(p_0) \cap q_k| \cdot w_k = 0 = a_k$ as required. If $a_k \neq 0$, we draw $\frac{a_k}{w_k}$ points on L resp. L' that are identified to give $\frac{a_k}{w_k}$ preimages of p_0 in q_k, thus $|\hat{\pi}^{-1}(p_0) \cap q_k| \cdot w_k = a_k$ holds in general. In particular, the degree of the cover is $\sum_{k=0}^{3g-3} a_k$.

Obviously, the map identifies the coefficient w_k of a term T_k with the weight of the edge q_k, therefore the contribution of a tuple to the $q^{2 \cdot \underline{a}}$-coefficient of the integral $I_{\Gamma,\Omega}(q_1, \ldots, q_{3g-3})$ (given by equation (26)) equals the contribution of the corresponding tropical cover to $N^{trop}_{\underline{a},\Gamma,\Omega}$ by definition 5.4.2. The statement follows. □

LEMMA 5.4.9. *The map of lemma 5.4.8 has a natural inverse and is a bijection.*

PROOF. One can reverse construction 5.4.6 in the obvious way. For any edge q_k which does not pass L resp. L', we set $a_k = 0$. Assume q_k connects the two vertices x_{k_1} and x_{k_2} and assume the order Ω satisfies $x_{k_1} < x_{k_2}$. Then it follows from definition 5.1.4 that we have to pick integration paths satisfying $\left|\frac{x_{k_1}}{x_{k_2}}\right| < 1$ and thus using lemma 5.3.2 the power series expansion of the corresponding constant term of the propagator (see theorem 5.3.1) then contains quotients $\frac{x_{k_1}}{x_{k_2}}$ as required. We pick the term $w_k \cdot \left(\frac{x_{k_1}}{x_{k_2}}\right)^{2w_k}$ for our tuple.

For any edge q_k that passes l_k times with weight w_k through L resp. L', we set $a_k = w_k \cdot l_k$. Obviously, w_k is a divisor of a_k and thus the term

$$w_k \cdot \left(\left(\frac{x_{k_1}}{x_{k_2}}\right)^{2w_k} + \left(\frac{x_{k_2}}{x_{k_1}}\right)^{2w_k} \right)$$

shows up in the $q_k^{2a_k}$-coefficient of the propagator as required (see theorem 5.3.1) and we can pick the summand corresponding to the orientation of our arrow as term for our tuple.

It is obvious that this inverse construction produces a bijection. □

Corollary 5.4.10. *A Feynman graph* Γ *satisfies* $N^{trop}_{\underline{a},\Gamma} = 0$ *for every* $\underline{a}$*, or (by theorem 5.2.6) equivalently,* $I_\Gamma(q_1, \dots, q_{3g-3}) = 0$ *if and only if* Γ *contains a bridge.*

Proof. One can view a tropical cover as a system of rivers flowing into each other without any source or sink, since the weights of the edges are positive and the balancing condition is satisfied. A graph with a bridge must have zero flow on the bridge, and thus cannot be the source of a tropical cover.

Alternatively, to get a nonzero contribution to the coefficient of $q^{2\cdot\underline{a}}$ in the integral $I_\Gamma(q_1, \dots, q_{3g-3}) = 0$, we must be given an order Ω and a tuple in $\mathcal{T}_{\underline{a},\Gamma,\Omega}$. But by lemma 5.4.8 such a tuple only exists if the balancing condition is satisfied at every vertex, thus the argument we just gave shows the coefficient is zero for a graph with a bridge.

Vice versa, we have to show that there exists a cover for every graph without a bridge. To see this, we give an algorithm below how to construct for a given bridgeless graph an orientation of the edges that satisfies the following: there is no cut into two connected components Γ_1 and Γ_2 for which all cut edges are oriented from Γ_1 to Γ_2. It is easy to see that for such an orientation, we can insert positive weights for the edges such that the balancing condition is satisfied at every vertex (we just add enough water to the system of rivers). Thus the statement follows from construction 5.4.11 and lemma 5.4.12 below. □

Construction 5.4.11. Let Γ be a bridgeless graph.

(1) Choose an arbitrary cycle and orient its edges in one direction. Also choose a reference vertex V on the cycle. Let K denote the set of vertices on the cycle, this is the set of "known vertices" that we will enlarge in the following steps.
(2) Let $U_1, \dots, U_s$ denote the connected components of the subgraph induced on the vertex set of Γ minus K. If $s \geq 1$, choose an arbitrary vertex $W \in U_1$. Since Γ is connected, there is a path from V to W and we can choose it such that it respects our so far fixed orientations for the edges. At some point, the path must leave the "known part" and enter U_1, call this edge E_1. Since E_1 is not a bridge, there must be at least a second edge E_2 connecting the known part to U_1. We go along E_1 into U_1 until we reach W, and then continue until we hit via E_2 the known part again. We orient the edges we follow on the way. We add the set of vertices we meet on the way to K and start again at (2).
(3) At each step described above, we increase the vertex set of the known part. If all vertices are known, we orient the remaining edges arbitrarily.

Lemma 5.4.12. *Given a bridgeless graph* Γ*, we can use construction 5.4.11 to orient the edges such that the following is satisfied:*

(1) Every vertex is contained in an oriented cycle that also contains the reference vertex V*.*
(2) There is no cut into two connected components Γ_1 *and* Γ_2 *such that all cut edges are oriented from* Γ_1 *to* Γ_2*.*

Proof. The first statement is obvious from the construction: we start with a cycle containing V and add oriented "handles". For the second statement, assume there was such a cut, and assume without restriction that V is in Γ_1. Choose an arbitrary vertex W in Γ_2. By (1), there is an oriented cycle containing W and V. This cycle must contain at least two cut edges which are thus oriented in opposite direction. □

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