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Abstract

If Ω is a domain in $\mathbb{R}^2$ and if $u : \Omega \to \mathbb{R}$ locally minimizes the energy

$$\int_\Omega \left[h_1\left(\left|(\nabla^2 u)_I\right|\right) + h_2\left(\left|(\nabla^2 u)_{II}\right|\right)\right]\, dx\,,$$

where $(\nabla^2 u)_I, (\nabla^2 u)_{II}$ denotes a decomposition of the Hessian matrix $\nabla^2 u$, then we prove the higher integrability and even the continuity of $\nabla^2 u$ under rather general assumptions imposed on the N-functions h_1, h_2.

DEDICATED TO PROF. N. URALTSEVA ON THE OCCASION OF HER JUBILEE

1 Introduction

In their paper [UU] Uraltseva and Urdaletova established the local Lipschitz regularity of bounded generalized solutions of certain degenerate, nonuniformly elliptic equations. In particular their result applies to bounded local minimizers of the variational integral

$$\int_\Omega \sum_{i=1}^n \left|\frac{\partial u}{\partial x_i}\right|^{m_i}\, dx\,, \tag{1.1}$$

Ω denoting a domain in $\mathbb{R}^n, n \geq 2$, provided the exponents m_i satisfy $m_i \geq 2$ together with

$$\max\{m_1, \ldots, m_n\} < 2m_i, \quad i = 1, \ldots, n\,. \tag{1.2}$$

The reader should note that Giaquinta's counterexample (see [Gi2]) involves a functional of the form (1.1) with $m_1 = \ldots = m_{n-1} = 2$, $m_n = 4$, which means that in the limit case of (1.2) unbounded local minimizers can exist (at least if n is large enough), whereas (1.2) together with the boundedness assumption leads to a higher degree of regularity (e.g. local boundedness of the gradient) for this restricted class of local minimizers u of (1.1). However, having a global minimization problem in mind, the hypothesis $u \in L^\infty(\Omega)$ is not so unnatural and usually follows from the maximum-principle. If

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the boundedness of u is not required a priori, then Fusco and Sbordone [FS] and Marcellini [Ma1], [Ma2] showed how to get regularity of u under stronger assumptions than (1.2): suppose that the range of anisotropy is limited through an inequality of the from

$$\max\{m_1, \dots, m_n\} < c(n) m_i, \quad i = 1, \dots, n\,, \tag{1.3}$$

for a suitable constant $c(n) > 1$ with $c(n) \to 1$ as $n \to \infty$. Then we still have that $|\nabla u| \in L^\infty_{\mathrm{loc}}(\Omega)$. In our recent paper [BFZ] we returned to the point of view of Uraltseva and Urdatelova and proved that their hypothesis $u \in L^\infty(\Omega)$ (or even $u \in L^\infty_{\mathrm{loc}}(\Omega)$) is a very strong assumption in the sense that it already implies higher regularity of u without further restrictions of the form (1.2) or (1.3). The purpose of the present note is the investigation of the regularity problem for the higher-order variant of (1.1) at least in a special case. To be precise let us consider the variational integral

$$I[u, \Omega] = \int_\Omega H(\nabla^2 u)\, dx\,, \tag{1.4}$$

where $\nabla^2 u = (\partial_\alpha \partial_\beta u)_{1 \le \alpha, \beta \le n}$ is the Hessian matrix of the function $u : \Omega \to \mathbb{R}$. We here already note that with similar arguments we can replace $\nabla^2 u$ by $\nabla^k u$ for some $k \ge 2$, and that it is also possible to include vectorial functions $u : \Omega \to \mathbb{R}^M, M \ge 2$. As a matter of fact a discussion of bounded local I-minimizers now seems to be artificial, since no maximum-principle is available for the higher order case, but as it will be outlined below it is possible to obtain regularity results without extra assumptions on u at least in the 2D-case. So let $n = 2$ and suppose that $\nabla^2 u$ is represented by the vector $(\partial_1 \partial_1 u, \partial_1 \partial_2 u, \partial_2 \partial_2 u) =: \xi$. With $(\nabla^2 u)_I$ and $(\nabla^2 u)_{II}$ we denote two arbitrary vectors in $\mathbb{R}^3$ formed of r respectively s entries of ξ filled up by 0, if necessary, where $r, s \in \{0, 1, 2, 3\}$ and where in case $r = 0$ or $s = 0$ we just have the zero vector in $\mathbb{R}^3$. The only requirement is the following: the set generated by the totality of all entries of $(\nabla^2 u)_I$ and $(\nabla^2 u)_{II}$ contains all entries of ξ, for example:

$$\begin{array}{lrl}
(\nabla^2 u)_I = (\partial_1 \partial_1 u, 0, 0), & (\nabla^2 u)_{II} = (0, \partial_1 \partial_2 u, \partial_2 \partial_2 u) & \text{or} \\
(\nabla^2 u)_I = \nabla^2 u, & (\nabla^2 u)_{II} = (0, \partial_1 \partial_2 u, 0) & \text{or} \\
(\nabla^2 u)_I = (\partial_1 \partial_1 u, \partial_1 \partial_2 u, 0), & (\nabla^2 u)_{II} = (0, \partial_1 \partial_2 u, \partial_2 \partial_2 u) & \text{or} \\
(\nabla^2 u)_I = \nabla^2 u, & (\nabla^2 u)_{II} = (0, 0, 0), & \text{etc.}
\end{array}$$

Returning to (1.4) we assume that the energy density H is of the form

$$H(\nabla^2 u) = h_1\left(\left|(\nabla^2 u)_I\right|\right) + h_2\left(\left|(\nabla^2 u)_{II}\right|\right)\,, \tag{1.5}$$

where for instance

$$h_i(t) = (\mu_i + t^2)^{\frac{m_i}{2}}, \quad i = 1, 2\,, \tag{1.6}$$

with $\mu_i \geq 0$ and exponents $m_i \geq 2$. A natural class for local I-minimizers then is the Sobolev space $W^2_{2,\mathrm{loc}}(\Omega)$ (see, e.g., [Ad] for a definition of these classes), and in [BF2] we proved:

Theorem 1.1. *Let $u \in W^2_{2,\mathrm{loc}}(\Omega)$ denote a local minimizer of the functional I from (1.4) with H defined according to (1.5). Suppose further that (1.6) holds together with*

$$\max\{m_1, m_2\} < 2\min\{m_1, m_2\}\,. \tag{1.7}$$

a) If the non-degenerate case μ_1, $\mu_2 > 0$ is considered, then we have $u \in C^{2,\alpha}(\Omega)$ for any $\alpha < 1$.

b) In the degenerate case we still have $u \in C^{1,\beta}(\Omega)$ for any $\beta < 1$.

Remark 1.1. *For $n = 2$ condition (1.2) introduced by Uraltseva and Urdaletova is equivalent to (1.7).*

Here we are going to improve the results of Theorem 1.1 by showing

Theorem 1.2. *The statements of Theorem 1.1 hold for any choices of exponents m_1, $m_2 \geq 2$ without requirering the bound (1.7).*

Theorem 1.2 will be a by-product of a more general result dealing with integrands H of splitting-type as described in (1.5) but replacing (1.6) by a larger class of functions h_1 and h_2. To be precise, let

$$H(E) := h_1\left(|(E)_I|\right) + h_2\left(|(E)_{II}|\right) \tag{1.8}$$

for symmetric (2×2)-matrices E with an obvious meaning of $(E)_{I,II}$. Suppose further that the functions h_1, $h_2 : [0,\infty) \to [0,\infty)$ are of class C^2 s.t. for $h = h_1$ and $h = h_2$ it holds

$$\left\{\begin{array}{l} h \text{ is strictly increasing and convex together with } h''(0) > 0 \text{ and} \\ \lim_{t\downarrow 0} \frac{h(t)}{t} = 0; \end{array}\right. \tag{A1}$$

there exists a constant $\overline{k} > 0$ such that $h(2t) \leq \overline{k}h(t)$ for all $t \geq 0$; (A2)

$$\left\{\begin{array}{l} \text{for an exponent } \omega \geq 0 \text{ and a constant } a \geq 0 \text{ it holds} \\ \frac{h'(t)}{t} \leq h''(t) \leq a(1+t^2)^{\frac{\omega}{2}}\,\frac{h'(t)}{t} \quad \text{for all } t \geq 0. \end{array}\right. \tag{A3}$$

Let us draw some conclusions from (A1)–(A3):

i) (A1) implies that $h(0) = 0 = h'(0)$ and $h'(t) > 0$ for $t > 0$. From (A3) it follows that $t \mapsto h'(t)/t$ is increasing, moreover we get $h(t) \geq h''(0)t^2/2$. In particular h is a N-function (see [Ad]) of at least quadratic growth.

ii) The ($\Delta 2$)-property stated in (A2) implies

$$h(t) \leq c(t^{\overline{m}} + 1)$$

for some exponent $\overline{m} \geq 2$, hence by the convexity of h

$$h'(t) \leq c(t^{\overline{m}-1} + 1),$$

where here and in the following "c" denotes a constant whose value may vary from line to line.

iii) Combining (A2) with the convexity of h we see that

$$\overline{k}^{-1} h'(t)t \leq h(t) \leq th'(t), \quad t \geq 0. \tag{1.9}$$

iv) For symmetric (2×2)-matrices Y, Z we have

$$\begin{aligned} &\min\left\{\frac{h_1'(|(Z)_I|)}{|(Z)_I|}, h_1''(|(Z)_I|)\right\} |(Y)_I|^2 \\ &\quad + \min\left\{\frac{h_2'(|(Z)_{II}|)}{|(Z)_{II}|}, h_2''(|(Z)_{II}|)\right\} |(Y)_{II}|^2 \\ &\leq D^2H(Z)(Y,Y) \leq \max\{\ldots\}\,|(Y)_I|^2 + \max\{\ldots\}\,|(Y)_{II}|^2 , \end{aligned}$$

so that by (A3)

$$\begin{aligned} &\frac{h_1'(|(Z)_I|)}{|(Z)_I|}|(Y)_I|^2 + \frac{h_2'(|(Z)_{II}|)}{|(Z)_{II}|}|(Y)_{II}|^2 \\ &\leq D^2H(Z)(Y,Y) \\ &\leq a(1+|(Z)_I|^2)^{\frac{\omega}{2}}\frac{h_1'(|(Z)_I|)}{|(Z)_I|}|(Y)_I|^2 \\ &\quad + a(1+|(Z)_{II}|^2)^{\frac{\omega}{2}}\frac{h_2'(|(Z)_{II}|}{|(Z)_{II}|}|(Y)_{II}|^2 , \end{aligned} \tag{1.10}$$

and for a suitable exponent $\overline{q} > 2$ it follows

$$c|Y|^2 \leq D^2H(Z)(Y,Y) \leq C(1+|Z|^2)^{\frac{\overline{q}-2}{2}}|Y|^2 , \tag{1.11}$$

the first inequality being a consequence of i).

Now our main result is

Theorem 1.3. *Let H satisfy (1.8) with functions h_1, h_2 for which (A1)–(A3) hold. Suppose further that $u \in W^2_{2,\mathrm{loc}}(\Omega)$ locally minimizes the functional I defined in (1.4). Then we have:*

a) $\nabla^2 u$ *belongs to the class* $L^p_{\mathrm{loc}}(\Omega;\mathbb{R}^{2\times 2})$ *for any finite* p*, in particular it holds* $u \in C^{1,\alpha}(\Omega)$ *for any* $0 < \alpha < 1$.

b) If $\omega < 2$ *in (A3), then we get* $u \in C^{2,\alpha}(\Omega)$ *for all* $\alpha < 1$.

Remark 1.2. *Of course Theorem 1.3 applies to the special choice of the functions h_i as stated in (1.6) provided $\mu_i > 0$, i.e. we obtain the $C^{2,\alpha}$-regularity result of Theorem 1.2 in the non-degenerate situation. To be precise, one has to replace h_i by $h_i - h_i(0)$ but this does not affect the arguments. The proof of Theorem 1.3 a) for the case that (1.6) holds with $\mu_i = 0$ is left to the reader.*

Remark 1.3. *Variational integrals of the form (1.4) defined on two-dimensional domains have some relation to the theory of elastic plates. For a discussion of this issue we refer to the paper [Fu], in which Theorem 1.3 a) is established for the isotropic energy $\int_\Omega h(|\nabla^2 u|)\,dx$ with h satisfying (A1)–(A3). Of course the results of this paper do not apply to the situation at hand, since now our energy density is of the splitting form (1.8) showing a completely different ellipticity behaviour in comparison to the isotropic case studied in [Fu].*

2 Higher integrability of $\nabla^2 u$

Here we are going to prove Theorem 1.3 a). So let u denote a local I-minimizer and fix domains $\Omega_1 \Subset \Omega_2 \Subset \Omega$. Proceeding as in [BF1], [BF2] we denote by $\overline{u}_m$ the mollification of u with radius $1/m$, $m \in \mathbb{N}$, in particular we have

$$\|\overline{u}_m - u\|_{W^2_2(\Omega_2)} \to 0$$

as $m \to \infty$, moreover it holds (compare (2.1) in [BF1])

$$I[\overline{u}_m, \Omega_2] \to I[u, \Omega_2]\,.$$

Recalling that on account of (1.11) the hypothesis (1.1) of [BF1] now holds for $p = 2$, $q = \overline{q}$ we may define

$$\rho_m := \|\overline{u}_m - u\|_{W^2_2(\Omega_2)} \left[\int_{\Omega_2} (1 + |\nabla^2 \overline{u}_m|^2)^{\frac{\overline{q}}{2}} \, dx\right]^{-1}$$

as well as the perturbed energy

$$I_m[w, \Omega_2] := \rho_m \int_{\Omega_2} (1 + |\nabla^2 w|^2)^{\frac{\overline{q}}{2}} \, dx + I[w, \Omega_2]$$

with density

$$H_m := \rho_m (1 + |\cdot|^2)^{\frac{\overline{q}}{2}} + H \, .$$

Finally we consider the unique solution u_m of

$$I_m[\cdot, \Omega_2] \to \min \ \text{ in } \overline{u}_m + \overset{\circ}{W}{}^2_{\overline{q}}(\Omega_2)$$

for which it was shown in [BF1]

$$u_m \rightharpoonup u \text{ in } W^2_2(\Omega_2) \, , \quad I_m[u_m, \Omega_2] \to I[u, \Omega_2] \tag{2.1}$$

as $m \to \infty$. Moreover we proved in [BF1] (compare the inequality stated after (2.13)) the validity of

$$\begin{aligned} &\int_{\Omega_2} \eta^6 D^2 H_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \partial_\alpha \nabla^2 u_m) \, dx \\ &\leq \ -\int_{\Omega_2} D^2 H_m(\nabla^2 u_m)(\partial_\alpha \nabla^2 u_m, \nabla^2 \eta^6 \partial_\alpha u_m + 2 \nabla \eta^6 \odot \nabla \partial_\alpha u_m) \, dx \, , \end{aligned} \tag{2.2}$$

where $\eta \in C^\infty_0(\Omega_2)$ is arbitrary and where we use summation convention w.r.t. greek indices repeated twice. In (2.2) "$\odot$" denotes the symmetric product of vectors, and we can justify (2.2) by an application of the difference quotient technique to the Euler equation satisfied by u_m. We note that the Caccioppoli-type inequality (2.2) also occurs in [BF2] (compare inequality (4.1)) being established along the same lines as in [BF1] but here we are going to exploit (2.2) in a completely different manner leading to the improvement of the result from [BF2], which we mentioned before.
For notational simplicity we will drop the index m, i.e. we write u, H, I in place of u_m, H_m, I_m, but the reader should keep in mind that we actually

work with an approximation. However, since we will prove estimates involving Ω_1 on the l.h.sides and with quantities like $I_m[u_m, \Omega_2]$ on the r.h.sides, uniform bounds on Ω_1 will be a consequence of (2.1).

Now after these preparations we integrate by parts on the r.h.s. of (2.2) in order to get

$$\begin{aligned} & \int_{\Omega_2} \eta^6 D^2 H(\nabla^2 u)(\partial_\alpha \nabla^2 u, \partial_\alpha \nabla^2 u)\, dx \\ & \quad \leq \int_{\Omega_2} DH(\nabla^2 u) : \partial_\alpha \left[\nabla^2 \eta^6 \partial_\alpha u + 2\nabla \eta^6 \odot \nabla \partial_\alpha u \right]\, dx\, . \end{aligned} \tag{2.3}$$

Note that this integration by parts is justified since the "critical term" occurring in $DH(\nabla^2 u) : \partial_\alpha[\ldots]$ is of the form $|DH(\nabla^2 u)||\nabla^3 u|$. But since the r.h.s. of (2.2) is finite (for each m), we deduce from (1.11) that $u(= u_m)$ is of class $W^3_{2,\mathrm{loc}}(\Omega_2)$ and since $n = 2$, it follows that $|\nabla^2 u|$ is in $L^t_{\mathrm{loc}}(\Omega_2)$ for any finite t. Recalling that $|DH|$ is bounded in terms of a suitable power, the local integrability of $|D^2 H(\nabla^2 u)||\nabla^3 u|$ follows (but at this stage not necessarily uniform in m). Let us discuss the r.h.s. of (2.3): from (1.8) we get

$$\begin{aligned} |\text{ r.h.s. of (2.3)}| &\leq c \int_{\Omega_2} \left\{ h_1'\left(|(\nabla^2 u)_I|\right) + h_2'\left(|(\nabla^2 u)_{II}|\right) \right\} \\ &\quad \cdot \left[|\nabla^3 \eta^6||\nabla u| + |\nabla^2 \eta^6||\nabla^2 u| + |\nabla \eta^6||\nabla^3 u| \right]\, dx \\ &= c \left(\int_{\Omega_2} \{\ldots\} |\nabla \eta^6||\nabla^3 u|\, dx \right. \\ &\quad \left. + \int_{\Omega_2} \{\ldots\} |\nabla^2 \eta^6||\nabla^2 u|\, dx + \int_{\Omega_2} \{\ldots\} |\nabla^3 \eta^6||\nabla u|\, dx \right) \\ &=: c(T_1 + T_2 + T_3)\, . \end{aligned}$$

To the terms T_i we apply Young's inequality:

$$T_1 \leq \varepsilon \int_{\Omega_2} \eta^6 |\nabla^3 u|^2\, dx + c(\varepsilon) \int_{\Omega_2} |\nabla \eta|^2 \{\ldots\}^2\, dx\, ,$$

where ε is arbitrary and where w.l.o.g. we assume $0 \leq \eta \leq 1$. On account of (1.11) the first item on the r.h.s. of the above inequality can be absorbed in the l.h.s. of (2.3), provided ε is small enough. Here we emphasize that the value of ε can be chosen independent of the approximation parameter m.

For T_2 we have

$$T_2 \le \int_{\Omega_2} |\nabla^2 u|^2 \, dx + \int_{\Omega_2} |\nabla^2 \eta^6|^2 \{\dots\}^2 \, dx$$

and by (2.1) the first integral on the r.h.s. is bounded independent of m. For T_3 we argue in a similar way by observing that $u_m - \overline{u}_m$ is in the space $\overset{\circ}{W}{}^2_2(\Omega_2)$, hence we can apply Poincare's inequality to get an uniform $L^2(\Omega_2)$-bound for $|\nabla u_m|$ in terms of the original energy. If we finally fix concentric discs $B_r(z) \subset B_R(z) \Subset \Omega_2$ and let $\eta = 1$ on $B_r(z)$, $\operatorname{spt} \eta \subset B_R(z)$, $|\nabla^l \eta| \le c(R-r)^{-l}$, $l = 1, 2$, then we obtain from the above estimates

$$\begin{aligned} &\int_{B_r(z)} D^2 H(\nabla^2 u)(\partial_\alpha \nabla^2 u, \partial_\alpha \nabla^2 u) \, dx \qquad (2.4) \\ &\le c\Bigg(I[u, \Omega_2] \\ &\qquad + (R-r)^{-\beta} \int_{B_R(z)} \left\{ h_1'\left(|(\nabla^2 u)_I|\right) + h_2'\left(|(\nabla^2 u)_{II}|\right)\right\}^2 \, dx \Bigg) . \end{aligned}$$

Here β is a suitable positive exponent and c denotes a positive constant both being independent of m. Let us have a closer look on the integrals involving $h'_{1,2}$ on the r.h.s. of (2.4): it holds

$$\begin{aligned} &\int_{B_R(z)} h_1'\left(|(\nabla^2 u)_I|\right)^2 \, dx \\ &= \int_{B_R(z) \cap [|(\nabla^2 u)_I| \le L]} \dots dx + \int_{B_R(z) \cap [|(\nabla^2 u)_I| \ge L]} \dots dx \\ &\le h_1'(L)^2 2\pi R^2 + cL^{-2} \int_{B_R(z) \cap [|(\nabla^2 u)_I| \ge L]} h_1^2\left(|(\nabla^2 u)_I|\right) \, dx \, , \end{aligned}$$

where we have used inequality (1.9) for h_1 and where $L > 0$ is arbitrary. Obviously the same estimate is valid for h_2 and if we choose

$$L = \lambda^{-1}(R-r)^{-\frac{\beta}{2}}$$

for a parameter $\lambda > 0$ and recall the estimate of h_i' in terms of the power $\overline{m} - 1$, then we get from (2.4) and the above inequalities

$$\begin{aligned} &\int_{B_r(z)} D^2 H(\nabla^2 u)(\partial_\alpha \nabla^2 u, \partial_\alpha \nabla^2 u)\, dx \qquad (2.5) \\ &\leq c\left[c(\lambda)(R-r)^{-\overline{\beta}} + \lambda^2 \int_{B_R(z)} \left\{ h_1^2\left(|(\nabla^2 u)_I)\right] + h_2^2\left(|(\nabla^2 u)_{II}|\right)\right\}\, dx\right] \end{aligned}$$

for a new positive exponent $\overline{\beta}$ and a constant c involving the energy of u on Ω_2.

Suppose now that we have fixed a disc $B_R(z) \Subset \Omega_2$. If $\rho \in (0, R)$, we then let $r := \frac{1}{2}(\rho + R)$ and choose $\eta \in C_0^\infty(B_r(z))$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_\rho(z)$, $|\nabla\eta| \leq c/(r-\rho)(= 2c/(R-\rho))$. Sobolev's inequality yields

$$\begin{aligned} &\int_{B_\rho(z)} \left[h_1\left(|(\nabla^2 u)_I|\right)^2 + h_2\left(|(\nabla^2 u)_{II}|\right)^2\right]\, dx \\ &\leq \int_{B_r(z)} \left[\left(\eta h_1\left(|(\nabla^2 u)_I|\right)\right)^2 + \left(\eta h_2\left(|(\nabla^2 u)_{II}|\right)\right)^2\right]\, dx \\ &\leq c\left[\int_{B_r(z)} |\nabla\eta| \left\{h_1\left(|(\nabla^2 u)_I|\right) + h_2\left(|(\nabla^2 u)_{II}|\right)\right\}\, dx\right. \\ &\qquad \left. + \int_{B_r(z)} h_1'\left(|(\nabla^2 u)_I|\right) |\nabla(\nabla^2 u)_I|\, dx + \int_{B_r(z)} h_2'\left(|(\nabla^2 u)_{II}|\right) |\nabla(\nabla^2 u)_{II}|\, dx\right]^2 \\ &\leq c(R-\rho)^{-2}\left(\int_{B_R(z)} H(\nabla^2 u)\, dx\right)^2 \\ &\qquad + c\left[\int_{B_r(z)} h_1'(|(\nabla^2 u)_I|)|\nabla(\nabla^2 u)_I|\, dx\right. \\ &\qquad \left. + \int_{B_r(z)} h_2'(|(\nabla^2 u)_{II}|)|\nabla(\nabla^2 u)_{II}|\, dx\right]^2. \end{aligned}$$

To the quantity $[\dots]^2$ we apply Hölder's ineqality in combination with (1.9):

$$
\begin{aligned}
[\dots]^2 \le & \int_{B_r(z)} \frac{h_1'(|(\nabla^2 u)_I|)}{|(\nabla^2 u)_I|} |\nabla(\nabla^2 u)_I|^2 \, dx \int_{B_r(z)} h_1'(|(\nabla^2 u)_I|)|(\nabla^2 u)_I| \, dx \\
& + \int_{B_r(z)} \frac{h_2'(|(\nabla^2 u)_{II}|)}{|(\nabla^2 u)_{II}|} |\nabla(\nabla^2 u)_{II}|^2 \, dx \int_{B_r(z)} h_2'(|(\nabla^2 u)_{II}|)|(\nabla^2 u)_{II}| \, dx \\
& \le c \int_{\Omega_2} H(\nabla^2 u) \, dx \\
& \quad \cdot \left\{ \int_{B_r(z)} \frac{h_1'(|(\nabla^2 u)_I|)}{|(\nabla^2 u)_I|} |\nabla(\nabla^2 u)_I|^2 \, dx \right. \\
& + \left. \int_{B_r(z)} \frac{h_2'(|(\nabla^2 u)_{II}|)}{|(\nabla^2 u)_{II}|} |\nabla(\nabla^2 u)_{II}|^2 \, dx \right\} .
\end{aligned}
$$

If we use the first inequality from (1.10) with the choices $Z = \nabla^2 u$ and $Y = \partial_1 \nabla^2 u, \partial_2 \nabla^2 u$ and add the results, then we obtain

$$
\{\dots\} \le \int_{B_r(z)} D^2 H(\nabla^2 u)(\partial_\alpha \nabla^2 u, \partial_\alpha \nabla^2 u) \, dx \, ,
$$

hence

$$
\begin{aligned}
& \int_{B_\rho(z)} \left[h_1 \left(|(\nabla^2 u)_I|\right)^2 + h_2 \left(|(\nabla^2 u)_{II}|\right)^2 \right] \, dx \\
& \quad \le \; c \left\{ (R-\rho)^{-2} + \int_{B_r(z)} D^2 H(\nabla^2 u)(\partial_\alpha \nabla^2 u, \partial_\alpha \nabla^2 u) \, dx \right\} ,
\end{aligned}
$$

and by applying (2.5) we find (w.l.o.g $\overline{\beta} \ge 2$)

$$
\begin{aligned}
& \int_{B_\rho(z)} \left[h_1 \left(|(\nabla^2 u)_I|\right)^2 + h_2 \left(|(\nabla^2 u)_{II}|\right)^2 \right] \, dx \qquad (2.6) \\
& \quad \le \; c \left[c(\lambda)(R-\rho)^{-\overline{\beta}} + \lambda^2 \int_{B_R(z)} \left\{ h_1(|(\nabla^2 u)_I|)^2 + h_2(|(\nabla^2 u)_{II}|)^2 \right\} \, dx \right]
\end{aligned}
$$

valid for all discs $B_\rho(z) \subset B_R(z) \Subset \Omega_2$ and any $\lambda > 0$. Choosing $\lambda = 1/\sqrt{2c}$, a well known lemma (see [Gi1], Lemma 3.1, p. 161) applies to (2.6) with the result that

$$
h_1(|(\nabla^2 u)_I|)^2 + h_2(|(\nabla^2 u)_{II}|)^2 \in L^1_{\mathrm{loc}}(\Omega_2)
$$

is true uniformly w.r.t. the hidden parameter m. But then (2.5) shows the same for $D^2H(\nabla u)(\partial_\alpha \nabla^2 u, \partial_\alpha \nabla^2 u)$, and (1.11) implies $u \in W^3_{2,\mathrm{loc}}(\Omega_2)$, which proves part a) of Theorem 1.3 by quoting Sobolev's embedding theorem one more time. $\square$

3 Hölder continuity of the second derivatives

Assume now that the hypothesis of Theorem 1.3 b) hold. Keeping the notation from Section 2 and referring again to [BF1], [BF2] we first observe that u (more precisely u_m) can be replaced on the r.h.s. of (2.2) by $u - P$, where P is a polynomial of degree ≤ 2. Letting

$$\Phi^2 := D^2H(\nabla^2 u)(\partial_\alpha \nabla^2 u, \partial_\alpha \nabla^2 u), \quad \sigma := DH(\nabla^2 u)$$

and choosing η such that $\eta = 1$ on $B_r(z_0)$, $\operatorname{spt}\eta \subset B_{2r}(z_0)$, $0 \leq \eta \leq 1$, $|\nabla^l \eta| \leq c/r^l$, $l = 1, 2$, for a disc $B_{2r}(z_0) \Subset \Omega_1$, we obtain from (2.2)

$$\begin{aligned} &\int_{B_{2r}(z_0)} \eta^6 \Phi^2 \, dx \qquad (3.1)\\ &\leq \; c \int_{B_{2r}(z_0)} |\nabla\sigma| \left[|\nabla^2 \eta^6||\nabla u - \nabla P| + |\nabla \eta^6||\nabla^2 u - \nabla^2 P| \right] \, dx \, . \end{aligned}$$

The Cauchy-Schwarz inequality applied to the bilinear form $D^2H(\nabla^2 u)$ implies

$$\begin{aligned} |\nabla\sigma|^2 &= D^2H(\nabla^2 u)(\partial_\alpha \nabla^2 u, \partial_\alpha \sigma) \\ &\leq D^2H(\nabla^2 u)(\partial_\alpha \nabla^2 u, \partial_\alpha \nabla^2 u)^{1/2} D^2H(\nabla^2 u)(\partial_\alpha \sigma, \partial_\alpha \sigma)^{1/2} \\ &\leq \Phi |D^2H(\nabla^2 u)|^{1/2} |\nabla\sigma| \, , \end{aligned}$$

i.e. $|\nabla\sigma| \leq \Phi |D^2H(\nabla^2 u)|^{1/2}$, and the second inequality from (1.10) gives

$$\begin{aligned} |D^2H(\nabla^2 u)|^{1/2} &\leq c \Bigg[(1 + |(\nabla^2 u)_I|^2)^{\frac{\omega}{4}} \sqrt{\frac{h_1'(|(\nabla^2 u)_I|)}{|(\nabla^2 u)_I|}} \\ &\qquad + (1 + |(\nabla^2 u)_{II}|^2)^{\frac{\omega}{4}} \sqrt{\frac{h_2'(|(\nabla^2 u)_{II}|)}{|(\nabla^2 u)_{II}|}} \;\Bigg] \\ &=: c \left[\widetilde{\Psi}_1 + \widetilde{\Psi}_2 \right] . \end{aligned}$$

Inserting these estimates into (3.1) we obtain by letting $\widetilde{\Psi} := \left(\widetilde{\Psi}_1^2 + \widetilde{\Psi}_2^2\right)^{1/2}$

$$\int_{B_r(z_0)} \Phi^2\, dx \;\leq\; c\left[\frac{1}{r}\int_{B_{2r}(z_0)} |\nabla^2 u - \nabla^2 P|\Phi\widetilde{\Psi}\, dx + \frac{1}{r^2}\int_{B_{2r}(z_0)} |\nabla u - \nabla P|\Phi\widetilde{\Psi}\, dx\right]. \tag{3.2}$$

Letting $\gamma = 4/3$ we can now follow the calculations in [BF1] leading from (2.18) to (2.21) in this reference, which means that after appropriate choice of P and proper applications of the Sobolev-Poincaré and the Poincaré inequality on the r.h.s. of (3.2) we obtain from (3.2) the basic estimate

$$\left[\fint_{B_r(z_0)} \Phi^2\, dx\right]^{\frac{1}{2}} \leq c\left[\fint_{B_{2r}(z_0)} (\Phi\widetilde{\Psi})^\gamma\, dx\right]^{\frac{1}{\gamma}}. \tag{3.3}$$

We recall one more time that (3.3) actually is valid for the approximations u_m, i.e. we have $\Phi = \Phi_m$, etc., but the constant c appearing in (3.3) is independent of m. Let us also note that during the derivation of (3.3) one needs the information that

$$|\nabla^3 u| \leq c\Phi \leq c\Phi\widetilde{\Psi}$$

which follows from (1.11). In order to continue as outlined after (2.21) in [BF1] we only have to check that

$$\exp(\beta\widetilde{\Psi}^2) \in L^1(\Omega_1) \tag{3.4}$$

holds for any $\beta > 0$, since $\Phi \in L^2(\Omega_1)$ (uniformly w.r.t. the index m) has already been shown in Section 2. Let us introduce the auxiliary functions

$$\Psi_1 := \int_0^{|(\nabla^2 u)_I|} \sqrt{\frac{h_1'(t)}{t}}\, dt, \quad \Psi_2 := \int_0^{|(\nabla^2 u)_{II}|} \sqrt{\frac{h_2'(t)}{t}}\, dt,$$

for which we have by the first inequality in (1.10)

$$|\nabla\Psi_1|^2 + |\nabla\Psi_2|^2 \leq c\Phi^2,$$

moreover (1.9) implies

$$\Psi_1^2 + \Psi_2^2 \leq cH(\nabla^2 u),$$

so that Ψ_1, Ψ_2 belong to $W^1_2(\Omega_1)$ and therefore $\Psi := (\Psi_1^2 + \Psi_2^2)^{1/2}$ is in the same space (uniform in m). Thus we can apply Trudinger's inequality (see [GT], Theorem 7.15) and find $\beta_0 > 0$ s.t. for discs $B_\rho \subset \Omega_1$ we have

$$\int_{B_\rho} \exp(\beta_0 \Psi^2)\, dx \le c(\rho)\,. \tag{3.5}$$

On the set $[|(\nabla^2 u)_I| \ge 1]$ it holds (recall (1.9))

$$\widetilde{\Psi}_1 \le c|(\nabla^2 u)_I|^{\frac{\omega}{2}-1} h_1(|(\nabla^2 u)_I|)^{\frac{1}{2}}\,,$$

whereas

$$\Psi_1 \ge \int_{|(\nabla^2 u)_I|/2}^{|(\nabla^2 u)_I|} \sqrt{\frac{h_1'(t)}{t}}\, dt \ge c h_1(|(\nabla^2 u)_I|)^{\frac{1}{2}}\,,$$

thus

$$\widetilde{\Psi}_1 \le c|(\nabla^2 u)_I|^{\frac{\omega}{2}-1} \Psi_1 \quad \text{on} \quad [|(\nabla^2 u)_I| \ge 1]\,.$$

From ii) after (A3) we obtain

$$\Psi_1 \le c h_1(|(\nabla^2 u)_I|^{\frac{1}{2}} \le c|(\nabla^2 u)_I|^{\frac{\overline{m}}{2}}\,,$$

and for $\delta > 0$ it follows

$$\widetilde{\Psi}_1 \le c \Psi_1^{1-\delta} |(\nabla^2 u)_I|^{\frac{\omega}{2}-1+\delta\frac{\overline{m}}{2}}$$

on $[|(\nabla^2 u)_I| \ge 1]$. Since we assume $\omega < 2$ we can fix $\delta > 0$ s.t.

$$\frac{\omega}{2} - 1 + \delta \frac{\overline{m}}{2} < 0$$

and Young's inequality gives for any $\mu > 0$

$$\widetilde{\Psi}_1^2 \le \mu \Psi_1^2 + c(\mu)$$

on the relevant set.
On $[|(\nabla^2 u)_I| \le 1]$ this inequality is immediate, and clearly the same arguments apply to $\Psi_2, \widetilde{\Psi}_2$, hence we have a.e.

$$\widetilde{\Psi}^2 \le \mu \Psi^2 + c(\mu)\,, \tag{3.6}$$

and (3.4) follows from (3.5) and (3.6) with $\mu := \beta_0/\beta$. Now the proof of Theorem 1.3 b) can be completed exactly with the same arguments as applied in [BF1], p. 361. □

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