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Abstract. The new systematization in measure and integration due to the author produced a version of the Kolmogorov projective limit theorem which is far more comprehensive than the previous ones. The present article is devoted to several consequences. In particular one obtains a topological version which applies to arbitrary Hausdorff spaces.

1. Introduction and Preliminaries

The present article is part of the author's new systematization in measure and integration, of which the latest introduction and account is in [13]. One of the fundamental points is the *projective limit theorem* of Kolmogorov [5]. Our new form of the theorem [9][10] is for *inner* $\bullet$ *premeasures* of mass one (=:prob), where this time we assume $\bullet = \sigma\tau$. The assertion is the one-to-one correspondence between the appropriate inner premeasures on an infinite product space $X = \prod_{t \in T} Y_t$ and the consistent families of such ones on the collection of all finite partial products $Y_p = \prod_{t \in p} Y_t$. Our *inner extension theorem* with the incorporation of $\bullet = \tau$ allows for the first time to overcome the barrier of *countably determined* subsets of X in a natural manner, and thus to arrive at an adequate concept and treatment of stochastic processes [9][10][11][12].

The present paper wants to develop three consequences of our Kolmogorov type projective limit theorem. The first consequence in section 3 is a *topological* version of the theorem which applies to *all Hausdorff spaces*. In contrast, the conventional theorems are restricted to Polish spaces or to Borel spaces, the Borel subspaces of Polish spaces, and in Rao [16] to Hausdorff spaces with *countable base*. The second consequence in section 4 is concerned with the frequent form of the conventional projective limit theorem which assumes inner regularity not for all members of the relevant consistent families but restricted to the simplest ones which live on the factors Y_t. In our framework this is not a natural assumption, but we show that our theorem is apt to include the situation.

In all these cases we first present the results in our natural form for inner $\bullet$ prob premeasures on X. Then we specialize to the conventional forms for measures on σ algebras. As a rule these domains consist of countably determined subsets of X, and thus are much too small in case of an uncountable

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index set T. As a consequence it must be expected that the inner regularity structure of the resultant measures cannot be expressed.

The third consequence in section 5 is devoted to the old method to obtain decent projective limit measures via *compactification* of X. For this extended matter we refer to Bogachev [2] Vol.II pp.447-448. It is obvious that our new systematization renders the method obsolete. However, we cannot resist to demonstrate that it can lead to situations which are best described as *compactification catastrophes*: the basic space X is turned into an inner null set!

The present section 1 continues with a few preliminaries, most of them previous results collected for convenience. Then section 2 recalls our Kolmogorov type projective limit theorem, combined with a certain variant which will be useful in the sequel.

Preliminaries on Set Systems. Most of the basic terms are as defined in [6][8][14]. A nonvoid set system $\mathfrak{S}$ on a nonvoid set X is called a *paving*. We define $\mathfrak{S}^\star \subset \mathfrak{S}^\sigma \subset \mathfrak{S}^\tau$ to consist of the *unions* of the nonvoid finite/countable/arbitrary subsystems of $\mathfrak{S}$, and $\mathfrak{S}_\star \subset \mathfrak{S}_\sigma \subset \mathfrak{S}_\tau$ to consist of the respective intersections. We also recall the *shorthand notation* $\bullet = \star\sigma\tau$. The first two remarks have obvious proofs.

1.1 Remark. *Let $H : X \to Y$ be a map between nonvoid sets X and Y, and $\mathfrak{B}$ be a paving in Y. Then*

$$H^{-1}\Big(\bigcup_{B\in\mathfrak{B}} B\Big) = \bigcup_{B\in\mathfrak{B}} H^{-1}(B) \quad \textit{and} \quad H^{-1}\Big(\bigcap_{B\in\mathfrak{B}} B\Big) = \bigcap_{B\in\mathfrak{B}} H^{-1}(B),$$
$$H^{-1}(\mathfrak{B}^\bullet) = (H^{-1}(\mathfrak{B}))^\bullet \quad \textit{and} \quad H^{-1}(\mathfrak{B}_\bullet) = (H^{-1}(\mathfrak{B}))_\bullet.$$

1.2 Remark. *Let the $\mathfrak{S}_l$ be pavings in $Y_l\,(l = 1, \cdots, n)$, and thus $\mathfrak{S} := \big(\prod_{l=1}^{n} \mathfrak{S}_l\big)^\star$ a paving in $Y := \prod_{l=1}^{n} Y_l$. If the $\mathfrak{S}_l$ are lattices/rings/algebras then $\mathfrak{S}$ is a lattice/ring/algebra as well.*

We turn to the *set-theoretical* notions of compactness initiated in Marczewski [15]. These notions are weaker and more flexible than topological compactness, and will be fundamental in the sequel. The paving $\mathfrak{S}$ is defined to be $\bullet$ *compact* iff each $\bullet$ subpaving of $\mathfrak{S}$ with intersection $\varnothing$ has a *finite* subpaving with intersection $\varnothing$. In case $\bullet = \star$ this is obvious for all $\mathfrak{S}$.

1.3 Examples. 1) In a Hausdorff topological space X the compact subsets form a τ compact paving Comp(X). 2) If the paving $\mathfrak{S}$ in X is $\bullet$ compact then $\mathfrak{S} \cup \{X\}$ is a $\bullet$ compact paving as well. This is a trivial remark, but its trivial nature comes to an abrupt end when one passes to infinite products, in particular to uncountable products.

We recall from [7] a few basic properties, of which 2) is the deepest one.

1.4 Properties. 1) *If the paving $\mathfrak{S}$ is $\bullet$ compact then $\mathfrak{S}_\bullet$ is $\bullet$ compact as well.* 2) *If the paving $\mathfrak{S}$ is $\bullet$ compact then $\mathfrak{S}^\star$ is $\bullet$ compact as well.* 3) *Let T be a nonvoid index set, and for each $t \in T$ let $\mathfrak{S}_t$ be a $\bullet$ compact paving in X_t. Then the product paving $\mathfrak{S} := \prod_{t\in T} \mathfrak{S}_t$ in $X := \prod_{t\in T} X_t$ is $\bullet$ compact as well.*

Preliminaries on Inner Premeasures. We start to recall from [9] the simple remark 1.4 and the important transformation theorem 3.10 with 3.12.

1.5 Remark. *Let $\mathfrak{S}$ be a lattice with $\varnothing$ in X. The inner $\bullet$ premeasures $\varphi : \mathfrak{S} \to [0, \infty[$ and the inner $\bullet$ premeasures $\psi : \mathfrak{S}_\bullet \to [0, \infty[$ are in one-to-one correspondence via $\varphi = \psi|\mathfrak{S}$. Moreover $\varphi_\bullet = \psi_\bullet = \psi_\star$.*

1.6 Theorem. *Let $H : X \to Y$ be a map between X and Y, and let $\mathfrak{S}$ in X and $\mathfrak{T}$ in Y be lattices with $\varnothing$ which fulfil*

$$H(\mathfrak{S}_\bullet) \subset \mathfrak{T}_\bullet \quad \textit{and} \quad H^{-1}(\mathfrak{T}_\bullet) \subset \mathfrak{S}\top\mathfrak{S}_\bullet.$$

If $\varphi : \mathfrak{S} \to [0, \infty[$ is an inner $\bullet$ premeasure and $\psi := \varphi_\bullet\big(H^{-1}(\cdot)\big)|\mathfrak{T} < \infty$, then $\psi : \mathfrak{T} \to [0, \infty[$ is an inner $\bullet$ premeasure as well. It satisfies $\psi_\bullet = \varphi_\bullet\big(H^{-1}(\cdot)\big)$ partout and $\mathfrak{C}(\psi_\bullet) = \{B \subset Y : H^{-1}(B) \in \mathfrak{C}(\varphi_\bullet)\}$.

1.7 Remark. *Let $H : X \to Y$ be a map between X and Y, and assume that the paving $\mathfrak{S}$ in X is $\bullet$ compact and fulfils $H^{-1}(\{b\}) \in \mathfrak{S}\top\mathfrak{S}_\bullet$ for all $b \in Y$. Then*

$$H\Big(\bigcap_{M\in\mathfrak{M}} M\Big) = \bigcap_{M\in\mathfrak{M}} H(M) \textit{ for all } \mathfrak{M} \subset \mathfrak{S} \textit{ nonvoid } \bullet \textit{ downward directed.}$$

Thus if $\mathfrak{S}$ is stable under finite intersections then $H(\mathfrak{S}_\bullet) \subset (H(\mathfrak{S})_\bullet$.

We add another simple remark which extends [7] 2.11 and has a routine proof. At last we recall the important recent result [14] 5.3.

1.8 Remark. *Let $\mathfrak{P}$ be a lattice with $\varnothing$ in X and $\mathfrak{S} = \mathfrak{P} \cup \{X\}$. The inner $\bullet$ prob premeasures $\vartheta : \mathfrak{P} \to [0, \infty[$ and the inner $\bullet$ prob premeasures $\varphi : \mathfrak{S} \to [0, \infty[$ with* $\sup(\varphi|\mathfrak{P}) = 1$ *are in one-to-one correspondence via $\vartheta = \varphi|\mathfrak{P}$. It fulfils $\vartheta_\bullet = \varphi_\bullet$.*

1.9 Theorem. *Assume that the set function $\varphi : \mathfrak{S} \to [0, \infty[$ on the lattice $\mathfrak{S}$ with $\varnothing$ can be extended to a content $\alpha : \mathfrak{A} \to [0, \infty]$ on a ring $\mathfrak{A} \supset \mathfrak{S}$ which is inner regular $\mathfrak{S}$, and that φ is downward $\bullet$ continuous at $\varnothing$. Then φ is an inner $\bullet$ premeasure, and $\Phi := \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an extention of α.*

2. The Projective Limit Theorem for Inner Premeasures

In the present section the situation is as follows: Let T be an infinite index set, and $I = I(T)$ consist of the nonvoid finite subsets $p, q, \cdots$ of T. We assume a family $(Y_t)_{t\in T}$ of nonvoid sets Y_t with product set $X = \prod_{t\in T} Y_t$, and the finite partial products $Y_p = \prod_{t\in p} Y_t$ for $p \in I$. Let $H_p : X \to Y_p$ and $H_{pq} : Y_q \to Y_p$ for $p \subset q$ in I denote the canonical projections.

Next we assume a family $(\mathfrak{K}_t)_{t\in T}$ of lattices $\mathfrak{K}_t$ in Y_t such that $\mathfrak{K}_t$ contains the finite subsets of Y_t and is $\bullet$ *compact*, where $\bullet = \sigma\tau$. We form the partial products $\mathfrak{K}_p = \big(\prod_{t\in p} \mathfrak{K}_t\big)^\star$ in Y_p for $p \in I$, which retain these properties in view of 1.2 and 1.4. The decisive construct is

$$\begin{aligned} \mathfrak{S} : &= \Big\{\prod_{t\in T} S_t : S_t \in \mathfrak{K}_t \cup \{Y_t\} \text{ with } S_t = Y_t \text{ for almost all } t \in T\Big\}^\star \\ &= \Big(\bigcup_{p\in I} H_p^{-1}\big(\prod_{t\in p} \mathfrak{K}_t\big) \cup \{X\}\Big)^\star = \Big(\bigcup_{p\in I} H_p^{-1}(\mathfrak{K}_p) \cup \{X\}\Big)^\star, \end{aligned}$$

which is a lattice in X with $\varnothing$ and X. Moreover $\mathfrak{S}$ is $\bullet$ *compact* in view of $\mathfrak{S} \subset \big(\prod_{t\in T}\mathfrak{K}_t \cup \{Y_t\}\big)^\star$ and 1.4. We add a few properties which have routine proofs.

2.1 Properties. 1) *For $p \subset q$ in I we have*

$H_{pq}^{-1}(\mathfrak{K}_p) \subset \mathfrak{K}_q \top \mathfrak{K}_q$ and hence $H_{pq}^{-1}((\mathfrak{K}_p)_\bullet) \subset \mathfrak{K}_q \top (\mathfrak{K}_q)_\bullet$ from 1.1*,*

$H_{pq}(\mathfrak{K}_q) \subset \mathfrak{K}_p$ and hence $H_{pq}((\mathfrak{K}_q)_\bullet) \subset (\mathfrak{K}_p)_\bullet$ from 1.7*.*

2) *For $p \in I$ we have*

$H_p^{-1}(\mathfrak{K}_p) \subset \mathfrak{S}$ and hence $H_p^{-1}((\mathfrak{K}_p)_\bullet) \subset \mathfrak{S}_\bullet$ from 1.1*,*

$H_p(\mathfrak{S}) \subset \mathfrak{K}_p \top \mathfrak{K}_p$ and hence $H_p(\mathfrak{S}_\bullet) \subset \mathfrak{K}_p \top (\mathfrak{K}_p)_\bullet$ from 1.7*.*

We come to the projective limit theorem. The essentials of the present version appeared for the first time in [9] 5.3, and the full assertion in [10] theorem 9. In the sequel we shall often abbreviate $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet) =: \Phi$, etc.

2.2 Theorem. *There is a one-to-one correspondence between*

the families $(\varphi_p)_{p\in I}$ of inner $\bullet$ prob premeasures $\varphi_p : \mathfrak{K}_p \to [0,\infty[$ which are consistent *in the sense that $\varphi_p = (\varphi_q)_\bullet(H_{pq}^{-1}(\cdot))|\mathfrak{K}_p$ for $p \subset q$ in I,*

and the inner $\bullet$ prob premeasures $\varphi : \mathfrak{S} \to [0,\infty[$,

via $\varphi_p = \varphi(H_p^{-1}(\cdot))|\mathfrak{K}_p$ for $p \in I$. It fulfils $(\varphi_p)_\bullet = \varphi_\bullet(H_p^{-1}(\cdot))$ partout and $\mathfrak{C}((\varphi_p)_\bullet) = \{B \subset Y_p : H_p^{-1}(B) \in \mathfrak{C}(\varphi_\bullet)\}$ for $p \in I$.

Moreover if $A \in \mathfrak{S}_\bullet$ and hence $H_p(A) \in \mathfrak{K}_p \top (\mathfrak{K}_p)_\bullet \subset \mathfrak{C}\big((\varphi_p)_\bullet\big)$ for $p \in I$ from 2.1.2) *then $\Phi(A) = \inf_{p\in I}(\Phi_p(H_p(A))$.*

After this we turn to the variant announced above. We need a few preparations.

2.3 Remark. *There is a one-to-one correspondence between*

the above consistent families $(\varphi_p)_{p\in I}$ of inner $\bullet$ prob premeasures $\varphi_p : \mathfrak{K}_p \to [0,\infty[$,

and the families $(\psi_p)_{p\in I}$ of inner $\bullet$ prob premeasures $\psi_p : (\mathfrak{K}_p)_\bullet \to [0,\infty[$, consistent *in the sense that $\psi_p = (\psi_q)_\bullet(H_{pq}^{-1}(\cdot))|(\mathfrak{K}_p)_\bullet$ for $p \subset q$ in I,*

via $\varphi_p = \psi_p|\mathfrak{K}_p$ for $p \in I$. It fulfils $(\varphi_p)_\bullet = (\psi_p)_\bullet$ for $p \in I$.

Proof. For each fixed $p \in I$ one obtains from 1.5 a one-to-one correspondence between the individual φ_p and ψ_p which fulfils $(\varphi_p)_\bullet = (\psi_p)_\bullet$. It remains to show that $(\varphi_p)_{p\in I}$ is consistent $\Leftrightarrow$ $(\psi_p)_{p\in I}$ is consistent. The implication $\Leftarrow$ is obvious. To see $\Rightarrow$ we conclude from 2.1.1) that 1.6 can be applied for $p \subset q$ to $H_{pq} : Y_q \to Y_p$ and to $\mathfrak{K}_q$ and $\mathfrak{K}_p$. Application to φ_q and φ_p furnishes $(\varphi_p)_\bullet = (\varphi_q)_\bullet(H_{pq}^{-1}(\cdot))$ partout and hence the assertion. $\square$

Next we form two variants of $\mathfrak{S}$, the set systems in X defined to be

$$\mathfrak{P} := \Big(\bigcup_{p\in I} H_p^{-1}\big(\prod_{t\in p}\mathfrak{K}_t\big)\Big)^\star = \Big(\bigcup_{p\in I} H_p^{-1}(\mathfrak{K}_p)\Big)^\star \quad\text{and}\quad \mathfrak{Q} := \Big(\bigcup_{p\in I} H_p^{-1}\big((\mathfrak{K}_p)_\bullet\big)\Big)^\star.$$

It is obvious that

$$\bigcup_{p\in I} H_p^{-1}\big(\prod_{t\in p}\mathfrak{K}_t\big) = \bigcup_{p\in I}\{\prod_{t\in T} S_t : S_t \in \mathfrak{K}_t \text{ for } t \in p \text{ and } S_t = Y_t \text{ for } t \in T \setminus p\}$$

is stable under finite intersections, and 1.1 then implies the same for the two subsequent set systems. Thus $\mathfrak{P}$ and $\mathfrak{Q}$ are lattices with $\varnothing$ in X and

$\mathfrak{P} \subset \mathfrak{Q}$. From 1.1 we see that $\mathfrak{Q} \subset \mathfrak{P}_\bullet$ and hence $\mathfrak{Q}_\bullet \subset \mathfrak{P}_\bullet$, so that $\mathfrak{P}_\bullet = \mathfrak{Q}_\bullet$. Moreover $\mathfrak{S} = \mathfrak{P} \cup \{X\}$ and hence $\mathfrak{S}_\bullet = \mathfrak{P}_\bullet \cup \{X\}$. It follows that $\mathfrak{P}$ and $\mathfrak{Q}$ are $\bullet$ *compact*. We need one more fact.

2.4 PROPERTY. *For nonvoid $M \subset T$ let $Y_M = \prod_{t\in M} Y_t$, and $H_M : X \to Y_M$ denote the canonical projection. For $\bullet = \sigma\tau$ then*

$$\mathfrak{Q}_\bullet = \Big(\Big(\bigcup_{M\bullet} H_M^{-1}\big(\prod_{t\in M}\mathfrak{K}_t\big)\Big)^\star\Big)_\bullet = \Big(\Big(\bigcup_{M\bullet} H_M^{-1}\big(\big((\prod_{t\in M}\mathfrak{K}_t)^\star\big)_\bullet\big)\Big)^\star\Big)_\bullet.$$

Proof. $\subset$) is clear from $\mathfrak{Q}_\bullet = \mathfrak{P}_\bullet$. $\supset$) For $A \in H_M^{-1}\big(\prod_{t\in M}\mathfrak{K}_t\big)$ we have $A = H_M^{-1}\big(\prod_{t\in M} S_t\big)$ with $S_t \in \mathfrak{K}_t$ for $t \in M$, thus $A = \prod_{t\in T} S_t$ with $S_t = Y_t$ for $t \in T \setminus M$. It follows that

$$A = \bigcap_{p\subset M}\big(\prod_{t\in T} S_t^p\big) \quad \text{with} \quad S_t^p = S_t \in \mathfrak{K}_t \text{ for } t \in p \text{ and } S_t^p = Y_t \text{ for } t \in T \setminus p.$$

Here $\prod_{t\in T} S_t^p \in H_p^{-1}\big(\prod_{t\in p}\mathfrak{K}_t\big) \subset \mathfrak{P} \subset \mathfrak{Q}$ and hence $A \in \mathfrak{Q}_\bullet$. Thus we obtain the first representation. The second one then follows from 1.1. □

2.5 REMARK. *There is a one-to-one correspondence between*

the inner $\bullet$ prob premeasures $\varphi : \mathfrak{S} \to [0,\infty[$,
the inner $\bullet$ prob premeasures $\vartheta : \mathfrak{P} \to [0,\infty[$,
and the inner $\bullet$ prob premeasures $\psi : \mathfrak{Q} \to [0,\infty[$,

via $\vartheta = \varphi|\mathfrak{P}$ and $\vartheta = \psi|\mathfrak{P}$. It fulfils $\varphi_\bullet = \vartheta_\bullet = \psi_\bullet$.

Proof. The correspondence $\varphi \mapsto \vartheta$ follows from 1.8, because the projective limit theorem 2.2 asserts for each $p \in I$ that

$$\sup\big(\varphi|H_p^{-1}(\mathfrak{K}_p)\big) = \sup\big(\varphi(H_p^{-1}(\cdot))|\mathfrak{K}_p\big) = \sup\varphi_p = 1,$$

and hence that $\sup(\varphi|\mathfrak{P}) = 1$. The correspondence $\psi \mapsto \vartheta$ is clear from $\mathfrak{P} \subset \mathfrak{Q}$ and $\mathfrak{P}_\bullet = \mathfrak{Q}_\bullet$ combined with 1.5. □

Now the variant of the projective limit theorem 2.2 reads as follows.

2.6 THEOREM. *There is a one-to-one correspondence between*

the above consistent families $(\psi_p)_{p\in I}$ of inner $\bullet$ prob premeasures $\psi_p : (\mathfrak{K}_p)_\bullet \to [0,\infty[$,

and the inner $\bullet$ prob premeasures $\psi : \mathfrak{Q} \to [0,\infty[$,

via $\psi_p = \psi(H_p^{-1}(\cdot))|(\mathfrak{K}_p)_\bullet$ for $p \in I$. It fulfils $(\psi_p)_\bullet = \psi_\bullet(H_p^{-1}(\cdot))$ partout and $\mathfrak{C}((\psi_p)_\bullet) = \{B \subset Y_p : H_p^{-1}(B) \in \mathfrak{C}(\psi_\bullet)\}$ for $p \in I$.

Moreover if $A \in \mathfrak{Q}_\bullet$ and hence $H_p(A) \in (\mathfrak{K}_p)_\bullet \top (\mathfrak{K}_p)_\bullet \subset \mathfrak{C}((\psi_p)_\bullet)$ for $p \in I$ then $\Psi(A) = \inf_{p\in I} \Psi_p(H_p(A))$.

Proof. We have one-to-one correspondences

from 2.3 between the $(\psi_p)_{p\in I}$ and the $(\varphi_p)_{p\in I}$,
from 2.2 between the $(\varphi_p)_{p\in I}$ and the φ,
from 2.5 between the φ and the ϑ and between the ϑ and the ψ.

The correspondences fulfil

$$\begin{aligned} \varphi_p &= \psi_p|\mathfrak{K}_p && \text{and} && (\varphi_p)_\bullet = (\psi_p)_\bullet \text{ for } p \in I,\\ \varphi_p &= \varphi(H_p^{-1}(\cdot))|\mathfrak{K}_p && \text{and} && (\varphi_p)_\bullet = \varphi_\bullet(H_p^{-1}(\cdot)) \text{ for } p \in I,\\ \vartheta &= \varphi|\mathfrak{P} && \text{and} && \vartheta_\bullet = \varphi_\bullet,\\ \vartheta &= \psi|\mathfrak{P} && \text{and} && \vartheta_\bullet = \psi_\bullet. \end{aligned}$$

Moreover $\mathfrak{C}((\varphi_p)_\bullet) = \{B \subset Y_p : H_p^{-1}(B) \in \mathfrak{C}(\varphi_\bullet)\}$. Thus for $p \in I$ we obtain $(\psi_p)_\bullet = \psi_\bullet(H_p^{-1}(\cdot))$ partout, and $H_p^{-1}((\mathfrak{K}_p)_\bullet) \subset \mathfrak{Q}$ from the definition of $\mathfrak{Q}$ implies that $\psi_p = \psi(H_p^{-1}(\cdot))|(\mathfrak{K}_p)_\bullet$. The final assertion then follows in the same manner. □

3. The Topological Projective Limit Theorem

The present section retains the situation of the previous one, but specialized as follows: The Y_t for $t \in T$ are assumed to be Hausdorff topological spaces, with the Y_p for $p \in I$ as well as X and the Y_M for nonvoid $M \subset T$ equipped with the product topologies. Then we assume $\mathfrak{K}_t = \mathrm{Comp}(Y_t)$ for $t \in T$ and $\bullet = \tau$. Thus [7] 2.4.2) asserts that

$$\Big(\big(\prod_{t\in M}\mathfrak{K}_t\big)^\star\Big)_\tau = \Big(\big(\prod_{t\in M}\mathrm{Comp}(Y_t)\big)^\star\Big)_\tau = \mathrm{Comp}(Y_M) \quad \text{for nonvoid } M \subset T,$$

in particular $(\mathfrak{K}_p)_\tau = \mathrm{Comp}(Y_p)$ for $p \in I$. From the definition of $\mathfrak{Q}$ and from 2.4 we obtain

$$\mathfrak{Q} = \big(\bigcup_{p\in I} H_p^{-1}(\mathrm{Comp}(Y_p))\big)^\star \quad \text{and} \quad \mathfrak{Q}_\tau = \Big(\big(\bigcup_{M\neq\varnothing} H_M^{-1}(\mathrm{Comp}(Y_M))\big)^\star\Big)_\tau.$$

After this the previous variant 2.6 has the immediate specialization which follows.

3.1 Theorem. *There is a one-to-one correspondence between*
the consistent families $(\psi_p)_{p\in I}$ *of Radon prob premeasures*
$\psi_p : \mathrm{Comp}(Y_p) \to [0,\infty[$,
and the inner τ *prob premeasures* $\psi : \mathfrak{Q} \to [0,\infty[$,
via $\psi_p = \psi(H_p^{-1}(\cdot))|\mathrm{Comp}(Y_p)$ *for* $p \in I$. *It fulfils* $(\psi_p)_\tau = \psi_\tau(H_p^{-1}(\cdot))$ *partout and* $\mathfrak{C}((\psi_p)_\tau) = \{B \subset Y_p : H_p^{-1}(B) \in \mathfrak{C}(\psi_\tau)\}$ *for* $p \in I$.

Moreover if $A \in \mathfrak{Q}_\tau$ *and hence* $H_p(A) \in \mathrm{Comp}(Y_p)\top\mathrm{Comp}(Y_p) \subset \mathfrak{C}((\psi_p)_\tau)$ *for* $p \in I$ *then* $\Psi(A) = \inf_{p\in I}\Psi_p(H_p(A))$.

We recall from our *inner extension theorem* and *localization principle* the basic fact that $\mathrm{Bor}(Y_p) \subset \mathfrak{C}((\psi_p)_\tau)$ and hence $H_p^{-1}(\mathrm{Bor}(Y_p)) \subset \mathfrak{C}(\psi_\tau)$ for $p \in I$. And we note that $H_{pq}^{-1}(\mathrm{Bor}(Y_p)) \subset \mathrm{Bor}(Y_q)$ for $p \subset q$ in I, because the projections $H_{pq} : Y_q \to Y_p$ are continuous. Thus the main part of the above theorem can be reformulated as follows.

3.2 Theorem. *There is a one-to-one correspondence between*
the families $(\beta_p)_{p\in I}$ *of Radon prob measures* $\beta_p : \mathrm{Bor}(Y_p) \to [0,\infty[$,
consistent *in the sense that* $\beta_p = \beta_q(H_{pq}^{-1}(\cdot))|\mathrm{Bor}(Y_p)$ *for* $p \subset q$ *in* I,
and the inner τ *prob premeasures* $\psi : \mathfrak{Q} \to [0,\infty[$,
via $\beta_p|\mathrm{Comp}(Y_p) = \psi(H_p^{-1}(\cdot))|\mathrm{Comp}(Y_p)$ *for* $p \in I$. *It fulfils* $H_p^{-1}(\mathrm{Bor}(Y_p)) \subset \mathfrak{C}(\psi_\tau)$ *and* $\beta_p = \Psi(H_p^{-1}(\cdot))|\mathrm{Bor}(Y_p)$ *for* $p \in I$.

However, it is quite clear that the full assertion cannot be obtained without the use of *inner premeasures* on the side of the product space X. In fact, to express the inner regularity behaviour of $\Psi = \psi_\tau|\mathfrak{C}(\psi_\tau)$ requires the immense set system $\mathfrak{Q}_\tau$. An implication in the style of the conventional projective limit theorems would read as follows.

3.3 Consequence. *Let $(\beta_p)_{p\in I}$ be a consistent family of Radon prob measures $\beta_p : \mathrm{Bor}(Y_p) \to [0,\infty[$ as above. Then there exists a unique prob measure $\beta : \mathfrak{B} \to [0,\infty[$ on the σ algebra $\mathfrak{B} := \mathrm{A}\sigma\big(\bigcup\limits_{p\in I} H_p^{-1}(\mathrm{Bor}(Y_p))\big)$ in X which fulfils $\beta_p = \beta(H_p^{-1}(\cdot))|\mathrm{Bor}(Y_p)$ for $p \in I$.*

Proof. We know that $\mathfrak{B} \subset \mathfrak{C}(\psi_\tau)$. Thus $\beta := \Psi|\mathfrak{B}$ is as required. To see the uniqueness of β note that for $p \subset q$ in I one has $H_p = H_{pq} \circ H_q$ and hence

$$H_p^{-1}(\mathrm{Bor}(Y_p)) = H_q^{-1}\big(H_{pq}^{-1}(\mathrm{Bor}(Y_p))\big) \subset H_q^{-1}(\mathrm{Bor}(Y_q)),$$

which implies that $\bigcup\limits_{p\in I} H_p^{-1}(\mathrm{Bor}(Y_p))$ is stable under finite intersections. Thus the uniqueness assertion follows from the classical uniqueness theorem [6] 3.1.σ). □

Nevertheless not even the full assertion 3.3 for arbitrary Hausdorff topological spaces Y_t $\forall t \in T$ seems to be in the literature. The most comprehensive partial assertion known to the author is in Rao [16] Cor.9 pp.429-430, where the Y_t are assumed to have countable bases. Note that in this case

$$\mathrm{Bor}(Y_p) = \mathrm{A}\sigma\big(\prod_{t\in p} \mathrm{Bor}(Y_t)\big) \quad \text{and hence} \quad \mathfrak{B} = \mathrm{A}\sigma\Big(\bigcup_{p\in I} H_p^{-1}\big(\prod_{t\in p} \mathrm{Bor}(Y_t)\big)\Big),$$

in view of the well-known facts [6] 13.15 and 1.11. The usual formulations are for Polish spaces Y_t, for example in Bauer [1] 35.3, or for Borel spaces Y_t, the Borel subspaces of Polish spaces, for example in Kallenberg [3] 6.16 and in Klenke [4] 14.36. These spaces are well-known to be metrizable topological spaces with countable bases on which all finite Borel measures are Radon measures.

We conclude with the remark that the immense domain $\mathfrak{C}(\psi_\tau)$ of our maximal prob measure $\Psi = \psi_\tau|\mathfrak{C}(\psi_\tau)$ contains the set systems

$$H_M^{-1}(\mathrm{Comp}(Y_M)) \subset \mathfrak{Q}_\tau \subset \mathfrak{C}(\psi_\tau) \quad \text{for nonvoid } M \subset T,$$

and hence for $M = T$ the set system $\mathrm{Comp}(X) \subset \mathfrak{Q}_\tau \subset \mathfrak{C}(\psi_\tau)$. But we do not claim that the system $\mathrm{Cl}(X)$ of the closed subsets of X, which after [7] 2.4.1) is

$$\mathrm{Cl}(X) = \Big(\big(\bigcup_{p\in I} H_p^{-1}\big(\prod_{t\in p} \mathrm{Cl}(Y_t)\big)\big)^\star\Big)_\tau = \Big(\big(\bigcup_{p\in I} H_p^{-1}(\mathrm{Cl}(Y_p)\big)^\star\Big)_\tau,$$

be contained in $\mathfrak{Q}\top\mathfrak{Q}_\tau \subset \mathfrak{C}(\psi_\tau)$, and neither that $\mathrm{Bor}(X)$ be contained in $\mathfrak{C}(\psi_\tau)$, as it would of course be desirable.

4. The Projective Limit Theorem under weakened Inner Regularity

The present section retains the situation of section 2. However, the actual modified set-up requires that the *inner premeasures* be abandoned on the side of the partial products Y_p $\forall p\in I$. In return we assume an additional family $(\mathfrak{A}_t)_{t\in T}$ of σ algebras $\mathfrak{A}_t\supset\mathfrak{K}_t$, and form the partial product σ algebras $\mathfrak{A}_p=\mathrm{A}\sigma\big(\underset{t\in p}{\Pi}\mathfrak{A}_t\big)\supset\mathfrak{K}_p$ in Y_p for $p\in I$. We note that $H_{pq}^{-1}(\mathfrak{A}_p)\subset\mathfrak{A}_q$ and hence $H_p^{-1}(\mathfrak{A}_p)\subset H_q^{-1}(\mathfrak{A}_q)$ for $p\subset q$ in I. The present projective limit theorem then reads as follows.

4.1 THEOREM. *Let $(\alpha_p)_{p\in I}$ be a family of prob measures $\alpha_p:\mathfrak{A}_p\rightarrow[0,\infty[$ which is* consistent *in the sense that $\alpha_p=\alpha_q(H_{pq}^{-1}(\cdot))|\mathfrak{A}_p$ for $p\subset q$ in I. Assume that $\alpha_{\{t\}}=:\alpha_t$ is inner regular $\mathfrak{K}_t$ for $t\in T$. Then the set functions $\varphi_p:=\alpha_p|\mathfrak{K}_p$ form a consistent family $(\varphi_p)_{p\in I}$ of inner $\bullet$ prob premeasures $\varphi_p:\mathfrak{K}_p\rightarrow[0,\infty[$, and its counterpart $\varphi:\mathfrak{S}\rightarrow[0,\infty[$ after* 2.2 *fulfils $H_p^{-1}(\mathfrak{A}_p)\subset\mathfrak{C}(\varphi_\bullet)$ and $\alpha_p=\Phi(H_p^{-1}(\cdot))|\mathfrak{A}_p$ for $p\in I$.*

4.2 CONSEQUENCE. *On the σ algebra*

$$\mathfrak{A}:=\mathrm{A}\sigma\Big(\bigcup_{p\in I}H_p^{-1}(\mathfrak{A}_p)\Big)=\mathrm{A}\sigma\Big(\bigcup_{p\in I}H_p^{-1}\big(\underset{t\in p}{\Pi}\mathfrak{A}_t\big)\Big)\subset\mathfrak{C}(\varphi_\bullet)$$

in X there exists a unique prob measure $\alpha:\mathfrak{A}\rightarrow[0,\infty[$ which fulfils $\alpha_p=\alpha(H_p^{-1}(\cdot))|\mathfrak{A}_p$ for $p\in I$. This is $\alpha=\Phi|\mathfrak{A}$.

The conventional projective limit theorems of the actual kind are of course restricted to the main assertion of the consequence 4.2 (and to $\bullet=\sigma$), for example in Bogachev [2] Vol.II theorem 7.7.1 (here without the uniqueness assertion). As before in 3.3, the prob measure $\alpha:\mathfrak{A}\rightarrow[0,\infty[$ receives no statement of inner regularity.

The proof of the above results requires the subsequent lemma.

4.3 LEMMA. *Let $(\mathfrak{B}_t)_{t\in T}$ be a family of algebras $\mathfrak{B}_t\supset\mathfrak{K}_t$, and form the partial products $\mathfrak{B}_p:=\big(\underset{t\in p}{\Pi}\mathfrak{B}_t\big)^\star$ in Y_p for $p\in I$* (which are algebras after 1.2 and fulfil $H_{pq}^{-1}(\mathfrak{B}_p)\subset\mathfrak{B}_q$ and hence $H_p^{-1}(\mathfrak{B}_p)\subset H_q^{-1}(\mathfrak{B}_q)$ as above). *Let $(\beta_p)_{p\in I}$ be a consistent family of prob contents $\beta_p:\mathfrak{B}_p\rightarrow[0,\infty[$. If the $\beta_{\{t\}}=:\beta_t$ are inner regular $\mathfrak{K}_t$ for $t\in T$, then the β_p are inner regular $\mathfrak{K}_p$ for all $p\in I$.*

Proof. i) Let $B\in\underset{t\in p}{\Pi}\mathfrak{B}_t$, that is $B=\underset{t\in p}{\Pi}B_t$ with $B_t\in\mathfrak{B}_t$, and $\varepsilon>0$. There exist $K_t\in\mathfrak{K}_t$ with $K_t\subset B_t$ and $\beta_t(B_t\setminus K_t)<\frac{\varepsilon}{\#(p)}$ for $t\in p$. For $K:=\underset{t\in p}{\Pi}K_t\in\underset{t\in p}{\Pi}\mathfrak{K}_t\subset\mathfrak{K}_p$ then $K\subset B$ and

$$B\setminus K\subset\bigcup_{s\in p}(B_s\setminus K_s)\times Y_{p\setminus\{s\}}=\bigcup_{s\in p}H_{\{s\}p}^{-1}(B_s\setminus K_s)\in\mathfrak{B}_p,$$

$$\beta_p(B\setminus K)\leqq\sum_{s\in p}\beta_p\big(H_{\{s\}p}^{-1}(B_s\setminus K_s)\big)=\sum_{s\in p}\beta_s(B_s\setminus K_s)<\varepsilon.$$

ii) Let $B\in\mathfrak{B}_p$, that is $B=\overset{n}{\underset{l=1}{\cup}}B^l$ with $B^l\in\underset{t\in p}{\Pi}\mathfrak{B}_t$, and $\varepsilon>0$. From i) we obtain $K^l\in\underset{t\in p}{\Pi}\mathfrak{K}_t$ with $K^l\subset B^l$ and $\beta_p(B^l\setminus K^l)<\frac{\varepsilon}{n}$. It follows that $K:=\overset{n}{\underset{l=1}{\cup}}K^l\in\mathfrak{K}_p$ with $K\subset B$ and $B\setminus K\subset\overset{n}{\underset{l=1}{\cup}}(B^l\setminus K^l)$. Therefore $\beta_p(B\setminus K)<\varepsilon$. $\square$

Proof of 4.1. i) We form $\mathfrak{B}_p:=\big(\underset{t\in p}{\Pi}\mathfrak{A}_t\big)^\star$ for $p\in I$, which after 1.2 is an algebra in Y_p with $\mathfrak{A}_p=\mathrm{A}\sigma(\mathfrak{B}_p)=\mathrm{R}\sigma(\mathfrak{B}_p)\supset\mathfrak{B}_p\supset\mathfrak{K}_p$. The restriction

$\alpha_p|\mathfrak{B}_p$ is a prob content on $\mathfrak{B}_p$ which is inner regular $\mathfrak{K}_p$ from the above lemma 4.3, and its restriction $\alpha_p|\mathfrak{K}_p = \varphi_p$ to $\mathfrak{K}_p$ is downward $\bullet$ continuous at $\varnothing$ since $\mathfrak{K}_p$ is $\bullet$ compact. Thus 1.9 asserts that $\varphi_p : \mathfrak{K}_p \to [0,\infty[$ is an inner $\bullet$ premeasure, and in fact an inner $\bullet$ prob premeasure since $\sup \varphi_p = \sup(\alpha_p|\mathfrak{B}_p) = 1$, and that $\Phi_p = (\varphi_p)_\bullet|\mathfrak{C}((\varphi_p)_\bullet)$ is an extension of $\alpha_p|\mathfrak{B}_p$. That means $\mathfrak{B}_p \subset \mathfrak{C}((\varphi_p)_\bullet)$ and $\alpha_p = \Phi_p$ on $\mathfrak{B}_p$. The classical uniqueness theorem [6] 3.1.σ) then implies that $\alpha_p = \Phi_p$ on $\mathfrak{A}_p \subset \mathfrak{C}((\varphi_p)_\bullet)$, so that Φ_p is an extension of α_p.

ii) Now $\alpha_p = \alpha_q(H_{pq}^{-1}(\cdot))|\mathfrak{A}_p$ for $p \subset q$ in I says that

$$(\varphi_p)_\bullet = (\varphi_q)_\bullet(H_{pq}^{-1}(\cdot)) \ \text{ on } \mathfrak{A}_p, \text{ in particular on } \mathfrak{K}_p,$$

and hence that the family $(\varphi_p)_{p\in I}$ is consistent. For its counterpart $\varphi : \mathfrak{S} \to [0,\infty[$ after 2.2 it follows that $H_p^{-1}(\mathfrak{A}_p) \subset H_p^{-1}(\mathfrak{C}((\varphi_p)_\bullet) \subset \mathfrak{C}(\varphi_\bullet)$, and for $A \in \mathfrak{A}_p$ we have

$$\alpha_p(A) = \Phi_p(A) = (\varphi_p)_\bullet(A) = \varphi_\bullet(H_p^{-1}(A)) = \Phi(H_p^{-1}(A)). \ \square$$

Proof of 4.2. We know that $H_p^{-1}(\mathfrak{A}_p) \subset \mathfrak{A} \subset \mathfrak{C}(\varphi_\bullet)$ for $p \in I$, and that $\alpha = \Phi|\mathfrak{A}$ is as required. Thus it remains to prove the uniqueness assertion. Let $\lambda : \mathfrak{A} \to [0,\infty[$ be a prob measure which fulfils $\alpha_p = \lambda(H_p^{-1}(\cdot))|\mathfrak{A}_p$ for $p \in I$. Then $\alpha = \lambda$ on $\bigcup_{p\in I} H_p^{-1}(\mathfrak{A}_p) \subset \mathfrak{A}$, which is stable under finite intersections and even an algebra. Thus the classical uniqueness theorem [6] 3.1.σ) implies that $\alpha = \lambda$. $\square$

5. The Compactification Catastrophe

The final section likewise retains the situation of section 2. But on top of it we put a kind of *abstract compactification* which reads as follows: We assume a family $(\tilde{Y}_t)_{t\in T}$ of sets $\tilde{Y}_t \supset Y_t$, and define as before the product set $\tilde{X}$ and the finite partial products $\tilde{Y}_p$ for $p \in I$ with the canonical projections $\tilde{H}_p$ and $\tilde{H}_{pq}$ for $p \subset q$ in I. Next we assume a family $(\tilde{\mathfrak{K}}_t)_{t\in T}$ of lattices $\tilde{\mathfrak{K}}_t \supset \mathfrak{K}_t$ in $\tilde{Y}_t$ with the previous properties to contain the finite subsets of $\tilde{Y}_t$ and to be $\bullet$ compact, and moreover with the properties

for $K \subset Y_t : K \in \tilde{\mathfrak{K}}_t \Rightarrow K \in \mathfrak{K}_t$ and $K \in (\tilde{\mathfrak{K}}_t)_\bullet \Rightarrow K \in (\mathfrak{K}_t)_\bullet$,

and $\tilde{Y}_t \in \tilde{\mathfrak{K}}_t$.

The last point $\tilde{Y}_t \in \tilde{\mathfrak{K}}_t$ is the decisive one: it is the reason that the transition to the new entities can be viewed as a *compactification* (a simple example is $Y_t = \mathbb{R}$ with $\mathfrak{K}_t = \text{Comp}(\mathbb{R})$ and $\tilde{Y}_t = \overline{\mathbb{R}}$ with $\tilde{\mathfrak{K}}_t = \text{Comp}(\overline{\mathbb{R}})$ in the usual topologies). As before we then form the partial products $\tilde{\mathfrak{K}}_p$ for $p \in I$ and the construct $\tilde{\mathfrak{S}}$ in $\tilde{X}$. Of course all these formations retain the previous properties. Moreover we introduce the *injections* $E_p : Y_p \to \tilde{Y}_p$ for $p \in I$, and note a few properties which have routine proofs.

5.1 Properties. *We have* $E_p^{-1}(B) = B \cap Y_p$ *for* $B \subset \tilde{Y}_p$. *For* $p \in I$ *moreover*

$$E_p^{-1}(\tilde{\mathfrak{K}}_p) \subset \mathfrak{K}_p \top \mathfrak{K}_p \text{ and hence } E_p^{-1}((\tilde{\mathfrak{K}}_p)_\bullet) \subset \mathfrak{K}_p \top (\mathfrak{K}_p)_\bullet \text{ from } 1.1,$$

and of course $E_p(\mathfrak{K}_p) \subset \tilde{\mathfrak{K}}_p$ *and* $E_p((\mathfrak{K}_p)_\bullet) \subset (\tilde{\mathfrak{K}}_p)_\bullet$.

After this we assume a consistent family $(\varphi_p)_{p\in I}$ of inner $\bullet$ prob premeasures $\varphi_p : \mathfrak{K}_p \to [0,\infty[$, and its counterpart $\varphi : \mathfrak{S} \to [0,\infty[$ after 2.2. Then

5.1 asserts that 1.6 can be applied for $p \in I$ to $E_p : Y_p \to \tilde{Y}_p$ and to $\mathfrak{K}_p$ and $\tilde{\mathfrak{K}}_p$. Application to φ_p furnishes the inner $\bullet$ prob premeasure $\tilde{\varphi}_p : \tilde{\mathfrak{K}}_p \to [0,\infty[$ defined to be $\tilde{\varphi}_p := (\varphi_p)_\bullet(E_p^{-1}(\cdot))|\tilde{\mathfrak{K}}_p$. It fulfils $(\tilde{\varphi}_p)_\bullet = (\varphi_p)_\bullet(E_p^{-1}(\cdot))$ partout, that is

$$(\tilde{\varphi}_p)_\bullet(B) = (\varphi_p)_\bullet(B \cap Y_p) \quad \text{for } B \subset \tilde{Y}_p.$$

It follows that the family $(\tilde{\varphi}_p)p \in I$ is consistent: For $p \subset q$ in I and $B \subset \tilde{Y}_p$ we have

$$\begin{aligned}((\tilde{\varphi}_q)_\bullet\big(\tilde{H}_{pq}^{-1}(B)\big) &= (\varphi_q)_\bullet\big(\tilde{H}_{pq}^{-1}(B) \cap Y_q\big) \\ &= (\varphi_q)_\bullet\big(\{u \in Y_q : \tilde{H}_{pq}(u) = H_{pq}(u) \in B \text{ and hence } \in B \cap Y_p\}\big) \\ &= (\varphi_q)_\bullet\big(H_{pq}^{-1}(B \cap Y_p)\big) = (\varphi_p)_\bullet(B \cap Y_p) = (\tilde{\varphi}_p)_\bullet(B),\end{aligned}$$

because $(\varphi_q)_\bullet(H_{pq}^{-1}(\cdot)) = (\varphi_p)_\bullet$ partout from 1.6 as seen in the proof of 2.3. Thus we obtain from 2.2 for the family $(\tilde{\varphi}_p)_{p\in I}$ the counterpart $\tilde{\varphi} : \tilde{\mathfrak{S}} \to [0,\infty[$.

Our aim is to look at this inner $\bullet$ prob premeasure $\tilde{\varphi} : \tilde{\mathfrak{S}} \to [0,\infty[$, produced via a certain *compactification*, as for its behaviour on X, in particular in comparison with the inner $\bullet$ prob premeasure $\varphi : \mathfrak{S} \to [0,\infty[$, our previous direct result from the projective limit theorem 2.2.

5.2 Proposition. *Assume that $S \in \tilde{\mathfrak{S}}_\bullet$ with $S \subset X$. Then there exist families $(K_t)_{t\in T}$ of subsets $K_t \in \mathfrak{K}_t$ such that $S \subset \prod_{t\in T} K_t =: K$. These families fulfil $\tilde{\varphi}_\bullet(S) \leqq \inf_{p\in I} \varphi_p\big(\prod_{t\in p} K_t\big)$* (note that in case $\bullet = \tau$ the second member is $= \varphi_\tau(K)$ with $K \in \mathfrak{S}_\tau$).

Proof. i) We see from 2.1 and from $\tilde{Y}_t \in \tilde{\mathfrak{K}}_t$ for $t \in T$ that

$$H_t(S) = \tilde{H}_t(S) \in \tilde{H}_t(\tilde{\mathfrak{S}}_\bullet) \subset \tilde{\mathfrak{K}}_t \top (\tilde{\mathfrak{K}}_t)_\bullet = (\tilde{\mathfrak{K}}_t)_\bullet,$$

where several times $\{t\}$ has been abbreviated into t. It follows that $H_t(S) \in (\mathfrak{K}_t)_\bullet$, and hence that $H_t(S) \subset K_t$ for some $K_t \in \mathfrak{K}_t$. Thus we obtain in fact $S \subset \prod_{t\in T} K_t$ for a family $(K_t)_{t\in T}$ as claimed. ii) From 2.2 we know that

$$\tilde{\varphi}_\bullet(S) = \inf_{p\in I} (\tilde{\varphi}_p)_\bullet(\tilde{H}_p(S)) = \inf_{p\in I} (\tilde{\varphi}_p)_\bullet(H_p(S)) = \inf_{p\in I} (\varphi_p)_\bullet(H_p(S)).$$

Thus if $S \subset \prod_{t\in T} K_t$ as above, then $H_p(S) \subset \prod_{t\in p} K_t \in \mathfrak{K}_p$ for $p \in I$ and hence the assertion. $\square$

5.3 Consequence. *Assume that*

$$(\circ) \quad \inf_{p\in I} \varphi_p\big(\prod_{t\in p} K_t\big) = 0 \quad \textit{for all families } (K_t)_{t\in T} \textit{ in } (\mathfrak{K}_t)_{t\in T}.$$

Then $\tilde{\varphi}_\bullet(S) = 0$ for all $S \in \tilde{\mathfrak{S}}_\bullet$ with $S \subset X$, and hence $\tilde{\varphi}_\bullet(X) = 0$.

Thus in the situation $(\circ)$ the present compactification ends up with the maximum possible disaster: the maximum possible contrast to $X \in \mathfrak{C}(\varphi_\bullet)$ and $\Phi(X) = 1$ (but note that we did not claim $X \in \mathfrak{C}(\tilde{\varphi}_\bullet)$). Of course as a rule $(\circ)$ is not fulfilled - it cannot happen when $Y_t \in \mathfrak{K}_t$ for all $t \in T$, and it has been seen in [12] section 4 to be not true for the two most prominent stochastic processes in case $\bullet = \tau$. However, there is the familiar product example that $(\circ)$ can occur for both $\bullet = \sigma\tau$ when T is uncountable.

5.4 Example. For each $t \in T$ let $\gamma_t : \mathfrak{K}_t \to [0,\infty[$ be an inner $\bullet$ prob premeasure which fulfils $\gamma_t < 1$ (obvious examples on $\mathbb{R}$ are those with unbounded support in the sense of [6] pp.94-95). For $p \in I$ let $\varphi_p = \prod\limits_{t\in p}\gamma_t$ be the product inner $\bullet$ prob premeasure of the respective γ_t in the sense of [7] section 1. It is clear that the family $(\varphi_p)_{p\in I}$ is consistent. Now fix a family $(K_t)_{t\in T}$ of sets $K_t \in \mathfrak{K}_t$. Since $\gamma_t(K_t) < 1$ for all $t \in T$ and T is uncountable there exists an uncountable subset $M \subset T$ and a positive number $c < 1$ such that $\gamma_t(K_t) \leqq c$ for $t \in M$. It follows that

$$\varphi_p\Big(\prod_{t\in p} K_t\Big) = \prod_{t\in p}\gamma_t(K_t) \leqq c^{\#(p)} \quad \text{for the } p \in I \text{ with } p \subset M,$$

and hence $\inf\limits_{p\subset M}\varphi_p\Big(\prod\limits_{t\in p} K_t\Big) = 0$. □

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