Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 235

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Saarbrücken 2009

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 235
submitted: 18^{th} of May, 2009

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Abstract

This paper is concerned with computable and guaranteed upper bounds of the difference between exact solutions of variational inequalities arising in the theory of viscous fluids and arbitrary approximations in the corresponding energy space. Such estimates (also called error majorants of functional type) have been derived for the considered class of nonlinear boundary value problems in [11] with the help of variational methods based on duality theory from convex analysis. In the present paper it is shown that error majorants can be derived in a different way by certain transformations of the variational inequalities that define generalized solutions. The error bounds derived by this techniques for the velocity function differ from those obtained by the variational method. These estimates involve only global constants coming from Korn and Friedrichs type inequalities, which are not difficult to evaluate in case of Dirichlet boundary conditions. For the case of mixed boundary conditions, we also derive another form of the estimate which contains only one constant coming from the following assertion: the L^2 norm of a vector valued function from $H^1(\Omega)$ in the factor-space generated by the equivalence with respect to rigid motions is bounded by the L^2 norm of the symmetric part of the gradient tensor. Since for some "simple" domains like squares or cubes, the constants in this inequality can be found analytically (or numerically), we obtain a unified form of an error majorant for any domain that admits a decomposition into such subdomains.

Key words. A posteriori estimates of functional type, variational inequalities, viscous incompressible fluids, generalized Newtonian fluids.
AMS 2000 Classification: 65N15,76A05,35J60

1 Introduction

The main focus of our investigation is to deduce guaranteed a posteriori estimates of the difference between the exact solution and its approximation obtained by various numerical schemes for a system of variational inequalities modelling the stationary and also slow flow of certain viscous incompressible fluids. To be precise, we consider a bounded domain $\Omega \subset \mathbb{R}^d, d = 2, 3$, with Lipschitz boundary $\Gamma = \Gamma_D \cup \Gamma_N$, where Γ_D and Γ_N are two measurable nonintersecting subsets of Γ and where the Dirichlet part Γ_D is of positive measure, whereas the Neumann part Γ_N is allowed to degenerate. The problem then is to find a velocity field $u = u(x)$ and a pressure function $p = p(x)$, $x \in \Omega$, satisfying the following relations

$$-\mathrm{div}\sigma = f - \nabla p \quad \text{in} \quad \Omega \tag{1.1}$$
$$\mathrm{div} u = 0 \quad \text{in} \quad \Omega \tag{1.2}$$
$$\sigma \in \partial\Pi(\varepsilon(u)) \quad \text{in} \quad \Omega \tag{1.3}$$
$$u = u_0 \quad \text{on} \quad \Gamma_D \tag{1.4}$$
$$\sigma n = F \quad \text{on} \quad \Gamma_N\,. \tag{1.5}$$

Acknowledgement. The second author's research was supported by DAAD.

Here σ is the deviatoric part of the stress tensor, and in the constitutive relation (1.3) we require that σ is an element of the subdifferential of the potential Π, which is given by

$$\Pi(\varepsilon) := \frac{\nu}{2}|\varepsilon|^2 + \pi(\varepsilon)\,, \tag{1.6}$$

ν denoting a positive constant (the viscosity coefficient). In (1.6) π is a convex function on the space $\mathbb{S}^d$ of all symmetric $(d \times d)$- matrices being of at most quadratic growth, i.e. for some $L > 0$ we have

$$|\pi(\varepsilon)| \leq L(|\varepsilon|^2 + 1), \ \varepsilon \in \mathbb{S}^d\,. \tag{1.7}$$

As usual $\varepsilon(u)$ is the symmetric gradient of u, and in (1.5) n represents the exterior normal to $\partial\Omega$. Finally, we assume that we are given functions u_0, f, F such that

$$\operatorname{div} u_0 = 0, f \in L^2(\Omega;\mathbb{R}^d), \ F \in L^2(\Gamma_N;\mathbb{R}^d)\,. \tag{1.8}$$

The equations (1.1) - (1.5) therefore model a mixed Dirichlet-Neumann boundary value problem for a generalized (incompressible) Newtonian fluid, whose specific properties are characterized through the potential Π defined in (1.6). In order to get a suitable weak formulation of the problem described through the equations (1.1) - (1.5), we have to introduce appropriate function spaces: let $V := H^1(\Omega;\mathbb{R}^d)$ denote the standard Sobolev space of vector valued functions $\Omega \to \mathbb{R}^d$ from $L^2(\Omega;\mathbb{R}^d)$ such that the first order weak derivatives are also square integrable on Ω. By V_0 we denote the subspace of V containing the functions vanishing on Γ_D, and $\mathcal{H}^1$ consists of all solenoidal fields from V_0.

Now, assuming in addition to (1.8) that u_0 is an element of V, a generalized solution of (1.1) - (1.5) is defined as the function $u \in u_0 + \mathcal{H}^1$ such that

$$J[u] = \inf\{J[v] : v \in u_0 + \mathcal{H}^1\}\,, \tag{1.9}$$

where

$$J[w] := \int_\Omega \left[\frac{\nu}{2}|\varepsilon(w)|^2 + \pi(\varepsilon(w)) - f \cdot w\right] dx - \int_{\Gamma_N} F \cdot w\, ds\,.$$

Since π is convex, we have the lower bound $\pi(\varepsilon) \geq \pi(0) + \varepsilon : \eta \quad$ for all $\varepsilon \in \mathbb{S}^d$, where $\eta \in \partial\pi(0)$. This together with (1.6) shows that Π is coercive, and the existence of a unique solution u of (1.9) follows from the lower semicontinuity of J together with suitable variants of Korn's and Poincarés inequality. The minimization problem (1.9) is equivalent to the following system of variational inequalities: to find a function $u \in u_0 + \mathcal{H}^1$ such that

$$\begin{aligned} &\int_\Omega \left[\nu\varepsilon(u) : \varepsilon(w-u) + \pi(\varepsilon(w)) - \pi(\varepsilon(u))\right] \, dx \\ &\qquad - \int_\Omega f \cdot (w-u) \, dx - \int_{\Gamma_N} F \cdot (w-u) \, ds \geq 0 \end{aligned} \tag{1.10}$$

holds for all $w \in u_0 + \mathcal{H}^1$. For a proof of this fact we refer to [8, 10, 16].

Numerical methods for nonlinear variational problems of type (1.9) (variational inequalities) have been studied by many authors (e.g., see [14]). In many practically interesting cases it is known that finite dimensional approximations of elliptic variational inequalities converge to the corresponding exact solutions provided that the approximation subspaces possess some additional properties. Thus, a certain sequence of approximations $\{v_k\} \in u_0 + \mathcal{H}^1$ converging to the minimizer u can be constructed numerically. An important question is how to guarantee that a desired accuracy is indeed achieved. The purpose of the present paper is to show that computable upper bounds for the error $\|\varepsilon(u-v)\|_\Omega$ ($\|\,.\,\|_\Omega$ denoting the L^2 norm w.r.t. Ω) can be derived from the variational inequality (1.10). The desired a posteriori error estimate is of the form

$$\|\varepsilon(u-v)\|_\Omega \leq \mathcal{M}(v, \mathcal{D}),$$

where $\mathcal{D}$ is the set of the problem data and where the functional $\mathcal{M}$ satisfies the following natural requirements:

i.) $\mathcal{M}$ is explicitly computable for any function v, and the quantity $\mathcal{M}$ inherits a clear physical meaning.

ii.) $\mathcal{M}(v, \mathcal{D})$ vanishes iff $v = u$; moreover $\mathcal{M}(v_k, \mathcal{D}) \to 0$ as $v_k \to u$ in the energy norm.

iii.) $\mathcal{M}(v, \mathcal{D})$ is a realistic upper bound for the error, which means that during the derivation of the estimate one should carefully try to avoid overestimation.

Such functional type a posteriori estimates in the setting of fluids have been derived successfully for many boundary value problems (see [20] for a systematic overview and further references).

The framework of the problem studied in the present note looks similar to the setting of the paper [11], let us therefore comment on the differences:

In [11] (as in [2, 3, 18, 19] and some other papers), the derivation of error majorants was based on variational techniques that follow from the principles of duality theory in convex analysis. In this paper we show that suitable majorants can be derived in a quite different way, namely by transformations of the underlying variational inequalities that define generalized solutions. Using this method, we obtain computable upper bounds of the difference between the exact solution and any admissible approximation for a wider class of dissipative potentials $\Pi(\varepsilon)$ than studied in [11].

Moreover, we here consider the case of a mixed boundary condition. The estimates derived in the present setting differ from those in [11], but have the same principal structure. We cannot prove that one estimate is better (more general) than another and believe that depending on a concrete problem one or another estimate may be preferable.

Our paper is organized as follows: in Section 2 we derive a first estimate for the error $\|\varepsilon(u-v)\|_\Omega$, when v is any approximation from $u_0 + \mathcal{H}^1$. This estimate involves one global constant $C(\Omega, \Gamma_N)$ resulting from an application of Friedrichs', Korn's and suitable trace inequalities valid for Ω (and Γ_N). In case of Dirichlet boundary conditions the constants in Friedrichs' and Korn's inequalities are easy to evaluate which leads to an guaranteed upper bound of the constant in the majorant. However, for mixed boundary conditions and complicated Ω (especially if $d = 3$) it might be difficult to find guaranteed and realistic upper bound for $C(\Omega, \Gamma_N)$. Therefore, in Section 3 we derive another form of the estimate, which contains only one constant C_R coming from the inequality (3.2). For some "standard" domains (e.g., for squares, cubes or triangles) C_R can be found analytically (or computed numerically). Thus, for all domains that admit a decomposition into such "elementary" subdomains a unified form of an error majorant containing only explicitly known constants is constructed. Section 4 is devoted to linearized models. In it, we derive upper estimates of the difference between u and the exact solution $u_{\mathcal{L}}$ of the problem linearized in a neighborhood of a certain given velocity function. In Section 5 we consider nonsolenoidal approximations v which makes it necessary to estimate the distance of v to the space of solenoidal fields. Finally, in Section 6 we present a short proof of rather elementary lemma which we need throughout our calculations.

2 Derivation Of Error Majorants From The Variational Inequality (1.10)

Let $u \in u_0 + \mathcal{H}^1$ denote the unique solution of (1.10) and consider any approximation v from the same class. We rewrite inequality (1.10) in the form

$$
\begin{aligned}
(2.1) \qquad & \int_\Omega [\nu\varepsilon(u-v) : \varepsilon(v-u) + \pi(\varepsilon(v)) - \pi(\varepsilon(u))] \, dx \\
& \geq \int_\Omega f \cdot (v-u) \, dx + \int_{\Gamma_N} F \cdot (v-u) \, ds - \int_\Omega \nu\varepsilon(v) : \varepsilon(v-u) \, dx
\end{aligned}
$$

and observe (π^* denoting the conjugate function of π) the validity of

$$
\int_\Omega [\pi(\varepsilon(u)) + \pi^*(\eta) - \eta : \varepsilon(u)] \, dx \geq 0
$$

for any $\eta \in L^2(\Omega; \mathbb{S}^d)$, hence

$$
\begin{aligned}
& \int_\Omega [\pi(\varepsilon(v)) - \pi(\varepsilon(u)) - \eta : \varepsilon(v-u)] \, dx \\
\leq & \int_\Omega [\pi(\varepsilon(v)) - \pi(\varepsilon(u)) + \pi^*(\eta) + \pi(\varepsilon(u)) - \eta : \varepsilon(v)] \, dx \\
= & \int_\Omega [\pi(\varepsilon(v)) + \pi^*(\eta) - \eta : \varepsilon(v)] \, dx \, .
\end{aligned}
$$

Letting

$$
D_\pi(\varepsilon(v), \eta) = \int_\Omega [\pi(\varepsilon(v)) + \pi^*(\eta) - \eta : \varepsilon(v)] \, dx
$$

we obtain from (2.1) in combination with the subsequent estimates

$$\begin{aligned}
\int_\Omega \nu|\varepsilon(u-v)|^2\,dx \le{}& D_\pi(\varepsilon(v),\eta) \\
&+\nu\int_\Omega \varepsilon(v):\varepsilon(v-u)\,dx + \int_\Omega f\cdot(u-v)\,dx \\
&+\int_{\Gamma_N} F\cdot(u-v)\,ds + \int_\Omega \eta:\varepsilon(v-u)\,dx\,.
\end{aligned}\tag{2.2}$$

Let us introduce the spaces

$$\begin{aligned}
Q^* &:= \left\{\tau\in L^2(\Omega;\mathbb{S}^d):\quad \mathrm{Div}\,\tau\in L^2(\Omega;\mathbb{R}^d)\,,\;\; \tau\,n\in L^2(\Gamma_N;\mathbb{R}^d)\right\}\,,\\
\widetilde{L}^2(\Omega) &:= \{q\in L^2(\Omega):\quad \int_\Omega q\,dx=0\}\,.
\end{aligned}$$

The elements of Q^* act as approximations of the tensor σ, and their divergence “Div” as well as their trace on Γ_N has to be understood in a generalized sense.

In the same spirit $\widetilde{L}^2(\Omega)$ serves as approximation space for the pressure p occurring in equation (1.1), and since we may assume $\int_\Omega p\,dx=0$, we have uniqueness for the pressure function p as an element of $\widetilde{L}^2(\Omega)$.

Choosing $\tau\in Q^*$ and $q\in\widetilde{L}^2(\Omega)$ we return to (2.2) and get the inequality ($\mathbf{1}$ being the unit element in $\mathbb{S}^d$)

$$\begin{aligned}
\int_\Omega \nu|\varepsilon(u-v)|^2\,dx \le{}& D_\pi(\varepsilon(v),\eta) + \int_\Omega(\mathrm{Div}\,\tau+f)\cdot(u-v)\,dx \\
&+\int_{\Gamma_N}(\tau n-F)\cdot(v-u)\,ds + \int_\Omega(\nu\varepsilon(v)+\eta-\tau-q\mathbf{1}):\;\varepsilon(v-u)\,dx\,,
\end{aligned}\tag{2.3}$$

where we have used the identity

$$\int_\Omega[\mathrm{Div}\,\tau\cdot w+\tau:\;\varepsilon(w)]\;dx = \int_{\Gamma_N}\tau n\cdot w\,ds\,,$$

being valid for functions $w\in V_0$, and the relation

$$\int_\Omega q\mathbf{1}:\;\varepsilon(w)\,dx=0\,,$$

which holds for all $w\in\mathcal{H}^1$. An upper bound for the r.h.s. of (2.3) is given by

$$\begin{aligned}
&\left(\|\mathrm{Div}\,\tau+f\|_\Omega^2+\|\tau n-F\|_{\Gamma_N}^2\right)^{1/2}\left(\|u-v\|_\Omega^2+\|u-v\|_{\Gamma_N}^2\right)^{1/2}\\
&\qquad+\left\|\frac{1}{\sqrt{\nu}}\left(\tau-\nu\varepsilon(v)+q\mathbf{1}-\eta\right)\right\|_\Omega\left\|\sqrt{\nu}\varepsilon(u-v)\right\|_\Omega + D_\pi(\varepsilon(v),\eta)\,,
\end{aligned}\tag{2.4}$$

moreover we have for any $w \in V_0$

$$\|w\|_\Omega + \|w\|_{\Gamma_N} \le C(\Omega,\Gamma_N)\|\varepsilon(w)\|_\Omega\,. \tag{2.5}$$

Here and later on $\|\,.\|_\Omega$ and $\|\,.\,\|_{\Gamma_N}$ denote L^2 norms of functions defined in Ω and Γ_N, respectively.
It is not difficult to see that (2.5) is a consequence of Friedrichs' and Korn's inequalities combined with the trace estimate valid for functions vanishing on Γ_D. Note that the positive constant $C(\Omega,\Gamma_N)$ occurring in (2.5) is determined just by the domain Ω and (part of) its boundary. Finally, we apply (2.5) to the function $w = u - v$, return to (2.3) and get after using (2.4) and applying Young's inequality twice

THEOREM 2.1 *Let u denote the exact solution of (1.10). Then for any α, $\beta \in (0,\infty)$ such that $\alpha+\beta < 2$, for all $v \in u_0 + \mathcal{H}^1$, $\eta \in L^2(\Omega;\mathbb{S}^d)$, $\tau \in Q^*$ and $q \in \widetilde{L}^2(\Omega)$ we have the error bound*

$$\begin{aligned} &\nu(2-\alpha-\beta)\|\varepsilon(u-v)\|^2_\Omega \\ &\le\; 2D_\pi(\varepsilon(v),\eta) + \frac{1}{\alpha}\left\|\frac{1}{\sqrt{\nu}}(\tau - \nu\varepsilon(v) - \eta + q\mathbf{1})\right\|^2_\Omega \\ &\quad+\frac{1}{\beta\nu}\,C(\Omega,\Gamma_N)^2\left(\|\mathrm{Div}\;\tau + f\|^2_\Omega + \|F-\tau n\|^2_{\Gamma_N}\right)\,. \end{aligned} \tag{2.6}$$

REMARK 2.1 *Let $\eta \in \partial\pi(\varepsilon(v))$, where $\partial\pi(\varepsilon(v))$ is the set of subgradients of π at $\varepsilon(v)$. Then $D_\pi(\varepsilon(v),\eta) = 0$ and (2.6) reduces to*

$$\begin{aligned} &\nu(2-\alpha-\beta)\|\varepsilon(u-v)\|^2_\Omega \\ &\le\; \frac{1}{\alpha}\left\|\frac{1}{\sqrt{\nu}}(\tau - \nu\varepsilon(v) - \eta + q\mathbf{1})\right\|^2_\Omega \\ &\quad+\frac{1}{\beta\nu}\,C(\Omega,\Gamma_N)^2\left(\|\mathrm{Div}\;\tau + f\|^2_\Omega + \|F-\tau n\|^2_{\Gamma_N}\right)\,. \end{aligned} \tag{2.7}$$

If the r.h.s. of (2.7) vanishes, then we must have

$$\begin{array}{ll} \tau = \nu\varepsilon(v) + \eta - q\mathbf{1} & a.e.\ \mathrm{in}\,\Omega\,, \\ \mathrm{Div}\;\tau + f = 0 & a.e.\ \mathrm{in}\,\Omega \quad \textit{and} \\ \tau n = F & a.e.\ \mathrm{on}\,\Gamma_N\,. \end{array}$$

Since $v \in u_0 + \mathcal{H}^1$ and since η belongs to the subdifferential of π, these relations mean that τ, v and q coincide with the exact stress, velocity and pressure, respectively. We also note that the quantity $D_\pi(\varepsilon(v),\sigma)$ vanishes if and only if the tensor σ is in $\partial\pi(\varepsilon(v))$ a.e., hence the r.h.s. of (2.6) vanishes only on the exact solution.

REMARK 2.2 *Estimate (2.6) as well as estimate (2.7) provide computable upper bounds for the error since all quantities on the right-hand sides are known or under our*

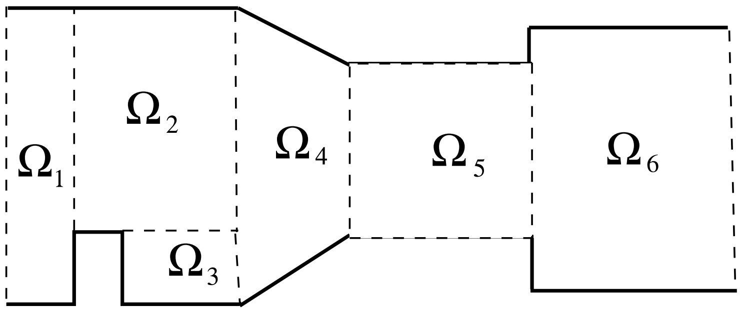

Figure 1: Decomposition of Ω into a collection of "simple" subdomains.

disposal. The global constant $C(\Omega, \Gamma_N)$, which is involved, can be approximately found by minimizing the expression

$$\int_\Omega |\varepsilon(w)|^2 \, dx \left(\int_\Omega |w|^2 \, dx + \int_{\Gamma_N} |w|^2 \, ds \right)^{-1}$$

over the space $V_0 \setminus \{0\}$. However, for domains having a complicated form, finding guaranteed and sharp upper bounds of $C(\Omega, \Gamma_N)$ may be a difficult task (because minimization of the above quotient on any finite dimensional subspace of $V_0 \setminus \{0\}$ gives only a lower bound of $C(\Omega, \Gamma_N)$). In the special (but important) case $\Gamma_N = \emptyset$, the constant $C(\Omega, \Gamma_N)$ depends only on Ω (so that we denote it by $C(\Omega)$) and satisfies the inequality $C(\Omega) \leq C_F(\Omega) C_K(\Omega)$, where $C_F(\Omega)$ and $C_K(\Omega)$ are the constants occurring in the Friedrichs inequality

$$\|w\|_\Omega \leq C_F(\Omega) \|\nabla w\|_\Omega$$

and in the Korn inequality

$$\|\nabla w\|_\Omega \leq C_K(\Omega) \|\varepsilon(w)\|_\Omega \, .$$

Since $C_F(\Omega) \leq C_F(\widehat{\Omega})$ for any domain $\widehat{\Omega}$ that contains Ω, we obtain an upper bound by taking some "simple" $\widehat{\Omega}$, for which the constant is known analytically. For functions vanishing on the boundary, the constant $C_K(\Omega)$ is known, so that the desired estimate for $C_F(\Omega)$ is easily computable. Regrettably, in the general case of mixed boundary conditions such a simple method is not applicable. In the next section, we discuss another (more sophisticated) modus operandi that bypasses these difficulties, and we show how to derive an error estimate, which does not involve any unknown constants for complicated domains.

3 Error Bounds Using Domain Decompositions

We assume that Ω, which still may have a rather complicated shape, can be decomposed into a collection of simple domains Ω_i like rectangles, squares or triangles, if $d = 2$, or cubes, etc., if the case $d = 3$ is considered (see Fig. 1).

Our aim is to derive an upper bound similar to (2.6) which instead of the constant $C(\Omega, \Gamma_N)$ will only involve constants associated to these simple domains Ω_i. We recall the following result: let G denote a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, with Lipschitz boundary and consider the space $R(G)$ of rigid motions, i.e. the kernel of the operator ε. By $[w]_G$ we denote the projection of a function $w \in H^1(G; \mathbb{R}^d)$ on this space, which means that $[w]_G$ satisfies the identity

$$\int_G (w - [w]_G) \cdot r \, dx = 0 \quad \forall r \in R(G)\,. \tag{3.1}$$

We have

LEMMA 3.1 *There exists a positive constant $C_R(G)$ such that the inequality*

$$\|w - [w]_G\|_G \le C_R(G)\|\varepsilon(w)\|_G \tag{3.2}$$

is true for any function $w \in H^1(G; \mathbb{R}^d)$.

Proof: Clearly (3.2) is equivalent to the estimate

$$\|w\|_G \le C_R(G)\|\varepsilon(w)\|_G \tag{3.3}$$

which is valid for any $w \in H^1(G; \mathbb{R}^d)$ orthogonal to $R(G)$. If a constant $C_R(G)$ with (3.3) does not exist, then we can find a sequence $\{w_k\}$ in $H^1(G; \mathbb{R}^d)$ orthogonal to $R(G)$ such that

$$\|w_k\|_G > k\|\varepsilon(w_k)\|_G \tag{3.4}$$

and if we pass to the scaled sequence $\widetilde{w}_k := w_k/\|w_k\|_G$, then we deduce from (3.4)

$$\|\varepsilon(\widetilde{w}_k)\|_G < 1/k\,. \tag{3.5}$$

By Korn's inequality the sequence $\{\widetilde{w}_k\}$ is bounded in the space $H^1(G; \mathbb{R}^d)$, hence $\widetilde{w}_k \rightharpoondown: w_0$ for some function w_0 in $H^1(G; \mathbb{R}^d)$. But (3.5) combined with the lower semicontinuity of $\|\varepsilon(\cdot)\|_G$ implies $\varepsilon(w_0) = 0$, hence $w_0 \in R(G)$. At the same time we have

$$\int_G \widetilde{w}_k \cdot r \, dx = 0 \quad \forall r \in R(G)\,,$$

and in conclusion this is also true for the limit w_0. This implies

$$\int_G w_0 \cdot w_0 \, dx = 0\,,$$

thus $w_0 = 0$, which contradicts $1 = \|\widetilde{w}_k\|_G$ and $\widetilde{w}_k \to w_0$ in $L^2(G; \mathbb{R}^d)$. $\square$

Assume that Ω is decomposed in N elementary subdomains, i.e.,

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i. \tag{3.6}$$

As in Section 2, we introduce a tensor function $\tau \in Q^*$, but now we assume that it satisfies two additional relations. The first one is

$$\tau n = F \quad \text{on } \Gamma_N \,. \tag{3.7}$$

We remark that (3.7) is not too restrictive (e.g., if F is a piecewise polynomial function, then (3.7) can be exactly satisfied with the help of piecewise polynomial approximations of τ). A more severe condition on τ is the requirement

$$\int_{\Omega_i} (\mathrm{Div}\ \tau + f) \cdot r \ dx = 0 \quad \forall r \in R(\Omega_i)\,,\ i = 1, ..., N\,, \tag{3.8}$$

which means that the residuals generated by τ are orthogonal to the spaces of rigid motions associated to each Ω_i. Obviously (3.8) can be rewritten as

$$\int_{\partial\Omega_i} (\tau n) \cdot r \ ds + \int_{\Omega_i} f \cdot r \ dx = 0 \quad \forall r \in R(\Omega_i), i = 1, ...N\,. \tag{3.9}$$

Assume now that (3.7) and (3.8) hold for $\tau \in Q^*$ and let us introduce the functional

$$\mathcal{L}_\tau(w) := \int_\Omega (\mathrm{Div}\ \tau + f) \cdot w \ dx\,.$$

If u and v are as in Theorem 2.1, then we have

$$\mathcal{L}_\tau(u - v) = \sum_{i=1}^{N} \int_{\Omega_i} (\mathrm{Div}\ \tau + f) \cdot (u - v - [u - v]_{\Omega_i}) \ dx\,,$$

and (3.2) (choosing $G = \Omega_i$) implies

$$\mathcal{L}_\tau(u - v) \le \sum_{i=1}^{N} C_R(\Omega_i) \|\mathrm{Div}\ \tau + f\|_{\Omega_i} \|\varepsilon(u - v)\|_{\Omega_i}\,.$$

Using this estimate in the calculations after (2.3), we obtain the following variant of (2.6):

THEOREM 3.1 *Under the assumptions and with the notation from Theorem 2.1 let us additionally suppose that (3.7) and (3.8) are valid for τ. Then it holds:*

$$\begin{aligned} \nu(2 - \alpha - \beta)\|\varepsilon(u - v)\|_\Omega^2 &\le 2D_\pi(\varepsilon(v), \eta) \\ &+ \frac{1}{\alpha} \left\| \frac{1}{\sqrt{\nu}} (\tau - \nu\varepsilon(v) - \eta + q\mathbf{1}) \right\|_\Omega^2 \\ &+ \frac{1}{\beta\nu} \sum_{i=1}^{N} C_R^2(\Omega_i) \, \|\mathrm{Div}\ \tau + f\|_{\Omega_i}^2\,, \end{aligned} \tag{3.10}$$

$C_R(\Omega_i)$ being defined in (3.3) for $G = \Omega_i$.

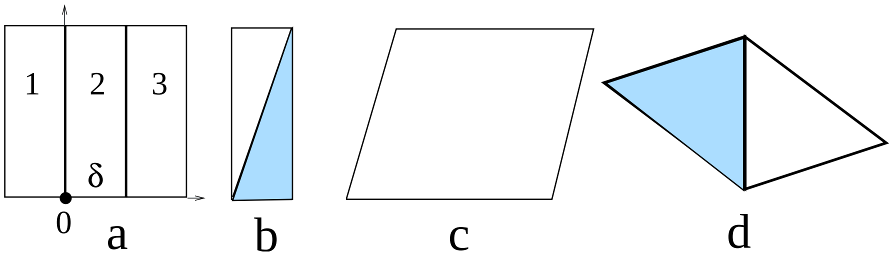

Figure 2: Evaluation of the constant C_R using symmetry arguments.

Corollary 3.1 *For the Stokes problem (i.e. $\pi = 0$ in (1.6)) we have the error estimate*

$$\begin{aligned}\nu(2-\alpha-\beta)\|\varepsilon(u-v)\|^2_\Omega \\ \leq\ & \frac{1}{\alpha}\left\|\frac{1}{\sqrt{\nu}}(\tau-\nu\varepsilon(v)+q\mathbf{1})\right\|^2_\Omega \\ & +\frac{1}{\beta\nu}\sum_{i=1}^{N} C_R^2(\Omega_i)\,\|\mathrm{Div}\,\tau+f\|^2_{\Omega_i}\end{aligned} \tag{3.11}$$

valid for $v \in u_0 + \mathcal{H}^1$, $q \in \widetilde{L}^2(\Omega)$, $\tau \in Q^*$ with (3.7) and (3.8) and for α, $\beta > 0$ s.t. $\alpha + \beta < 2$.

REMARK 3.1 *Estimate (3.11) can be put in a nicer form by observing that*

$$\begin{aligned}\inf_{\alpha,\beta>0,\,\alpha+\beta<2}&\left\{\frac{1}{\alpha(2-\alpha-\beta)}A_1^2+\frac{1}{\beta(2-\alpha-\beta)}A_2^2\right\} \\ \leq\ & (A_1+A_2)^2, \quad A_1, A_2 \geq 0\,,\end{aligned}$$

which follows by choosing $\alpha := A_1/(A_1+A_2)$ and $\beta := A_2/(A_1+A_2)$. Therefore (3.11) implies

$$\begin{aligned}\|\nu\varepsilon(u-v)\|_\Omega \ \leq\ & \|\tau-\nu\varepsilon(v)+q\mathbf{1}\|_\Omega \\ & +\left(\sum_{i=1}^{N} C_R(\Omega_i)^2\|\mathrm{Div}\,\tau+f\|^2_{\Omega_i}\right)^{1/2}\end{aligned}$$

as an error estimate for the pure Stokes problem being valid for $v \in u_0 + \mathcal{H}^1$, $q \in \widetilde{L}^2(\Omega)$ and for $\tau \in Q^*$ with (3.7) and (3.8).

REMARK 3.2 *The idea of how to apply (3.10) and (3.11) is of course not the computation of N different constants for N different domains Ω_i. For example, if $d = 2$ and if Ω is decomposed into squares Ω_i, then we have the bound*

$$C_R(\Omega_i) \leq \ \mathit{diam}\ (\Omega_i) C_R(\Omega_0), \tag{3.12}$$

where Ω_0 is the unit square. Thus a reasonable application of (3.10) and (3.11) is possible in all cases, where the domains Ω_i are obtained from one "standard domain" by scaling. Moreover, similar estimates are easy to prove for any rectangle and for any triangle having $\pi/2$ angle using symmetry argumentation. Indeed, assume that w^1 is a vector valued function orthogonal to R defined in the rectangle 1 (see Fig. 2 a). This means that

$$\int_{\Omega_1} w^1 \cdot e_k dx = 0, \quad e_1 = \{1,0\},\ e_2 = \{0,1\},\ e_3 = \{-x_2, x_1\}$$

Set $w^2 = w^1(-x_1, x_2)$ in Ω_2. Then the function w^{12} defined in $\Omega_{12} = \Omega_1 + \Omega_2$ as w^i in Ω^i, is orthogonal to R as well. Since w^{12} is continuous at $x_1 = 0$, we observe that

$$\|w^{12}\|_{\Omega_{12}} = 2\|w^1\|_{\Omega_1} \leq C_R(\Omega_{12})\|\varepsilon(w)\|_{\Omega_{12}} = 2C_R(\Omega_{12})\|\varepsilon(w)\|_{\Omega_1}$$

which means that $C_R(\Omega_{12})$ is valid for the inequality related to Ω_1.
We can continue this process and define on Ω_3 the function $w^3 = w^1(x_1 + 2\delta, x_2)$. It is easy to see that ($\xi_1 = x_1 - 2\delta$)

$$\int_{\Omega_3} w^3 \cdot e_3 dx_1 dx_2 = \int_{\Omega_1} w^1 \cdot (-x_2, \xi_1 + 2\delta) d\xi_1 dx_2 = 0,$$

hence

$$\|w^{123}\|_{\Omega_{123}} = 3\|w^1\|_{\Omega_1} \leq C_R(\Omega_{123})\|\varepsilon(w)\|_{\Omega_{123}} = 3C_R(\Omega_{123})\|\varepsilon(w)\|_{\Omega_1}$$

which means that $C_R(\Omega_{123})$ is valid for the inequality related to Ω_1.
In other words: if we have a constant related to the rectangle $(0, d_1) \times (0, d_2)$ then it is valid for a smaller rectangle $(0, d_1/k) \times (0, d_2)$, $k \in \mathbb{N}$. Using symmetry with respect to the diagonal, one can establish analogous estimates for triangles with $\pi/2$ angles (see Fig. 2 b).
By the transformation $\xi_1 = x_1 + ax_2$, we can evaluate the constant for a parallelogram (see Fig. 2 c) and then by symmetry argumentation for an arbitrary triangle (see Fig. 2 d).

4 Error Estimates For A Linearized Model

Linearization is a common way of finding approximate solutions of nonlinear problems. In this section, we consider a problem linearized in a neighborhood of a given function v. The corresponding linearized problem reads as follows: find $u_{\mathcal{L}}$, $\sigma_{\mathcal{L}}$, and $p_{\mathcal{L}}$ satisfying the system

$$-\mathrm{div}\sigma_{\mathcal{L}} = f - \nabla p_{\mathcal{L}} \quad \text{in} \quad \Omega \tag{4.1}$$
$$\mathrm{div} u_{\mathcal{L}} = 0 \quad \text{in} \quad \Omega \tag{4.2}$$
$$\sigma_{\mathcal{L}} = (\nu + \pi'(\varepsilon(v)))\varepsilon(u_{\mathcal{L}}) \quad \text{in} \quad \Omega \tag{4.3}$$
$$u_{\mathcal{L}} = u_0 \quad \text{on} \quad \Gamma_D \tag{4.4}$$
$$\sigma_{\mathcal{L}} n = F \quad \text{on} \quad \Gamma_N\,. \tag{4.5}$$

Here π' denotes the Gateaux derivative of π (if π is differentable) or an element of the corresponding subdifferential. In the variational form this problem is to minimize on $u_0 + \mathcal{H}^1$ the functional

$$J_{\mathcal{L}}[w] := \int_\Omega \left[\frac{\nu}{2}|\varepsilon(w)|^2 + \pi'(\varepsilon(v)) : \varepsilon(w) - f \cdot w\right] dx - \int_{\Gamma_N} F \cdot w \, ds + c(v),$$

where $c(v) = \int_\Omega (\pi(\varepsilon(v)) - \pi'(\varepsilon(v)) : \varepsilon(v)) \, dx$. We apply (2.3) with

$$v = u_{\mathcal{L}}, \; q = p_{\mathcal{L}}, \; \tau = \tau_L := \nu\varepsilon(u_{\mathcal{L}}) + \pi'(\varepsilon(v)) : \varepsilon(u_{\mathcal{L}}) - p_{\mathcal{L}}\mathbf{1}.$$

Since $\operatorname{Div} \tau_L + f = 0$ and $\tau_L n = F$ on Γ_N, we obtain

$$\int_\Omega \nu|\varepsilon(u - u_{\mathcal{L}})|^2 \, dx \leq D_\pi(\varepsilon(u_{\mathcal{L}}), \eta) + \int_\Omega (\eta - \pi'(\varepsilon(v))) : \; \varepsilon(u_{\mathcal{L}} - u) \, dx \,, \tag{4.6}$$

which leads to upper error bounds for the linearized model in different forms, namely

$$\int_\Omega \nu|\varepsilon(u - u_{\mathcal{L}})|^2 dx \leq \inf_{\substack{\rho>0,\\ \eta \in L^2(\Omega;\mathbb{S}^d)}} \left\{ \left(\frac{1}{2} + \frac{4\rho^2+1}{8\rho}\right) \left\|\frac{1}{\sqrt{\nu}}(\eta - \pi'(\varepsilon(v))\right\|_\Omega^2 + \left(1 + \frac{1}{2\rho}\right) D_\pi(\varepsilon(u_{\mathcal{L}}), \eta) \right\}, \tag{4.7}$$

$$\|\sqrt{\nu}\varepsilon(u - u_{\mathcal{L}})\|_\Omega \leq \inf_{\eta \in L^2(\Omega;\mathbb{S}^d)} \left\{ \frac{1}{2} \left\|\frac{1}{\sqrt{\nu}}(\eta - \pi'(\varepsilon(v))\right\|_\Omega + \sqrt{D_\pi(\varepsilon(u_{\mathcal{L}}), \eta) + \frac{1}{4}\left\|\frac{1}{\sqrt{\nu}}(\eta - \pi'(\varepsilon(v))\right\|_\Omega^2} \right\}, \tag{4.8}$$

and

$$\|\sqrt{\nu}\varepsilon(u - u_{\mathcal{L}})\|_\Omega \leq \left\|\frac{1}{\sqrt{\nu}}(\pi'(\varepsilon(v)) - \pi'(\varepsilon(u_{\mathcal{L}}))\right\|_\Omega . \tag{4.9}$$

From (4.9) it follows that if v (the function that generates the linearized problem) coincides with the solution of (4.1)–(4.5), then $u_{\mathcal{L}}$ is a generalized solution of the original problem (1.1)–(1.5). Moreover, the difference between u and $u_{\mathcal{L}}$ is controlled by the difference between $\varepsilon(v)$ and $\varepsilon(u_{\mathcal{L}})$ evaluated in terms of the nonlinear term gradient.
One more estimate follows from (4.6), if we use the inequality

$$\int_\Omega (\eta - \pi'(\varepsilon(v))) : \; \varepsilon(u_{\mathcal{L}} - u) \, dx \leq \int_\Omega \left(\frac{1}{2\nu\varrho}|\eta - \pi'(\varepsilon(v))|^2 + \frac{\nu\varrho}{2}|\varepsilon(u_{\mathcal{L}} - u)|^2\right) dx,$$

where $\varrho(x) \in (0,2)$. We then obtain an estimate in terms of the weighted norm

$$\int_\Omega \nu\left(1-\frac{\varrho}{2}\right)|\varepsilon(u-u_{\mathcal{L}})|^2dx \leq \inf_{\substack{\rho>0,\\ \eta\in L^2(\Omega;\mathbb{S}^d)}} \left\{\frac{1}{2}\left\|\frac{1}{\sqrt{\nu\varrho}}(\eta-\pi'(\varepsilon(v))\right\|^2_\Omega + D_\pi(\varepsilon(u_{\mathcal{L}}),\eta)\right\}. \tag{4.10}$$

Typically, $u_{\mathcal{L}}$ is unknown and instead we have an approximation $v_{\mathcal{L}} \in V_0 + u_0$. Since

$$D_\pi(\varepsilon(u_{\mathcal{L}}),\eta) = D_\pi(\varepsilon(v_{\mathcal{L}}),\eta) + \int_\Omega (\pi(\varepsilon(u_{\mathcal{L}})) - \pi(\varepsilon(v_{\mathcal{L}})) + \eta : (\varepsilon(v_{\mathcal{L}}) - \varepsilon(u_{\mathcal{L}}))dx, \tag{4.11}$$

we arrive at the estimate

$$\|\sqrt{\nu}\varepsilon(u-u_{\mathcal{L}})\|^2_\Omega \leq \inf_{\substack{\rho>0,\\ \eta\in L^2(\Omega;\mathbb{S}^d)}} \left\{\left(\frac{1}{2}+\frac{4\rho^2+1}{8\rho}\right)\left\|\frac{1}{\sqrt{\nu}}(\eta-\pi'(\varepsilon(v))\right\|^2_\Omega + \left(1+\frac{1}{2\rho}\right)D_\pi(\varepsilon(v_{\mathcal{L}}),\eta) + \left(1+\frac{1}{2\rho}\right)\int_\Omega (\pi(\varepsilon(u_{\mathcal{L}})) - \pi(\varepsilon(v_{\mathcal{L}})) + \eta : (\varepsilon(v_{\mathcal{L}}) - \varepsilon(u_{\mathcal{L}}))dx\right\}. \tag{4.12}$$

Our further analysis is based on Lemma 6.1, which implies the estimate

$$\int_\Omega (\pi(\varepsilon(u_{\mathcal{L}})) - \pi(\varepsilon(v_{\mathcal{L}})) + \eta : (\varepsilon(v_{\mathcal{L}}) - \varepsilon(u_{\mathcal{L}}))dx \leq R(u_{\mathcal{L}}-v_{\mathcal{L}}) := c_1\|\varepsilon(u_{\mathcal{L}}-v_{\mathcal{L}})\|_\Omega + c_2\|\varepsilon(u_{\mathcal{L}}-v_{\mathcal{L}})\|^2_\Omega,$$

where $c_1 = 4L\|\varepsilon(v_{\mathcal{L}})\|_\Omega + \|\eta\|_\Omega + 2$ and $c_2 = 4L$. Hence, (4.12) yields the estimate

$$\|\sqrt{\nu}\varepsilon(u-u_{\mathcal{L}})\|^2_\Omega \leq \inf_{\substack{\rho>0,\\ \eta\in L^2(\Omega;\mathbb{S}^d)}} \left\{\left(\frac{1}{2}+\frac{4\rho^2+1}{8\rho}\right)\left\|\frac{1}{\sqrt{\nu}}(\eta-\pi'(\varepsilon(v)))\right\|^2_\Omega + \left(1+\frac{1}{2\rho}\right)D_\pi(\varepsilon(v_{\mathcal{L}}),\eta) + \left(1+\frac{1}{2\rho}\right)R(u_{\mathcal{L}}-v_{\mathcal{L}})\right\} \tag{4.13}$$

We note that the problem (4.1)–(4.5) belongs to the class of generalized Stokes problems considered in [20] where computable upper bounds of the error $\|\varepsilon(u_{\mathcal{L}}-v_{\mathcal{L}})\|_\Omega$ are derived. These estimates provide upper bounds of $R(u_{\mathcal{L}}-v_{\mathcal{L}})$ expressed in terms of the problem data and $v_{\mathcal{L}}$. Since the first two terms in the right hand side of (4.13) are explicitly computable, we observe that this estimate gives a computable and guaranteed bound of the difference between solutions of problems (1.1)–(1.5) and (4.1)–(4.5).

If we set $\eta = \pi'(v_{\mathcal{L}})$, then (4.13) leads to the estimate

$$(4.14)\quad \|\sqrt{\nu}\varepsilon(u-u_{\mathcal{L}})\|_\Omega^2 \le \inf_{\substack{\rho>0,\\ \eta\in L^2(\Omega;\mathbb{S}^d)}} \left\{\left(\frac{1}{2}+\frac{4\rho^2+1}{8\rho}\right)\left\|\frac{1}{\sqrt{\nu}}(\pi'(v_{\mathcal{L}})-\pi'(\varepsilon(v))\right\|_\Omega^2 + \left(1+\frac{1}{2\rho}\right)\int_\Omega (\pi'(\varepsilon(v_{\mathcal{L}}))-\pi'(\varepsilon(u_{\mathcal{L}})) : (\varepsilon(v_{\mathcal{L}})-\varepsilon(u_{\mathcal{L}}))dx\right\},$$

which may give a better result provided that the second term in the right hand side is estimated from above. For example, for potentials with power growth $\pi(\varepsilon) = k|\varepsilon|^\alpha$, $\alpha \in (1,2]$, we know that (see the proof in [11])

$$(4.15)\quad \int_\Omega (\pi'(\varepsilon(v+w))-\pi'(\varepsilon(v))) : \varepsilon(w)\,dx \le 2^{2-\alpha}(3-\alpha)k\alpha|\Omega|^{1-\frac{\alpha}{2}}\|\varepsilon(w)\|_\Omega^\alpha.$$

By (4.14) and (4.15), we arrive at the estimate

$$(4.16)\quad \|\sqrt{\nu}\varepsilon(u-u_{\mathcal{L}})\|_\Omega^2 \le \inf_{\substack{\rho>0,\\ \eta\in L^2(\Omega;\mathbb{S}^d)}} \left\{\left(\frac{1}{2}+\frac{4\rho^2+1}{8\rho}\right)\|\pi'(v_{\mathcal{L}})-\pi'(\varepsilon(v)\|_\Omega^2 + 2^{1-\alpha}(3-\alpha)\frac{k\alpha(2\rho+1)}{\rho}|\Omega|^{1-\frac{\alpha}{2}}\|\varepsilon(v_{\mathcal{L}}-u_{\mathcal{L}})\|_\Omega^\alpha.\right\},$$

REMARK 4.1 *By the triangle inequality we have*

$$\|\sqrt{\nu}\varepsilon(u-v_{\mathcal{L}})\|_\Omega \le \|\sqrt{\nu}\varepsilon(u-u_{\mathcal{L}})\|_\Omega + \|\sqrt{\nu}\varepsilon(v_{\mathcal{L}}-u_{\mathcal{L}})\|_\Omega,$$

hence the difference between $v_{\mathcal{L}}$ and u can be estimated by (4.12), (4.13) and by known estimates for the corresponding linearized model.

5 Nonsolenoidal Approximations

Up to now approximations v of the exact solution u from the space $u_0 + \mathcal{H}^1$ have been considered, which in particular means that v satisfies the differential constraint $\operatorname{div} v = 0$. Here we are going to remove this restriction, which makes it necessary to construct a solenoidal field v in a controllable neighborhood of a given field $\hat{v}$ whose divergence might be different from zero., This can be achieved with the help of the following well-known result (see, e.g. [1, 17, 13]):

LEMMA 5.1 *There exists a positive constant $\overline{C} = \overline{C}(\Omega)$ depending on Ω such that for any function $\phi \in L^2(\Omega)$ satisfying*

$$(5.1)\quad \int_\Omega \phi\,dx = 0$$

we find a field $w = w(\phi) \in H^1(\Omega;\mathbb{R}^d)$, $w = 0$ on $\partial\Omega$, such that $\operatorname{div} w = \phi$ *and*

$$(5.2)\quad \|\nabla w\|_\Omega \le \overline{C}\|\phi\|_\Omega\,.$$

REMARK 5.1 *The constant* $\overline{C}$ *occurring in Lemma 5.1 is given by* $1/C_{LBB}$*, where* C_{LBB} *is the constant in the Ladyzhenskaya-Babuška-Brezzi condition*

$$\inf_w \inf_q \frac{\int_\Omega q \operatorname{div} w \; dx}{\|\nabla w\|_\Omega \|q\|_\Omega} \geq C_{LBB} \,. \tag{5.3}$$

Here q *ranges in* $\widetilde{L}^2(\Omega) \setminus \{0\}$*, and* w *is taken from* $H^1(\Omega;\mathbb{R}^d) \setminus \{0\}$ *with zero trace on* $\partial\Omega$*. For a discussion of (5.3) we refer to e.g., [4], bounds for* C_{LBB} *are given in e.g., [5, 7, 15].*

Let us now consider $\hat{v} \in u_0 + V_0$ such that

$$\int_\Omega \operatorname{div} \hat{v} \; dx = 0 \,. \tag{5.4}$$

In the case $\Gamma_N = \emptyset$ (5.4) automatically follows from our assumption $\operatorname{div} u_0 = 0$. From (5.4) we get (5.1) with the choice $\phi := -\operatorname{div} \hat{v}$, and we can use the lemma to find $w = w(\phi)$ as indicated. Then $v := \hat{v} + w$ is in $u_0 + \mathcal{H}^1$ and therefore we can apply (2.2) to this choice of the field v (observing $w = 0$ on $\partial\Omega$), i.e.

$$\begin{aligned}
&\int_\Omega \nu \left|\varepsilon(u - \hat{v} - w)\right|^2 \; dx \\
&= \int_\Omega \nu \left|\varepsilon(u - \hat{v})\right|^2 \; dx - 2\int_\Omega \nu\varepsilon(u - \hat{v}) : \varepsilon(w) \; dx + \int_\Omega \nu \left|\varepsilon(w)\right|^2 \; dx \\
&\leq \int_\Omega \nu\varepsilon(\hat{v} + w) : \varepsilon(\hat{v} + w - u) \; dx + D_\pi(\varepsilon(\hat{v} + w), \eta) \\
&\quad + \int_\Omega f \cdot (u - \hat{v} - w) \; dx + \int_{\Gamma_N} F \cdot (u - \hat{v}) \; ds \\
&\quad + \int_\Omega \eta : \varepsilon(\hat{v} + w - u) \; dx \,.
\end{aligned}$$

Collecting terms we arrive at

$$\begin{aligned}
\int_\Omega \nu \left|\varepsilon(u - \hat{v})\right|^2 \; dx &\leq D_\pi(\varepsilon(\hat{v}), \eta) \\
&+ \int_\Omega \left[\pi(\varepsilon(\hat{v} + w)) - \pi(\varepsilon(\hat{v}))\right] \; dx + \int_\Omega \eta : \varepsilon(\hat{v} - u) \; dx \\
&+ \int_\Omega f \cdot (u - \hat{v} - w) \; dx + \int_{\Gamma_N} F \cdot (u - \hat{v}) \; ds \\
&+ \int_\Omega \nu\varepsilon(\hat{v}) : \varepsilon(\hat{v} + w - u) \; dx + \int_\Omega \nu\varepsilon(w) : \varepsilon(u - \hat{v}) \; dx \,.
\end{aligned} \tag{5.5}$$

At the same time it holds for $\tau \in L^2(\Omega;\mathbb{S}^d)$, $q \in \widetilde{L}^2(\Omega)$

$$
\begin{aligned}
&\int_\Omega \nu\varepsilon(\hat{v}) : \varepsilon(\hat{v}+w-u)\ dx \\
&\quad = \int_\Omega (\nu\varepsilon(\hat{v}) + \eta - \tau - q\mathbf{1}) : \varepsilon(\hat{v}-u)\ dx \\
&\qquad + \int_\Omega (\tau + q\mathbf{1} - \eta) : \varepsilon(\hat{v}-u)\ dx + \int_\Omega \nu\varepsilon(\hat{v}) : \varepsilon(w)\ dx\,,
\end{aligned}
$$

which gives in combination with (5.5)

$$
\begin{aligned}
&\int_\Omega \nu\,|\varepsilon(u-\hat{v})|^2\ dx \\
&\quad \le D_\pi(\varepsilon(\hat{v}),\eta) + \int_\Omega [\pi(\varepsilon(\hat{v}+w)) - \pi(\varepsilon(\hat{v}))]\ dx \\
&\qquad + \int_\Omega f\cdot(u-\hat{v}-w)\ dx + \int_{\Gamma_N} F\cdot(u-\hat{v})\ ds \\
&\qquad + \int_\Omega (\nu\varepsilon(\hat{v}) + \eta - \tau - q\mathbf{1}) : \varepsilon(\hat{v}-u)\ dx \\
&\qquad + \int_\Omega \tau : \varepsilon(\hat{v}-u)\ dx + \int_\Omega q \operatorname{div}\hat{v}\ dx \\
&\qquad + \int_\Omega \nu\varepsilon(w) : \varepsilon(u-\hat{v})\ dx + \int_\Omega \nu\varepsilon(\hat{v}) : \varepsilon(w)\ dx\,.
\end{aligned}
\tag{5.6}
$$

We have as before (for $\tau \in Q^*$)

$$
\int_\Omega \tau : \varepsilon(\hat{v}-u)\ dx = -\int_\Omega \operatorname{Div}\tau\cdot(\hat{v}-u)\ dx + \int_{\Gamma_N} \tau n\cdot(\hat{v}-u)\ ds
$$

so that (5.6) turns into

$$
\begin{aligned}
&\int_\Omega \nu\,|\varepsilon(u-\hat{v})|^2\ dx \\
&\quad \le D_\pi(\varepsilon(\hat{v}),\eta) + \int_\Omega (\operatorname{Div}\tau + f)\cdot(u-\hat{v})\ dx \\
&\qquad + \int_{\Gamma_N} (\tau n - F)\cdot(\hat{v}-u)\ ds \\
&\qquad + \int_\Omega (\nu\varepsilon(\hat{v}) + \eta - \tau - q\mathbf{1}) : \varepsilon(\hat{v}-u)\ dx \\
&\qquad + \int_\Omega [\pi(\varepsilon(\hat{v}+w)) - \pi(\varepsilon(\hat{v}))]\ dx - \int_\Omega f\cdot w\ dx \\
&\qquad + \int_\Omega q\operatorname{div}\hat{v}\ dx + \int_\Omega \nu\varepsilon(w) : \varepsilon(u-\hat{v})\ dx \\
&\qquad + \int_\Omega \nu\varepsilon(\hat{v}) : \varepsilon(w)\ dx.
\end{aligned}
\tag{5.7}
$$

For the difference

$$\pi(\varepsilon(\hat{v}+w)) - \pi(\varepsilon(\hat{v})) \leq \pi'(\varepsilon(\hat{v})) : \varepsilon(w) + (\pi'(\varepsilon(\hat{v}+w)) - \pi'(\varepsilon(\hat{v})) : \varepsilon(w)$$

we have

$$\int_\Omega \pi(\varepsilon(\hat{v}+w)) - \pi(\varepsilon(\hat{v}))\ dx \leq \int_\Omega \pi'(\varepsilon(\hat{v})) : \varepsilon(w)dx +$$
$$\int_\Omega |\pi'(\varepsilon(\hat{v}+w)) - \pi'(\varepsilon(\hat{v})| : |\varepsilon(w)| dx\,.$$

Note that $\operatorname{div}(\hat{v}+w) = 0$ and

$$\int_\Omega q \operatorname{div} \hat{v}\ dx = -\int_\Omega q \operatorname{div} w\ dx.$$

This yields

$$\begin{aligned}
&\int_\Omega [\pi(\varepsilon(\hat{v}+w)) - \pi(\varepsilon(\hat{v}))]\, dx - \int_\Omega f \cdot w\ dx + \int_\Omega q \operatorname{div} \hat{v}\ dx \\
&+ \int_\Omega \nu\varepsilon(w) : \varepsilon(u-\hat{v})\ dx + \int_\Omega \nu\varepsilon(\hat{v}) : \varepsilon(w)\ dx \leq \int_\Omega (\pi'(\varepsilon(\hat{v})) + \nu\varepsilon(\hat{v}) - q\mathbf{1} - \tau) : \varepsilon(w)dx + \\
&\int_\Omega (\pi'(\varepsilon(\hat{v}+w)) - \pi'(\varepsilon(\hat{v}))) : (\varepsilon(w))dx - \int_\Omega (\operatorname{Div}\, \tau + f) \cdot w\, dx + \int_\Omega \nu\varepsilon(u-\hat{v}) : \varepsilon(w)\ dx.
\end{aligned}$$

Note further that

$$\begin{aligned}
&\int_\Omega (\pi'(\varepsilon(\hat{v})) + \nu\varepsilon(\hat{v}) - q\mathbf{1} - \tau) : \varepsilon(w)dx \leq \overline{C}\|\pi'(\varepsilon(\hat{v})) + \nu\varepsilon(\hat{v}) - q\mathbf{1} - \tau\|_\Omega \|\operatorname{div}\hat{v}\|_\Omega, \\
&\int_\Omega (\operatorname{Div}\, \tau + f) \cdot w\, dx \leq \overline{C} C_{\Omega,\Gamma_N} \|\operatorname{Div}\, \tau + f\|_\Omega \|\operatorname{div}\hat{v}\|_\Omega, \\
&\int_\Omega \nu\varepsilon(u-\hat{v}) : \varepsilon(w)\ dx \leq \frac{\gamma\nu}{2}\|\varepsilon(u-\hat{v})\|^2 + \frac{\overline{C}\nu}{2\gamma}\|\operatorname{div}\hat{v}\|^2_\Omega\,.
\end{aligned}$$

We see that these terms represent "second order" errors. For example, the first integral is estimated by the product of the "constitutive relation" error and the error caused by a possible violation of the divergence-free condition.
For the term with $\pi'(\varepsilon(\hat{v}+w)) - \pi'(\varepsilon(\hat{v}))$ we use Lemma 6.1 again and deduce a general (but rough) estimate

$$\int_\Omega |\pi'(\varepsilon(\hat{v}+w)) - \pi'(\varepsilon(\hat{v}))||\varepsilon(w)| dx \leq 4L(\hat{c} + \|\varepsilon(w)\|_\Omega)\|\varepsilon(w)\|_\Omega \leq$$
$$4L\overline{C}(\hat{c} + \overline{C}\|\operatorname{div}\hat{v}\|_\Omega)\|\operatorname{div}\hat{v}\|_\Omega,$$

where $\hat{c} = |\Omega|^{1/2} + \|\varepsilon(\hat{v})\|_\Omega$.
We arrive at the following result

THEOREM 5.1 *Let $u \in u_0 + \mathcal{H}^1$ denote the exact solution of (1.10). Then, for any α, β, $\gamma > 0$ such that $\alpha + \beta + \gamma < 2$, for all $\hat{v} \in u_0 + V_0$ and for arbitrary choices of $\eta \in L^2(\Omega; \mathbb{S}^d)$, $\tau \in Q^*$ and $q \in \widetilde{L}^2(\Omega)$, we have the error estimate*

$$
\begin{aligned}
(5.8) \qquad & \nu(2 - \alpha - \beta - \gamma)\|\varepsilon(u - \hat{v})\|_\Omega^2 \\
& \leq \; 2D_\pi(\varepsilon(\hat{v}), \eta) + \frac{1}{\alpha}\left\|\frac{1}{\sqrt{\nu}}(\tau - \nu\varepsilon(\hat{v}) - \eta - q\mathbf{1})\right\|_\Omega^2 \\
& \quad + \frac{1}{\beta\nu} C(\Omega, \Gamma_N)^2 \left(\|\mathrm{Div}\ \tau + f\|_\Omega^2 + \|F - \tau n\|_{\Gamma_N}^2\right) + G(\hat{v}, q, \gamma),
\end{aligned}
$$

where

$$
\begin{aligned}
G(\hat{v}, q, \gamma) := & \frac{\nu}{\gamma}\overline{C}(\Omega)\|\,\mathrm{div}\,\hat{v}\|_\Omega^2 + \\
& + \overline{C}(\Omega)\|\,\mathrm{div}\,\hat{v}\|_\Omega \Big\{ \|\pi'(\varepsilon(\hat{v})) + \nu\varepsilon(\hat{v}) - q\mathbf{1} - \tau\|_\Omega + C_{\Omega,\Gamma_N}\|\mathrm{Div}\ \tau + f\|_\Omega + \\
& \qquad 4L\hat{c} + 4L\overline{C}\|\mathrm{div}\hat{v}\|_\Omega) \Big\}.
\end{aligned}
$$

REMARK 5.2 *If a particular form of π is given, then with high probability this estimate can be improved. For example, for potentials with power growth we use (4.15) and obtain*

$$
\int_\Omega (\pi'(\varepsilon(\hat{v} + w)) - \pi'(\varepsilon(\hat{v}))) : (\varepsilon(w))dx \leq 2^{2-\alpha}(3 - \alpha)k\alpha|\Omega|^{1-\alpha/2}\overline{C}^\alpha\|\mathrm{div}\hat{v}\|_\Omega^\alpha.
$$

In this case we have a better upper bound with

$$
\begin{aligned}
G(\hat{v}, q, \gamma) := & \frac{\nu}{\gamma}\overline{C}(\Omega)\|\,\mathrm{div}\,\hat{v}\|_\Omega^2 + \\
& + \overline{C}(\Omega)\|\,\mathrm{div}\,\hat{v}\|_\Omega \Big\{ \|\pi'(\varepsilon(\hat{v})) + \nu\varepsilon(\hat{v}) - q\mathbf{1} - \tau\|_\Omega + C_{\Omega,\Gamma_N}\|\mathrm{Div}\ \tau + f\|_\Omega + \\
& \qquad 2^{2-\alpha}(3 - \alpha)k\alpha|\Omega|^{1-\alpha/2}\overline{C}^{\alpha-1}\|\mathrm{div}\hat{v}\|_\Omega^{\alpha-1} \Big\}.
\end{aligned}
$$

REMARK 5.3 *Clearly (5.8) reduces to (2.6) if* $\mathrm{div}\,\hat{v} = 0$.

REMARK 5.4 *If we consider a domain decomposition (3.6) and if τ satisfies (3.7) and (3.8), then (5.8) holds, provided we replace on the r.h.s. the term*

$$
\frac{1}{\beta\nu} C(\Omega, \Gamma_N)^2 \left(\|\mathrm{Div}\ \tau + f\|_\Omega^2 + \|F - \tau n\|_{\Gamma_N}^2\right)
$$

by the quantity

$$
\frac{1}{\beta\nu}\sum_{i=1}^N C_R^2(\Omega_i)\|\ \mathrm{Div}\ \tau + f\|_{\Omega_i}^2.
$$

6 Appendix

LEMMA 6.1 *Let $\psi : \mathbb{R}^d \to \mathbb{R}$, $d \geq 2$, denote a nonnegative convex function satisfying the relation*

$$|\psi(\xi)| \leq L|\xi|^2 + \lambda,\ \xi \in \mathbb{R}^d \tag{6.1}$$

with $L > 0$ and $\lambda \geq 0$. Then it holds

$$|\psi(x+y) - \psi(x)| \leq d_1|y| + d_2|y|^2, \qquad d_1 := 4L|x| + 2\sqrt{L\lambda}, \quad d_2 := 4L. \tag{6.2}$$

Proof.
By convexity of ψ we find that for any ξ and ζ

$$\psi'(\xi) \cdot \zeta \leq \psi(\xi + \zeta) - \psi(\xi).$$

We choose $\zeta := te$, where t is a positive parameter and $|e| = 1$. Then (6.1) implies

$$\psi'(\xi) \cdot e \leq (L|\xi + te|^2) + \lambda - \psi(\xi))/t.$$

Note that the right hand side of the above inequality attains its minimum (for fixed ξ) if we choose $t = \sqrt{\frac{\lambda - \psi(\xi)}{L} + |\xi|^2}$. This yields an upper bound for the directional derivative

$$\psi'(\xi) \cdot e \leq 2L\left(|\xi| + \sqrt{\frac{\lambda - \psi(\xi)}{L} + |\xi|^2}\right). \tag{6.3}$$

We apply (6.3) to estimate the difference

$$\psi(x+y) - \psi(x) \leq \psi'(x+y) \cdot y = \psi'(x+y) \cdot \left(\frac{y}{|y|}\right)|y| \leq \tag{6.4}$$

$$2L\left||x+y| + \sqrt{\frac{\lambda - \psi(x+y)}{L} + |x+y|^2}\right||y| \leq 4L\left||x| + |y| + \sqrt{\frac{\lambda}{4L}}\right||y|.$$

□

Corollary 6.1 *From (6.3) it follows that the gradient of ψ at ξ is bounded by the quantity $4L|\xi| + 2\sqrt{L\lambda}$. In particular, if $\lambda = L$, then the gradient is bounded by $2L(2|\xi| + 1)$, and this bound can be improved to $2L(|\xi| + 1)$ if $\psi(\xi)$ depends just on the modulus of ξ.*

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