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Abstract

If $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^M$ locally minimizes the functional $\int_\Omega h(|\nabla u|)\,dx$ with h such that $\frac{h'(t)}{t} \le h''(t) \le c(t+t^2)^\omega \frac{h'(t)}{t}$ for all $t \ge 0$, then u is locally Lipschitz independent of the value of $\omega \ge 0$.

1 Introduction

We consider local minima $u : \Omega \to \mathbb{R}^M, M \ge 1$, defined on an open set $\Omega \subset \mathbb{R}^n, n \ge 2$, of the variational integral

$$I[u,\Omega] = \int_\Omega H(\nabla u)\,dx \tag{1.1}$$

and want to establish interior regularity results like the local boundedness of ∇u or even the local Hölder continuity of the first derivatives. As a matter of fact - when dealing with the vector case $M \ge 2$ - we have to assume that H is of special structure in the sense that

$$H(Z) = h(|Z|), \quad Z \in \mathbb{R}^{nM}\,, \tag{1.2}$$

for a function $h : [0,\infty) \to [0,\infty)$ of class C^2. Integrands of this particular form with essential contributions to the question of interior regularity have been studied by many authors: the case $h(t) = t^p$ with $p \ge 2$ was investigated first by Uhlenbeck [Uh] and later extended by Giaquinta and Modica [GM], more general functions h are the subject of Marcellini's work (see [Ma1-3] and also [MP]), and the case of nearly linear growth, i.e. $h(t) = t\ln(1+t)$, is due to Mingione and Siepe [MS]. Here we are going to extend the recent work [ABF] under the following hypotheses imposed on h:

(A1) h is strictly increasing and convex together with $h''(0) > 0$ and $\lim\limits_{t\downarrow 0} \frac{h(t)}{t} = 0\,;$

(A2) there exists a constant $\overline{k} > 0$ such that $h(2t) \le \overline{k}\,h(t)$ for all $t \ge 0\,;$

(A3) $\left\{ \begin{array}{l} \text{for an exponent } \omega \ge 0 \text{ and a constant } a \ge 0 \text{ it holds} \\ \frac{h'(t)}{t} \le h''(t) \le a(1+t^2)^{\frac{\omega}{2}}\,\frac{h'(t)}{t} \quad \text{for all } t \ge 0\,. \end{array} \right.$

¿From (A1) it follows that $h(0)=0=h'(0)$, $h'(t)>0$ for all $t>0$. The first inequality in (A3) shows that $t\mapsto \frac{h'(t)}{t}$ is increasing, moreover we get

$$h(t)\geq \frac{1}{2}h''(0)t^2,\ t\geq 0\,, \tag{1.3}$$

and (1.3) means that we have a problem of at least quadratic growth. The ($\Delta 2$)-property formulated as condition(A2) shows by elementary calculations that there exists an exponent $m\in[2,\infty)$ and a constant c such that

$$h(t)\leq c(t^m+1)\,,$$

and by convexity $h'(t)$ grows at most as t^{m-1}. Note that (A1) together with (1.3) gives that h is a N-function in the sense of [Ad, Section 8.2], for which

$$\overline{k}\,h'(t)t\leq h(t)\leq th'(t),\ t\geq 0\,, \tag{1.4}$$

holds. This inequality is a simple consequence of (A2) and the convexity of h. We finally observe that according to (1.2)

$$\begin{aligned}&\min\ \left\{h''(|Z|),\ \frac{h'(|Z|)}{|Z|}\right\}|Y|^2\leq D^2H(Z)(Y,Y)\leq\\ &\max\ \{\dots\}\,|Y|^2,\ Y,\ Z\in\mathbb{R}^{nM}\,,\end{aligned}$$

hence by (A3)

$$\frac{h'(|Z|)}{|Z|}|Y|^2\leq D^2H(Z)(Y,Y)\leq a(1+|Z|^2)^{\frac{\omega}{2}}\,\frac{h'(|Z|)}{|Z|}|Y|^2\,, \tag{1.5}$$

in particular we can find an exponent $\overline{q}$ such that

$$D^2H(Z)(Y,Y)\leq c(1+|Z|^2)^{\frac{\overline{q}-2}{2}}\,|Y|^2\,. \tag{1.6}$$

As outlined in e.g. [BF1] or [ABF] this inequality can be used to introduce a local regularization.

The reader should also note the following fact: if the function h satisfies (A1), (A2) together with $h'(t)/t\leq h''(t)$ for all $t\geq 0$, and if in addition we know $h''(t)\leq ct^s$ for large values of t and for some exponent s, then we get the second inequality in (A3) letting $\omega:=s$.

Definition 1.1. *Let $u\in W^1_{1,\mathrm{loc}}(\Omega;\mathbb{R}^M)$. u is said to be a local minimizer of the functional I from (1.1) if for any subdomain Ω' with compact closure in Ω it holds $I[u,\Omega']<\infty$ as well as $I[u,\Omega']\leq I[v,\Omega']$ for any $v\in W^1_{1,\mathrm{loc}}(\Omega;\mathbb{R}^M)$ such that* $\mathrm{spt}(u-v)\Subset\Omega'$.

For a definition of the Sobolev classes $W^k_{p,\mathrm{loc}}(\Omega;\mathbb{R}^M)$ and related spaces we refer to [Ad]. Note that local minima actually are elements of the Orlicz-Sobolev class $W^1_{h,\mathrm{loc}}(\Omega;\mathbb{R}^M)$.

Now we can state our main result

Theorem 1.1. *Suppose that (A1-3) and (1.2) hold. Let $u \in W^1_{1,\mathrm{loc}}(\Omega;\mathbb{R}^M)$ denote a local minimizer of the functional I from (1.1). Then ∇u is a locally bounded function.*
If in addition we have the Hölder condition

$$\text{(A4)} \qquad |D^2H(P) - D^2H(Q)| \le c\left(1 + |P|^2 + |Q|^2\right)^{\frac{\overline{q}-2-\gamma}{2}} |P-Q|^\gamma$$

for some $\gamma \in (0,1)$ and with $\overline{q}$ from (1.6) valid for all P, $Q \in \mathbb{R}^{nM}$, then ∇u is Hölder continuous.

We remark that Theorem 1.1 requires no restriction on the (finite) number $\omega \ge 0$. Anyhow, we need the second inequality in (A3) not only for having (1.6), it is also used in Section 3. Let us further remark that Theorem 1.1 follows from the results in [ABF] provided $\omega < 2$ and u is locally bounded.

2 Higher integrability of the gradient

Let the hypotheses (A1-3) of Theorem 1.1 hold and consider a local $I[\cdot,\Omega]$-minimizer $u \in W^1_{1,\mathrm{loc}}(\Omega;\mathbb{R}^M)$. The following calculations can be justified by working with a suitable local regularization as outlined for example in [BF1]. The smoothness of the approximate solutions can be deduced from [GM] and [Ca], where the local boundedness and the weak differentiability of their gradients is stated.

Lemma 2.1. *∇u is in the space $L^t_{\mathrm{loc}}(\Omega;\mathbb{R}^{nM})$ for any finite exponent t.*

Remark 2.1. *More precisely Lemma 2.1 means that the derivatives of the approximations have this integrability property uniformly w.r.t. the approximation parameter which can be shown in the same way as outlined below.*

Proof of Lemma 2.1: Let $s \ge 0$, $\Gamma := 1 + |\nabla u|^2$ and consider $\eta \in C^\infty_0(\Omega)$. From now on we use summation convention w.r.t. indices repeated twice. Starting from the equation

$$0 = \int_\Omega \partial_\alpha\left(DH(\nabla u)\right) : \nabla\left(\eta^2 \Gamma^{\frac{s}{2}} \partial_\alpha u\right)\, dx$$

we obtain

$$\begin{aligned}
&\int_\Omega D^2H(\nabla u)\,(\partial_\alpha \nabla u, \partial_\alpha \nabla u)\,\eta^2\Gamma^{\frac{s}{2}}\,dx\\
&= \;-\int_\Omega \partial_\alpha(DH(\nabla u)) : \left[\partial_\alpha u \otimes \nabla\left(\eta^2\Gamma^{\frac{s}{2}}\right)\right]\,dx\\
&= \;-\int_\Omega D^2H(\nabla u)\left(\partial_\alpha \nabla u, \partial_\alpha u \otimes \nabla\Gamma^{\frac{s}{2}}\right)\eta^2\,dx\\
&\qquad -\int_\Omega \partial_\alpha(DH(\nabla u)) : \left[\partial_\alpha u \otimes \nabla\eta^2\right]\Gamma^{\frac{s}{2}}\,dx\\
&= \;-\int_\Omega D^2H(\nabla u)\left(\partial_\alpha \nabla u, \partial_\alpha u \otimes \nabla\Gamma^{\frac{s}{2}}\right)\eta^2\,dx\\
&\qquad +\int_\Omega DH(\nabla u) : \partial_\alpha\left\{\left[\partial_\alpha u \otimes \nabla\eta^2\right]\Gamma^{\frac{s}{2}}\right\}\,dx =: -T_1 + T_2\,.
\end{aligned}$$

Here " : " is the scalar product of matrices, " $\otimes$ " stands for the tensor product of vectors. (1.5) gives

$$\int_\Omega \frac{h'(|\nabla u|)}{|\nabla u|}\Gamma^{\frac{s}{2}}|\nabla^2 u|^2\eta^2\,dx \le -T_1 + T_2\,. \tag{2.1}$$

¿From the structure condition (1.2) we infer $T_1 \ge 0$, since we have the formula

$$D^2H(\nabla u)(\partial_\alpha \nabla u, \partial_\alpha u \otimes \nabla\Gamma) = a_{\alpha\beta}\partial_\alpha\Gamma\partial_\beta\Gamma$$

with coefficients

$$a_{\alpha\beta} := \frac{1}{2}\delta_{\alpha\beta}\frac{h'(|\nabla u|)}{|\nabla u|} + \frac{1}{2}\left[h''(|\nabla u|) - \frac{h'(|\nabla u|)}{|\nabla u|}\right]\frac{\partial_\alpha u \cdot \partial_\beta u}{|\nabla u|^2}$$

generating an elliptic matrix. Therefore (2.1) implies

$$\int_\Omega \eta^2\frac{h'(|\nabla u|)}{|\nabla u|}\Gamma^{\frac{s}{2}}|\nabla^2 u|^2\,dx \le T_2\,, \tag{2.2}$$

and it remains to estimate T_2: using $|DH(Z)| \le h'(|Z|)$ we get

$$\begin{aligned}
|T_2| \;\le\; &\int_\Omega 2h'(|\nabla u|)|\nabla^2 u|\eta|\nabla\eta|\Gamma^{\frac{s}{2}}\,dx\\
&+\int_\Omega 2h'(|\nabla u|)|\nabla u|\eta|\nabla\eta|s\Gamma^{\frac{s}{2}-1}|\nabla u||\nabla^2 u|\,dx\\
&+\int_\Omega h'(|\nabla u|)|\nabla u|\Gamma^{\frac{s}{2}}|\nabla^2\eta^2|\,dx =: S_1 + S_2 + S_3\,,
\end{aligned}$$

and by (1.4) it follows

$$S_3 \le c\int_\Omega h(|\nabla u|)\Gamma^{\frac{s}{2}}|\nabla^2\eta^2|\,dx\,,$$

where c denotes a constant which may vary from line and which may also depend on s. Obviously it holds

$$\begin{aligned} S_1+S_2 &\le c(s)\int_\Omega h'(|\nabla u|)\eta|\nabla^2 u||\nabla\eta|\Gamma^{\frac{s}{2}}\,dx \\ &= c(s)\int_\Omega\left(\frac{h'(|\nabla u|)}{|\nabla u|}\right)^{1/2}\eta|\nabla^2 u|\Gamma^{\frac{s}{4}}\Gamma^{\frac{s}{4}}|\nabla\eta|\,(h'(|\nabla u|)|\nabla u|)^{1/2}\,dx\,, \end{aligned}$$

and Young's inequality gives for any $\varepsilon>0$ (using (1.4))

$$S_1+S_2\le\varepsilon\int_\Omega\eta^2\frac{h'(|\nabla u|)}{|\nabla u|}|\nabla^2 u|^2\Gamma^{\frac{s}{2}}\,dx+c(\varepsilon,s)\int_\Omega|\nabla\eta|^2 h(|\nabla u|)\Gamma^{\frac{s}{2}}\,dx\,.$$

Collecting our estimates and returning to (2.2), we find after appropriate choice of ε

$$\begin{aligned} &\int_\Omega\eta^2\frac{h'(|\nabla u|)}{|\nabla u|}|\nabla^2 u|^2\Gamma^{\frac{s}{2}}\,dx \\ &\le c(s)\int_\Omega h(|\nabla u|)\Gamma^{\frac{s}{2}}\left[|\nabla\eta|^2+|\nabla^2\eta^2|\right]\,dx\,. \end{aligned}\tag{2.3}$$

For $s=0$ the r.h.s. of (2.3) is just the energy of u on $\mathrm{spt}(\eta)$, hence for

$$\Psi_0:=\int_0^{|\nabla u|}\left(\frac{h'(t)}{t}\right)^{1/2}\,dt$$

we have by (2.3)

$$\nabla\Psi_0\in L^2_{\mathrm{loc}}(\Omega;\mathbb{R}^n)\,.\tag{2.4}$$

¿From the monotonicity of $t\mapsto h'(t)/t$ and from (1.4) it follows

$$h(|\nabla u|)\le c\Psi_0^2\,,\tag{2.5}$$

$$\Psi_0\le ch(|\nabla u|)^{1/2}\,.\tag{2.6}$$

Clearly (2.6) implies together with (2.4) that

$$\Psi_0\in W^1_{2,\mathrm{loc}}(\Omega)\,.\tag{2.7}$$

If $n=2$, then (2.7) shows $\Psi_0^t\in L^1_{\mathrm{loc}}(\Omega)$ for any $t<\infty$, and the claim of Lemma 2.1 follows from (2.5) and (1.3). So let us suppose that $n\ge 3$. In this case (2.5) and (2.7) give $h(|\nabla u|)^{\frac{n}{n-2}}\in L^1_{\mathrm{loc}}(\Omega)$, so that by (1.3) $|\nabla u|^{2\frac{2}{n-2}}h(|\nabla u|)\in L^1_{\mathrm{loc}}(\Omega)$, and we see that the r.h.s. of (2.3) is finite for the choice $s_1:=\frac{4}{n-2}$. Letting

$$\Psi_1:=\int_0^{|\nabla u|}\left[\frac{h'(t)}{t}t^{s_1}\right]^{1/2}\,dt$$

we get from (2.3)

$$\nabla\Psi_1 \in L^2_{\mathrm{loc}}(\Omega)\,, \tag{2.8}$$

and using a variant of (2.6) valid for Ψ_1, (2.8) shows in analogy to (2.7) that

$$\Psi_1 \in W^1_{2,\mathrm{loc}}(\Omega)\,.$$

Suppose now by induction that we have finiteness of the r.h.s. of (2.3) for a number s_k. Then as before

$$\Psi_k := \int_0^{|\nabla u|} \left(\frac{h'(t)}{t} t^{s_k}\right)^{1/2} dt \in W^1_{2,\mathrm{loc}}(\Omega)\,,$$

and if we observe $\Psi_k \geq c\,(h(|\nabla u|)|\nabla u|^{s_k})^{1/2}$, we obtain

$$h\left(|\nabla u|\right)|\nabla u|^{\frac{4}{n-2}}|\nabla u|^{s_k\frac{n}{n-2}} \overset{(1.3)}{\leq} ch(|\nabla u|)^{\frac{n}{n-2}}|\nabla u|^{s_k\frac{n}{n-2}} \leq c\Psi_k^{\frac{2n}{n-2}}\,,$$

so that the r.h.s. of (2.3) is also finite by Sobolev's theorem for $s_{k+1} := \frac{4}{n-2} + s_k\frac{n}{n-2}$. Since $s_k \to \infty$ (starting with $s_0 = 0$, $s_1 = \frac{4}{n-2}$), the claim of Lemma 2.1 is established.

□

3 Interior gradient bounds

We use the notation from the previous section and recall that actually we work with a sequence of regularizations. We claim

Lemma 3.1. *Under the hypotheses (A1-3) the function ∇u is locally bounded.*

Proof: The local boundedness of ∇u follows via De Giorgi-type arguments as applied in [Bi], Theorem 5.22, or [ABF]. For $k > 0$ and balls $B\rho(x_0) \subset \Omega$ let $A(k,\rho) := \{x \in B_\rho(x_0) : \Gamma > k\}$. Then we have for $\eta \in C_0^\infty(B_\rho(x_0))$

$$0 = \int_{B_\rho(x_0)} D^2H(\nabla u)\left(\partial_\alpha\nabla u, \nabla\left[\eta^2\partial_\alpha u \max(\Gamma - k, 0)\right]\right) dx$$

which by the structure condition (1.2) turns into the estimate

$$\int_{A(k,\rho)} \eta^2 a_{\alpha\beta}\partial_\alpha\Gamma\partial_\beta\Gamma\, dx \leq -2\int_{A(k,\rho)} a_{\alpha\beta}\partial_\alpha\Gamma\partial_\beta\eta\eta(\Gamma - k)\, dx\,. \tag{3.1}$$

On the r.h.s. of (3.1) we can apply the Cauchy-Schwarz inequality to the symmetric bilinear form induced by $(a_{\alpha\beta})$ with the result

$$\int_{A(k,\rho)} \eta^2 a_{\alpha\beta}\partial_\alpha\Gamma\partial_\beta\Gamma\, dx \leq c\int_{A(k,\rho)} a_{\alpha\beta}\partial_\alpha\eta\partial_\beta\eta(\Gamma - k)^2\, dx\,. \tag{3.2}$$

Let $r < \hat{r}$, $B_{\hat{r}}(x_0) \Subset \Omega$, and choose $\eta = 1$ on $B_r(x_0)$, $0 \leq \eta \leq 1$, $\operatorname{spt}\eta \subset B_{\hat{r}}(x_0)$, $|\nabla\eta| \leq c/(\hat{r} - r)$. From

$$\int_{A(k,r)} (\Gamma - k)^{\frac{n}{n-1}}\, dx \leq \int_{B_{\hat{r}}(x_0)} \big(\eta[\Gamma - k]^+\big)^{\frac{n}{n-1}}\, dx$$

($[\dots]^+$ denoting the positive part) and Sobolev's theorem we get

$$\int_{A(k,r)} (\Gamma - k)^{\frac{n}{n-1}}\, dx \leq c\left[I_1^{\frac{n}{n-1}} + I_2^{\frac{n}{n-1}}\right], \tag{3.3}$$

where

$$\begin{aligned} I_1^{\frac{n}{n-1}} &:= \left[\int_{A(k,\hat{r})} |\nabla\eta|(\Gamma - k)\, dx\right]^{\frac{n}{n-1}} \leq c(\hat{r} - r)^{-\frac{n}{n-1}}\left[\int_{A(k,\hat{r})} (\Gamma - k)\, dx\right]^{\frac{n}{n-1}}, \\ I_2^{\frac{n}{n-1}} &:= \left[\int_{A(k,\hat{r})} \eta|\nabla\Gamma|\, dx\right]^{\frac{n}{n-1}}. \end{aligned}$$

For $k \geq 1$ we have on $A(k, \hat{r})$

$$\frac{h'(|\nabla u|)}{|\nabla u|} \leq c\Gamma^{\frac{\omega-2}{2}} h(|\nabla u|), \qquad h''(|\nabla u|) \overset{\text{(A3)}}{\leq} c\Gamma^{\frac{\omega-2}{2}} h(|\nabla u|),$$

$$\frac{h'(|\nabla u|)}{|\nabla u|} \geq c\Gamma^{-1} h(|\nabla u|), \qquad h''(|\nabla u|) \overset{\text{(A3)}}{\geq} c\Gamma^{-1} h(|\nabla u|).$$

If we use these inequalities in combination with the ellipticity estimate

$$\begin{aligned} &\frac{1}{2} \quad \min \quad \left\{\frac{h'(|\nabla u|)}{|\nabla u|},\ h''(|\nabla u|)\right\} |\tau|^2 \leq a_{\alpha\beta}\tau_\alpha\tau_\beta \leq \\ &\frac{1}{2} \quad \max \quad \{\dots\}\, |\tau|^2,\ \tau \in \mathbb{R}^n, \end{aligned}$$

we find after applying Hölder's inequality

$$
\begin{aligned}
I_2^{\frac{n}{n-1}} &= \left[\int_{A(k,\hat r)} \eta|\nabla\Gamma| h(|\nabla u|)^{1/2}\Gamma^{-1/2}\Gamma^{1/2}h(|\nabla u|)^{-1/2}\,dx\right]^{\frac{n}{n-1}}\\
&\le \left[\int_{A(k,\hat r)} \eta^2|\nabla\Gamma|^2 h(|\nabla u|)\Gamma^{-1}\,dx\right]^{\frac12\frac{n}{n-1}}\\
&\quad\cdot\left[\int_{A(k,\hat r)} \Gamma h(|\nabla u|)^{-1}\,dx\right]^{\frac12\frac{n}{n-1}}\\
&\le c\left[\int_{A(k,\hat r)} a_{\alpha\beta}\partial_\alpha\Gamma\partial_\beta\Gamma\eta^2\,dx\right]^{\frac12\frac{n}{n-1}}\\
&\quad\cdot\left[\int_{A(k,\hat r)} \Gamma h(|\nabla u|)^{-1}\,dx\right]^{\frac12\frac{n}{n-1}}\\
&\overset{(3.2)}{\le} c\left[\int_{A(k,\hat r)} a_{\alpha\beta}\partial_\alpha\eta\partial_\beta\eta(\Gamma-k)^2\,dx\right]^{\frac12\frac{n}{n-1}}\\
&\quad\cdot\left[\int_{A(k,\hat r)} \Gamma h(|\nabla u|)^{-1}\,dx\right]^{\frac12\frac{n}{n-1}}\\
&\le c(\hat r-r)^{-\frac{n}{n-1}}\left[\int_{A(k,\hat r)} (\Gamma-k)^2\Gamma^{\frac{\omega-2}{2}}h(|\nabla u|)\,dx\right]^{\frac12\frac{n}{n-1}}\\
&\qquad\cdot\left[\int_{A(k,\hat r)} \Gamma h(|\nabla u|)^{-1}\,dx\right]^{\frac12\frac{n}{n-1}}.
\end{aligned}
$$

Another application of Hölder's inequality yields

$$
\begin{aligned}
I_1^{\frac{n}{n-1}} &\le c(\hat r-r)^{-\frac{n}{n-1}}\left[\int_{A(k,\hat r)} (\Gamma-k)^2\Gamma^{\frac{\omega-2}{2}}h(|\nabla u|)\,dx\right]^{\frac12\frac{n}{n-1}}\\
&\qquad\cdot\left[\int_{A(k,\hat r)} \Gamma^{\frac{2-\omega}{2}}h(|\nabla u|)^{-1}\,dx\right]^{\frac12\frac{n}{n-1}},
\end{aligned}
$$

and therefore we get returning to (3.3)

$$
\begin{aligned}
&\int_{A(k,r)} (\Gamma-k)^{\frac{n}{n-1}}\,dx \le c(\hat r-r)^{-\frac{n}{n-1}}\\
&\quad\cdot\left[\int_{A(k,\hat r)} (\Gamma-k)^2\Gamma^{\frac{\omega-2}{2}}h(|\nabla u|)\,dx\right]^{\frac12\frac{n}{n-1}}\left[\int_{A(k,\hat r)} \Gamma h(|\nabla u|)^{-1}\,dx\right]^{\frac12\frac{n}{n-1}}.
\end{aligned}
\tag{3.4}
$$

Since by (A2) $h(t)\le ct^m$ on $[1,\infty)$ for some exponent $m\ge 2$ and since we have (1.3), we can choose exponents q and $\mu>0$ such that

$$(1+t^2)^{\frac{\omega-2}{2}}h(t)\le c(1+t^2)^{\frac{q-2}{2}},$$

$(1+t^2)h(t)^{-1}(\leq \text{ const}) \leq c(1+t^2)^{\frac{\mu}{2}}$ for $t \geq 1$. But then (3.4) exactly is inequality (24) in Lemma 5.23 of [Bi] and we can follow the calculations from p.158 of [Bi] (using Lemma 2.1) to get the claim of Lemma 3.1.

□

Having shown the local boundedness of ∇u, we fix $\Omega' \Subset \Omega$ and a number K s.t. $|\nabla u| \leq K$ on Ω'. Then - following the construction of Mingione and Siepe [MS] - we can find an integrand $F : \mathbb{R}^{nM} \to \mathbb{R}$ such that $F(Z) = H(Z)$ for $|Z| \leq 2K$ and $D^2F(Z) \hat{=} (1+|Z|^2)^{\frac{s}{2}}$ for some $s \geq 2$. If D^2H satisfies the Hölder condition (A4), then so does F, and the $C^{1,\alpha}$-regularity of u follows from [GM], Theorem 3.1. Further details are given in [BF2], end of Section 2.1.

□

Remark 3.1. *Going through the arguments of this section and of the previous one it is not hard to show that Theorem 1.1 remains valid if the first inequality in (A3) is replaced by the slightly weaker requirement* $\bar{\varepsilon}h'(t)/t \leq h''(t)$ *for all* $t \geq 0$ *and for some number* $\bar{\varepsilon} \in (0,1)$. *We leave the details to the reader.*

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