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MEASURE AND INTEGRATION: THE BASIC EXTENSION THEOREMS

HEINZ KÖNIG

Abstract. The present article returns to the new foundations of measure and integration due to the author. In this development the basic extension procedures lead from the so-called outer and inner premeasures to their unique maximal extensions. The initial version was for *extended real valued* set functions. In the sequel we want to achieve a major simplification, in that we develop the procedures - with no loss in the essentials - in the traditional frame of *nonnegative* set functions. The final section then will obtain an important extension theorem in the inner theories.

The present article is devoted to the fundamentals of the new systematization in measure and integration developed in the author's work of recent years. This work consists of the book [4] of 1997 and of numerous subsequent articles, of which [6][7][9][10] are survey articles. Its foundational part is made up of parallel outer and inner extension theories, which lead from so-called outer and inner premeasures to their unique maximal extensions, both times in three parallel procedures $\bullet = \star\sigma\tau$, the *finite, sequential*, and *nonsequential* ones. The result is a comprehensive edifice which contains the relevant theories of the 20th century as immediate specializations, and created a unification which was able to remove quite some notorious drawbacks. The most important topics include Daniell-Stone and Riesz representation theorems, finite and infinite products, projective limits, and applications to stochastic processes. In the course of time it became clear that the inner extension procedures are much more fundamental than the outer ones.

The present article now has the aim to remove a certain obstacle within the actual presentation of the systematization. In the book [4] the foundational extension procedures had been set up in an unconventional frame which was more involved than the traditional one - in the belief that it would be the scope of the future: for outer premeasures with values in $]-\infty,\infty]$ instead of $[0,\infty]$, and for inner premeasures with values in $[-\infty,\infty[$ instead of $[0,\infty[$. It is plain that this set-up requires certain unconventional concepts and quite some additional expenditure, despite the lucky fact that in the set-up the outer and inner procedures turned out to be equivalent.

However, it soon became clear that the most fundamental applications remained within the traditional frame. Therefore the author thinks that

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the foundational extension procedures should be developed within this traditional frame. This will be done in the present paper. The presentation will be much shorter than before, and of course incorporate the progress achieved in the meantime. It will be *ab ovo* and complete, except that we shall not reproduce the old proofs of a few isolated assertions which remain the same. We treat the outer and inner situations in parallel as in [4] and in the survey article [6], in spite of the particular importance of the inner one. And as before we shall confine the new proofs to one of the two situations when both of them follow the same method.

All this will be done in sections 1-4. As an application the final section 5 will then present a useful new extension theorem in the inner $\bullet$ theories and its consequences.

1. Preliminaries on Set Systems and Set Functions

The entire paper assumes a nonvoid set X. The set X carries the set systems under consideration.

Set Systems. A nonvoid set system $\mathfrak{S}$ in X is called a *paving.* We recall the familiar particular kinds called *lattice/ring/algebra.* Such a paving is called a *σ lattice/σ ring/σ algebra* iff it is stable under countable unions and intersections.

For a paving $\mathfrak{S}$ we define $\mathfrak{S}^\star \subset \mathfrak{S}^\sigma \subset \mathfrak{S}^\tau$ to consist of the *unions* of the finite/countable/arbitrary subpavings of $\mathfrak{S}$, and $\mathfrak{S}_\star \subset \mathfrak{S}_\sigma \subset \mathfrak{S}_\tau$ to consist of the respective *intersections.* We shall use the *shorthand notation* $\bullet = \star\sigma\tau$, to mean that the symbol $\bullet$ in a fixed context be read as one and the same of the symbols $\star/\sigma/\tau$ or of the words *finite/countable/arbitrary*, like variables are in constant use all over mathematics. If $\mathfrak{S}$ is a lattice then $\mathfrak{S}^\bullet$ and $\mathfrak{S}_\bullet$ are lattices as well.

A paving $\mathfrak{S}$ is called

upward directed iff for any $A, B \in \mathfrak{S}$ there exists $S \in \mathfrak{S}$ with $A, B \subset S$,
downward directed iff for any $A, B \in \mathfrak{S}$ there exists $S \in \mathfrak{S}$ with $A, B \supset S$.

For $\mathfrak{S}$ and $A \subset X$ we define

$\mathfrak{S} \uparrow A$ to mean that $\mathfrak{S}$ is upward directed with union $= A$,
$\mathfrak{S} \uparrow\supset A$ to mean that $\mathfrak{S}$ is upward directed with union $\supset A$,

and likewise $\mathfrak{S} \downarrow A$ and $\mathfrak{S} \downarrow\subset A$. We start to recall [6] 2.1.

1.1 Lemma. *Let $\mathfrak{S}$ be a lattice.*

Out) *Let $\mathfrak{M} \subset \mathfrak{S}^\bullet$ be a $\bullet$ paving with $\mathfrak{M} \uparrow A$. Then there exists a $\bullet$ paving $\mathfrak{N} \subset \mathfrak{S}$ with $\mathfrak{N} \uparrow A$ such that each $N \in \mathfrak{N}$ is contained in some $M \in \mathfrak{M}$.*

Inn) *Let $\mathfrak{M} \subset \mathfrak{S}_\bullet$ be a $\bullet$ paving with $\mathfrak{M} \downarrow A$. Then there exists a $\bullet$ paving $\mathfrak{N} \subset \mathfrak{S}$ with $\mathfrak{N} \downarrow A$ such that each $N \in \mathfrak{N}$ contains some $M \in \mathfrak{M}$.*

Next we define for two pavings $\mathfrak{S}$ and $\mathfrak{T}$ the *transporter* $\mathfrak{S}\top\mathfrak{T} := \{A \subset X : S \in \mathfrak{S} \Rightarrow A \cap S \in \mathfrak{T}\}$. The set system $\mathfrak{S}\top\mathfrak{T}$ can be void and thus need not be a paving; but in case $\varnothing \in \mathfrak{T}$ one has $\varnothing \in \mathfrak{S}\top\mathfrak{T}$. And for a paving $\mathfrak{S}$ we define $\mathfrak{S}\perp$ to consist of the complements $A' := X \setminus A$ of the members $A \in \mathfrak{S}$.

Set-theoretical Compactness. The set-theoretical notions of compactness initiated in Marczewski [11] are weaker and more flexible than topological compactness, and will be fundamental in the present systematization. The paving $\mathfrak{S}$ is defined to be $\bullet$ *compact* iff each $\bullet$ subpaving $\mathfrak{M} \subset \mathfrak{S}$ with intersection $= \varnothing$ has some finite subpaving with intersection $= \varnothing$. In case $\bullet = \star$ this condition is trivially fulfilled for all $\mathfrak{S}$.

1.2 Examples. 1) In a Hausdorff topological space X the compact subsets form a τ compact paving $\mathrm{Comp}(X)$. 2) If the paving $\mathfrak{S}$ in X is $\bullet$ compact then $\mathfrak{S} \cup \{X\}$ is $\bullet$ compact as well. This is a trivial remark, but its trivial nature comes to an abrupt end when one passes to infinite products, in particular to uncountable products.

We cannot resist to list a few properties which are fundamental but will not be needed in the present paper. Therefore we only refer to their proofs in [5] 2.5 and 2.6.

1.3 Properties. 1) *If the paving $\mathfrak{S}$ is $\bullet$ compact then $\mathfrak{S}_\bullet$ is $\bullet$ compact as well.* 2) *If the paving $\mathfrak{S}$ is $\bullet$ compact then $\mathfrak{S}^\star$ is $\bullet$ compact as well.* 3) *Let I be a nonvoid index set, and for each $t \in I$ let X_t be a nonvoid set and $\mathfrak{S}_t$ be a $\bullet$ compact paving in X_t. Then the product paving $\mathfrak{S} := \{\prod_{t\in I} S_t : S_t \in \mathfrak{S}_t \ \forall t \in I\}$ in the product set $X := \prod_{t\in I} X_t$ is $\bullet$ compact as well.*

Set Functions. Let $\mathfrak{S}$ be a paving in X. A set function $\varphi : \mathfrak{S} \to \overline{\mathbb{R}} := [-\infty, \infty]$ is called *isotone* iff $\varphi(A) \leqq \varphi(B)$ for all $A \subset B$ in $\mathfrak{S}$. The present subsection assumes an isotone set function $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$. Later on we shall restrict ourselves to set functions φ with values in $[0, \infty]$ or in $[0, \infty[$. A remarkable application of the more comprehensive situation is the extended Choquet capacitability theorem [4] 10.5 and [6] 2.5.

The set function φ is defined to be *upward/downward $\bullet$ continuous* iff

$\sup_{M\in\mathfrak{M}} \varphi(M) = \varphi(A)$ for all $\bullet$ subpavings $\mathfrak{M} \subset \mathfrak{S}$ with $\mathfrak{M} \uparrow A \in \mathfrak{S}$ resp.
$\inf_{M\in\mathfrak{M}} \varphi(M) = \varphi(A)$ for all $\bullet$ subpavings $\mathfrak{M} \subset \mathfrak{S}$ with $\mathfrak{M} \downarrow A \in \mathfrak{S}$.

In case $\bullet = \star$ these conditions are trivially fulfilled for all φ, and in case $\bullet = \sigma$ are equivalent to the more familiar conditions

$\lim_{l\to\infty} \varphi(S_l) = \varphi(A)$ for all sequences $(S_l)_l$ in $\mathfrak{S}$ with $S_l \uparrow A \in \mathfrak{S}$ resp.
$\lim_{l\to\infty} \varphi(S_l) = \varphi(A)$ for all sequences $(S_l)_l$ in $\mathfrak{S}$ with $S_l \downarrow A \in \mathfrak{S}$.

We define φ to be *almost upward/almost downward $\bullet$ continuous* if the above conditions are required but for the $\mathfrak{M}$ with $\varphi|\mathfrak{M} > -\infty$ in the upward case and $\varphi|\mathfrak{M} < \infty$ in the downward case. The *almost downward* behaviour is familiar from traditional measure theory. One also defines all these properties *at an individual* $A \in \mathfrak{S}$ and *at an individual subpaving of* $\mathfrak{S}$.

Next the set function φ is called *outer regular/inner regular* $\mathfrak{M}$ for a subpaving $\mathfrak{M} \subset \mathfrak{S}$ iff

$\varphi(A) = \inf\{\varphi(M) : M \in \mathfrak{M} \text{ with } M \supset A\}$ for all $A \in \mathfrak{S}$ resp.
$\varphi(A) = \sup\{\varphi(M) : M \in \mathfrak{M} \text{ with } M \subset A\}$ for all $A \in \mathfrak{S}$,

with the usual conventions $\inf \varnothing := \infty$ and $\sup \varnothing := -\infty$. Once more one also defines these properties *at an individual* $A \in \mathfrak{S}$ and *at an individual subpaving of* $\mathfrak{S}$.

After these conventional concepts we turn to the $\bullet = \star\sigma\tau$ *envelopes* for the isotone set functions $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$, which dominate the new systematization. The *outer/inner* $\bullet$ *envelopes* $\varphi^\bullet : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ and $\varphi_\bullet : \mathfrak{P}(X) \to \overline{\mathbb{R}}$ for φ are defined to be

$\varphi^\bullet(A) = \inf\{\sup_{M\in\mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \bullet \text{ paving with } \mathfrak{M} \uparrow\supset A\}$ resp.

$\varphi_\bullet(A) = \sup\{\inf_{M\in\mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S} \bullet \text{ paving with } \mathfrak{M} \downarrow\subset A\}$.

In case $\bullet = \star$ one comes back to the *crude envelopes*

$\varphi^\star(A) = \inf\{\varphi(S) : S \in \mathfrak{S} \text{ with } S \supset A\}$ resp.

$\varphi_\star(A) = \sup\{\varphi(S) : S \in \mathfrak{S} \text{ with } S \subset A\}$,

which fulfil $\varphi_\star \leqq \varphi^\star$, and in case $\bullet = \sigma$ one has the simpler reformulations

$\varphi^\sigma(A) = \inf\{\lim_{l\to\infty} \varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ increasing with } \bigcup_{l=1}^{\infty} S_l \supset A\}$ resp.

$\varphi_\sigma(A) = \sup\{\lim_{l\to\infty} \varphi(S_l) : (S_l)_l \text{ in } \mathfrak{S} \text{ decreasing with } \bigcap_{l=1}^{\infty} S_l \subset A\}$.

The envelopes $\varphi^\bullet$ and $\varphi_\bullet$ are isotone, and fulfil $\varphi^\star \geqq \varphi^\sigma \geqq \varphi^\tau$ and $\varphi_\star \leqq \varphi_\sigma \leqq \varphi_\tau$. We recall the basic properties collected in [6] 2.2.

1.4 Properties. *Assume that* $\mathfrak{S}$ *is a lattice.*

Out) 1) $\varphi^\bullet$ *is outer regular* $\mathfrak{S}^\bullet$. 2) *For* $A \in \mathfrak{S}$ *one has* $\varphi^\bullet(A) \leqq \varphi(A)$, *and* $\varphi^\bullet(A) = \varphi(A) \Leftrightarrow \varphi$ *is upward* $\bullet$ *continuous at* A. *In particular* $\varphi^\star|\mathfrak{S} = \varphi$. 3) *If* φ *is upward* $\bullet$ *continuous then* $\varphi^\bullet|\mathfrak{S}^\bullet = \varphi_\star|\mathfrak{S}^\bullet$, *and this set function is upward* $\bullet$ *continuous as well.* 4) *If* φ *is upward* $\bullet$ *continuous and* $\{A \in \mathfrak{S}^\bullet : \varphi^\bullet(A) < \infty\} \subset \mathfrak{S}$ *then* $\varphi^\bullet = \varphi^\star$.

Inn) 1) $\varphi_\bullet$ *is inner regular* $\mathfrak{S}_\bullet$. 2) *For* $A \in \mathfrak{S}$ *one has* $\varphi_\bullet(A) \geqq \varphi(A)$, *and* $\varphi_\bullet(A) = \varphi(A) \Leftrightarrow \varphi$ *is downward* $\bullet$ *continuous at* A. *In particular* $\varphi_\star|\mathfrak{S} = \varphi$. 3) *If* φ *is downward* $\bullet$ *continuous then* $\varphi_\bullet|\mathfrak{S}_\bullet = \varphi^\star|\mathfrak{S}_\bullet$, *and this set function is downward* $\bullet$ *continuous as well.* 4) *If* φ *is downward* $\bullet$ *continuous and* $\{A \in \mathfrak{S}_\bullet : \varphi_\bullet(A) > -\infty\} \subset \mathfrak{S}$ *then* $\varphi_\bullet = \varphi_\star$.

Modular Set Functions. Let $\mathfrak{S}$ be a lattice. A set function $\varphi : \mathfrak{S} \to]-\infty, \infty]$ is called

modular iff $\varphi(A \cup B) + \varphi(A \cap B) = \varphi(A) + \varphi(B)$ for all $A, B \in \mathfrak{S}$,

submodular iff $\varphi(A \cup B) + \varphi(A \cap B) \leqq \varphi(A) + \varphi(B)$ for all $A, B \in \mathfrak{S}$,

supermodular iff $\varphi(A \cup B) + \varphi(A \cap B) \geqq \varphi(A) + \varphi(B)$ for all $A, B \in \mathfrak{S}$.

In case $\varnothing \in \mathfrak{S}$ the set function φ is defined to be a *content* iff it is isotone with $\varphi(\varnothing) = 0$, and hence $\varphi : \mathfrak{S} \to [0, \infty]$, and modular. If $\mathfrak{S}$ is a *ring* then $\varphi : \mathfrak{S} \to [0, \infty]$ is a content iff $\varphi \not\equiv \infty$ and $\varphi(A \cup B) = \varphi(A) + \varphi(B)$ for all disjoint pairs $A, B \in \mathfrak{S}$. We recall [6] 2.8, and present a new proof for the assertions 2).

1.5 Properties. *Assume that* $\mathfrak{S}$ *is a lattice with* $\varnothing \in \mathfrak{S}$ *and* $\varphi : \mathfrak{S} \to [0, \infty]$ *an isotone set function with* $\varphi(\varnothing) = 0$.

Out) 0) $\varphi^\bullet(\varnothing) = 0$. 1) φ *submodular* $\Rightarrow \varphi^\bullet$ *submodular.* 2) φ *submodular* $\Rightarrow \varphi^\sigma$ *and* φ^τ *are upward* σ *continuous.*

Inn) 0) $\varphi_\bullet(\varnothing) \geqq 0$, *and* $\varphi_\bullet(\varnothing) = 0 \Leftrightarrow \varphi$ *is downward* $\bullet$ *continuous at* $\varnothing$. 1) φ *supermodular* $\Rightarrow \varphi_\bullet$ *supermodular.* 2) φ *supermodular* $\Rightarrow \varphi_\sigma$ *and* φ_τ *are almost downward* σ *continuous.*

Proof of Out.2). Let $(A_n)_n$ be an increasing sequence in X with $A_n \uparrow A$ and $c := \lim_{n\to\infty} \varphi^\bullet(A_n) \leqq \varphi^\bullet(A)$. To be shown is $\varphi^\bullet(A) \leqq c$, so that we can assume that $c < \infty$ and hence $\varphi^\bullet(A_n) < \infty\ \forall n \in \mathbb{N}$. We fix $\varepsilon > 0$, and then for each $n \in \mathbb{N}$ a $\bullet$ subpaving $\mathfrak{M}(n) \subset \mathfrak{S}$ with

$$\mathfrak{M}(n) \uparrow\supset A_n \quad\text{and}\quad \sup_{S\in\mathfrak{M}(n)} \varphi(S) \leqq \varphi^\bullet(A_n) + \frac{\varepsilon}{2^{n+1}}.$$

i) We claim for $n \in \mathbb{N}$ that

$$\varphi(S_1 \cup \cdots \cup S_n) \leqq \varphi^\bullet(A_n) + \varepsilon\big(1 - \frac{1}{2^n}\big) \text{ for all } S_l \in \mathfrak{M}(l)\,(l = 1, \cdots, n).$$

This is clear for $n = 1$, and the induction step $1 \leqq n \Rightarrow n + 1$ proceeds as follows: The set system

$$\{(S_1 \cup \cdots \cup S_n) \cap S_{n+1} : S_l \in \mathfrak{M}(l)\,(l = 1, \cdots, n+1)\} \subset \mathfrak{S}$$

is a $\bullet$ paving with $\uparrow\supset A_n$. Thus there exist $S_l^\circ \in \mathfrak{M}(l)\,(l = 1, \cdots n+1)$ with

$$\varphi\big((S_1^\circ \cup \cdots \cup S_n^\circ) \cap S_{n+1}^\circ\big) \geqq \varphi^\bullet(A_n) - \frac{\varepsilon}{2^{n+2}}.$$

Now for $S_l \in \mathfrak{M}(l)\,(l = 1, \cdots, n+1)$ there are $T_l \in \mathfrak{M}(l)$ with $S_l \cup S_l^\circ \subset T_l$. Then the submodularity of φ and the induction hypothesis furnish

$$\begin{aligned}
&\varphi(S_1 \cup \cdots \cup S_n \cup S_{n+1}) + \varphi^\bullet(A_n) - \frac{\varepsilon}{2^{n+2}}\\
&\leqq \varphi\big((T_1 \cup \cdots \cup T_n) \cup T_{n+1}\big) + \varphi\big((T_1 \cup \cdots \cup T_n) \cap T_{n+1}\big)\\
&\leqq \varphi(T_1 \cup \cdots \cup T_n) + \varphi(T_{n+1})\\
&\leqq \varphi^\bullet(A_n) + \varepsilon\big(1 - \frac{1}{2^n}\big) + \varphi^\bullet(A_{n+1}) + \frac{\varepsilon}{2^{n+2}},
\end{aligned}$$

and hence the assertion for $n + 1$. ii) The set system

$$\{S_1 \cup \cdots \cup S_n : S_l \in \mathfrak{M}(l)\,(l = 1, \cdots, n) \text{ and } n \in \mathbb{N}\} \subset \mathfrak{S}$$

is a $\bullet$ paving and upward directed $\uparrow\supset A$. Therefore

$$\varphi^\bullet(A) \leqq \sup\{\varphi(S_1 \cup \cdots \cup S_n) : S_l \in \mathfrak{M}(l)\,(l = 1, \cdots, n) \text{ and } n \in \mathbb{N}\},$$

which after i) is $\leqq c + \varepsilon$. $\square$

The Satellites in the Inner Situation. The last point in the present preliminaries is specific for the inner situation. Its importance will become clear in the final section, in particular in connection with *set-theoretical compactness*. Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$ and $\varphi : \mathfrak{S} \to [0, \infty[$ be isotone with $\varphi(\varnothing) = 0$. We define for nonvoid $B \subset X$ the *satellites* $\varphi_\bullet^B : \mathfrak{P}(X) \to [0, \infty]$ of the inner $\bullet$ envelopes $\varphi_\bullet$ to be

$$\varphi_\bullet^B(A) := \sup\{\inf_{M\in\mathfrak{M}} \varphi(M) : \mathfrak{M} \subset \mathfrak{S}\ \bullet \text{ paving with } \mathfrak{M} \downarrow\subset A \text{ and } M \subset B\ \forall M \in \mathfrak{M}\}.$$

Thus $\varphi_\star(A) = \varphi_\star^B(A) \leqq \varphi_\sigma^B(A) \leqq \varphi_\tau^B(A)$ for $A \subset B$. We recall the properties in [4] 6.29.

1.6 Properties. 1) *$\varphi_\bullet^B(A)$ is isotone in A and in B.*
2) $\varphi_\bullet = \sup_{B\in\mathfrak{S}} \varphi_\bullet^B = \varphi_\bullet^X$. *Moreover $\varphi(B) = \varphi_\bullet^B(X)$ for $B \in \mathfrak{S}$.*
3) *φ supermodular $\Rightarrow$ $\varphi_\bullet^B$ supermodular.*
4) *φ downward $\bullet$ continuous $\Rightarrow$ $\varphi_\bullet^B(A) = \varphi_\bullet(A)$ for $A \subset B$ and $B \in \mathfrak{S}$.*

2. Preliminaries on the Carathéodory Class

For a set function $\vartheta : \mathfrak{P}(X) \to [0,\infty]$ with $\vartheta(\varnothing) = 0$ the Carathéodory *class* [2] is defined to be

$$\mathfrak{C}(\vartheta) := \{A \subset X : \vartheta(M) = \vartheta(M \cap A) + \vartheta(M \cap A') \text{ for all } M \subset X\};$$

its members are called *measurable* ϑ. Beyond $\vartheta(\varnothing) = 0$ the class $\mathfrak{C}(\vartheta) \subset \mathfrak{P}(X)$ is defined after [4] section 4, but the explicit definition will not be needed in the sequel. We note some basic properties.

2.1 Properties. *Assume that $\vartheta(\varnothing) = 0$. 0) $\vartheta|\mathfrak{C}(\vartheta)$ is a content on the algebra $\mathfrak{C}(\vartheta)$.*

Out) *Assume that ϑ is isotone, submodular, and upward σ continuous. Then $\mathfrak{C}(\vartheta)$ is a σ algebra, and $\vartheta|\mathfrak{C}(\vartheta)$ is a measure.*

Inn) *Assume that ϑ is isotone, supermodular, inner regular $[\vartheta < \infty] := \{M \subset X : \vartheta(M) < \infty\}$, and almost downward σ continuous. Then $\mathfrak{C}(\vartheta)$ is a σ algebra, and $\vartheta|\mathfrak{C}(\vartheta)$ is a measure.*

Proof of Inn). 1) We prove that $\mathfrak{C}(\vartheta)$ is a σ algebra. Let $(A_l)_l$ be a sequence in $\mathfrak{C}(\vartheta)$ with $A_l \downarrow A \subset X$. To be shown is $\vartheta(M) \leqq \vartheta(M \cap A) + \vartheta(M \cap A')$ for all $M \subset X$, and it suffices to do this for $\vartheta(M) < \infty$. Now

$$\vartheta(M) = \vartheta(M \cap A_l) + \vartheta(M \cap A_l') \leqq \vartheta(M \cap A_l) + \vartheta(M \cap A') \text{ for } l \in \mathbb{N},$$

and $\vartheta(M \cap A_l) \leqq \vartheta(M) < \infty$ implies that $\vartheta(M \cap A_l) \downarrow \vartheta(M \cap A)$, and hence the assertion.

2) Let $(A_l)_l$ and A be in $\mathfrak{C}(\vartheta)$ with $A_l \uparrow A$. To be shown is $\vartheta(A_l) \uparrow \vartheta(A)$. We fix a real $c < \vartheta(A)$, and then an $M \subset A$ with $c < \vartheta(M) < \infty$. Then $M \cap A_l' \downarrow M \cap A' = \varnothing$ and hence $\vartheta(M \cap A_l') \downarrow 0$. Thus $\vartheta(M) = \vartheta(M \cap A_l) + \vartheta(M \cap A_l')$ implies that $\lim\limits_{l\to\infty} \vartheta(A_l) \geqq \lim\limits_{l\to\infty} \vartheta(M \cap A_l) = \vartheta(M) > c$. □

We turn to the main point for the present context.

2.2 Lemma. Out) *Assume that $\vartheta : \mathfrak{P}(X) \to [0,\infty]$ is isotone and submodular, and let $T \subset X$ with $\vartheta(T) < \infty$. Then each $A \subset X$ fulfils the implication*

$$\begin{aligned} &\vartheta(T) \geqq \vartheta(T \cap A) + \vartheta(T \cap A') \\ \Rightarrow\ &\vartheta(M) \geqq \vartheta(M \cap A) + \vartheta(M \cap A') \text{ for all } M \subset T. \end{aligned}$$

Inn) *Assume that $\vartheta : \mathfrak{P}(X) \to [0,\infty]$ is isotone and supermodular, and let $T \subset X$ with $\vartheta(T) < \infty$. Then each $A \subset X$ fulfils the implication*

$$\begin{aligned} &\vartheta(T) \leqq \vartheta(T \cap A) + \vartheta(T \cap A') \\ \Rightarrow\ &\vartheta(M) \leqq \vartheta(M \cap A) + \vartheta(M \cap A') \text{ for all } M \subset T. \end{aligned}$$

Proof of Inn). For $M \subset T$ we have

$$\begin{aligned} \vartheta(T) + 2\vartheta(M) &\leqq \big(\vartheta(T \cap A) + \vartheta(M)\big) + \big(\vartheta(T \cap A') + \vartheta(M)\big) \\ &\leqq \Big(\vartheta\big((T \cap A) \cup M\big) + \vartheta(M \cap A)\Big) + \Big(\vartheta\big(T \cap A') \cup M\big) + \vartheta(M \cap A')\Big) \\ &\leqq \big(\vartheta(T) + \vartheta(M)\big) + \big(\vartheta(M \cap A) + \vartheta(M \cap A')\big), \end{aligned}$$

where the last $\leqq$ results from the combination of the two first terms in the previous line, and hence the assertion. □

Now for a paving $\mathfrak{T}$ in X we recall from [4] p.4 the notation $\sqsubset \mathfrak{T} := \{S \subset X : S \subset \text{some } T \in \mathfrak{T}\}$.

2.3 Proposition. Out) *Let $\vartheta : \mathfrak{P}(X) \to [0,\infty]$ be isotone and submodular, and assume that the paving $\mathfrak{T}$ in X with $\vartheta|\mathfrak{T} < \infty$ is upward directed and fulfils*

$$\vartheta(M) = \sup_{T\in\mathfrak{T}} \vartheta(M\cap T) \text{ for all } M \subset X \text{ with } \vartheta(M) < \infty,$$

that is $\vartheta|[\vartheta < \infty]$ is inner regular $\sqsubset \mathfrak{T}$. Then each $A \subset X$ satisfies the implication

$$\begin{aligned} &\vartheta(T) \geqq \vartheta(T\cap A) + \vartheta(T\cap A') \text{ for all } T \in \mathfrak{T} \\ \Rightarrow\ &\vartheta(M) \geqq \vartheta(M\cap A) + \vartheta(M\cap A') \text{ for all } M \subset X, \end{aligned}$$

and hence $A \in \mathfrak{C}(\vartheta)$ when $\vartheta(\varnothing) = 0$.

Inn) *Let $\vartheta : \mathfrak{P}(X) \to [0,\infty]$ be isotone and supermodular, and assume that the paving $\mathfrak{T}$ in X with $\vartheta|\mathfrak{T} < \infty$ fulfils*

$$\vartheta(M) = \sup_{T\in\mathfrak{T}} \vartheta(M\cap T) \text{ for all } M \subset X,$$

that is ϑ is inner regular $\sqsubset \mathfrak{T}$. Then each $A \subset X$ satisfies the implication

$$\begin{aligned} &\vartheta(T) \leqq \vartheta(T\cap A) + \vartheta(T\cap A') \text{ for all } T \in \mathfrak{T} \\ \Rightarrow\ &\vartheta(M) \leqq \vartheta(M\cap A) + \vartheta(M\cap A') \text{ for all } M \subset X, \end{aligned}$$

and hence $A \in \mathfrak{C}(\vartheta)$ when $\vartheta(\varnothing) = 0$.

Proof of Out). To be shown is the assertion in case $\vartheta(M) < \infty$. Now let $P, Q \in \mathfrak{T}$, and $T \in \mathfrak{T}$ with $P \cup Q \subset T$. Then from 2.2.Out)

$$\begin{aligned} \vartheta(M) \geqq \vartheta(M\cap T) &\geqq \vartheta(M\cap T\cap A) + \vartheta(M\cap T\cap A') \\ &\geqq \vartheta(M\cap A\cap P) + \vartheta(M\cap A'\cap Q), \end{aligned}$$

and hence $\vartheta(M) \geqq \vartheta(M\cap A) + \vartheta(M\cap A')$ from the initial assumption. $\square$

We continue with an important consequence of 2.3. We note that former versions of it had been useful, for example [4] 6.17 in [4] section 18 and [8] 1.8 in [8] section 2.

2.4 Theorem. Out) *Let $\vartheta : \mathfrak{P}(X) \to [0,\infty]$ be isotone $\vartheta(\varnothing) = 0$ and submodular, and assume that the paving $\mathfrak{T}$ in X with $\vartheta|\mathfrak{T} < \infty$ fulfils $[\vartheta < \infty] \subset (\sqsubset \mathfrak{T})$. If the set function $\psi : \mathfrak{P}(X) \to [0,\infty]$ is isotone $\psi(\varnothing) = 0$ with $\vartheta \leqq \psi$ and $\vartheta|\mathfrak{T} = \psi|\mathfrak{T}$, then $\vartheta|\mathfrak{C}(\vartheta)$ is an extension of $\psi|\mathfrak{C}(\psi)$.*

Inn) *Let $\vartheta : \mathfrak{P}(X) \to [0,\infty]$ be isotone $\vartheta(\varnothing) = 0$ and supermodular, and assume that the paving $\mathfrak{T}$ in X with $\vartheta|\mathfrak{T} < \infty$ fulfils ϑ inner regular $\sqsubset \mathfrak{T}$. If the set function $\psi : \mathfrak{P}(X) \to [0,\infty]$ is isotone with $\psi \leqq \vartheta$ and $\psi|\mathfrak{T} = \vartheta|\mathfrak{T}$, then $\vartheta|\mathfrak{C}(\vartheta)$ is an extension of $\psi|\mathfrak{C}(\psi)$.*

Proof of Inn). Fix $A \in \mathfrak{C}(\psi)$. For $T \in \mathfrak{T}$ we have

$$\vartheta(T) = \psi(T) = \psi(T\cap A) + \psi(T\cap A') \leqq \vartheta(T\cap A) + \vartheta(T\cap A') \leqq \vartheta(T) < \infty,$$

hence = partout. Thus $A \in \mathfrak{C}(\vartheta)$ from 2.3.Inn). And $\psi(T\cap A) = \vartheta(T\cap A)$ for $T \in \mathfrak{T}$ implies that

$$\psi(A) \geqq \sup_{T\in\mathfrak{T}} \psi(T\cap A) = \sup_{T\in\mathfrak{T}} \vartheta(T\cap A) = \vartheta(A),$$

and hence $\psi(A) = \vartheta(A)$. $\square$

3. The Outer and Inner Premeasures

The present section assumes a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$.

3.1 Definition. Out) Let $\varphi : \mathfrak{S} \to [0,\infty]$ be isotone with $\varphi(\varnothing) = 0$. We call φ an *outer* $\bullet$ *premeasure* iff it can be extended to a content $\alpha : \mathfrak{A} \to [0,\infty]$ on a ring $\mathfrak{A} \supset \mathfrak{S}^\bullet$ such that

α is outer regular $\mathfrak{S}^\bullet$ and
$\alpha|\mathfrak{S}^\bullet$ is upward $\bullet$ continuous.

These set functions α are called the *outer* $\bullet$ *extensions* of φ. Note that the outer $\bullet$ premeasures are modular and upward $\bullet$ continuous.

Inn) Let $\varphi : \mathfrak{S} \to [0,\infty[$ be isotone with $\varphi(\varnothing) = 0$. We call φ an *inner* $\bullet$ *premeasure* iff it can be extended to a content $\alpha : \mathfrak{A} \to [0,\infty]$ on a ring $\mathfrak{A} \supset \mathfrak{S}_\bullet$ such that

α is inner regular $\mathfrak{S}_\bullet$ and
$\alpha|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous (note that $\alpha|\mathfrak{S}_\bullet < \infty$).

These set functions α are called the *inner* $\bullet$ *extensions* of φ. Note that the inner $\bullet$ premeasures are modular and downward $\bullet$ continuous.

After this definition we are faced with the tasks to *characterize* the outer/inner $\bullet$ premeasures φ, and for these φ to *describe* the collection of the outer/inner $\bullet$ extensions.

3.2 Proposition. Out) *Let $\varphi : \mathfrak{S} \to [0,\infty]$ be isotone with $\varphi(\varnothing) = 0$. Then each outer $\bullet$ extension of φ is a restriction of $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$.*

Inn) *Let $\varphi : \mathfrak{S} \to [0,\infty[$ be isotone with $\varphi(\varnothing) = 0$. Then each inner $\bullet$ extension of φ is a restriction of $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$.*

Proof of Inn). Let $\alpha : \mathfrak{A} \to [0,\infty]$ be an inner $\bullet$ extension of φ. 1) We claim that $\alpha = \varphi_\bullet|\mathfrak{A}$. In fact, from 1.4.Inn.2)3)1) we have $\alpha = \varphi_\bullet$ on $\mathfrak{S}$, hence on $\mathfrak{S}_\bullet$, and hence on $\mathfrak{A}$. 2) It remains to prove that $\mathfrak{A} \subset \mathfrak{C}(\varphi_\bullet)$. Fix $A \in \mathfrak{A}$. For $M \in \mathfrak{S}_\bullet \subset \mathfrak{A}$ we have $\alpha(M) = \alpha(M \cap A) + \alpha(M \cap A')$, and hence $\varphi_\bullet(M) = \varphi_\bullet(M \cap A) + \varphi_\bullet(M \cap A')$ from 1). From $\leqq$ and 1.4.Inn.1) thus $\varphi_\bullet(M) \leqq \varphi_\bullet(M \cap A) + \varphi_\bullet(M \cap A')$ for all $M \subset X$, and hence $=$ for all $M \subset X$ after 1.5.Inn.0)1). □

We conclude from 3.2 and from definition 3.1 that

each outer $\bullet$ premeasure φ is a restriction of its $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$,
each inner $\bullet$ premeasure φ is a restriction of its $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$.

The next step will then be as follows.

3.3 Proposition. Out) *Let $\varphi : \mathfrak{S} \to [0,\infty]$ be isotone with $\varphi(\varnothing) = 0$, such that $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ is an extension of φ. Then $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ is an outer $\bullet$ extension of φ.*

Inn) *Let $\varphi : \mathfrak{S} \to [0,\infty[$ be isotone with $\varphi(\varnothing) = 0$, such that $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an extension of φ. Then $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an inner $\bullet$ extension of φ.*

We start with the outer situation which needs a little interlude. Let $\varphi : \mathfrak{S} \to [0,\infty]$ be isotone with $\varphi(\varnothing) = 0$. We have reason to introduce the condition

($\bullet$) $\quad \varphi^\bullet(M) = \sup\{\varphi^\bullet(M \cap S) : S \in [\varphi < \infty]\}$ for all $M \in [\varphi^\bullet < \infty]$,

which is of an *inner regular* kind. We note at once that its rôle is limited to the case $\bullet = \tau$.

3.4 Remark. *Condition* ($\bullet$) *is fulfilled in case* $\bullet = \star$ *for all* φ, *and in case* $\bullet = \sigma$ *for all submodular* φ.

Proof. $\bullet = \star$) is obvious. $\bullet = \sigma$) Let $M \in [\varphi^\sigma < \infty]$, that is $\varphi^\sigma(M) < \infty$. Then there exists a sequence $(S_l)_l$ in $\mathfrak{S}$ with $S_l \uparrow\supset M$ and $\lim\limits_{l\to\infty} \varphi(S_l) < \infty$, so that $(S_l)_l$ is in $[\varphi < \infty]$. Now $S_l \cap M \uparrow M$ and hence $\varphi^\sigma(S_l \cap M) \uparrow \varphi^\sigma(M)$ from 1.5.Out.2). The assertion follows. □

3.5 Lemma. *Let* φ *be submodular and upward* $\bullet$ *continuous, and*

$$\varphi^\bullet(V) \geqq \varphi(S) + \varphi^\bullet(V \setminus S) \text{ for all } S \in \mathfrak{S} \text{ and } V \in \mathfrak{S}^\bullet \text{ with } S \subset V.$$

Then φ *fulfils* ($\bullet$).

Proof. Let $M \in [\varphi^\bullet < \infty]$, that is $\varphi^\bullet(M) < \infty$. There exists $V \in \mathfrak{S}^\bullet$ with $V \supset M$ and $\varphi^\bullet(V) < \infty$, and hence a $\bullet$ paving $\mathfrak{P} \subset [\varphi < \infty]$ with $\mathfrak{P} \uparrow V$. For $P \in \mathfrak{P}$ we have

$$\varphi^\bullet(M) - \varphi^\bullet(M \cap P) \leqq \varphi^\bullet(M \cap P') \leqq \varphi^\bullet(V \cap P') \leqq \varphi^\bullet(V) - \varphi(P),$$

since $\varphi^\bullet$ is submodular and from the assumption. Thus $\varphi^\bullet(V) = \sup\limits_{P\in\mathfrak{P}} \varphi(P)$ implies the assertion. □

Proof of 3.3.Out). In view of 1.4.Out.1)3) it remains to prove that $\mathfrak{S}^\bullet \subset \mathfrak{C}(\varphi^\bullet)$. We have $\mathfrak{S} \subset \mathfrak{C}(\varphi^\bullet)$ and hence the final assumption in 3.5, so that ($\bullet$) holds true. Thus the assumptions in 2.3.Out) are fulfilled for $\vartheta := \varphi^\bullet$ and $\mathfrak{T} := [\varphi < \infty]$. Now 2.3.Out) asserts that each $A \subset X$ with

$$\varphi^\bullet(T) \geqq \varphi^\bullet(T \cap A) + \varphi^\bullet(T \cap A') \text{ for all } T \in [\varphi < \infty]$$

is in $\mathfrak{C}(\varphi^\bullet)$. So fix $A \in \mathfrak{S}^\bullet$, and let $\mathfrak{M} \subset \mathfrak{S}$ be a $\bullet$ paving with $\mathfrak{M} \uparrow A$. For $T \in [\varphi < \infty]$ and $M \in \mathfrak{M}$ then

$$\varphi^\bullet(T) = \varphi^\bullet(T \cap M) + \varphi^\bullet(T \cap M') \geqq \varphi^\bullet(T \cap M) + \varphi^\bullet(T \cap A'),$$

since $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ is a content on the algebra $\mathfrak{C}(\varphi^\bullet) \supset \mathfrak{S}$. Thus $\varphi^\bullet(T \cap A) = \sup\limits_{M\in\mathfrak{M}} \varphi^\bullet(T \cap M)$ from 1.4.Out.3) implies the assertion. □

Proof of 3.3.Inn). The assumption implies that φ is downward $\bullet$ continuous. In view of 1.4.Inn.1)3) it remains to prove that $\mathfrak{S}_\bullet \subset \mathfrak{C}(\varphi_\bullet)$. Now the assumptions in 2.3.Inn) are fulfilled for $\vartheta := \varphi_\bullet$ and $\mathfrak{T} := \mathfrak{S}$. Thus 2.3.Inn) asserts that each $A \subset X$ with

$$\varphi_\bullet(T) \leqq \varphi_\bullet(T \cap A) + \varphi_\bullet(T \cap A') \text{ for all } T \in \mathfrak{S}$$

is in $\mathfrak{C}(\varphi_\bullet)$. So fix $A \in \mathfrak{S}_\bullet$, and let $\mathfrak{M} \subset \mathfrak{S}$ be a $\bullet$ paving with $\mathfrak{M} \downarrow A$. For $T \in \mathfrak{S}$ and $M \in \mathfrak{M}$ then

$$\varphi_\bullet(T) = \varphi_\bullet(T \cap M) + \varphi_\bullet(T \cap M') \leqq \varphi_\bullet(T \cap M) + \varphi_\bullet(T \cap A'),$$

since $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is a content on the algebra $\mathfrak{C}(\varphi_\bullet) \supset \mathfrak{S}$. Thus $\varphi_\bullet(T \cap A) = \inf\limits_{M\in\mathfrak{M}} \varphi_\bullet(T \cap M)$ from 1.4.Inn.3) implies the assertion. □

Thus we arrive at our first main result.

3.6 Theorem. Out) *For* $\varphi : \mathfrak{S} \to [0, \infty]$ *isotone* $\varphi(\varnothing) = 0$ *the following are equivalent.*

1) φ *is an outer* $\bullet$ *premeasure.*
2) $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ *is an extension of* φ.
3) $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ *is an outer* $\bullet$ *extension of* φ.

In this case it follows that $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ *is the unique maximal outer* $\bullet$ *extension of* φ*. For* $\bullet = \sigma\tau$ *this* $\varphi^\bullet|\mathfrak{C}(\varphi^\bullet)$ *is a measure on the* σ *algebra* $\mathfrak{C}(\varphi^\bullet)$*.*

Inn) *For* $\varphi : \mathfrak{S} \to [0,\infty[$ *isotone* $\varphi(\varnothing) = 0$ *the following are equivalent.*

1) φ *is an inner* $\bullet$ *premeasure.*
2) $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ *is an extension of* φ*.*
3) $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ *is an inner* $\bullet$ *extension of* φ*.*

In this case it follows that $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ *is the unique maximal inner* $\bullet$ *extension of* φ*. For* $\bullet = \sigma\tau$ *this* $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ *is a measure on the* σ *algebra* $\mathfrak{C}(\varphi_\bullet)$*.*

In fact, we have the implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 1) and the maximality assertions from 3.2 and 3.3. The two final assertions result from 2.1 combined with the two final assertions 2) in 1.5.

We conclude with a pair of particular results in which the two theories show quite different behaviour.

3.7 Proposition. Out) *Let* $\varphi : \mathfrak{S} \to [0,\infty]$ *be isotone with* $\varphi(\varnothing) = 0$ *and submodular. For* $A \subset X$ *then* $\varphi^\bullet(A) = 0 \Longrightarrow A \in \mathfrak{C}(\varphi^\bullet)$*.*

Inn) *Let* $\varphi : \mathfrak{S} \to [0,\infty[$ *be isotone with* $\varphi(\varnothing) = 0$ *as well as supermodular and downward* $\bullet$ *continuous at* $\varnothing$*. For* $A \subset X$ *then* $\varphi_\bullet(A) = \varphi_\bullet(X) < \infty \Longrightarrow A \in \mathfrak{C}(\varphi_\bullet)$*.*

Proof of Out). $\varphi^\bullet$ is submodular by 1.5.1.Out). For $M \subset X$ we have $\varphi^\bullet(M) = \varphi^\bullet(M) + \varphi^\bullet(\varnothing) \leqq \varphi^\bullet(M \cap A) + \varphi^\bullet(M \cap A')$ with $\varphi^\bullet(M \cap A) = 0$, thus $\geqq$ and hence $=$.

Proof of Inn). $\varphi_\bullet$ is supermodular by 1.5.1.Inn). For $M \subset X$ we have $\varphi_\bullet(M \cup A) + \varphi_\bullet(M \cap A) \geqq \varphi_\bullet(M) + \varphi_\bullet(A)$, which in view of $\varphi_\bullet(A) = \varphi_\bullet(M \cup A) = \varphi_\bullet(X) < \infty$ implies that $\varphi_\bullet(M) \leqq \varphi_\bullet(M \cap A) \leqq \varphi_\bullet(M \cap A) + \varphi_\bullet(M \cap A')$. Hence combined with $\varphi_\bullet(M) = \varphi_\bullet(M) + \varphi_\bullet(\varnothing) \geqq \varphi_\bullet(M \cap A) + \varphi_\bullet(M \cap A')$ we obtain $=$. $\square$

We recall that [9] theorem 4.4 describes an example of an inner τ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ with $\Phi(X) = \varphi_\tau(X) = 1$ and of an $A \subset X$ with $\varphi_\tau(A) = \varphi_\tau(A') = 0$. Of course $A \notin \mathfrak{C}(\varphi_\tau)$. Thus in case Inn) the counterpart of Out) is not true, even for inner τ premeasures. Then the *complementation theorem* [6] 4.6 implies that the opposite concern, in case Out) the counterpart of Inn), is not true either, even for outer τ premeasures.

4. The Characterization Theorems

The present section will prove the basic characterization results for the outer and inner $\bullet$ premeasures. There will be certain differences between the two situations. For the outer situation we recall the condition ($\bullet$) introduced in the previous section. As before let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$.

4.1 Outer Characterization Theorem. *For* $\varphi : \mathfrak{S} \to [0,\infty]$ *isotone* $\varphi(\varnothing) = 0$ *the following are equivalent.*

0) φ *is an outer* $\bullet$ *premeasure.*
1) φ *is submodular and upward* $\bullet$ *continuous and fulfils* ($\bullet$)*, and* $\varphi(B) \geqq \varphi(A) + \varphi^\bullet(B \setminus A)$ *for all* $A \subset B$ *in* $\mathfrak{S}$*.*
1') φ *fulfils* ($\bullet$)*, and* $\varphi(B) = \varphi(A) + \varphi^\bullet(B \setminus A)$ *for all* $A \subset B$ *in* $\mathfrak{S}$*.*

2) *φ is submodular and upward $\bullet$ continuous, and*
$\varphi^\bullet(V) \geqq \varphi(A) + \varphi^\bullet(V \setminus A)$ *for all* $A \in \mathfrak{S}$ *and* $V \in \mathfrak{S}^\bullet$ *with* $A \subset V$.
2') $\varphi^\bullet(V) = \varphi(A) + \varphi^\bullet(V \setminus A)$ *for all* $A \in \mathfrak{S}$ *and* $V \in \mathfrak{S}^\bullet$ *with* $A \subset V$.

Proof. First of all we note that the conditions 1') and 2') both imply the assertion that φ is upward $\bullet$ continuous. In fact, both 1') for $A = \varnothing$ and 2') for $A = V \in \mathfrak{S}$ combined with 1.5.Out.0) read $\varphi(B) = \varphi^\bullet(B)$ for $B \in \mathfrak{S}$, so that 1.4.Out.2) implies the assertion. Therefore in the subsequent proof the conditions 1') and 2') will be read so as to contain this assertion.

Now the proof will consist of the four parts

$$0) \Rightarrow 2'), 2') \Rightarrow 2) \text{ and } 1') \Rightarrow 1), 2) \Rightarrow 1), 1) \Rightarrow 0).$$

The two middle parts furnish on the one hand $2') \Rightarrow 2) \Rightarrow 1)$, and hence $2') \Rightarrow 2')1) \Rightarrow 2')(\bullet) \Rightarrow 1')$ and therefore on the other hand $2') \Rightarrow 1') \Rightarrow 1)$. These two chaines $2') \Rightarrow 2) \Rightarrow 1)$ and $2') \Rightarrow 1') \Rightarrow 1)$ combined with the other two parts $0) \Rightarrow 2')$ and $1) \Rightarrow 0)$ lead to the desired equivalences.

$0) \Rightarrow 2')$ is obvious from $1) \Rightarrow 3)$ in 3.6.Out).

$2') \Rightarrow 2)$ and $1') \Rightarrow 1)$ We have to conclude that φ is submodular from the assumption that $\varphi(B) = \varphi(A) + \varphi^\bullet(B \setminus A)$ for all $A \subset B$ in $\mathfrak{S}$. For this proof we can assume $A, B \in \mathfrak{S}$ with $\varphi(A), \varphi(B) < \infty$. But then in fact

$$\begin{aligned}\varphi(A \cup B) - \varphi(A) &= \varphi^\bullet\big((A \cup B) \setminus A\big) \\ &= \varphi^\bullet\big(B \setminus (A \cap B)\big) = \varphi(B) - \varphi(A \cap B).\end{aligned}$$

$2) \Rightarrow 1)$ To be shown is $(\bullet)$. But this has been proved in 3.5.

$1) \Rightarrow 0)$ We have to prove 3.6.Out.2), that is $\mathfrak{S} \subset \mathfrak{C}(\varphi^\bullet)$. In view of $(\bullet)$ the assumptions in 2.3.Out) are fulfilled for $\vartheta := \varphi^\bullet$ and $\mathfrak{T} := [\varphi < \infty]$. Thus 2.3.Out) asserts that each $A \subset X$ with

$$\varphi^\bullet(T) \geqq \varphi^\bullet(T \cap A) + \varphi^\bullet(T \cap A') \text{ for all } T \in \mathfrak{T}$$

is in $\mathfrak{C}(\varphi^\bullet)$. But for the $A \in \mathfrak{S}$ this is fulfilled by the last condition in 1). It follows that $\mathfrak{S} \subset \mathfrak{C}(\varphi^\bullet)$. $\square$

We turn to the inner situation.

4.2 Inner Characterization Theorem. *For* $\varphi : \mathfrak{S} \to [0, \infty[$ *isotone* $\varphi(\varnothing) = 0$ *the following are equivalent.*

0) *φ is an inner $\bullet$ premeasure.*
1) *φ is supermodular and downward $\bullet$ continuous, and*
$\varphi(B) \leqq \varphi(A) + \varphi_\bullet(B \setminus A)$ *for all* $A \subset B$ *in* $\mathfrak{S}$.
1') $\varphi(B) = \varphi(A) + \varphi_\bullet(B \setminus A)$ *for all* $A \subset B$ *in* $\mathfrak{S}$.
2) *φ is supermodular and downward $\bullet$ continuous at $\varnothing$, and*
$\varphi(B) \leqq \varphi(A) + \varphi_\bullet^B(B \setminus A)$ *for all* $A \subset B$ *in* $\mathfrak{S}$.
2') $\varphi(B) = \varphi(A) + \varphi_\bullet^B(B \setminus A)$ *for all* $A \subset B$ *in* $\mathfrak{S}$.

Proof. First of all we note that condition 1') implies the assertion that φ is downward $\bullet$ continuous, and condition 2') implies the assertion that φ is downward $\bullet$ continuous at $\varnothing$. In fact, 1') for $A = \varnothing$ reads $\varphi(B) = \varphi_\bullet(B)$ for $B \in \mathfrak{S}$, and 2') for $A = B$ reads $\varphi_\bullet^B(\varnothing) = 0$ for $B \in \mathfrak{S}$, so that the assertions follow from 1.4.Inn.2) and 1.6.2). Therefore in the subsequent proof the conditions 1') and 2') will be read so as to contain these assertions.

Now the proof will consist of the linear sequence of implications $0) \Rightarrow 1') \Rightarrow 2') \Rightarrow 2) \Rightarrow 1) \Rightarrow 0)$.

0) $\Rightarrow$ 1′) is obvious from 1) $\Rightarrow$ 2) in 3.6.Inn).

1′) $\Rightarrow$ 2′) follows from 1.6.4).

2′) $\Rightarrow$ 2) Condition 2’) with 1.6.1) implies that φ is supermodular.

2) $\Rightarrow$ 1) To be shown is that φ is downward $\bullet$ continuous. For $A \subset B$ in $\mathfrak{S}$ we have from 1.6.3)2)

$$\begin{aligned}\varphi_\bullet^B(A) + \varphi(B) &\leqq \varphi_\bullet^B(A) + \varphi(A) + \varphi_\bullet^B(B \setminus A)\\ &\leqq \varphi(A) + \varphi_\bullet^B(B) + \varphi_\bullet^B(\varnothing) \leqq \varphi(A) + \varphi(B),\end{aligned}$$

and hence $\varphi_\bullet^B(A) \leqq \varphi(A)$. Thus $\varphi_\bullet(A) \leqq \varphi(A)$ and hence $\varphi_\bullet(A) = \varphi(A)$ for all $A \in \mathfrak{S}$.

1) $\Rightarrow$ 0) We have to prove 3.6.Inn.2), that is $\mathfrak{S} \subset \mathfrak{C}(\varphi_\bullet)$. The assumptions in 2.3.Inn) are fulfilled for $\vartheta := \varphi_\bullet$ and $\mathfrak{T} := \mathfrak{S}$. Thus 2.3.Inn) asserts that each $A \subset X$ with

$$\varphi_\bullet(T) \leqq \varphi_\bullet(T \cap A) + \varphi_\bullet(T \cap A') \quad \text{for all } T \in \mathfrak{S}$$

is in $\mathfrak{C}(\varphi_\bullet)$. But for $A \in \mathfrak{S}$ this is fulfilled by the last condition in 1). Thus $\mathfrak{S} \subset \mathfrak{C}(\varphi_\bullet)$. □

4.3 Addendum. *If $\mathfrak{S}$ is $\bullet$ compact, then all isotone set functions $\varphi : \mathfrak{S} \to \overline{\mathbb{R}}$ are downward $\bullet$ continuous at $\varnothing$.* In fact, a $\bullet$ paving $\mathfrak{M} \subset \mathfrak{S}$ with $\mathfrak{M} \downarrow \varnothing$ has $\varnothing \in \mathfrak{M}_\star$ and hence $\varnothing \in \mathfrak{M}$, so that $\inf\limits_{M \in \mathfrak{M}} \varphi(M) = \varphi(\varnothing)$. For $\bullet$ compact domains $\mathfrak{S}$ therefore condition 2) in the inner characterization theorem 4.2 becomes much simpler. This fact is the basic reason for the importance of the satellite envelopes $\varphi_\bullet^B$.

We conclude with an important consequence.

4.4 Theorem (Localization Principle). Out) *Let $\varphi : \mathfrak{S} \to [0,\infty]$ be an outer $\bullet$ premeasure. Then $[\varphi < \infty]\top\mathfrak{C}(\varphi^\bullet) \subset \mathfrak{C}(\varphi^\bullet)$.*

Inn) *Let $\varphi : \mathfrak{S} \to [0,\infty[$ be an inner $\bullet$ premeasure. Then $\mathfrak{S}\top\mathfrak{C}(\varphi_\bullet) \subset \mathfrak{C}(\varphi_\bullet)$.*

Proof of Out). Let $A \in [\varphi < \infty]\top\mathfrak{C}(\varphi^\bullet)$, that is $A \cap S \in \mathfrak{C}(\varphi^\bullet)$ for all $S \in [\varphi < \infty]$. Thus

$$\varphi^\bullet(S) = \varphi^\bullet(S \cap A) + \varphi^\bullet(S \cap A') \quad \text{for } S \in [\varphi < \infty].$$

In view of ($\bullet$) the assumptions in 2.3.Out) are fulfilled for $\vartheta := \varphi^\bullet$ and $\mathfrak{T} := [\varphi < \infty]$. It follows that $A \in \mathfrak{C}(\varphi^\bullet)$.

Proof of Inn). Let $A \in \mathfrak{S}\top\mathfrak{C}(\varphi_\bullet)$, that is $A \cap S \in \mathfrak{C}(\varphi_\bullet)$ for all $S \in \mathfrak{S}$. Thus

$$\varphi_\bullet(S) = \varphi_\bullet(S \cap A) + \varphi_\bullet(S \cap A') \quad \text{for } S \in \mathfrak{S}.$$

Now the assumptions in 2.3.Inn) are fulfilled for $\vartheta := \varphi_\bullet$ and $\mathfrak{T} := \mathfrak{S}$. It follows that $A \in \mathfrak{C}(\varphi_\bullet)$. □

5. Continuation of the Inner Theories

We start with an application of 2.4.Inn).

5.1 Lemma. *Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$ and $\varphi : \mathfrak{S} \to [0,\infty[$ be isotone with $\varphi(\varnothing) = 0$ and supermodular. Then*

σ) φ downward σ continuous $\Rightarrow$ $\varphi_\sigma|\mathfrak{C}(\varphi_\sigma)$ is an extension of $\varphi_\star|\mathfrak{C}(\varphi_\star)$,

τ) φ downward τ continuous $\Rightarrow$ $\varphi_\tau|\mathfrak{C}(\varphi_\tau)$ is an extension of $\varphi_\sigma|\mathfrak{C}(\varphi_\sigma)$.

Proof. The above 2.4.Inn) will be applied
in case σ) to $\vartheta := \varphi_\sigma$ and $\psi := \varphi_\star$,
in case τ) to $\vartheta := \varphi_\tau$ and $\psi := \varphi_\sigma$,
and both times to $\mathfrak{T} := \mathfrak{S}$. Then ϑ is supermodular by 1.5.Inn.1), and hence both times all assumptions in 2.4.Inn) are fulfilled. It follows that $\vartheta|\mathfrak{C}(\vartheta)$ is an extension of $\psi|\mathfrak{C}(\psi)$, which is the assertion. □

We note that there is an earlier partial result in [4] 6.24 and 6.25.

5.2 Theorem ($\bullet = \star\sigma\tau$). *Assume that $\varphi : \mathfrak{S} \to [0,\infty[$ on the lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ can be extended to a content $\alpha : \mathfrak{A} \to [0,\infty]$ on a ring $\mathfrak{A} \supset \mathfrak{S}$ such that α is inner regular $\mathfrak{S}$, and that φ is downward $\bullet$ continuous at $\varnothing$. Then φ is an inner $\bullet$ premeasure, and $\Phi := \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an extension of α.*

Proof. By assumption φ is an inner $\star$ premeasure, and α is a restriction of $\varphi_\star|\mathfrak{C}(\varphi_\star)$. From $\varphi_\star(A) \leqq \varphi_\bullet^B(A)$ for $A \subset B$ and 4.2 it follows that φ is an inner $\bullet$ premeasure. Now 5.1 implies that $\varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$ is an extension of $\varphi_\star|\mathfrak{C}(\varphi_\star)$ and hence of α. □

5.3 Remark. *Assume that $\varphi : \mathfrak{S} \to [0,\infty[$ on the lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ can be extended to a content $\alpha : \mathfrak{A} \to [0,\infty]$ on a ring $\mathfrak{A} \supset \mathfrak{S}_\bullet$ such that α is inner regular $\mathfrak{S}_\bullet$, and that φ is downward $\bullet$ continuous at $\varnothing$. Then $\alpha|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous, so that α is an inner $\bullet$ extension of φ. Thus φ is an inner $\bullet$ premeasure.*

This is a remarkable fortification of the definition 3.1.Inn).

Proof. We know from [4] 8.12 =[8] 1.2 that $\alpha|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous at $\varnothing$. Thus 5.2 can be applied to $\psi := \alpha|\mathfrak{S}_\bullet < \infty$ and α. It follows that ψ is an inner $\bullet$ premeasure and hence is downward $\bullet$ continuous. □

We have to note that in cases $\bullet = \sigma\tau$ the above 5.2 comes close to the brink of truth: Under the assumption of 5.2 the set function φ extends to a content $\alpha : \mathfrak{A} \to [0,\infty]$ on a ring $\mathfrak{A} \supset \mathfrak{S}$ which is inner regular $\mathfrak{S}$, and α in turn extends to the *measure* Φ on the σ algebra $\mathfrak{C}(\varphi_\bullet)$ which is *inner regular* $\mathfrak{S}_\bullet$. However, it can happen that φ cannot be extended to a *measure* on a σ algebra which is *inner regular* $\mathfrak{S}$. This has been asserted in essence in [10] section 3 and will be confirmed with example 5.4 below. We want to remark that [10] section 3 also disclaims an implication in opposite direction which is in obvious connection with the present context: An inner $\bullet$ premeasure $\varphi : \mathfrak{S} \to [0,\infty[$ for some $\bullet = \sigma\tau$ need not have an extension to a content $\alpha : \mathfrak{A} \to [0,\infty]$ on a ring $\mathfrak{A} \supset \mathfrak{S}$ which is *inner regular* $\mathfrak{S}$, that is need not be an inner $\star$ premeasure. We recall an example from [4] 6.32 in 5.5 below.

5.4 Example. On $X = \mathbb{R}$ let $\mathfrak{S}$ consist of $\varnothing$ and of the intervals $[0,s]$ with $0 < s < \infty$. Thus $\mathfrak{S}$ is totally ordered under inclusion and hence a lattice. Define $\varphi : \mathfrak{S} \to [0,\infty[$ to be $\varphi(\varnothing) = 0$ and $\varphi(S) = 1$ for the other $S \in \mathfrak{S}$, that is $\varphi = \delta_0|\mathfrak{S}$. One verifies that $\varphi_\bullet = \delta_0$ for $\bullet = \sigma\tau$. Moreover $\varphi(B) = \varphi(A) + \varphi_\star(B \setminus A)$ for $A \subset B$ in $\mathfrak{S}$, so that 4.2 for $\bullet = \star$ implies that the assumption of 5.2 is fulfilled. Now assume that φ extends to a measure $\alpha : \mathfrak{A} \to [0,\infty]$ on a σ algebra $\mathfrak{A} \supset \mathfrak{S}$. Then $\{0\} \in \mathfrak{A}$ with $\alpha(\{0\}) = 1$. Thus it is obvious that α is not inner regular $\mathfrak{S}$.

5.5 Example. Let $X = \mathbb{R}$ and $\mathfrak{S} = \mathrm{Op}(\mathbb{R})$ for the usual norm, and $\varphi = \delta_0|\mathfrak{S}$. One verifies that $\varphi_\bullet = \delta_0$ for $\bullet = \sigma\tau$, so that φ is an inner $\bullet$ premeasure. But we have $\varphi_\star(\{0\}) = 0$, which implies that the condition

$\varphi(B) = \varphi(A) + \varphi_\star(B \setminus A)$ for $A \subset B$ in $\mathfrak{S}$ is violated for $0 \in B$ and $A = B \setminus \{0\}$. Thus φ is not an inner $\star$ premeasure.

5.6 Remark. Next we want to compare the present context with the version of the fundamental inner extension theorem in the beautiful book [12] of David Pollard, which is Appendix A theorem $< 12 >$ for $\bullet = \sigma$ and problem [1] for $\bullet = \tau$. In the present notations he assumes a lattice $\mathfrak{S}$ with $\varnothing \in \mathfrak{S}$ and a set function $\varphi : \mathfrak{S} \to [0, \infty[$ which is an inner $\star$ premeasure and downward $\bullet$ continuous at $\varnothing$. After 4.2 and the above 5.5 this assumption is *strictly stronger* than to require that φ be an inner $\bullet$ premeasure. He then forms the envelope $\tilde{\varphi} := \varphi^\star|\mathfrak{S}_\bullet$. Thus in the present terms $\tilde{\varphi} = \varphi_\bullet|\mathfrak{S}_\bullet$ and hence $\tilde{\varphi}_\star = \varphi_\bullet$ from 1.4.Inn.3)1), so that he obtains the present maximal inner $\bullet$ extension of φ in the form $\tilde{\varphi}_\star|\mathfrak{C}(\tilde{\varphi}_\star) = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$. - Now let us return to the above theorem 5.2. It asserts that under the cited assumption of Pollard [12] the maximal inner $\star$ extension $\varphi_\star|\mathfrak{C}(\varphi_\star)$ of φ is a restriction of $\tilde{\varphi}_\star|\mathfrak{C}(\tilde{\varphi}_\star) = \varphi_\bullet|\mathfrak{C}(\varphi_\bullet)$. This contradicts the first sentence of his Appendix A section 4. However, we have to note that the result 5.2 requires a certain extra effort.

5.7 Specialization of 5.2 ($\bullet = \tau$). *Let $\mathfrak{A}$ be an algebra, and assume that X carries a topology* $\mathrm{Op}(X)$ *such that the open subsets in $\mathfrak{A}$ form a basis of* $\mathrm{Op}(X)$ (which means that the lattice $\mathfrak{P} := \mathfrak{A} \cap \mathrm{Cl}(X) \subset \mathfrak{A}$ satisfies $(\mathfrak{P}_\tau)\perp = (\mathfrak{P}\perp)^\tau = (\mathfrak{A} \cap \mathrm{Op}(X))^\tau = \mathrm{Op}(X)$ or $\mathfrak{P}_\tau = \mathrm{Cl}(X)$). *Let $\alpha : \mathfrak{A} \to [0, \infty]$ be a content which fulfils*

α is inner regular $\mathfrak{S} := \{S \in \mathfrak{A} \cap \mathrm{Cl}(X) : \alpha(S) < \infty\}$ and
$\alpha|\mathfrak{S} < \infty$ is downward τ continuous at $\varnothing$.

Then $\varphi := \alpha|\mathfrak{S}$ is an inner τ premeasure such that $\Phi := \varphi_\tau|\mathfrak{C}(\varphi_\tau)$ extends α, and Φ has a domain $\mathfrak{C}(\varphi_\tau) \supset \mathrm{Bor}(X)$ and is inner regular $\mathrm{Cl}(X)$.

In fact, we have $\mathfrak{P} = \mathfrak{A} \cap \mathrm{Cl}(X) \subset \mathfrak{S}\top\mathfrak{S}$ and hence $\mathrm{Cl}(X) = \mathfrak{P}_\tau \subset \mathfrak{S}\top\mathfrak{S}_\tau \subset \mathfrak{C}(\varphi_\tau)$ in view of 4.4.Inn).

The above specialization 5.7 is in close connection with certain important extension theorems in the literature. In this point the author wants to thank Vladimir Bogachev and David Fremlin for their hints and comments.

1) The particular case $\alpha < \infty$ extends Bogachev [1] Vol.II theorem 7.3.2(ii), which assumes that the topology $\mathrm{Op}(X)$ be Hausdorff and regular. In fact it turns out that these two assumptions can be dispensed with. The remark in [1] Vol.II p.443 that the second assumption cannot be avoided somewhat misses the case, because the cited counterexample in Fremlin [3] 419H does not assume α to be inner regular $\mathfrak{A} \cap \mathrm{Cl}(X)$ but merely to be inner regular $\mathfrak{A} \cap \mathrm{Bor}(X)$.

2) The present result 5.7 is in close connection with Fremlin [3] 415L: Here α is assumed to be a measure on a σ algebra, and in place of the present $\mathfrak{S} = \{S \in \mathfrak{A} \cap \mathrm{Cl}(X) : \alpha(S) < \infty\}$ inner regularity is required with respect to the smaller lattice

$$\{S \in \mathfrak{A} \cap \mathrm{Cl}(X) : \alpha(U) < \infty \text{ for some } U \in \mathfrak{A} \cap \mathrm{Op}(X) \text{ with } S \subset U\}.$$

In return then the resultant extension Φ proves to be a *quasi-Radon* measure, for example from [6] 4.9.

Both times the present result 5.7 has the fundamental advantage that it offers an *explicit representation* for the extension in question - in accordance with the decided purpose of the entire enterprise.

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