Constructive Hopf's theorem: or how to untangle closed planar curves

by

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ABSTRACT

We consider the classification of polygons (i.e. closed polygonal paths) in which, essentially, two polygons are equivalent if one can be continously transformed into the other without causing two adjacent edges to overlap at some moment. By a theorem of Hopf (for dimension 1, applied to polygons), this amounts to counting the winding number of the polygons. We show that a quadratic number of elementary steps suffices to transform between any two equivalent polygons. Furthermore, this sequence of elementary steps, although quadratic in number, can be described and found in linear time. In order to get our constructions, we give a direct proof of Hopf's result.

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1 Why a circle differs from a figure-of-eight

First consider closed planar curves that are smooth. It is intuitively clear that there are no smooth transformations from a figure-of-eight into a circle without introducing a 'kink' at some intermediate moment.

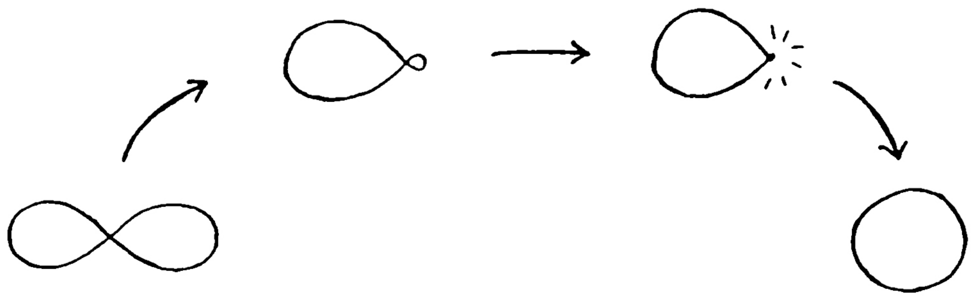

Figure 1. Appearance of a kink

We will say two closed planar curves are equivalent if one is transformable to the other in a kink-free manner and consider the problem of classifying the equivalence classes.

This problem has been completely solved, and we describe its mathematical formulation: let $C : S^1 \to E^2$ be the natural parameterization (by arc-length) of a smooth closed curve, where S^1 is the unit circle. We then define $\omega_C : S^1 \to S^1$ where $\omega_C(t)$ is defined to be the orientation of the directed tangent at the point $C(t)$. The notion of kink-free transformation is captured by saying that two curves C and D are *equivalent* if ω_C and ω_D are homotopic. The *winding number* (or degree) of C is defined to be

$$\int_{S^1} d\omega_C.$$

For instance the winding number of the figure-of-eight and the circle are 0 and 2π, respectively. Then a famous theorem of Hopf (for dimension 1) says that C and D are equivalent if and only if they have the same winding number. See [2,1].

The purpose of this paper is to give a constructive version of Hopf's theorem for dimension one, in order to give quantitative and complexity information implicit in the result.

The rest of this paper is organized as follows. In section 2, we convert Hopf's theorem to the polygonal setting. In section 3, we introduce a normal form for polygons. In section 4, we give a constructive proof of Hopf's theorem for polygons. In section 5, we use the insights of the proof to develop a linear time algorithm. We conclude in section 6.

2 Classification of Polygons

In order to make our problem concrete (computational), we will turn this into a problem on closed polygonal paths (which we will simply call a *polygon*). Since polygons have kinks at their vertices, we reformulate kink-free transformations as transformations of polygons that do not cause two adjacent edges to overlap at some point. This is made precise next.

Definition: A *path* Π is specified by a sequence

$$\Pi = < v_1, v_2, \ldots, v_n >, n \geq 2$$

of points which we call *vertices*. The *initial* and *final* vertices are v_1 and v_n, respectively. We require $v_i \neq v_{i+1}$ for $i = 1, \ldots, n-1$. The edges of the path are the line segments $[v_i, v_{i+1}]$ for $i = 1, \ldots, n-1$. A *closed* path is one of the form

$$\Pi = < v_1, \ldots, v_n, v_{n+1} >, n \geq 3$$

such that $v_{n+1} = v_1$. Two closed paths Π, Π' are said to *combinatorially equivalent* if one sequence can be obtained from the other by a cyclic shift, possibly followed by a reversal. More precisely, if $\Pi = < v_1, \ldots, v_n, v_1 >$, then Π' is either equal to

$$< v_i, \ldots, v_n, v_1, \ldots, v_{i-1}, v_i >$$

for some $i = 1, \ldots, n$, or the reverse of this sequence. A *polygon* P on n vertices is defined as the combinatorial equivalence class of some closed path $\Pi = < v_1, \ldots, v_n, v_1 >$. We will express the polygon as

$$P = (v_1, \ldots, v_n).$$

So P can also be written as $(v_2, v_3, \ldots, v_n, v_1)$, say. For this paper, we further require that polygons satisfy the following local condition:

(C) For any consecutive triple v_{i-1}, v_i, v_{i+1}, if the three vertices are collinear, then v_i lies strictly between the other two,

$$v_i = \alpha v_{i-1} + (1-\alpha) v_{i+1}$$

for some $0 < \alpha < 1$.

(Here, as throughout the paper, arithmetic on indices of vertices of a polygon P is modulo n, the length of the sequence P.) Note that (C) in particular prevents $v_i = v_j$ for $|i-j| = 1$ or 2. However, we allow $v_i = v_j$ for $|i-j| > 2$ and in fact edges may even coincide. The reason for (C) is that the equivalence we seek allows local transformations that do not create kinks, and if (C) fails then it sometimes becomes ambiguous whether a kink is introduced.

The transformations we allow are of three types:

(T0) Insertion. We may transform $P = (v_1, \ldots, v_n)$ to

$$Q = (v_1, \ldots, v_i, u, v_{i+1}, \ldots, v_n), (i = 1, \ldots, n)$$

where u is a point lying strictly between v_i and v_{i+1}.

(T1) Deletion. We may transform $P = (v_1, \ldots, v_n)$ to

$$Q = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n), (i = 1, \ldots, n)$$

provided v_{i-1}, v_i, v_{i+1} are collinear.

To introduce the last type of transformation, we need a definition. Relative to any vertex v_i, we define two *forbidden cones* (at v_{i-1} and v_{i+1}, respectively): the forbidden cone at v_{i-1} is bounded by the two rays emanating from v_{i-1}, one ray directed towards v_{i-2} and the other directed away from v_{i+1}. Of the two choices of cones bounded by these rays, we choose the one that does not contain v_i. The forbidden cone at v_{i+1} is similarly defined, being bounded by the two rays emanating from v_{i+1} and directed towards v_{i+2} and away from v_{i-1}, respectively. Each cone is a closed region so it includes the bounding rays. This definition applies for all $n \geq 3$. We are now ready for the third transformation type.

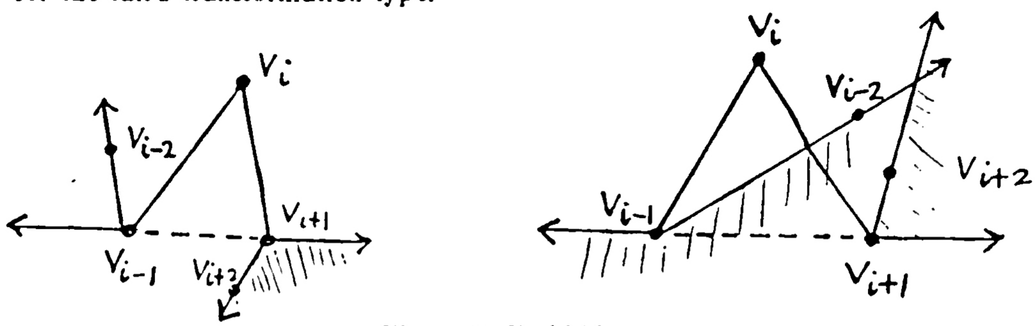

Figure 2. Forbidden cones

(T2) Translation. We may transform $P = (v_1, \ldots, v_n)$ to

$$Q = (v_1, \ldots, v_{i-1}, u, v_{i+1}, \ldots, v_n), (i = 1, \ldots, n)$$

where u is any point not in the union of the two forbidden cones of v_i. In otherwords, we replace the point v_i by u.

Definition: We say that two polygons P, Q are *equivalent* if one can be transformed to the other by a finite sequence of operations of types T0,T1,T2.

It is not hard to see that the relation is a true equivalence relation. Of course, we know from Hopf's theorem (applied to polygons) that it is sufficient to look at the winding number of a polygon. However, given two equivalent polygons P and Q, it is unclear how one can

(A) transform P into Q by a sequence of (T0-T2) transformations,

(B) bound the number of transformation steps needed, and

(C) give an efficient algorithm to find a sequence of such transformation steps.

Indeed, standard proofs of Hopf's theorem yield no constructive information to answer these questions.

We answer these questions (A-C) by defining a unique normal form (i.e. representative) of each equivalence class, and showing that a quadratic number of steps suffices to reduce a polygon into the normal form. The algorithm to find these steps runs in linear time. In particular, this solves the problem of transforming between two equivalent polygons P, Q since the normal form is unique and the steps are reversible: first convert P to the normal form and then reverse the steps from the normal form to Q.

To think about what might be desirable, we note (see Fig. 3) that the triangle and the bow-tie are obvious candidates for normal forms. Perhaps less convincingly, the 5-point star (5-star) also seems like a good candidate for a normal form.

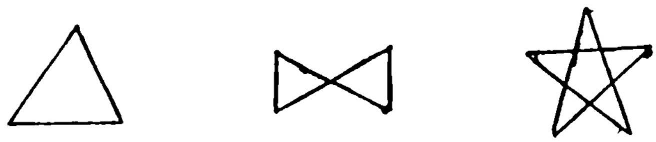

Figure 3. The triangle, bowtie and 5-star

The next figure illustrates a sequence of transformations to reduce a Victoria Cross polygon to one with fewer vertices.

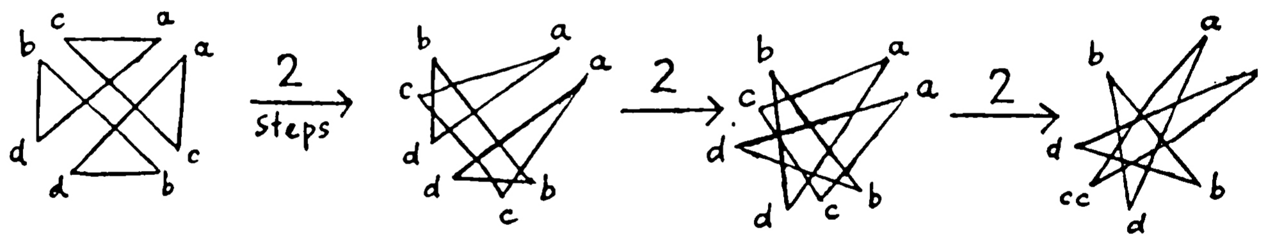

Figure 4. Reduction of the Victoria Cross

Remark. Any smooth closed curve can be approximated by a polygon and the transformations can discretized as a series of our (T0-2) transformations. So in some sense we have also solved the corresponding question for smooth curves.

3 Star Polygons

Definition: A polygon is *reducible* if it is equivalent to one with fewer vertices. It is *irreducible* otherwise.

One can check that the triangle, bow-tie and 5-star cannot be transformed by (T1) or (T2) transformations into any polygon with fewer vertices. But it turns out that even with (T0) transformations (which insert new vertices) the result is true: these are irreducible polygons. Since they have different number of vertices, it follows that they are inequivalent to each other. (Of course we can conclude this at once if we apply Hopf's theorem.) In fact, all polygons on 3, 4 and 5 vertices are equivalent to these three candidates using only (T2) transformations. One may also check that there are no irreducible polygons on 6 vertices!

The 5-star is also equivalent to the polygons in Fig. 5.

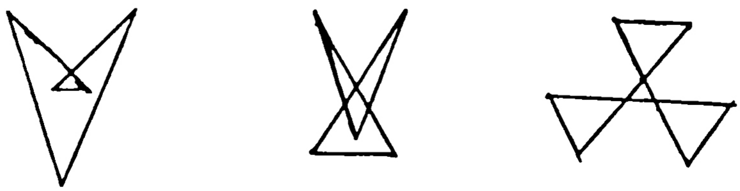

Figure 5. Polygons equivalent to the 5-star: the Fox, the Rabbit and Radioactive Sign

Note that the first polygon in Fig. 5 has the minimal number of self-intersections among its equivalence class, so perhaps it is a better choice of a normal form than the 5-star. This indicates that the choice of a good normal form is not obvious. Our first result, though simple, helps to narrow our choices considerably.

Theorem 1 *Every polygon P can be transformed by (T2) transformations into a polygon Q all of whose vertices are distinct and lie on a circle.*

Proof. Let C be a circle that contains all the vertices of P in its interior. For each vertex v of P, we may move v onto C using a (T2) transformation: this follows from the observation that the non-forbidden region relative to v always contain an infinite cone K at v. We may move v in any direction inside K until we reach C. Furthermore, we can make sure that we avoid any vertex already on C. Q.E.D.

Henceforth, we assume that polygons have their vertices on some circle. Among the polygons on a circle, we define a particularly nice class.

Definition: A path $\Pi = < v_1, \ldots, v_n >$ is called a *star path* if the edges $e_i = [v_i, v_{i+1}]$ (for each $i = 1, \ldots, n-1$) intersect each of the edges

$$e_1, e_2, \ldots, e_{i-1}.$$

Here the edges are closed line segments and so e_i ($i \geq 2$) always intersects e_{i-1}. A polygon $P = (v_1, \ldots, v_n)$ is called an *n-star* if for some choice of an *initial* vertex v_i, $i = 1, \ldots, n$, the path

$$\Pi_i = < v_i, \ldots, v_n, v_1, \ldots, v_{i-1} >$$

is a star path.

This terminology agrees with what we had called the 5-star before. The triangle is a 3-star and the bow-tie a 4-star. The following figure shows the next few n-stars.

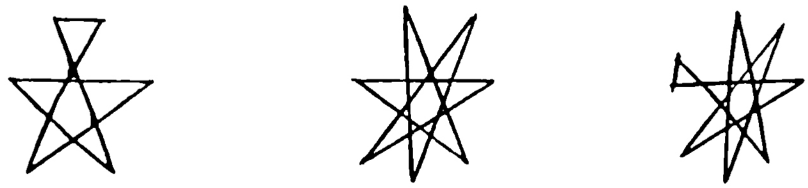

Figure 6. 6-, 7-, and 8-stars

The following lemma is easy:

Lemma 2
1. For $n = 4$ or for odd values of n, if a polygon $P = (v_1, \ldots, v_n)$ is an n-star then for every choice of initial vertex v_i, the path $\Pi_i = < v_i, \ldots, v_n, v_1, \ldots, v_{i-1} >$ is a star path. (In other words, the definition of a n-star does not depend on the choice of the initial vertex in these cases.)
2. For all other cases of n, there is a unique choice of an initial vertex v_i which makes Π_i a star path.

Lemma 3 *Let n, m be odd positive integers or equal to 4. If $n \neq m$, then the n-star and the m-star are inequivalent.*

Proof. One checks that the winding number of the 4-star is 0 and for each positive integer k, the $(2k+1)$-star has winding number $\pm 2k\pi$ (where the sign corresponds to a choice direction for the polygon). The result then follows from the fact that the winding number of a polygon is unchanged by any transformation of types (T0-2). **Q.E.D.**

This lemma supplies us with an infinite list of inequivalent polygons. We next prove that this list exhausts all the equivalence classes.

4 A Canonical Form for Polygons

We now set out to prove

Theorem 4 (Canonical Form) *Every polygon can be transformed by a sequence of (T1) and (T2) transformations into an n-star, for some n that is either odd or equal to 4.*

Corollary 5 *An n-star is irreducible if and only if $n = 4$ or n is odd.*

Proof. Suppose that an n-star is irreducible. Then the theorem implies that n must be 4 or odd. Conversely, let $n = 4$ or odd. If an n-star is reducible, then the theorem shows that it is reducible to an m-star for some $m < n$ where $m = 4$ or odd. This contradicts the previous lemma that the n- and m-stars are inequivalent. **Q.E.D.**

We prove the canonical form theorem by a sequence of lemmas.

A polygon that can (resp. cannot) be transformed to one with fewer vertices using just (T1) and (T2) transformations will be called *semi-reducible* (resp. *semi-irreducible*).

We need a useful notation. Henceforth, we write only the *indices* (i.e. subscripts) of vertices to denote the vertices. So we write $P = (1, 2, \ldots, n)$ for $n \geq 3$. Next, if $u_1, u_2, \ldots, u_k$ ($k \geq 3$) are indices (not necessarily consecutive) we shall write

$$u_1 < u_2 < \cdots < u_k$$

to mean that as we traverse the circle of P in a clockwise direction, starting from u_1, we will meet the indices $u_1, u_2, \ldots, u_k$ in this order.

The following simple fact is very useful:

Lemma 6 *Suppose $P = (1, \ldots, n)$ ($n \geq 4$) is such that the pair of edges $[1, 2]$ and $[3, 4]$ do not intersect, and also the pair $[2, 3]$ and $[4, 5]$ do not intersect. Then P is equivalent to $(1, 2, 4, 5, \ldots, n)$ by a (T1) transformation. In other words, we may delete index 3.*

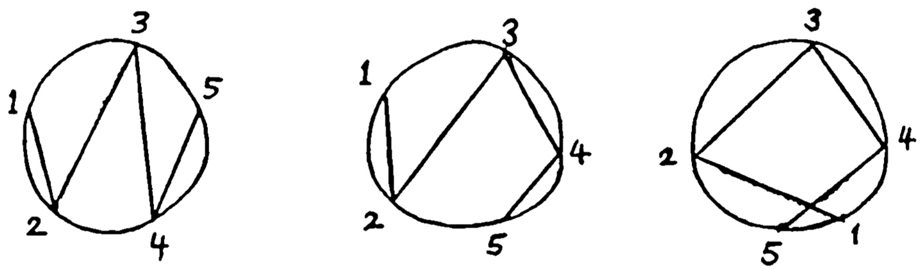

Figure 7. Can delete index 3

This is the only form of deletion of vertices used in our proof. Henceforth, whenever we delete vertices it is by appeal to this lemma.

We say that $P = (1, 2, \ldots, n)$ *contains an N-shape* if $n \geq 4$ and for some choice of index i, we have

$$i < i + 1 < i + 3 < i + 2$$

or

$$i < i + 2 < i + 3 < i + 1.$$

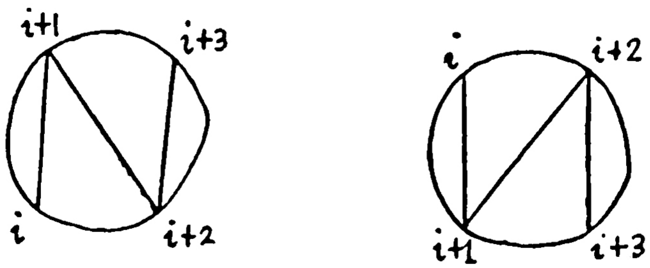

Figure 8. An N-shape

Lemma 7 *A semi-irreducible polygon $P = (1, 2, \ldots, n)$ does not contain an N-shape unless $n = 4$.*

Proof. By way of contradiction, assume P has an N-shape. By symmetry, assume $1 < 2 < 4 < 3$.

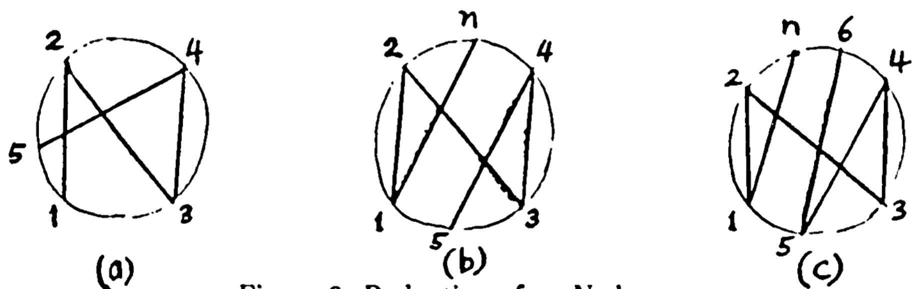

Figure 9. Reduction of an N-shape

The result is true for $n = 4$, so suppose $n \geq 5$. Since P is semi-irreducible, by the previous lemma, the edge $[4, 5]$ must intersect $[2, 3]$; hence $3 < 5 < 2$. Similarly, $2 < n < 3$. This shows that $n \neq 5$ so assume $n \geq 6$. If $1 < 5 < 2$ (Fig. 9a) then we can translate index 2 so that $1 < 2 < 5$ (this translation can occur because $2 < n < 3$). Then we can delete 3. Therefore, we have $3 < 5 < 1$. By symmetry, we have $2 < n < 4$. The situation is shown in Fig. 9b.

If $n = 6$ then it is easy to see that P is semi-reducible. Otherwise, consider the location of index 6. There are two cases. First suppose $n < 6 < 4$ (Fig. 9c). Then we may translate index 5 so that $2 < 5 < 6$. Then we can delete index 3. In the other case, $4 < 6 < n$, we can translate index 4 so that $2 < 4 < n$. Next translate index 3 so that $2 < 4 < 3 < n$. Now we may delete index 2. Q.E.D.

Corollary 8 *An n-star is semi-reducible if n even and not equal to 4.*

Proof. Let n be even, $n \neq 4$, and let $P = (1, \ldots, n)$ be an n-star. Then the vertices $n-2, n-1, n, 1$ forms an N-shape. Q.E.D.

We say that $P = (1, 2, \ldots, n)$ *contains an U-shape* if $n \geq 4$ and for some choice of index i, we have

$$i < i+1 < i+2 < i+3$$

or

$$i < i+3 < i+2 < i+1.$$

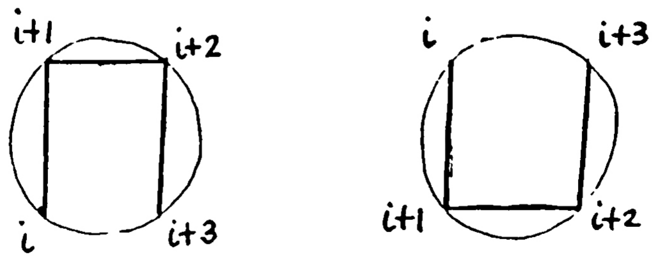

Figure 10. An U-shape

Lemma 9 *A semi-irreducible polygon* $P = (1, 2, \ldots, n)$ *cannot contain a U-shape.*

Proof. The result is true for $n = 3, 4$ and 5, so assume $n \geq 6$. By the previous lemma, we know that P does not contain an N-shape. Suppose indices $(2, 3, 4, 5)$ forms a U-shape as in the figure, $2 < 5 < 4 < 3$.

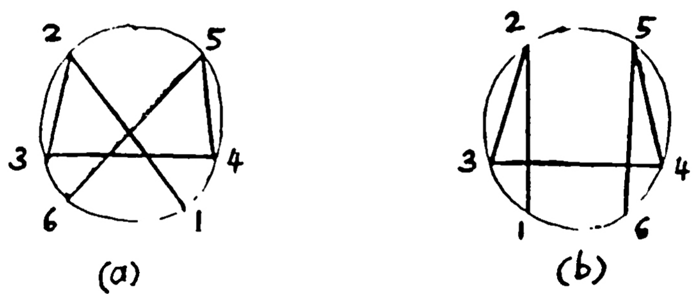

Figure 11. The elimination of U-shape

Since index 3 cannot be deleted, we have $4 < 1 < 3$, and similarly, since index 4 cannot be deleted, $4 < 6 < 3$. Suppose the relative position of indices 1 and 6 satisfies

$$4 < 1 < 6 < 3$$

as in Fig. 11a. Then we may translate index 3 so that $1 < 3 < 6$ and then translate index 4 so that $1 < 4 < 3 < 6$. Then we may delete index 3 (or 4), contradicting the semi-irreducibility of P. Hence we may assume the situation of Fig. 11b, with $4 < 6 < 1 < 3$.

Consider index n. If $2 < n < 1$ then $(n, 1, 2, 3)$ forms an N-shape which implies P is semi-reducible. Hence we have $1 < n < 2$. (In particular, this means $n \neq 6$, so $n \geq 7$.) So choose the smallest index v from $7,\ldots,n$ such that $1 < v < 5$. Such a choice of v exists since, *a fortiori*, $1 < n < 5$. We now use induction on v to prove the following claim: we may transform P such that indices $1,\ldots,5$ remain fixed but index 6 satisfies

$$4 < 1 < 6 < 3.$$

Clearly $v > 6$. If $v = 7$ then $(4,5,6,7)$ forms an N-shape which is a contradiction. If $v = 8$ then note that $v - 1 < 6 < 1 < v$. We can translate 6 so that $1 < 6 < v$; next translate 3 so that $1 < 6 < 3 < v$. Now we may delete index 4. Hence assume $v \geq 8$.

Consider indices $v - 1, v - 2$ and $v - 3$. These vertices, by definition of v, all lie on the arc clockwise from index 5 to index 1. Since there are no N-shapes in P, we have $v - 1 < v - 2 < 1$. There are two cases for $v - 3$. If $v - 1 < v - 2 < v - 3$ then there are 2 possibilities for $v - 4$: either (a) $v - 3 < v - 4 < 1$ or (b) $v - 4 < v - 2 < v - 3$. If (a) holds then $v - 2$ can be deleted. Hence (b) holds, in which case we may translate $v - 3$ so that $1 < v - 3 < v$. Now, for this transformed polygon, we can apply the induction hypothesis (replace v by $v - 3$). The other case for $v - 3$ is $v - 3 < v - 1 < v - 2$. Then we may translate $v - 2$ so that $1 < v - 3 < v$ and again we may apply the induction hypothesis (replace v by $v - 2$).

This completes the proof of the claim. But the claim transforms P into the shape in Fig. 11a, which we already show is a contradiction. **Q.E.D.**

We now give the last lemma.

Lemma 10 *Let* $P = (1,2,\ldots,n)$ *be any semi-irreducible polygon. Then* $\Pi =< 1,2,\ldots,n >$ *is a star path.*

Proof. We now know that P has no N- and no U-shapes. We will show that $\Pi_v =< 1,2,\ldots,v >$ ($v = 3,\ldots,n-1$) is a star path implies Π_{v+1} is a star path. Consider the situation in Fig. 12 (so $1 < 3 < 2$).

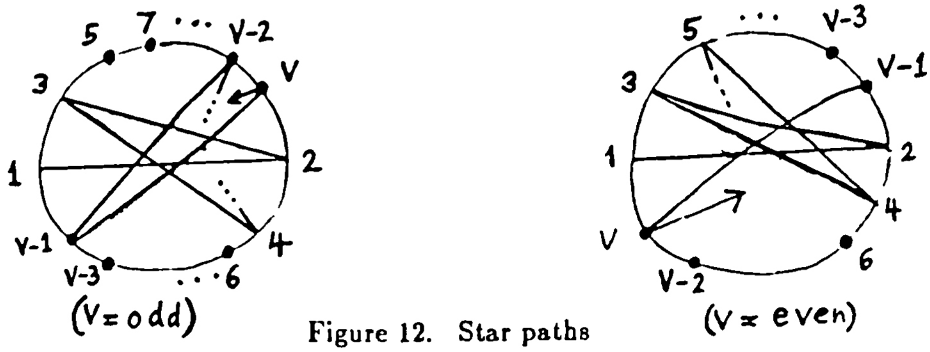

Figure 12. Star paths

The result $v = 3$ follows from the previous two lemmas since if Π_4 is not a star path, then it forms a U- or an N-shape. The same is true when $v = 4$, so let $v \geq 5$.

First suppose v is odd. If Π_{v+1} is not a star path then $1 < v + 1 < v - 2$. Then we can move index $v - 1$ so that $1 < v - 1 < 3$. In fact, we can repeat this for $v - 3, v - 5, \ldots, 4$ (in this order) so that $1 < v - 1 < v - 3 < \cdots < 4 < 3$. Now $(1,2,3,4)$ is a U-shape, contradiction.

Suppose v is even. Then $2 < v + 1 < v$ and we move indices $n-1, n-3, \ldots, 5$ (in this order) so that $1 < 5 < 7 < \cdots < n-1 < 4$. Now $(2,3,4,5)$ is a U-shape. Q.E.D.

The main theorem now follows: given any polygon P, we can reduce it by (T1) and (T2) transformations until it is semi-irreducible. Then by the last lemma, the result must be an n-star. By the corollary to lemma 7, n cannot be even and not equal to 4. This proves the theorem.

5 Algorithm

The proof of the canonical form theorem contains an implicit quadratic time algorithm to transform a polygon to its normal form. We now give a linear time algorithm to construct the sequence of these quadratic transformation steps - this apparent paradox will be clarified below.

Since transforming a polygon so that its vertices all lie on a circle is a linear time process, we may assume that the input polygon $P = (v_1, \ldots, v_n)$ is already in this form. We further assume that the circle is the unit circle centered at the origin, with $v_1 = (-1, 0)$ and $v_2 = (1, 0)$.

The algorithm processes the input vertices in order. In the generic situation, the vertices $v_1, \ldots, v_{j-1}$ have been processed and the polygon has been transformed into an equivalent polygon

$$P_j = (u_1, \ldots, u_{i-1}, v_j, \ldots, v_n).$$

Furthermore, we assume that

$$< u_1, \ldots, u_{i-1} >$$

forms a star path. The *current vertex* being processed is v_j although our algorithm may look ahead slightly, up to v_{j+3}. It is a realtime algorithm in the sense that each vertex takes $O(1)$ time to process, and it outputs $O(j)$ transformation steps with the processing of each vertex v_j. This apparent paradox (that $O(j)$ steps can be described in $O(1)$ processing time) arises because the $O(j)$ steps involve translating blocks of $O(j)$ vertices of a star path. Such transformations can be described in $O(1)$ time. Thus P_j can be obtained from P by applying the entire sequence of $O(j^2)$ transformation steps output up to now.

An interesting feature of the algorithm is that it uses only $O(1)$ runtime memory. More precisely, at the moment of processing v_j, we only need in the active memory the values of the indices i, j; the values of the vertices

$$v_j, \ldots, v_{j+3};$$

and the sign (i.e. left or right) of the turns

$$(u_{i-2}, u_{i-1}, v_j) \text{ and } (u_{i-1}, v_j, v_{j+1}).$$

We will call the latter the *sign information*. In our algorithm, we must be careful to show how we can reconstruct this sign information despite the fact that we do not explicitly store the u's.

For simplicity, in our description we will not explicitly say what (T1,T2) transformation steps will be output. But each step of the algorithm will be given justification and the reader can easily deduce the transformation steps needed.

The relation $i \leq j$ always hold (since we do not create new vertices). Therefore it is unambigous to refer to the vertices by their indices, as long as it is clear whether an index is less than i or at least j. We may now begin and assume that $i - 1 \geq 3$ and $j \geq 4$. To initialize, we may let $u_i = v_i$ for $i = 1, 2, 3$. Without loss of generality, let $1 < 3 < 2$ (i.e. $(1, 2, 3)$ is a left turn).

There are 4 cases to describe in the processing of vertex v_j: (u_{i-2}, u_{i-1}, v_j) is either a left turn or a right turn, and i is either odd or even. First assume $i = odd$ (see Fig. AA).

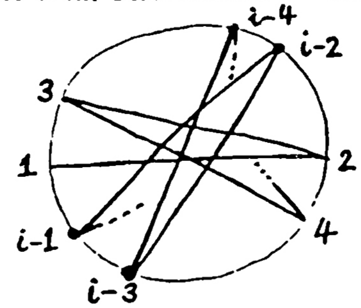

Figure AA. Case i =odd

Case A. $(i-2, i-1, j)$ is a right turn.
Case A1. $(i-1, j, j+1)$ is a left turn. Decrease i.

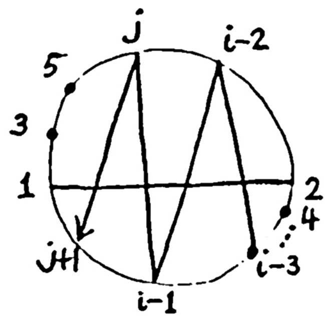

Figure BB. Case A1

Justification: See Fig. BB. We may delete $i-1$. To reconstruct the sign information, note that the turns $(i-3, i-2, j)$ and $(i-2, j, j+1)$ are both left turns.

Case A2. $(i-1, j, j+1)$ is a right turn. We have some subcases.

Case A21. $j < j+1 < 2$. Decrement i by 1.

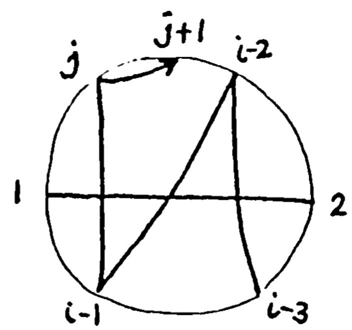

Figure CC. Case A21

Justification: We may assume that $j < j+1 < i-2$ since we can move the odd u-vertices (i.e. $u_3, u_5, \ldots, u_{i-2}$) clockwise as close to index 2 as desired.[3] This is shown in Fig. CC. We can then delete $i-1$ and note that $(i-3, i-2, j)$ and $(i-2, j, j+1)$ are left and right turns, respectively.

Case A22. $1 < j < 2 < j+1$. Decrease i by 1.

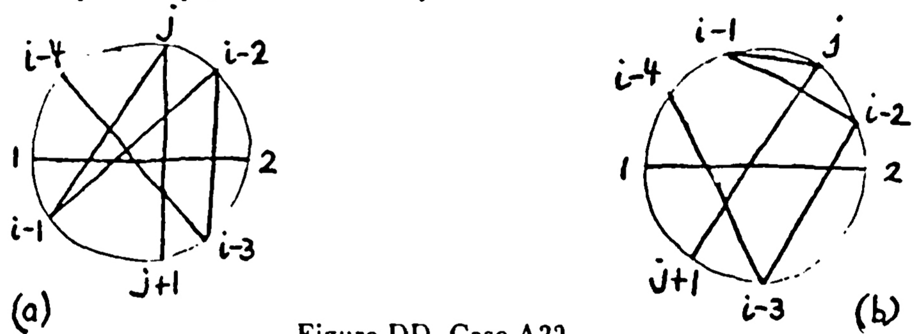

Figure DD. Case A22

Justification: We may assume $i-4 < j < i-2$ and $i-3 < j+1 < i-1$ since the u's can be moved appropriately. Then we are as in Fig. DD(a). We may now move $i-1$ such that $i-4 < i-1 < j$ (Fig. DD(b)), and then delete $i-2$. Now $(1, \ldots, i-3, i-1)$ is a star path and $(i-3, i-1, j)$ and $(i-1, j, j+1)$ are both right turns.

Case A23. $j < 1 < 2 < j+1$. So $(i-1, j, j+1)$ is a right turn (Fig. EE)

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[3] Observe that this 'assumption' actually require a sequence of $O(j)$ transformation steps to realize, but it is easily described in $O(1)$ time since we are transforming a consecutive sequence of vertices in a star path. This clarifies an earlier remark.

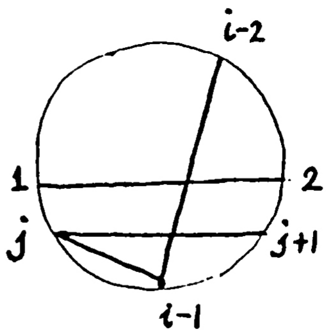


Figure EE. Case A23

Case A231. $1 \leq j+2 < j$. Decrease i by 1.
Justification: We can move j such that $1 < j < 2$ which brings us into case A22.
Case A232. $j < j+2 < 1$. So $(j, j+1, j+2)$ is a left turn, see Fig. FF.

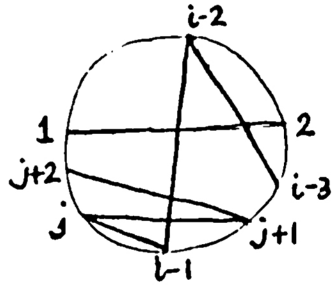


Figure FF. Case A232

Case A2321. $(j+1, j+2, j+3)$ is a right turn. Increment j by 2.
Justification: We may delete $j+1$ and j (in that order). Also $(i-2, i-1, j+2)$ and $(i-1, j+2, j+3)$ are both right turns.
Case A2322. $(j+1, j+2, j+3)$ is a left turn. Increment j by 3.

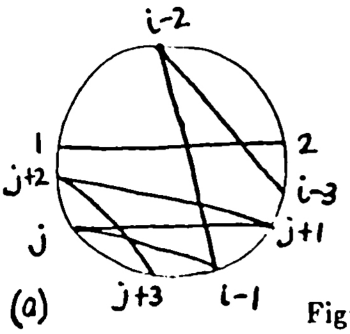
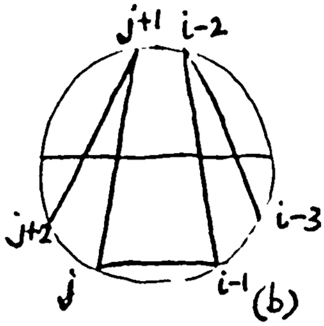


Figure GG. Case A2322



Justification: See Fig. GG(a). We move $j+1$ counterclockwise so that $1 < j+1 < i-2$, as in Fig. GG(b). Now we may delete $i-1, j$ and $j+1$, in that order. Then $(1, \ldots, i-2, j+2)$ is a star path. Also $(i-2, j+2, j+3)$ is a left turn. We can determine the sign of the turn $(j+2, j+3, j+4)$ by looking at $j+4$.

Case B. $(i-2, i-1, j)$ is a left turn. Then $j \leq n$.

Case B1. $i-2 < j < 2$. Increment i and j by 1 each. See Fig. HH.

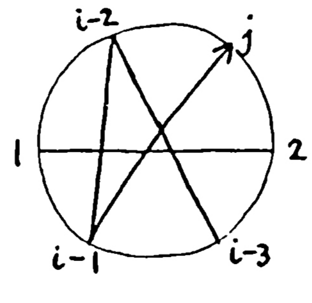


Figure HH. Case B1

Justification: $(1, \ldots, i-1, j)$ is a star path.

Case B2. $2 < j < i-1$.

Case B21. $(i-1, j, j+1)$ is a right turn. Increment j by 1.

Justification: By moving the even u-vertices (i.e. $u_2, \ldots, u_{i-3}$) if necessary, we may assume that $2 < i-3 < j$, as in Fig. JJ. Now delete $i-1$ and $(1, \ldots, i-2, j)$ is a star path. Also $(i-2, j, j+1)$ is a right turn.

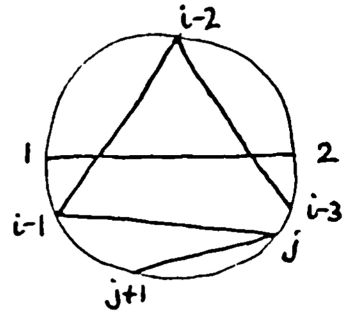


Figure JJ. Case B21

Case B22. $(i-1, j, j+1)$ is a left turn.

Case B221. $j < j+1 < 1$. See Fig. KK

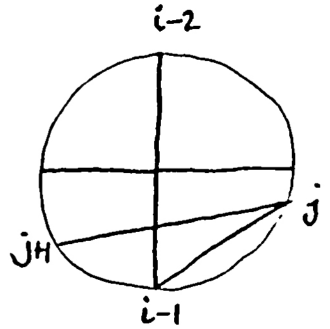


Figure KK. Case B221

Case B2211. $j < j+2 < 2$. Increment both i and j by 1.
Justification: We can move j counterclockwise so that $i-2 < j < 2$. This reduces to case B1.
Case B2212. $2 < j+2 < j$. See Fig. LL.

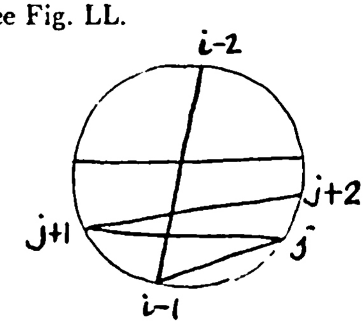
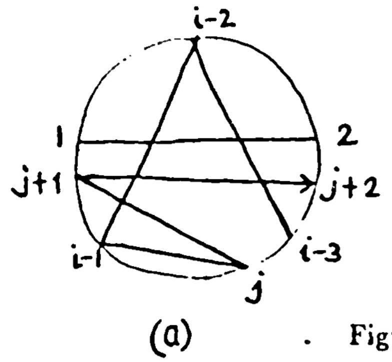


Figure LL. Case B2212

Case B22121. $(j+1, j+2, j+3)$ is a left turn. Increment j by 2.
Justification: We delete $j+1$ and then j. The turns $(i-2, i-1, j+2)$ and $(i-1, j+2, j+3)$ are both left turns.
Case B22122. $(j+1, j+2, j+3)$ is a right turn. Increment i by 1 and j by 2.



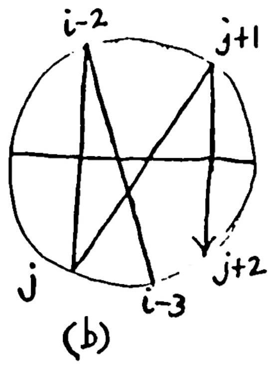
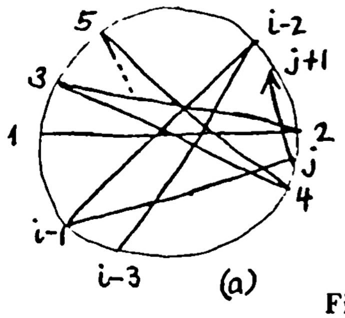


Figure MM. Case B22122

Justification: We may assume that $i-3 < j < i-1$ (Fig. MM(a)). Then we can move $j+1$ clockwise so that $i-2 < j+1 < 2$ and delete $i-1$. The sequence $(1, \ldots, i-2, j, j+1)$ is a star path, Fig. MM(b).

Case B222. $1 < j+1 < 2$. Increment i by 1 and j by 2.



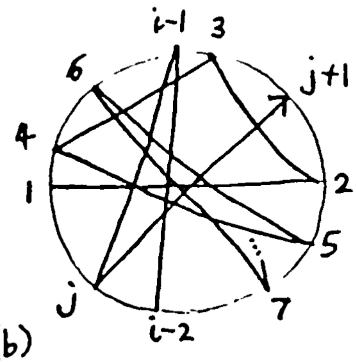


Figure NN. Case B222

Justification: See Fig. NN(a). We may assume that $3 < j+1 < 2$. Now move the vertices $i-2, i-4, \ldots, 7, 5$ (in that order) clockwise between 2 and j, i.e.,

$$2 < 5 < 7 < \cdots < i-2 < j.$$

Similarly move the vertices $i-1, i-3, \ldots, 6, 4$ (in that order) clockwise between 1 and $j+1$, i.e.,

$$1 < 4 < 6 < \cdots < i-1 < j+1.$$

See Fig. NN(b). Now we delete index 3 and obverve that $(1, 2, 4, 5, \ldots, i-1, j, j+1)$ is a star path. Obviously we can deduce the sign of the turns $(j, j+1, j+2)$ and $(j+1, j+2, j+3)$.

Case B223. $2 < j+1 < j$. Increase j by 1.

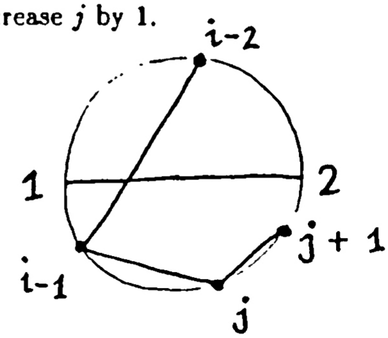


Figure OO. Case B223

Justification: We can delete j in this situation. This completes subcase B and hence the case $i = odd$. The case $i = even$ is similar and we leave it to the reader.

6 Conclusion

We have given an elementary proof of Hopf's theorem, resulting in a linear algorithm to "untangle" any polygon. We emphasize that the true contribution of this work the construction of the transformation steps: checking if two polygons are equivalent is in itself a trivial process of keeping a cumulative sum of the angles turned. The key insight comes from putting the vertices on a circle, thereby reducing polygons to combinatorial objects (the cyclic permutation of their vertices around the circle).

Our proof shows incidentally: (1) It suffices to use (T1,T2) transformations to make a polygon irreducible and (2) any two equivalent irreducible polygons are inter-transformable using only (T2) transformations.

We can generalize the problem to the classification of polygonal paths. The transformations of vertices not near the endpoints are as before, with the following important modification: the transformation must not let an edge cross either endpoints of the path. Without this requirement, the classification becomes trivial. At the endpoints, we require that the end edges never overlap the imaginary edge connecting both endpoints. The methods of this paper ought to suffice for this classification.

We pose as a problem for further research to reduce the higher dimensional version of Hopf's theorem to a constructive, algorithmic form as we have done here.

References

[1] H. Hopf. Abbildungsklassen n-dimensionaler mannigfaltigkeiten. *Math. Annalen*, 96:209–224, 1927.

[2] Solomon Lefschetz. *Introduction to Topology*. Princeton University Press, 1949.