An efficient

ELL (1) — Parser Generator

Reinhold Heckmann
Fachbereich 10
Universität des Saarlandes
Technischer Bericht Nr. A 85 / 04
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Reinhold Heckmann
Universität des Saarlandes
6600 Saarbrücken
Bundesrepublik Deutschland

ABSTRACT

Extended context-free grammars also called regular right part grammars allow for compact and readable descriptions of the syntax of programming languages. Recursion in conventional context-free grammars can in most cases be replaced by iteration. Parser tables of predictive LL-parsers for extended grammars can be generated very efficiently if the length s of the look-ahead is 1. The generation time is proportional to the size of the parser table, that is the product of grammar size and number of terminal symbols.
0. Introduction

Extended context-free grammars are grammars where the non-terminal symbols produce regular expressions over terminal and non-terminal symbols instead of strings. They are introduced because they are more compact and more readable than conventional ones. In addition, they allow for the replacement of some left or right recursion (e.g. for parameter lists in most programming languages) by a "star expression" using the Kleene-star "\(*\)". Left factoring - often needed to obtain an LL-grammar - can also be done easily without introducing additional non-terminals when regular right parts are allowed.

Predictive LL-parsers for extended grammars can be generated very efficiently if the length of the look-ahead is 1. Naturally, a simplified version of this algorithm can be used to generate LL-parsers also for conventional LL(1)-grammars more efficiently than the algorithm given in [1].

Heilbrunner [3] used another approach; he transformed extended grammars into conventional ones and generated the parser for the resulting grammar. But this parser finds the complete structure of the input under the new grammar, even though much of this structure may be unnecessary for reconstructing the parse for the original grammar.

Purdom and Brown [6] show how to generate LR-parsers directly from extended context-free grammars. Another less efficient algorithm to generate LL-parsers directly from extended grammars can be found in [4].

This paper is a summary of [2]; some less important lemmata and almost all proofs have been omitted here. The absence of proofs in this paper should not suggest that the stated theorems are obvious or easy to prove; some of the omitted proofs are very extensive.

In a predictive LL-parser, first- and follow-sets are needed to obtain the look-ahead entries in the parsing table. Thus in chapters 1 through 15 we investigate the computation of these sets.

The theory in chapters 1 through 9 and 15 treats the general case of an arbitrary look-ahead length; its main results are theorems on how the first- and follow-sets can be deduced from each reduced extended grammar, and criteria for the existence of parsers for a given grammar.
In chapter 1, the fundamental operations occurring often in the formulas later on, are introduced, and their main properties are listed.

In chapter 2 we recall the definition of regular expressions and chapter 3 introduces extended grammars and the regular derivation.

An important tool to investigate a grammar is a graph associated with it, representing the relations between the terminal and non-terminal symbols and the structure of the produced regular expressions. It is described in chapter 4; here, we also introduce some terms and abbreviations denoting diverse subsets of the vertex set of the graph and successors of a given vertex.

In chapter 5, the first-sets are defined and it is stated that they are least fixed points of a system of set equations.

The regular derivation of chapter 3 is not adequate to define the follow-sets, thus in chapter 6 we define another kind of derivation called K-derivation. Chapter 7 treats the relations between these two derivations.

In contrast to first-sets which can also be computed for non-reduced grammars, it is necessary that the grammar is reduced when the follow-sets are computed. Therefore we state in chapter 8 how to transform a non-reduced grammar into an equivalent reduced one.

In chapter 9, the follow-sets are defined and we see that they are also the least fixed point of a system of set equations.

In chapters 10 through 14 we give the efficient computation of first- and follow-sets in linear time relative to the product of the number of terminals and the size of the grammar.

In chapter 10, a less efficient algorithm, the general goal to improve it by eliminating the outmost iteration and the first steps in this direction are given.

Chapter 11 gives the overall-structure of the efficient algorithm, and chapters 12 through 14 its parts.

The real computation of first- and follow-sets in chapters 12 and 13 needs a precomputation postponed to chapter 14. It finds the non-terminals which derive the empty word and those which derive nothing at all at the same time; its complexity is proportional to the size of the
grammar and it may also be used independently from the task of generating an LL-parser.

Chapter 15 treats the ELL(s) conditions guaranteeing that an LL-parser with look-ahead length s exists for a given grammar; and chapter 16 gives the structure of the ELL(1)-parser, describes how to generate it and compares it with the conventional LL-parser in [1]. In chapter 17, it is proved correct and we see that the time and the space the generated parser needs is linear in the length of the input the parser has to analyze.

Finally we give a table indicating which chapters of [2] correspond to the chapters here for readers who want to look there for proofs and further details:

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<th>Chapters here:</th>
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1. Operations on words and languages

To reason concisely about first- and follow-sets and equations between them, we introduce the first-mapping "F", the first-operation "o" and the first-closure "O". "o" and "O" are the image of concatenation and Kleene-closure "*" under the first-mapping (see theorem 1.8).

Let V denote a non-empty set (the alphabet) and V* the set of words over V and let s be a fixed natural number, the length of look-ahead of the future ELL-parser.

The first-function maps a word onto the substring consisting of its first s symbols. It is simply denoted by "F" without explicitly specifying the parameter s.
**Definition (1.1):**

\[ F: V^* \to V^*, \quad F(\varepsilon) := \varepsilon, \]

\[ F(\alpha_1 \ldots \alpha_l) := \begin{cases} 
\alpha_1 \ldots \alpha_l & \text{if } l \leq s \\
\alpha_1 \ldots \alpha_s & \text{if } l \geq s 
\end{cases} \]

for all \( l \in \mathbb{N} \), \( \alpha_1, \ldots, \alpha_l \in V \).

(\( \varepsilon \) denotes the empty word and \( \mathbb{N} \) the set of natural numbers.)

The behavior of concatenation under the mapping \( F \) is described by the partially defined first-operation "\( o \)":

**Definition (1.2):**

Let \( V^o = \{ w \in V^* \text{ such that } |w| \leq s \} \)
and \( o: V^o \times V^o \to V^o \) be the operation defined by \( u \circ v = F(uv) \).

\((V^o, o)\) is a monoid with neutral element \( \varepsilon \) and \( F \) is a homomorphism from the monoid \((V^*, \cdot)\) to \((V^o, o)\).

Now "\( F \)" and "\( o \)" can be obviously extended to languages:

**Definition (1.3):**

\[ F: \mathcal{P}(V^*) \to \mathcal{P}(V^*) \quad \text{(powerset of } V^*), \]

for \( L \subseteq V^* \): let \( F(L) = \{ F(w) \text{ where } w \in L \} \).

\[ o: \mathcal{P}(V^o) \times \mathcal{P}(V^o) \to \mathcal{P}(V^o), \]

for \( L, M \subseteq V^o \), let \( L \circ M = \{ u \circ v \text{ where } u \in L \text{ and } v \in M \} \).

We now give the main properties of the operation "\( o \)":

**Prop. (1.4):**

For all \( L, M, K \subseteq V^o \) holds:

- (a) \( L \circ M \neq \emptyset \iff L \neq \emptyset \) and \( M \neq \emptyset \);
- (b) \( \varepsilon \in L \circ M \iff \varepsilon \in L \) and \( \varepsilon \in M \);
- (c) \( L \circ \{ \varepsilon \} = \{ \varepsilon \} \circ L = L \);
- (d) \( L \circ (M \circ K) = (L \circ M) \circ K \);

In the particular case \( s = 1 \), the operation "\( o \)" can be replaced by conventional operations of set theory:
Lemma (1.5):

\[
L \circ M = \begin{cases} 
\emptyset & \text{if } M = \emptyset \\
L & \text{if } M \neq \emptyset \text{ and } \emptyset \text{ not in } L \\
(L - \{\epsilon\}) \cup M & \text{if } M \neq \emptyset \text{ and } \epsilon \text{ in } L 
\end{cases}
\]

For \( s = 1 \): \( L \circ M \subset L \cup M \) if \( s = 1 \). Therefore, \( L \circ M \) contains only words which are already in \( L \) or in \( M \), permitting the efficient computation of first- and follow-sets (see chapter 10 through 13). If \( s > 1 \) this property may not be true, e.g. \( s = 2 \) and \( \{a\} \circ \{b\} = \{ab\} \).

In analogy to

\[
L^* = \bigcup_{n=0}^{\infty} L^n
\]

we want to obtain a closure operator in the monoid \((V^0, \circ)\).

Definition (1.6):

For \( L \subseteq V^0 \): Let \( P_0(L) = \{\epsilon\} \) and \( P_{n+1}(L) = P_n(L) \circ L \);

then let \( L^0 = \bigcup_{n=0}^{\infty} P_n(L) \).

The new definition of \( V^0 \) is compatible with the old one \( (V^0 = \{w \in V^* \text{ such that } lwL \leq s\}) \).

The closure can also be obtained by a finite union:

Theorem (1.7):

\[
L^0 = \bigcup_{n=0}^{s} P_n(L).
\]

In particular for \( s = 1 \), \( L^0 = \{\emptyset\} \cup L \) holds.

The main result of this chapter is the following theorem which characterizes the behavior of the regular operations union, concatenation and closure under the first-mapping:

Theorem (1.8):

For all \( L, M \subseteq V^* \):

(a) \( F(L \cup M) = F(L) \cup F(M) \);

(b) \( F(L \cdot M) = F(L) \circ F(M) \);

(c) \( F(L^*) = (F(L))^0 \).
2. **Regular expressions**

In this chapter, we give a definition of regular expressions suitable for our purposes and some further terms concerning them.

**Definition (2.1):**
Let \( M = \{ (', ')', '/', '\ast' \} \) be a set of metasymbols disjoint from \( V \).

The set \( \text{RE}(V) \) of regular expressions over the alphabet \( V \) is the least subset of \( (V \cup M)^* \) such that

(a) \( \varepsilon \in \text{RE}(V) \);
(b) \( V \in \text{RE}(V) \);
(c) \( R, Q \in \text{RE}(V) \) implies \( RQ \in \text{RE}(V) \);
(d) \( R_1, \ldots, R_n \in \text{RE}(V) \) where \( n > 1 \) implies \( (R_1 / \ldots / R_n) \in \text{RE}(V) \);
(e) \( R \in \text{RE}(V) \) implies \( (R)^\ast \in \text{RE}(V) \).

Two regular expressions are called equal iff they are equal as strings over \( V \cup M \). Thus \((a/b)\) and \((b/a)\) are different although they generate the same language \((a, b)\) (see chapter 3).

Due to rules (a) and (c) of the definition, \( \text{RE}(V) \) is a submonoid of \((V \cup M)^\ast\). A regular expression \( R \) is called **reducible** if it is the product of two non-empty expressions:

\[
R = Q_1 Q_2 \text{ where } Q_1, Q_2 \text{ in } \text{RE}(V) - \{\varepsilon\}.
\]

It is called **prime** if it is different from \( \varepsilon \) and not reducible. Then we can prove that each non-empty expression can be written as product of uniquely defined prime expressions.

3. **Extended grammar and regular derivation**

In extended grammars, the non-terminal symbols produce regular expressions over terminals and non-terminals instead of strings of these symbols (see Def. 3.1). Since a grammar should generate a language, we must define what the language associated with a regular expression is. This is done by introducing a derivation on regular expressions (Def. 3.2). The language generated by an expression then is the set of terminal words derivable from it (Def. 3.3). Theorem (3.4) lists the main properties of the language mapping; it just transforms the syntactical operations "/", concatenation and "\ast" on regular expressions into the
regular operations union, product and closure on languages.

**Definition (3.1):**
An *extended context-free grammar* (sometimes called regular right part grammar) is a quadruple $G = (V_N, V_T, p, S)$.

- $V_N$: a finite non-empty set (set of non-terminal symbols)
- $V_T$: a finite non-empty set disjoint from $V_N$ (set of terminal symbols)
- $p$: $V_N \rightarrow RE$ where $RE := RE (V_N \cup V_T)$; a mapping associating a regular expression with each non-terminal symbol
- $S \in V_N$: the start symbol of the grammar.

For $A \in V_N$, $p(A)$ is called the regular expression produced by $A$.

The productions may be written as a mapping because non-terminal symbols not occurring on the left side of at least one production can be omitted, and different productions with the same left side can be combined using alternation "|".

Usually, regular expressions are introduced together with the regular sets they denote. The elementary derivation steps induced by a production $A \rightarrow R$ are then declared as $A \rightarrow w$ where $w$ is an element of the regular set denoted by $R$ (see [4]).

The problem connected to this approach is that for any non-terminal $A$, there may be an infinite number of strings directly derivable from $A$, and that no LL-parser may be able to decide which one to select. Take $R = (a)^* (b/c)$ as an example.

Therefore we prefer a different notion of derivation whose steps are finer and which can be simulated by a LL-parser (if the grammar satisfies some conditions).

**Definition (3.2):**
"$\rightarrow$" is a relation in RE called *regular derivation* (shorter R-derivation). It is recursively defined by:

(a) $\varepsilon \rightarrow R$ and $a \rightarrow R$ where $a \in V_T$ are always false.
(b) $A \in V_N$: $A \rightarrow R$ iff $R = p(A)$;
(c) $(R_1 / \ldots / R_n) \rightarrow R$ iff $R = R_i$ for an $i \in \{1, \ldots, n\}$;
(d) \((Q)^* \rightarrow R\)  
iff \(R = \varepsilon\) or \(R = Q (Q)^*\).

(e) Let \(Q = R_1 \ldots R_n\), \(n > 1\), where all the \(R_i\) are prime.

\(Q \rightarrow R\)  
iff there is an \(i \in \{1, \ldots, n\}\) such that

\[ R = R_1 \ldots R_{i-1} R' R_{i+1} \ldots R_n \] 
and

\[ R_1, \ldots, R_{i-1} \in V_T \text{ and } R_i \rightarrow R'. \]

Note that the definition is recursive only in the last case, where \(R_1 \ldots R_n \rightarrow R\) is traced back to \(R_i \rightarrow R'\) which can be decided without further recursion since \(R_i\) is prime. The derivation is leftmost due to the condition "\(R_1, \ldots, R_{i-1} \in V_T\)".

Later, we shall see that \(R\)-derivation is suitable to define first-sets, but we need another kind of derivation to define follow-sets (see chapter 6).

Definition (3.3):

Let "\(\*\)" be the reflexive transitive closure of "\(\rightarrow\)". The language generated by a regular expression in \(RE\) is defined by

\[ L: RE \rightarrow P (V_T^*) \]  
for \(R \in RE\) let \(L(R) = \{w \in V_T^* \text{ such that } R \* \rightarrow w\}\).

The language described by the whole grammar is \(L(S)\).

The following theorem states that the mapping "generated language" of this paper just transforms the syntactical operations on regular expressions into the regular operations on languages; therefore it is the same as the commonly defined one. Besides, we shall see that \(L(R)\) for an expression \(R\) not containing non-terminal symbols is just the regular set denoted by \(R\).

Theorem (3.4):

(a) \(L(\varepsilon) = \{\varepsilon\}\);

(b) For \(a \in V_T\): \(L(a) = \{a\}\);

(c) For \(A \in V_N\): \(L(A) = L(\, p(A) \, )\);

(d) \(L(\, R_1 / \ldots / R_n \, ) = L(R_1) \cup \ldots \cup L(R_n)\);

(e) \(L(R_1 \ldots R_n) = L(R_1) \ast \ldots \ast L(R_n)\);

(f) \(L(\, (R)^* \, ) = L(R)^*\).

Now, we could define the first-set of an expression \(R\) as \(F(L(R))\), but we prefer to define the first-sets for the nodes of the grammar graph introduced in the following chapter.
4. The grammar graph

This chapter associates a graph with each grammar; it is the representation of the grammar in the parser generator, thus first- and follow-sets will be defined for its nodes (the idea is taken from [5]). At first, the structure of the grammar graph for a given fixed grammar $G$ will be described informally; then an example will follow. Finally some notations are introduced concerning subsets of the set of nodes and successors of nodes.

Informal description of the grammar graph:
We associate a $\text{structure tree } ST(R)$ with each $R \in RE$.

(a) $R = \epsilon$ : $ST(R) =$

(b) $R = a \in V_T$ : $ST(R) =$

(c) $R = A \in V_N$ : $ST(R) =$

(d) $R = (R_1 / \ldots / R_n) :$ $ST(R) =$

(e) $R = R_1 \ldots R_n, n > 1 :$ $ST(R) =$

where all $R_i$ are prime

(f) $R = (Q)^* :$ $ST(R) =$

Vertices are labeled with regular expressions in cases (d), (e), and (f) only in the presentation of the algorithm. The implementation only requires labels standing for "/", "+", or "*".

If $V_N = \{A_1, \ldots, A_r\}$, the graph is created as follows:

1) Build the structure trees $ST(p(A_1)), \ldots, ST(p(A_r))$.

2) For all $A \in V_N$: Draw an edge from each node marked by $A$ to the root
of ST(p(A)).

3) Add a special node $k_0$ labeled with S and an edge from $k_0$ to the root of ST(p(S)).

Example for a grammar graph:

$V_N = \{S, A\}$; $V_T = \{b, c\}$;

Productions: $S \to (A)* b$ and $A \to S (c / / A)$.

$k_0$ has no predecessor, the roots of the trees ST(p(A)) for $A \in V_N$ have as many predecessors as there are occurrences of A in the right sides of the productions. All the other nodes have exactly one predecessor.

Let $K$ denote the set of the vertices in the grammar graph and $M: K \rightarrow RE$ the labeling function such that $M(k)$ is the regular expression k is marked by.

The cardinality of $K$ is in the order of the size of the grammar, more exactly $|K| \leq |G| + 1$ holds where $|G|$ denotes the size of the extended context-free grammar G defined as the sum of the number of its productions and the numbers of terminals, non-terminals and metasymbols '(', ')', '/', '*' occurring on the right hand sides of its productions.

$K$ can be divided into several classes:

$K_{\text{eps}} = \{k \in K : M(k) = \varepsilon\}$ 

"$\varepsilon$ - nodes"

$K_{\text{term}} = \{k \in K : M(k) \in V_T\} = K_T$

"terminal nodes"

$K_{\text{non}} = \{k \in K : M(k) \in V_N\}$

"non-terminal nodes"

$K_{\text{alt}} = \{k \in K : M(k) = \{R_1/\ldots/R_n\}\}$

"alternative nodes"

$K_{\text{prod}} = \{k \in K : M(k) = R_1\ldots R_n, n>1, \text{all } R_i \text{ prime}\}$

"product nodes"

$K_{\text{star}} = \{k \in K : M(k) = (R)^*\}$

"star nodes"

The successors of a given node $k$ are called $s_1(k)$, $s_2(k)$, ... from left to right, and their number is denoted by $N(k)$. If there is only one
successor (that is if k is a non-terminal or star node), the index '1' can be omitted.

5. The first-sets

We start with the definition of the first-sets for nodes in the grammar graph. After that, theorem (5.2) gives us the equations to compute the first-set of a node from the sets of its successors, and theorem (5.3) tells us that the first-sets are the least solution of the system of these equations relative to set inclusion.

The first-set of a node in the grammar graph is the set of the beginnings of the words in the language generated from the regular expression the node is marked by.

Definition (5.1):

\[ F_i : K \rightarrow P(V_1^0) \quad (\text{powerset of } V_1^0) \]

Let \( F_i(k) = F_iM(k) \) for all \( k \in K \).

\[ M \Krightarrow L \Krightarrow RE \Krightarrow P(V_1^*) \Krightarrow P(V_1^0) \]

M is the marking function (Chapter 4), L the generated language (Def. 3.3), F the first-function on languages (Def. 1.3).

The first-set of a node can be computed from the first-sets of its successors:

Theorem (5.2): (First-equations)
(a) For \( k \in K_{eps} \cup K_{term} \): \( F_i(k) = \{ M(k) \} \)

The first-set of a \( \varepsilon \)-node is \( \{ \varepsilon \} \), and the first-set of a terminal node marked by 'a' is \( \{ a \} \).

(b) For \( k \in K_{non} \): \( F_i(k) = F_i(su(k)) \)

The first-set of a non-terminal node is the same as the first-set of its successor.

(c) For \( k \in K_{alt} \): \( F_i(k) = F_i(su_1(k)) \cup \ldots \cup F_i(su_{N(k)}(k)) \)

The first-set of an alternative node is the union of the sets of its successors.

(d) For \( k \in K_{prod} \): \( F_i(k) = F_i(su_1(k)) \circ \ldots \circ F_i(su_{N(k)}(k)) \)

The first-set of a product node is the first-product (see chapter 1) of the sets of its successors.
(e) For \( k \in K_{\text{star}} \):
\[
F_i(k) = \left( F_i(su(k)) \right)^0
\]
The first-set of a star node is the first-closure of the set of its successor.

The theorem is proved by (3.4) and (1.8).

Systems of set equations like the one of theorem (5.2) in general have more than one solution.

**Theorem (5.3):**

The first-sets are the least solution of the system of first-equations relative to set inclusion.

From (5.3) we see by applying a fixed point theorem that "Fi" can be computed by iterative applications of the first-equations starting from the configuration where the empty set is associated with each node. In chapter 10 we shall see that we can compute the first-sets more quickly without this iteration if the look-ahead length is 1.

6. **The K-derivation**

In the previous chapter we saw that the notion of R-derivation from chapter 3 was adequate to define first-sets. Now we try to define the follow-sets using R-derivation and show that this definition is inadequate. In the example showing this, sequences of words not consisting of symbols, but of nodes of the grammar graph are obtained. These sequences belonging to R-derivation paths become K-derivation paths by definition (6.3). In definition (6.4), the languages generated by K-derivation from words of nodes are defined, and the marking function and the first-set mapping are extended to words of nodes.

What is the formal definition of the follow-set of a vertex?

A first approach is

\[
F_0(k) = \begin{cases} 
\emptyset, & \text{if } S \rightarrow R M(k) Q \text{ is not a rule} \\
\bigcup \{FL(Q) \mid S \rightarrow R M(k) Q \} & \text{otherwise}
\end{cases}
\]

An example will show the inadequacy of this definition:
Example 6a1: \(V_N = \{S\}; \quad V_T = \{a, b, c, d\}; \quad \rho(S) = a (ab / bc) d.\)

Graph:

- \(k_0\) \(S\) \(\rightarrow\) \(a (ab / bc) d\) \(k_1\)
- \(k_2\) \(a\)
- \(k_3\) \((ab / bc)\)
- \(k_4\) \(d\)
- \(k_5\) \(ab\)
- \(k_6\) \(bc\)
- \(k_7\) \(a\)
- \(k_8\) \(b\)
- \(k_9\) \(b\)
- \(k_{10}\) \(c\)

What is \(\text{Fo}(k_5)\)?

The various derivations from \(S\) are:
- \(S \rightarrow S\),
- \(S \rightarrow a (ab/bc) d\),
- \(S \rightarrow a a b d\),
- \(S \rightarrow a b c d\). Hence we obtain \(\text{Fo}(k_5) = \{d, c\}\) by the definition above, although only the derivation \(S \rightarrow a a b d\) involves \(k_5\) and thus we want to have \(\text{Fo}(k_5) = \{d\}\). Additionally, these follow-sets are hard to compute because the substring \'ab\' in the word \'abcd\' results from \(k_2\) and \(k_9\) which are not local to \(k_5\).

To make precise the correspondence between a derivation path and the vertices it involves, consider the following pictures:

**Diagram 6.2:**

\(S \leftrightarrow k_0\)
- \(\downarrow\)
- \(a (ab / bc) d \leftrightarrow k_1 k_2 k_3 k_4\)
- \(\downarrow\)
- \(a a b d \leftrightarrow k_2 k_5 k_4\)

\(S \leftrightarrow k_0\)
- \(\downarrow\)
- \(a (ab / bc) d \leftrightarrow k_1 k_2 k_3 k_4\)
- \(\downarrow\)
- \(a b c d \leftrightarrow k_2 k_6 k_4\)

The relation indicated by \(\leftrightarrow\) depicts the fact that the labels of the vertices on the right side written down consecutively result in the expression on the left side. Here, the order of the vertices corresponding to an expression is important, thus tuples or words of nodes are associated with each expression. The set of all such words is \(K^*\).
The $R$-derivation path $S \to a (ab / bc) d \to a a b d$ corresponds to a sequence of words of vertices, that is

$$k_0, k_1, k_2 k_3 k_4, k_2 k_5 k_4, k_2 k_7 k_8 k_4.$$  

This sequence becomes a derivation path from $k_0$ to the terminal word $k_2 k_7 k_8 k_4$ if a suitable derivation on $K^*$ is introduced. Due to this $K$-derivation we will later obtain $Fo(k_c) = \{d\}$.

To make the following definition of $K$-derivation more transparent, we shall add to each case of the exact definition a supplement where a node marked by the regular expression $R$ is denoted by $[R]$. These additions cannot replace the exact definition since there are in general many nodes having the same marking.

**Definition (6.3):** Let "$\rightarrow$" be a relation on $K^*$, the $K$-derivation.

**Derivations from a single node:**

- $k \in K_{\text{term}}$: terminal nodes do not derive anything
- $k \in K_{\text{eps}}$: $k \rightarrow \varepsilon$ \quad $[\varepsilon] \rightarrow \varepsilon$
- $k \in K_{\text{nont}}$: $k \rightarrow su(k)$ \quad $[A] \rightarrow [\text{regular expression produced by } A]$
- $k \in K_{\text{prod}}$: $k \rightarrow su_i(k) \ldots su_n(k)$ \quad $[R_1 \ldots R_n] \rightarrow [R_1] \ldots [R_n]$
- $k \in K_{\text{alt}}$: $k \rightarrow su_i(k)$ for each $i \in \{1, \ldots, N(k)\}$ \quad $[(R_1 / \ldots / R_n)] \rightarrow [R_i]$
- $k \in K_{\text{star}}$: $k \rightarrow \varepsilon$ and $k \rightarrow su(k)$ \quad $[(R)^*] \rightarrow \varepsilon$ or $\rightarrow [R] [(R)^*]$

**Derivations from a word:**

$u \rightarrow v$ iff there are $t \in K_T^*$, $k \in K$ and $x$, $w \in K^*$ such that

$$u = t k w, \quad v = t x w \quad \text{and} \quad k \rightarrow x.$$  

Therefore $u \rightarrow v$ is possible only if $u$ does not belong to $K_T^*$; at the transition from $u$ to $v$, the leftmost non-terminal component of $u$ is replaced by $x$.

**Definition (6.4):**

Let "$\rightarrow^{**}$" be the reflexive transitive closure of "$\rightarrow$" and

$L'(u) = \{t \in K_T^* \text{ such that } u \rightarrow^{**} t\}$ be the language generated from a word of nodes $u$.  

Extension of \( M \) and \( F_i \) to \( K^* \):

\( M: K^* \rightarrow RE \), let \( M \) be defined by

\[ M(\varepsilon) = \varepsilon, M(k_1 \cdots k_n) = M(k_1) \cdots M(k_n) \]

\( F_i: K^* \rightarrow P(V_i^0), F_i(u) = FLM(u) \) for \( u \in K^* \).

Then for all \( u, v \in K^* \):

\[ L'(u \cdot v) = L'(u) \cdot L'(v), \quad M(u \cdot v) = M(u) \cdot M(v) \]

and \( F_i(u \cdot v) = F_i(u) \circ F_i(v) \) hold.

7. Relations between \( R^- \) and \( K^* \) derivation

\( R^- \) and \( K^* \) derivation are similar but they are defined on different domains; \( R^- \) derivation on \( RE \) which is a monoid according to (2.1), but has also a more complex structure, \( K^* \) derivation on \( K^* \) which is also a monoid, but not more. The marking function forms a monoid homomorphism from \( K^* \) into \( RE \) and it is a derivation homomorphism, too:

**Theorem (7.1):**

For all \( u, v \in K^* \), \( u \not\rightarrow v \) implies \( M(u) \not\rightarrow M(v) \).

The theorem states that each \( K^* \) derivation path can be transformed in a \( R^- \) path by means of the marking function \( M \). The inverse does not hold; take the example of chapter 6. Here \( M(k_8) = b \not\rightarrow M(k_9) = b \) holds, but \( k_8 \not\rightarrow k_9 \) does not.

The following theorem gives a restricted inversion of the claim of theorem (7.1):

**Theorem (7.2):**

For all \( u \in K^* \) and \( R \in RE \), \( M(u) \not\rightarrow R \) implies the existence of a word \( v \in K^* \) such that \( M(v) = R \) and \( u \not\rightarrow v \).

It implies that each \( R^- \) derivation path starting from the marking of a word of nodes can be simulated by a \( K^* \) path starting from this word. Theorem (7.1) and (7.2) are illustrated by diagram (6.2). Note that a \( K^* \) derivation path is in general longer than the corresponding \( R^- \) path.

Languages defined by \( K^* \) derivation are essentially the same as those defined by \( R^- \) derivation:
Theorem (7.3):

For all $u \in K^*$, $LM(u) = ML'(u)$ holds.
(The mapping "$M'$" on the right hand side of the formula is extended to
the powerset of $K^*$.)

Due to this theorem, the first-sets are also definable by means of
the $K$-derivation as $Fi(k) = FML'(k)$.

8. **Non-reduced grammars**

Up to this point, the considered grammars did not have to be reduced,
the theorems of the first-sets also hold for non-reduced grammars. But
to compute the follow-sets, the grammar must be reduced (see chapter 9).
Therefore we now treat the transformation of a non-reduced grammar into
an equivalent reduced one with respect to the generated language.

A non-extended grammar is commonly said to be reduced if all non-
terminal symbols are productive (i.e. derive terminal words) and reach-
able (i.e. occur in a sentential form derived from the start symbol).
Here, these terms are defined for nodes in the grammar graph using $K$-
derivation:

**Definition (8.1):**

A vertex $k$ is called **unproductive** if $L'(k) = \emptyset$ (or equivalent $LM(k) = \emptyset$ or $Fi(k) = \emptyset$).

It is $K$-**unreachable** if there are no words $u, v \in K^*$ such that $k_0 \Rightarrow \cdots u k v$.

It is $G$-**unreachable** if there is no path in the grammar graph from $k_0$ to $k$.

A grammar is **reduced** if there are no unproductive or $K$-unreachable
nodes in its graph.

If a node is $G$-unreachable, then it is also $K$-unreachable. The
inverse does not generally hold:
Example: Let $V_T = \{b\}$, $V_N = \{S, A, B\}$, $p(S) = A \ B$, $p(A) = A$, $p(B) = b$.

Graph: $k_0 \rightarrow A \ B \ k_1 \\
      A \ k_2 \ \ \ \ \ \ k_3 \ B \\
      ST(p(S)) \ \ \ \ ST(p(A)) \ \ \ \ ST(p(B))$

$(k_0, k_1, k_3, k_5)$ is a graph path from $k_0$ to $k_5$. The derivations of $k_0$ are $k_0 \rightarrow k_1 \rightarrow k_2 \rightarrow k_3 \rightarrow k_4 \ k_3 \rightarrow \ldots$, thus there are no words $u, v \in K^*$ such that $k_0 \rightarrow^* u \ k_5 \ v$.

If there are no unproductive vertices in the graph, G- and K-unreachability is the same which allows for finding out the unreachable nodes by graph algorithms.

We now give an algorithm to transform a grammar into an equivalent reduced one. Starting from the original grammar, compute at first the predicate "Fi(k) $\neq \emptyset$" for all nodes k of the grammar graph by the algorithm given in chapter 14. Then delete all subtrees of the structure trees in the graph which have unproductive roots, and finally eliminate all structure trees having unreachable roots.

The resulting graph represents a reduced grammar. At its nodes, it has the same languages and first-sets as the original graph.

9. The follow-sets

In this chapter, the follow-set of a node is defined by means of K-derivation; theorem (9.2) gives the equations to compute the follow-set of a vertex from the sets of its predecessors, and theorem (9.3) states that the follow-sets are the least solution of these equations.

**Definition (9.1):**
Follow-sets $\text{Fo}: K \rightarrow P(V_1^0)$:
For each node $k \in K$ let $\text{Fo}(k) = \{ x \in V_1^0 \mid$ there are $u, v \in K^*$ such that $k_0 \rightarrow^* u \ k \ v$ and $x \in \text{Fi}(v) \}$,
or shortly $\text{Fo}(k) = \bigcup_{k_0 \rightarrow^* u \ k \ v} \text{Fi}(v)$.
For the follow-rules, we need some further notations:
\[ K_{\text{root}} = \text{set of roots of the structure trees (including } k_0), \]
\[ K_{\text{int}} = K - K_{\text{root}} = \text{set of internal vertices in the trees}, \]
\[ \text{prs}(k) \text{ for } k \in K_{\text{root}} = \text{set of all predecessors of } k, \]
\[ \text{pr}(k) \text{ for } k \in K_{\text{int}} = \text{the unique predecessor of } k. \]

**Theorem (9.2):** (Follow-equations)
If the considered grammar is reduced, then
(a) \( F_0(k_0) = \{ \epsilon \}. \)
   The follow-set of the start node is \( \{ \epsilon \}. \)
(b) If \( k \in K_{\text{root}} - \{ k_0 \}: F_0(k) = \bigcup_{k' \in \text{prs}(k)} F_0(k'). \)
   The follow-set of a root is the union of the sets of its predecessors. These are non-terminal nodes marked by the non-terminal that produces the regular expression the root belongs to.
(c) If \( k \in K_{\text{int}} \) and \( \text{pr}(k) \in K_{\text{alt}}: F_0(k) = F_0(\text{pr}(k)). \)
   The follow-set of a successor of an alternative node is the same as the set of the node itself.
(d) If \( k \in K_{\text{int}} \) and \( \text{pr}(k) \in K_{\text{star}}: F_0(k) = F_0(\text{pr}(k)) \circ F_0(\text{pr}(k)). \)
   The follow-set of the only successor \( k \) of a star node is obtained by the first-product "o" of the first-set and the follow-set of the star node.
(e) If \( k \in K_{\text{int}} \) and \( \text{pr}(k) \in K_{\text{prod}} \) and \( k = \text{su}_i(\text{pr}(k)): \\
   F_0(k) = F_0(\text{su}_{i+1}(\text{pr}(k))) \circ \ldots \circ F_0(\text{su}_{N(\text{pr}(k))}(\text{pr}(k))) \circ F_0(\text{pr}(k)). \)
   \( F_0(\text{rs}(k)) \)
   The follow-set of a successor of a product node is the first-product of the first-set of the right siblings of the node - "\( F_0(\text{rs}(k))" \) - and the follow-set of the product node. \( F_0(\text{rs}(k)) \) is \( \{ \epsilon \} \) for the rightmost successor of the product node.

Remark:
The follow-equations do not hold in general if the grammar is not reduced. Take the example in chapter 8. Here \( F_0(k_3) = \{ \epsilon \} \) holds, but \( F_0(k_5) = \emptyset \) since there are no words \( u, v \in K^* \) such that \( k_0 \Rightarrow^* u k_5 v \). This is a contradiction to equation (b).

**Theorem (9.3):**
The follow-sets are the least solution of the system of set-equations of theorem (9.2) relative to inclusion.
Theorem (9.3) implies that the follow-sets can also be computed by iterative applications of a system of set equations. There is an algorithm explained in chapters 10 through 14 without this iteration if the length of look-ahead is 1.

10. **Simple algorithms to compute first- and follow-sets**

From chapter 5 and chapter 9 we can derive simple algorithms to compute first- and follow-sets. In this chapter we present these algorithms and give the fundamental idea to improve them if the look-ahead length is 1. In the following four chapters the improved algorithms are explained.

In the algorithms and the comments belonging to them, we use the expressions "Fi(k)" and "Fo(k)" to denote the abstract first- and follow-set of the node k, and "FI[k]" and "FO[k]" for the variables containing the current values of first- and follow-sets during the algorithms. If the algorithms are correct, FI[k] = Fi(k) and FO[k] = Fo(k) must hold after their termination.

Now the simple algorithms follow in a notation similar to PASCAL:

**Algorithm (10.1):** (First-computation for all look-ahead values)

```pascal
var S: set of terminals (including $\varepsilon$);
CHANGE: boolean;

begin for all k in K do FI[k] := $\emptyset$; (* initialization *)
repeat CHANGE := false;
for all k in K do begin
  case k is in
  K term: S := {M(k)};
  K eps : S := {$\varepsilon$};
  K non: S := FI[su(k)];
  K alt : S := FI[su_1(k)] U ... U FI[su_N(k)(k)];
  K prod: S := FI[su_1(k)] o ... o FI[su_N(k)(k)];
  K star: S := FI[su(k)]^0;
  end; (* case *)
  CHANGE := CHANGE or (S \neq FI[k]);
  FI[k] := S
  end (* for *)
end
```
until not CHANGE
end;

In the algorithm, the first-equations of theorem (5.2) are evaluated iteratively beginning from empty sets until no further changes occur. In the following we shall write such algorithms like this:

ALGORITHM (10.1):
(First-set computation)

for all \( k \) in \( K \) do \( FI[k] := 0 \);
evaluate the equations
given in theorem (5.2)
until no further changes occur

ALGORITHM (10.2):
(Follow-set computation)

for all \( k \) in \( K \) do \( FO[k] := 0 \);
evaluate the equations
given in theorem (9.2)
until no further changes occur

If the length of look-ahead is 1, we may replace the first-closure "\( FI[\text{su}(k)]^0 \)" by "\( \{E\} \cup FI[\text{su}(k)] \)" due to theorem (1.7). The time complexity of the set operations "\( U \)" and "\( o \)" then is \( O(|V_T|) \) since all sets occurring here are subsets of \( V_T \cup \{E\} \). Thus the complexity of the for-loop containing the case-statement is \( O(|K| \cdot |V_T|) \) and therefore the whole algorithms need time \( O(c \cdot |K| \cdot |V_T|) \) where \( c \) is the number of executions of the outmost loop. Since there are grammars where the number \( c \) is about \( |V_T| \) or even about \( |K| \), the complexity of the algorithms can be significantly reduced to \( O(|K| \cdot |V_T|) = O(|G| \cdot |V_T|) \) by removing the outmost iteration ("fixed point iteration").

This is our goal for the remaining of this chapter and the following ones. It is reached by eliminating the first-product "\( o \)" from the algorithms so that "\( U \)" is the only remaining operation, and traversing the graph while observing dependencies among the sets to be united.

By lemma (1.5), we obtain

\[
L \circ M = \begin{cases} 
0 & \text{if } M = 0 \\
L & \text{if } M \neq 0 \text{ and } E \text{ not in } L \\
(L - \{E\}) \cup M & \text{if } M \neq 0 \text{ and } E \text{ in } L 
\end{cases}
\]

Substitution of "\( o \)" in the algorithms by the right hand side of this law would yield equations only containing "\( - \{E\} \)" and union as operations, but at each node the choice which operation is to do would depend on the current values of the operands of the former occurrences of "\( o \)". But to remove fixed-point iteration we need set equations merely containing union and merely depending on the actual node but not on the
current values of first- or follow-sets. We call such equations invariant (towards fixed-point iteration).

As a first step in this direction we introduce an easier operation "o'":

**Definition (10.3):**

Let \( o' : P(V^0_T) \times P(V^0_T) \rightarrow P(V^0_T) \) be the operation defined by

\[
L \ o' \ M = \begin{cases} 
L & \text{if } \varepsilon \text{ not in } L \\
(L - \{\varepsilon\}) \cup M & \text{if } \varepsilon \text{ in } L.
\end{cases}
\]

"o'" has the same properties as "o" (see Prop. 1.4) except of (a).

We can now simplify the algorithms:

**Theorem (10.4):**

Let the considered grammar be reduced. If in the algorithms (10.1) and (10.2) the operation "o" is replaced by "o'", they remain correct.

Now the problem to obtain invariant follow-equations is solved since the decision which case of definition (10.3) is to be chosen only depends on the left operand of "o'", and from the follow-equations we see that in the follow-set computation these left operands are always first-sets. If we assume that the first-set computation is finished when the follow-set computation is begun, we can decide for each node of the grammar graph before the fixed-point iteration starts which case of definition (10.3) is valid for this node. The resulting algorithm (12.2) is given at the beginning of the next but one chapter; it will there be further improved by eliminating the fixed point-iteration.

But we have not yet invariant first-equations since there are first-sets being left operands of "o'" (see Alg. 10.1). They become invariant if we precompute before first-computation starts which first-sets contain \( \varepsilon \) since this is just the information needed in Def. (10.3). This precomputation is postponed until chapter 14; in the next but one chapter we give the first- and follow-algorithms achieved so far and improve them further. Chapter 11 gives the general idea of this improvement.
11. **Ideas to improve the algorithms**

The previous chapter ended with algorithms to compute first- and follow-sets using fixed-point iteration of invariant equations. For each node there is a first- and a follow-equation containing union as the only operation, and not depending on the current value of the sets during the iteration. First- and follow-sets are the least solutions of these systems of equations which are listed at the beginning of the following chapter.

In this chapter and in the next two chapters, we explain how to solve these equations without fixed-point iteration. Remember that the grammar graph consists of structure trees linked by edges from some leaves to roots of trees. Therefore we can reduce the number of equations by combining all equations in a tree to one equation for the root containing only first- and follow-sets of other roots on the right hand side.

This is done in the next chapter, and in the following one, we solve these systems of root equations by a graph algorithm and thus obtain the first- and follow-sets at the roots of the trees in the grammar graph. Then it is easy to compute the sets at the other nodes. The resulting algorithm for the first- and follow-set computation will be as follows:

1) Compute which first-sets contain \( \epsilon \) (chapter 14);
2) Generate the first-equations for the roots (chapter 12);
   - Compute the first-sets at the roots (chapter 13);
   - Compute the other first-sets (chapter 13);
3) Generate the follow-equations for the roots (chapter 12);
   - Compute the follow-sets at the roots (chapter 13);
   - Compute the other follow-sets (chapter 13);

The complexity of part (1) is \( O(|K|) \), all the other computation steps need time \( O(|V_T^1| \cdot |K|) = O(|V_T^1| \cdot |G_1|) \).

It would also be possible to apply the graph algorithm of chapter 13 used to compute the sets at the roots, directly to all equations without having reduced their number. But this would be less efficient than the method proposed here since the graph algorithm is rather complex.
12. **Reducing the number of equations**

In this chapter we start with first- and follow-algorithms using fixed-point iteration of invariant equations which were found in chapter 10. Then a common form for all equations is obtained, and the equations in the structure trees are combined to one equation for the root.

Let $E_p(k) = (E \in F_i(k))$ be the precomputed predicate (see chapter 14) and $E_p[k]$ the corresponding program variable. Knowing $E_p(k)$, the sets $F_i'(k) = F_i(k) - \{E\}$ are still to be computed.

**Algorithm (12.1):**

(first-computation using fixed-point iteration of invariant equations; compare with algorithm (10.1))

for all $k \in K$ do $F_i'[k] := \emptyset$;

evaluate the following equations

$$F_i'[k] = \begin{cases} \emptyset & \text{if } k \in K_{eps} \\ \{M(k)\} & \text{if } k \in K_{term} \\ F_i'[\text{su}(k)] & \text{if } k \in K_{non} U K_{star} \\ F_i'[\text{su}_4(k)] U ... U F_i'[\text{su}_n(k)] & \text{if } k \in K_{prod} U K_{alt} \\ \end{cases}$$

until no further changes occur.

For alternative nodes, the number $n(k)$ is the number of all successors of $k$, and for product nodes $k$, $n(k)$ is a number between 1 and the number of all successors depending on the $E_p$-values of the successors of $k$ according to the definition of "o".

**Algorithm (12.2):**

(follow-computation using fixed-point iteration of invariant equations; compare with theorem (9.2))

For all $k \in K$ do $F_0[k] := \emptyset$;

for each successor $k$ of a product node compute the first-set of its right siblings $F_i(rs(k))$ and derive $F_i'(rs(k)) = F_i(rs(k)) - \{E\}$ from it;

evaluate the following equations

(a) $F_0[k^0] = \{E\}$,

(b) If $k$ is root of a structure tree: $F_0[k] = \bigcup_{k \in prs(k)} F_0[k']$,

(c) If $k$ is successor of an alternative node $k'$:

$$F_0[k] = F_0[k']$$
(d) If $k$ is successor of a star node $k'$:
\[ \text{FO}[k] = \text{FI}'[k'] \cup \text{FO}[k'], \]

(e) If $k$ is successor of a product node $k'$:
\[ \text{FO}[k] = \text{FI}'(\text{rs}(k)) \cup \text{FO}[k'], \]
\[ \text{FO}[k] = \text{FI}'(\text{rs}(k)) \]

until no further changes occur.

Now we look for a common form of first- and follow-equations. In all equations, the set of a node results from some constant sets - e.g. \( \{M(k)\} \) in the first-equations or the first-sets in the follow-equations - and the sets of some successors or predecessors of the node. If we write "f" for "FI'" resp. "Fo", we obtain as common form for all equations

**Equations (12.3):**
\[ f(k) = g(k) \cup \bigcup_{k' \in h(k)} f(k') \]

where $g(k)$ denotes the constant set and $h(k)$ is the set of those successors/predecessors of $k$ whose $\text{FI}'$/follow-sets are united together with the constant set to build the $\text{FI}'$/follow-set of $k$. From the invariant equations of algorithm (12.1) and (12.2), we obtain the sets $g(k)$ and $h(k)$:

(a) From the $\text{FI}'$-equations:

\[ g(k) = \begin{cases} 
\{M(k)\} & \text{if } k \text{ is terminal node} \\
\emptyset & \text{otherwise} 
\end{cases} \]

and

\[ h(k) = \begin{cases} 
\emptyset & \text{if } k \text{ is terminal or } \epsilon \text{-node} \\
\{su(k)\} & \text{if } k \text{ is non-terminal or star node} \\
\{su_1(k), \ldots, su_n(k)\} & \text{if } k \text{ is alternative or product node} 
\end{cases} \]

(b) From the follow-equations:

\[ g(k) = \begin{cases} 
\{\epsilon\} & \text{in case (a)} \\
\emptyset & \text{in case (b) and (c)} \\
\text{FI}'(pr(k)) & \text{in case (d)} \\
\text{FI}'(\text{rs}(k)) & \text{in case (e)} 
\end{cases} \]

and
\[ h(k) = \begin{cases} 
\emptyset & \text{in case (a)} \\
\text{prs}(k) & \text{in case (b)} \\
\{pr(k)\} & \text{in case (c), (d) and (e1)} \\
\emptyset & \text{in case (e2)} 
\end{cases} \]

Now the equations for the individual nodes in the trees are combined to one equation for the root of the tree. Thus we shall reduce the number of set equations from \(|K|\) to \(|K_{\text{root}}| = |V_N| + 1\).

In the following, let \(T(r)\) be the tree associated with the root \(r\), that is the structure tree of the regular expression on the right hand side of the production belonging to \(r\).

**Reducing the number of \(F_i\)-equations**:

To combine the first-equations of \(T(r)\), take equation (12.3) for the root \(r\) and substitute \(f(k')\) in it by the right hand side of the equation for \(k'\) if \(k'\) is not a non-terminal node. Repeat that until an equation is obtained having only first-sets of non-terminal nodes which are leaves of \(T = T(r)\) on the right side. Formally let \(T'\) be the subtree of \(T\) which contains all those nodes that influence the first-set of the root; those are the nodes \(k \in T\) that are reachable from \(r\) by a path \((r = k_1, k_2, \ldots, k_L = k)\) where \(k_{i+1} \in h(k_i)\) for all \(1 \leq i < L\). Then we obtain the equation

\[ F_i'(r) = \bigsqcup_{k' \in K_{\text{nont}} \text{ and leaf of } T'} \left( \bigsqcup_{k \in T'} g(k) \right) U F_i'(k') \]

by the described combination process.

Since \(F_i'(k) = F_i'(su(k))\) holds for non-terminal nodes \(k\), we can substitute \(F_i'(k')\) where \(k'\) is a non-terminal leaf of \(T'\) by \(F_i'(r')\) where \(r' = su(k')\) is a root of another structure tree. Then we obtain

**Equation (12.4)**:

\[ F_i'(r) = G(r) U \bigsqcup_{r' \in H(r)} F_i'(r') \]

where \(G(r) = \bigsqcup_{k \in T'} g(k) = \{M(k) \mid k \text{ is terminal node in } T'\}\) and \(H(r) = \{ r' \in K_{\text{root}} \mid \text{a predecessor of } r' \text{ is leaf of } T' \}\).
Equation (12.4) claims how the first-set of a root depends on the first-sets of the other roots. Since we want to solve the resulting system of root equations by a graph algorithm in the next chapter, we represent it as a digraph called Fi'-graph whose vertices are the elements of $K_{\text{root}}$ and which contains an edge $(r, r')$ iff $r'$ is in $H(r)$. Each vertex $r$ of the graph is marked by the set $G(r)$.

The Fi'-graph can be constructed by going through $T'(r)$ for each root $r \in K_{\text{root}}$: by looking at the EP values we can decide which successors of the actual node belong to $T'$. The algorithm reaches each node at most once; the union in it (to compute $G(r)$) has complexity $O(|V_T|)$ since $G(r) \subseteq V_T^O = \{E\} \cup V_T$ holds. Therefore the total complexity is $O(|K| + |K_T| \cdot |V_T|) = O(|V_T| \cdot |K|)$.

**Reducing the number of follow-equations:**

The follow-equations for roots $r \in K_{\text{root}} - \{k_0\}$ look like

\[ F_0(r) = \bigcup_{k' \in \text{prs}(r)} F_0(k') \quad (1) \]

where the nodes $k'$ are non-terminal nodes and leaves of some trees. For all nodes except the roots, the follow-equations look like

\[ F_0(k) = g(k) \quad (2) \quad \text{or} \quad F_0(k) = g(k) \cup F_0(\text{pr}(k)) \quad (3). \]

If we substitute $F_0(k')$ in (1) by the right hand side of the equation for $F_0(k')$, we can do so until we reach an $F_0(k)$ where $k$ is a node whose equation looks like (2) or $k$ is the root of the tree. Thus we obtain

**Equation (12.5):**

\[ F_0(r) = G(r) \cup \bigcup_{r' \in \text{H}(r)} F_0(r') \]

where $G(r) \subseteq V_T^O$ is the union of all sets $g(k)$ obtained by the substitutions and $H(r) \subseteq K_{\text{root}}$ is the set of all roots we reach. The resulting system of equations is represented as a digraph called Fo-graph; it is defined like the Fi'-graph and can be computed by an algorithm given in [2] whose complexity is $O(|V_T| \cdot |K|)$. 


13. **Solving the root equations**

In the previous chapter we obtained the systems of set equations (12.4) and (12.5) implying that the first- resp. follow-set of a root $r$ of a structure tree in the grammar graph can be computed from the union of a constant set $G(r)$ associated with the root and the first- resp. follow-sets of some other roots building the set $H(r)$. If we write "f" for "Fi" and "Fo", these systems look like

\[(13.1) \quad f(r) = G(r) \cup \bigcup_{r' \in H(r)} f(r') \text{ for all roots } r \in K_{\text{root}}\]

where $G(r) \subseteq V_T^0 = \{E\} \cup V_T$ is a set of terminal symbols and $H(r) \subseteq K_{\text{root}}$ is a set of roots.

Since there is no other difference between the first-system and the follow-system except in the sets $G(r)$ and $H(r)$, the same algorithm can be used to solve both systems. Thus we refer in the following to one system of set equations (13.1).

It is represented as a labeled digraph $D = (V, E, G)$. Its vertex set $V$ is the set $K_{\text{root}}$ of roots of the structure trees in the grammar graph, its edge set is $E = \{(r, r') \mid r' \in H(r)\} \subseteq V^2$, and the vertices $r \in V$ are labeled with the sets $G(r)$.

The equations of system (13.1)

\[f(r) = G(r) \cup \bigcup_{(r', r') \in E} f(r') \]

are in general mutually recursive, thus they cannot be solved by a single pass through $V$. By partitioning the digraph $D$ into strongly connected components we can find an algorithm to solve system (13.1) efficiently.

Two vertices $r_1, r_2 \in V$ are called strongly connected iff there are paths in $D$ from $r_1$ to $r_2$ and from $r_2$ to $r_1$. It is easy to prove that this is an equivalence relation. Its classes are called strongly connected components of the graph $D$.

By compressing the components in one vertex, we obtain the digraph $D' = (V', E', G')$ where $V'$ is the set of strongly connected components of $D$, $E' = \{(C, C') \in V^2 \mid C \neq C' \text{ and there are } r \in C, r' \in C' \text{ such that } \}$. 

\((r, r') \in E\) is the set of all edges between different components and the label of a component is the union of the labels of all vertices in it: \(G'(C) = \bigsqcup_{r \in C} G(r)\).

From system (13.1) we can infer that strongly connected nodes have the same \(f\)-set. If we write \(f(C)\) for the common \(f\)-set of the nodes in the component \(C\), we obtain as set equations

\[
(13.2) \quad f(C) = G'(C) \cup \bigcup_{(C, C') \in E'} f(C')
\]

Although system (13.2) looks like (13.1), there is an important difference: digraph \(D'\) contains no cycles! Therefore we can solve system (13.2) directly without fixed-point iteration if the components are considered in a suitable sequence. There is an algorithm to compute the strongly connected components of any digraph \((V, E)\) in time \(O(|V|+|E|)\) (see [7]). In addition, the algorithm produces them suitably, since it does not output a component \(C\) before all end vertices of edges beginning in \(C\) were output in other components. The resulting algorithm is as follows:

**Algorithm (13.3):**

Repeat

(a) call the graph algorithm to compute the next strongly connected component \(C\);

(b) compute \(G'(C)\) by uniting all sets \(G(r)\) of vertices \(r\) in \(C\);

(c) compute \(f(C)\) by equation (13.2) (the sets \(f(C')\) are already known);

(d) let \(f(r) = f(C)\) for all vertices \(r\) in \(C\)

until all components are found.

The complexity of the graph algorithm is \(O(|V|+|E|) = O(|K_{root}|+|E|)\), the total number of unions in (b) is at most \(|V| = |K_{root}|\) and in (c) at most \(|E'| \leq |E|\); step (d) can be omitted if pointers are used. To each edge \((r, r')\) in \(E\), there is a path in the grammar graph from the root \(r\) to the root \(r'\). This path must involve a non-terminal leaf, therefore \(|E| \leq |K_{nont}|\) holds and \(|K_{root}|+|E| \leq |K_{root}|+|K_{nont}| \leq 2|K|. Each union takes time \(O(|V_t|)\), thus the whole algorithm has complexity \(O(|V_t|+|K|)\).
When the first- resp. follow-sets at the roots are known, they are still to be computed for the other nodes in the grammar graph. The first-sets can be obtained by going bottom-up through each structure tree and the follow-sets by going top-down. Both computations need time \( O(|V_t| \cdot |K|) \), too.

14. The Computation of "Ep"

In this chapter, an algorithm will be given to compute which first-sets contain the empty word \( \varepsilon \), at first informally, later formally. This information is needed for the efficient computation of first-sets given in the previous chapters. By a slight extension of the algorithm, we can also compute which nodes are unproductive, that is a part of the test whether the grammar is reduced.

The basis of the algorithm is the valuation of the Ep-equations given below by fixed-point iteration. If the nodes are considered in a suitable sequence, only one iteration step is needed to obtain "Ep". Therefore the complexity of the algorithm will be \( O(|K|) \).

For the description of the algorithm let \( EP(k) = (\varepsilon \text{ in } F_i(k)) \) be the value it must compute, and \( EP[k] \) be the actual value of a boolean variable associated with \( k \). At the beginning, let \( EP[k] = 'false' \) for all \( k \in K \), and at the end, \( EP = Ep \) must hold if the algorithm is correct.

The algorithm we start from is as follows:

**Algorithm (14.1): (Computation of "Ep" using fixed-point iteration)**

For all \( k \) in \( K \) do \( EP[k] := \textit{false} \);

evaluate the following equations

\[
EP[k] = \begin{cases} 
\text{true} & \text{if } k \in K_{\text{star}} \cup K_{\text{eps}} \\
\text{false} & \text{if } k \in K_{\text{term}} \\
EP[su(k)] & \text{if } k \in K_{\text{nont}} \\
EP[su_{1}(k)] \text{ or } \ldots \text{ or } EP[su_{N(k)}(k)] & \text{if } k \in K_{\text{alt}} \\
EP[su_{1}(k)] \text{ and } \ldots \text{ and } EP[su_{N(k)}(k)] & \text{if } k \in K_{\text{prod}} 
\end{cases}
\]

until no further changes occur.

Fixed-point theory implies that the EP value of a vertex can only change from 'false' to 'true'; if it is ever 'true' for a vertex it will never change later on. In addition, the value of a node can only change
if the value of one of its successors has changed. Inversely, if the value of a node has changed we must test whether the values of its predecessors must be changed. Internal nodes have exactly one predecessor, therefore the test can immediately be executed. At roots, we must enter the predecessors in a worklist to test them one by one. This worklist is initialized with all nodes whose first-sets are known to contain $E$, namely the $E$ and the star nodes.

In a main loop a node is taken from the worklist as actual node. Then an inner loop is entered, and the EP value of the actual node is changed to 'true'. If it is a root, its predecessors are entered into the worklist and the inner loop is terminated. Otherwise, we must test whether the EP value of the only predecessor remains 'false' or has already been 'true' (then the inner loop will be terminated) or is to be changed from 'false' to 'true' (then the predecessor becomes the actual node and the inner loop will be repeated). If the inner loop terminates, the main loop is repeated and a new vertex is taken from the worklist. The main loop and also the whole algorithm are terminated if the worklist is empty.

To prove the algorithm correct and calculate its complexity, note that

1) if a node is ever taken from the worklist, it will never again be added to it,

2) if a vertex $k$ becomes actual node, $E_p(k)$ is 'true', but EP$[k]$ has still been 'false' and is changed to 'true',

3) each node having 'true' as $E_p$ value will become actual node at some time.

Now we consider the decision in the inner loop more precisely. Let $k$ be the actual node which is an internal node in a tree, and let $k'$ be its predecessor. Then $k'$ is either an alternative or a star or a product node. EP$[k]$ has just changed to 'true'.

Case $k'$ is alternative node:
Then $E_p(k') = 'true'$ holds. If EP$[k']$ is already 'true', the inner loop is terminated; if EP$[k']$ is still 'false', $k'$ becomes actual
node and the inner loop is repeated.

Case k' is star node:
Then k' either is still in the worklist, then EP[k'] is 'false', or it is already taken from it, EP[k'] is 'true' and the inner loop is to be terminated. If EP[k'] is still 'false', we terminate it, too, because k' is then still in the worklist.

Case k' is product node:
Since EP[k] was 'false' till now, EP[k'] is always 'false' in this case. It has to be changed to 'true' iff the EP values of all successors of k' are 'true'. To avoid the large-scale investigation of the successors of k', we associate a number NF[k'] denoting the quantity of successors having 'false' as EP value with each product node k'. At the beginning, all EP values are 'false', therefore NF[k'] must equal the number of all successors N(k'). When a product successor k is actual node, the number NF[k'] is decremented by one; if it is still greater than 0, the inner loop is terminated, otherwise k' becomes actual node and the inner loop is repeated.

Now the final algorithm follows in a notation similar to PASCAL:

Algorithm (14.2):
begin
for all k in K do begin
  EP[k] := false;  (* This part can already be done *)
  if k is product node
    then NF[k] := N(k)  (* during the construction of the *)
  end;  (* for *)
T := K_eps U K_star;  (* graph in the machine. *)
while T ≠ Ø do begin
  k := arbitrary element of T;  T := T - {k};
  repeat EP[k] := true;  (* Beginning of the inner loop. *)
    if k in K_root then begin
      stop := true;
      T := T U prs(k)  (* The predecessors of the root k *)
    end
  end
(* T denotes the worklist. It is *)
(* filled with ε and star nodes. *)
(* Beginning of the main loop. *)
(* 'stop' indicates whether the *)
(* inner loop is to be terminated.*)
(* are added to the worklist. *)
(* This part can already be done *)
(* during the construction of the *)
(* graph in the machine. *)
(* *)
(* *)
(* T denotes the worklist. It is *)
(* filled with ε and star nodes. *)
(* Beginning of the main loop. *)
(* 'stop' indicates whether the *)
(* inner loop is to be terminated.*)
(* are added to the worklist. *)
else begin $k' := \text{pr}(k)$; (* $k$ is internal node, *)
case $k'$ is in (* $k'$ its predecessor. *)

  $K_{\text{alt}} : \text{stop} := E\text{P}[k']$;
  $K_{\text{star}} : \text{stop} := \text{true}$;
  $K_{\text{prod}} : \text{begin NF}[k'] := \text{NF}[k'] - 1$; \text{stop} := \text{NF}[k'] > 0 \text{ end}
end; (* case *)

  if not \text{stop} then $k := k'$ (* $k'$ becomes actual node. *)
end (* else *) (* *

  until \text{stop} (* End of the inner loop. *)
end (* while *) (* End of the main loop. *)

end;

Since each node becomes actual node at most once, the inner loop is repeated at most $|K|$ times. All operations in it are of complexity $O(1)$ except the assignment (*). But each node is also added to $T$ at most once, therefore the total complexity of all executions of (*) is $O(|K|)$, too.

By a slight extension of the algorithm we can also compute the 'non-empty' predicate $\text{Ne}(k) = (\text{Fi}(k) \neq \emptyset)$ needed for the test whether the grammar is reduced. "$\text{Ne}$" is the least fixed point of some $\text{Ne}$-equations equal to the $\text{Ep}$-equations except $\text{Ne}(k) = \text{true}$' for terminal nodes. In addition, $\text{Ep} \preceq \text{Ne}$ holds since $\epsilon \in \text{Fi}(k)$ implies $\text{Fi}(k) \neq \emptyset$. Therefore we can compute "$\text{Ep}$" together with a part of "$\text{Ne}$". If the worklist is empty, it is refilled by the set of terminal nodes and the algorithm is started again to compute the remainder of "$\text{Ne}$".

15. The $\text{ELL}(s)$ conditions

After having discussed in the previous chapters how the first- and follow-sets can be efficiently computed if the length of look-ahead is 1, we now consider the $\text{ELL}(s)$ conditions for general look-ahead length $s$ before introducing a predictive $\text{ELL}(1)$ parser in the next chapter.

As in conventional grammars, we must distinguish between weak and strong $\text{ELL}(s)$ conditions. In addition, we can define $\text{ELL}(s)$ either by $R$- or by $K$-derivation. Hence, many different $\text{ELL}(s)$ classes result.

In this thesis, we restrict ourselves to the $\text{KELL}(s)$ and the strong $\text{KELL}(s)$ property; for "very strong" $\text{KELL}(s)$ conditions and $\text{RELL}(s)$ and
strong RELL(s) properties and the relations between them, we refer to [2].

**Definition (15.1):** (KELL(s) and SKELL(s))

(a) A grammar is **KELL(s)** if for all \( u, v_1, v_2, w \in K^* \) and \( k \in K \)

\[
\begin{align*}
&\Rightarrow^* u v_1 w \\
&k_0 \Rightarrow^* u k w \\
&\text{and } v_1 \neq v_2 \text{ implies } \\
&\Rightarrow^* u v_2 w
\end{align*}
\]

\( \text{Fi}(v_1 w) \) and \( \text{Fi}(v_2 w) \) are disjoint.

(b) A grammar is strongly KELL(s) or **SKELL(s)**

if for all \( u_1, u_2, v_1, v_2, w_1, w_2 \in K^* \) and \( k \in K \)

\[
\begin{align*}
&\Rightarrow^* u_1 k w_1 \Rightarrow^* u_1 v_1 w_1 \\
&k_0 \Rightarrow^* u_2 k w_2 \Rightarrow^* u_2 v_2 w_2 \\
&\text{and } v_1 \neq v_2 \text{ implies } \\
&\text{Fi}(v_1 w_1) \text{ and Fi}(v_2 w_2) \text{ are disjoint.}
\end{align*}
\]

**Theorem (15.2):**

Each SKELL(s) grammar is KELL(s). The inverse implication is not generally true if \( s > 1 \), but KELL(1) and SKELL(1) are equivalent.

The example showing the inverse is not true for \( s = 2 \) is essentially the same as the one showing LL(2) \( \neq \) SLL(2) for conventional grammars.

There is a local criterion for SKELL(s) using first- and follow-sets:

**Theorem (15.3):** A reduced grammar has the SKELL(s) property iff

(a) for all \( k \in K_{\text{alt}} \), \( i, j \in \{1, \ldots, \text{N}(k)\} \) where \( i \neq j \),

\[
\text{Fi}(s_u(i)(k)) \circ \text{Fo}(k) \text{ and Fi}(s_u(j)(k)) \circ \text{Fo}(k) \text{ are disjoint and}
\]

(b) for all \( k \in K_{\text{star}} \), \( \text{Fi}(s_u(k)) \circ \text{Fo}(k) \) and \( \text{Fo}(k) \) are disjoint.

If \( s = 1 \), (b) can be replaced by:

(b') For all \( k \in K_{\text{star}} \), \( \text{Fi}(s_u(k)) \) and \( \text{Fo}(k) \) are disjoint and \( \text{Fi}(s_u(k)) \)

does not contain \( \varepsilon \).

Alternative and star nodes are the only nodes that are able to derive more than one word of nodes (see definition 6.3). An alternative node \( k \)
may derive each of its successors \( s_u(k) \), \( s_u(k) \), ..., and the successor
su_i(k) is to be chosen if the look-ahead is in the set Fi(su_i(k)) \circ Fo(k). A star node k may either derive \varepsilon or su(k) k. The derivation to \varepsilon is to be selected if the look-ahead is in the follow-set of k, and the other if it is in the first-set Fi(su(k)) of the successor of k.

Finally we introduce some kind of unambiguity. Other kinds can be found in [2].

**Definition (15.4):**

A grammar is called "strongly K-unambiguous" or SKUA iff

\[
\begin{align*}
& \xymatrix{ \varepsilon_1 \ar[r] & t_1 \ar[r]^* & \varepsilon_T \vspace{0.5cm} \\
& k_0 \ar[r]^* & u \ar[r] & \varepsilon_T \\
& \xymatrix{ \varepsilon_2 \ar[r] & t_2 \ar[r]^* & \varepsilon_T }
\end{align*}
\quad\text{where } u_1 \neq u_2 \text{ implies } M(t_1) \neq M(t_2).
\]

KELL(s) implies SKUA, and this unambiguity implies that there is exactly one K-derivation path from k_0 to a terminal word of nodes marked by a given word in the language of the grammar. This path is the one found by the ELL(1)-parser of the following chapter.

**Summary of chapter 15: (for reduced grammars)**

Criterion (15.3 a and b) \(\iff\) SKELL(s) \(\iff\) KELL(s) \(\iff\) SKUA

criterion (15.3 a and b') \(\iff\) SKELL(1) \(\iff\) KELL(1)

16. **The ELL(1)-Parser**

In this chapter, we first describe by means of an example how the ELL(1) parser works, then list its possible actions. It follows an algorithm to generate the parser. Then we give an example for a parser table and finally compare our parser with the predictive LL-parser in [1]. In chapter 17, the generated parser is proved correct and its time and space complexity is considered.

The parser finds a K-derivation path from k_0 to a terminal word of nodes whose marking is the word to be analyzed by the parser (if such a path exists).
Let \( w \) be this word. We shall write \( u.v\# \) to denote the parsing situation where \( w = u.v, \) \( u \) is already consumed by the parser and the first symbol of \( v\# \) is the look-ahead symbol; \( \# \) is a pseudo terminal symbol indicating the end of the input.

This situation can be represented by an item \( t.xk\# \) where \( t \) is a terminal word of nodes in \( K_T^* \) and \( x \) is a word in \( K^* \) such that \( M(t) = u \) and \( k_0 \geq \star \) \( t \) \( x. \) \( k\# \) is a pseudo node not belonging to \( K \) considered as terminal node marked by \( \# \), thus its first-set is \( \{\#\} \). The parser does not contain the word \( t \), but it has stored the word \( xk\# \) in its stack. A similar parser for conventional grammars is the predictive LL(1)-parser given in [1].

Since we introduced \( \# \) and \( k\# \), we must replace \( \varepsilon \) in the follow-sets by \( \# \). In the first-sets, \( \varepsilon \) remains \( \varepsilon \).

An example will show the working of the parser:

**Example (16.1):** Let \( V_N = \{\text{sum}\} \), \( V_T = \{\text{smd, } +, -\} \) ("smd" = "summand") and \( p(\text{sum}) = \text{smd } (\text{+/-} \text{ smd })* \). The parser shall analyze the input string 'smd + smd'.

Grammar graph:

```
  sum  k_0  k_2  smd
    |         |         |
    +--- k_1 +---
    |         |         |
  ( +/- smd )*  ( +/- smd )*
    |         |         |
  smd  k_3   ( +/- smd )
    |         |         |
  k_4         k_5
    |         |         |
  ( +/- smd  k_6
    |         |         |
  k_7 +      - k_8
  +         -
  + s    - smd
```

In the example, we shall write a node marked by \( R \) as \([R]\) (but note that there are two different nodes \([\text{smd}]\) in our example!).
The nodes stored in the stack will be represented by integers; thus the stack ST is an array of integers. Let TOP be the index of the top of stack.

The parser action table associates an action code and at most one integer parameter with each pair of node number and look-ahead symbol. We now enumerate the possible parser actions and describe how the parser executes them.

**Expansion** and **select** both with parameter J:

The actual node is changed to J.

\[\text{begin } \text{ST} \left[ \text{TOP} \right] := J \text{ end;}\]

**Shift** without parameter:

The scanner reads a new look-ahead symbol and the actual node is popped from the stack.

\[\text{begin } \text{SCAN; } \text{TOP} := \text{TOP} - 1 \text{ end;}\]

**E-shift** without parameter:

No symbol is read and the actual node is popped.

\[\text{begin } \text{TOP} := \text{TOP} - 1 \text{ end;}\]

**Star expansion** with parameter J:

The node J is pushed onto the stack.

\[\text{begin } \text{TOP} := \text{TOP} + 1; \text{ ST} \left[ \text{TOP} \right] := J \text{ end;}\]
**Product expansion** with parameter N:

The actual node is a product node which shall be replaced by the list of its successors. To avoid the necessity of more than one parameter, the numbers are given to the nodes in such a manner that if a product node \( k \) has number \( j \), its successors have the following numbers \( j + 1, \ldots, j + N(k) \). This is possible since all successors of product nodes have prime markings and thus are never again product nodes. The number of successors \( N(k) \) then is the parameter of product expansion.

```plaintext
var I, J: integer;
begin J := ST [TOP];
   for I := 0 to N - 1 do ST [TOP + I] := J + N - I;
   TOP := TOP + N - 1
end;
```

There still are two further actions, namely "accept" and "error", both without parameters. The action "accept" can be considered as a special kind of "shift" involving \# and \( k_\# \).

Now an algorithm follows to generate the parser action table from the grammar graph and the first- and follow-sets. The table is here represented by a two-dimensional array

\[ T: \text{array} [0 .. |K|, \{\#, \star\} U V_T] \text{ of } (\text{action, parameter}). \]

Naturally, it is also possible to implement the table by nested case-statements and the action codes by procedure calls.

**Algorithm (16.2):**

Initialize the table by 'error' at each place.

Let at first \( T(|K|, \#) = (\text{accept}) \). (Let \( |K| \) be the number of \( k_\# \).)

Then pass through the grammar graph. Let \( k \) be the actual node and \( i \) its number; in addition, let \( j \) be the number of upper nodes and \( j_r \) the number of lower nodes.

**Case k is non-terminal node:**

\[ T(i, ch) = (\text{expansion, } j) \text{ for all } ch \in F_t(k) \cup F_o(k). \]

**Case k is terminal node:** \( T(i, M(k)) = (\text{shift}). \)

**Case k is e node:**

\[ T(i, ch) = (\text{e-shift}) \text{ for all } ch \in F_t(k) \cup F_o(k) = F_o(k). \]

**Case k is product node:**

\[ T(i, ch) = (\text{product expansion, } N(k)) \text{ for all } ch \in F_t(k) \cup F_o(k). \]
Case $k$ is alternative node:
For $r := 1$ to $N(k)$ do $T(i, ch) = (\text{select, } j_r)$ for all $ch \in \text{Fi} \left( s_u(r)(k) \right) \cup \text{Fo}(k)$. If a conflict arises at this, the grammar is not ELL(1).

Case $k$ is star node:
Let $T(i, ch) = (\star \text{ expansion, } j)$ for all $ch \in \text{Fi} \left( s_u(k) \right)$ and $T(i, ch) = (\varepsilon \text{-shift})$ for all $ch \in \text{Fo}(k)$. If a conflict arises at this or if $\varepsilon$ is in $\text{Fi} \left( s_u(k) \right)$, the grammar is not ELL(1).

The parser driver works as follows:

\begin{align*}
\text{ST}[0] & := \text{IKL} ; \\
\text{TOP} & := 1 ; \text{ST}[1] := 0 ; \\
\text{repeat} & \text{ execute the action of } T \left( \text{ST} \left[ \text{TOP} \right], \text{look-ahead symbol} \right) \text{ until 'accept' or 'error' are called.}
\end{align*}

As an example we give now the parser table belonging to the grammar of example (16.1).

At first some information about the nodes in the grammar graph:

<table>
<thead>
<tr>
<th>number</th>
<th>marking</th>
<th>class</th>
<th>first-set</th>
<th>follow-set</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>sum</td>
<td>non-terminal</td>
<td>{smd}</td>
<td>{#}</td>
</tr>
<tr>
<td>1</td>
<td>smd</td>
<td>product</td>
<td>{smd}</td>
<td>{#}</td>
</tr>
<tr>
<td>2</td>
<td>smd</td>
<td>terminal</td>
<td>{smd}</td>
<td>{+, -, #}</td>
</tr>
<tr>
<td>3</td>
<td>(+/-) smd*</td>
<td>star</td>
<td>{+, -, \varepsilon}</td>
<td>{#}</td>
</tr>
<tr>
<td>4</td>
<td>(+/-) smd</td>
<td>product</td>
<td>{+, -}</td>
<td>{+, -, #}</td>
</tr>
<tr>
<td>5</td>
<td>(+/-)</td>
<td>alternative</td>
<td>{+, -}</td>
<td>{smd}</td>
</tr>
<tr>
<td>6</td>
<td>smd</td>
<td>terminal</td>
<td>{smd}</td>
<td>{+, -, #}</td>
</tr>
<tr>
<td>7</td>
<td>+</td>
<td>terminal</td>
<td>{+}</td>
<td>{smd}</td>
</tr>
<tr>
<td>8</td>
<td>-</td>
<td>terminal</td>
<td>{-}</td>
<td>{smd}</td>
</tr>
<tr>
<td>9</td>
<td>#</td>
<td>pseudo</td>
<td>{#}</td>
<td></td>
</tr>
</tbody>
</table>
Parser table:

<table>
<thead>
<tr>
<th></th>
<th>smd</th>
<th>+</th>
<th>-</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Expansion</td>
<td>Error</td>
<td>Error</td>
<td>Error</td>
</tr>
<tr>
<td>0</td>
<td>Product exp. 2</td>
<td>Error</td>
<td>Error</td>
<td>Error</td>
</tr>
<tr>
<td>1</td>
<td>Shift</td>
<td>Error</td>
<td>Error</td>
<td>Error</td>
</tr>
<tr>
<td>2</td>
<td>Error</td>
<td>Star exp. 4</td>
<td>Star exp. 4</td>
<td>Shift</td>
</tr>
<tr>
<td>3</td>
<td>Error</td>
<td>Product exp. 2</td>
<td>Product exp. 2</td>
<td>Error</td>
</tr>
<tr>
<td>4</td>
<td>Error</td>
<td>Select</td>
<td>Select</td>
<td>Error</td>
</tr>
<tr>
<td>5</td>
<td>Shift</td>
<td>Error</td>
<td>Error</td>
<td>Error</td>
</tr>
<tr>
<td>6</td>
<td>Error</td>
<td>Shift</td>
<td>Error</td>
<td>Error</td>
</tr>
<tr>
<td>7</td>
<td>Error</td>
<td>Error</td>
<td>Shift</td>
<td>Error</td>
</tr>
<tr>
<td>8</td>
<td>Error</td>
<td>Error</td>
<td>Error</td>
<td>Accept</td>
</tr>
<tr>
<td>9</td>
<td>Error</td>
<td>Error</td>
<td>Error</td>
<td>Error</td>
</tr>
</tbody>
</table>

If a conventional grammar were directly written as extended grammar, all productions from a non-terminal would be combined into one whose structure tree would generally look as follows:

```
  alternative node
    /
   /   /
 product node  ...  product node
      /
     /
    ...  ...
```

terminal or non-terminal nodes

The parser of Aho and Ullman in [1] has only two actions: 'shift' and 'expand'. While their action 'shift' is like ours, their action 'expand' starts from a non-terminal, pops it from the stack, selects the correct production by look-ahead and then pushes the produced string onto the stack. This would be done by our parser in three actions: 'expansion', 'select' and 'product expansion'. This splitting is inevitable because alternative expressions may be involved in other alternatives.

Since only strings of terminals and non-terminals are pushed on the stack of Aho and Ullman, their parser table entries need only be simple symbols while our entries are numbers denoting nodes standing for regular expressions. Thus our table directly produced from an extended grammar, is greater than the table of Aho and Ullman. On the other hand extended grammars are usually easier to construct and comprehend.
17. The correctness and complexity of the ELL(1) parser

At first, we prove that the generated ELL(1)-parser works correct, and then we consider the time and space complexity of this parser.

Theorem (17.1):
Let G be a reduced KELL(1) grammar. Then the parser generated from G is correct, i.e. it always terminates and accepts a word if and only if it is in L(G).

Proof: If an input word w belongs to L(G), there is a unique K-derivation path from k₀ to a word of nodes u such that M(u) = w since w ∈ L(G) = L(S) = LM(k₀) = ML'(k₀) (see chapter 7). The parser then executes as many actions as the length of this derivation path and accepts w at the end.
If w is not in L(G), it can be written as w# = w₁ t w₂ where w₁ is a prefix of a word in L(G) and t is a terminal (or #) such that w₁ t is not a valid prefix. The parser reads w₁ and then has a node k on the top of its stack whose look-ahead set Fi(k) o Fo(k) does not contain the next input symbol t. The analysis terminates since 'error' is called in this situation.

Now we shall see that the time and the space the parser needs to analyze an input is linear in the length of this input. At first, we define these terms exactly:

Definition (17.2):
(a) Let the time for the product expansion of a node k be N(k) and the time for any other action be 1. The time the parser needs to analyze an input is the sum of the times of the actions it must execute until 'accept' or 'error' are called.
(b) The space it needs is the maximal number of stack components occupied during the parsing.

Theorem (17.3):
The parser generated for the grammar G needs time O(c·n) and space O(|G|·n) to analyze an input word of n symbols where c depends only on G but is not in any polynomial relation to |G|. 
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References: