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Abstract

In this note, it is proved that multiplier algebras of analytic reproducing kernel Hilbert spaces which are compatible with the action of the torus group possess Kraus' completely contractive approximation property (CCAP) and, consequently, have the Property S_σ. Our results apply in particular to the usual reproducing kernel Hilbert spaces on bounded symmetric domains.

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1 Introduction

Since the pioneering work of Grothendieck many different versions of approximation properties for Banach and operator spaces have been considered. The common ground of these properties is the question whether or not there exists, over a given Banach or operator space X, a net of finite rank operators V_i, possibly bounded in norm or cb norm, such that $\lim_i V_i(x) = x$ holds in a prescribed topology for all $x \in X$.

In this note, we shall be concerned with the so-called *σ-weak completely contractive approximation property* (CCAP) as introduced by J. Kraus in [10]: A σ-weakly closed subspace $\mathcal{S}$ of $B(G, H)$ (G, H Hilbert spaces) is said to have the CCAP if there exists a net $(V_i)_i$ of completely contractive σ-weakly continuous finite-rank operators on $\mathcal{S}$ such that $\lim_i V_i(S) = S$ holds σ-weakly for all $S \in \mathcal{S}$.

In [10] it is shown that the dual algebra generated by an injective weighted shift has the CCAP. In particular this means that $H^\infty(\mathbb{D})$, identified with the multiplier algebra of the Hardy space $H^2(\mathbb{D})$, has the CCAP. It is the main result of this note to prove, more generally, that multiplier spaces of holomorphic reproducing kernel Hilbert spaces have the CCAP whenever the underlying reproducing kernel Hilbert spaces are *circular*. More precisely, let $D \subset \mathbb{C}^d$ be a circular domain (that is, $e^{it}D \subset D$ holds for all $t \in \mathbb{R}$) containing the origin. Then a reproducing kernel Hilbert space $\mathcal{H} \subset \mathcal{O}(D)$ is called circular if it contains the constant functions and if the reproducing kernel $K_{\mathcal{H}}$ of $\mathcal{H}$ satisfies $K_{\mathcal{H}}(e^{it}z, e^{it}w) = K_{\mathcal{H}}(z, w)$ for all $z, w \in D$ and $t \in \mathbb{R}$. This class of spaces includes the usual reproducing kernel spaces on bounded symmetric domains (see for instance [7] or [1]), which attracted a lot of attention in recent time. In particular, the Hardy and Bergman space over the d-dimensional unit ball, and also the Arveson-Drury space (cf. [3]

for details) are circular in the above sense. The key step in the proof of our main result (Theorem 3.3) is the observation that every multiplier between circular reproducing kernel Hilbert spaces can be approximated pointwise by a sequence of polynomials which is bounded in multiplier norm.
As an immediate consequence, multiplier algebras of circular reproducing kernel Hilbert spaces have the CCAP, and therefore have Kraus' Property S_σ (cf. [9] and [10]). This can be used to prove a tensor product formula (Corollary 3.4) which generalizes the well-known equality $H^\infty(\mathbb{D}, T) = H^\infty(\mathbb{D})\overline{\otimes}T$, where $H^\infty(\mathbb{D}, T)$ is the space of all bounded holomorphic functions on $\mathbb{D}$ with values in a σ-weakly closed subspace T of $B(\mathcal{D}, \mathcal{E})$, and $H^\infty(\mathbb{D})\overline{\otimes}T$ denotes the normal spatial tensor product, that is, the σ-weak closure of $H^\infty(\mathbb{D}) \otimes T$ in $B(H^2(\mathbb{D}) \otimes \mathcal{D}, H^2(\mathbb{D}) \otimes \mathcal{E})$. A generalization of this formula in the setting of strictly pseudoconvex domains can be found in the recent paper [6].

2 Reproducing kernel Hilbert spaces

In this preliminary section, we are going to recapitulate some basic facts about reproducing kernel Hilbert spaces (see [2] for further reference). A Hilbert space $\mathcal{H}$ of functions on a set X with values in a Hilbert space $\mathcal{E}$ is called a *reproducing kernel Hilbert space* if the point evaluations $\delta_z : \mathcal{H} \to \mathcal{E}\ ,\ f \mapsto f(z)$, are continuous for all $z \in X$. Equivalently, there exists a unique function $K_\mathcal{H} : X \times X \to B(\mathcal{E})$ (the *reproducing kernel* of $\mathcal{H}$) with the property that the functions $K_\mathcal{H}(\cdot, z)x : X \to \mathcal{E}$ belong to $\mathcal{H}$ and that $\langle f, K_\mathcal{H}(\cdot, z)x\rangle = \langle f(z), x\rangle$ holds for all $f \in \mathcal{H}, z \in X$ and $x \in \mathcal{E}$. Moreover, it is well known that $K_\mathcal{H}$ is a positive definite $B(\mathcal{E})$-valued function and that conversely every positive definite function is the reproducing kernel of a uniquely determined reproducing kernel Hilbert space. Hence it makes sense to define, for a given scalar reproducing kernel Hilbert space $\mathcal{H} \subset \mathbb{C}^X$ and a Hilbert space $\mathcal{E}$, the so-called *inflation* $\mathcal{H}_\mathcal{E}$ as the reproducing kernel Hilbert space associated with the positive definite function $K_\mathcal{H} \cdot 1_\mathcal{E}$. It is not hard to see that $\mathcal{H}_\mathcal{E}$ coincides with the Hilbert space tensor product $\mathcal{H} \otimes \mathcal{E}$ via canonical identification.
Given two reproducing kernel Hilbert spaces $\mathcal{G}, \mathcal{H} \subset \mathbb{C}^X$ and Hilbert spaces $\mathcal{D}, \mathcal{E}$, the *multiplier space* $\mathcal{M}(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ is defined as the collection of all functions $\phi : X \to B(\mathcal{D}, \mathcal{E})$ with the property that the pointwise product $\phi \cdot f$ belongs to $\mathcal{H}_\mathcal{E}$ for all $f \in \mathcal{G}_\mathcal{D}$. Furthermore, we write $\mathcal{M}_T(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ for the space of all multipliers taking values in a prescribed subspace T of $B(\mathcal{D}, \mathcal{E})$. As a consequence of the closed graph theorem, the multiplication operator $M_\phi : \mathcal{G}_\mathcal{D} \to \mathcal{H}_\mathcal{E}\ ,\ f \mapsto \phi \cdot f$, associated with a multiplier ϕ, is continuous. This induces a seminorm on $\mathcal{M}(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ by setting $\|\phi\|_\mathcal{M} = \|M_\phi\|$. This seminorm

is a norm if we assume that $\mathcal{G}$ and $\mathcal{H}$ contain the constant functions, since in this case the assignment $\phi \mapsto M_\phi$ is one-to-one. In this situation, we shall often regard $\mathcal{M}(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ or, more generally, $\mathcal{M}_T(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ as an (operator) subspace of $B(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$.

Lemma 2.1. *Suppose that $\mathcal{G}, \mathcal{H} \subset \mathbb{C}^X$ are reproducing kernel Hilbert spaces containing the constant functions and that $\mathcal{D}, \mathcal{E}$ are Hilbert spaces.*

(a) A function $\psi : X \to B(\mathcal{D}, \mathcal{E})$ belongs to $\mathcal{M}(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ with $\|\psi\|_\mathcal{M} \leq 1$ if and only if the function $X \times X \to B(\mathcal{E})$, $(z, w) \mapsto K_\mathcal{H}(z, w)1_\mathcal{E} - K_\mathcal{G}(z, w)\psi(z)\psi(w)^*$, is positive definite.

(b) Let T be a σ-weakly closed subspace of $B(\mathcal{D}, \mathcal{E})$. Then $\mathcal{M}_T(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ is σ-weakly closed in $B(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$.

(c) A bounded net $(\psi_i)_i$ in $\mathcal{M}(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ converges σ-weakly to $\psi \in \mathcal{M}(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ if and only if $(\psi_i(z))_i$ converges weakly to $\psi(z)$ for all $z \in X$.

(d) The equality $\mathcal{M}(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E}) = \mathcal{M}(\mathcal{G}, \mathcal{H})\overline{\otimes} B(\mathcal{D}, \mathcal{E})$ holds, up to canonical identification.

Proof. Part (a) is Corollary 3.5 in [5]. Assertion (b) is proved in [4], and (c) is checked by a straightforward calculation. To prove (d), one first verifies by use of (a) that for every multiplier $\psi \in \mathcal{M}(\mathcal{G}, \mathcal{H})$ and every $T \in B(\mathcal{D}, \mathcal{E})$, the function $\psi \cdot T$ belongs to $\mathcal{M}(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ with $M_{\psi \cdot T} = M_\psi \otimes T$. Then (b) yields the inclusion $\mathcal{M}(\mathcal{G}, \mathcal{H})\overline{\otimes} B(\mathcal{D}, \mathcal{E}) \subset \mathcal{M}(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$. Conversely, fix $\psi \in \mathcal{M}(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ and choose nets $(P_i)_i$ and $(Q_j)_j$ of finite-rank projections approximating σ-weakly the identities on $\mathcal{D}$ and $\mathcal{E}$, respectively. Then again by (a), the functions $\psi_{i,j} = Q_j \cdot \psi \cdot P_i$ belong to $\mathcal{M}(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ with $\|\psi_{i,j}\|_\mathcal{M} \leq \|\psi\|_\mathcal{M}$. Moreover, each $\psi_{i,j}$ belongs to the algebraic tensor product $\mathcal{M}(\mathcal{G}, \mathcal{H}) \otimes B(\mathcal{D}, \mathcal{E})$. In fact, for rank-one projections $P = u \otimes u$ and $Q = v \otimes v$, we have $Q\psi(z)P = \langle \psi(z)u, v\rangle(v \otimes u)$ for all $z \in X$ and by (a), the function $X \to \mathbb{C}$, $z \mapsto \langle \psi(z)u, v\rangle$, belongs to $\mathcal{M}(\mathcal{G}, \mathcal{H})$. Since $\lim_{i,j} \psi_{i,j}(z) = \psi(z)$ weakly for all $z \in X$, an application of (c) proves the claim. □

3 Main results

Throughout this section, $D \subset \mathbb{C}^d$ denotes a circular domain containing the origin. For the reader's convenience, we start by recapitulating some facts about the Fejér kernels

$$F_N : [-\pi, \pi] \to [0, \infty) \ , \ F_N(t) = \frac{1}{N} \sum_{n=0}^{N-1} D_n(t) = \frac{1}{N} \frac{\left(\sin \frac{Nt}{2}\right)^2}{\left(\sin \frac{t}{2}\right)^2} \quad (N \geq 1),$$

where

$$D_n : [-\pi, \pi] \to \mathbb{C} \ , \ D_n(t) = \sum_{\nu=-n}^{n} e^{i\nu t} \quad (n \geq 0)$$

is the nth-order Dirichlet kernel. It is well known (see Lemma 2.2 in [8]) that $\int_{-\pi}^{\pi} F_N(t)\ dt = 2\pi$, and that $(F_N)_N$ converges to zero uniformly outside $(-\delta, \delta)$, for every $0 < \delta < \pi$. The following lemma probably is well known. Since we are unable to find an exact reference, we include a proof.

Lemma 3.1. *Suppose that X is a Banach space and that $u \in \mathcal{O}(D, X)$ is a holomorphic function. Then the functions*

$$u_N : D \to X \ , \ u_N(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(e^{it}z) F_N(t)\ dt \tag{1}$$

are polynomials (of degree at most $N-1$) with coefficients in X. Furthermore,

$$\lim_N u_N(z) = u(z) \quad \textit{(in the norm of } X\textit{)}$$

holds for all $z \in D$.

Proof. First of all, u_N obviously is analytic. By the definition of the Fejér kernel, we see that

$$D^\alpha u_N(z) = \frac{1}{2\pi N} \sum_{n=0}^{N-1} \sum_{\nu=-n}^{n} \int_{-\pi}^{\pi} D^\alpha u(e^{it}z) e^{i(|\alpha|+\nu)t}\ dt$$

holds for all $z \in D$ and $\alpha \in \mathbb{N}_0^d$. By Cauchy's Theorem, the integrals on the right-hand side of the above expression are zero whenever $|\alpha| \geq N$. Since D is connected, this shows that u_N is a polynomial of degree at most $N-1$. It remains to prove that $(u_N)_N$ converges pointwise to u. So, given $z \in D$ and $\epsilon > 0$, we choose $\delta > 0$ such that $\|u(e^{it}z) - u(z)\| < \epsilon$ holds for all $|t| < \delta$. Using the remarks preceding this lemma, we obtain that

$$\begin{aligned}
\|u_N(z) - u(z)\| &\leq \frac{1}{2\pi} \int_{|t|\leq\delta} \|u(e^{it}z) - u(z)\| F_N(t)\ dt \\
&\quad + \frac{1}{2\pi} \int_{\delta\leq|t|\leq\pi} \|u(e^{it}z) - u(z)\| F_N(t)\ dt \\
&\leq \epsilon + M \sup\{F_N(t)\ ;\ \delta \leq |t| \leq \pi\}
\end{aligned}$$

for all $N \geq 1$, where $M = \sup\{\|u(e^{it}z) - u(z)\|\ ;\ |t| \leq \pi\}$. As indicated earlier, the sequence $(F_N)_N$ converges to zero uniformly on compact subsets of $[-\pi, \pi] \backslash \{0\}$, which completes the proof. $\square$

Remark 3.2. *We point out that the polynomials u_N in the above lemma can be computed explicitly: If $u(z) = \sum_\alpha c_\alpha z^\alpha$ is the Taylor expansion of u on a neighbourhood of 0, then a straightforward calculation reveals that*

$$u_N(z) = \sum_{|\alpha|<N} c_\alpha \frac{N - |\alpha|}{N} z^\alpha$$

holds for all $z \in \mathbb{C}$ and all $N \geq 1$. Since we shall not make further use of this observation, a proof is omitted.

The main result now reads as follows.

Theorem 3.3. *Suppose that $\mathcal{G}, \mathcal{H} \subset \mathcal{O}(D)$ are circular reproducing kernel Hilbert spaces, that $\mathcal{D}, \mathcal{E}$ are Hilbert spaces, and that T is a norm closed subspace of $B(\mathcal{D}, \mathcal{E})$. Then there exists a sequence of complete contractions V_N on $\mathcal{M}_T(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ such that for every $\phi \in \mathcal{M}_T(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$, $V_N\phi$ is a polynomial of degree at most $N-1$ with coefficients in T for all N, and such that $\lim_N (V_N\phi)(z) = \phi(z)$ holds for all $z \in D$ in the norm topology of $B(\mathcal{D}, \mathcal{E})$. If T is σ-weakly closed in $B(\mathcal{D}, \mathcal{E})$, then $\lim_N V_N\phi = \phi$ holds σ-weakly for all $\phi \in \mathcal{M}_T(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$. If in addition $\mathcal{D}$ and $\mathcal{E}$ are separable, then the mappings V_N are σ-weakly continuous.*

Proof. For $\phi \in \mathcal{M}_T(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$, we write $\phi_t : D \to B(\mathcal{D}, \mathcal{E})$, $\phi_t(z) = \phi(e^{it}z)$. By Lemma 3.1, the functions

$$\phi_N : D \to B(\mathcal{D}, \mathcal{E}) \ , \ \phi_N(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_t(z) F_N(t) \ dt \quad (N \geq 1)$$

are polynomials of degree at most $N-1$ with coefficients in T, and the sequence $(\phi_N)_N$ approximates ϕ pointwise in the norm topology of $B(\mathcal{D}, \mathcal{E})$. We claim that the functions ϕ_N are multipliers with $\|\phi_N\|_\mathcal{M} \leq \|\phi\|_\mathcal{M}$. In fact, Lemma 2.1(a) and the circular symmetry of $K_\mathcal{G}$ and $K_\mathcal{H}$ imply that the functions ϕ_t belong to $\mathcal{M}_T(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ with $\|\phi_t\|_\mathcal{M} \leq \|\phi\|_\mathcal{M}$. So the mapping $\mathbb{R} \to B(\mathcal{G}_\mathcal{D}, \mathcal{H}_\mathcal{E})$, $t \mapsto M_{\phi_t}$, is well defined and moreover weakly continuous, since the functions of the form $K_\mathcal{H}(\cdot, z)x$ form a total subset of $\mathcal{H}_\mathcal{E}$ and since the family $\{M_{\phi_t} \ , \ t \in \mathbb{R}\}$ is bounded. Thus it makes sense to define operators

$$A_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} M_{\phi_t} F_N(t) \ dt \quad (N \geq 1),$$

as weak integrals. They obviously satisfy

$$\|A_N\| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|M_{\phi_t}\| F_N(t) \ dt \leq \frac{1}{2\pi} \|\phi\|_\mathcal{M} \int_{-\pi}^{\pi} F_N(t) \ dt = \|\phi\|_\mathcal{M}.$$

Furthermore,

$$\begin{aligned}\langle (A_N f)(z), x\rangle &= \langle A_N f, K_{\mathcal{H}}(\cdot, z)x\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle M_{\phi_t} f, K_{\mathcal{H}}(\cdot, z)x\rangle F_N(t)\ dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle \phi_t(z) f(z), x\rangle F_N(t)\ dt = \langle \phi_N(z) f(z), x\rangle\end{aligned}$$

holds for all $f \in \mathcal{G}_{\mathcal{D}}$, $z \in D$ and $x \in \mathcal{E}$. This shows that $\phi_N \in \mathcal{M}_T(\mathcal{G}_{\mathcal{D}}, \mathcal{H}_{\mathcal{E}})$ and that $A_N = M_{\phi_N}$ holds for all $N \geq 1$. It remains to show that the contractive operators

$$V_N : \mathcal{M}_T(\mathcal{G}_{\mathcal{D}}, \mathcal{H}_{\mathcal{E}}) \to \mathcal{M}_T(\mathcal{G}_{\mathcal{D}}, \mathcal{H}_{\mathcal{E}})\ ,\ \phi \mapsto \phi_N \quad (N \geq 1)$$

actually are complete contractions. To see this, let us write $V_N = V_N[\mathcal{D}, \mathcal{E}]$, and observe that the kth amplification of $V_N[\mathcal{D}, \mathcal{E}]$ coincides with the mapping $V_N[\mathcal{D}^k, \mathcal{E}^k]$, which of course is contractive by what we have shown so far.

Now suppose that T is σ-weakly closed in $B(\mathcal{D}, \mathcal{E})$. Then by Lemma 2.1(b), the space $\mathcal{M}_T(\mathcal{G}_{\mathcal{D}}, \mathcal{H}_{\mathcal{E}})$ is σ-weakly closed in $B(\mathcal{G}_{\mathcal{D}}, \mathcal{H}_{\mathcal{E}})$. The assertion that $(V_N\phi)_N$ converges σ-weakly towards ϕ for every $\phi \in \mathcal{M}_T(\mathcal{G}_{\mathcal{D}}, \mathcal{H}_{\mathcal{E}})$ follows immediately from Lemma 2.1(c). If in addition $\mathcal{D}$ and $\mathcal{E}$ are separable Hilbert spaces, then also $\mathcal{G}_{\mathcal{D}}$ and $\mathcal{H}_{\mathcal{D}}$ are separable Hilbert spaces, and $\mathcal{M}_T(\mathcal{G}_{\mathcal{D}}, \mathcal{H}_{\mathcal{E}})$ is the dual of a separable Banach space. By the Krein-Smulian theorem, the σ-weak continuity of the operators V_N can be proved by showing that every V_N is sequentially σ-weakly continuous on bounded sets. So suppose that $(\phi_k)_k$ is a bounded sequence in $\mathcal{M}_T(\mathcal{G}_{\mathcal{D}}, \mathcal{H}_{\mathcal{E}})$ converging σ-weakly to some $\phi \in \mathcal{M}_T(\mathcal{G}_{\mathcal{D}}, \mathcal{H}_{\mathcal{E}})$. By Lemma 2.1(c), this means exactly that the sequence $(\phi_k(z))_k$ converges weakly to $\phi(z)$ for all $z \in D$. Fix $z \in D$ and $x \in \mathcal{D}$, $y \in \mathcal{E}$. From

$$\|\phi_k(e^{it}z)\| \leq \|\phi_k\|_{\mathcal{M}} \|\mathbf{1}\|_{\mathcal{G}} K_{\mathcal{H}}(e^{it}z, e^{it}z)^{\frac{1}{2}} = \|\phi_k\|_{\mathcal{M}} \|\mathbf{1}\|_{\mathcal{G}} K_{\mathcal{H}}(z, z)^{\frac{1}{2}}$$

$(t \in \mathbb{R}, k \in \mathbb{N})$ we deduce that

$$\begin{aligned}\lim_k \langle (V_N\phi_k)(z)x, y\rangle &= \lim_k \int_{-\pi}^{\pi} \langle \phi_k(e^{it}z)x, y\rangle F_N(t)\ dt \\ &= \int_{-\pi}^{\pi} \langle \phi(e^{it}z)x, y\rangle F_N(t)\ dt = \langle (V_N\phi)(z)x, y\rangle\end{aligned}$$

holds. By Lemma 2.1(c) we conclude that $\lim_k V_N\phi_k = V_N\phi$ σ-weakly, thus proving the claimed σ-weak continuity of V_N. □

Theorem 3.3 shows in particular that, for circular reproducing kernel Hilbert spaces $\mathcal{G}$ and $\mathcal{H}$, the space $\mathcal{M}(\mathcal{G},\mathcal{H}) \cap \mathbb{C}[z]$ is sequentially σ-weakly dense in $\mathcal{M}(\mathcal{G},\mathcal{H})$. If $\mathcal{M}(\mathcal{G},\mathcal{H})$ contains the coordinate functions, then $\mathbb{C}[z]$ is sequentially σ-weakly dense in $\mathcal{M}(\mathcal{G},\mathcal{H})$.
As a second consequence, Theorem 3.3 implies that $\mathcal{M}(\mathcal{G},\mathcal{H})$ has Kraus' Property S_σ. Recall that a σ-weakly closed subspace $\mathcal{S}$ of $B(G,H)$ is said to have Property S_σ if, for all Hilbert spaces D, E and all σ-weakly closed subspaces T of $B(D,E)$, the Fubini product $F(\mathcal{S},T)$ coincides with the normal spatial tensor product $\mathcal{S}\overline{\otimes}T$. The Fubini product $F(\mathcal{S},T)$ can be defined as

$$F(\mathcal{S},T) = \{X \in \mathcal{S}\overline{\otimes}B(D,E)\ ;\ R_\lambda(X) \in T \text{ for all } \lambda \in \mathcal{S}_*\}$$

(cf. [10], p.119). Here, for $\lambda \in \mathcal{S}_*$ (the space of all σ-weakly continuous functionals on $\mathcal{S}$), the right slice mapping $R_\lambda : \mathcal{S}\overline{\otimes}B(D,E) \to B(D,E)$ is the unique σ-weakly continuous operator satisfying $R_\lambda(S \otimes T) = \lambda(S)T$ for all $S \in \mathcal{S}$ and $T \in B(D,E)$.

Corollary 3.4. *For circular reproducing kernel Hilbert spaces $\mathcal{G},\mathcal{H} \subset \mathcal{O}(D)$, the dual space $\mathcal{M}(\mathcal{G},\mathcal{H})$ has the completely contractive σ-weak approximation property (CCAP). In particular, $\mathcal{M}(\mathcal{G},\mathcal{H})$ has Property S_σ. For every σ-weakly closed subspace T of $B(\mathcal{D},\mathcal{E})$, the tensor product formula*

$$\mathcal{M}(\mathcal{G},\mathcal{H})\overline{\otimes}T = \mathcal{M}_T(\mathcal{G}_\mathcal{D},\mathcal{H}_\mathcal{E})$$

holds.

Proof. It is clear from Theorem 3.3 that $\mathcal{M}(\mathcal{G}_\mathcal{D},\mathcal{H}_\mathcal{E})$ has the CCAP. That the CCAP implies Property S_σ is Theorem 2.10 in [10]. Furthermore, the inclusion $\mathcal{M}(\mathcal{G},\mathcal{H})\overline{\otimes}T \subset \mathcal{M}_T(\mathcal{G}_\mathcal{D},\mathcal{H}_\mathcal{E})$ is obvious, since the right-hand side is σ-weakly closed. Suppose conversely that $\phi \in \mathcal{M}_T(\mathcal{G}_\mathcal{D},\mathcal{H}_\mathcal{E})$. Then, by Lemma 2.1(d), $\phi \in \mathcal{M}(\mathcal{G},\mathcal{H})\overline{\otimes}B(\mathcal{D},\mathcal{E})$, and $R_{\lambda_z}(\phi) = \phi(z) \in T$ holds for all $z \in D$, where $\lambda_z \in \mathcal{M}(\mathcal{G},\mathcal{H})_*$ is the point evaluation at $z \in D$. By a simple Hahn-Banach argument, the set of all λ_z is norm total in $\mathcal{M}(\mathcal{G},\mathcal{H})_*$, which shows that $R_\lambda(\phi) \in T$ for all $\lambda \in \mathcal{M}(\mathcal{G},\mathcal{H})_*$. This means (cf. [10], p.119) that $\phi \in F(\mathcal{M}(\mathcal{G},\mathcal{H}),T)$. Since $\mathcal{M}(\mathcal{G},\mathcal{H})$ has Property S_σ, the claim is proved. □

Finally, we point out that the equality $\mathcal{M}(\mathcal{G},\mathcal{H})\overline{\otimes}T = \mathcal{M}_T(\mathcal{G}_\mathcal{D},\mathcal{H}_\mathcal{E})$ can be deduced directly from Theorem 3.3 if we suppose in addition that all polynomials belong to $\mathcal{M}(\mathcal{G},\mathcal{H})$. In fact, in this case, the polynomials $V_N\phi$ associated with a multiplier $\phi \in \mathcal{M}_T(\mathcal{G}_\mathcal{D},\mathcal{H}_\mathcal{E})$ obviously belong to the algebraic tensor product $\mathcal{M}(\mathcal{G},\mathcal{H}) \otimes T$.

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