Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 200

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Saarbrücken 2007

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 200
submitted: 11.09.2007

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Abstract

We combine a maximum principle for vector-valued mappings established by D'Ottavio, Leonetti and Musciano [DLM] with regularity results from [BF3] and prove the Hölder continuity of the first derivatives for local minimizers $u\colon \Omega \to \mathbb{R}^N$ of splitting-type variational integrals provided Ω is a domain in $\mathbb{R}^2$.

Roughly speaking, anisotropic variational integrals $\int_\Omega F(\nabla u)\,\mathrm{d}x$ with integrand $F(\nabla u)$ of (p,q)-growth for exponents $p \leq q$ are characterized through a condition of the form

$$a\big[|\nabla u|^p - 1\big] \leq F(\nabla u) \leq A\big[|\nabla u|^q + 1\big]$$

allowing different growth rates of F w.r.t. the partial derivatives $\partial_i u$. If p and q are too far apart, then the counterexample of Giaquinta [Gi2] shows that in general nothing can be said about the smoothness of weak local minimizers, and this even concerns the scalar case. Starting from this example Marcellini, Fusco and Sbordone, Acerbi and Fusco and Esposito, Leonetti and Mingione exhibited relations between p and q guaranteeing the interior (partial) regularity of local minima. We mention the papers [Ma1], [Ma2], [AF], [FS], [ELM1], [ELM2], [BF3] and [BFZ], where the reader will find further references.

A special situation occurs if the integrand F has an additive decomposition into parts depending in an uniform convex way on submatrices of the Jacobian matrix ∇u. Then the question of regularity has been discussed recently in [BF3] and [BFZ], and in this note we want to have a closer look at the case of functions $u = u(x_1, x_2)$ depending on two variables and the corresponding integrands $F(\nabla u) = f(\partial_1 u) + g(\partial_2 u)$, with the result that local minima are always smooth independent of the growth rates of f and g (just assuming $2 \leq p \leq q$), and this is also true for vectorial minimizers.

Let us now give a precise statement of our setting. For an open set $\Omega \subset \mathbb{R}^2$ we consider a vector function $u\colon \Omega \to \mathbb{R}^N$ which locally minimizes the energy

$$I[u,\Omega] = \int_\Omega F(\nabla u)\,\mathrm{d}x \tag{1}$$

with density F being of splitting-type, i.e. we have the decomposition

$$F(Z) = f(Z_1) + g(Z_2)\,, \qquad Z = (Z_1 Z_2) \in \mathbb{R}^{2N}\,, \tag{2}$$

with functions f, g: $\mathbb{R}^N \to [0,\infty)$ of class C^2 satisfying the conditions

$$\lambda(1+|\xi|^2)^{\frac{p-2}{2}}|\eta|^2 \leq D^2 f(\xi)(\eta,\eta) \leq \Lambda(1+|\xi|^2)^{\frac{p-2}{2}}|\eta|^2 \tag{3}$$
$$\lambda(1+|\xi|^2)^{\frac{q-2}{2}}|\eta|^2 \leq D^2 g(\xi)(\eta,\eta) \leq \Lambda(1+|\xi|^2)^{\frac{q-2}{2}}|\eta|^2 \tag{4}$$

AMS Subject Classification: 49 N 60
Keywords: local minimizers, interior regularity, anisotropic energies, two-dimensional problems

valid for all ξ, $\eta \in \mathbb{R}^N$. Here λ, Λ denote positive constants, and the exponents p and q are such that

$$2 \le p \le q < \infty\,. \tag{5}$$

Note that (3) and (4) imply the strict convexity of F in the sense that

$$\lambda |Y|^2 \le D^2F(Z)(Y,Y) \le \Lambda(1+|Z|^2)^{\frac{q-2}{2}}|Y|^2 \tag{6}$$

for all Y, $Z \in \mathbb{R}^{2N}$, moreover, we deduce from (3) and (4) the growth condition

$$a|Z|^p - b \le F(Z) \le A|Z|^q + B \tag{7}$$

for all $Z \in \mathbb{R}^{2N}$ with suitable constants a, $A > 0$, b, $B \ge 0$. W.r.t. (7) we clarify the notion of a local minimizer: u should belong to the Sobolev class $W^1_{p,loc}(\Omega;\mathbb{R}^N)$ (see [Ad] for a definition of these spaces) together with

$$I[u,\Omega'] < \infty\,, \qquad I[u,\Omega'] \le I[v,\Omega']$$

for any open subset Ω' of Ω with compact closure in Ω and all functions $v \in W^1_{p,loc}(\Omega;\mathbb{R}^N)$ such that $\mathrm{spt}(u-v) \subset \Omega'$.

The main concern of our note now is the question of interior $C^{1,\alpha}$-regularity of such local minima u, but let us start with an overview of the known results.

If the scalar case is considered, then Marcellini [Ma1], Theorem A, and later Fusco and Sbordone, [FS], Theorem 4.1, proved the local boundedness of ∇u under very general growth and convexity conditions imposed on F (see, e.g., (1.6) and (1.7) in [Ma1]), which will be satisfied if we require the validity of (2)–(5). Moreover, if $N = 1$ and if (2)–(5) hold, then it is standard to get $u \in C^{1,\alpha}(\Omega)$ from $\nabla u \in L^\infty_{loc}(\Omega;\mathbb{R}^n)$, compare e.g. [Gi1].

Let us pass to the case $N > 1$ assuming from now on that F satisfies (2), (3) and (4). If (5) is replaced by the condition

$$2 \le p \le q < 2p\,, \tag{5'}$$

then we proved in [BF2] the desired $C^{1,\alpha}$-result by the way improving the regularity theorem outlined in [BF1] which can be applied on account of (6), provided (5) holds with the additional restriction that $q < 4$. Very recently (see [BF3], Theorem 1 (c)) we succeeded to establish the smoothness of u just working with inequality (5), i.e. no additional relation between the exponents p and q is required, but assuming

$$u \in L^\infty_{loc}(\Omega;\mathbb{R}^N) \tag{8}$$

together with the structural condition

$$f(\xi) = \tilde{f}(|\xi|)\,, \quad g(\xi) = \tilde{g}(|\xi|)\,, \quad \xi \in \mathbb{R}^N\,, \tag{9}$$

with increasing functions $\tilde{f}$, $\tilde{g}$. As stated in Remark 4 (b) of [BF3], the hypotheses (8) and (9) can be dropped if $p > 2$, and here we will show that this is partially true in the limit case $p = 2$, more precisely we have:

THEOREM 1. *Let (2)–(5) hold. If* $p = 2$*, then we assume that for* $\varphi := f$ *and for* $\varphi := g$ *it holds:*

$$\varphi(\xi) \geq \varphi(\xi^1 \dots 0 \dots \xi^N) \quad \text{with equality iff } \xi^i = 0\,, \tag{10}$$

$i = 1, \dots, N$*,* $\xi \in \mathbb{R}^N$*. Let* $u \in W^1_{p,loc}(\Omega; \mathbb{R}^N)$ *denote a local minimizer of the energy defined in (1). Then* u *is of class* $C^{1,\alpha}(\Omega; \mathbb{R}^N)$ *for any* $0 < \alpha < 1$*.*

REMARK 1. *If we have (3) and (4), and if we require*

$$\frac{\partial \varphi}{\partial \xi^i}(\xi^1 \dots 0 \dots \xi^N) = 0\,, \quad i = 1, \dots, N\,, \;\; \xi \in \mathbb{R}^N\,, \tag{10'}$$

for the functions $\varphi := f$ *and* $\varphi := g$*, then (10) is an immediate consequence of the strict convexity of* f *and* g *expressed through the first inequalities stated in (3) and (4). Conversely, (10) directly implies (10').*

REMARK 2. *Note that (9) gives the validity of (10) provided* $\tilde{f}$ *and* $\tilde{g}$ *are strictly increasing which is assumed in [DLM]. Condition (10) allows more flexibility in comparison to (9): for example,*

$$f_0(\xi) := \sum_{i=1}^N \alpha_i (\xi^i)^2\,, \qquad \xi \in \mathbb{R}^N\,,$$

with weights $\alpha_i > 0$ *in general does not depend on* $|\xi|$*, and the same is true for the choice*

$$g_0(\xi) := (1 + f_0(\xi))^{\frac{q}{2}}\,.$$

Clearly f_0 *and* g_0 *fulfill (10).*

REMARK 3. *An inspection of the proof will show that we can replace* f *by a function depending on the whole gradient, provided that (3) holds now for* ξ*,* $\eta \in \mathbb{R}^{2N}$ *and – in case* $p = 2$ *– the appropriate version of (10) is true. However, if the* q*-part* g *depends on* ∇u *and is strictly convex in the sense of (4) with* ξ*,* $\eta \in \mathbb{R}^{2N}$*, then Theorem 1 follows from [BF1] without assumption (10), since* $D^2F(Z)$ *is estimated from above and below by* $(1+|Z|^2)^{(q-2)/2}$ *so that we have an isotropic convexity condition which means that we can replace the exponent* s *on the l.h.s. of inequality (1) in [BF1] by the exponent* q*.*

REMARK 4. *The hypotheses (2)–(4) can be replaced by an appropriate ellipticity condition imposed on* F *(see (11) of [BF3]) together with a suitable variant of (10).*

REMARK 5. *The novelty of Theorem 1 concerns the case* $p = 2$*.*

Proof of Theorem 1. Let the assumptions of Theorem 1 hold, and consider a local minimizer $u \in W^1_{p,loc}(\Omega; \mathbb{R}^N)$. We fix $x_0 \in \Omega$ and a radius R_0 for which $B_{R_0}(x_0) \Subset \Omega$. From [Mo], Theorem 3.6.1 (c), we get $u_{|\partial B_r(x_0)} \in W^1_p(\partial B_r(x_0); \mathbb{R}^N)$ for almost all $r < R_0$, in particular we find a radius $R > 0$ such that $2R < R_0$ together with $u_{|\partial B_{2R}(x_0)} \in W^1_p(\partial B_{2R}(x_0; \mathbb{R}^N)$. Thus there is a real number $t > 0$ with the property

$$|u^i| \leq t \quad \text{on } \partial B_{2R}(x_0)\,, \tag{11}$$

$i = 1, \ldots, N$. Now we follow [DLM] and show that (11) implies

$$|u^i| \le t \quad \text{on } B_{2R}(x_0)\,, \tag{12}$$

$i = 1, \ldots, N$: it is enough to consider $i = 1$ for proving (12). We have on account of (11) that

$$v := \begin{cases} (\min[u^1, t], u^2, \ldots, u^N) & \text{on} \quad B_{2R}(x_0)\,, \\ u & \text{on} \quad \Omega - B_{2R}(x_0) \end{cases}$$

is of class $W^1_{p,loc}(\Omega; \mathbb{R}^n)$ together with

$$\begin{aligned} I[v, B_{2R}(x_0)] &= \int_{B_{2R}(x_0)\cap[u^1\le t]} F(\nabla u)\,\mathrm{d}x + \int_{B_{2R}(x_0)\cap[u^1\ge t]} F(0\,\nabla u^2 \ldots \nabla u^N)\,\mathrm{d}x \\ &=: T_1 + T_2\,. \end{aligned}$$

Clearly

$$T_1 \le \int_{B_{2R}(x_0)} F(\nabla u)\,\mathrm{d}x < \infty\,.$$

For T_2 we observe (see (10))

$$\begin{aligned} F(0\nabla u^2 \ldots \nabla u^N) &= f(0\,\partial_1 u^2 \ldots \partial_1 u^N) + g(0\,\partial_2 u^2 \ldots \partial_2 u^N) \\ &\le f(\partial_1 u) + g(\partial_2 u) \;=\; F(\nabla u)\,, \end{aligned}$$

hence also T_2 is finite by the properties of local minimizers. Thus v is an admissible comparison function which means

$$\begin{aligned} I[u, B_{2R}(x_0)] &\le I[v, B_{2R}(x_0)] \quad \Leftrightarrow \\ \int_{B_{2R}(x_0)\cap[u^1\ge t]} F(\nabla u)\,\mathrm{d}x &\le \int_{B_{2R}(x_0)\cap[u^1\ge t]} F(0\,\nabla u^2 \ldots \nabla u^N)\,\mathrm{d}x\,. \end{aligned}$$

¿From (10) it follows that $\nabla u^1 = 0$ a.e. on $B_{2R}(x_0) \cap [u^1 \ge t]$, hence

$$\nabla \Phi = 0\,, \qquad \Phi := \max[u^1 - t, 0]\,.$$

The function Φ is in the space $\overset{\circ}{W}{}^1_p(B_{2R}(x_0))$ so that $\Phi \equiv 0$ on $B_{2R}(x_0)$. This proves $u^1 \le t$ on $B_{2R}(x_0)$, and the lower bound $u^1 \ge -t$ is established in an analogous way.

Now let $B := B_R(x_0)$ and define $(u)_\varepsilon$, $\delta = \delta(\varepsilon)$, F_δ, I_δ and u_δ as in [BF3] w.r.t. the disc B. From (12) it follows that

$$|(u^i)_\varepsilon| \le t \quad \text{on } B\,,$$

$i = 1, \ldots, N$, and the same calculations as above turn this estimate for the boundary data into the uniform bound

$$\|u^i_\delta\|_{L^\infty(B)} \le t\,,$$

$i = 1, \ldots, N$. According to Remark 4 (b) of [BF3] we can therefore carry out all the arguments of this reference ending up with $\partial_\gamma u \in C^{0,\alpha}(B; \mathbb{R}^N)$, $\gamma = 1, 2$, $0 < \alpha < 1$. Since the center x_0 of the disc is arbitrary, the claim of Theorem 1 follows. $\square$.

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