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Abstract

We consider local minimizers $u : \mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ of anisotropic variational integrals of (p,q)-growth with exponents $2 \le p \le q \le \min\{2+p, p\frac{n}{n-2}\}$. If the integrand is of splitting-type, then partial C^1-regularity of u is established.

1 Introduction

In this paper we prove a partial regularity result for vector-valued functions u: $\mathbb{R}^n \supset \Omega \to \mathbb{R}^N$ defined on an open subset Ω of $\mathbb{R}^n$, $n \ge 3$, which locally minimize a strictly convex variational integral

$$I[u,\Omega] = \int_\Omega F(\nabla u)\,\mathrm{d}x \tag{1.1}$$

with energy density $F : \mathbb{R}^{nN} \to [0,\infty)$ being of anisotropic (p,q)-growth, i.e. we have the following estimate giving an upper and a lower bound for the growth of F

$$a|Z|^p - b \le F(Z) \le A|Z|^q + B \quad \forall\, Z \in \mathbb{R}^{nN} \tag{1.2}$$

with exponents $1 < p \le q < \infty$ and with constants a, $A > 0$, b, $B \ge 0$. In accordance with (1.2) we say that a function u from the local Sobolev-class $W^1_{p,\mathrm{loc}}(\Omega;\mathbb{R}^N)$ (see [Ad] for a definition of these spaces) is a local minimizer of the functional I from (1.1) if and only if $I[u,\Omega'] < \infty$ and $I[u,\Omega'] \le I[v,\Omega']$ hold for all $v \in W^1_{p,\mathrm{loc}}(\Omega;\mathbb{R}^N)$ s.t. $\mathrm{spt}(u-v) \subset \Omega'$, where Ω' is any subdomain of Ω with compact closure in Ω. For the investigation of the partial regularity properties of such local minima one has to replace (1.2) by a stronger condition, for example one can consider F of class C^2 satisfying the anisotropic ellipticity estimate

$$\lambda(1+|Z|^2)^{\frac{p-2}{2}}|Y|^2 \le D^2F(Z)(Y,Y) \le \Lambda(1+|Z|^2)^{\frac{q-2}{2}}|Y|^2 \tag{1.3}$$

valid for all Y, $Z \in \mathbb{R}^{nN}$, λ, Λ denoting positive constants. Clearly (1.3) implies estimate (1.2), moreover, it follows from the first inequality in (1.3) that F is a strictly convex function. Assuming the validity of (1.2) together with the first inequality in (1.3) Passarelli Di Napoli and Siepe [PS] proved for a local minimizer u the existence of an open subset Ω_0 with full Lebesgue-measure such that u is of class $C^{1,\alpha}(\Omega_0;\mathbb{R}^N)$ for any $0 < \alpha < 1$, provided we have $2 \le p \le q < \min\{p+1, \frac{np}{n-1}\}$, whereas in [BF1] the authors established this result by working with hypothesis (1.3) and the weaker bound $1 < p \le q < p\frac{n+2}{n}$ imposed on the exponents. We also mention the paper [AF] of Acerbi and Fusco where partial regularity is shown for a special class of integrands. We refer to [BF2] for a detailed discussion of examples, where the methods of Acerbi and Fusco lead to better results in comparison to the exponent bounds stated above.

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A particular situation occurs if in addition to (1.3) the integrand F is of special structure in the sense that ($Z = (Z_1, \ldots, Z_n) \in \mathbb{R}^{nN}$)

$$F(Z) = G(|Z_1|, |Z_2|, \ldots, |Z_n|) \tag{1.4}$$

for a function G which is increasing w.r.t. each argument. In fact, condition (1.4) implies the validity of a maximum-principle (see [DLM] or [BF2]), and therefore it makes sense to discuss local minimizers from the space $L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ with the result (see [Bi], p.145, Corollary 5.6) that partial regularity holds if we require the inequality $1 < p \le q < \min\{p + 2, p\frac{n}{n-2}\}$ for the admissible range of anisotropy. We wish to emphasize that in a similar setting Esposito, Leonetti and Mingione [ELM] showed higher integrability results for local minima u from the space $L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$.

The main purpose of the present paper is to prove partial regularity for a class of integrands to which the previous results do not directly apply, since (1.3) is violated, but for which the lack of ellipticity is compensated by an additional property which means that they are in a sense decomposable into elliptic parts of different growth rates. For example, let us look at the density

$$F_{p,q}(\nabla u) = (1 + |\tilde{\nabla} u|^2)^{p/2} + (1 + |\partial_n u|^2)^{q/2}$$

with exponents $2 \le p \le q$. Here we have abbreviated $\tilde{\nabla} u := (\partial_1 u, \ldots, \partial_{n-1} u)$. Note that $F_{p,q}$ is of the same type as the examples studied by Giaquinta [Gi1] and later on by Hong [Ho]. Obviously $F_{p,q}$ does not satisfy (1.3), we just have the inequality

$$c|Y|^2 \le D^2 F_{p,q}(Z)(Y, Y) \le C(1 + |Z|^2)^{\frac{q-2}{2}} |Y|^2 \,,$$

so that according to [Bi], Corollary 5.6, partial regularity of a local minimizer $u \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ holds in the 3D-case if $q < 4$. In order to improve this result in a more general setting we also assume that our density F is of splitting-type which means that we can write

$$F(Z) = f(\tilde{Z}) + g(Z_n) \,, \tag{1.5}$$

$Z = (Z_1, \ldots, Z_n)$, $\tilde{Z} = (Z_1, \ldots, Z_{n-1})$, $Z_1, \ldots, Z_n \in \mathbb{R}^N$, w ith C^2-functions $f : \mathbb{R}^{(n-1)N} \to [0, \infty)$, $g : \mathbb{R}^N \to [0, \infty)$ satisfying for all Y, $Z \in \mathbb{R}^{nN}$

$$\begin{aligned} \lambda(1 + |\tilde{Z}|^2)^{\frac{p-2}{2}} |\tilde{Y}|^2 \le D^2 f(\tilde{Z})(\tilde{Y}, \tilde{Y}) \le \Lambda(1 + |\tilde{Z}|^2)^{\frac{p-2}{2}} |\tilde{Y}|^2 \,, \\ \lambda(1 + |Z_n|^2)^{\frac{q-2}{2}} |Y_n|^2 \le D^2 g(Z_n)(Y_n, Y_n) \le \Lambda(1 + |Z_n|^2)^{\frac{q-2}{2}} |Y_n|^2 \,. \end{aligned} \tag{1.6}$$

Condition (1.4) is replaced by

$$f(X_1, \ldots, X_{n-1}) = \hat{f}(|X_1|, \ldots, |X_{n-1}|), \quad g(X_n) = \hat{g}(|X_n|) \tag{1.7}$$

with $\hat{g}$ increasing and $\hat{f}$ increasing w.r.t. each argument. Note that (1.7) implies the maximum-principle, moreover, (1.6) and (1.7) occur in the paper [BF3] as sufficient hypotheses for the higher integrability of the gradient of locally bounded local minima provided that $q \le 2p + 2$. Now we can state our main result:

THEOREM 1.1. *Suppose that F satisfies (1.5)-(1.7) with exponents $2 \leq p \leq q$, and let $u \in W^1_{p,\mathrm{loc}}(\Omega; \mathbb{R}^N)$ denote a local minimizer of the energy defined in (1.1). Assume further that $u \in L^\infty_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ and that*

$$(H1) \quad q \leq p + 2\,,$$
$$(H2) \quad q \leq p\frac{n}{n-2}$$

are valid. Then there is an open subset Ω_0 of Ω such that $|\Omega - \Omega_0| = 0$ ($|\cdot| =$ Lebesgue-measure) and $u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N)$ for all $\alpha \in (0,1)$.

REMARK 1.1. *Our theorem extends [Bi], Corollary 5.6, to the case of splitting functionals allowing even equality in the conditions imposed on the exponents.*

REMARK 1.2. *As discussed in [BF3] our results are not limited to the specific decomposition (1.5), for example, we can consider the case*

$$F(Z) = \sum_{i=1}^{n} (1 + |Z_i|^2)^{p_i/2}$$

with exponents $p_i \geq 2$. Then we define $p := \min\{p_i\}$, $q := \max\{p_i\}$ and require the validity of $(H1), (H2)$ for these choices. In the same spirit we can replace (1.5)–(1.7) by (1.4) together with the ellipticity condition

$$\lambda[(1 + |\tilde{X}|^2)^{\frac{p-2}{2}} |\tilde{Z}|^2 + (1 + |X_n|^2)^{\frac{q-2}{2}} |Z_n|^2] \leq D^2F(X)(Z,Z) \leq \Lambda[\ldots]\,.$$

REMARK 1.3. *The structural condition (1.7) just enters through the fact that we need the uniform boundedness of the solutions of some approximate problems. For $p > n$ it can be dropped by Sobolev's embedding theorem.*

REMARK 1.4. *If $n = 2$ or if $N = 1$, then we can prove $\Omega_0 = \Omega$ even without $(H1)$ and $(H2)$, we refer to [BF3] and [BFZ].*

REMARK 1.5. *W.r.t. the results of [BF3] it would be desirable to remove $(H2)$ from Theorem 1.1 and to replace $(H1)$ by the weaker bound $q \leq 2p + 2$. But this is still open.*

Our paper is organized as follows: in Section 2 we collect some preliminary material based on the work [BF3]. The proof of Theorem 1.1 using a blow-up argument is presented in Section 3.

2 Preliminary results

Let the assumptions of Theorem 1.1 hold but with the exception that $(H1, 2)$ are replaced by the weaker bound

$$(H3) \quad q \leq 2p + 2\,.$$

Proceeding as in [BF3] we fix a ball $B := B_R(x_0)$ with compact closure in Ω and consider an exponent $\tilde{q} > q$.

For $\varepsilon > 0$ let $(u)_\varepsilon$ denote the mollification of u and define

$$\delta := \delta(\varepsilon) := 1/(1 + \varepsilon^{-1} + \|(\nabla u)_\varepsilon\|^{2\tilde{q}}_{L^{\tilde{q}}(B)}) .$$

Moreover, with

$$F_\delta(Z) := \delta(1 + |Z|^2)^{\tilde{q}/2} + F(Z), \quad Z \in \mathbb{R}^{nN} ,$$

we define $u_\delta \in W^1_{\tilde{q}}(B; \mathbb{R}^N)$ as the unique solution of the problem $I_\delta[w, B] := \int_B F_\delta(\nabla w)\, \mathrm{d}x \to \min$ in $(u)_\varepsilon + \overset{\circ}{W}{}^1_{\tilde{q}}(B; \mathbb{R}^N)$. The following properties of this approximation can be found in [BF2], [BF3]:

LEMMA 2.1. *a) We have as $\varepsilon \to 0$:*
$u_\delta \rightharpoondown u$ in $W^1_p(B; \mathbb{R}^N)$; $\delta \int_B (1 + |\nabla u_\delta|^2)^{\tilde{q}/2}\, \mathrm{d}x \to 0$; $\int_B F(\nabla u_\delta)\, \mathrm{d}x \to \int_B F(\nabla u)\, \mathrm{d}x$.

b) $\|u_\delta\|_{L^\infty(B)}$ is bounded independent of ε.

c) ∇u_δ is in the space $L^\infty_{\mathrm{loc}} \cap W^1_{2,\mathrm{loc}}(B; \mathbb{R}^{nN})$.

REMARK 2.1. *a) We have to regularize with some exponent $\tilde{q} > q$ in order to deduce c) of Lemma 2.1 from the work [GM] of Giaquinta and Modica.*

b) Part b) of Lemma 2.1 is the only place where the structural condition (1.7) enters. Any condition implying b) of the lemma could replace (1.7).

In [BF3] we proved:

LEMMA 2.2. *a) (Caccioppoli-type inequality) For any $\eta \in C^\infty_0(B)$ and any $\gamma \in \{1, \dots, n\}$ we have*

$$\int_B \eta^2 D^2 F_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\, \mathrm{d}x \le c \int_B D^2 F_\delta(\nabla u_\delta)(\nabla \eta \otimes \partial_\gamma u_\delta, \nabla \eta \otimes \partial_\gamma u_\delta)\, \mathrm{d}x.$$

b) For any radius $\rho < R$ it holds

$$\int_{B_\rho(x_0)} |\tilde{\nabla} u_\delta|^{p+2}\, \mathrm{d}x + \int_{B_\rho(x_0)} |\partial_n u_\delta|^{q+2}\, \mathrm{d}x \le c(\rho) < \infty .$$

In a) and b) the constants are uniform in ε.

Now we will make use of Lemma 2.2 to derive the following aulixiary result

LEMMA 2.3. *Let $\tilde{\Psi} := (1 + |\tilde{\nabla} u|^2)^{p/4}$ and $\Psi^{(n)} := (1 + |\partial_n u|^2)^{q/4}$. Then we have $\tilde{\Psi}$, $\Psi^{(n)} \in W^1_{2,\mathrm{loc}}(\Omega)$, moreover, u belongs to the space $W^2_{2,\mathrm{loc}}(\Omega; \mathbb{R}^N)$.*

Proof. From Lemma 2.2 a) we get using the estimates (1.6) for $D^2 f$, $D^2 g$ (no summation w.r.t. $\gamma = 1, \ldots, n$)

$$\begin{aligned}
&\delta \int_B \eta^2 (1+|\nabla u_\delta|^2)^{\frac{\tilde q-2}{2}} |\partial_\gamma \nabla u_\delta|^2 \,\mathrm{d}x \\
&\quad + \int_B \eta^2 (1+|\tilde\nabla u_\delta|^2)^{\frac{p-2}{2}} |\partial_\gamma \tilde\nabla u_\delta|^2 \,\mathrm{d}x + \int_B \eta^2 (1+|\partial_n u_\delta|^2)^{\frac{q-2}{2}} |\partial_\gamma \partial_n u_\delta|^2 \,\mathrm{d}x \\
&\quad \le c \Bigg[\delta \int_B |\nabla\eta|^2 (1+|\nabla u_\delta|^2)^{\frac{\tilde q-2}{2}} |\partial_\gamma u_\delta|^2 \,\mathrm{d}x + \int_B |\nabla\eta|^2 (1+|\tilde\nabla u_\delta|^2)^{\frac{p-2}{2}} |\partial_\gamma u_\delta|^2 \,\mathrm{d}x \\
&\qquad + \int_B |\nabla\eta|^2 (1+|\partial_n u_\delta|^2)^{\frac{q-2}{2}} |\partial_\gamma u_\delta|^2 \,\mathrm{d}x \Bigg] ,
\end{aligned}$$

η denoting a test function with support in B. With an obvious meaning of $\tilde\Psi_\delta$ and $\Psi_\delta^{(n)}$ the inequality from above gives

$$\begin{aligned}
&\int_B \Big[|\nabla^2 u_\delta|^2 + |\nabla \tilde\Psi_\delta|^2 + |\nabla \Psi_\delta^{(n)}|^2 \Big] \eta^2 \,\mathrm{d}x \\
&\le\ c \Bigg[\delta \int_B |\nabla\eta|^2 (1+|\nabla u_\delta|^2)^{\tilde q/2} \,\mathrm{d}x + \int_B |\nabla\eta|^2 (1+|\tilde\nabla u_\delta|^2)^{p/2} \,\mathrm{d}x \\
&\quad + \int_B |\nabla\eta|^2 (1+|\partial_n u_\delta|^2)^{q/2} \,\mathrm{d}x + \int_B |\nabla\eta|^2 (1+|\tilde\nabla u_\delta|^2)^{\frac{p-2}{2}} |\partial_n u_\delta|^2 \,\mathrm{d}x \\
&\quad + \int_B |\nabla\eta|^2 (1+|\partial_n u_\delta|^2)^{\frac{q-2}{2}} |\tilde\nabla u_\delta|^2 \,\mathrm{d}x \Bigg] =: c \sum_{i=1}^{5} T_i \,. \qquad (2.1)
\end{aligned}$$

From Lemma 2.1 a) we deduce that T_1, T_2, T_3 can be bounded independent of ε. Let us look at T_4 : if $p = 2$, then we are done. Otherwise we use Young's inequality and get

$$(1+|\tilde\nabla u_\delta|^2)^{\frac{p-2}{2}} |\partial_n u_\delta|^2 \le c \Big\{ (1+|\tilde\nabla u_\delta|^2)^{\frac{p}{2}} + |\partial_n u_\delta|^{\frac{p}{2}} \Big\} ,$$

hence T_4 is also bounded independent of ε. For discussing T_5 we can also assume that $q > 2$. Then

$$(1+|\partial_n u_\delta|^2)^{\frac{q-2}{2}} |\tilde\nabla u_\delta|^2 \le c \Big\{ (1+|\partial_n u_\delta|^2)^{\frac{q+2}{2}} + |\tilde\nabla u_\delta|^{2\frac{q+2}{4}} \Big\} ,$$

and the uniform boundedness of T_5 is a consequence of Lemma 2.2 b) and our hypothesis (H3). Returning to (2.1) it is shown that

$$\int_{B_\rho} \Big[|\nabla^2 u_\delta|^2 + |\nabla \tilde\Psi_\delta|^2 + |\nabla \Psi_\delta^{(n)}|^2 \Big] \,\mathrm{d}x \le c(\rho) \qquad (2.2)$$

for any ball $B_\rho = B_\rho(x_0)$, $\rho < R$. This implies $u \in W^2_{2,\mathrm{loc}}(\Omega; \mathbb{R}^N)$ and for a subsequence

$$\nabla u_\delta \to \nabla u \text{ in } L^2_{\mathrm{loc}}(B; \mathbb{R}^{nN}) \text{ and a.e. on } B\,. \qquad (2.3)$$

At the same time (2.2) gives

$$\begin{array}{lll} \tilde\Psi_\delta \rightharpoondown: \tilde\nu, & \Psi_\delta^{(n)} \rightharpoondown: \nu^{(n)} & \text{in } W^1_{2,\mathrm{loc}}(B)\,, \\ \tilde\Psi_\delta \to \tilde\nu, & \Psi_\delta^{(n)} \to \nu^{(n)} & \text{a.e. on } B\,, \end{array}$$

but with the pointwise convergence stated in (2.3) we must have $\tilde\nu = \tilde\Psi$, $\nu^{(n)} = \Psi^{(n)}$ which proves Lemma 2.3. □

For later purposes we observe that Lemma 2.2 a) implies (summation w.r.t. $\gamma = 1, \dots, n$)

$$\int_B \eta^2\Big[|\nabla^2 u_\delta|^2+|\nabla\tilde\Psi_\delta|^2+|\nabla\Psi_\delta^{(n)}|^2\Big]\,\mathrm{d}x \le c\int_B D^2F_\delta(\nabla u_\delta)(\nabla\eta\otimes\partial_\gamma u_\delta, \nabla\eta\otimes\partial_\gamma u_\delta)\,\mathrm{d}x\,, \quad (2.4)$$

and by lower semicontinuity it holds

$$\int_B \eta^2\Big[|\nabla^2 u|^2+|\nabla\tilde\Psi|^2+|\nabla\Psi^{(n)}|^2\Big]\,\mathrm{d}x \le \liminf_{\varepsilon\to 0}\Big[\text{l.h.s. of (2.4)}\Big]\,. \quad (2.5)$$

We have ($F^{(\tilde q)}(Z) := (1+|Z|^2)^{\tilde q/2}$)

$$\begin{aligned} \text{r.h.s. of (2.4)} \;=\; & c\Big[\int_B \delta D^2F^{(\tilde q)}(\nabla u_\delta)(\nabla\eta\otimes\partial_\gamma u_\delta, \nabla\eta\otimes\partial_\gamma u_\delta)\,\mathrm{d}x \\ & + \int_B D^2F(\nabla u_\delta)(\nabla\eta\otimes\partial_\gamma u_\delta, \nabla\eta\otimes\partial_\gamma u_\delta)\,\mathrm{d}x\Big]\,, \end{aligned}$$

and by Lemma 2.1 the first integral on the r.h.s. vanishes as $\varepsilon\to 0$, whereas (2.3) implies

$$\omega_\delta := D^2F(\nabla u_\delta)(\nabla\eta\otimes\partial_\gamma u_\delta, \nabla\eta\otimes\partial_\gamma u_\delta) \to \omega := D^2F(\nabla u)(\nabla\eta\otimes\partial_\gamma u, \nabla\eta\otimes\partial_\gamma u)$$

a.e. as $\varepsilon\to 0$. Note that on account of the integrability properties of $\tilde\nabla u$, $\partial_n u$ we know that $\int_B \omega\,\mathrm{d}x < \infty$. We therefore get from (2.4), (2.5) together with the considerations from above the limit inequality

$$\int_B \eta^2\Big[|\nabla^2 u|^2+|\nabla\tilde\Psi|^2+|\nabla\Psi^{(n)}|^2\Big]\,\mathrm{d}x \le c\int_B D^2F(\nabla u)(\nabla\eta\otimes\partial_\gamma u, \nabla\eta\otimes\partial_\gamma u)\,\mathrm{d}x \quad (2.6)$$

(again summation w.r.t. γ), provided we can show $\int_B \omega_\delta^s\,\mathrm{d}x \le \text{const} < \infty$ for some exponent $s>1$, since then $\omega_\delta \rightharpoondown: \omega'$ weakly in $L^s(B)$, hence $\int_B\omega_\delta\,\mathrm{d}x \to \int_B\omega'\,\mathrm{d}x$, but $\omega=\omega'$ on account of $\omega_\delta\to\omega$ a.e. By the structure of D^2F we have

$$\begin{aligned} \omega_\delta \;\le\; & c|\nabla\eta|^2\Big[(1+|\tilde\nabla u_\delta|^2)^{p/2}+(1+|\partial_n u_\delta|^2)^{q/2} \\ & +(1+|\tilde\nabla u|^2)^{\frac{p-2}{2}}|\partial_n u_\delta|^2+(1+|\partial_n u_\delta|^2)^{\frac{q-2}{2}}|\tilde\nabla u_\delta|^2\Big] \\ =:\; & c|\nabla\eta|^2\sum_{i=1}^4 S_i\,, \end{aligned}$$

and from the uniform local integrability of $|\partial_n u_\delta|^{q+2}$ and $|\tilde{\nabla} u_\delta|^{p+2}$ it follows that

$$\int_B \Big(|\nabla\eta|^2[S_1+S_2+S_3]\Big)^s \,\mathrm{d}x \le \text{ const } < \infty$$

for exponents $s > 1$ close to 1 provided we replace (H3) by the stronger condition

$$(H3)' \qquad q < 2p+2\ :$$

let us look for example at S_4 assuming w.l.o.g. that $q > 2$. Then

$$(1+|\partial_n u_\delta|^2)^{s\frac{q-2}{2}}|\tilde{\nabla} u_\delta|^{2s} \le c\Big[(1+|\partial_n u_\delta|^2)^{\frac{q+2}{2}} + |\tilde{\nabla} u_\delta|^{2st}\Big]\,,$$

where

$$t := \Big(\frac{q+2}{s(q-2)}\Big)' = \frac{(q+2)/s(q-2)}{\frac{q+2}{s(q-2)}-1} = \frac{q+2}{q+2-s(q-2)}\,.$$

Obviously $t = t(s) \to \frac{q+2}{4}$ as $s \downarrow 1$, hence $2st \to \frac{q+2}{2}$ as $s \downarrow 1$, and since $q < 2p+2$, it follows $2st \le p+2$ at least for exponents s very close to 1. This proves inequality (2.6).

□

In a similar way still assuming $(H3)'$ we obtain the following variant of (2.6) valid for arbitrary matrices $Q = (Q^1, \dots, Q^n) \in \mathbb{R}^{nN}$ and for all $\eta \in C_0^\infty(B)$:

$$\int_B \eta^2\Big[|\nabla^2 u|^2 + |\nabla\tilde{\Psi}|^2 + |\nabla\Psi^{(n)}|^2\Big]\,\mathrm{d}x \le c\int_B D^2F(\nabla u)\Big(\nabla\eta\otimes[\partial_\gamma u - Q^\gamma], \nabla\eta\otimes[\partial_\gamma u - Q^\gamma]\Big)\,\mathrm{d}x\,. \tag{2.7}$$

3 Proof of Theorem 1.1

We will apply a blow-up argument. To this purpose let us assume that the hypotheses of Theorem 1.1 are valid and define for balls $B_r(x) \Subset \Omega$ the excess function

$$E(x,r) := \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^2\,\mathrm{d}y + \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^q\,\mathrm{d}y\,,$$

where $(\cdot)_{x,r}$ and $\fint_{B_r(x)} \dots \mathrm{d}y$ denote the mean value of a function w.r.t. to $B_r(x)$. Note that $(H1)$ together with the higher integrability of $\tilde{\nabla} u$ implies that $E(x,r)$ is well-defined. The claim of Theorem 1.1 then is a consequence of the following

LEMMA 3.1. *Fix $L > 0$ and a subdomain $\Omega' \Subset \Omega$. Then there is a constant $C_*(L)$ such that for every $\tau \in (0, 1/4)$ one can find a number $\varepsilon = \varepsilon(L,\tau)$ with the following property: if $B_r(x) \Subset \Omega'$ and if*

$$|(\nabla u)_{x,r}| \le L\,, \quad E(x,r) \le \varepsilon\,, \tag{3.1}$$

then

$$E(x,\tau r) \le C_*(L)\tau^2 E(x,r)\,. \tag{3.2}$$

The proof of Lemma 3.1 originates in the works of Giusti and Miranda [GiuMi] and Evans [Ev], where it is also outlined how to deduce the desired partial regularity result from Lemma 3.1. A sketch of this routine procedure is also given in [Bi], Lemma 3.40. We divide the proof of Lemma 3.1 into several steps.

Step 1. Scaling Let us suppose that the claim of Lemma 3.1 is wrong. Assume further that a number $L > 0$ is fixed, the corresponding constant $C_*(L)$ will be chosen in Step 2. Then, for some $\tau > 0$, there is a sequence of balls $B_{r_m}(x_m) \Subset \Omega'$ such that (compare (3.1) and (3.2))

$$
\begin{aligned}
|(\nabla u)_{x_m,r_m}| &\leq L,\ E(x_m,r_m) =: \lambda_m^2 \to 0, \quad \text{as } m \to \infty\,, && (3.3)\\
E(x_m,\tau r_m) &> C_* \tau^2 \lambda_m^2\,. && (3.4)
\end{aligned}
$$

With $a_m := (u)_{x_m,r_m}$, $A_m := (\nabla u)_{x_m,r_m}$ we let

$$
u_m(z) := \frac{1}{\lambda_m r_m}\Big[u(x_m + r_m z) - a_m - r_m A_m z\Big], \quad z \in B_1 := B_1(0)
$$

and obtain from (3.3)

$$
|A_m| \leq L\,, \quad \fint_{B_1} |\nabla u_m|^2 \, \mathrm{d}z + \lambda_m^{q-2} \fint_{B_1} |\nabla u_m|^q \, \mathrm{d}z = 1\,, \tag{3.5}
$$

whereas (3.4) implies

$$
\fint_{B_\tau} |\nabla u_m - (\nabla u_m)_{0,\tau}|^2 \, \mathrm{d}z + \lambda_m^{q-2} \fint_{B_\tau} |\nabla u_m - (\nabla u_m)_{0,\tau}|^q \, \mathrm{d}z > C_* \tau^2\,. \tag{3.6}
$$

Using (3.5) and passing to subsequences we find

$$
\begin{aligned}
A_m \to: A\,, \quad u_m \rightharpoondown: \overline{u} \quad &\text{in} \quad W^1_2(B_1;\mathbb{R}^N)\,, \quad (\overline{u})_{0,1} = 0\,, \quad (\nabla \overline{u})_{0,1} = 0\,, && (3.7)\\
\lambda_m \nabla u_m \to 0 \quad &\text{in} \quad L^2(B_1;\mathbb{R}^{nN}) \quad \text{and a.e. on } B_1\,, && (3.8)\\
\lambda_m^{1-2/q} \tilde{\nabla} u_m \rightharpoondown 0 \quad &\text{in} \quad L^q(B_1;\mathbb{R}^{(n-1)N})\,, && (3.9)\\
\lambda_m^{1-2/q} \partial_n u_m \rightharpoondown 0 \quad &\text{in} \quad L^q(B_1;\mathbb{R}^N)\,. && (3.10)
\end{aligned}
$$

Note that (3.5) first implies that the left-hand sides of (3.8)–(3.10) must have (weak) limits but these limits are equal to zero on account of $\nabla u_m \rightharpoondown \nabla \overline{u}$ in $L^2(B_1;\mathbb{R}^{nN})$. Note also that for (3.9) and (3.10) we clearly require $q > 2$.

Step 2. Limit equation We claim the validity of

$$
\int_{B_1} D^2 F(A)(\nabla \overline{u}, \nabla \varphi) \, \mathrm{d}z = 0 \quad \forall \varphi \in C^\infty_0(B_1;\mathbb{R}^N)\,. \tag{3.11}
$$

In fact, the local minimality of u implies after scaling the equation

$$
\int_{B_1} DF(A_m + \lambda_m \nabla u_m) : \nabla \varphi \, \mathrm{d}z = 0\,,
$$

hence

$$\int_{B_1} \frac{1}{\lambda_m}\Big[DF(A_m+\lambda_m\nabla u_m)-DF(A_m)\Big]:\nabla\varphi\,\mathrm{d}z=0$$

or equivalently

$$\int_{B_1}\int_0^1 D^2F(A_m+s\,\lambda_m\nabla u_m)(\nabla u_m,\nabla\varphi)\,\mathrm{d}s\,\mathrm{d}z=0\,.$$

Thus we arrive at

$$\begin{aligned}&\int_{B_1} D^2F(A_m)(\nabla u_m,\nabla\varphi)\,\mathrm{d}z\\ &= -\int_{B_1}\int_0^1\Big[D^2F(A_m+s\,\lambda_m\nabla u_m)-D^2F(A_m)\Big](\nabla u_m,\nabla\varphi)\,\mathrm{d}s\,\mathrm{d}z\,. \qquad (3.12)\end{aligned}$$

By (3.7) the l.h.s. of (3.12) converges towards the l.h.s. of (3.11). We discuss the r.h.s. of (3.12): given $\varepsilon>0$, we can find $\delta=\delta(\varepsilon)$ such that

$$\int_A |\nabla\varphi|^2\,\mathrm{d}z\le\varepsilon\,, \qquad (3.13)$$

whenever A is a measurable subset of B_1 such that $\mathcal{L}^n(A)\le\delta$. (3.8) implies the existence of a set $S\subset B_1$ with the properties $\mathcal{L}^n(B_1-S)\le\delta(\varepsilon)$ and

$$\lambda_m\nabla u_m \rightrightarrows 0 \text{ on } S\,. \qquad (3.14)$$

(3.7) together with (3.14) then shows

$$\begin{aligned}&\Big|\int_S\int_0^1\Big[D^2F(A_m+s\,\lambda_m\nabla u_m)-D^2F(A_m)\Big](\nabla u_m,\nabla\varphi)\,\mathrm{d}s\,\mathrm{d}z\Big|\\ &\le \sup_{S\times[0,1]}\Big|[\ldots]\Big|\Big(\int_{B_1}|\nabla u_m|^2\,\mathrm{d}z\Big)^{1/2}\Big(\int_{B_1}|\nabla\varphi|^2\,\mathrm{d}z\Big)^{1/2}\\ &\to 0 \text{ as } m\to\infty\,.\end{aligned}$$

On the other hand we observe

$$\begin{aligned}T &:= \Bigg|\int_{B_1-S}\int_0^1[\ldots](\nabla u_m,\nabla\varphi)\,\mathrm{d}s\,\mathrm{d}z\Bigg|\\ &\le c\int_{B_1-S}\int_0^1\Big[\Big(1+|A_m+s\,\lambda_m\nabla u_m|^2\Big)^{\frac{q-2}{2}}+1\Big]|\nabla u_m||\nabla\varphi|\,\mathrm{d}s\,\mathrm{d}z\\ &\le c\int_{B_1-S}\Big[|\nabla u_m||\nabla\varphi|+\lambda^{q-2}|\nabla u_m|^{q-1}|\nabla\varphi|\Big]\,\mathrm{d}z\\ &\le c\Bigg[\Bigg(\int_{B_1}|\nabla u_m|^2\,\mathrm{d}z\Bigg)^{1/2}\Bigg(\int_{B_1-S}|\nabla\varphi|^2\,\mathrm{d}z\Bigg)^{1/2}\\ &\quad+\Bigg(\int_{B_1}\lambda_m^{\frac{q}{q-1}(q-2)}|\nabla u_m|^q\,\mathrm{d}z\Bigg)^{1-1/q}\Bigg(\int_{B_1}|\nabla\varphi|^q\,\mathrm{d}z\Bigg)^{1/q}\Bigg]\le\ldots\end{aligned}$$

$$\dots \underset{(3.13)}{\le} c\Bigg[\sqrt{\varepsilon}\Bigg(\int_{B_1}|\nabla u_m|^2\,\mathrm{d}z\Bigg)^{1/2} + \Bigg(\int_{B_1}\lambda_m^{\frac{q}{q-1}(q-2)}|\nabla u_m|^q\,\mathrm{d}z\Bigg)^{1-1/q}\Bigg(\int_{B_1}|\nabla\varphi|^q\,\mathrm{d}z\Bigg)^{1/q}\Bigg]$$

and the λ_m-term vanishes on account of (3.9). This gives

$$\limsup_{m\to\infty} T \le c\sqrt{\varepsilon}\,.$$

Altogether it is shown that

$$\limsup_{m\to\infty} |\text{r.h.s. of (3.12)}| \le c\sqrt{\varepsilon}\,,$$

and since ε is arbitrary, the limit equation (3.11) follows. Since (3.11) is an elliptic system with constant coefficients and ellipticity constants just depending on L (and p, q) we have according to [Gi2] the Campanato-estimate

$$\fint_{B_\tau}|\nabla\overline{u}-(\nabla\overline{u})_{0,\tau}|^2\,\mathrm{d}z \le C^*\tau^2\fint_{B_1}|\nabla\overline{u}-(\nabla u)_{0,1}|^2\,\mathrm{d}z \tag{3.15}$$

with $C^* = C^*(L)$. Note that $(\nabla u)_{0,1} = 0$, moreover, $\fint_{B_1}|\nabla u_m|^2\,\mathrm{d}z \le 1$ (recall (3.5)) implies $\fint_{B_1}|\nabla\overline{u}|^2\,\mathrm{d}z \le 1$, hence (3.15) gives

$$\fint_{B_\tau}|\nabla\overline{u}-(\nabla\overline{u})_{0,\tau}|^2\,\mathrm{d}z \le C^*\tau^2\,. \tag{3.16}$$

So if we let $C_* := 2C^*$, then (3.16) is in contradiction to (3.6) provided we can show in addition to (3.7), (3.9) and (3.10) that

$$\begin{aligned}
\nabla u_m \to \nabla\overline{u} &\quad \text{in } L^2_{\mathrm{loc}}(B_1;\mathbb{R}^{nN})\,, &(3.17)\\
\lambda_m^{1-2/q}\tilde{\nabla} u_m \to 0 &\quad \text{in } L^q_{\mathrm{loc}}(B_1;\mathbb{R}^{(n-1)N})\,, &(3.18)\\
\lambda_m^{1-2/q}\partial_n u_m \to 0 &\quad \text{in } L^q_{\mathrm{loc}}(B_1;\mathbb{R}^{N}) &(3.19)
\end{aligned}$$

are valid.

Step 3. Proof of (3.17)–(3.19) Let

$$\begin{aligned}
\tilde{\Psi}_m(z) &:= \lambda_m^{-1}\Big[\Big(1+|\tilde{A}_m+\lambda_m\tilde{\nabla}u_m|^2\Big)^{\frac{p}{4}} - \Big(1+|\tilde{A}_m|^2\Big)^{\frac{p}{4}}\Big]\,,\\
\Psi_m^{(n)}(z) &:= \lambda_m^{-1}\Big[\Big(1+|A_m^{(n)}+\lambda_m\partial_n u_m|^2\Big)^{\frac{q}{4}} - \Big(1+|A_m^{(n)}|^2\Big)^{\frac{q}{4}}\Big]
\end{aligned}$$

for $z \in B_1$. We have

$$\begin{aligned}
\nabla^2 u_m(z) &= r_m\lambda_m^{-1}\nabla^2 u(x_m+r_m z)\,,\\
\nabla\tilde{\Psi}_m(z) &= r_m\lambda_m^{-1}\nabla\tilde{\Psi}(x_m+r_m z)\,,\\
\nabla\Psi_m^{(n)}(z) &= r_m\lambda_m^{-1}\nabla\Psi^{(n)}(x_m+r_m z)\,,
\end{aligned}$$

and if we choose $\eta \in C_0^\infty(B_1)$ we deduce from (2.7) (by letting $Q := A_m$) the inequality

$$\begin{aligned}
&\int_{B_1} \eta^2 \Big[|\nabla^2 u_m|^2 + |\nabla \tilde{\Psi}_m|^2 + |\nabla \Psi_m^{(n)}|^2 \Big] \, \mathrm{d}z \\
&\quad \leq \ c \int_{B_1} D^2 F(\lambda_m \nabla u_m + A_m)(\nabla \eta \otimes \partial_\gamma u_m, \nabla \eta \otimes \partial_\gamma u_m) \, \mathrm{d}z \, . \qquad (3.20)
\end{aligned}$$

Suppose now that $\eta = 1$ on B_ρ, $\operatorname{spt} \eta \subset B_r$ for some $r \in (\rho, 1)$ and $0 \leq \eta \leq 1$. From (3.20) we then obtain

$$\int_{B_\rho} \Big[|\nabla^2 u_m|^2 + |\nabla \tilde{\Psi}_m|^2 + |\nabla \Psi_m^{(n)}|^2 \Big] \, \mathrm{d}z \leq c(\rho) < \infty \qquad (3.21)$$

with $c(\rho)$ being independent of m provided the r.h.s. of (3.20) can be bounded in an appropriate way: according to the structure of D^2F we have

$$\Big| D^2 F(\lambda_m \nabla u_m + A_m) \Big| \leq c \Big(1 + \lambda_m^{q-2} |\nabla u_m|^{q-2} \Big) \, ,$$

hence

$$|\text{r.h.s. of } (3.20)| \leq c(r-\rho)^{-2} \Big[\int_{B_r} |\nabla u_m|^2 \, \mathrm{d}z + \lambda_m^{q-2} \int_{B_r} |\nabla u_m|^q \, \mathrm{d}z \Big] \, ,$$

and if we use (3.5), inequality (3.21) follows, and the local strong convergence (3.17) is immediate.

To prove (3.19) we fix $\rho \in (0,1)$, a number $M \gg 1$ and let $U_m := U_m(M, \rho) := B_\rho \cap [\lambda_m |\partial_n u_m| \leq M]$. Then we get from (3.17) and the smoothness of $\overline{u}$

$$\begin{aligned}
\int_{U_m} \lambda_m^{q-2} |\partial_n u_m|^q \, \mathrm{d}z \ &\leq \ c \Bigg[\lambda_m^{q-2} \int_{U_m} |\partial_n u_m - \partial_n \overline{u}|^q \, \mathrm{d}z + \lambda_m^{q-2} \int_{U_m} |\partial_n \overline{u}|^q \, \mathrm{d}z \Bigg] \\
&\leq \ c \Bigg[\lambda_m^{q-2} \int_{U_m} \Big(|\partial_n u_m|^{q-2} + |\partial_n \overline{u}|^{q-2} \Big) |\partial_n u_m - \partial_n \overline{u}|^2 \, \mathrm{d}z \\
&\qquad + \lambda_m^{q-2} \int_{U_m} |\partial_n \overline{u}|^q \, \mathrm{d}z \Bigg] \to 0
\end{aligned}$$

as $m \to \infty$. On the other hand, for M large enough and for $z \in B_\rho - U_m$ it holds

$$|\Psi_m^{(n)}(z)| \geq c \lambda_m^{-1} \lambda_m^{\frac{q}{2}} |\partial_n u_m(z)|^{\frac{q}{2}} \, ,$$

i.e.

$$\lambda_m^{q-2} |\partial_n u_m(z)|^q \leq c \Psi_m^{(n)}(z)^2 \, .$$

Thus (3.21) gives by Sobolev's embedding theorem

$$\begin{aligned}
\int_{B_\rho - U_m} \lambda_m^{q-2} |\partial_n u_m|^q \, \mathrm{d}z \ &\leq \ c \int_{B_\rho - U_m} \Psi_m^{(n)}(z)^2 \, \mathrm{d}z \\
&\leq \ c \Big(\int_{B_\rho} |\Psi_m^{(n)}(z)|^{\frac{2n}{n-2}} \, \mathrm{d}z \Big)^{1 - \frac{2}{n}} |B_\rho - U_m|^{2/n} \to 0
\end{aligned}$$

as $m \to \infty$, since $|B_\rho - U_m| \to 0$ on account of (3.8), and we have established (3.19). Note that our calculation used the fact that actually

$$\|\Psi_m^{(n)}\|_{W_2^1(B_\rho)} \leq c(\rho)$$

for all $\rho < 1$. The bound for $\|\Psi_m^{(n)}\|_{L^2(B_\rho)}$ follows from the definition of $\Psi_m^{(n)}$ together with (3.5).

Up to now we have not used our assumption $(H2)$. This hypothesis is needed for the proof of (3.18): let $\tilde{U}_m := B_\rho \cap [\lambda_m |\tilde{\nabla} u_m| \leq M]$. Then as above we get

$$\int_{\tilde{U}_m} \lambda_m^{q-2} |\tilde{\nabla} u_m|^q \, \mathrm{d}z \to 0$$

as $m \to \infty$. On the set $B_\rho - \tilde{U}_m$ we estimate

$$|\tilde{\Psi}_m(z)| \geq c \lambda_m^{-1} \lambda_m^{\frac{p}{2}} |\tilde{\nabla} u_m(z)|^{\frac{p}{2}} ,$$

i.e.

$$\lambda_m^{q-2} |\tilde{\nabla} u_m(z)|^q \leq c \lambda_m^{2\frac{q}{p}-2} |\tilde{\Psi}_m(z)|^{2\frac{q}{p}} .$$

If $q = p$, then we combine (3.21) with Sobolev's inequality (note that also $\|\tilde{\Psi}_m\|_{L^2(B_\rho)} \leq c(\rho)$ $\forall \rho < 1$) to see

$$\begin{aligned} \int_{B_\rho - \tilde{U}_m} \lambda_m^{q-2} |\tilde{\nabla} u_m(z)|^q \, \mathrm{d}z &\leq c \int_{B_\rho - \tilde{U}_m} |\tilde{\Psi}_m(z)|^2 \, \mathrm{d}z \\ &\leq c \Big(\int_{B_\rho - \tilde{U}_m} |\tilde{\Psi}_m|^{2\frac{n}{n-2}} \, \mathrm{d}z \Big)^{1-\frac{2}{n}} |B_\rho - \tilde{U}_m|^{2/n} \to 0 \end{aligned}$$

as $m \to \infty$ (recall (3.8)). If $q > p$, then $\lambda_m^{2\frac{q}{p}-2} \to 0$, and the claim (3.18) follows provided that $2\frac{q}{p} \leq 2\frac{n}{n-2}$ which is a consequence of $(H2)$.

In order to complete our proof we have to show that

$$\int_{B_\rho} |\tilde{\Psi}_m|^2 \, \mathrm{d}z \leq c(\rho) \quad \forall \rho < 1 \tag{3.22}$$

which by the definition of $\tilde{\Psi}_m$ will follow from

$$\int_{B_\rho} \lambda_m^{p-2} |\tilde{\nabla} u_m|^2 \, \mathrm{d}z \leq c(\rho) \, . \tag{3.23}$$

Let $\eta \in C_0^\infty(B_1)$. Then we have that

$$\int_{B_1} DF(A + \lambda_m \nabla u_m) : \nabla(\eta^2 [u_m - \overline{u}]) \, \mathrm{d}z = 0 \, ,$$

i.e.

$$
\begin{aligned}
0 &= \int_{B_1} \frac{1}{\lambda_m}\Big[DF(\lambda_m \nabla u_m + A_m) - DF(A_m + \lambda_m \nabla \overline{u})\Big] : \nabla(\eta^2[u_m - \overline{u}])\,\mathrm{d}z \\
&\quad + \int_{B_1} \frac{1}{\lambda_m}\Big[DF(A_m + \lambda_m \nabla \overline{u}) - DF(A_m)\Big] : \nabla\Big(\eta^2[u_m - \overline{u}]\Big)\,\mathrm{d}z \\
&=: \; T_m^1 + T_m^2 \,.
\end{aligned}
$$

Observing

$$
\begin{aligned}
\frac{1}{\lambda_m}\Big[DF(A_m + \lambda_m \nabla \overline{u}) - DF(A_m)\Big] &= \int_0^1 D^2F(A_m + s\,\lambda_m \nabla \overline{u})(\nabla \overline{u}, \cdot) \\
&\to D^2 f(A)(\nabla \overline{u}, \cdot)
\end{aligned}
$$

uniformly on $\operatorname{spt}\eta$ and recalling (3.17) we get

$$
\lim_{m\to\infty} T_m^2 = 0\,. \tag{3.24}
$$

(3.24) implies $\lim\limits_{m\to\infty} T_m^1 = 0$ or equivalently

$$
\begin{aligned}
0 = \lim_{m\to\infty} \Bigg[&\int_{B_1} \int_0^1 D^2F\Big(A_m + \lambda_m \nabla \overline{u} + s\lambda_m(\nabla u_m - \nabla \overline{u})\Big) \\
&\quad (\nabla u_m - \nabla \overline{u}, \nabla u_m - \nabla \overline{u})\eta^2\,\mathrm{d}s\,\mathrm{d}z \\
&+ \int_{B_1} \int_0^1 D^2F(\ldots)(\nabla u_m - \nabla \overline{u}, \nabla \eta \otimes (u_m - \overline{u}))\, 2\eta\,\mathrm{d}s\,\mathrm{d}z \Bigg]\,.
\end{aligned} \tag{3.25}
$$

If we apply the Cauchy-Schwarz and Young's inequality in the second integral of (3.25) we see that for any $\varepsilon > 0$

$$
\begin{aligned}
[\ldots] \;\geq\; & (1-\varepsilon) \int_{B_1} \int_0^1 D^2F(\ldots)(\nabla u_m - \nabla \overline{u}, \nabla u_m - \nabla \overline{u})\eta^2\,\mathrm{d}s\,\mathrm{d}z \\
& - c(\varepsilon) \int_{B_1} \int_0^1 D^2F(\ldots)(\nabla \eta \otimes (u_m - \overline{u}), \nabla \eta \otimes (u_m - \overline{u}))\,\mathrm{d}s\,\mathrm{d}z\,,
\end{aligned}
$$

therefore (3.25) will give

$$
0 = \lim_{m\to\infty} \int_{B_1} \int_0^1 D^2F(\ldots)(\nabla u_m - \nabla \overline{u}, \nabla u_m - \nabla \overline{u})\eta^2\,\mathrm{d}s\,\mathrm{d}z \tag{3.26}
$$

as soon as we can show

$$
0 = \lim_{m\to\infty} \int_{B_1} \int_0^1 D^2F(\ldots)\Big(\nabla \eta \otimes (u_m - \overline{u}), \nabla \eta \otimes (u_m - \overline{u})\Big)\,\mathrm{d}s\,\mathrm{d}z\,. \tag{3.27}
$$

Note that (3.26) combined with the structure of D^2F will justify (3.23) and hence (3.22). Thus it remains to show (3.27). We have

$$|D^2F(\dots)| \le c\Big(1+|\dots|^2\Big)^{\frac{q-2}{2}} \le c\Big(1+\lambda_m^{q-2}|\nabla u_m|^{q-2}\Big)$$

and therefore

$$\Big|\int_{B_1}\int_0^1 D^2F(\dots)\Big(\nabla\eta\otimes(u_m-\overline{u}),\nabla\eta\otimes(u_m-\overline{u})\Big)\,\mathrm{d}s\,\mathrm{d}z\Big|$$

$$\le\ c\|\nabla\eta\|^2_{L^\infty(B_1)}\int_{\operatorname{spt}\eta}\Big[|u_m-\overline{u}|^2+\lambda_m^{q-2}|\nabla u_m|^{q-2}|u_m-\overline{u}|^2\Big]\,\mathrm{d}z$$

which leads to the discussion of

$$\int_{\operatorname{spt}\eta}\lambda_m^{q-2}|\nabla u_m|^{q-2}|u_m-\overline{u}|^2\,\mathrm{d}z =: \xi_m\,.$$

Höldars's inequality gives

$$\xi_m \le \Big(\int_{B_1}\lambda_m^{q-2}|\nabla u_m|^q\,\mathrm{d}z\Big)^{1-2/q}\Big(\int_{\operatorname{spt}\eta}\lambda_m^{q-2}|u_m-\overline{u}|^q\,\mathrm{d}z\Big)^{2/q},$$

the first integral being bounded by (3.5). If $q=2$, then the second integral vanishes as $m\to\infty$. If $q>2$, then the same is true if we can show that

$$v_m := \lambda_m^{1-2/q}u_m \to 0 \quad \text{in } L^q(B_1;\mathbb{R}^N)\,. \tag{3.28}$$

To this purpose we observe that $(v_m)_{0,1}=0$ so that (3.5) in combination with Poincaré's inequality implies $\sup_m\|v_m\|_{W^1_q(B_1)}<\infty$, hence $v_m \rightharpoondown: \overline{v}$ in $W^1_q(B_1;\mathbb{R}^N)$ and $v_m\to\overline{v}$ in $L^q(B_1;\mathbb{R}^N)$. (3.9) and (3.10) give $\nabla\overline{v}=0$, and since $(\overline{v})_{0,1}=0$, we must have (3.28). Note that with (3.26) we actually have shown that (3.23) can be replaced by $\lambda_m^{1-2/p}\tilde{\nabla}u_m\to 0$ in $L^p_{\mathrm{loc}}(B_1;\mathbb{R}^{(n-1)N})$. Altogether the proof of Theorem 1.1 is complete. □

REMARK 3.1. *In order to prove Theorem 1.1 under less restrictive assumptions concerning p and q we could try to replace the excess function $E(x,r)$ by the more natural one*

$$\tilde{E}(x,r) := \fint_{B_r(x)}|\nabla u-(\nabla u)_{x,r}|^2\,\mathrm{d}y + \fint_{B_r(x)}|\tilde{\nabla}u-(\tilde{\nabla}u)_{x,r}|^p\,\mathrm{d}y + \fint_{B_r(x)}|\partial_n u-(\partial_n u)_{x,r}|^q\,\mathrm{d}y\,.$$

But in this new setting we could not verify estimate (3.21) with the help of (3.20) since it is not obvious how to bound the right-hand side of (3.20) using the equation $\fint_{B_1}|\nabla u_m|^2\,\mathrm{d}z+\lambda_m^{p-2}\fint_{B_1}|\tilde{\nabla}u_m|^p\,\mathrm{d}z+\lambda_m^{q-2}\fint_{B_1}|\partial_n u_m|^q\,\mathrm{d}z=1$ *which now replaces (3.5).*

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