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Darya Apushkinskaya and Nina Uraltseva

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Darya Apushkinskaya

Saarland University
Department of Mathematics
P.O. Box 15 11 50
66041 Saarbrücken
Germany
darya@math.uni-sb.de

Nina Uraltseva

St. Petersburg State University
Department of Mathematics
Universitetsky prospekt, 28 (Peterhof)
198504 St. Petersburg
Russia
uraltsev@pdmi.ras.ru

Edited by
FR 6.1 – Mathematik
Universität des Saarlandes
Postfach 15 11 50
66041 Saarbrücken
Germany

Fax: + 49 681 302 4443
e-Mail: preprint@math.uni-sb.de
WWW: http://www.math.uni-sb.de/

Abstract

For weak solutions of the two-phase obstacle problem

$$\Delta u = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}} \quad \text{in } B_1^+, \qquad \lambda^\pm \geqslant 0, \quad \lambda^+ + \lambda^- > 0,$$

satisfying the non-zero Dirichlet condition on the flat part of ∂B_1^+, we obtain the optimal regularity, i.e., we show that u is a W^2_∞-function.

1 Introduction

We consider a weak solution of the obstacle-problem-like equation

$$\Delta u = \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}} \quad \text{in } B_1^+ := \{x : |x| < 1, x_1 > 0\}, \tag{1}$$

satisfying the boundary condition

$$u = \varphi \quad \text{on } \Pi_1 := \{x : |x| \leqslant 1, x_1 = 0\}, \tag{2}$$

where Δ is the Laplacian, λ^+ and λ^- are non-negative constants such that $\lambda^+ + \lambda^- > 0$, and χ_E is the characteristic function of the set E. The Dirichlet data φ is supposed to satisfy the following conditions:

$$\varphi \text{ is a } W^3_\infty - \text{function}, \tag{3}$$

$$\exists\, L > 0 \text{ such that } |D'\varphi(x)| \leqslant L|\varphi(x)|^{2/3} \quad \forall x \in \Pi_1. \tag{4}$$

Observe that if the boundary data φ is non-negative (non-positive) then the solution u is so too, and we arrive at the classical one-phase obstacle problem. It is well-known (see [Je]) that the solution of the one-phase obstacle problem with $C^{2,\alpha}$ boundary data is a W^2_∞-function up to the boundary, and this regularity is optimal.
The L_∞-estimates of the second derivatives $D^2 u$ near Π_1 for solutions of the two-phase problem (1)-(2) are of main interest of this paper. Now we can state our main result.

Theorem. *Let u be a solution of the problem (1)-(2), with a function φ satisfying the assumptions (3) and (4). Suppose also that* $\sup\limits_{B_1^+} |u| \leqslant M$.
Then for any $\delta \in (0,1)$ there exists a positive constant C completely defined by n, M, $\lambda^\pm$, δ, L, and by the norm of φ in the Sobolev space $W^3_\infty(\Pi_1)$ such that

$$\operatorname*{ess\,sup}_{B^+_{1-\delta}} |D^2 u| \leqslant C.$$

Throughout this article we use the following notation:
$x = (x_1, x') = (x_1, x_2, \dots, x_n)$ are points in $\mathbb{R}^n$, $n \geqslant 2$, with the Euclidean norm $|x|$.
χ_E denotes the characteristic function of the set $E \subset \mathbb{R}^n$;
∂E stands for the boundary of the set E;
$\| \cdot \|_{p,E}$ denotes the norm in $L_p(E)$.
$v_+ = \max\{v, 0\}$;
$B_r(x^0)$ denotes the open ball in $\mathbb{R}^n$ with center x^0 and radius r;
$B_r^+(x^0) = \{x \in B_r(x^0) : x_1 > 0\}$; $B_r = B_r(0)$; $B_r^+ = B_r \cap \{x_1 > 0\}$.
$\Pi = \{(x, t) \in \mathbb{R}^{n+1} : x_1 = 0\}$; $\Pi_r = \Pi \cap B_r$.
D_i denotes the differential operator with respect to x_i; $Du = (D_1 u, D' u) = (D_1 u, D_2 u, \dots, D_n u)$ is the gradient of the function u; D_ν stands for the operator of differentiation along the direction $\nu \in \mathbb{R}^n$, i.e., $|\nu| = 1$ and

$$D_\nu u = \sum_{i=1}^{n} \nu_i D_i u;$$

$D^2 = D(D)$ denotes the Hessian.
We adopt the convention that the index τ runs from 2 to n. We also adopt the convention regarding summation with respect to repeated indices.
We use letters N, L, and C (with or without indices) to denote various constants. To indicate that, say, C depends on some parameters, we list them in the parentheses: $C(\dots)$. We will write $C(\varphi)$ to indicate that C is defined by the Sobolev-norms of φ.
For a C^1-function u defined in B_1^+, we introduce the following sets:
$\Omega^\pm(u) = \{x \in B_1^+ : \pm u(x) > 0\}$;
$\Lambda(u) = \{x \in B_1^+ : u(x) = |Du(x)| = 0\}$;
$\Gamma(u) = \partial\{x \in B_1^+ : u(x) \neq 0\} \cap B_1^+$ is the free boundary. We emphasize that in the two-phase case we do not have the property that the gradient vanishes on the free boundary, as it was in the classical one-phase case; this causes difficulties.
$\Gamma^0(u) = \Gamma(u) \cap \Lambda(u)$; $\Gamma^*(u) = \Gamma(u) \setminus \Gamma^0(u)$. We observe that $\Gamma^*(u)$ is $C^{1,\alpha}$-surface for any $\alpha < 1$.
From now on we suppose that $\sup\limits_{B_1^+} |u| \leqslant M$. Together with the assumptions

(3) it provides for any $\delta \in (0,1)$ the following estimates for u:

$$\|D^2u\|_{q,B^+_{1-\delta}} \leqslant N_1(q,M,\delta,\varphi), \qquad \forall q<\infty, \tag{5}$$

$$\sup_{B^+_{1-\delta}} |Du| \leqslant N_2(M,\delta,\varphi), \tag{6}$$

$$\frac{|Du(x)-Du(y)|}{|x-y|^\alpha} \leqslant N_3(\alpha,M,\delta,\varphi), \qquad \forall \alpha \in (0,1). \tag{7}$$

Observe that the constants $N_1 - N_3$ depend on W^2_∞-norm of φ.

Now we formulate an important tool used to prove Main Theorem. This is the celebrated monotonicity formula due to H.W. Alt, L.A. Caffarelli, and A. Friedman (see [ACF]).

Lemma 1. *Let x^0 be a point in $\mathbb{R}^n$, and let h_1 and h_2 be non-negative, sub-harmonic, continuous functions in the unit ball $B_1(x^0)$, satisfying*

$$h_1(x^0)=h_2(x^0)=0, \qquad h_1(x)\cdot h_2(x)=0 \text{ in } B_1(x^0).$$

Then the functional

$$\Phi(r,x^0,h_1,h_2) := \frac{1}{r^4}\int\limits_{B_r(x^0)} \frac{|Dh_1|^2}{|x-x^0|^{n-2}}dx \int\limits_{B_r(x^0)} \frac{|Dh_2|^2}{|x-x^0|^{n-2}}dx$$

is monotone increasing in r, $0<r<1$.

2 Estimates of the tangential gradient near the boundary

Lemma 2. *Let u be a solution of Equation (1), and let e be a direction in $\mathbb{R}^n$. Then for $x \in B^+_1$ we have*

$$(i) \qquad \Delta(D_e u(x)) = (\lambda^+ + \lambda^-)\frac{D_e u(x)}{|Du(x)|}\mathcal{H}^{n-1}\lfloor\Gamma^*(u),$$

$$(ii) \qquad \Delta|u(x)| = \lambda^+\chi_{\Omega^+(u)} + \lambda^-\chi_{\Omega^-(u)} + 2|Du(x)|\mathcal{H}^{n-1}\lfloor\Gamma^*(u),$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure of the surface $\Gamma^*(u)$.

Proof. For a proof of part (i) we refer the reader to (the proof of) Lemma 2 in [U1]. Part (ii) follows from direct computation. Indeed, for any test-function

$\eta \in C_0^\infty(\Omega)$ the value of the distribution $\Delta|u|$ on η equals

$$\langle \Delta|u|, \eta\rangle := \int\limits_{\Omega^+(u)\cup\Omega^-(u)} |u|\Delta\eta dx = \int\limits_{\Omega^+(u)} u\Delta\eta dx - \int\limits_{\Omega^-(u)} u\Delta\eta dx$$
$$= \int\limits_{\Omega^+(u)} (\Delta u)\eta dx - \int\limits_{\Omega^-(u)} (\Delta u)\eta dx + 2\int\limits_{\Gamma^*(u)} (D_\gamma u)\eta dx,$$

where $\gamma = \gamma(x)$ is the unit normal to $\Gamma^*(u)$ chosen outward w.r.t. the set $\Omega^-(u)$, i.e., $\gamma(x) = \frac{Du(x)}{|Du(x)|}$. Application Eq. (1) to the right-hand side of the above identity finishes the proof. □

Lemma 3. *Let the assumptions of Theorem hold. Then for arbitrary small* $\delta > 0$ *there exists constant* N_δ *such that*

$$|D_\tau u(x) - D_\tau\varphi(x')| \leqslant N_\delta x_1, \qquad \textit{for } x \in B^+_{1-\delta},\ \tau = 2,\dots,n. \tag{8}$$

The constant N_δ *is completely defined by* δ, n, M, L, $\lambda^\pm$ *and by the norm of* φ *in the Sobolev space* $W^3_\infty(\Pi_1)$.

Proof. We fix $\delta \in (0, 1/2)$ and $\tau \in \mathbb{N}$, $2 \leqslant \tau \leqslant n$.
Consider in the cylinder $Q_\delta = \{x \in \mathbb{R}^n : 0 < x_1 < \sqrt{\delta},\ |x'| < 1-\delta\}$, the auxiliary functions

$$v^\pm(x) = \pm(D_\tau u(x) - D_\tau\varphi(x')) + |u(x)| - |\varphi(x')|,$$

and the barrier function

$$w(x) = N_4\left(\frac{x_1}{\sqrt{\delta}} - \frac{x_1^2}{2\delta}\right) + N_5\left((|x'| - 1 + 2\delta))_+\right)^2$$

with positive constants N_4 and N_5 which will be chosen later.
It is easy to see that the inequalities

$$v^\pm(x) \leqslant w(x) \quad \text{in} \quad Q_\delta \tag{9}$$

together with (6) imply the desired estimate (8). Therefore, it remains only to verify (9).
To prove (9), first we observe that $v^\pm(x) \leqslant w(x)$ for all $x \in \Lambda(u) \cap Q_\delta$. Indeed, for a point $y \in \Lambda(u) \cap Q_\delta$ elementary computation combining with

the inequality (7) for $\alpha = 1/2$, give

$$|\varphi(y')| \leqslant \int_0^{y_1} |D_1 u(t,y')|dt = \int_0^{y_1} |D_1 u(y_1,y') - D_1 u(t,y')|dt$$
$$\leqslant N_3 \int_0^{y_1} (y_1 - t)^{1/2} dt \leqslant N_3 y_1^{3/2}. \tag{10}$$

Taking into account the assumption (4) and the inequality (10) we arrive at

$$v^{\pm}(y) \leqslant |D_\tau \varphi(y)| \leqslant L N_3^{2/3} y_1 \leqslant w(y) \qquad \forall y \in \Lambda(u) \cap Q_\delta,$$

if N_1 is chosen so that $N_1 \geqslant 2\sqrt{\delta} L N_3^{2/3}$.
Now we consider the sets $D^{\pm} := Q_\delta \cap \{x : v^{\pm}(x) > w(x)\}$. According to the above arguments $D^{\pm}$ have no intersections with $\Lambda(u)$. If we show that $D^{\pm}$ are empty then the proof of (9) is complete. Suppose, towards a contradiction, that at least one of the sets $D^{\pm}$ is non-empty.
It is obvious that an appropriate choice of the constants N_4 and N_5 guarantees the inequality

$$v^{\pm} \leqslant w \quad \text{on} \quad \partial Q_\delta. \tag{11}$$

We emphasize also that the assumption (3) provides the estimate $\sup\limits_{Q_\delta} \Delta(D_\tau \varphi) \leqslant N_6$, whereas the assumptions (3) and (4) guarantee $\sup\limits_{Q_\delta} \Delta|\varphi| \leqslant N_7$, where the constants N_6 and N_7 are defined by the W^3_∞-norm and W^2_∞-norm of φ, respectively.
Next, the direct computation in combination with the above estimates for $\Delta(D_\tau \varphi)$ and $\Delta|\varphi|$, and the equalities from Lemma 2 yield

$$\Delta(v^{\pm} - w)\big|_{D^{\pm}} \geqslant -N_6 - N_7 + \frac{N_4}{\delta} - 2nN_5 + \sigma^{\pm} \mathcal{H}^{n-1} \lfloor \Gamma^*(u) \cap D^{\pm},$$

where the measure densities $\sigma^{\pm}$ are defined by the formula

$$\sigma^{\pm}(x) = 2|Du(x)| \pm \lambda \frac{D_\tau u(x)}{|Du(x)|}, \qquad \lambda := \lambda^+ + \lambda^-.$$

We claim that $\sigma^{\pm} \geqslant 0$ on $\Gamma^*(u) \cap D^{\pm}$, respectively. Indeed, it is suffices to show that for $x \in \Gamma^*(u) \cap D^{\pm}$ we have

$$2|Du(x)|^2 + \lambda \left(\pm D_\tau \varphi(x') + |\varphi(x')| + \frac{N_4}{2\sqrt{\delta}} x_1 \right) \geqslant 0. \tag{12}$$

Suppose that

$$2|D_1u(x)|^2 < \lambda|D_\tau\varphi(x')|; \tag{13}$$

otherwise (12) is proved. Arguing in the same way as in deriving (10) we get the estimate

$$\begin{aligned} |\varphi(x')| &\leqslant \int_0^{x_1} |D_1u(t,x')|dt \leqslant \int_0^{x_1} |D_1u(x_1,x') - D_1u(t,x')|dt + |D_1u(x)|x_1 \\ &\leqslant N_3(x_1)^{3/2} + |D_1u(x)|x_1. \end{aligned} \tag{14}$$

If $N_3(x_1)^{3/2} < |D_1u(x)|x_1$ then the inequalities (4),(13) and (14) imply

$$|\varphi(x')| \leqslant 2|D_1u(x)|\,x_1 < 2\sqrt{\lambda|D_\tau\varphi(x')|}\,x_1 \leqslant 2\sqrt{\lambda L}|\varphi(x')|^{1/3}\,x_1,$$

and, consequently, $|D_\tau\varphi(x')| \leqslant L|\varphi(x')|^{2/3} \leqslant 2L\sqrt{\lambda L}\,x_1$. From here, increasing N_4 if it is necessary, we arrive at (12).
In the other case, i.e., if $|D_1u(x)|x_1 \leqslant N_3(x_1)^{3/2}$, the inequalities (4) and (14) guarantee that

$$|D_\tau\varphi(x')| \leqslant L|\varphi(x')|^{2/3} \leqslant (2N_3)^{2/3}L\,x_1.$$

Again, increasing N_4 if it is necessary, we arrive at (12).
Now we are able to conclude that

$$\Delta(v^\pm - w)\big|_{D^\pm} \geqslant -N_6 - N_7 + \frac{N_4}{\delta} - 2nN_5 \geqslant 0, \tag{15}$$

provided by the choice of N_4 large enough.
Thanks to (11) and (15) we can apply the comparison principle to the functions $v^\pm$ and w on the sets $D^\pm$, respectively, and deduce the inequalities

$$v^\pm(x) \leqslant w(x) \quad \text{in} \quad D^\pm = Q_\delta \cap \{x : v^\pm(x) > w(x)\},$$

which give the desired contradiction with our assumption that $D^\pm \neq \emptyset$ and complete the proof. □

3 Boundary estimates of the second derivatives

Lemma 4. *Let the assumptions of Theorem hold, let an arbitrary $\delta \in (0,1)$ be fixed, and let x^0 be an arbitrary point in $B^+_{1-\delta}$. Then*

$$\frac{1}{R^2}\int_{B_R(x^0)} \frac{|D^2u(x)|}{|x-x^0|^{n-2}}dx \leqslant C_\delta, \tag{16}$$

where R is defined by the formula

$$R := \begin{cases} \delta/2, & \text{if} \quad x_1^0 > \delta/2 \\ x_1^0/2, & \text{otherwise}, \end{cases} \tag{17}$$

and C_δ depends on the same arguments as the constant $N_{\delta/2}$ from Lemma 3.

Proof. First of all, we observe that it is enough to show that

$$\frac{1}{R^2} \int\limits_{B_R(x^0)} \frac{|D(D_\tau u)|^2}{|x - x^0|^{n-2}} dx \leqslant C_\delta, \tag{18}$$

for any tangential direction τ, since we can find the derivative $D_1 D_1 u$ from Eq.(1).
Each of the derivatives $D_\tau u$, $\tau = 2, \ldots, n$, satisfies the integral identity

$$\int\limits_{B_1^+} D(D_\tau u(x)) D\eta(x) dx = \int\limits_{B_1^+} f D_\tau \eta(x) dx, \qquad \forall \eta \in \overset{\circ}{W}{}_2^1(B_1^+), \tag{19}$$

where $f := \lambda^+ \chi_{\{u>0\}} - \lambda^- \chi_{\{u<0\}}$. Suppose now that we are given a point $x^0 \in B_{1-\delta}^+$ with some $\delta \in (0,1)$ and $x_1^0 \leqslant \delta/2$.
In this case we set $\eta = \zeta^2 G(D_\tau u - D_\tau \varphi)$, where $\zeta \in C_0^\infty(B_{2R}(x^0))$ is a cut-off function such that $\zeta = 1$ on $B_R(x^0)$ and

$$|D\zeta| \leqslant \frac{N_8(n)}{R}, \qquad |D^2 \zeta| \leqslant \frac{N_8(n)}{R^2},$$

while G is defined by the formula $G(x) = \min\{|x - x^0|^{2-n}, \beta^{2-n}\}$ for some small β. Plugging η into (19) we obtain

$$\begin{aligned} \int\limits_{B_1^+} |D(D_\tau u)|^2 \zeta^2 G dx = &- \int\limits_{B_1^+} f D_\tau(D_\tau \varphi) \zeta^2 G dx + \int\limits_{B_1^+} f (D_\tau u - D_\tau \varphi) D_\tau(\zeta^2 G) dx \\ &- \int\limits_{B_1^+} (D_\tau u - D_\tau \varphi) D(D_\tau u) D(\zeta^2 G) dx \\ &+ \int\limits_{B_1^+} \left[f D_\tau(D_\tau u) + D(D_\tau \varphi) D(D_\tau u) \right] \zeta^2 G dx \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Our next objective is to estimate these four integrals. For I_1 from (3) it follows that

$$I_1 \leqslant \sup_{B_{2R}(x^0)} |f| \sup_{B_{2R}(x^0)} |D_\tau(D_\tau\varphi)| \int\limits_{B_{2R}(x^0)} \zeta^2 G dx \leqslant N_9(n,\lambda^\pm,\varphi)R^2.$$

Observe that due to Lemma 3 we have $|D_\tau u - D_\tau\varphi| \leqslant 2N_{\delta/2}R$ in $B_{2R}(x^0)$. Hence

$$I_2 \leqslant \sup_{B_{2R}(x^0)} |f| \sup_{B_{2R}(x^0)} |D_\tau u - D_\tau\varphi| \int\limits_{B_{2R}(x^0)} D_\tau(\zeta^2 G) dx \leqslant N_{10}(n,M,\delta,\lambda^\pm,\varphi)R^2.$$

Further, we transform I_3 into $I_3 \pm \int\limits_{B_1^+} (D_\tau u - D_\tau\varphi) D(D_\tau\varphi) D(\zeta^2 G) dx$, apply integration by parts, and take into account Lemma 3. As a result we get

$$I_3 = \int\limits_{B_{2R}(x^0)\setminus B_\beta(x^0)} \frac{1}{2}(D_\tau u - D_\tau\varphi)^2 \Delta(\zeta^2 G) dx + \frac{n-2}{\beta^{n-1}} \int\limits_{\partial B_\beta(x^0)} \frac{1}{2}(D_\tau u - D_\tau\varphi)^2 dx$$
$$- \int\limits_{B_1^+} (D_\tau u - D_\tau\varphi) D(D_\tau\varphi) D(\zeta^2 G) dx \leqslant N_{11}(n) N_{\delta/2}^2 R^2 + N_{12}(n,\varphi) N_{\delta/2} R^2.$$

Finally, using $|f D_\tau(D_\tau u) + D(D_\tau\varphi) D(D_\tau u)| \leqslant \frac{1}{2}|D(D_\tau u)|^2 + |f|^2 + |D(D_\tau\varphi)|^2$, we obtain

$$I_4 \leqslant \frac{1}{2} \int\limits_{B_1^+} |D(D_\tau u)|^2 \zeta^2 G dx + N_{13}(n,\lambda^\pm,\varphi)R^2.$$

Thus, collecting all inequalities, we arrive at

$$\int\limits_{B_1^+} |D(D_\tau u)|^2 \zeta^2 \widetilde{G} dx \leqslant N_{14}(n,M,\delta,\lambda^\pm,\varphi)R^2.$$

Letting $\beta \to 0$ we obtain (18) and, consequently, (16).

Turning to the case $x_1^0 > \delta/2$ we note that similar to (16) estimate

$$\frac{4}{\delta^2} \int\limits_{B_{\delta/2}(x^0)} \frac{|D^2 u(x)|^2}{|x - x^0|^{n-2}} dx \leqslant C_\delta$$

follows easily from the Hölder inequality and (5). □

Proof of Theorem. Let $\delta \in (0,1)$ be fixed, let $x^0 \in \Omega^+(u) \cup \Omega^-(u)$ with $|x^0| < 1-\delta$, let $\nu = \frac{Du(x^0)}{|Du(x^0)|}$, and let a direction $e \in \mathbb{R}^n$ be orthogonal to ν. Since $D_e u(x^0) = 0$, it follows that

$$C(n)|D(D_e u)(x^0)|^4 \leqslant \lim_{r\to 0} \Phi(r, x^0, (D_e u)_+, (D_e u)_-).$$

On the other hand, according to Lemma 1, we have the inequality

$$\Phi(r, x^0, (D_e u)_+, (D_e u)_-) \leqslant \Phi(R, x^0, (D_e u)_+, (D_e u)_-),$$

where R is defined by formula (17). Application of Lemma 4 enable us to estimate the right-hand side of the last relation by the constant C_δ^2. This means that we obtained the estimate of all the derivatives $D(D_e u)(x^0)$ with $e \perp \nu$. It is evident that the derivative $D_\nu D_\nu u(x^0)$ can be now estimated from Eq. (1).
Since the Lebesgue measure of $\Gamma(u)$ is zero (see [W]), it remains only to note that the obtained estimate of the second derivatives at the point x^0 does not depend on $\mathrm{dist}(x^0, \Gamma(u))$, as well as on x_1^0. This finishes the proof. □

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