Universität des Saarlandes

Fachrichtung 6.1 – Mathematik

Preprint Nr. 169

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Saarbrücken 2006

Fachrichtung 6.1 – Mathematik
Universität des Saarlandes

Preprint No. 169
submitted: 22.03.2006

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Abstract

We discuss the weak form of the Ramberg/Osgood equations for nonlinear elastic materials on a 3-dimensional domain and show that the stress tensor is Hölder continuous on an open subset whose complement is of Lebesgue-measure zero. We also give an estimate for the Hausdorff-dimension of the singular set.

1 Introduction

In this paper we investigate the smoothness properties of weak solutions of the Ramberg/Osgood equations defined on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$. To be precise, we fix $q \in (2, \infty)$, define the conjugate exponent $p = q/(q-1)$, and consider the spaces (see[GS], [Kn] or [FuS])

$$\begin{aligned} L^{q,2}(\Omega) &:= \left\{\sigma : \Omega \to \mathbb{S}^3 : \sigma^D \in L^q(\Omega),\, \operatorname{tr}\sigma \in L^2(\Omega)\right\}, \\ U^{p,2}(\Omega) &:= \left\{v : \Omega \to \mathbb{R}^3 : u \in L^p(\Omega),\, \varepsilon^D(u) \in L^p(\Omega),\, \operatorname{div} u \in L^2(\Omega)\right\}, \end{aligned}$$

where

$$\begin{aligned} \mathbb{S}^3 &:= \text{space of all symmetric } 3\times 3\text{-matrices}, \\ \sigma^D &:= \sigma - \frac{1}{3}\operatorname{tr}\sigma\mathbf{1} \quad \text{for } \sigma \in \mathbb{S}^3, \\ \varepsilon(u) &:= \frac{1}{2}(\nabla u + \nabla u^T) \quad \text{for } u\colon \Omega \to \mathbb{R}^3. \end{aligned}$$

The spaces $L^{q,2}(\Omega)$, $U^{p,2}(\Omega)$ are normed in a standard way, we again refer to [GS], [Kn] or [FuS] for details. Finally, we let $U_0^{p,2}(\Omega)$ denote the closure of $C_0^\infty(\Omega;\mathbb{R}^3)$ in $U^{p,2}(\Omega)$ w.r.t. the corresponding norm.

DEFINITION 1.1. *A pair $(\sigma, u) \in L^{q,2}(\Omega) \times U^{p,2}(\Omega)$ is called a weak solution of the Ramberg/Osgood equations iff*

$$\int_\Omega \left[A\sigma + \alpha|\sigma^D|^{q-2}\sigma^D\right] : \tau \,\mathrm{d}x = \int_\Omega \varepsilon(u) : \tau \,\mathrm{d}x \tag{1.1}$$

and

$$\int_\Omega \sigma : \varepsilon(v)\,\mathrm{d}x = \int_\Omega f \cdot v \,\mathrm{d}x \tag{1.2}$$

hold for any $(\tau, v) \in L^{q,2}(\Omega) \times U_0^{p,2}(\Omega)$.

In (1.1), (1.2) the symbols ":" and "·" denote the scalar products in $\mathbb{S}^3$ and $\mathbb{R}^3$, respectively. A is a symmetric linear operator $\mathbb{S}^3 \to \mathbb{S}^3$ such that

$$A\sigma : \sigma \geq \lambda|\sigma|^2 \tag{1.3}$$

AMS Subject Classification: 74B20, 74G40, 49N60, 35J60
Keywords: nonlinear elastic materials, Ramberg/Osgood equations, regularity

for all $\sigma \in \mathbb{S}^3$ and a constant $\lambda > 0$. The volume forces $f\colon \Omega \to \mathbb{R}^3$ are assumed to be in the space $L^q(\Omega)$, and $\alpha > 0$ is a given parameter. If we introduce the potential

$$W(\tau) := \frac{1}{2} A\tau : \tau + \frac{\alpha}{q} |\tau^D|^q, \qquad \tau \in \mathbb{S}^3,$$

then (1.1) is equivalent to

$$\varepsilon(u) = DW(\sigma). \tag{1.4}$$

The above equations were first introduced by Ramberg and Osgood [OR] as constitutive relations describing the behaviour of aluminium alloys. Generally speaking, (1.1) and (1.2) are adequate for physically nonlinear elastic materials with constitutive law of power-law type. The physical and historical background of the subject is carefully explained in the thesis [Kn], where also the existence of weak solutions is established. Moreover, a large part of the work [Kn] is devoted to the investigation of the local regularity of the stress and the strain tensor, for example weak differentiability and higher integrability results are discussed (partially up to the boundary).

In our recent paper [BF] we gave a slight improvement of these regularity results by showing using methods developed in [BFZ] that the tensors σ and $\varepsilon(u)$ satisfy a local Hölder condition provided the case of plane domains is considered. Here we are going to discuss the 3D-case for which we will prove

THEOREM 1.1. *Let all the hypotheses stated before hold and suppose that $(\sigma, u) \in L^{q,2}(\Omega) \times U^{p,2}(\Omega)$ is a weak solution in the sense of Definition 1.1. Suppose further that*

$$f \in L^\infty_{\mathrm{loc}}(\Omega) \tag{1.5}$$

and

$$p > 6/5 \tag{1.6}$$

hold. Then there is an open subset Ω_0 of Ω of full Lebesgue measure such that σ and $\varepsilon(u)$ are locally Hölder continuous on Ω_0.

REMARK 1.1. *It will be shown that*

$$\Omega_0 = \Bigg\{ x \in \Omega : \limsup_{r\to 0} |(\sigma)_{x,r}| < \infty \text{ and } \liminf_{r\to 0} \fint_{B_r(x)} |\sigma - (\sigma)_{x,r}|^2 \,\mathrm{d}y + \fint_{B_r(x)} |\sigma^D - (\sigma^D)_{x,r}|^q \,\mathrm{d}y = 0 \Bigg\}.$$

Here $(\cdot)_{x,r}$ and $\fint_{B_r(x)} \cdot \,\mathrm{d}y$ denote mean values over the ball $B_r(x)$. Note that in this description of Ω_0 only the stress tensor σ is considered. In fact, we will first prove the continuity of σ on Ω_0, the continuity of $\varepsilon(u)$ then follows from equation (1.4).

REMARK 1.2. *Condition (1.6) means $q < 6$, and we can avoid this restriction in Theorem 1.2 (but not for free). Technically (1.6) enters via the fact that we need compactness of the embedding $\overset{\circ}{W}{}^{1,p}(\Omega) \hookrightarrow L^2(\Omega)$, where $\overset{\circ}{W}{}^{1,p}(\Omega)$ is the standard Sobolev space of functions with zero trace, see, e.g. [Ad].*

REMARK 1.3. *From the description of the regular set* Ω_0 *given in Remark 1.1 it is immediate that* $\mathcal{L}^3(\Omega - \Omega_0) = 0$. *Using results of [Kn] we will actually have that* $\mathcal{H}^{1+\varepsilon}(\Omega - \Omega_0) = 0$ *for any* $\varepsilon > 0$, $\mathcal{H}^{1+\varepsilon}$ *being the* $(1+\varepsilon)$*-dimensional Hausdorff-measure.*

REMARK 1.4. *Of course we can consider domains* $\Omega \subset \mathbb{R}^d$ *with* $d \geq 4$ *and get the result of Theorem 1.1 under the condition* $p > 2d/(d+2)$.

Next we like to discuss how to get rid of the unpleasant assumption that $p > 6/5$. This can be achieved by increasing the regularity of the function f and by allowing a different regular set Ω_0^* (still open and of full $\mathcal{L}^3$-measure), where now also the mean oscillation of $\varepsilon(u)$ comes into play so that $\Omega_0^* \subset \Omega_0$, Ω_0 denoting the set defined in Remark 1.1. To be precise, we first observe that according to [Kn], Lemma 2.18, the function u from a pair $(\sigma, u) \in L^{q,2}(\Omega) \times U^{p,2}(\Omega)$ of solutions to (1.1) and (1.2) is in the space $W^{2,r}_{\mathrm{loc}}(\Omega)$, $r := 3p/(p+1)$. Obviously $r < 2$ and by Sobolev's embedding theorem it follows $\nabla u \in L^{3r/(3-r)}(\Omega)$, i.e. $\nabla u \in L^{3p}_{\mathrm{loc}}(\Omega)$, which gives $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ so that the definition of the set Ω_0^* in the next result makes sense.

THEOREM 1.2. *Let the assumptions of Theorem 1.1 hold with the exception that (1.5) is replaced by*

$$f \in W^{1,\infty}_{\mathrm{loc}}(\Omega) \tag{1.7}$$

and that $p \in (1,2)$ *is arbitrary. Then, if* $(\sigma, u) \in L^{q,2}(\Omega) \times U^{p,2}(\Omega)$ *is a weak solution according to Definition 1.1, the set*

$$\begin{aligned} \Omega_0^* \;:=\; \Bigg\{ & x \in \Omega : \limsup_{r \to 0} |(\sigma)_{x,r}| < \infty \;\; \textit{and} \\ & \liminf_{r \to 0} \Bigg[\fint_{B_r(x)} |\sigma - (\sigma)_{x,r}|^2 \, \mathrm{d}y + \fint_{B_r(x)} |\sigma^D - (\sigma^D)_{x,r}|^q \, \mathrm{d}y \\ & + \fint_{B_r(x)} |\varepsilon(u) - (\varepsilon(u))_{x,r}|^2 \, \mathrm{d}y \Bigg] = 0 \Bigg\} \end{aligned}$$

is open (and of full Lebesgue-measure). Moreover, σ *and* $\varepsilon(u)$ *are locally Hölder continuous on* Ω_0^*.

REMARK 1.5. *According to (1.4) the quantities* $\varepsilon(u)$ *and* σ *are related through* $\varepsilon(u) = DW(\sigma)$ *but this formula does not imply that a point* x *from the set* Ω_0 *defined in Remark 1.1 belongs to* Ω_0^*, *i.e. the vanishing of the mean oscillation of* σ *at* $x \in \Omega$ *does not necessarily imply the same for* $\varepsilon(u)$. *Hence the "new regular set"* Ω_0^* *is a strict subset of* Ω_0, *and involves the stress tensor as well as the strain tensor.*

REMARK 1.6. *At the end of Section 5 we will discuss the size of the singular set* $\Omega - \Omega_0^*$ *with the result that* $\mathcal{H}^{3/(p+1)}(\Omega - \Omega_0^*) = 0$, *and from the definition of* Ω_0^* *a priori no better estimate is available. This also shows (besides the stronger requirement (1.7) concerning* f*) that the removal of the condition* $p > 6/5$ *is not for free.*

REMARK 1.7. *As outlined in [Kn], Section 1.3.2, under appropriate boundary conditions the function* u *from a pair* (σ, u) *of weak solutions of (1.1) and (1.2) is a minimizer* *of the total deformation energy E, whose leading part is given by $E(v) = \int_\Omega W_{\mathrm{el}}(\varepsilon(v))\,\mathrm{d}x$. Here W_{el} denotes the stored energy density for Ramberg/Osgood materials being defined as the conjugate function of W, i.e.*

$$W_{\mathrm{el}}(\varepsilon) := W^*(\varepsilon) := \sup_{\tau\in\mathbb{S}^3}(\varepsilon : \tau - W(\tau)), \qquad \varepsilon \in \mathbb{S}^3.$$

In Lemma 1.22 of [Kn] the growth of W_{el} is analyzed leading to the inequalities

$$c_0 + c_1|\operatorname{tr}\varepsilon|^2 + c_2|\varepsilon^D|^p \le W_{\mathrm{el}}(\varepsilon) \le c_3|\operatorname{tr}\varepsilon|^2 + c_4|\varepsilon^D|^p \tag{1.8}$$

with constants $c_0 \in \mathbb{R}$, $c_i > 0$, $i = 1, \ldots, 4$. In his paper [Se] on the theory of plastic deformations with power-law hardening, Seregin – besides other things – considers the minimization problem in $U^{p,2}(\Omega)$ (plus boundary conditions) for the energy (K_0 a positive constant)

$$J[v] = \int_\Omega \Big[\frac{K_0}{2}(\operatorname{div} v)^2 + g_0(|\varepsilon^D(v)|)\Big]\,\mathrm{d}x$$

with a function g_0 of class C^2 such that $g''(t)$ behaves like $(1+t^2)^{(p-2)/2}$ for some $p \in (1,2)$. Theorem 1.3 of [Se] then gives partial C^1-regularity of a J-minimizer $\overline{u}$, and the growth of g'' implies an estimate like (1.8) for the density of the energy J. So one might think of applying Seregin's result to the E-minimizer u. But in general, neither W_{el} can be computed explicitely, nor does $D^2W_{\mathrm{el}}(\varepsilon)(\overline{\varepsilon},\overline{\varepsilon})$ behave like $(\operatorname{tr}\overline{\varepsilon})^2 + (1+|\varepsilon^D|^2)^{(p-2)/2}|\overline{\varepsilon}^D|^2$, which is true for the density of J, and which of course is strongly used throughout Seregin's proof. We wish to remark that only in the exceptional case, that the tensor A is a constant multiple of the identity, our situation can be reduced to the setting studied in [Se]: if for example

$$W(\tau) = \frac{\beta}{2}|\tau|^2 + \frac{\alpha}{q}|\tau^D|^q, \qquad \beta > 0,$$

then

$$W_{\mathrm{el}}(\varepsilon) = \frac{1}{2}\frac{1}{3\beta}(\operatorname{tr}\varepsilon)^2 + \Big[\frac{1}{2}\beta\psi(|\varepsilon^D|)^2 + \alpha(1-q^{-1})\psi(|\varepsilon^D|)^q\Big]$$

with $\psi := \varphi^{-1}$, $\varphi(t) := \beta t + \alpha t^{q-1}$, and (1.1) of [Se] holds for all $t \ge 0$. For $A \ne \beta\mathbf{1}$ it is impossible to get such a representation of $W_{\mathrm{el}}(\varepsilon)$, and from the general formula for D^2W_{el} (see, e.g. [Kn], (A.12)), i.e. from the identity $D^2W^*(\varepsilon) = (D^2W(DW^*(\varepsilon)))^{-1}$, it is not clear how to derive a suitable ellipticity estimate for D^2W_{el}.

REMARK 1.8. *From the proofs of Theorems 1.1 and 1.2 it will become evident that relation (1.1) can be replaced by the slightly more general equation $\varepsilon(u) = DF(\sigma)$ for a potential $F\colon \mathbb{S}^3 \to \mathbb{R}$ having the form $F(\sigma) = F_1(\sigma) + F_2(\sigma^D)$ with C^2-functions F_1, F_2 for which*

$$\lambda_1|\tau|^2 \le D^2F_1(\sigma)(\tau,\tau) \le \overline{\lambda}_1|\tau|^2,$$
$$\lambda_2|\tau^D|^2(\kappa + |\sigma^D|^2)^{\frac{q-2}{2}} \le D^2F_2(\sigma^D)(\tau^D,\tau^D) \le \overline{\lambda}_2|\tau^D|^2(\kappa + |\sigma^D|^2)^{\frac{q-2}{2}}$$

holds with constants λ_1, $\overline{\lambda}_1$, λ_2, $\overline{\lambda}_2 > 0$ and another constant $\kappa \ge 0$.

REMARK 1.9. *The statement of Theorem 1.2 holds in the case that $\Omega \subset \mathbb{R}^d$ with dimension $d \ge 4$.*

Our paper is organized as follows: the proof of Theorem 1.1 uses a blow-up argument which is presented in Section 2 and in Section 3. The iteration of this process is shortly sketched in Section 4 and finally leads to the formula for the regular set given in Remark 1.1. Moreover, Section 4 contains the estimate for the Hausdorff-dimension of the singular set. In Section 5 we will prove Theorem 1.2 by indicating the changes which have to be carried out in Sections 2 and 3.

2 Blow-up: scaling and properties of the weak limit

Let the assumptions of Theorem 1.1 hold and for technical simplicity replace (1.5) by the stronger condition that $f \in L^\infty(\Omega)$ – otherwise we consider a domain Ω' with compact closure in Ω. Crucial for the proof of Theorem 1.1 is the following

LEMMA 2.1. *Given a positive number L, define the constant $C_* = C_*(L)$ according to (2.25). Then, for any $\tau \in (0,1)$ there exists $\varepsilon = \varepsilon(\tau, L) > 0$ such that the validity of*

$$|(\sigma)_{x,r}| \le L \quad \text{and} \quad E(x,r) \le \varepsilon \tag{2.1}$$

for some ball $B_r(x) \Subset \Omega$ implies the estimate

$$\tilde{E}(x,\tau r) \le C_* \tau^2 E(x,r). \tag{2.2}$$

Here we have abbreviated

$$\begin{aligned} \tilde{E}(x,r) &:= \fint_{B_r(x)} |\sigma - (\sigma)_{x,r}|^2 \,\mathrm{d}y + \fint_{B_r(x)} |\sigma^D - (\sigma^D)_{x,r}|^q \,\mathrm{d}y, \\ E(x,r) &:= \tilde{E}(x,r) + r^{2\mu}. \end{aligned}$$

In the definition of $E(x,r)$ the exponent μ denotes any number in $(0,1)$ which will be fixed from now on. Actually, the quantity ε will also depend on this parameter.

The *proof of Lemma 2.1* argues by contradiction. So let us suppose that the statement is wrong. Then, for $L > 0$ fixed and for some $\tau \in (0,1)$ there exists a sequence of balls $B_{r_m}(x_m) \Subset \Omega$ such that

$$|(\sigma)_{x_m,r_m}| \le L\,, \quad E(x_m,r_m) =: \lambda_m^2 \to 0 \tag{2.3}$$

as $m \to \infty$ but

$$\tilde{E}(x_m,\tau r_m) > C_* \tau^2 \lambda_m^2. \tag{2.4}$$

We introduce the scaled functions ($z \in B_1 = B_1(0)$)

$$\begin{aligned} \sigma_m(z) &:= \frac{1}{\lambda_m}(\sigma(x_m + r_m z) - \omega_m), \quad \omega_m := (\sigma)_{x_m,r_m}, \\ u_m(z) &:= \frac{1}{\lambda_m r_m}\big(u(x_m + r_m z) - (u)_{x_m,r_m} - r_m(\varepsilon(u))_{x_m,r_m} z\big) \end{aligned}$$

and get from (2.3)

$$\fint_{B_1} |\sigma_m|^2 \,\mathrm{d}z + \lambda_m^{q-2} \fint_{B_1} |\sigma_m^D|^q \,\mathrm{d}z + \frac{r_m^{2\mu}}{\lambda_m^2} = 1, \tag{2.5}$$

whereas (2.4) implies

$$\fint_{B_\tau} |\sigma_m - (\sigma_m)_{0,\tau}|^2 \,\mathrm{d}z + \lambda_m^{q-2} \fint_{B_\tau} |\sigma_m^D - (\sigma_m^D)_{0,\tau}|^q \,\mathrm{d}z \geq C_* \tau^2. \tag{2.6}$$

Thus, after passing to subsequences, we deduce from (2.3) and (2.5)

$$\begin{aligned} \omega_m &\to: \overline{\omega} \quad \text{in } \mathbb{S}^3; && (2.7)\\ \sigma_m &\rightharpoondown: \overline{\sigma} \quad \text{in } L^2(B_1) \quad \text{and} \quad \lambda_m \sigma_m \to 0 \quad \text{a.e.}; && (2.8)\\ \lambda_m^{1-\frac{2}{q}} \sigma_m^D &\rightharpoondown 0 \quad \text{in } L^q(B_1). && (2.9) \end{aligned}$$

For (2.9) we observe the boundedness of $\lambda_m^{1-2/q}\sigma_m^D$ in $L^q(B_1)$, hence $\lambda_m^{1-2/q}\sigma_m^D \rightharpoondown: \tau$ in $L^q(B_1)$, but the first part of (2.8) gives $\tau = 0$. We also claim that

$$u_m \rightharpoondown: \overline{u} \quad \text{in } W^{1,p}(B_1), \quad \overline{u} \in U^{p,2}(B_1) \quad \text{and} \quad \operatorname{div} u_m \rightharpoondown \operatorname{div} \overline{u} \quad \text{in } L^2(B_1). \tag{2.10}$$

To prove (2.10) we observe that from (1.4) it follows that ($\varepsilon_m := (\varepsilon(u))_{x_m, r_m}$)

$$\begin{aligned} \varepsilon(u_m)(z) &= \frac{1}{\lambda_m}\big[A(\lambda_m\sigma_m(z) + \omega_m) + \alpha|\lambda_m\sigma_m^D(z) + \omega_m^D|^{q-2}(\lambda_m\sigma_m^D(z) + \omega_m^D) - \varepsilon_m\big]\\ &= A\sigma_m(z) + \frac{1}{\lambda_m}\big[DW_0(\lambda_m\sigma_m^D(z) + \omega_m^D) - (DW_0(\lambda_m\sigma_m^D + \omega_m^D))_{0,1}\big], \end{aligned}$$

where $W_0(\tau) := \frac{\alpha}{q}|\tau|^q$. Since $\operatorname{tr}[\dots] = 0$, it follows $\operatorname{div} u_m = \operatorname{tr}(A\sigma_m)$, hence (see (2.8))

$$\sup_m \int_{B_1} (\operatorname{div} u_m)^2 \,\mathrm{d}z < \infty. \tag{2.11}$$

Now we write

$$\begin{aligned} \varepsilon(u_m)(z) &= A\sigma_m(z) + \frac{1}{\lambda_m}\big[DW_0(\lambda_m\sigma_m^D(z) + \omega_m^D) - DW_0(\omega_m^D)\big]\\ &\quad + \frac{1}{\lambda_m}\big[DW_0(\omega_m^D) - (DW_0(\lambda_m\sigma_m^D + \omega_m^D))_{0,1}\big]\\ &=: A\sigma_m(z) + T_1 + T_2,\\ T_1 &= \frac{1}{\lambda_m}\int_0^1 D^2W_0(\omega_m^D + s\lambda_m\sigma_m^D(z))(\lambda_m\sigma_m^D(z), \cdot)\,\mathrm{d}s,\\ T_2 &= -\frac{1}{\lambda_m}\fint_{B_1}\big[DW_0(\lambda_m\sigma_m^D(z) + \omega_m^D) - DW_0(\omega_m^D)\big]\,\mathrm{d}z. \end{aligned}$$

We have the formula

$$D^2W_0(\eta)(\theta,\tau) = \alpha|\eta^D|^{q-2}\theta^D : \tau^D + \alpha(q-2)|\eta^D|^{q-4}(\eta^D : \theta^D)(\eta^D : \tau^D), \tag{2.12}$$

η, θ, $\tau \in \mathbb{S}^3$, thus

$$|T_1| \leq c\int_0^1 |\omega_m^D + s\lambda_m\sigma_m^D(z)|^{q-2}|\sigma_m^D(z)|\,\mathrm{d}s,$$

and by (2.7) and (2.9) we get that $T_1 = T_1^m(z)$ stays bounded in $L^p(B_1)$ uniformly w.r.t. m. With a similar argument we deduce that $T_2 = T_2^m$ is a bounded sequence in $\mathbb{S}^3$. Recalling (2.11), (2.10) is established.

We remark that up to now the condition (1.6) has not been used. This is also true for

PROPOSITION 2.1. *(limit equations)*
The weak limits $\overline{u}$ and $\overline{\sigma}$ satisfy

$$\int_{B_1} D^2W(\overline{\omega})(\overline{\sigma},\tau)\,\mathrm{d}z = \int_{B_1} \varepsilon(\overline{u}) : \tau\,\mathrm{d}z, \tag{2.13}$$

$$\int_{B_1} \overline{\sigma} : \varepsilon(w)\,\mathrm{d}z = 0 \tag{2.14}$$

for any $\tau \in L^2(B_1)$ and all $w \in \overset{\circ}{W}{}^{1,2}(B_1)$. Moreover, $\varepsilon(\overline{u})$ is in the space $L^2(B_1)$.

Proof of Proposition 2.1. Consider $\tau \in L^{q,2}(B_1)$ and observe that (1.1) implies after scaling

$$\int_{B_1} \frac{1}{\lambda_m}\big[DW(\lambda_m\sigma_m + \omega_m) - (DW(\sigma))_{x_m,r_m}\big] : \tau\,\mathrm{d}z = \int_{B_1} \varepsilon(u_m) : \tau\,\mathrm{d}z, \tag{2.15}$$

where we have replaced $\varepsilon_m = (\varepsilon(u))_{x_m,r_m}$ by $(DW(\sigma))_{x_m,r_m}$. We claim

$$\lim_{m\to\infty} \int_{B_1} \varepsilon(u_m) : \tau\,\mathrm{d}z = \int_{B_1} \varepsilon(\overline{u}) : \tau\,\mathrm{d}z. \tag{2.16}$$

In fact, we may write

$$\int_{B_1} \varepsilon(u_m) : \tau\,\mathrm{d}z = \frac{1}{3}\int_{B_1} \operatorname{div} u_m \operatorname{tr}\tau\,\mathrm{d}z + \int_{B_1} \varepsilon^D(u_m) : \tau^D\,\mathrm{d}z,$$

and (2.16) follows from (2.10). For discussing the l.h.s. of (2.15) we observe

$$\begin{aligned}
&\int_{B_1} \frac{1}{\lambda_m}\big[DW(\lambda_m\sigma_m + \omega_m) - (DW(\sigma))_{x_m,r_m}\big] : \tau\,\mathrm{d}z\\
&= \int_{B_1} \frac{1}{\lambda_m}\big[DW(\lambda_m\sigma_m + \omega_m) - DW(\omega_m)\big] : \tau\,\mathrm{d}z\\
&\quad + \int_{B_1} \frac{1}{\lambda_m}\big[DW(\omega_m) - (DW(\sigma))_{x_m,r_m}\big] : \tau\,\mathrm{d}z =: T_1^* + T_2^*,
\end{aligned}$$

where we have

$$\begin{aligned}
T_1^* &= \frac{1}{\lambda_m}\int_{B_1}\int_0^1 D^2W(\omega_m + s\lambda_m\sigma_m)(\lambda_m\sigma_m,\tau)\,\mathrm{d}s\,\mathrm{d}z\\
&= \int_{B_1} D^2W(\omega_m)(\sigma_m,\tau)\,\mathrm{d}z\\
&\quad + \int_{B_1}\int_0^1 \big[D^2W(\omega_m + s\lambda_m\sigma_m) - D^2W(\omega_m)\big]\,\mathrm{d}s(\sigma_m,\tau)\,\mathrm{d}z =: T_3^* + T_4^*.
\end{aligned}$$

By (2.7) and (2.8) we have

$$\lim_{m\to\infty} T_3^* = \int_{B_1} D^2W(\overline{\omega})(\overline{\sigma},\tau)\,\mathrm{d}z. \tag{2.17}$$

Suppose that we are given $\delta > 0$. Then there exists $\delta' = \delta'(\delta)$ such that $\mathcal{L}^3(A) \le \delta'$ for a measurable subset $A \subset B_1$ implies

$$\int_A |\tau|^2 \,\mathrm{d}z \le \delta. \tag{2.18}$$

Since by (2.8) $\lambda_m \sigma_m \to 0$ a.e. we may apply Egoroff's theorem to find a subset S of B_1 such that $\mathcal{L}^3(B_1 - S) \le \delta'$ and $\lambda_m \sigma_m \to 0$ uniformly on S. This implies

$$\begin{aligned}
&\left| \int_S \int_0^1 \left[D^2 W(\omega_m + s\lambda_m \sigma_m) - D^2 W(\omega_m) \right] \mathrm{d}s (\sigma_m, \tau) \,\mathrm{d}z \right| \\
&\quad \le \sup_S \left| \int_0^1 \left[D^2 W(\omega_m + s\lambda_m \sigma_m) - D^2 W(\omega_m) \right] \mathrm{d}s \right| \int_{B_1} |\sigma_m||\tau| \,\mathrm{d}z \\
&\quad =: \theta_m \to 0 \quad \text{as } m \to \infty
\end{aligned}$$

since $\int_{B_1} |\sigma_m||\tau| \,\mathrm{d}z$ stays bounded. Moreover, recalling (2.12), we have

$$\begin{aligned}
&\int_{B_1 - S} \int_0^1 \left[D^2 W(\omega_m + s\lambda_m \sigma_m) - D^2 W(\omega_m) \right] \mathrm{d}s (\sigma_m, \tau) \,\mathrm{d}z \\
&\quad = \alpha \int_{B_1 - S} \int_0^1 \left[|\omega_m^D + s\lambda_m \sigma_m^D|^{q-2} - |\omega_m^D|^{q-2} \right] \mathrm{d}s \sigma_m^D : \tau^D \,\mathrm{d}z \\
&\qquad + \alpha(q-2) \int_{B_1 - S} \int_0^1 \ldots \,\mathrm{d}s \,\mathrm{d}z,
\end{aligned}$$

where the integrals on the r.h.s. are of the same type. If we split the integrand in $\alpha \int_{B_1 - S} \int_0^1 \ldots \,\mathrm{d}s \,\mathrm{d}z$, then we get the integrals

$$\int_{B_1 - S} |\sigma_m^D||\tau^D| \,\mathrm{d}z, \qquad \lambda_m^{q-2} \int_{B_1} |\sigma_m^D|^{q-1} |\tau^D| \,\mathrm{d}z$$

as an upper bound for the α-term. Since by (2.9)

$$\lambda_m^{q-2} \int_{B_1} |\sigma_m^D|^{q-1} |\tau^D| \,\mathrm{d}z \to 0,$$

we get (recall (2.8) and (2.18))

$$\begin{aligned}
&\limsup_{m \to \infty} \left| \int_{B_1 - S} \int_0^1 \left[D^2 W(\omega_m + s\lambda_m \sigma_m) - D^2 W(\omega_m) \right] \mathrm{d}s (\sigma_m, \tau) \,\mathrm{d}z \right| \\
&\qquad \le c \left[\int_{B_1 - S} |\tau|^2 \,\mathrm{d}z \right]^{\frac{1}{2}} \le c\sqrt{\delta},
\end{aligned}$$

and since δ is arbitrary, it is shown that

$$\lim_{m \to \infty} T_4^* = 0. \tag{2.19}$$

It remains to discuss the sequence $T_2^* = T_2^{*m} \in \mathbb{S}^3$. We have

$$\begin{aligned} T_2^* &= \int_{B_1} \tau \,\mathrm{d}z : \frac{1}{\lambda_m}\big[DW(\omega_m) - (DW(\omega_m + \lambda_m \sigma_m))_{0,1}\big] \\ &= -\fint_{B_1}\int_0^1 D^2W(\omega_m + s\lambda_m\sigma_m)(\sigma_m, \tau_0)\,\mathrm{d}s\,\mathrm{d}z, \end{aligned}$$

where $\tau_0 := \int_{B_1} \tau \,\mathrm{d}y$, hence

$$\begin{aligned} T_2^* &= -\fint_{B_1}\int_0^1 \big[D^2W(\omega_m + s\lambda_m\sigma_m) - D^2W(\omega_m)\big]\,\mathrm{d}s(\sigma_m, \tau_0)\,\mathrm{d}z \\ &\quad -\fint_{B_1}\int_0^1 D^2W(\omega_m)(\sigma_m, \tau_0)\,\mathrm{d}s\,\mathrm{d}z =: (a) + (b). \end{aligned}$$

The quantity (a) corresponds to T_4^* if we replace τ by the constant tensor τ_0 in this expression, therefore $\lim_{m\to\infty}(a) = 0$. Moreover,

$$(b) = -D^2W(\omega_m)\left(\fint_{B_1} \sigma_m \,\mathrm{d}z, \tau_0\right) = 0,$$

and if we combine (2.15), (2.16), (2.17) and (2.19) with the latter convergences, (2.13) is established with the restriction that τ is from the space $L^{q,2}(B_1)$. But (2.13) implies $\varepsilon(\overline{u}) = D^2W(\overline{\omega})(\overline{\sigma}, \cdot)$. Now, since the tensor $D^2W(\overline{\omega})$ is constant and strictly elliptic (see (1.3) and (2.12)), and since $\overline{\sigma}$ is in $L^2(B_1)$, it follows that $\varepsilon(\overline{u})$ also belongs to this space, and by approximation (2.13) is valid for $\tau \in L^2(B_1)$.

For proving (2.14) let us consider $w \in \overset{\circ}{W}{}^{1,2}(B_1)$. Then we get from (1.2) (note: $U_0^{p,2}(B_1) \supset \overset{\circ}{W}{}^{1,2}(B_1)$)

$$\int_{B_1} \sigma_m : \varepsilon(w)\,\mathrm{d}z = \lambda_m^{-1} r_m \int_{B_1} w \cdot f(x_m + r_m z)\,\mathrm{d}z,$$

hence

$$\begin{aligned} \left|\int_{B_1} \sigma_m : \varepsilon(w)\,\mathrm{d}z\right| &\le \|f\|_{L^\infty(\Omega)} \frac{r_m}{\lambda_m}\int_{B_1} |w|\,\mathrm{d}z \\ &\to 0 \quad \text{as } m \to \infty, \end{aligned}$$

since $r_m^2\lambda_m^{-2} = r_m^{2\mu}\lambda_m^{-2} r_m^{2-2\mu} \to 0$ by (2.5). (Note that on account of $\lambda_m^2 \to 0$ and $r_m^{2\mu}\lambda_m^{-2} \le 1$ it clearly holds that $r_m \to 0$.) On the other hand (2.8) gives

$$\int_{B_1} \sigma_m : \varepsilon(w)\,\mathrm{d}z \to \int_{B_1} \overline{\sigma} : \varepsilon(w)\,\mathrm{d}z,$$

hence (2.14) follows. □

In order to continue with the proof of Lemma 2.1 let us define the linear operator $\mathcal{A}$: $\mathbb{S}^3 \to \mathbb{S}^3$ through the equation

$$\mathcal{A}\eta : \overline{\eta} = D^2W(\overline{\omega})(\eta, \overline{\eta}), \qquad \eta, \overline{\eta} \in \mathbb{S}^3.$$

Then (2.13) implies $\varepsilon(\overline{u}) = \mathcal{A}\overline{\sigma}$, and we get from (2.14)

$$\int_{B_1} \mathcal{A}^{-1}\varepsilon(\overline{u}) : \varepsilon(w)\,\mathrm{d}z = 0 \quad \text{for all } w \in \overset{\circ}{W}{}^{1,2}(B_1). \tag{2.20}$$

But (2.20) is a linear elliptic system with constant coefficients depending on $\overline{\omega}$ whose upper and lower ellipticity bounds are determined by L (and λ, q). Quoting Campanato-type estimates stated for example in [GM] or [FuS], we get

$$\fint_{B_\tau} |\varepsilon(\overline{u}) - (\varepsilon(\overline{u}))_{0,\tau}|^2\,\mathrm{d}z \le C_1(L)\tau^2 \fint_{B_1} |\varepsilon(\overline{u}) - (\varepsilon(\overline{u}))_{0,1}|^2\,\mathrm{d}z, \tag{2.21}$$

where $C_1(L)$ is a constant depending on L and where $(\varepsilon(\overline{u}))_1 = \lim_{m\to\infty}(\varepsilon(u_m))_{0,1} = 0$. Retransformation of (2.21) shows

$$\fint_{B_\tau} |\overline{\sigma} - (\overline{\sigma})_{0,\tau}|^2\,\mathrm{d}z \le C_2(L)\tau^2 \fint_{B_1} |\overline{\sigma} - (\overline{\sigma})_{0,1}|^2\,\mathrm{d}z, \tag{2.22}$$

where as above $(\overline{\sigma})_1 = 0$. Suppose now that we know the strong convergences

$$\sigma_m \to \overline{\sigma} \quad \text{in } L^2_{\mathrm{loc}}(B_1), \qquad \lambda_m^{1-\frac{2}{q}}\sigma_m^D \to 0 \quad \text{in } L^q_{\mathrm{loc}}(B_1). \tag{2.23}$$

(For the proof of (2.23) we will make use of $p > 6/5$.) Then, by lower semicontinuity, we first observe that (2.5) and (2.8) imply

$$\fint_{B_1} |\overline{\sigma}|^2\,\mathrm{d}z \le 1, \tag{2.24}$$

whereas (2.23) together with the choice

$$C_*(L) := 2C_2(L) \tag{2.25}$$

turns (2.6) into

$$\fint_{B_\tau} |\overline{\sigma} - (\overline{\sigma})_{0,\tau}|^2\,\mathrm{d}z \ge 2C_2(L)\tau^2.$$

This clearly is in contradiction to (2.22) if we use (2.24) on the r.h.s. of (2.22). Thus the proof of Lemma 2.1 is complete with the exception that (2.23) has to be established which is done in the next section. □

3 Blow-up: strong convergences of the scaled sequence

First we note that the the limit function $\overline{u}$ introduced in Proposition 2.1 actually is of class $C^\infty(B_1)$, and since $\overline{\sigma} = \mathcal{A}^{-1}\varepsilon(\overline{u})$, the same is true for $\overline{\sigma}$. We return to (2.15) choosing $\tau := \eta^2(\sigma_m - \overline{\sigma}) \in L^{q,2}(B_1)$ with $\eta \in C_0^\infty(B_1)$, $0 \le \eta \le 1$. We get

$$\begin{aligned}
&\int_{B_1} \frac{1}{\lambda_m}\big[DW(\lambda_m\sigma_m + \omega_m) - DW(\lambda_m\overline{\sigma} + \omega_m)\big] : \eta^2(\sigma_m - \overline{\sigma})\,\mathrm{d}z\\
&+ \int_{B_1} \frac{1}{\lambda_m}\big[DW(\lambda_m\overline{\sigma} + \omega_m) - (DW(\sigma))_{x_m,r_m}\big] : \eta^2(\sigma_m - \overline{\sigma})\,\mathrm{d}z\\
&= \int_{B_1} \varepsilon(u_m) : \eta^2(\sigma_m - \overline{\sigma})\,\mathrm{d}z,
\end{aligned}$$

or equivalently ($X_m := \omega_m + \lambda_m\overline{\sigma} + s\lambda_m(\sigma_m - \overline{\sigma})$)

$$\begin{aligned}
&\int_{B_1}\int_0^1 D^2W(X_m)(\sigma_m - \overline{\sigma}, \sigma_m - \overline{\sigma})\eta^2 \,\mathrm{d}s\,\mathrm{d}z \\
&= \; -\int_{B_1} \frac{1}{\lambda_m}\big[DW(\lambda_m\overline{\sigma} + \omega_m) - (DW(\sigma))_{x_m,r_m}\big] : \eta^2(\sigma_m - \overline{\sigma})\,\mathrm{d}z \\
&\quad + \int_{B_1} \varepsilon(u_m) : \eta^2(\sigma_m - \overline{\sigma})\,\mathrm{d}x =: -I_1 + I_2. \qquad (3.1)
\end{aligned}$$

We have by (2.14)

$$\begin{aligned}
I_2 &= \int_{B_1} \varepsilon(\eta^2 u_m) : (\sigma_m - \overline{\sigma})\,\mathrm{d}z - \int_{B_1} (u_m \odot \nabla\eta^2) : (\sigma_m - \overline{\sigma})\,\mathrm{d}z \\
&= \int_{B_1} \varepsilon(\eta^2 u_m) : \sigma_m\,\mathrm{d}z - \int_{B_1} (u_m \odot \nabla\eta^2) : (\sigma_m - \overline{\sigma})\,\mathrm{d}z \\
&=: J_1 - J_2,
\end{aligned}$$

and as in the proof of (2.14)

$$|J_1| \leq \|f\|_{L^\infty(\Omega)}\frac{r_m}{\lambda_m}\int_{B_1}\eta^2|u_m|\,\mathrm{d}z \to 0 \quad \text{as } m \to \infty,$$

whereas $J_2 \to 0$ follows from $\sigma_m \rightharpoondown \overline{\sigma}$ in $L^2(B_1)$ together with $u_m \to \overline{u}$ in $L^2(B_1)$ (recall (1.6)), hence

$$\lim_{m\to\infty} I_2 = 0. \qquad (3.2)$$

We discuss I_1:

$$\begin{aligned}
-I_1 &= -\frac{1}{\lambda_m}\int_{B_1}\big[DW(\omega_m) - (DW(\sigma))_{x_m,r_m}\big] : \eta^2(\sigma_m - \overline{\sigma})\,\mathrm{d}z \\
&\quad -\frac{1}{\lambda_m}\int_{B_1}\big[DW(\omega_m + \lambda_m\overline{\sigma}) - DW(\omega_m)\big] : \eta^2(\sigma_m - \overline{\sigma})\,\mathrm{d}z \\
&=: -H_1 - H_2,
\end{aligned}$$

where we have

$$\begin{aligned}
H_2 &= \int_{B_1}\eta^2\int_0^1 D^2W(\omega_m + s\lambda_m\overline{\sigma})\,\mathrm{d}s(\overline{\sigma}, \sigma_m - \overline{\sigma})\,\mathrm{d}z \\
&\to 0 \quad \text{as } m \to \infty,
\end{aligned}$$

which follows from $D^2W(\omega_m + s\lambda_m\overline{\sigma}) \to D^2W(\overline{\omega})$ uniformly on $[0,1] \times \operatorname{spt}\eta$ and $\sigma_m \rightharpoondown \overline{\sigma}$ in $L^2(B_1)$. Moreover,

$$H_1 = \int_{B_1}\eta^2(\sigma_m - \overline{\sigma})\,\mathrm{d}z : \frac{1}{\lambda_m}\big[DW(\omega_m) - (DW(\sigma))_{x_m,r_m}\big] \to 0$$

since again $\sigma_m \rightharpoondown \sigma$ in $L^2(B_1)$ and $\lambda_m^{-1}[\dots]$ can be discussed with the same arguments as used for the quantity T_2^* after (2.19). This shows

$$\lim_{m\to\infty} I_1 = 0. \qquad (3.3)$$

Combining (3.1)–(3.3) we find

$$\lim_{m\to\infty}\int_{B_1}\int_0^1 D^2W(X_m)(\sigma_m-\overline{\sigma},\sigma_m-\overline{\sigma})\eta^2\,\mathrm{d}s\,\mathrm{d}z=0. \tag{3.4}$$

The formula for D^2W together with (3.4) first shows that

$$\int_{B_1}\eta^2 A(\sigma_m-\overline{\sigma}):(\sigma_m-\overline{\sigma})\,\mathrm{d}z\to 0,$$

hence $\sigma_m\to\overline{\sigma}$ in $L^2_{\mathrm{loc}}(B_1)$ as $m\to\infty$. From formula (2.12), (3.4) and the definition of X_m we also get

$$\lim_{m\to\infty}\int_{B_1}\int_0^1\eta^2\big|\omega_m^D+\lambda_m\overline{\sigma}^D+s\lambda_m(\sigma_m^D-\overline{\sigma}^D)\big|^{q-2}|\sigma_m^D-\overline{\sigma}^D|^2\,\mathrm{d}s\,\mathrm{d}z=0. \tag{3.5}$$

Now we use the elementary inequality (τ, $\overline{\tau}\in\mathbb{S}^3$)

$$\int_0^1|\tau+s\overline{\tau}|^{q-2}\,\mathrm{d}s\ge\Big(\frac{1}{4}\Big)^{q-1}|\overline{\tau}|^{q-2}$$

in order to deduce from (3.5)

$$\lim_{m\to\infty}\int_{B_1}\eta^2\lambda_m^{q-2}|\sigma_m^D-\overline{\sigma}^D|^q\,\mathrm{d}z=0.$$

The local boundedness of $\overline{\sigma}$ then finally shows $\lambda_m^{1-2/q}\sigma_m^D\to 0$ in $L^q_{\mathrm{loc}}(B_1)$ and (2.23) is established. □

4 Iteration and proof of Theorem 1.1

Let us fix $\mu\in(0,1)$. We introduce the set

$$\Omega_0=\Big\{x\in\Omega:\ \limsup_{r\to 0}|(\sigma)_{x,r}|<\infty\,\text{and}\,\liminf_{r\to 0}\tilde{E}(x,r)=0\Big\},$$

and consider $x_0\in\Omega_0$. Let $L:=2\limsup_{r\to 0}|(\sigma)_{x_0,r}|$ and calculate $C_*=C_*(L)$ according to (2.25). Finally, τ is defined through $C_*\tau^2=1/2$, and by enlarging $C_*(L)$ we may assume that $\theta:=\tau^{2\mu}$ is in $(0,1/2)$. With L and τ fixed we can calculate $\varepsilon=\varepsilon(\tau,L)$ as in Lemma 2.1. Next we choose a radius R according to

$$\tilde{E}(x_0,R)+R^{2\mu}=E(x_0,R)<\overline{\varepsilon}^2,\qquad |(\sigma)_{x_0,R}|<\frac{2}{3}L, \tag{4.1}$$

where $\overline{\varepsilon}$ is determined by the requirement

$$\overline{\varepsilon}^2\le\min\Big\{\frac{1}{4},\frac{1-2\theta}{2}\Big\}\varepsilon^2,\qquad \tau^{-\frac{3}{2}}\sum_{i=0}^{\infty}2^{-\frac{i}{2}}\frac{1}{\sqrt{1-2\theta}}\overline{\varepsilon}<\frac{L}{3}. \tag{4.2}$$

Then we have

PROPOSITION 4.1. *If (4.1) holds with $\overline{\varepsilon}$ according to (4.2), then for any $k \in \mathbb{N}$*

$$\tilde{E}(x_0, \tau^k R) \le 2^{-k}\tilde{E}(x_0, R) + \sum_{j=1}^{k} 2^{-j}\theta^{k-j}R^{2\mu}. \tag{4.3}$$

Proof. If $k = 1$, then (4.1) and (4.2) clearly imply (2.1), thus by (2.2)

$$\tilde{E}(x_0, \tau R) \le C_* \tau^2 E(x_0, R) = \frac{1}{2}(\tilde{E}(x_0, R) + R^{2\mu}),$$

hence we have (4.3) for $k = 1$.

The inductive step is carried out exactly as in [FL], Proposition 5.2, using the inequality

$$|(\sigma)_{x_0,\tau^{l+1}R}| \le |(\sigma)_{x_0,R}| + \tau^{-\frac{3}{2}} \sum_{i=0}^{l} \tilde{E}(x_0, \tau^i R)^{\frac{1}{2}}.$$

We also note that in [FL] during the inductive step the following inequality is established (see (5.8) of [FL])

$$\tilde{E}(x_0, \tau^k R) \le 2^{-k}\Big[\tilde{E}(x_0, R) + \frac{1}{1-2\theta}R^{2\mu}\Big]. \tag{4.4}$$

Now it is standard to show (see, e.g. [Gi]) how to get from (4.4) the estimate

$$\tilde{E}(x_0, r) \le c\Big(\frac{r}{R}\Big)^{\beta}\big[\tilde{E}(x_0, R) + R^{2\mu}\big] \tag{4.5}$$

for some exponent $\beta \in (0,1)$ and for radii $r \le R$. Recall that (4.5) is valid under the hypothesis (4.1). But (4.1) clearly holds for centers $\overline{x}_0$ close to x_0 (with R fixed), thus (4.5) is valid for all centers $\overline{x}_0 \in B_\rho(x_0)$, $\rho \ll 1$. In particular we get

$$⨍_{B_r(y)} |\sigma - (\sigma)_{y,r}|^2 \,\mathrm{d}x \le cr^{\beta}$$

for $y \in B_\rho(x_0)$ and $r \le R$, hence $\sigma \in C^{0,\beta/2}(B_\rho(x_0))$. The continuity of σ on $B_\rho(x_0)$ implies $B_\rho(x_0) \subset \Omega_0$, hence the set Ω_0 is open, and we haved proved the Hölder continuity of σ on Ω_0. Finally, $\mathcal{L}^3(\Omega - \Omega_0) = 0$ is immediate, and the continuity of $\varepsilon(u)$ on Ω_0 follows from (1.4).

In order to give the better estimate for the singular set stated in Remark 1.3, we observe that in [Kn] the following weak differentiability results are established:

$$\sigma \in W^{1,2}_{\mathrm{loc}}(\Omega), \quad |\sigma^D|^{\frac{q}{2}} \in W^{1,2}_{\mathrm{loc}}(\Omega) \quad \text{and} \quad |\sigma^D|^{\frac{q-2}{2}}\nabla\sigma^D \in L^2_{\mathrm{loc}}(\Omega). \tag{4.6}$$

From (4.6) it follows that $F := |\sigma^D|^{(q-2)/2}\sigma^D$ also is in the space $W^{1,2}_{\mathrm{loc}}(\Omega)$. The weak differentiability of σ implies (see [Gi], Theorem 2.1, p. 100)

$$\mathcal{H}^{1+\varepsilon}\Big(\Big\{x \in \Omega : \limsup_{r\to 0} |(\sigma)_{x,r}| = \infty\Big\}\Big) = 0$$

for any $\varepsilon > 0$. Moreover,

$$⨍_{B_r(x)} |\sigma - (\sigma)_{x,r}|^2 \,\mathrm{d}y \le cr^2 ⨍_{B_r(x)} |\nabla\sigma|^2 \,\mathrm{d}y \to 0$$

as $r \to 0$ for $\mathcal{H}^1$-a.a. $x \in \Omega$ ([Gi], Theorem 2.2, p. 101) and

$$\begin{aligned}
\fint_{B_r(x)} |\sigma^D - (\sigma^D)_{x,r}|^q \,\mathrm{d}y &\le \fint_{B_r(x)} |\sigma^D - \xi|^q \,\mathrm{d}y \\
&\le c \fint_{B_r(x)} |F - |\xi|^{\frac{q-2}{2}}\xi|^2 \,\mathrm{d}y \\
&\le cr^2 \fint_{B_r(x)} |\nabla F|^2 \,\mathrm{d}y \to 0
\end{aligned}$$

as $r \to 0$ again for $\mathcal{H}^1$ a.a. $x \in \Omega$, provided we choose ξ in such a way that $|\xi|^{(q-2)/2}\xi = (F)_{x,r}$. This implies $\mathcal{H} - \dim(\Omega - \Omega_0) \le 1$ and completes the proof of Theorem 1.1 and of the subsequent remarks. □

5 Proof of Theorem 1.2

Assume that the hypotheses of Theorem 1.2 hold, and define the new excess functions on balls $B_r(x) \subset \Omega$

$$\begin{aligned}
\tilde{e}(x,r) &:= \tilde{E}(x,r) + \fint_{B_r(x)} |\varepsilon(u) - (\varepsilon(u))_{x,r}|^2 \,\mathrm{d}y, \\
e(x,r) &:= \tilde{e}(x,r) + r^{2\mu}
\end{aligned}$$

with $0 < \mu < 1$ and with E, $\tilde{E}$ according to Lemma 2.1. Then we claim that Lemma 2.1 remains true with e replacing E and $\tilde{e}$ replacing $\tilde{E}$. If this is not the case, then we get (2.3) and (2.4) for the modifies excess functions, and if we define σ_m, u_m as before, (2.5) has to be replaced by

$$\fint_{B_1} |\sigma_m|^2 \,\mathrm{d}z + \lambda_m^{q-2} \fint_{B_1} |\sigma_m^D|^q \,\mathrm{d}z + \fint_{B_1} |\varepsilon(u_m)|^2 \,\mathrm{d}z + \frac{r_m^{2\mu}}{\lambda_m^2} = 1, \tag{5.1}$$

whereas (2.6) now reads

$$\fint_{B_\tau} |\sigma_m - (\sigma_m)_{0,\tau}|^2 \,\mathrm{d}z + \lambda_m^{q-2} \fint_{B_\tau} |\sigma_m^D - (\sigma_m^D)_{0,\tau}|^q \,\mathrm{d}z + \fint_{B_\tau} |\varepsilon(u_m) - (\varepsilon(u_m))_{0,\tau}|^2 \,\mathrm{d}z > C_* \tau^2. \tag{5.2}$$

(2.7)–(2.9) remain unchanged, and (2.10) can obviously be written as

$$u_m \rightharpoondown: \overline{u} \quad \text{in } W^{1,2}(B_1) \tag{5.3}$$

which follows directly from $\fint_{B_1} |\varepsilon(u_m)|^2 \,\mathrm{d}z \le 1$. The proof of the limit equations stated in Proposition 2.1 requires no changes, and as outlined at the end of Section 2 we will arrive at a contradiction to (5.2) if besides the strong convergences (2.23) we can show that

$$\varepsilon(u_m) \to \varepsilon(\overline{u}) \quad \text{in } L^2_{\mathrm{loc}}(B_1) \tag{5.4}$$

is true. Note that from (5.3) we already get

$$u_m \to \overline{u} \quad \text{in } L^2(B_1), \tag{5.5}$$

and therefore

$$\lim_{m\to\infty} \int_{B_1} (u_m \odot \nabla\eta^2) : (\sigma_m - \overline{\sigma}) \, \mathrm{d}z = 0,$$

which gives (3.2) without the hypothesis $p > 6/5$. This means that the modified assumption (5.1) closes the gap in the proof of (2.23) for exponents $p \le 6/5$ but at the same time requires the proof of a second strong convergence leading to the desired contradiction. We like to remark explicitely that all the other calculations needed for (2.23) remain unchanged – they just use the convergences (2.7) – (2.9).

In order to derive (5.4), we will apply a scaled version of the Caccioppoli-type inequality (summation w.r.t. i)

$$\int_\Omega \varphi^2 D^2W(\sigma)(\partial_i\sigma, \partial_i\sigma) \, \mathrm{d}x \le -\int_\Omega \partial_i\sigma : (\nabla\varphi^2 \odot \partial_i\tilde{u}) \, \mathrm{d}x + \int_\Omega \partial_i f \cdot \partial_i\tilde{u}\varphi^2 \, \mathrm{d}x \tag{5.6}$$

valid for any $\varphi \in C_0^\infty(\Omega)$. Here we have abbreviated $\tilde{u}(x) = u(x) - Px$ for an arbitrary matrix $P \in \mathbb{R}^{3\times 3}$. The proof of (5.6) is postponed to the end of this section.

Now let $\psi \in C_0^\infty(B_1)$ and choose $P := (\varepsilon(u))_{x_m,r_m} + \lambda_m R$ for a rigid motion R. From (5.6) with $\varphi(x) = \psi((x - x_m)/r_m)$ we get using the boundedness of $|\nabla f|$ (w.l.o.g. we assume that f is in the global space $W^{1,\infty}(\Omega)$)

$$\begin{aligned}
&\int_{B_1} \psi^2 D^2W(\lambda_m\sigma_m + \omega_m)(\partial_i\sigma_m, \partial_i\sigma_m) \, \mathrm{d}z \\
&\quad\le \int_{B_1} |\nabla\psi^2||\nabla\sigma_m||\nabla u_m - R| \, \mathrm{d}z + c\frac{r_m^2}{\lambda_m}\int_{B_1} |\nabla u_m - R| \, \mathrm{d}z \\
&\quad\le c\Bigg[\delta\int_{B_1} \psi^2|\nabla\sigma_m|^2 \, \mathrm{d}z + \frac{1}{\delta}\int_{B_1} |\nabla\psi|^2|\nabla u_m - R|^2 \, \mathrm{d}z \\
&\qquad + \frac{r_m^2}{\lambda_m}\left(\int_{B_1} |\nabla u_m - R|^2 \, \mathrm{d}z + 1\right)\Bigg],
\end{aligned}$$

where $\delta \in (0, 1)$ is arbitrary. For δ small enough, the first term on the r.h.s. can be absorbed in the l.h.s., hence

$$\begin{aligned}
&\int_{B_1} \psi^2 D^2W(\lambda_m\sigma_m + \omega_m)(\partial_i\sigma_m, \partial_i\sigma_m) \, \mathrm{d}z \\
&\quad\le c\left[\|\nabla\psi\|_\infty^2 \int_{B_1} |\nabla u_m - R|^2 \, \mathrm{d}z + \frac{r_m^2}{\lambda_m}\int_{B_1} |\nabla u_m - R|^2 \, \mathrm{d}z + \frac{r_m^2}{\lambda_m}\right],
\end{aligned} \tag{5.7}$$

and we have proved

$$\int_{B_\rho} D^2W(\lambda_m\sigma_m + \omega_m)(\partial_i\sigma_m, \partial_i\sigma_m) \, \mathrm{d}z \le c(\rho) < \infty \tag{5.8}$$

with $c(\rho)$ independent of m for any $\rho \in (0, 1)$ provided that we can bound $\int_{B_1} |\nabla u_m - R|^2 \, \mathrm{d}z$. But by Korn's inequality we have

$$\|\nabla(u_m - Rz)\|_2 \le c\big[\|u_m - Rz\|_2 + \|\varepsilon(u_m)\|_2\big],$$

moreover(see [FuS], Lemma 3.0.3 ii), p. 137)

$$\inf_R \|u_m - Rz\|_2 \le c\|\varepsilon(u_m)\|_2,$$

where the infimum is taken w.r.t. all rigid motions. So if we apply these observations on the r.h.s. of (5.7) and use $r_m^2/\lambda_m \to 0$ together with $\fint_{B_1} |\varepsilon(u_m)|^2 \, \mathrm{d}z \le 1$, then (5.8) follows. Note that (5.8) implies $(0 < \rho < 1)$

$$\int_{B_\rho} \left[|\nabla\sigma_m|^2 + |\lambda_m\sigma_m^D + \omega_m^D|^{q-2}|\nabla\sigma_m^D|^2\right] \mathrm{d}z \le c(\rho), \tag{5.9}$$

and that (5.9) together with $\fint_{B_1} |\sigma_m|^2 \, \mathrm{d}z \le 1$ gives

$$\sup_m \|\sigma_m\|_{W^{1,2}(B_\rho)} \le c(\rho) < \infty,$$

so that the known strong convergence $\sigma_m \to \overline{\sigma}$ in $L^2_{\mathrm{loc}}(B_1)$ (see (2.23)) is established in a different way.

After these preparations we now turn to the proof of (5.4). First we use $\sigma_m \to \overline{\sigma}$ a.e. and the formula (see after (2.11))

$$\varepsilon(u_m) = A\sigma_m + T_1 + T_2 \tag{5.10}$$

to establish

$$\varepsilon(u_m) \to \varepsilon(\overline{u}) \quad \text{a.e. on } B_1. \tag{5.11}$$

Obviously

$$T_1 \to D^2W_0(\overline{\omega}^D)(\overline{\sigma}^D, \cdot)$$

a.e. (recall (2.7) and $\lambda_m\sigma_m^D(z) \to 0$ for a.a. $z \in B_1$), moreover

$$\begin{aligned} T_2 &= -\frac{1}{\lambda_m}\fint_{B_1}\left[DW_0(\omega_m^D + \lambda_m\sigma_m^D) - DW_0(\omega_m^D)\right] \mathrm{d}z \\ &= -\fint_{B_1}\int_0^1 D^2W_0(\omega_m^D + s\lambda_m\sigma_m^D)(\sigma_m^D, \cdot)\, \mathrm{d}s\, \mathrm{d}z \\ &= -\fint_{B_1}\int_0^1 \left[D^2W_0(\omega_m^D + s\lambda_m\sigma_m^D)(\sigma_m^D, \cdot) - D^2W_0(\omega_m^D)(\sigma_m^D, \cdot)\right] \mathrm{d}s\, \mathrm{d}z \\ &\quad -\fint_{B_1} D^2W_0(\omega_m^D)(\sigma_m^D, \cdot)\, \mathrm{d}z, \end{aligned}$$

and the last integral vanishes on account of $(\sigma_m)_{0,1} = 0$. If $\tau_0 \in \mathbb{S}^3$ is arbitrary, then $T_2 : \tau_0$ exactly corresponds to the quantity T_2^* introduced after (2.19), hence

$$\lim_{m\to\infty} T_2 : \tau_0 = 0$$

for any τ_0, thus $T_2 \to 0$. Recalling the decomposition (5.10) we have shown

$$\varepsilon(u_m) \to A\overline{\sigma} + D^2W_0(\overline{\omega}^D)(\overline{\sigma^D}, \cdot) = \varepsilon(\overline{u})$$

a.e. (see (2.13)) and (5.11) follows.

To proceed further, we claim that

$$\sup_m \int_{B_\rho} |\varepsilon(u_m)|^3 \,\mathrm{d}z \le c(\rho) < \infty \tag{5.12}$$

for any $\rho < 1$. First, from $\sup_m \|\sigma_m\|_{W^{1,2}(B_\rho)} \le c(\rho)$ and Sobolev's embedding theorem we find

$$\sup_m \int_{B_\rho} |\sigma_m|^6 \,\mathrm{d}z \le c(\rho) < \infty. \tag{5.13}$$

Second, T_2 is just a sequence of tensors in $\mathbb{S}^3$ with limit zero, hence $|T_2| \le 1$ for $m \gg 1$, and according to the decomposition (5.10) we have to show that

$$\sup_m \int_{B_\rho} |T_1|^3 \,\mathrm{d}z \le c(\rho) < \infty. \tag{5.14}$$

Let

$$\varphi_m = \frac{1}{\lambda_m}\big[|\omega_m^D + \lambda_m \sigma_m^D|^{\frac{q}{2}} - |\omega_m^D|^{\frac{q}{2}}\big].$$

Then

$$\begin{aligned} |\varphi_m| &\le c\big[|\sigma_m^D| + \lambda_m^{\frac{q-2}{2}} |\sigma_m^D|^{\frac{q}{2}}\big], \\ |\nabla\varphi_m| &\le c|\omega_m^D + \lambda_m\sigma_m^D|^{\frac{q-2}{2}} |\nabla\sigma_m^D|, \end{aligned}$$

and by (5.9) we see

$$\sup_m \int_{B_\rho} |\nabla\varphi_m|^2 \,\mathrm{d}z \le c(\rho), \tag{5.15}$$

whereas (5.1) implies

$$\sup_m \int_{B_\rho} |\varphi_m|^2 \,\mathrm{d}z \le c(\rho), \tag{5.16}$$

and therefore we deduce from (5.15), (5.16)

$$\sup_m \|\varphi_m\|_{W^{1,2}(B_\rho)} \le c(\rho) < \infty,$$

so that in the end

$$\sup_m \int_{B_\rho} |\varphi_m|^6 \,\mathrm{d}z \le c(\rho) < \infty. \tag{5.17}$$

From the definition of T_1 we get

$$\begin{aligned} |T_1| &\le \int_0^1 \big|D^2 W_0(\omega_m^D + s\lambda_m\sigma_m^D)\big| |\sigma_m^D| \,\mathrm{d}s \\ &\le c\int_0^1 |\omega_m^D + s\lambda_m\sigma_m^D|^{q-2} |\sigma_m^D| \,\mathrm{d}s \\ &\le c\big[|\omega_m^D|^{q-2}|\sigma_m^D| + \lambda_m^{q-2}|\sigma_m^D|^{q-1}\big] \\ &\le c\big[|\sigma_m^D| + \lambda_m^{q-2}(|\sigma_m^D|^q + 1)\big], \end{aligned} \tag{5.18}$$

where we used the boundedness of ω_m. Fix a large number M and let $U_m := \{z \in B_\rho : \lambda_m|\sigma_m^D(z)| \le M\}$. On $B_\rho - U_m$ we have (using $|\omega_m| \le L$)

$$\varphi_m(z) \ge \frac{c}{\lambda_m}\lambda_m^{\frac{q}{2}}|\sigma_m^D|^{\frac{q}{2}} = c\lambda_m^{\frac{q}{2}-1}|\sigma_m^D|^{\frac{q}{2}},$$

so that

$$\int_{B_\rho - U_m} \varphi_m^6 \,\mathrm{d}z \ge c\int_{B_\rho - U_m} \left[\lambda_m^{q-2}|\sigma_m^D|^q\right]^3 \mathrm{d}z,$$

which by (5.17) implies

$$\int_{B_\rho - U_m} \left[\lambda_m^{q-2}|\sigma_m^D|^q\right]^3 \mathrm{d}z \le c(\rho).$$

Since by (5.13)

$$\int_{B_\rho} |\sigma_m|^6 \,\mathrm{d}z \le c(\rho),$$

the estimate (5.18) yields the bound

$$\int_{B_\rho - U_m} |T_1|^3 \,\mathrm{d}z \le c(\rho). \tag{5.19}$$

On U_m we have (see again (5.18))

$$\begin{aligned} |T_1| &\le c\int_0^1 |\omega_m^D + s\lambda_m\sigma_m^D|^{q-2}|\sigma_m^D| \,\mathrm{d}s \\ &\le c(L+M)^{q-2}|\sigma_m^D|, \end{aligned}$$

hence

$$\int_{U_m} |T_1|^3 \,\mathrm{d}z \le c\int_{B_\rho} |\sigma_m^D|^3 \,\mathrm{d}z \le c(\rho),$$

where (5.13) is used to get the latter inequality. This together with (5.19) finally implies (5.14) and this leads us to (5.12), and we may combine (5.11) and (5.12) with Vitali's theorem to get (5.4). Therefore the blow-up procedure is complete as soon as we have established the Caccioppoli-type inequality (5.6).

We fix a coordinate direction e_i, $i \le 3$, a number $h \ne 0$ and let $\Delta_h\rho(x) = \frac{1}{h}(\rho(x+he_i) - \rho(x))$ denote the difference quotient of the function ρ. With φ as in (5.6) we obtain from (1.1) $\Delta_h DW(\sigma) = \Delta_h\varepsilon(u)$, and in consequence

$$\int_\Omega \Delta_h DW(\sigma) : \Delta_h\sigma\varphi^2 \,\mathrm{d}x = \int_\Omega \Delta_h\varepsilon(u) : \Delta_h\sigma\varphi^2 \,\mathrm{d}x. \tag{5.20}$$

It is easy to show that $\Delta_h DW(\sigma) : \Delta_h\sigma \ge 0$, moreover $\sigma \in W^{1,2}_{\mathrm{loc}}(\Omega)$ implies $\Delta_h\sigma \to \partial_i\sigma$ in $L^2_{\mathrm{loc}}(\Omega)$ and a.e. (for a subsequence). From Lemma 2.17 of [Kn] we deduce the weak differentiability of $DW(\sigma)$, hence a.e.

$$\Delta_h DW(\sigma) \to \partial_i DW(\sigma) = D^2W(\sigma)(\partial_i\sigma, \cdot).$$

Thus we can apply Fatou's lemma on the l.h.s. of (5.20) with the result (summation w.r.t. i)

$$\int_\Omega \varphi^2 D^2W(\partial_i\sigma, \partial_i\sigma) \,\mathrm{d}x \le \liminf_{h\to 0}\int_\Omega \Delta_h\varepsilon(u) : \Delta_h\sigma\varphi^2 \,\mathrm{d}x. \tag{5.21}$$

Next we use (1.2) with $v := \Delta_{-h}(\varphi^2\Delta_h\tilde{u})$. Note that v is in $U_0^{p,2}(\Omega)$ since $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$, thus

$$\int_\Omega \Delta_h\sigma : \varepsilon(\varphi^2\Delta_h\tilde{u})\,\mathrm{d}x = \int_\Omega \Delta_h f \cdot \Delta_h\tilde{u}\varphi^2\,\mathrm{d}x.$$

This implies

$$\int_\Omega \Delta_h\varepsilon(u) : \Delta_h\sigma\varphi^2\,\mathrm{d}x = \int_\Omega \Delta_h f \cdot \Delta_h\tilde{u}\varphi^2\,\mathrm{d}x - \int_\Omega \Delta_h\sigma : \nabla\varphi^2 \odot \Delta_h\tilde{u}\,\mathrm{d}x, \tag{5.22}$$

and again we recall u, $\sigma \in W^{1,2}_{\mathrm{loc}}(\Omega)$ to see that the r.h.s. of (5.22) converges to

$$\int_\Omega \partial_i f \cdot \partial_i\tilde{u}\varphi^2\,\mathrm{d}x - \int_\Omega \partial_i\sigma : \nabla\varphi^2 \odot \partial_i\tilde{u}\,\mathrm{d}x.$$

With (5.21) inequality (5.6) follows, and we have proved the appropriate version of Lemma 2.1.

The iteration procedure from Section 4 does not change, but the estimate of the singular set has to be adjusted: in Ω_0^* we additionally have to consider points $x \in \Omega$ such that

$$\liminf_{\rho\to 0} \fint_{B_\rho(x)} |\varepsilon(u) - (\varepsilon(u))_{x,\rho}|^2\,\mathrm{d}y = 0.$$

As stated before Theorem 1.2 the function u is in the space $W^{2,r}_{\mathrm{loc}}(\Omega)$, $r = 3p/(p+1)$, and $r < 2$. This implies

$$\begin{aligned}\left[\int_{B_\rho(x)} |\varepsilon(u) - (\varepsilon(u))_{x,\rho}|^2\,\mathrm{d}y\right]^{\frac{1}{2}} &\leq \left[\int_{B_\rho(x)} |\varepsilon(u) - (\varepsilon(u))_{x,\rho}|^{s(r)}\,\mathrm{d}y\right]^{\frac{1}{s(r)}} \mathcal{L}^3(B_\rho(x))^{\frac{1}{2}-\frac{1}{s(r)}}\\ &\leq \left[\int_{B_\rho(x)} |\nabla^2 u|^r\,\mathrm{d}y\right]^{\frac{1}{r}} \mathcal{L}^3(B_\rho(x))^{\frac{1}{2}-\frac{1}{s(r)}},\end{aligned}$$

$s(r)$ denoting the Sobolev exponent of r, i.e. $s(r) = 3p$. We get

$$\begin{aligned}\fint_{B_\rho(x)} |\varepsilon(u) - (\varepsilon(u))_{x,\rho}|^2\,\mathrm{d}y &\leq \mathcal{L}^3(B_\rho(x))^{-1}\mathcal{L}^3(B_\rho(x))^{1-\frac{2}{s(r)}} \left[\int_{B_\rho(x)} |\nabla^2 u|^r\,\mathrm{d}y\right]^{\frac{2}{r}}\\ &= c\rho^{-\frac{2}{p}}\left[\int_{B_\rho(x)} |\nabla^2 u|^r\,\mathrm{d}y\right]^{\frac{2}{r}}\\ &= c\left[\rho^{-\frac{r}{p}}\int_{B_\rho(x)} |\nabla^2 u|^r\,\mathrm{d}y\right]^{\frac{2}{r}},\end{aligned}$$

and by [Gi], Theorem 2.2, p. 101, we see

$$\fint_{B_\rho(x)} |\varepsilon(u) - (\varepsilon(u))_{x,\rho}|^2\,\mathrm{d}y \to 0$$

for $\mathcal{H}^{3/(p+1)}$-a.a. $x \in \Omega$. □

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