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Abstract

In the calculus of variations one prominent problem is minimizing anisotropic integrals with a (p,q)-elliptic density $F:\mathbb{R}^n\supset\Omega\to\mathbb{R}^N$. The best known sufficient bound for regularity of solutions is $q<p\,(n+2)/n$. On the other hand, if we allow an additional x-dependence of the density we have the much weaker result $q<p\,(n+1)/n$. If one additionally imposes the local boundedness of the minimizer, then these bounds can be improved to $q<p+2$ and $q<p+1$. In this paper we give natural assumptions for F closing the gap between the autonomous and non-autonomous situation.

1 Introduction

In these paper we investigate regularity results for local minimizers of functionals

$$J[w]:=\int_\Omega F(\cdot,\nabla w)\,dx \tag{1.1}$$

with a function $F:\overline{\Omega}\times\mathbb{R}^{nN}\to[0,\infty)$ and a domain $\Omega\subset\mathbb{R}^n$. Before Esposito, Leonetti und Mingione found rather surprising counterexamples (see [ELM]), it was a wide-spread meaning, that the theorems valid for the autonomous situation extend to the case of x-dependence are portable and therefore most authors ignored x-dependence for technical simplification of their proofs. We are interested in anisotropic growth conditions. In the following we assume

$$\lambda(1+|Z|^2)^{\frac{p-2}{2}}\,|Q|^2\le D_P^2F(x,Z)(Q,Q)\le\Lambda(1+|Z|^2)^{\frac{q-2}{2}}\,|Q|^2 \tag{1.2}$$

for all $Z,Q\in\mathbb{R}^{nN}$ and all $x\in\overline{\Omega}$ with positive constants λ,Λ and exponents $1<p\le q<\infty$. From (1.2) we can deduce

$$\begin{aligned}|D_PF(x,Z)|&\le\Lambda_1(1+|Z|^2)^{\frac{q-1}{2}}\text{ and}\\ c_1\,|Z|^p-c_2\le F(x,Z)&\le c_3\,|Z|^q+c_4\end{aligned}$$

with further constants Λ_1,c_1,c_2,c_3 and c_4. Furthermore we assume for all $Z\in\mathbb{R}^{nN}$ and all $x\in\overline{\Omega}$

$$|\partial_\gamma D_PF(x,Z)|\le\Lambda_2(1+|Z|^2)^{\frac{q-1}{2}} \tag{1.3}$$

with $\Lambda_2>0$ and $\gamma\in\{1,...,n\}$. In [ELM] Esposito, Leonetti and Mingione examine the Lavrentiev gap functional which is defined as

$$\mathcal{L}:=\inf_{u_0+W_0^{1,q}(B,\mathbb{R}^N)}J-\inf_{u_0+W_0^{1,p}(B,\mathbb{R}^N)}J$$

on a ball $B \Subset \Omega$ with boundary data $u_0 \in W^{1,p}(B, \mathbb{R}^N)$. The results of the studies from [ELM] provide the sharpness of the bound

$$q < p\,\frac{n+\alpha}{n}$$

for higher integrability of solutions (assuming that $D_P F(x,Z)$ is α-Hölder continuous with respect to x) and without this condition they have examples for the Lavrentiev-phenomenon. If (1.2) and (1.3) (with at least continuous derivatives) hold together with

$$q < p\,\frac{n+1}{n}, \tag{1.4}$$

Bildhauer and Fuchs proved full $C^{1,\alpha}$-regularity for $N = 1$ or $n = 2$ and partial regularity in the general vector case. This statement is in accordance with the results of [ELM]. Under several structure conditions Bildhauer and Fuchs can improve the last result to full regularity (see [BF1]). Without x-dependence we know from [BF3] that the better bound

$$q < p\,\frac{n+2}{n} \tag{A1}$$

is sufficient for regularity. Having a look at the proof in [BF1], two main differences to the autonomous case become obvious. The first obstacle is that the standard-regularization u_δ does not converge against the minimum u without (1.4). Thus we work with a regularization from below which is based on a construction from [CGM] (see [BF1], section 3, for its presentation). This regularization F_M ($M \gg 1$) has the properties

- $F_M(x,P) \le F(x,P)$ for all $P \in \mathbb{R}^{nN}$;
- for $|P| \le M$ is $F_M(x,P) = F(x,P)$;
- $F_M(x,P)$ growth isotropic: i.e.

  $$\overline{a}\,|P|^p - \overline{b} \le F_M(x,P) \le A_M\,|P|^p + B_M \tag{1.5}$$

  for all $P \in \mathbb{R}^{nN}$ with uniform constants $\overline{a} > 0$, $\overline{b} \in \mathbb{R}$ and constants A_M and B_M depending on M.
- $F_M(x,P)$ is uniform (p,q)-elliptic, which means we have for $Z, Q \in \mathbb{R}^{nN}$ and $\gamma \in \{1, ..., n\}$

  $$\begin{aligned} \overline{\lambda}(1+|Z|^2)^{\frac{p-2}{2}}\,|Q|^2 \le D_P^2 F_M(x,Z)(Q,Q) \le \Lambda_3(1+|Z|^2)^{\frac{q-2}{2}}\,|Q|^2, \\ |\partial_\gamma D_P F_M(x,Z)| \le \Lambda_3(1+|Z|^2)^{\frac{q-1}{2}} \end{aligned} \tag{1.6}$$

  with constants $\overline{\lambda}, \Lambda_3 > 0$.

In order to introduce this regularization we have to assume

$$F(x,P) = g(x,|P|). \tag{A2}$$

If (A2) holds, then (1.1) reads as

$$\begin{aligned} \lambda(1+t^2)^{\frac{p-2}{2}} &\le \frac{g'(x,t)}{t} \le \Lambda(1+t^2)^{\frac{q-2}{2}}, \\ \lambda(1+t^2)^{\frac{p-2}{2}} &\le g''(x,t) \le \Lambda(1+t^2)^{\frac{q-2}{2}}. \end{aligned} \tag{A3}$$

The second obstacle in the proof of [BF1] is estimating the term

$$\int \partial_\gamma D_P F(\cdot, \nabla u) : \partial_\gamma \nabla u \, dx.$$

Motivated from [Bi] (section 4.2.2.2) we now require

$$|\partial_\gamma g''(x,t)| \le \Lambda_4 \left[g''(x,t)(1+t^2)^{\frac{\epsilon}{2}} + (1+t^2)^{\frac{p+q}{4}-1} \right] \tag{A4}$$

for all $(x,t) \in \overline{\Omega} \times [0,\infty)$ and $\gamma \in \{1,...,n\}$ with $0 \le \epsilon \ll 1$ in order to handle this integral. All of our assumptions have to be satisfied by the regularization function F_M uniformly in M and this can be achieved by requiring

$$|\partial^2_\gamma g''(x,t)| \le \Lambda_5 (1+t^2)^{\frac{q-2}{2}}. \tag{A5}$$

Additionally to (1.5) and (1.6) we now get for the sequence F_M, constructed in [BF1] (section 3), the following estimates:

Lemma 1.1 *Under the conditions for g above ((A2)-(A5) and continuity of the derivatives) we have for all $\gamma \in \{1,...,n\}$ and uniform in $x \in \Omega$*

- *$F_M(x,P)$ is p-elliptic, i.e. for $Z, Q \in \mathbb{R}^{nN}$ is*

  $$\begin{aligned} \overline{\lambda}(1+|Z|^2)^{\frac{p-2}{2}} |Q|^2 \le D^2_P F_M(x,Z)(Q,Q) \le \Lambda_M (1+|Z|^2)^{\frac{p-2}{2}} |Q|^2, \\ |\partial_\gamma D_P F_M(x,Z)| \le \Lambda_M (1+|Z|^2)^{\frac{p-1}{2}} \end{aligned}$$

  with a uniform constant $\overline{\lambda}$ and a constant Λ_M depending on M.

- *For all $P, Z \in \mathbb{R}^{nN}$ it holds*

  $$\begin{aligned} \left|\partial^2_\gamma D_P F_M(x,Z)\right| &\le \Lambda_6 (1+|Z|^2)^{\frac{q-1}{2}}, \\ \left|\partial_\gamma D^2_P F_M(x,Z)(P,Z)\right| &\le \Lambda_6 \left|D^2_P F_M(x,Z)(P,Z)\right| (1+|Z|^2)^{\frac{\epsilon}{2}} \\ &\quad + \Lambda_6 (1+|Z|^2)^{\frac{p+q-2}{4}} |P| \end{aligned}$$

  uniform in M with $\Lambda_6 \ge 0$.

The proof of Lemma 1.1 is presented in section 2.
In the isotropic situation, full regularity in the general vector case under modulus-dependence is only possible, if we know

$$\left|D^2F(x,P) - D^2F(x,Q)\right| \leq c(1+|P|^2+|Q|^2)^{\frac{q-2-\alpha}{2}} |P-Q|^\alpha \qquad \text{(A6)}$$

for all $P, Q \in \mathbb{R}^{nN}$, all $x \in \overline{\Omega}$ with $\alpha \in (0,1)$ (a first theorem in this context can be found in [Uh]). Therefore we need such a condition in the anisotropic case, too. With these preparations we define the regularization u_M of the problem (1.1) as the unique minimizer of

$$J_M[w] = \int_B F_M(\cdot, \nabla w)\, dx$$

in $u + W_0^{1,p}(B, \mathbb{R}^N)$ with a ball $B = B_{2R} \Subset \Omega$. This is the solution of a isotropic problem and so we get

Lemma 1.2 • *∇u_M belongs to the space $L^t_{loc}(B, \mathbb{R}^{nN})$ for $t = pn/(n-2)$ if $n \geq 3$ and t arbitrary if $n = 2$.*

- *u_M belongs to the space $W^{2,2}_{loc} \cap W^{1,\infty}_{loc}(B, \mathbb{R}^N)$ if $n = 2$ or $N = 1$.*
- *$\Gamma_M^{\frac{p-2}{4}} |\nabla^2 u_M|$ belongs to the space $L^2_{loc}(B)$ and $\nabla^2 u_M \in L^t_{loc}(B, \mathbb{R}^{n^2N})$ for $t := \min\{p, 2\}$.*
- *u_M is in $W^{1,p}(B, \mathbb{R}^N)$ uniformly bounded.*
- *If we have $u \in L^\infty_{loc}(\Omega, \mathbb{R}^N)$ then $\nabla u_M \in L^{p+2}_{loc}(B, \mathbb{R}^{nN})$ and $\sup_M \|u_M\|_\infty < \infty$.*

The first and the third part can be found for example in [BF1] (Lemma 2.3 and 2.5 for $p = q$). The second statement follows from [BF1] Thm. 1.1 (which is a classical result for $N = 1$, see [LU]). For the uniform boundedness of u_M in $W^{1,p}(B, \mathbb{R}^N)$ we see by (1.5)

$$\begin{aligned}\int_B |\nabla u_M|^p\, dx &\leq c\left[\int_B F_M(\cdot, \nabla u_M)\, dx + 1\right] \leq c\left[\int_B F_M(\cdot, \nabla u)\, dx + 1\right] \\ &\leq c\left[\int_B F(\cdot, \nabla u)\, dx + 1\right] \leq c.\end{aligned}$$

The Poincaré-inequality supplies

$$\|u_M\|_{L^p(B)} \leq c\left[\|\nabla(u_M - u)\|_{L^p(B)} + \|u\|_{L^p(B)}\right]$$

$$\leq c\left[\|\nabla u_M\|_{L^p(B)} + \|u\|_{W^{1,p}(B)}\right] \leq c.$$

$\sup_M \|u_M\|_\infty < \infty$ is received by quoting the maximum-principle of [DLM], since we assume (A2). For the last statement we quote [BF1] (proof of Lemma 2.8 for $\alpha = 0$). Note that it is not necessary to have Lipschitz-regularity for the regularization in [BF1] to proof this. Hence, one can choose there $q \geq p+2$ (therefore we do not need (A6) at this point). $\square$

In the following, we formulate the new results valid for local minimizers of (1.1):

THEOREM 1.1 *Under the assumptions (A1)-(A5), where all involved derivatives are assumed to be continuous, we have:*

(a) for a local minimizer $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ of (1.1) there is an open subset Ω_0 with full Lebesgue-measure such that u belongs to the space $C^{1,\alpha}(\Omega_0, \mathbb{R}^N)$ for any $\alpha \in (0,1)$.

(b) If we have additionally for $n \geq 5$ one of the following conditions

$$(i) \qquad q < p\,\frac{n-1}{n-2} \tag{A7}$$

$$(ii)\; g'(x,t) \leq c g''(x,t)(1+t^2)^{\frac{\omega}{2}} \text{ for } \omega < \left(\frac{pn}{n-2} - q\right) + 1 \tag{A8}$$

for all $t \geq 0$ as well as (A6) if $n \geq 3$ and $N \geq 2$, then any local minimizer $u \in W^{1,p}_{loc}(\Omega, \mathbb{R}^N)$ of (1.1) belongs to the space $C^{1,\alpha}(\Omega, \mathbb{R}^N)$ for all $\alpha < 1$.

Remark 1.3 • *Considering the situation $F(x,Z) = f(x)g(|Z|)$ with two C^2-functions f and g, where f is strictly positive and g shares suitable anisotropic growth conditions, it is easy to see, that F satisfies all assumptions necessary for Theorem 1.1. But even in this case there is no possibility to argue any simpler than in our proof of higher integrability (see Lemma 2.1). The critical point is the integral*

$$\int \eta^2 \partial_\gamma f g'(|\nabla u|)\frac{\nabla u}{|\nabla u|} : \partial_\gamma \nabla u \, dx.$$

Since the estimate $g'(|\nabla u|) \leq c(1+|\nabla u|^2)^{(q-1)/2}$ is not sharp enough, one must integrate by parts to handle this term.

- *The motivation for allowing small $\epsilon > 0$ in (A4) is the minimization of*

$$\int_\Omega \left(1 + |\nabla w|^2\right)^{\frac{f(x)}{2}} \, dx$$

for $f \in C^2$. The density satisfies the growth condition (A4) for every $\epsilon > 0$.

Remark 1.4

- *The condition (A7) is necessary to ensure the uniform boundedness of the integral*

$$\int (1 + |\nabla u_M|^2)^{q-\frac{p}{2}} \ln(1 + |\nabla u_M|^2) \, dx.$$

The higher integrability of Theorem 2.1 provides (A7), which follows from (A1) for $n \leq 4$. Alternatively, if we assume (A8) we have another possibility to argue at the critic point.

- *In the proof of full regularity we return our problem to an integral with isotropic growth conditions, so that we get the claim from (A2) and (A6).*

- *Under assumption (A7) or (A8) we get Theorem 1.1 a) directly from the proof of part b). There we show local boundedness of ∇u without (A6), which is the reason why we can quote another time the results of the isotropic situation to get partial regularity (and full regularity for $n = 2$ or $N = 1$).*

- *Fuchs [Fu] shows regularity for minimizers of variational integrals which depend on the modulus of the gradient, too. He gets an arbitrary range of anisotropy. His results are dedicated to the autonomous situation, but with a little modification and adaption of the hypotheses they extend to the case of x-dependence, too. In contrast to our assumption they are very restrictive, which can easily be seen by consideration of the functions g_1 and g_3 from of 6. In both cases the functions must be strictly increasing, convex and must satisfy $\lim_{t\to 0} g(x,t)/t = 0$. Indeed, our functions do not necessarily perform a Δ_2-condition. Considering the hypothesis ($a \geq 0$)*

$$\frac{g'(x,t)}{t} \leq g''(x,t) \leq a(1+t^2)^{\frac{\omega}{2}} \frac{g'(x,t)}{t},$$

with an arbitrary $\omega \geq 0$, claimed for in [Fu], the second inequality is surely always fulfilled. But the first inequality is very restrictive. Even *if we allow constants smaller than 1 this is in general false in our context. Fuchs does not only need this condition for having superquadratic growth, which is consistent with $p \geq 2$ in our case. He requires the monotonicity of $t \mapsto g'(\cdot, t)/t$, which he needs for (2.5) and (2.6) of [Fu]. Another possibility for this argumentation is a Δ_2-condition for $g'(x,t)$. In case of g_3 this is false again.*

- *Obviously we can choose $\omega > 1$ in (A8), where the difference $\omega - 1$ is the margin between q and the maximal integrability which we get from Lemma 2.1.*

For locally bounded minimizers we get dimensionless conditions between p and q: Without x-dependence Bildhauer and Fuchs proved full regularity for $N = 1$ and in the general vector case with structure conditions under the assumption (see [BF3])

$$q < p + 2, \tag{A9}$$

whereas the non-autonomous situation requires the much more restrictive bound (see [BF1])

$$q < p + 1.$$

Our new statement is

THEOREM 1.2 *We assume (A2)-(A5), where all involved derivatives are assumed to be continuous and (A6) if $n \geq 3$ and $N \geq 2$. Then any local minimizer $u \in W^{1,p}_{loc} \cap L^{\infty}_{loc}(\Omega, \mathbb{R}^N)$ of (1.1) belongs to the space $C^{1,\alpha}(\Omega, \mathbb{R}^N)$ for all $\alpha < 1$, if we have (A9) and*

$$g'(x,t) \leq c g''(x,t)(1+t^2)^{\frac{\omega}{2}} \text{ for } \omega < (p+2-q)+1. \tag{A10}$$

Remark 1.5 • *Analogous to (A8) we can allow $\omega > 1$ in (A10), where the difference $\omega - 1$ is again the range between q and the maximal integrability $p + 2$.*

2 Regularization & higher integrability

Following [BF1] we consider for a fixed number $M \gg 1$ a cut-off function $\eta \in C^1([0,\infty))$ with $\eta \equiv 1$ on $[0, 3/2M]$, $\eta \equiv 0$ on $[2M, \infty)$ and $\left|\eta'\right| \le c/M$. We define

$$h(x,t) := \eta(t) + (1-\eta(t))\lambda \frac{(1+t^2)^{\frac{p-2}{2}}}{g''(x,t)}.$$

Then h is a continuous function with the following properties:

- $h(x,M) = \eta(M) = 1$ for all $x \in \overline{\Omega}$;
- for all $(x,t) \in \overline{\Omega} \times [0,\infty)$ follow $0 \le h(x,t) \le 1$;
- $g''(x,t)h(x,t) \ge \lambda(1+t^2)^{\frac{p-2}{2}}$ and $g''(x,t)h(x,t) \le c(M)(1+t^2)^{\frac{p-2}{2}}$.

After these preparations we define

$$g_M(x,t) := \begin{cases} g(x,t), & \text{for } 0 \le t \le M \\ g(x,M) + g'(x,M)(t-M) + \int\limits_M^t \int\limits_M^\rho g''(x,\tau)h(x,\tau)d\tau d\rho, & \text{for } t > M \end{cases}$$

and finally $F_M(x,Z) := g_M(x,|Z|)$.

Proof of Lemma 1.1: Obviously only the case $|Z| > M$ is of interest for the growth estimates. Assuming this we get

$$\begin{aligned} D_P F_M(x,Z) &= g'(x,M)\frac{Z}{|Z|} + \int_M^{|Z|} g''(x,\tau)h(x,\tau)d\tau \frac{Z}{|Z|}, \\ \partial_\gamma D_P F_M(x,Z) &= \partial_\gamma g'(x,M)\frac{Z}{|Z|} + \int_M^{|Z|} \partial_\gamma \left[g''(x,\tau)h(x,\tau)\right] d\tau \frac{Z}{|Z|}, \\ \partial_\gamma^2 D_P F_M(x,Z) &= \partial_\gamma^2 g'(x,M)\frac{Z}{|Z|} + \int_M^{|Z|} \partial_\gamma^2 \left[g''(x,\tau)h(x,\tau)\right] d\tau \frac{Z}{|Z|}. \end{aligned}$$

By (A4) and the definition of h we see

$$\begin{aligned} |\partial_\gamma D_P F_M(x,Z)| &\le c(M) + \left| \int_M^{|Z|} \partial_\gamma g''(x,\tau)\eta(\tau)d\tau \right| \\ &\le c(M) + \int_M^{2M} |\partial_\gamma g''(x,\tau)|d\tau \le c(M) \\ &\le c(M)(1+|Z|^2)^{\frac{p-1}{2}}. \end{aligned}$$

For the uniform bound of $\partial_\gamma^2 D_P F_M$ we obtain in the same way from (A4) and (A5)

$$
\begin{aligned}
\left|\partial_\gamma^2 D_P F_M(x,Z)\right| &\le c(1+M^2)^{\frac{q-1}{2}} + \int_M^{|Z|} |\partial_\gamma^2 g''(x,\tau)\eta(\tau)|d\tau \\
&\le c\left\{(1+|Z|^2)^{\frac{q-1}{2}} + \frac{1}{M}\int_M^{2M}(1+\tau^2)^{\frac{q-1}{2}}d\tau\right\} \\
&\le c\left\{(1+|Z|^2)^{\frac{q-1}{2}} + (1+M^2)^{\frac{q-1}{2}}\right\}.
\end{aligned}
$$

Because of $|Z| > M$ we finally receive

$$
\left|\partial_\gamma^2 D_P F_M(x,Z)\right| \le c(1+|Z|^2)^{\frac{q-1}{2}}.
$$

The uniform growth conditions of $\partial_\gamma D_P F_M$ and the uniform (p,q)-ellipticity are already established in [BF1]. We only have to show that $D_P^2 F$ is p-elliptic. We have

$$
\begin{aligned}
D_P^2 F_M(x,Z)(P,P) &= \frac{g'(x,M)}{|Z|}\left[|P|^2 - \frac{|Z:P|^2}{|Z|^2}\right] \\
&\quad + \int_M^{|Z|} g''(x,\tau)h(x,\tau)d\tau \frac{1}{|Z|}\left[|P|^2 - \frac{|Z:P|^2}{|Z|^2}\right] \\
&\quad + g''(x,|Z|)h(x,|Z|)\frac{|Z:P|^2}{|Z|^2} \\
&:= T_1 + T_2 + T_3.
\end{aligned}
$$

For the first term we see by (A3) in case $p \ge 2$

$$
T_1 \le \frac{g'(x,M)}{M}|P|^2 \le c(M)|P|^2 \le c(M)(1+|Z|^2)^{\frac{p-2}{2}}|P|^2
$$

and for $p < 2$

$$
\begin{aligned}
T_1 &\le c(M)\frac{(1+|Z|^2)^{\frac{2-p}{2}}}{|Z|}(1+|Z|^2)^{\frac{p-2}{2}}|P|^2 \\
&\le c(M)(1+|Z|^2)^{\frac{p-2}{2}}|P|^2.
\end{aligned}
$$

For T_2 we conclude from the properties of $g''h$ (remember $M \ge 1$)

$$
T_2 \le c(M)(1+|Z|^2)^{\frac{p-2}{2}}|P|^2
$$

and in an analogous way

$$T_3 \le c(M)(1+|Z|^2)^{\frac{p-2}{2}}|P|^2$$

which shows the p-ellipticity.
Considering the derivatives $\partial_\gamma D_P^2 F_M$ we get

$$\begin{aligned}\partial_\gamma D_P^2 F_M(x,Z)(P,Q) = {} & \frac{\partial_\gamma g'(x,M)}{|Z|}\left[P:Q-\frac{(Z:P)(Z:Q)}{|Z|^2}\right] \\ & + \int_M^{|Z|} \partial_\gamma\left[g''(x,\tau)h(x,\tau)\right]d\tau \frac{1}{|Z|}\left[P:Q-\frac{(Z:P)(Z:Q)}{|Z|^2}\right] \\ & + \partial_\gamma\left[g''(x,|Z|)h(x,|Z|)\right]\frac{(Z:P)(Z:Q)}{|Z|^2}\end{aligned}$$

for arbitrary $P, Q, Z \in \mathbb{R}^{nN}$. In case $Q = Z$ we see by (A4)

$$\begin{aligned} & \left|\partial_\gamma D_P^2 F_M(x,Z)(P,Z)\right| \\ = {} & \left|\partial_\gamma\left[g''(x,|Z|)h(x,|Z|)\right]Z:P\right| = \left|\eta(|Z|)\partial_\gamma g''(x,|Z|)\right|\left|Z:P\right| \\ \le {} & c\left[\left|\eta(|Z|)g''(x,|Z|)\right|(1+|Z|^2)^{\frac{\epsilon}{2}}\left|Z:P\right| + (1+|Z|^2)^{\frac{p+q}{4}-1}|Z|\,|P|\right] \\ \le {} & c\left[\left|g''(x,|Z|)h(x,|Z|)Z:P\right|(1+|Z|^2)^{\frac{\epsilon}{2}} + (1+|Z|^2)^{\frac{p+q-2}{4}}|P|\right] \\ = {} & c\left[\left|D_P^2 F_M(x,Z)(P,Z)\right|(1+|Z|^2)^{\frac{\epsilon}{2}} + (1+|Z|^2)^{\frac{p+q-2}{4}}|P|\right]\end{aligned}$$

which finally proves Lemma 1.1. □

Lemma 2.1 *Under the assumptions of Theorem 1.1 we get for local minimizers $u \in W^{1,p}_{loc}(\Omega,\mathbb{R}^N)$ of (1.1)*

$$\nabla u \in \begin{cases} L^{\frac{pn}{n-2}}_{loc}(\Omega,\mathbb{R}^{nN}) & \text{if } n \ge 3 \\ L^s_{loc}(\Omega,\mathbb{R}^{nN}), & \text{for all } s<\infty, \text{ if } n=2. \end{cases}$$

Also u belongs to the space $W^{2,t}_{loc}(\Omega,\mathbb{R}^N)$ for $t := \min\{p,2\}$.

For the proof of Lemma 2.1 we need a Caccioppoli-type inequality:

Lemma 2.2 *There is a constant $c > 0$ independent from M such that*

$$\int_B \eta^2 \Gamma_M^{\frac{p-2}{2}}\left|\nabla^2 u_M\right|^2 dx \le c\,\|\nabla\eta\|_\infty^2 \int_{\operatorname{spt}\nabla\eta} \Gamma_M^{\frac{q}{2}} dx + c\int_{\operatorname{spt}\eta}\Gamma_M^{\frac{q}{2}}dx$$

for all $\eta \in C_0^1(B)$.

Proof: The growth of D_P^2 implies (sum over γ)

$$\int\limits_B \eta^2 \Gamma_M^{\frac{p-2}{2}} \left|\nabla^2 u_M\right|^2 dx \le c \int\limits_B \eta^2 D_P^2 F_M(\cdot, \nabla u_M)(\partial_\gamma \nabla u_M, \partial_\gamma \nabla u_M) dx$$

$$\le c \Bigg[-2 \int\limits_B \eta D_P^2 F_M(\cdot, \nabla u_M)(\partial_\gamma \nabla u_M, \partial_\gamma u_M \otimes \nabla \eta) dx$$

$$- 2 \int\limits_B \eta \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \partial_\gamma u_M \otimes \nabla \eta dx$$

$$- \int\limits_B \eta^2 \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \partial_\gamma \nabla u_M dx \Bigg]$$

$$:= I_1 + I_2 + I_3,$$

where we used the method of difference quotients for the second inequlity (see [BF2], Lemma 3.1, for details) and Lemma 1.2 (part 3). Using Young-inequality and Lemma 1.1 we can calculate all terms except for I_3. For this one we write

$$I_3 = \int\limits_B \partial_\gamma \left\{ \eta^2 \partial_\gamma D_P F_M(\cdot, \nabla u_M) \right\} : \nabla u_M dx$$

$$= \int\limits_B \eta^2 \partial_\gamma^2 D_P F_M(\cdot, \nabla u_M) : \nabla u_M dx$$

$$+ \int\limits_B \eta^2 \partial_\gamma D_P^2 F_M(\cdot, \nabla u_M)(\partial_\gamma \nabla u_M, \nabla u_M) dx$$

$$+ \int\limits_B \partial_\gamma D_P F_M(\cdot, \nabla u_M) : \nabla u_M \partial_\gamma \eta^2 dx$$

$$:= I_3^1 + I_3^2 + I_3^3.$$

Lemma 1.1 (part 2) delivers

$$I_3^1 \le c \int\limits_{\operatorname{spt} \eta} \Gamma_M^{\frac{q}{2}} dx$$

and from (1.6) we deduce

$$I_3^3 \le c \left\|\nabla \eta\right\|_\infty \int\limits_{\operatorname{spt} \nabla\eta} \Gamma_M^{\frac{q}{2}} dx \le c \left\|\nabla \eta\right\|_\infty^2 \int\limits_{\operatorname{spt} \nabla\eta} \Gamma_M^{\frac{q}{2}} dx + c \int\limits_{\operatorname{spt} \eta} \Gamma_M^{\frac{q}{2}} dx.$$

For I_3^2 we conclude from Lemma (1.2), part 2,

$$I_3^2 \le c \int_B \eta^2 \left| D_P^2 F_M(\cdot, \nabla u_M)(\partial_\gamma \nabla u_M, \nabla u_M) \right| (1 + |\nabla u_M|^2)^{\frac{\epsilon}{2}}\, dx$$

$$+ c \int_B \eta^2 \Gamma_M^{\frac{p+q-2}{4}} \left| \nabla^2 u_M \right| dx.$$

We can bound the first integral by

$$\tau \int_B \eta^2 D_P^2 F_M(\cdot, \nabla u_M)(\partial_\gamma \nabla u_M, \partial_\gamma \nabla u_M) dx$$

$$+ c(\tau) \int_B \eta^2 D_P^2 F_M(\cdot, \nabla u_M)(\nabla u_M, \nabla u_M)(1 + |\nabla u_M|^2)^{\epsilon} dx.$$

If we know

$$\epsilon < \frac{1}{2} \left(p\, \frac{n+2}{n} - q \right),$$

we can increase q to $q + 2\epsilon$ w.l.o.g. Now we can absorb the first term and bound the second one by

$$c \int_{\operatorname{spt} \eta} \Gamma_M^{\frac{q}{2}} dx.$$

For arbitrary $\tau > 0$ we obtain by Young's inequality

$$\int_B \eta^2 \Gamma_M^{\frac{p+q-2}{4}} \left| \nabla^2 u_M \right| dx \le \tau \int_B \eta^2 \Gamma_M^{\frac{p-2}{2}} \left| \nabla^2 u_M \right|^2 dx + c(\tau) \int_{\operatorname{spt} \eta} \Gamma_M^{\frac{q}{2}} dx$$

which we handle conventionally. □

Proof of Lemma 2.1: We follow the lines of [BF2]. We chose $\eta \in C_0^\infty(B_{r+\rho})$ with $\eta \equiv 1$ on B_r and $\|\nabla \eta\|_\infty \le c/\rho$ (asking for $0 < r \le R' < 2R$ and $0 < \rho \le R' - r$). For $\alpha := pn/2(n-2)$ $(n \ge 3)$ and $h_M := \Gamma_M^{\alpha(n-2)/2n}$ we clearly get by Lemma 2.2

$$\int_{B_r} \Gamma_M^\alpha dx \le c \left\{ \frac{1}{\rho^2} \int_{B_{r+\rho} - B_r} \Gamma_M^{\frac{q}{2}} dx + \int_{B_{r+\rho}} \Gamma_M^{\frac{q}{2}} dx \right\}^{\frac{n}{n-2}}. \tag{2.1}$$

Since $p < q < 2\alpha$ (compare (A1)), there is a $\theta \in (0,1)$ such that

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{2\alpha}. \tag{2.2}$$

Using interpolation inequality (compare [GT], 7.9, p. 146) we obtain

$$\|\nabla u_M\|_q \le \|\nabla u_M\|_p^{\theta} \|\nabla u_M\|_{2\alpha}^{1-\theta},$$

where the norms are taken over $B_{r+\rho} - B_r$. It follows

$$\begin{aligned} \frac{1}{\rho^2} \int\limits_{B_{r+\rho}-B_r} \Gamma_M^{\frac{q}{2}} dx \le & \frac{c}{\rho^2} \left[\int\limits_{B_{r+\rho}-B_r} \Gamma_M^{\frac{p}{2}} dx \right]^{\frac{\theta q}{p}} \left[\int\limits_{B_{r+\rho}-B_r} \Gamma_M^{\alpha} dx \right]^{\frac{(1-\theta)q}{2\alpha}} \\ & + \frac{c}{\rho^2} \int\limits_{B_{2R}} \Gamma_M^{\frac{p}{2}} dx. \end{aligned} \tag{2.3}$$

Now (A1) shows

$$(1-\theta)\frac{q}{p} = \frac{n}{2}\left(\frac{q}{p} - 1\right) < 1$$

and so we can find positive exponents β_1 and β_2 such that

$$\frac{1}{\rho^2} \int\limits_{B_R} \Gamma_M^{\frac{q}{2}} dx \le \frac{c}{\rho^{\beta_1}} \left[\int\limits_{B_{2R}} \Gamma_M^{\frac{p}{2}} dx \right]^{\beta_2} + \left[\int\limits_{B_{r+\rho}-B_r} \Gamma_M^{\alpha} dx \right]^{\frac{n-2}{n}}. \tag{2.4}$$

Apply this into (2.1), for suitable $\beta_3, \beta_4 > 0$ we see

$$\int\limits_{B_r} \Gamma_M^{\alpha} dx \le \frac{c}{\rho^{\beta_3}} \left[\int\limits_{B_{2R}} \Gamma_M^{\frac{p}{2}} dx \right]^{\beta_4} + c \left[\int\limits_{B_{r+\rho}} \Gamma_M^{\frac{q}{2}} dx \right]^{\frac{n-2}{n}} + c \int\limits_{B_{r+\rho}-B_r} \Gamma_M^{\alpha} dx$$

and the "hole filling technique" provides for a $\widetilde{\vartheta} \in (0,1)$

$$\int\limits_{B_r} \Gamma_M^{\alpha} dx \le \frac{c}{\rho^{\beta_3}} \left[\int\limits_{B_{2R}} \Gamma_M^{\frac{p}{2}} dx \right]^{\beta_4} + c \left[\int\limits_{B_{r+\rho}} \Gamma_M^{\frac{q}{2}} dx \right]^{\frac{n-2}{n}} + \widetilde{\vartheta} \int\limits_{B_{r+\rho}} \Gamma_M^{\alpha} dx. \tag{2.5}$$

Exactly as in (2.3) we can calculate the second term and can follow

$$\int_{B_r} \Gamma_M^{\alpha} dx \leq \frac{c}{\rho^{\beta_3}} \left[\int_{B_{2R}} \Gamma_M^{\frac{p}{2}} dx \right]^{\beta_4} + \vartheta \int_{B_{r+\rho}} \Gamma_M^{\alpha} dx \tag{2.6}$$

for a $\vartheta \in (0,1)$. From a well-known Lemma by Giaquinta (see [Gi], Lemma 5.1, p. 81) we deduce

$$\int_{B_r} \Gamma_M^{\alpha} dx \leq \frac{c}{(R'-r)^{\beta_3}} \left[\int_{B_{2R}} \Gamma_M^{\frac{p}{2}} dx \right]^{\beta_4},$$

and therefore the uniform boundedness of ∇u_M in $L^{2\alpha}_{loc}(B, \mathbb{R}^{nN})$. In case $n = 2$ we argue similarly (see [Bi] or [BF2]) for an arbitrary $\alpha > \frac{1}{2}\frac{p^2}{2p-q}$. To transfer the integrability to the solution u we have to show the convergence $u_M \to u$. For $p \geq 2$ we get the uniform boundedness of $\nabla^2 u_M$ in L^2_{loc} by a combination of Lemma 2.2 and the uniform $W^{1,q}_{loc}(B, \mathbb{R}^N)$-bound of u_M. In case $p < 2$ it appears by Young-inequality, the uniform $W^{1,p}_{loc}(B, \mathbb{R}^N)$-bound of u_M and Lemma 2.2 for an arbitrary $r < 2R$

$$\begin{aligned} \int_{B_r} |\nabla^2 u_M|^p dx &= \int_{B_r} \Gamma_M^{\frac{p-2}{4}} |\nabla^2 u_M|^p \Gamma_M^{\frac{2-p}{4}} dx \\ &\leq \int_{B_r} \Gamma_M^{\frac{p-2}{2}} |\nabla^2 u_M|^2 dx + \int_{B_r} \Gamma_M^{\frac{p}{2}} dx \leq c(r). \end{aligned}$$

In both cases u_M is uniformly bounded in $W^{2,t}_{loc}(B, \mathbb{R}^N)$ for $t := \min\{2, p\}$, so we can follow after passing to a subsequence

$$\begin{gathered} u_M \rightharpoonup: v \text{ in } W^{2,t}_{loc}(B, \mathbb{R}^N) \text{ and} \\ \nabla u_M \to \nabla v \text{ almost everywhere on } B \end{gathered}$$

for a function $v \in W^{2,t}_{loc}(B, \mathbb{R}^N)$. So we achieve

$$F_M(\cdot, \nabla u_M) \to F(\cdot, \nabla v) \text{ almost everywhere on } B$$

and by the Lemma of Fatou and the minimality of u_M we see

$$\begin{aligned} \int_B F(\cdot, \nabla v) dx &\leq \liminf_{M \to \infty} \int_B F_M(\cdot, \nabla u_M) dx \leq \liminf_{M \to \infty} \int_B F_M(\cdot, \nabla u) dx \\ &\leq \int_B F(\cdot, \nabla u) dx \end{aligned}$$

on account of $F_M \leq F$. By uniqueness of local minimizers of (1.1) this implies $u = v$ and so the result of Lemma 2.1. □

3 Partial regularity

A first preparation for proving Theorem 1.1 a) is

Lemma 3.1 *Let* $H_M := \Gamma_M^{\frac{p}{4}}$*,* $\Gamma := 1 + |\nabla u|^2$ *and* $H := \Gamma^{\frac{p}{4}}$*. Than we have*

- $H \in W^{1,2}_{loc}(B)$*,*
- $H_M \rightharpoonup H$ *in* $W^{1,2}_{loc}(B)$ *for* $M \to \infty$ *and*
- $\nabla u_M \to \nabla u$ *almost everywhere* B *for* $M \to \infty$*.*

Proof: The second statement we get from Lemma 2.2 and 2.1 under observation of

$$|\nabla H_M| \le \Gamma_M^{\frac{p-2}{4}} |\nabla^2 u_M|.$$

This shows the first part directly, whereas the third point was shown at the end of the last section.
By lower semicontinuity of the map $v \mapsto \|\nabla v\|_2^2$ for $v \in W^{1,2}$ and Lemma 2.2 we obtain

$$\begin{aligned} \int_B \eta^2 |\nabla H|^2 \, dx &\le \liminf_{M\to\infty} \int_B \eta^2 |\nabla H_M|^2 \, dx \\ &\le \liminf_{M\to\infty} c \left\{ \|\nabla\eta\|_\infty^2 \int_{\operatorname{spt}\nabla\eta} \Gamma_M^{\frac{q}{2}} dx + \int_{\operatorname{spt}\eta} \Gamma_M^{\frac{q}{2}} dx \right\}. \end{aligned}$$

Since $q < pn/(n-2)$ we see by the higher integrability of Γ_M, the convergence $\Gamma_M^{\frac{q}{2}} \to \Gamma^{\frac{q}{2}}$ almost everywhere on B (see Lemma 3.1) and the Vitali convergence theorem $\Gamma_M^{\frac{q}{2}} \to \Gamma^{\frac{q}{2}}$ in $L^1(\operatorname{spt}\eta)$ and thus

Lemma 3.2 *For* $\eta \in C_0^\infty(B)$ *and arbitrary balls* $B \Subset \Omega$ *we have*

$$\int_B \eta^2 |\nabla H|^2 \, dx \le c \, \|\nabla\eta\|_\infty^2 \int_{\operatorname{spt}\nabla\eta} \Gamma^{\frac{q}{2}} dx + c \int_{\operatorname{spt}\eta} \Gamma^{\frac{q}{2}} dx.$$

In case $n = 2$ an analogous argumentation is possible.
We define if $q \ge 2$

$$E^+(x,r) := \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^q \, dy + \fint_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^2 \, dy$$

and in the other situation

$$E^{-}(x,r) := \fint_{B_r(x)} |V(\nabla u) - V((\nabla u)_{x,r})|^2 \, dy,$$

$$V(\xi) := (1+|\xi|^2)^{\frac{q-2}{4}} \xi \text{ for } \xi \in \mathbb{R}^{nN}.$$

So we get

LEMMA 3.3 *Fix $L > 0$. Then there exists a constant $C^*(L)$ such that for every $\tau \in (0, 1/4)$ there is an $\epsilon = \epsilon(\tau, L) > 0$ satisfying: if $B_r \Subset B_R$ and we have*

$$|(\nabla u)_{x,r}| \le L, \quad E(x,r) + r^{\gamma^*} \le \epsilon$$

then

$$E(x, \tau r) \le C^* \tau^2 [E(x,r) + r^{\gamma^*}].$$

Here $\gamma^* \in (0,2)$ is an arbitrary number.

We follow the lines of [BF2] and so the only part which needs a comment is the uniform bound of $\int_{B_\rho} |\nabla \psi_m|^2 \, dx$ for $\rho < 1$ (the function ψ_m is defined in [BF2]). For $\Theta(Z) := (1+|Z|^2)^{\frac{p}{4}}$ $(Z \in \mathbb{R}^{nN})$ we see

$$\begin{aligned} \int_{B_\rho} |\nabla \psi_m(z)|^2 \, dz &= \int_{B_\rho} |D\Theta(A_m + \lambda_m \nabla u_m(z)) : \nabla^2 u_m(z)|^2 \, dz \\ &= r_m^{-n} \frac{r_m^2}{\lambda_m^2} \int_{B_{\rho r_m}(x_m)} |\nabla H|^2 \, dz \\ &\le c(\rho)\, r_m^2 \lambda_m^{-2} \fint_{B_{r_m}(x_m)} \Gamma^{\frac{q}{2}} \, dz, \end{aligned} \tag{3.1}$$

where $\lambda_m^2 := E(x_m, r_m) + r_m^2$. Furthermore we receive for $q \ge 2$

$$\begin{aligned} \fint_{B_{r_m}(x_m)} \Gamma^{\frac{q}{2}} \, dz &\le c \left[1 + \fint_{B_{r_m}(x_m)} |\nabla u|^q \, dz \right] \\ &\le c \left[1 + \fint_{B_{r_m}(x_m)} |\nabla u - (\nabla u)_{x_m, r_m}|^q \, dz + \fint_{B_{r_m}(x_m)} |(\nabla u)_{x_m, r_m}|^q \, dz \right] \end{aligned}$$

$$\leq cE(x_m, r_m) + c(L).$$

If $q < 2$ we define for $t \in (1, \infty)$ and $\xi \in \mathbb{R}^{nN}$

$$V_t(\xi) := (1 + |\xi|^2)^{\frac{t-2}{4}} \xi \text{ and } H_t(\xi) := (1 + |\xi|^2)^{\frac{t}{2}}.$$

By Lemma 2.3 of [Ha] then we have

$$\left| \sqrt{H_t(\xi_1)} - \sqrt{H_t(\xi_2)} \right| \leq c \left| V_t(\xi_1) - V_t(\xi_2) \right|.$$

Now we can calculate

$$\begin{aligned} \fint_{B_{r_m}(x_m)} \Gamma^{\frac{q}{2}} \, dz &\leq \fint_{B_{r_m}(x_m)} \left[\left| \sqrt{H_q(\nabla u)} - \sqrt{H_q((\nabla u)_{x_m, r_m})} \right| + \left| \sqrt{H_q((\nabla u)_{x_m, r_m})} \right| \right]^2 dz \\ &\leq c \fint_{B_{r_m}(x_m)} \left| V_q(\nabla u) - V_q((\nabla u)_{x_m, r_m}) \right|^2 dz + c(L) \\ &= cE(x_m, r_m) + c(L) \end{aligned}$$

In both cases we obtain

$$\int_{B_\rho} |\nabla \psi_m(z)|^2 \, dz \leq c(\rho) \left[r_m^2 + r_m^2 \lambda_m^{-2} c(L) \right].$$

Recalling the choice of γ^* we have $r_m^2 \lambda_m^{-2} \to 0$ for $m \to$ and the boundedness of $\int_{B_\rho} |\nabla \psi_m|^2 \, dx$ follows. Now the proof can be completed as in [BF2]. □

4 Full regularity

Here we consider the standard regularization: u_δ is defined as the unique minimizer of ($F_\delta(Z) = F(Z) + \delta(1 + |Z|^2)^{\frac{\widetilde{q}}{2}}$)

$$I_\delta[w, B] := \int_B F_\delta(\nabla w) dx \tag{4.1}$$

in $(u)_\epsilon + W_0^{1,\widetilde{q}}(B)$ for $B \Subset \Omega$ and $\widetilde{q} > q$. Thereby $(u)_\epsilon$ is the mollification of u with parameter ϵ and

$$\delta = \delta(\epsilon) := \frac{1}{1 + \epsilon^{-1} + \|\nabla (u)_\epsilon\|_{L^{\widetilde{q}}(B)}^{2\widetilde{q}}}.$$

For u_δ we obtain, since u is a $W_{loc}^{1,q}$-minimizer by Lemma 2.1 (compare [BF1], Lemma 2.1 and 2.7):

Lemma 4.1 • *As $\epsilon \to 0$ we have: $u_\delta \rightharpoondown u$ in $W^{1,p}(B,\mathbb{R}^N)$,*

$$\delta \int_B \left(1+|\nabla u_\delta|^2\right)^{\frac{\tilde{q}}{2}} dx \to 0; \quad \int_B F(\nabla u_\delta)dx \to \int_B F(\nabla u)dx;$$

- $\nabla u_\delta \in W^{1,2}_{loc} \cap L^\infty_{loc}(\Omega,\mathbb{R}^N)$.
- *If we have $u \in L^\infty_{loc}(\Omega,\mathbb{R}^N)$ then we get* $\sup_\delta \|u_\delta\|_\infty < \infty$.

For the last statement we need (A2) and quote [DLM]. Exactly as in the proof of Lemma 2.1 we can show

$$\nabla u_\delta \in L^{\frac{pn}{n-2}}_{loc}(B,\mathbb{R}^{nN}) \quad \text{uniformly.} \tag{4.2}$$

Note, that we can chose $\widetilde{q}$ arbitrary near to q. Therefore we can replace in all of our estimations q by $\widetilde{q}$ and especially (A1) stays true. For showing Theorem 1.1 b) i) we define

$$\tau(k,r) := \int\limits_{A(k,r)} \Gamma_\delta^{\frac{\nu}{2}} (\omega_\delta - k)^2 dx$$

where $A(k,r) := B_r \cap [\omega_\delta > k]$ for balls $B_r \Subset B$ with the choice

$$\frac{\nu}{2} := \widetilde{q} - \frac{p}{2} < \frac{1}{2}\frac{pn}{n-2}.$$

Lemma 2.1 guarantees together with (A7) the uniform boundedness of τ. Following the argumentation of [Bi] (p. 65 ff.) we see

$$\tau(h,r) \le X_M \int\limits_{A(h,\widehat{r})} \left|\nabla\left\{\eta\Gamma_\delta^{\frac{p}{4}}(\omega_\delta - h)\right\}\right|^2 dx,$$

$$X_\delta := \left[\int\limits_{A(h,r)} \Gamma_\delta^{\frac{\chi}{\chi-1}\beta} dx\right]^{\frac{\chi-1}{\chi}}$$

for $\beta := (\nu - p)/2$ and $\chi := n/(n-2)$ if $n \ge 3$. To estimate the second term on the r.h.s. we have to handle the integrals

$$\int\limits_{A(h,\widehat{r})} |\partial_\gamma D_P F(\cdot,\nabla u_\delta) : \nabla\left\{\eta^2 \partial_\gamma u_\delta [\omega_\delta - h]\right\}| \, dx,$$

$$\int_{A(h,\widehat{r})} |\partial_\gamma D_P F(\cdot, \nabla u_\delta) : \nabla \left\{\eta^2 \partial_\gamma u_\delta [\omega_\delta - h]^2\right\}| \, dx$$

additionally to the terms in [Bi], p. 62. So we get for the first integral the three integrals

$$\begin{aligned}
& 2 \int_{A(h,\widehat{r})} \eta |\partial_\gamma D_P F(\cdot, \nabla u_\delta)||\nabla \eta||\nabla u_\delta|(\omega_\delta - h) \, dx \\
&+ \int_{A(h,\widehat{r})} \eta^2 |\partial_\gamma D_P F(\cdot, \nabla u_\delta)||\nabla^2 u_\delta|(\omega_\delta - h) \, dx \\
&+ \int_{A(h,\widehat{r})} \eta^2 |\partial_\gamma D_P F(\cdot, \nabla u_\delta)||\nabla u_\delta||\nabla \omega_\delta| \, dx \\
:=& S_1 + S_2 + S_3.
\end{aligned}$$

Considering S_2 we obtain for an arbitrary $\tau > 0$

$$\begin{aligned}
S_2 \le & c \int_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{\frac{q-1}{2}} |\nabla^2 u_\delta|(\omega_\delta - h) \, dx \\
\le & \tau \int_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{\frac{p-2}{2}} |\nabla^2 u_\delta|^2 (\omega_\delta - h)^2 \, dx \\
& + c(\tau) \int_{A(h,\widehat{r})} \Gamma_\delta^{\frac{\nu}{2}} \, dx
\end{aligned}$$

where the first term can be observed in the same way as in [Bi]. Furthermore we have

$$\begin{aligned}
S_1 \le & c \int_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{\frac{q}{2}} |\nabla \eta|(\omega_\delta - h) \, dx \\
\le & c \int_{A(h,\widehat{r})} \Gamma_\delta^{\frac{\nu}{2}} |\nabla \eta|^2 (\omega_\delta - h)^2 \, dx + c \int_{A(h,\widehat{r})} \Gamma_\delta^{\frac{\nu}{2}} \, dx
\end{aligned}$$

by making use of Young's inequality. For S_3 one estimates

$$S_3 \le c \int_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{\frac{q}{2}} |\nabla \omega_\delta| \, dx$$

$$\leq \tau \int\limits_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{\frac{p}{2}} |\nabla \omega_\delta|^2 \, dx + c(\tau) \int\limits_{A(h,\widehat{r})} \Gamma_\delta^{\frac{\nu}{2}} \, dx$$

for all $\tau > 0$, which allows an absorption for another time.
After estimating

$$\int\limits_{A(h,\widehat{r})} |\partial_\gamma D_P F(\cdot, \nabla u_\delta) : \nabla \left\{ \eta^2 \partial_\gamma u_\delta [\omega_\delta - h]^2 \right\} | \, dx$$

in the same way, we finally have showed

$$\tau(h,r) \leq c X_\delta \left[\int\limits_{A(h,\widehat{r})} |\nabla \eta|^2 \Gamma_\delta^{\frac{\nu}{2}} (\omega_\delta - h)^2 \, dx + c\tau(h,\widehat{r}) + \int\limits_{A(h,\widehat{r})} \Gamma_\delta^{\frac{\nu}{2}} \, dx \right].$$

Assuming w.l.o.g. $h - k \leq 1$ now we have

$$\tau(h,r) \leq \frac{c}{(\widehat{r} - r)^2 (h-k)^2} X_\delta \tau(k, \widehat{r}). \tag{4.3}$$

The choice of ν and (A7) deliver

$$\nu < \frac{pn}{n-2},$$

or equivalently

$$\frac{\chi}{\chi - 1} \beta < \frac{\nu}{2}.$$

In case $n = 2$ we can achieve this, if we choose χ big enough. Now we receive

$$X_\delta \leq \left[\int\limits_{A(h,\widehat{r})} \Gamma_\delta^{\frac{\nu}{2}} \, dx \right]^{\frac{2}{n}} \leq \frac{c}{(h-k)^{\frac{4}{n}}} \tau(k,\widehat{r})^{\frac{2}{n}}. \tag{4.4}$$

Combining (4.3) and (4.4) we see

$$\tau(h,r) \leq \frac{c}{(\widehat{r} - r)^2 (h-k)^{2+\frac{4}{n}}} \tau(k,\widehat{r})^{1+\frac{2}{n}},$$

and $u \in W^{1,\infty}_{loc}(\Omega, \mathbb{R}^N)$ follows by [St], Lemma 5.1, as in [Bi]. So we locally have a variational problem with isotropic growth conditions and by (A6) the claim follows (compare [BF1], Lemma 2.7).

For showing Theorem 1.1 b) ii) we similarly use DeGiorgi-arguments but with other estimates at the critical point. For a technical simplification we assume $\epsilon = 0$ in (A4). So we define (with an obvious meaning of Γ_δ and ω_δ)

$$\tau(h,r) := \int\limits_{A(h,r)} \Gamma_\delta^{\frac{\nu}{2}} (\omega_\delta - k)^2 dx$$

for $\nu := q + \omega - 1$.

By (A8) and (4.2) we get the uniform boundedness of τ for another time. We obtain by decomposition $\nu = \nu_1 + \nu_2$ for $\nu_1 := (n-2)\nu/n$, $\nu_2 := 2\nu/n$ and $\chi := n/(n-2)$

$$\begin{aligned}
\tau(h,r) &= \int\limits_{A(h,r)} \Gamma_\delta^{\frac{\nu_1}{2}} (\omega_\delta - h)^2 \Gamma_\delta^{\frac{\nu_2}{2}}\, dx \\
&\le \left[\int\limits_{A(h,r)} \Gamma_\delta^{\frac{\nu_1}{2}\chi} (\omega_\delta - h)^{2\chi}\, dx \right]^{\frac{1}{\chi}} \left[\int\limits_{A(h,r)} \Gamma_\delta^{\frac{\chi}{\chi-1}\frac{\nu_2}{2}}\, dx \right]^{\frac{\chi-1}{\chi}} \\
&\le X_\delta \left[\int\limits_{A(h,\widehat{r})} \left\{ \eta \Gamma_\delta^{\frac{p}{4}} (\omega_\delta - h) \right\}^{2\chi} dx \right]^{\frac{1}{\chi}}.
\end{aligned}$$

Note, that $\nu_1 < p$ since $\nu < pn/(n-2)$. Here X_δ stands for the second integral in the second line of the estimation and $\eta \in C_0^\infty(B_{\widehat{r}})$ is a suitable cut-off function. By Sobolev's inequality we have

$$\tau(k,r) \le X_\delta \int\limits_{A(h,\widehat{r})} \left| \nabla \left\{ \eta \Gamma_\delta^{\frac{p}{4}} (\omega_\delta - h) \right\} \right|^2 dx. \tag{4.5}$$

By the choice of ν_2 we have

$$X_\delta = \left[\int\limits_{A(h,r)} \Gamma_\delta^{\frac{\nu}{2}}\, dx \right]^{\frac{2}{n}} \le \frac{1}{(h-k)^{\frac{4}{n}}} \tau(k,\widehat{r})^{\frac{2}{n}}.$$

For the remaining integral we can argue as before in the proof of part b) i). During the calculation of this integral we have to estimate the terms S_2 and S_3 as in the proof of part b) i). We receive from (A4)

$$|\partial_\gamma g'(x,t)| \le c \left[g'(x,t) + (1+t^2)^{\frac{p+q-2}{4}} \right]. \tag{4.6}$$

Assuming w.l.o.g. $\epsilon = 0$ we get

$$
\begin{aligned}
|\partial_\gamma g'(x,t)| &\leq \int_0^1 |\partial_\gamma g''(x,st)| t ds + |\partial_\gamma g''(x,0)| \\
&\leq c \int_0^1 \left[g''(x,st)t + (1+(st)^2)^{\frac{p+q}{4}-1} t \right] ds + c \\
&\leq c \left[g'(x,t) + (1+t^2)^{\frac{p+q-2}{4}} \right],
\end{aligned}
$$

here we quote [AF], Lemma 2.1, if $p+q<4$. By (4.6) we have

$$
\begin{aligned}
S_2 \leq & c \int\limits_{A(h,\widehat{r})} \eta^2 g'(\cdot,|\nabla u_\delta|)|\nabla^2 u_\delta|(\omega_\delta - h)\, dx \\
&+ c \int\limits_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{\frac{p+q-2}{4}} |\nabla^2 u_\delta|(\omega_\delta - h)\, dx.
\end{aligned}
$$

For the second term we obtain the upper bound

$$
\tau \int\limits_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{\frac{p-2}{2}} |\nabla^2 u_\delta|^2 (\omega_\delta - h)\, dx + c(\tau) \int\limits_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{\frac{q}{2}} (\omega_\delta - h)\, dx
$$

for all $\tau > 0$ and argue is in part i). Finally we receive for the first integral $\mathcal{S}_2^1$ by Young's inequality

$$
\begin{aligned}
S_2^1 \leq & \tau \int\limits_{A(h,\widehat{r})} \eta^2 \frac{g'(\cdot,|\nabla u_\delta|)}{\Gamma_\delta^{\frac{\omega}{2}}} |\nabla^2 u_\delta|^2 (\omega_\delta - h)\, dx \\
&+ c(\tau) \int\limits_{A(h,\widehat{r})} \eta^2 g'(\cdot,|\nabla u_\delta|) \Gamma_\delta^{\frac{\omega}{2}} (\omega_\delta - h)\, dx.
\end{aligned}
$$

We want to absorb the first one. First we estimate

$$
\begin{aligned}
D^2F(x,P)(X,X) &= \frac{g'(x,|P|)}{|P|} \left(|X|^2 - \frac{(P:X)^2}{|P|^2} \right) + g''(x,|P|) \frac{(P:X)^2}{|P|^2} \\
&\geq \min \left\{ g''(\cdot,|P|), \frac{g'(\cdot,|P|)}{|P|} \right\} |X|^2
\end{aligned}
$$

for all $P, X \in \mathbb{R}^{nN}$. Finally we get by (A8)

$$
\frac{g'(\cdot,|\nabla u_\delta|)}{\Gamma_\delta^{\frac{\omega}{2}}} |\nabla^2 u_\delta|^2 \leq c \min \left\{ g''(\cdot,|\nabla u_\delta|), \frac{g'(\cdot,|\nabla u_\delta|)}{|\nabla u_\delta|} \right\} |\nabla^2 u_\delta|^2
$$

$$
\begin{aligned}
&\le c D_P^2 F(\cdot, \nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \\
&\le c D_P^2 F_\delta(\cdot, \nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta),
\end{aligned} \tag{4.7}
$$

where the sum is taken over $\gamma \in \{1, ..., n\}$. Considering S_2^1, the remaining term is

$$
\mathcal{S} := \int\limits_{A(h,\widehat{r})} \Gamma_\delta^{\frac{q-1+\omega}{2}} (\omega_\delta - h)\, dx \le \tau(h, \widehat{r}) + \int\limits_{A(h,\widehat{r})} \Gamma_\delta^{\frac{\nu}{2}}\, dx \tag{4.8}
$$

by (A8). For the examination of S_3 we argue similar, using (4.6)

$$
\begin{aligned}
S_3 \le c &\int\limits_{A(h,\widehat{r})} \eta^2 g'(\cdot, |\nabla u_\delta|) \Gamma_\delta^{\frac{1}{2}} |\nabla \omega_\delta|\, dx \\
&+ c \int\limits_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{\frac{p+q}{4}} |\nabla \omega_\delta|\, dx \\
&=: S_3^1 + S_3^2.
\end{aligned}
$$

So we can follow

$$
S_3^2 \le \tau \int\limits_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{\frac{p}{2}} |\nabla \omega_\delta|^2\, dx + c(\tau) \int\limits_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{\frac{q}{2}}\, dx
$$

and absorb the first term. For $\mathcal{S}_3^1$ we get by Young's inequality

$$
\begin{aligned}
S_3^1 \le \tau &\int\limits_{A(h,\widehat{r})} \eta^2 \frac{g'(\cdot, |\nabla u_\delta|)}{\Gamma_\delta^{\frac{\omega}{2}}} \Gamma_\delta |\nabla \omega_\delta|^2\, dx \\
&+ c(\tau) \int\limits_{A(h,\widehat{r})} \eta^2 g'(\cdot, |\nabla u_\delta|) \Gamma_\delta^{\frac{\omega}{2}}\, dx.
\end{aligned}
$$

The second integral is bounded by $\mathcal{S}$ from (4.8), whereas we obtain by (4.7)

$$
c \int\limits_{A(h,\widehat{r})} \eta^2 D_P^2 F_\delta(\cdot, \nabla u_\delta)(e_i \otimes \nabla \omega_\delta, e_i \otimes \nabla \omega_\delta) \Gamma_\delta\, dx
$$

as a upper bound for the first one (compare [Bi], (32), together with (A2)). Now it is possible to end up the proof like before (compare [Bi] for more details).

5 Locally bounded minimizers

In this section we prove Theorem 1.2. In a first step we show

Lemma 5.1 *Under the assumptions of Theorem 1.2 we have* $\nabla u \in L^{p+2}_{loc}(\Omega, \mathbb{R}^{nN})$.

Proof: We show the uniform boundedness of ∇u_M in $W^{1,t}_{loc} \cap L^{p+2}_{loc}(B, \mathbb{R}^{nN})$ $(t := \min\{2, p\})$. Then the claim of Lemma 5.1 follows as at the end of section 2. We fix $\eta \in C_0^\infty(B_r)$ with $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_\rho \Subset B_r \Subset B$ and $|\nabla \eta| \le c/(r-\rho)^{-1}$ and get by integrating by parts, following the lines of [Bi], p. 155, (let $h \equiv 1$, $k = 1$ and $\alpha = 0$)

$$\int_{B_\rho} \Gamma_M^{\frac{p+2}{2}} \, dx \le c \int_{B_r} \eta^2 \Gamma_M^{\frac{p-2}{2}} |\nabla^2 u_M|^2 \, dx + \frac{1}{4} \int_{B_r} \eta^2 \Gamma_M^{\frac{p+2}{2}} \, dx$$

To calculate the only critical integral we use Lemma 2.2 and see for a suitable $\beta > 0$

$$\begin{aligned} c \int_{B_r} \eta^2 \Gamma_M^{\frac{p-2}{2}} |\nabla^2 u_M|^2 \, dx &\le c(r-\rho)^{-2} \int_{B_r} \Gamma_M^{\frac{q}{2}} \, dx \\ &\le c(r-\rho)^{-\beta} + \frac{1}{4} \int_{B_r} \Gamma_M^{\frac{p+2}{2}} \, dx \end{aligned} \tag{5.1}$$

because $q < p + 2$, using Young's inequality. Finally we obtain

$$\int_{B_\rho} \Gamma_M^{\frac{p+2}{2}} \, dx \le c(r-\rho)^{-\beta} + \frac{1}{2} \int_{B_r} \Gamma_M^{\frac{p+2}{2}} \, dx. \tag{5.2}$$

By [Gi], Lemma 5.1, p. 81, one can follow from (5.2) the uniform boundedness of $\nabla u_M \in L^{p+2}_{loc}(B, \mathbb{R}^{nN})$. Now we can deduce from (5.1) the uniform boundedness of ∇u_M in $W^{1,t}_{loc}(B, \mathbb{R}^{nN})$. □

So the existence of a $W^{1,q}_{loc}$-minimizer is proved and therefore we can work with the δ-regularization from the proof of Theorem 1.1 b) ii) and Lemma 4.1. Analogous to Lemma 5.1, we can show

$$\nabla u_\delta \in L^{p+2}_{loc}(B, \mathbb{R}^{nN}) \text{ uniformly,} \tag{5.3}$$

since the additional term $\delta \int_B \Gamma_\delta^{\frac{\tilde{q}}{2}} \, dx$ is not critical by Lemma 4.1, part 1. Now we can show

Lemma 5.2 *Under the assumptions of Theorem 1.2 we have* $\nabla u_\delta \in L^t_{loc}(\Omega, \mathbb{R}^{nN})$ *for all* $t < \infty$ *uniformly in* δ.

Proof: We quote for another time [Bi] and get for $\alpha \geq 0$ and $k \gg 1$

$$\int_B \eta^{2k} \Gamma_\delta^{\frac{p+\alpha+2}{2}}\, dx \leq c(\eta) + c \int_B \eta^{2k} \Gamma_\delta^{\frac{p-2+\alpha}{2}} |\nabla^2 u_\delta|^2\, dx. \tag{5.4}$$

For calculating the $|\nabla^2 u_\delta|$-integral we need an inequality of Caccioppoli-type:

Lemma 5.3 *Under the assumptions of Theorem 1.2 for all $\alpha \geq 0$ and all $\eta \in C_0^\infty(B)$ there is a constant $c = c(\alpha, \eta)$ independent of δ with*

$$\int_B \eta^{2k} \Gamma_\delta^{\frac{p-2+\alpha}{2}} |\nabla^2 u_\delta|^2\, dx \leq c \left[1 + \int_B \eta^{2k-2} \Gamma_\delta^{\frac{p+\alpha}{2}}\, dx \right].$$

Proof: Inserting $\phi := \partial_\gamma u_\delta \Gamma_\delta^{\frac{\alpha}{2}} \eta^{2k}$ and summing up over $\gamma \in \{1, ..., n\}$, we only have one term of interest (compare [BF5], p. 326):

$$\int_B \left| \partial_\gamma D_P F(\cdot, \nabla u_\delta) : \nabla \left\{ \partial_\gamma u_\delta \Gamma_\delta^{\frac{\alpha}{2}} \eta^{2k} \right\} \right| dx. \tag{5.5}$$

Using (1.3) and (4.6) one obtains the three integrals

$$\begin{aligned} R_1 &:= \int_B |\nabla \eta| \Gamma_\delta^{\frac{q+\alpha}{2}} \eta^{2k-1}\, dx, \\ R_2 &:= \int_B \eta^{2k} g'(\cdot, |\nabla u_\delta|) |\nabla^2 u_\delta| \Gamma_\delta^{\frac{\alpha}{2}}\, dx, \\ R_3 &:= \int_B \eta^{2k} \Gamma_\delta^{\frac{p-2+q+2\alpha}{4}} |\nabla^2 u_\delta|\, dx. \end{aligned}$$

For $k \gg 1$ we get by Young's inequality

$$R_1 \leq \tau \int_B \eta^{2k} \Gamma_\delta^{\frac{p+\alpha+2}{2}}\, dx + c(\eta, \tau),$$

since $q < p + 2$. By (5.4) we can handle the r.h.s. for τ small enough. Considering R_3 we see

$$R_3 \leq \tau \int_B \eta^{2k} \Gamma_\delta^{\frac{p-2+\alpha}{2}} |\nabla^2 u_\delta|^2\, dx + c(\tau) \int_B \eta^{2k} \Gamma_\delta^{\frac{q+\alpha}{2}}\, dx.$$

We absorb the first integral by (5.4) and see

$$\int_B \eta^{2k} \Gamma_\delta^{\frac{q+\alpha}{2}}\, dx \leq \tau' \int_B \eta^{2k} \Gamma_\delta^{\frac{p+2+\alpha}{2}}\, dx + c(\tau')$$

for the second one by Young's inequality and $q < p+2$. Obviously, the critical term is R_2: For this integral we get

$$R_2 \le \tau \int_B \eta^{2k} \frac{g'(\cdot, |\nabla u_\delta|)}{\Gamma_\delta^{\frac{\omega}{2}}} |\nabla^2 u_\delta|^2 \Gamma_\delta^{\frac{\alpha}{2}}\, dx + c(\tau) \int_B \eta^{2k} g'(\cdot, |\nabla u_\delta|) \Gamma_\delta^{\frac{\omega+\alpha}{2}}\, dx.$$

The first term on the r.h.s. can be absorbed in the l.h.s. of (5.5) for $\tau \ll 1$ if we note (4.7) and (A10). For the remaining integral we have the estimate

$$\int_B \Gamma_\delta^{\frac{q+\omega-1+\alpha}{2}} \le \tau' \int_B \Gamma_\delta^{\frac{p+\alpha+2}{2}}\, dx + c(\tau')$$

if we use Young's inequality and $\omega < (p+2-q)+1$. This ends up the proof of Lemma 5.3 and we have showed

$$\int_B \eta^{2k} \Gamma_\delta^{\frac{p+\alpha+2}{2}}\, dx \le c(\eta) \left[1 + \int_B \eta^{2k-2} \Gamma_\delta^{\frac{p+\alpha}{2}}\, dx \right] \tag{5.6}$$

by (5.4). Iteratively we can follow the claim of Lemma 5.2. As a starting point one can choose $\alpha = 2$ by Lemma 5.3. □

In the next step we show

Lemma 5.4 *Under the assumptions of Theorem 1.2 ∇u_δ is uniformly bounded in $L^\infty_{loc}(\Omega, \mathbb{R}^{nN})$.*

The claim of Theorem 1.2 follows by reducing our problem to a variational integral with isotropic growth.

Proof of Lemma 5.4: We consider the function

$$\tau(h,r) := \int\limits_{A(h,r)} \Gamma_\delta^{q-\frac{p}{2}+1} (\Gamma_\delta - h)^2\, dx$$

where $A(h,r) := B_r \cap [\Gamma_\delta > h]$ and proof

$$\tau(h,r) \le \frac{c}{(\hat{r}-r)^{\frac{n}{n-1}\frac{1}{s}} (h-k)^{\frac{n}{n-1}\frac{1}{s}\frac{1}{t}}} \tau(k,\hat{r})^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\left[1+\frac{1}{t}\right]}$$

for $0 < k < h$ and $0 < r < \hat{r} < R$ with $s,t > 1$ and c independent of $h, k, r, \hat{r}, \delta$. For an arbitrary $s > 1$ we calculate (compare [Bi], p. 157, for details)

$$\tau(h,r) \le c(s) \left[I_1^{\frac{n}{n-1}\frac{1}{s}} + I_2^{\frac{n}{n-1}\frac{1}{s}} \right]. \tag{5.7}$$

By a later choice of $t > 1$ we get

$$I_1^{\frac{n}{n-1}\frac{1}{s}} \leq \frac{c}{(\widehat{r}-r)^{\frac{n}{n-1}\frac{1}{s}}(h-k)^{\frac{n}{n-1}\frac{1}{s}\frac{1}{t}}}\tau(k,\widehat{r})^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\left[1+\frac{1}{t}\right]}. \tag{5.8}$$

For I_2 one obtains with sum over $j \in \{1,...,N)$

$$I_2^{\frac{n}{n-1}} \leq c\left[\int_{A(h,\widehat{r})} \eta^2 D^2 F_\delta(\cdot,\nabla u_\delta)(e_j\otimes\nabla\Gamma_\delta, e_j\otimes\nabla\Gamma_\delta)dx\right]^{\frac{1}{2}\frac{n}{n-1}} \times$$
$$\frac{c}{(h-k)^{\frac{n}{n-1}\frac{1}{t}}}\tau(k,\widehat{r})^{\frac{1}{2}\frac{n}{n-1}\frac{1}{t}}. \tag{5.9}$$

For the term in brackets we show

Lemma 5.5 *Suppose the assumptions of Theorem 1.2. Then there is a constant $c > 0$ for which we have the estimate*

$$\int_{A(h,\widehat{r})} \eta^2 D^2F(\nabla u_\delta)\left(e_j\otimes\nabla\Gamma_\delta, e_j\otimes\nabla\Gamma_\delta\right)\,dx \leq \frac{c}{(\widehat{r}-r)^2(h-k)^2}\tau(k,\widehat{r})$$

for all $\eta \in C_0^\infty(B_{\widehat{r}}(x_0))$.

Proof We choose the test function $\varphi := \eta^2\partial_\gamma u_\delta[\Gamma_\delta - h]^+$ and get

$$Q_1 := \int_{A(h,\widehat{r})} \eta\Gamma_\delta^{\frac{q}{2}}|\nabla\eta|(\Gamma_\delta - h)\,dx$$
$$Q_2 := \int_{A(h,\widehat{r})} \eta^2\Gamma_\delta^{\frac{q-1}{2}}|\partial_\gamma\nabla u_\delta|(\Gamma_\delta - h)\,dx$$
$$Q_3 := \int_{A(h,\widehat{r})} \eta^2\Gamma_\delta^{\frac{q}{2}}|\nabla\Gamma_\delta|\,dx.$$

as the interesting terms (see [BF1]). It follows

$$Q_1 \leq \frac{c}{(\widehat{r}-r)(h-k)}\int_{A(h,\widehat{r})} \Gamma_\delta^{q-\frac{p}{2}+1}(\Gamma_\delta - h)^2\,dx$$
$$\leq \frac{c}{(\widehat{r}-r)^2(h-k)^2}\tau(k,\widehat{r}),$$

since we can assume w.l.o.g. $h-k \le 1$. The Young's inequality delivers

$$Q_2 \le \tau \int\limits_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{\frac{p-2}{2}} |\partial_\gamma \nabla u_\delta|^2 (\Gamma_\delta - h)\, dx$$

$$+ c(\tau) \int\limits_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{q-\frac{p}{2}} (\Gamma_\delta - h)\, dx.$$

The growth of $D_P^2 F_\delta$ allows us to absorb the first term (compare [BF1], (2.41)). The other term can be bounded by

$$\frac{c}{(\widehat{r}-r)^2 (h-k)^2} \tau(k,\widehat{r}).$$

For Q_3 we obtain by Young's inequality

$$Q_3 \le \tau \int\limits_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{\frac{p-2}{2}} |\nabla \Gamma_\delta|^2\, dx + c(\tau) \int\limits_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{q-\frac{p}{2}+1}\, dx.$$

The inequality

$$\int\limits_{A(h,\widehat{r})} \eta^2 \Gamma_\delta^{q-\frac{p}{2}+1}\, dx \le \frac{c}{(h-k)^2} \tau(k,\widehat{r}) \le \frac{c}{(\widehat{r}-r)^2 (h-k)^2} \tau(k,\widehat{r})$$

ends up the proof of Lemma 5.5.
By recapitulating (5.7)-(5.9) and Lemma 5.5 we have

$$\tau(h,r) \le \frac{c}{(\widehat{r}-r)^{\frac{n}{n-1}\frac{1}{s}} (h-k)^{\frac{n}{n-1}\frac{1}{s}\frac{1}{t}}} \tau(k,\widehat{r})^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\left[1+\frac{1}{t}\right]}.$$

Since s,t are arbitrary numbers in $(1,\infty)$, we can achieve

$$\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\left[1+\frac{1}{t}\right] > 1$$

and the claim follows by [St], Lemma 5.1. □

6 Examples

Let $\eta \in C^\infty([0,\infty),[0,1])$ and $\theta > 0$ such that

$$\eta(t) = \begin{cases} 0, & \text{on } [4k\theta, (4k+1)\theta] \\ 1, & \text{on } [(4k+2)\theta, (4k+3)\theta] \end{cases}$$

for all $k \in \mathbb{N}_0$. We define for $1 < p \leq q < \infty$ and $t \geq 0$

$$g_\theta(t) := \int_0^t \int_0^\rho \left[\eta(\tau)(1+\tau^2)^{\frac{p-2}{2}} + (1-\eta(\tau))(1+\tau^2)^{\frac{q-2}{2}}\right] d\tau d\rho$$

Then we have

Lemma 6.1 *The function g_θ satisfies the condition (A3).*

Whereas the estimates for g_θ'' are trivial, we need for handling g_θ' the inequality

$$(1+t^2)^{\frac{2-p}{2}} \frac{1}{t} \int_0^t (1+\tau^2)^{\frac{p-2}{2}} \, d\tau \geq c \tag{6.1}$$

for a $c > 0$, which delivers directly the first of the two requested estimates. We see

$$\lim_{t\to 0}(1+t^2)^{\frac{2-p}{2}} \frac{1}{t} \int_0^t (1+t^2)^{\frac{p-2}{2}} = 1$$

$$\text{and } \lim_{t\to\infty}(1+t^2)^{\frac{2-p}{2}} \frac{1}{t} \int_0^t (1+\tau^2)^{\frac{p-2}{2}} = \frac{1}{p-1},$$

which proves (6.1). By similar arguments we get

$$(1+t^2)^{\frac{2-q}{2}} \frac{1}{t} \int_0^t (1+\tau^2)^{\frac{q-2}{2}} \, d\tau \leq c$$

and thereby the estimate from above. A trivial way to receive a spatial dependence is

$$g_1(x,t) := \alpha(x) g_\theta(t)$$

for a C^2-function α which is bigger than $\epsilon_0 > 0$. Now it is easy to see, that g_1 satisfies the assumptions (A3), (A4) for $\epsilon = 0$ and (A5).
A possibility which is not as trivial is

$$g_2(x,t) := (1+t^2)^{\frac{f(x)}{2}}$$

for a C^2-function f satisfying $f(x) > 1$ for all $x \in \overline{\Omega}$. Than we obtain

Lemma 6.2 *The function g_2 satisfies (A3), (A4) for any $\epsilon > 0$, (A5) and if $f(x) \geq 2$ on $\overline{\Omega}$ we have (A6) for*

$$F_2(x,P) := g_2(x,|P|).$$

Proof: W.l.o.g. let $f(x) > 2$. Note

$$\begin{aligned} g_8'(x,t) &= f(x)(1+t^2)^{\frac{f(x)}{2}-1}t \\ \text{and } g_8''(x,t) &= f(x)(f(x)-2)(1+t^2)^{\frac{f(x)}{2}-2}t^2 + f(x)(1+t^2)^{\frac{f(x)}{2}-1}. \end{aligned}$$

So we see (A3) by $p := \inf f$ and $q := \sup f$. Furthermore we get

$$\begin{aligned} \partial_\gamma g_8''(x,t) &= 2\partial_\gamma f(x)\left[f(x)-1\right](1+t^2)^{\frac{f(x)}{2}-2}t^2 \\ &\quad + \frac{1}{2}f(x)(f(x)-2)\partial_\gamma f(x)\ln(1+t^2)(1+t^2)^{\frac{f(x)}{2}-2}t^2 \\ &\quad + \partial_\gamma f(x)(1+t^2)^{\frac{f(x)}{2}-1} \\ &\quad + \frac{1}{2}f(x)\partial_\gamma f(x)\ln(1+t^2)(1+t^2)^{\frac{f(x)}{2}-1}. \end{aligned}$$

We can bound $\partial_\gamma g_8''$ by

$$\ln(1+t^2)(1+t^2)^{\frac{f(x)}{2}-1}.$$

Using the estimate

$$\ln(1+t^2) \le c(\epsilon)(1+t^2)^{\frac{\epsilon}{2}}$$

takes (A4). If we derive $\partial_\gamma g_8''(x,t)$ once again we receive (A5) for $q := \sup f + 2\epsilon$. Now we prove (A6):

$$\begin{aligned} DF_8(x,P) &= g'(x,|P|)\frac{P}{|P|}, \\ D^2F_8(x,P) &= \frac{g'(x,|P|)}{|P|}\left(I - \frac{P\otimes P}{|P|^2}\right) + g''(x,|P|)\frac{P\otimes P}{|P|^2} \\ &= f(x)(1+|P|^2)^{\frac{f(x)}{2}-1}I + f(x)(f(x)-2)(1+|P|^2)^{\frac{f(x)}{2}-2}P\otimes P. \end{aligned}$$

Thereby we have for $P, Q \in \mathbb{R}^{nN}$

$$\begin{aligned} |D^2F_8(x,P) - D^2F_8(x,Q)| &\le f(x)\sqrt{n}\left|(1+|P|^2)^{\frac{f(x)}{2}-1} - (1+|Q|^2)^{\frac{f(x)}{2}-1}\right| \\ &\quad + f(x)|f(x)-2|\left|(1+|P|^2)^{\frac{f(x)}{2}-2}P\otimes P - (1+|Q|^2)^{\frac{f(x)}{2}-2}Q\otimes Q\right| \\ &=: \alpha_1 + \alpha_2. \end{aligned}$$

Estimating α_1 and α_2 seperatly and using [AF], Lemma 2.1, if $2 < f(x) < 3$ we obtain ($i \in \{1,2\}$)

$$\alpha_i \le c(1+|P|^2+|Q|^2)^{\frac{f(x)-2-\gamma}{2}}|P-Q|^\gamma$$

for an arbitrary $\gamma \in (0,1)$. This shows for $q := \sup f$ the requested Hölder-condition for D^2F.
Due to a local argumentation it is possible to choose p and q arbitrarily near and to quote older regularity results for minimizers of $\int_\Omega g_2(\cdot, |\nabla w)|\, dx$. This motivates the following construction: let

$$g_3(x,t) := \int_0^t \int_0^\rho \left[\eta(\tau)(1+\tau^2)^{\frac{p+f(x)-2}{2}} + (1-\eta(\tau))(1+\tau^2)^{\frac{q+f(x)-2}{2}}\right] d\tau d\rho$$

for an η as in the definition of g_θ and a function $f \in C^2(\overline{\Omega}, \mathbb{R}_0^+)$. Here we can produce an arbitrarily wide range of anisotropy with a non-trivial x-dependence. Following the arguments of the two examples before, it is easy to see, that all of our assumptions are satisfied.

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