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Abstract

A new shock filter model designed to sharpen numerically diffused discontinuities in a conservative fashion is presented. Besides the description of the modeling, the discussion includes a mathematically rigorous validation with respect to the meaning of the model as well as a presentation of some numerical results.
Keywords: conservation laws, shock filter, finite difference methods
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1 Introduction

Within this paper, we develop a new shock filter model designed to enhance numerically smeared discontinuities occuring in the context of the approximation of conservation laws. The emphasis is laid in this work on the description of the modeling process and on the mathematically rigorous validation of the model. We also give numerical tests indicating the potential of the approach. The most important feature of the new shock filter is its divergence form: it is based on anti-diffusion rather than on propagation by the local curvature like the classical shock filter

$$u_t = - |u_x| \operatorname{signum}(u_{xx}) \tag{1}$$

proposed in [5]. Since the filter (1) is not in divergence form, it is *a priori* problematic to employ that filter in the context of hyperbolic conservation laws where the conservation property is of fundamental importance, especially in the context of shock approximation, see e.g. [4]. Concerning shock filters within the field of mathematical image analysis, the main issue is the regularisation of (1) because of its sensitivity to noise; see e.g. [2] for a useful discussion.

In this work we give a mathematically justified starting point for further developments of shock filtering methods. An earlier work concerned with such schemes approximating conservation laws contains algorithms derived largely on a heuristic basis on the discrete level [1]. Other attempts relying on the use of strategies from image processing were focused on the use of diffusion filters in order to reduce spurious oscillations, see e.g. [3] and the references therein.

Since this paper represents the beginning of our developments, we focus on the scalar one-dimensional case; applications of the developed model with respect to systems of equations and higher dimensions are under consideration.

The content of this paper is as follows. We model the filter and give it a sound mathematical basis. Finally, we briefly discuss some numerical results.

2 Derivation and rigorous validation of the model

The setting for the modeling process is given by taking into account one-dimensional scalar *Riemann problems*, i.e., we consider approximations of discontinuous solutions obtained for initial conditions

$$u_0(x) = \begin{cases} U_l : x \leq x_0 \\ U_r : x > x_0 \end{cases} . \tag{2}$$

The consideration of Riemann problems is very important for the development of algorithms for hyperbolic conservation laws. As it is well-known, the numerical approximation using a shock capturing and numerically very cheap first-order scheme shows the effect of abundant *numerical diffusion* when approximating a moving shock. See e.g. [4] for more information on these topics.

For the shortness of the presentation, we give the modeling process of our new shock filter for the case corresponding to such a Riemann problem with $U_l > U_r$. The case $U_l < U_r$ can be addressed in an analogous fashion.
We define the function

$$H\left(a\right) = \begin{cases} 0 : a \leq 0 \\ 1 : a > 0 \end{cases} . \tag{3}$$

This function is essentially the Heaviside function; note, however, the definition of $H(0)$ which plays a significant role in the following.
Let t_1 and t_2 be arbitrarily chosen but fixed times with $t_1 < t_2$. We also use an arbitrarily chosen but fixed one-dimensional spatial interval $[x_1,\, x_2]$, $x_1 < x_2$.
As usual, the total amount of a quantity $u \equiv u\left(x,\, t\right)$ within $[x_1,\, x_2]$ at a fixed time $\tilde{t} \geq 0$ is given by

$$\int_{x_1}^{x_2} u\left(x,\, \tilde{t}\right)\, dx. \tag{4}$$

The sought filter shall induce a bounded negative flux when the data gradient is negative. If the status of locally planar data is achieved, especially in the direction of this induced flux, the process shall stop. This ingredient is necessary in order to stabilise the process. We investigate the change of the quantity u within $[x_1,\, x_2]$ which reads

$$\frac{d}{dt}\int_{x_1}^{x_2} u\left(x,\, t\right)\, dx = \left[-H\left(-u_x\left(x_1,\, t\right)\right)\right] - \left[-H\left(-u_x\left(x_2,\, t\right)\right)\right]. \tag{5}$$

The form of equation (5) is known as the integral form of a conservation law, see e.g. [4]. The conservation of the property u is ensured since fluxes only take place at the boundaries of $[x_1,\, x_2]$. Note the influence of the definition of $H(0)$: if $u_x(\tilde{x}, \cdot) = 0$ holds no change takes place at $\tilde{x}$.
Integration over the time interval of interest $[t_1,\, t_2]$ yields

$$\begin{aligned} &\int_{x_1}^{x_2} u\,(x,\, t_2)\; dx - \int_{x_1}^{x_2} u\,(x,\, t_1)\; dx \\ &= \int_{t_1}^{t_2} \left[-H\left(-u_x\left(x_1,\, t\right)\right)\right] - \left[-H\left(-u_x\left(x_2,\, t\right)\right)\right]\, dt. \end{aligned} \tag{6}$$

Let us for the moment assume that H is smooth. Assuming then also the smoothness of u, we could derive

$$\int_{x_1}^{x_2} \int_{t_1}^{t_2} \frac{\partial}{\partial t} u\,(x,\, t)\; dtdx = -\int_{t_1}^{t_2} \int_{x_1}^{x_2} \frac{\partial}{\partial x} \left[-H\left(-u_x\left(x,\, t\right)\right)\right]\, dxdt, \tag{7}$$

i.e., because of the arbitrary choice of t_1, t_2, x_1, x_2 we would arrive at the p.d.e.

$$u_t(x,t) = H\left(-u_x(x,t)\right)_x. \tag{8}$$

Motivated by the fact that we started from the integral form of a conservation law (5), we want to keep equation (8) as our *filter model* and understand the solution u of (8) in a *weak sense* as a solution of

$$\begin{aligned} &\int_0^{\infty} \int_{-\infty}^{\infty} \left[u(x,t)\phi_t(x,t) - H\left(-u_x(x,t)\right)\phi_x(x,t)\right] dx\, dt \\ &= -\int_{-\infty}^{\infty} u(x,0)\phi(x,0)\, dx, \end{aligned} \tag{9}$$

where we employ the test function $\phi \in C_0^\infty\left(\mathbf{R} \times \mathbf{R}_+;\, \mathbf{R}\right)$. However, we need to make sure that the filter model (8) is not a meaningless equation since we used an unjustified (however, usual) smoothness assumption within its motivation.
In fact, (9) can be read as a distributional interpretation of (8). Solutions for piecewise differentiable initial data are described by the following Theorem.

Theorem 2.1 *Let $u_0 : \mathbf{R} \to \mathbf{R}$ be a piecewise differentiable function. Then there exists a unique function $u : \mathbf{R}\times[0,T] \to \mathbf{R}$ with the following properties:*

1. *The function u is piecewise differentiable in $\mathbf{R} \times [0,T]$, where regions of differentiability are separated by differentiable curves.*

2. *For every (x,t) with $\lim_{\xi\to x+0} u(\xi,t) \neq \lim_{\xi\to x-0} u(\xi,t)$, the function $u(\cdot,t)$ is non-increasing or non-decreasing within an open neighborhood of x.*

3. *The function u is a solution of (8) in a distributional sense.*

4. *If u_0 is continuous at x, then it also holds $\lim_{t\downarrow 0} u(x,t) = u_0(x)$.*

Furthermore, the following statements are true:

5. *Assume the curve $C : x = s(t)$ separates a region where u is constant in x from a region where u is decreasing in x with $u_x < 0$. Then u has a discontinuity along C where at each point (x,t), $x = s(t)$, the jump height and the slope of s are inversely proportional:*

$$s'(t) \times \left(\lim_{\xi\to x+0} u(\xi,t) - \lim_{\xi\to x-0} u(\xi,t) \right) = -1 \ .$$

6. *If $(x,t) \in \mathbf{R} \times [0,T]$ does not lie on a curve C as described in 5, u is constant w.r.t. t in (x,t).*

Proof Let $u(x,t)$ be a function with the properties 1–3.
Consider now a point (x_0,t_0) such that u is differentiable in an open neighbourhood of (x_0,t_0), and $u_x(x_0,t_0) \neq 0$. Since, as derivative of a constant, the right-hand side of (8) vanishes, the equation is a usual p.d.e., thus $u_t(x_0,t_0) = 0$.
Similarly we find $u_t(x_0,t_0) = 0$ if u is constant in x in a neighbourhood of (x_0,t_0). Consequently, the dynamics of u is concentrated at the boundaries of constant regions. By hypothesis, these boundaries are differentiable curves.
Assume now (x_0,t_0) is located on a differentiable curve $C : x = s(t)$, where s is strictly monotonically increasing, $s(t_0) = x_0$, and we have $u_x(x,t) = 0$ for $x < s(t)$, $u_x(x,t) < 0$ for $x > s(t)$ within a small open neighbourhood N of (x_0,t_0). Then $H(-u_x(x,t))$ is piecewise constant in N with a jump along C. Evaluating derivatives of single-layer potentials [6], we find that

$$(H(-u_x(x,t)))_x = \cos\vartheta \times \delta(x - s(t))$$

where $\vartheta = \arctan(s'(t))$ is the angle between the normal on C and the x-axis, and where δ is a one-dimensional delta distribution, see Fig. 1a. We have therefore

$$(H(-u_x(x,t)))_x = \frac{1}{\sqrt{1 + (s'(t))^2}} \times \delta(x - s(t)) \ . \qquad (10)$$

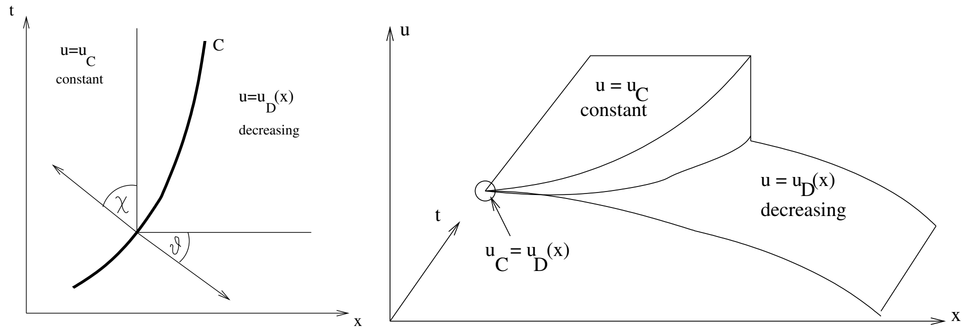

Figure 1: Left (a): A curve C separating constant and decreasing regions of u w.r.t. x, and the angles ϑ, χ involved in the distributional derivatives, see text. Right (b): Breaking of continuity at a local extremum.

Simultaneously, the left-hand side of (8) is the distributional derivative of a function that may have a jump along C but is smooth in $N \setminus C$. We find thus

$$u_t(x,t) = \cos\chi \times \left(\lim_{\tau\to t+0} u(x,\tau) - \lim_{\tau\to t-0} u(x,\tau) \right) \times \delta(t - s^{-1}(x))$$

where $\chi = \operatorname{arccot}(s'(t))$ is the angle between the normal on C and the y-axis, compare again Fig. 1. For the jump-height factor, note that within N, $u(x,t)$ for fixed x takes not more than two different values: The value u_{C} in the constant area left of C and a value $u_{\mathrm{D}}(x)$ in the decreasing area right of C. Consequently,

$$u_t(x,t) = \frac{s'(t)}{\sqrt{1+(s'(t))^2}}(u_{\mathrm{C}} - u_{\mathrm{D}}(x)) \times \delta(t - s^{-1}(x)) \qquad (11)$$

holds. Equalling the right-hand sides of (10) and (11) at (x_0, t_0) gives

$$\frac{s'(t)}{\sqrt{1+(s'(t))^2}}(u_{\mathrm{C}} - u_{\mathrm{D}}(x)) = \frac{1}{\sqrt{1+(s'(t))^2}}$$

as condition for a solution of (8), and therefore

$$s'(t) = \frac{1}{u_{\mathrm{C}} - u_{\mathrm{D}}(x)} \,. \qquad (12)$$

Provided that $u_{\mathrm{C}} > u_{\mathrm{D}}(x)$ holds in N, we see that (12) describes a discontinuity which for increasing t moves in positive x direction. It extends a constant plateau (maximum) while diminishing a decreasing slope right of

the discontinuity. Since the velocity at which the discontinuity moves is inverse proportional to the height of the jump, the integral of u over x grows in t at constant rate.
The argument can be extended to the case when the tangent of C in (x_0, t_0) is in x direction. Only then we can, and must, have $u_\mathrm{C} = u_\mathrm{D}(x_0)$. Since u_D is decreasing in x, it cannot equal u_C elsewhere in a neighbourhood of x_0. Thus, if for some t_0 the transition between a local maximum or inflection point and the decreasing slope is continuous, a discontinuity will start out from that point at infinite initial speed and instantaneously break up the continuity, see Fig. 1b.
In the symmetric situation when a decreasing slope lies left of a minimum or inflection point w.r.t. x, analogous arguments imply that for t increasing a discontinuity travels in negative x direction, obeying eq. (12). This process reduces the integral of u over x at constant rate w.r.t. t.
Finally, a downward jump between two constant plateaus leads to $u_t = 0$ and is therefore not moved. Increasing slopes w.r.t. x, too, are not affected by (8) since $H(-u_x)$ equals 1 for both extrema and increasing intervals.
Consequently, solutions of (8) as characterised in the Theorem must have all claimed properties. Their existence is obvious which concludes the proof. ■

Considering (8) as evolution in t, its action can be summarized as follows: Each decreasing slope linking a local maximum with a local minimum is successively eliminated by two discontinuity waves which travel from the maximum to the right and from the minimum to the left, extending the extrema to the region between them. One discontinuity increases the integral over u at constant rate while the other one decreases it at the same rate, making the entire action of the filter on the slope conservative. The process stops when both shocks merge into one single jump between the maximum and the minimum.

3 Numerical tests

We have applied our filter as a corrector step in sequence with numerical schemes for conservation laws as documented by the two following test cases. For the first numerical test, we have incorporated a simple discretisation of our filter within a standard TVD scheme, see e.g. [4]. We approximate the well-known non-linear *Burgers equation* $u_t + (u^2/2)_x = 0$ using a square signal as initial condition. In our example, the location of the moving shock front is given exactly at a cell border. By Figure 2, we demonstrate the effect of the filtering. In contrast to the numerical solution obtained by a classical TVD scheme even on a very fine mesh, the shock is captured as exact as the

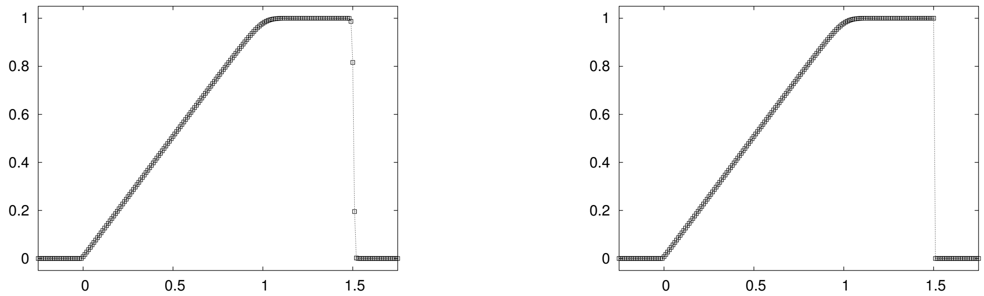

Figure 2: Numerical solutions obtained (left) with a classical TVD scheme and (right) by the same scheme featuring additionally our shock filter.

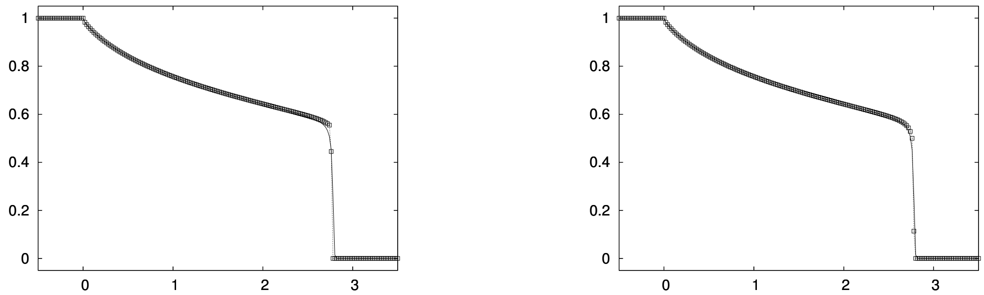

Figure 3: Comparison of numerical solutions obtained by use of the Upwind scheme (lines) and shock filtering schemes (square symbols): on the left hand side using an accurate representation of the inflection point and on the right hand side a rough approximation.

mesh enables when using in addition our shock filter. In case the exact shock location is not identical with a cell border, there is at most one grid point defining the shock location.
A second numerical test is concerned with the *Buckley-Leverett equation* $u_t + [u^2/(u^2 + a(1-u)^2)]_x = 0$, $a := 1/2$, featuring a non-convex flux function yielding a *mixed-wave solution* of the classical Riemann problem, i.e., a left state $u_L = 1$ and a right state $u_R = 0$, separated at $x = 0$, which we use as initial condition. Here, we have coupled a numerical routine estimating the location of the *inflection point* of the flux function with the shock filter, underlying we use the first-order Upwind scheme. By Figure 3 (left), we see the numerical solution obtained for an accurate representation of the latter value, whereas by Figure 3 (right) we illuminate the effect of a rough approximation (setting this value as 1/2): the resulting algorithm yields a slightly improved resolution of the shock compared to the unfiltered signal computed by the Upwind scheme, but it is not unstable, showing the robustness of the approach.

Note that the application of the filter relies on the use of a suitable indicator. In the first test case this not an issue due to the convexity of the flux, while in the second test we have used as indicated an estimator of the inflection point. In the case of systems, we conjecture that similar procedures can be used relying on the physical properties of flow discontinuities, see e.g. [4] for a discussion of such properties for the case of the Euler equations of gas dynamics.

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