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Abstract

We consider the stationary Stokes problem for a class of power-law fluids and prove functional type a posteriori error estimates for the difference of the exact solution and any admissible function from the energy class.

1 Introduction

We consider a stationary and also slow flow of a power-law fluid in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. More precisely, we are looking for a velocity field $u\colon \Omega \to \mathbb{R}^n$ and a pressure function $p\colon \Omega \to \mathbb{R}$ such that the following nonlinear system of partial differential equations is satisfied:

$$-\operatorname{div} \sigma = f - \nabla p \quad \text{in } \Omega; \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega; \tag{1.2}$$

$$\sigma = D\pi\,(\varepsilon(u)) \quad \text{in } \Omega; \tag{1.3}$$

$$u = u_0 \quad \text{on } \partial\Omega. \tag{1.4}$$

Here $\varepsilon(u)$ is the symmetric gradient of u, the tensor σ represents the deviatoric part of the stress-tensor, $f\colon \Omega \to \mathbb{R}^n$ is a given system of volume forces, and $u_0\colon \Omega \to \mathbb{R}^n$ denotes a fixed boundary datum such that $\operatorname{div} u_0 = 0$ in Ω. In the constitutive relation (1.3), π is a power-law potential, i.e. we assume that for some $\alpha \in (1, \infty)$

$$\pi(\varepsilon) = \frac{1}{\alpha}|\varepsilon|^\alpha. \tag{1.5}$$

In the case $\alpha = 2$ the problem (1.1)–(1.4) reduces to the Stokes-problem for a Newtonian fluid (see, e.g. [La] or [Ga]), the physical relevance of the general power-law model (1.5) is discussed in the monographs [AM] and [BAH].

It is well-known (see, e.g. [DL] or [FS]) how to give a weak formulation of the problem (1.1)–(1.4) in the correct function spaces (depending on the value of α). So let $u_0 \in W^1_\alpha(\Omega; \mathbb{R}^n)$ be given such that $\operatorname{div} u_0 = 0$ and assume further that $f \in L^{\alpha^*}(\Omega; \mathbb{R}^n)$, where $\alpha^* = \alpha/\alpha - 1$ is the exponent conjugate to α. (For a definition of the Lebesgue- and Sobolev-spaces, L^{α^*}, W^1_α, etc., we refer to [Ad].) Let $\overset{\circ}{V}_\alpha$ denote the closure of all smooth

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solenoidal vectorfields with compact support in Ω w.r.t. the norm of $W^1_\alpha(\Omega;\mathbb{R}^n)$. Then $u \in u_0 + \overset{\circ}{V}_\alpha$ is termed a weak solution of (1.1)–(1.4) if and only if

$$\int_\Omega \sigma(u) : \varepsilon(w)\, dx = \int_\Omega f \cdot w\, dx \quad \text{for all}\, w \in \overset{\circ}{V}_\alpha, \tag{1.6}$$

where $\sigma(u) := D\pi\big(\varepsilon(u)\big)$. We remark that (1.6) is the Euler equation for the functional

$$J[v] = \int_\Omega \Big[\pi\big(\varepsilon(v)\big) - f \cdot v\Big]\, dx, \quad v \in u_0 + \overset{\circ}{V}_\alpha,$$

and since J is strictly convex, continuous and coercive on the space $u_0 + \overset{\circ}{V}_\alpha$, the minimization problem

$$(\mathcal{P}) \qquad J\,[v] \rightsquigarrow \min \ \text{on}\ u_0 + \overset{\circ}{V}_\alpha$$

has a unique solution u whose smoothness is discussed for example in [FS].

The main purpose of our paper is to obtain explicitely computable upper bounds for the difference between this exact solution u and any "approximation" v from the energy class $u_0 + \overset{\circ}{V}_\alpha$ which in the optimal case take the form

$$\|\varepsilon(u - v)\|_{L^\alpha} \le \mathcal{M}(v, f, \alpha, \Omega, u_0), \tag{1.7}$$

where $\mathcal{M}$ is a non-negative functional depending on the problem data such as f, α, Ω, u_0 and which vanishes if and only if $v = u$. Of course, an estimate like (1.7) is of practical importance provided that:

i.) the value of $\mathcal{M}$ can be explicitly computed for any admissible v;

ii.) $\mathcal{M}(v_k, \cdot) \to 0$ as $v_k \to u$ in the energy space;

iii.) $\mathcal{M}$ provides a realistic upper bound for the quantity $\|\varepsilon(v - u)\|_{L^\alpha}$.

Estimates of the form (1.7) sharing the properties i.)–iii.) are known as functional type a posteriori error estimates. Their clear advantage is that they do not refer to any concrete numerical scheme and therefore give explicit control for the accuracy of an approximation regardless of the way in which such an approximation has been constructed. We like to mention that functional type a posteriori error estimates for various settings have already been established in the papers [Re1]–[Re4] and [BR], error estimates for other classes of generalized Newtonian fluids recently have been treated in [FR], where mainly lower order perturbations of a Newtonian fluid where considered.

Our paper is organized as follows: in Section 2 we give a precise formulation and a proof of our results for the case $\alpha \ge 2$, i.e. we will study several variants of the principal estimate (1.7). The case $1 < \alpha < 2$ is studied in Section 3 and requires the use of different methods: in the subquadratic case it is impossible to find a natural upper bound for the quantity $\|\varepsilon(u-v)\|_{L^\alpha}$ that makes it fully controllable as in the case $\alpha \geq 2$. We therefore pass to the dual variational problem $(\mathcal{P})^*$ with unique solution σ and establish suitable estimates for the error $\|\sigma - \overline{\sigma}\|_{L^{\alpha^*}}$, where $\overline{\sigma}$ denotes an admissible comparison tensor.

2 Error estimates for the case $\alpha \geq 2$

First, we have to introduce some notation: let $Y := L^\alpha(\Omega;\mathbb{R}^{n\times n})$ and $Y^* := L^{\alpha^*}(\Omega;\mathbb{R}^{n\times n})$. The dual variational problem associated to problem $(\mathcal{P})$ is the maximization problem

$$(\mathcal{P})^* \qquad\qquad J^*[\sigma] \to \max \text{ in } Y^*,$$

where in our case J^* is given by

$$\begin{aligned} J^*[\sigma] &:= \begin{cases} \displaystyle\int_\Omega \Big[\varepsilon(u_0) : \sigma - \frac{1}{\alpha^*}\,|\sigma|^{\alpha^*} - f\cdot u_0\Big]\,dx, & \text{if} \quad \sigma \in Q_f, \\ \qquad\qquad -\infty, & \text{if} \quad \sigma \neq Q_f, \end{cases} \\ Q_f &:= \Big\{\sigma \in Y^* : \int_\Omega \sigma : \varepsilon(w)\,dx = \int_\Omega f\cdot w\,dx \quad \text{for all } w \in \overset{\circ}{V}_\alpha\Big\}. \end{aligned}$$

For a general definition of the dual variational problem and for further information on convex analysis we refer the reader to [ET], where one can also find the following facts.

If $u \in u_0 + \overset{\circ}{V}_\alpha$ and $\sigma \in Q_f$ denote the unique solutions to the problems $(\mathcal{P})$ and $(\mathcal{P})^*$, respectively, then we have

$$\text{(2.1)} \qquad \inf_{u_0+\overset{\circ}{V}_\alpha} J = J[u] = J^*[\sigma] = \sup_{Y^*} J^*,$$

$$\text{(2.2)} \qquad \sigma = |\varepsilon(u)|^{\alpha-2}\,\varepsilon(u) \quad \text{a.e. in } \Omega,$$

$$\text{(2.3)} \qquad \varepsilon(u) = |\sigma|^{\alpha^*-2}\,\sigma \quad \text{a.e. in } \Omega.$$

Now we can state our first result:

THEOREM 2.1. *Let $\alpha \geq 2$. With the notation introduced above we have for any $v \in u_0 + \overset{\circ}{V}_\alpha$ and for any $\kappa \in Y^*$ with α^*-summable divergence and for any $\beta > 0$ the estimate*

$$\text{(2.4)} \qquad \big\|\varepsilon(v-u)\big\|^\alpha_{L^\alpha} \leq \alpha\; 2^{\alpha-1}\Big(\mathcal{M}_1\big[\varepsilon(v),\kappa,\beta\big] + \mathcal{M}_2\big[\kappa,\beta\big]\Big),$$

where

$$\begin{aligned} \mathcal{M}_1\big[\varepsilon(v),\kappa,\beta\big] &:= D_\alpha\big[\varepsilon(v),\kappa\big] + \frac{\beta^\alpha}{\alpha}\,\big\||\kappa|^{\alpha^*-2}\kappa - \varepsilon(v)\big\|^\alpha_{L^\alpha}, \\ \mathcal{M}_2[\kappa,\beta] &:= C_\alpha(\Omega)^{\alpha^*}\Big[\frac{1}{\alpha^*\beta^{\alpha^*}} + 2^{2-\alpha^*}(3-\alpha^*)\Big]\big\|f + \operatorname{div}\kappa\big\|^{\alpha^*}_{L^{\alpha^*}}, \end{aligned}$$

and where the functional $D_\alpha : Y \times Y^* \to \mathbb{R}_0^+$ is defined as follows:

$$D_\alpha[\varkappa, \kappa] := \int_\Omega \Big[\frac{1}{\alpha}\,|\varkappa|^\alpha + \frac{1}{\alpha^*}\,|\kappa|^{\alpha^*} - \varkappa : \kappa\Big]\; dx.$$

REMARK 2.1. *i) The constant $C_\alpha(\Omega)$ is defined in formula (2.15) as the product of the constants appearing in Korn's and Poincaré's inequalities.*

ii) Clearly the right-hand side of (2.4) vanishes if and only if

$$\begin{cases} |\kappa|^{\alpha^*-2}\,\kappa = \varepsilon(v) & \textit{and} \\ \operatorname{div}\,\kappa + f = 0 & \textit{a.e. in } \Omega. \end{cases} \tag{2.5}$$

By uniqueness, (2.5) is equivalent to $v = u$ together with $\kappa = \sigma$.

iii) The functional D_α evidently is nonnegative and zero if and only if $\kappa = |\varkappa|^{\alpha-2}\varkappa$, $\varkappa = |\kappa|^{\alpha^*-2}\kappa$, thus $D_\alpha[\varepsilon(v), \kappa]$ is a measure for the error in the duality relations (2.2) and (2.3). $\mathcal{M}_1$ measures the same quantity, whereas $\mathcal{M}_2$ controls the deviation from the equilibrium equation.

(iv) If $v_k \to u$ and $\kappa_k \to \sigma$ with respect to the norms of the corresponding spaces, then it is easy to see that the right–hand side of (2.4) goes to zero.

Proof of Theorem 2.1: According to [MM1] we have the following inequality valid in case $\alpha \geq 2$ for arbitrary tensor-valued functions τ_1, $\tau_2 \in Y$

$$\int_\Omega \Big[\Big|\frac{\tau_1 + \tau_2}{2}\Big|^\alpha + \Big|\frac{\tau_1 - \tau_2}{2}\Big|^\alpha\Big]\; dx \;\leq\; \frac{1}{2}\,\|\tau_1\|_{L^\alpha}^\alpha + \frac{1}{2}\,\|\tau_2\|_{L^\alpha}^\alpha. \tag{2.6}$$

Using (2.6) we get for any $v \in u_0 + \overset{\circ}{V}_\alpha$

$$\begin{aligned} & J[v] + J[u] - 2\,J\Big[\frac{u+v}{2}\Big] \\ = \;& \int_\Omega \Big[\pi\big(\varepsilon(v)\big) + \pi\big(\varepsilon(u)\big) - 2\pi\Big(\frac{\varepsilon(u) + \varepsilon(v)}{2}\Big)\Big[\; dx \\ = \;& \frac{2}{\alpha}\int_\Omega \Big[\frac{1}{2}\,\big|\varepsilon(v)\big|^\alpha + \frac{1}{2}\,\big|\varepsilon(u)\big|^\alpha - \Big\{\frac{|\varepsilon(u) + \varepsilon(v)|}{2}\Big\}^\alpha\Big]\; dx \\ \geq \;& \frac{2}{\alpha}\int_\Omega \Big|\frac{\varepsilon(v-u)}{2}\Big|^\alpha\; dx = \frac{1}{\alpha\, 2^{\alpha-1}}\,\big\|\varepsilon(v-u)\big\|_{L^\alpha}^\alpha. \end{aligned}$$

The J-minimality of u implies

$$J[v] + J[u] - 2\,J\Big[\frac{u+v}{2}\Big] \leq J[v] - J[u],$$

hence

$$\|\varepsilon(v-u)\|_{L^\alpha}^{\alpha} \le \alpha\, 2^{\alpha-1}\big(J[v]-J[u]\big). \tag{2.7}$$

On the right-hand side of (2.7) we apply (2.1) and get

$$\|\varepsilon(v-u)\|_{L^\alpha}^{\alpha} \le \alpha\, 2^{\alpha-1}\big(J[v]-J^*[\sigma]\big) \le \alpha\, 2^{\alpha-1}\big(J[v]-J^*[\tau]\big) \tag{2.8}$$

being valid for any $v \in u_0 + \overset{\circ}{V}_\alpha$ and any $\tau \in Q_f$. Observing

$$\begin{aligned}
J[v]-J^*[\tau] &= \int_\Omega \Big[\frac{1}{\alpha}\,\big|\varepsilon(v)\big|^\alpha + \frac{1}{\alpha^*}|\tau|^{\alpha^*} - \varepsilon(u_0):\tau - f\cdot(v-u_0)\Big]\,dx\\
&= \int_\Omega \Big[\frac{1}{\alpha}\big|\varepsilon(v)\big|^\alpha + \frac{1}{\alpha^*}\big|\tau\big|^{\alpha^*} - \varepsilon(v):\tau\Big]\,dx\\
&= D_\alpha\big[\varepsilon(v),\tau\big],
\end{aligned}$$

inequality (2.8) can be rewritten as

$$\|\varepsilon(v-u)\|_{L^\alpha}^{\alpha} \le \alpha\, 2^{\alpha-1} D_\alpha\big[\varepsilon(v),\tau\big] \tag{2.9}$$

for functions v and tensors τ as above. Of course estimate (2.9) suffers from the fact that τ has to satisfy the differential equation $\operatorname{div}\tau + f = 0$ which might be hard to verify for a concrete approximation. (The same is true for the constraint $\operatorname{div} v = 0$ but this obstacle will be removed in a second step after having finished the proof of Theorem 2.1.)

So let κ denote any tensor from Y^* with the property that $\operatorname{div}\kappa \in L^{\alpha^*}(\Omega;\mathbb{R}^n)$. Then we have for any choices of $\tau \in Q_f$ and $v \in u_0 + \overset{\circ}{V}_\alpha$

$$\begin{aligned}
J[v]-J^*[\tau] &= D_\alpha\big[\varepsilon(v),\kappa\big] + \int_\Omega \Big[\frac{1}{\alpha^*}\,\big|\tau\big|^{\alpha^*} - \frac{1}{\alpha^*}\,|\kappa|^{\alpha^*} + \varepsilon(v):(\kappa-\tau)\Big]\,dx\\
&\le D_\alpha\big[\varepsilon(v),\kappa\big] + \int_\Omega \Big[|\tau|^{\alpha^*-2}\,\tau:(\tau-\kappa) + \varepsilon(v):(\kappa-\tau)\Big]\,dx\\
&= D_\alpha\big[\varepsilon(v),\kappa\big] + \int_\Omega \big(|\kappa|^{\alpha^*-2}\,\kappa - \varepsilon(v)\big):(\tau-\kappa)\,dx\\
&\qquad + \int_\Omega \big(|\tau|^{\alpha^*-2}\,\tau - |\kappa|^{\alpha^*-2}\,\kappa\big):(\tau-\kappa)\,dx.
\end{aligned}$$

In order to control the last integral on the right–hand side we use the inequality

$$\Big(\frac{a}{|a|^\Theta} - \frac{b}{|b|^\theta}\Big)\cdot(a-b) \le 2^\Theta(1+\Theta)\,|a-b|^{2-\Theta} \tag{2.10}$$

valid for all $a, b \in \mathbb{R}^l, l \geq 1$, and any choice of $\Theta \in [0,1)$, a proof being presented e.g. in [BR]. Inequality (2.10) provides the bound

$$\int\limits_\Omega \left(|\tau|^{\alpha^*-2}\,\tau - |\kappa|^{\alpha^*-2}\,\kappa\right) : (\tau-\kappa)\;dx \leq 2^{2-\alpha^*}(3-\alpha^*)\int\limits_\Omega |\tau-\kappa|^{\alpha^*}\,dx,$$

and we deduce

$$\tag{2.11} \begin{aligned} J[v] - J^*[\tau] \leq D_\alpha\big[\varepsilon(v),\kappa\big] \quad &+ \quad \big\|\,|\kappa|^{\alpha^*-2}\,\kappa - \varepsilon(v)\big\|_{L^\alpha}\big\|\tau-\kappa\big\|_{L^{\alpha^*}} \\ &+ \quad 2^{2-\alpha^*}(3-\alpha^*)\,\|\tau-\kappa\|^{\alpha^*}_{L^{\alpha^*}} \end{aligned}$$

valid for all v, τ, κ as specified before. Of course we like to use (2.11) as an upper bound for the right-hand side of (2.8), and in order to do so we now select $\tau \in Q_f$ in such a way that we can control the norm $\|\tau-\kappa\|_{L^{\alpha^*}}$ by a quantity which measures how far κ is away from satisfying the equation div $\kappa + f = 0$.
So again, fix $\kappa \in Y^*$ such that div $\kappa \in L^{\alpha^*}(\Omega;\mathbb{R}^n)$ and let τ denote the solution of the minimum problem

$$\|\tau-\kappa\|_{L^{\alpha^*}} \longrightarrow \min \text{ in } Q_f.$$

Letting $Q_{\tilde f} := \{\lambda \in Y^* : \int_\Omega \lambda : \varepsilon(w)\;dx = \int_\Omega \tilde f \cdot w\;dx \text{ for all } w \in \overset{\circ}{V}_\alpha\}$, where $\tilde f := f + \text{div }\kappa$, it is immediate that

$$\inf_{\tau' \in Q_f} \frac{1}{\alpha^*}\,\|\tau'-\kappa\|^{\alpha^*}_{L^{\alpha^*}} = -\sup_{\lambda \in Q_{\tilde f}}\Big[-\frac{1}{\alpha^*}\,\|\lambda\|^{\alpha^*}_{L^{\alpha^*}}\Big],$$

and the problem on the right-hand side can be seen as a dual problem, more precisely, we have

$$\tag{2.12} \sup_{\lambda \in Q_{\tilde f}}\Big[-\frac{1}{\alpha^*}\,\|\lambda\|^{\alpha^*}_{L^{\alpha^*}}\Big] = \inf_{w \in \overset{\circ}{V}_\alpha}\int\limits_\Omega \Big[\frac{1}{\alpha}|\varepsilon(w)|^\alpha - \tilde f \cdot w\Big]\,dx.$$

For any $w \in \overset{\circ}{V}_\alpha$ it holds

$$\tag{2.13} \begin{aligned} &\int\limits_\Omega \Big[\frac{1}{\alpha}\,|\varepsilon(w)|^\alpha - \tilde f \cdot w\Big]\;dx \\ \geq\;& \frac{1}{\alpha}\,K_\alpha^{-\alpha}(\Omega)\|\nabla w\|^\alpha_{L^\alpha} - \|\tilde f\|_{L^{\alpha^*}}\|\nabla w\|_{L^\alpha}P_\alpha(\Omega) \\ \geq\;& \inf_{t\geq 0}\Big[K_\alpha^{-\alpha}(\Omega)\frac{1}{\alpha}t^\alpha - P_\alpha(\Omega)\|\tilde f\|_{L^{\alpha^*}}\,t\Big], \end{aligned}$$

where $P_\alpha(\Omega)$ is the constant in Poincaré's inequality and $K_\alpha(\Omega)$ denotes the constant in Korn's inequality, i.e. $K_\alpha(\Omega)$ is the smallest number s.t.

$$K_\alpha^\alpha(\Omega)\int\limits_\Omega |\varepsilon(w)|^\alpha\;dx \geq \int\limits_\Omega |\nabla w|^\alpha\,dx \quad \text{for all } w \in \overset{\circ}{V}_\alpha\,.$$

For this version of the classical Korn's inequality we refer the reader to [MM2]. The infimum on the right–hand side of (2.13) is attained for the value

$$t_0 = K_\alpha(\Omega)^{\alpha^*} P_\alpha(\Omega)^{\frac{1}{\alpha-1}} \, \|\tilde{f}\|_{L^{\alpha^*}}^{\frac{1}{\alpha-1}},$$

and by inserting t_0 we deduce

$$\inf_{w\in \overset{\circ}{V}_\alpha} \int_\Omega \Big[\frac{1}{\alpha}\,|\varepsilon(w)|^\alpha - \tilde{f}\cdot w\Big]\,dx \geq -\frac{1}{\alpha^*}\,\Big[P_\alpha(\Omega)\,K_\alpha(\Omega)\,\|\tilde{f}\|_{L^{\alpha^*}}\Big]^{\alpha^*}. \tag{2.14}$$

We let

$$C_\alpha(\Omega) := P_\alpha(\Omega)K_\alpha(\Omega). \tag{2.15}$$

Then, by combining (2.12), (2.13) and (2.14), we get

$$\inf_{\tau'\in Q_f} \frac{1}{\alpha^*}\|\tau' - \kappa\|_{L^{\alpha^*}}^{\alpha^*} \leq \frac{1}{\alpha^*}\,C_\alpha(\Omega)^{\alpha^*}\,\|f + \operatorname{div}\,\kappa\|_{L^{\alpha^*}}^{\alpha^*}. \tag{2.16}$$

Recalling (2.8), using (2.11) for estimating the right–hand side of (2.8), taking the infimum w.r.t $\tau \in Q_f$ on the r.h.s. of (2.11) and estimating the resulting quantity with the help of (2.16), we finally see that

$$\begin{aligned} \big\|\varepsilon(v-u)\big\|_{L^\alpha}^\alpha \leq \alpha\, 2^{\alpha-1}\Big[& D_\alpha[\varepsilon(v),\kappa] \\ & + C_\alpha(\Omega)\,\big\||\kappa|^{\alpha^*-2}\,\kappa - \varepsilon(v)\big\|_{L^\alpha}\,\|f+\operatorname{div}\,\kappa\|_{L^{\alpha^*}} \\ & + 2^{2-\alpha^*}(3-\alpha^*)C_\alpha(\Omega)^{\alpha^*}\,\|f+\operatorname{div}\,\kappa\|_{L^{\alpha^*}}^{\alpha^*}\Big]. \end{aligned} \tag{2.17}$$

Note that (2.17) is valid for any $v \in u_0 + \overset{\circ}{V}_\alpha$ and all $\kappa \in Y^*$ such that div $\kappa \in L^{\alpha^*}(\Omega;\mathbb{R}^n)$. Using Young's inequality, i.e. (a, $b \geq 0$, $\beta > 0$)

$$ab \leq \frac{\beta^\alpha}{\alpha}a^\alpha + \frac{1}{\alpha^*\beta^{\alpha^*}}b^{\alpha^*},$$

inequality (2.4) follows from (2.17), the proof of Theorem 2.1 is complete. □

Up to now we considered approximations v from the energy space $u_0 + \overset{\circ}{V}_\alpha$, in particular div $v = 0$ is required. If we drop this condition, then we have to estimate the distance of v to the set of solenoidal vectorfields which can be done with the help of the following

LEMMA 2.1. *(see, e.g. [LS], [Pi] or [Ga]) There exists a positive constant $\overline{C}_\alpha(\Omega)$ depending on Ω such that for any function $\phi \in L^\alpha(\Omega)$ satisfying $\int_\Omega \phi\, dx = 0$ we can find a vectorfield $u_\phi \in \overset{\circ}{W}{}^1_\alpha(\Omega;\mathbb{R}^n)$ such that*

$$\operatorname{div}\, u_\phi = \phi \text{ and } \|Du_\phi\|_{L^\alpha} \leq \overline{C}_\alpha(\Omega)\|\phi\|_{L^\alpha}. \tag{2.18}$$

Given a function $\overline{v} \in u_0 + \overset{\circ}{W}{}^1_\alpha(\Omega;\mathbb{R}^n)$we apply the lemma with $\phi := \operatorname{div} \overline{v}$ (note that $\int_\Omega \operatorname{div} \overline{v}\, dx = \int_\Omega \operatorname{div} u_0\, dx = 0$) and find u_ϕ with (2.18), in particular $v := \overline{v} - u_\phi$ is in $u_0 + \overset{\circ}{V}_\alpha$ so that (2.4) is valid. We obtain

$$\begin{aligned} \big\|\varepsilon(u-\overline{v})\big\|_{L^\alpha} &\le \big\|\varepsilon(u-v)\big\|_{L^\alpha} + \big\|\varepsilon(v-\overline{v})\big\|_{L^\alpha} \\ &\le \big\|\varepsilon(u-v)\big\|_{L^\alpha} + \rho\,(\overline{v}), \end{aligned}$$

where $\rho\,(\overline{v}) := \overline{C}_\alpha(\Omega)\|\operatorname{div} \overline{v}\|_{L^\alpha}$, and (2.4) implies

$$\big\|\varepsilon(u-\overline{v})\big\|_{L^\alpha} \le \rho(\overline{v}) + \alpha^{1/\alpha}2^{1-1/\alpha}\Big(\mathcal{M}_1\big[\varepsilon(v),\tau,\beta\big] + \mathcal{M}_2[\tau,\beta]\Big)^{1/\alpha} \tag{2.19}$$

with $\beta > 0$ and $\tau \in Y^*$ with α^*-summable divergence. Obviously we do not have to change $\mathcal{M}_2$. We discuss the quantity $D_\alpha\big[\varepsilon(v),\tau\big]$ occurring in $\mathcal{M}_1$: from the convexity of the potential π we get

$$\begin{aligned} D_\alpha\big[\varepsilon(v),\tau\big] &= \int_\Omega \Big[\frac{1}{\alpha}\,\big|\varepsilon(v)\big|^\alpha + \frac{1}{\alpha^*}\,|\tau|^{\alpha^*} - \varepsilon(v) : \tau\Big]\, dx \\ &\le \int_\Omega \Big[\frac{1}{\alpha}\big|\varepsilon(\overline{v})\big|^\alpha + \frac{1}{\alpha^*}|\tau|^{\alpha^*} - \varepsilon(\overline{v}) : \tau\Big]\, dx \\ &\quad + \int_\Omega \Big(\big|\varepsilon(v)\big|^{\alpha-2}\varepsilon(v) - \tau\Big) : \varepsilon(v-\overline{v})\, dx \\ &= D_\alpha\big[\varepsilon(\overline{v}),\tau\big] + \int_\Omega \Big(\big|\varepsilon(\overline{v})\big|^{\alpha-2}\varepsilon(\overline{v}) - \tau\Big) : \varepsilon(v-\overline{v})\ dx \\ &\quad + \int_\Omega \Big(\big|\varepsilon(v)\big|^{\alpha-2}\varepsilon(v) - \big|\varepsilon(\overline{v})\big|^{\alpha-2}\varepsilon(\overline{v})\Big) : \varepsilon(v-\overline{v})\ dx \\ &=: D_\alpha\big[\varepsilon(\overline{v}),\tau\big] + T_1 + T_2. \end{aligned} \tag{2.20}$$

Clearly

$$\begin{aligned} T_1 &\le \big\|\varepsilon(v-\overline{v})\big\|_{L^\alpha}\big\||\varepsilon(\overline{v})|^{\alpha-2}\varepsilon(\overline{v}) - \tau\big\|_{L^{\alpha^*}} \\ &\le \rho(\overline{v})\big\||\varepsilon(\overline{v})|^{\alpha-2}\,\varepsilon(\overline{v}) - \tau\big\|_{L^{\alpha^*}} \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} T_2 &\le \big\|\varepsilon(v-\overline{v})\big\|_{L^\alpha}\Big\{\big\||\varepsilon(v)|^{\alpha-1}\big\|_{L^{\alpha^*}} + \big\||\varepsilon(\overline{v})|^{\alpha-1}\big\|_{L^{\alpha^*}}\Big\} \\ &\le \rho(\overline{v})\Big\{\Big(\int_\Omega |\varepsilon(v)|^\alpha\, dx\Big)^{\frac{\alpha-1}{\alpha}} + \Big(\int_\Omega |\varepsilon(\overline{v})|^\alpha\, dx\Big)^{\frac{\alpha-1}{\alpha}}\Big\} \\ &\le \rho(\overline{v})\ \Big\{\big\|\varepsilon(\overline{v})\big\|^{\alpha-1}_{L^\alpha} + \Big[\big\|\varepsilon(\overline{v})\big\|_{L^\alpha} + \rho(\overline{v})\Big]^{\alpha-1}\Big\}. \end{aligned} \tag{2.22}$$

Let us finally look at the term $\big\||\tau|^{\alpha^*-2}\tau-\varepsilon(v)\big\|_{L^\alpha}$ which is also a part of $\mathcal{M}_1$: here we just observe

$$\big\||\tau|^{\alpha^*-2}\tau-\varepsilon(v)\big\|_{L^\alpha}\le\big\||\tau|^{\alpha^*-2}\tau-\varepsilon(\overline{v})\big\|_{L^\alpha}+\rho\,(\overline{v}).$$

Putting together (2.19)–(2.22) and the latter estimate we have established the following result:

THEOREM 2.2. *Let $\alpha\ge 2$. With the notation introduced above we have for any $\overline{v}\in u_0+\overset{\circ}{W}{}^1_\alpha(\Omega;\mathbb{R}^n)$, for any $\tau\in Y^*$ such that* div $\tau\in L^{\alpha^*}(\Omega;\mathbb{R}^n)$ *and for any $\beta>0$ the inequality*

$$\begin{aligned}
&\big\|\varepsilon(u-\overline{v})\big\|_{L^\alpha}\le\rho\,(\overline{v})\\
&\quad+\ \alpha^{1/\alpha}\,2^{1-1/\alpha}\Big(\mathcal{M}_1\big[\varepsilon(\overline{v}),\tau,\beta\big]+\mathcal{M}_2[\tau,\beta]\\
&\quad+\ \rho(\overline{v})\,\Big\{\big\||\varepsilon(\overline{v})|^{\alpha-2}\varepsilon(\overline{v})-\tau\big\|_{L^{\alpha^*}}+\big\|\varepsilon(\overline{v})\big\|^{\alpha-1}_{L^\alpha}\\
&\qquad+\Big[\big\|\varepsilon(\overline{v})\big\|_{L^\alpha}+\rho\,(\overline{v})\Big]^{\alpha-1}\Big\}\\
&\quad+\ 2^{\alpha-1}\frac{\beta^\alpha}{\alpha}\,\Big[\rho(\overline{v})^\alpha+\big\||\tau|^{\alpha^*-2}\tau-\varepsilon(\overline{v})\big\|^\alpha_{L^\alpha}\Big]\Big)^{1/\alpha}.
\end{aligned}\tag{2.23}$$

The reader should note that the factor $2^{\alpha-1}$ in front of the last item on the right-hand side of (2.23) comes from the estimate

$$\Big(\big\||\tau|^{\alpha^*-2}\tau-\varepsilon(\overline{v})\big\|_{L^\alpha}+\rho\,(\overline{v})\Big)^\alpha\le 2^{\alpha-1}\Big\{\big\||\tau|^{\alpha^*-2}\tau-\varepsilon(\overline{v})\big\|^\alpha_{L^\alpha}+\rho\,(\overline{v})^\alpha\Big\}.$$

We also like to remark that apart of the more complicated form the principal structure of the estimate (2.23) is the same as for solenoidal fields: the right-hand side is a combination of the terms

$$\begin{aligned}
&\big\|f+\mathrm{div}\ \tau\big\|_{L^{\alpha^*}},\quad \big\||\varepsilon(\overline{v})|^{\alpha-2}\varepsilon(\overline{v})-\tau\big\|_{L^{\alpha^*}},\\
&\big\||\tau|^{\alpha^*-2}\tau-\varepsilon(\overline{v})\big\|_{L^\alpha},\quad D_\alpha\big[\varepsilon(\overline{v}),\tau\big]\quad\text{and}\quad\rho\,(\overline{v}),
\end{aligned}$$

and the right-hand side of (2.23) vanishes if and only if all these terms are equal to zero, which means that $\overline{v}$ and τ must coincide with the exact solutions u and σ of the problems $(\mathcal{P})$ and $(\mathcal{P})^*$, respectively. Of course it would be desirable to avoid the quantity $\big\|\varepsilon(\overline{v})\big\|_{L^\alpha}$ in estimate (2.23).

3 Error estimates for the case $1<\alpha<2$

First we recall that our results concerning the relations between the problem $(\mathcal{P})$ and $(\mathcal{P})^*$ stated in the beginning of Section 2 are valid for any choice of $\alpha>1$.

THEOREM 3.1. *Let $1<\alpha<2$ and let $\sigma\in Q_f$ (we use the notation from Section 2) denote the unique solution of the problem $(\mathcal{P})^*$. Given any vectorfield $v\in u_0+\overset{\circ}{V}_\alpha$, an* *arbitrary tensor $\tau \in Y^*$ such that* div $\tau \in L^{\alpha^*}(\Omega;\mathbb{R}^n)$ *and a number $\beta > 0$ we have the error estimate*

$$\|\tau - \sigma\|^{\alpha^*}_{L^{\alpha^*}} \le \mathcal{N}_1\big[\varepsilon(v), \tau, \beta\big] + \mathcal{N}_2[\tau, \beta] \tag{3.1}$$

with $\mathcal{N}_1, \mathcal{N}_2$ defined as follows:

$$\begin{aligned}\mathcal{N}_1\big[\varepsilon(v), \tau, \beta\big] &:= \alpha^* 4^{\alpha^*-1}\Big[D_\alpha\big[\varepsilon(v), \tau\big] + \frac{\beta}{2}\big\||\tau|^{\alpha^*-2}\tau - \varepsilon(v)\big\|^2_{L^\alpha}\Big],\\ \mathcal{N}_2\big[\tau, \beta\big] &:= \alpha^* 4^{\alpha^*-1}\Big[\frac{1}{2\beta} + (\alpha^*-1)\big(E(\tau) + 2\|\tau\|_{L^{\alpha^*}}\big)^{\alpha^*-2}\Big]E^2(\tau) + 2^{\alpha^*-1}E^{\alpha^*}(\tau).\end{aligned}$$

Here D_α has the same meaning as in Theorem 2.1 and we let

$$E(\tau) := C_\alpha(\Omega)\big\|f + \text{div } \tau\big\|_{L^{\alpha^*}},$$

i.e. (see (2.16)) $E(\tau)$ measures the distance of τ to Q_f.

REMARK 3.1. *Our previous Remark 2.1 extends to the present situation with obvious modifications.*

Proof of Theorem 3.1: We consider the functional

$$(-J^*)[\tau] = \int_\Omega \Big[-\varepsilon(u_0) : \tau + \frac{1}{\alpha^*}|\tau|^{\alpha^*} + f\cdot u_0\Big]\, dx$$

which is uniformly convex. As in Section 2 we get for any $\tau' \in Q_f$ (using (2.6) with α replaced by α^*)

$$\begin{aligned}&(-J^*)[\tau'] + (-J^*)[\sigma] - 2(-J^*)\big[\frac{\tau'+\sigma}{2}\big]\\ &= \frac{1}{\alpha^*}\int_\Omega \Big[|\tau'|^{\alpha^*} + |\sigma|^{\alpha^*} - 2\Big(\frac{|\tau'+\sigma|}{2}\Big)^{\alpha^*}\Big]\, dx\\ &\ge \frac{1}{\alpha^*\, 2^{\alpha^*-1}}\,\|\tau' - \sigma\|^{\alpha^*}_{L^{\alpha^*}},\end{aligned}$$

and the maximality of σ implies for any $\tau' \in Q_f$

$$\|\tau' - \sigma\|^{\alpha^*}_{L^{\alpha^*}} \le \alpha^*\, 2^{\alpha^*-1}\Big[J^*[\sigma] - J^*[\tau']\Big]. \tag{3.2}$$

By (2.1) we have for all $v \in u_0 + \overset{\circ}{V}_\alpha$

$$J^*[\sigma] \le J[u] \le J[v]$$

and as demonstrated in Section 2 (see the calculations after (2.8)) (3.2) turns into the estimate

$$\|\tau' - \sigma\|^{\alpha^*}_{L^{\alpha^*}} \leq \alpha^* \, 2^{\alpha^*-1} D_\alpha\big[\varepsilon(v), \tau'\big] \tag{3.3}$$

being true for τ' and v as above. As before we like to replace $\tau' \in Q_f$ by a tensor τ as specified in Theorem 3.1. Using

$$\|\sigma - \tau\|^{\alpha^*}_{L^{\alpha^*}} \leq 2^{\alpha^*-1}\Big[\|\sigma - \tau'\|^{\alpha^*}_{L^{\alpha^*}} + \|\tau' - \tau\|^{\alpha^*}_{L^{\alpha^*}}\Big]$$

we deduce from (3.3)

$$\|\sigma - \tau\|^{\alpha^*}_{L^{\alpha^*}} \leq \alpha^* \, 4^{\alpha^*-1} \; D_\alpha\big[\varepsilon(v), \tau'\big] + 2^{\alpha^*-1}\|\tau' - \tau\|^{\alpha^*}_{L^{\alpha^*}}. \tag{3.4}$$

As in Section 2 we have

$$\begin{aligned} D_\alpha\big[\varepsilon(v), \tau'\big] \;&\leq\; D_\alpha\big[\varepsilon(v), \tau\big] + \int\limits_\Omega \big(|\tau|^{\alpha^*-2}\,\tau - \varepsilon(v)\big) : (\tau' - \tau)\; dx \\ &\quad + \int\limits_\Omega \big(|\tau'|^{\alpha^*-2}\,\tau' - |\tau|^{\alpha^*-2}\,\tau\big) : (\tau' - \tau)\; dx. \end{aligned} \tag{3.5}$$

In order to continue we need the following elementary inequality: consider a, $b \in \mathbb{R}^l$ for some $l \geq 1$. Then we have

$$\big(|a|^{\alpha^*-2}a - |b|^{\alpha^*-2}b\big) \cdot (a - b) \leq (\alpha^* - 1)|a - b|^2\big(|a| + |b|\big)^{\alpha^*-2}. \tag{3.6}$$

In fact, letting $V(\xi) := |\xi|^{\alpha^*-2}\xi \cdot (a - b)$, $\xi \in \mathbb{R}^l$, we see that

$$\begin{aligned} V(a) - V(b) \;&=\; \int_0^1 \frac{d}{dt} V(b + t(a - b))\, dt \\ &\leq\; (\alpha^* - 1)|a - b|^2 \int_0^1 |t(a - b) + b|^{\alpha^*-2}\, dt \\ &=\; (\alpha^* - 1)|a - b|^2 \int_0^1 |ta + (1 - t)b|^{\alpha^*-2}\, dt \\ &\leq\; (\alpha^* - 1)|a - b|^2\big(|a| + |b|\big)^{\alpha^*-2}, \end{aligned}$$

where of course we used that $\alpha^* > 2$. We apply (3.6) to the second integral on the right-hand side of (3.5) and get

$$\begin{aligned}
&\int_\Omega \big(|\tau'|^{\alpha^*-2}\tau' - |\tau|^{\alpha^*-2}\tau\big) : (\tau' - \tau)\, dx \\
&\leq (\alpha^* - 1)\int_\Omega |\tau' - \tau|^2\big(|\tau'| + |\tau|\big)^{\alpha^*-2}\, dx \\
&\leq (\alpha^* - 1)\Bigg[\int_\Omega |\tau' - \tau|^{\alpha^*}\, dx\Bigg]^{\frac{2}{\alpha^*}}\Bigg[\int_\Omega \big(|\tau'| + |\tau|\big)^{\alpha^*}\, dx\Bigg]^{\frac{\alpha^*-2}{\alpha^*}} \\
&= (\alpha^* - 1)\|\tau' - \tau\|^2_{L^{\alpha^*}}\||\tau'| + |\tau|\|^{\alpha^*-2}_{L^{\alpha^*}} \\
&\leq (\alpha^* - 1)\|\tau' - \tau\|^2_{L^{\alpha^*}}\Big[\|\tau'\|_{L^{\alpha^*}} + \|\tau\|_{L^{\alpha^*}}\Big]^{\alpha^*-2} \\
&\leq (\alpha^* - 1)\|\tau' - \tau\|^2_{L^{\alpha^*}}\Big[\|\tau - \tau'\|_{L^{\alpha^*}} + 2\|\tau\|_{L^{\alpha^*}}\Big]^{\alpha^*-2}.
\end{aligned}$$

With Hölder's and Young's inequality we get for the first integral on the right-hand side of (3.5)

$$\begin{aligned}
\int_\Omega \big[|\tau|^{\alpha^*-2}\tau - \varepsilon(v)\big] : (\tau' - \tau)\, dx &\leq \||\tau|^{\alpha^*-2}\tau - \varepsilon(v)\|_{L^\alpha}\|\tau' - \tau\|_{L^{\alpha^*}} \\
&\leq \frac{\beta}{2}\||\tau|^{\alpha^*-2}\tau - \varepsilon(v)\|^2_{L^\alpha} + \frac{1}{2\beta}\|\tau' - \tau\|^2_{L^{\alpha^*}}.
\end{aligned}$$

Collecting our estimates and returning to (3.4) it is shown that

$$\begin{aligned}
\|\sigma - \tau\|^{\alpha^*}_{L^{\alpha^*}} \leq\ & \alpha^* 4^{\alpha^*-1}\Big[D_\alpha[\varepsilon(v), \tau] + \frac{\beta}{2}\||\tau|^{\alpha^*-2}\tau - \varepsilon(v)\|^2_{L^\alpha}\Big] \\
& + \alpha^* 4^{\alpha^*-1}\Big[\frac{1}{2\beta}\|\tau' - \tau\|^2_{L^{\alpha^*}} \\
& \qquad + (\alpha^* - 1)\|\tau' - \tau\|^2_{L^{\alpha^*}}\big(\|\tau' - \tau\|_{L^{\alpha^*}} + 2\|\tau\|_{L^{\alpha^*}}\big)^{\alpha^*-2}\Big] \\
& + 2^{\alpha^*}\|\tau' - \tau\|^{\alpha^*}_{L^{\alpha^*}}.
\end{aligned}$$

Observing that $\inf_{\tau' \in Q_f}\|\tau - \tau'\|_{L^{\alpha^*}} \leq E(\tau)$, the claim of Theorem 3.1 follows. $\square$.

As in Section 2 we discuss the situation if the approximation v is a non-solenoidal vectorfield, i.e. we consider $\overline{v} \in u_0 + \overset{\circ}{W}{}^1_\alpha(\Omega; \mathbb{R}^n)$. From (3.1) we get

$$\begin{aligned}
\|\tau - \sigma\|^{\alpha^*}_{L^{\alpha^*}} \leq\ & \mathcal{N}_1\big[\varepsilon(\overline{v}), \tau, \beta\big] + \mathcal{N}_2[\tau, \beta] + \alpha^* 4^{\alpha^*-1}\Big(D_\alpha\big[\varepsilon(v), \tau\big] - D_\alpha\big[\varepsilon(\overline{v}), \tau\big]\Big) \\
& + \frac{\beta}{2}\Big(\big\||\tau|^{\alpha^*-2}\tau - \varepsilon(v)\big\|^2_{L^\alpha} - \big\||\tau|^{\alpha^*-2}\tau - \varepsilon(\overline{v})\big\|^2_{L^\alpha}\Big),
\end{aligned}$$

where $v \in u_0 + \overset{\circ}{V}_\alpha$ is defined as in Section 2. We have (see the calculations starting with

(2.20))

$$D_\alpha\big[\varepsilon(v),\tau\big]-D_\alpha\big[\varepsilon(\overline{v}),\tau\big] \;\le\; \rho(\overline{v})\Big\{\big\||\varepsilon(\overline{v})|^{\alpha-2}\varepsilon(\overline{v})-\tau\big\|_{L^{\alpha^*}} + \big\|\varepsilon(\overline{v})\big\|^{\alpha-1}_{L^{\alpha^*}}+\Big[\big\|\varepsilon(\overline{v})\big\|_{L^\alpha}+\rho(\overline{v})\Big]^{\alpha-1}\Big\}$$

and

$$\big\||\tau|^{\alpha^*-2}\tau-\varepsilon(v)\big\|^2_{L^\alpha}\le 2\big\||\tau|^{\alpha^*-2}\tau-\varepsilon(\overline{v})\big\|^2_{L^\alpha}+2\rho(\overline{v})^2.$$

Putting together these estimates we get

THEOREM 3.2. *Let $\alpha\in(1,2)$. With the notation introduced in Theorem 3.1 we have for any $\beta>0$, for all $\tau\in Y^*$ such that* $\operatorname{div}\tau\in L^{\alpha^*}(\Omega;\mathbb{R}^n)$ *and for all $\overline{v}\in u_0+\overset{\circ}{W}{}^1_\alpha(\Omega;\mathbb{R}^n)$ the estimate*

$$\begin{aligned}\|\tau-\sigma\|^{\alpha^*}_{L^{\alpha^*}} \;\le\;& \mathcal{N}_1\big[\varepsilon(\overline{v}),\tau,\beta\big]+\mathcal{N}_2(\tau,\beta)\\ &+\alpha^*\,4^{\alpha^*-1}\Big\{\big\|\,|\varepsilon(\overline{v})|^{\alpha-2}\varepsilon(\overline{v})-\tau\big\|_{L^{\alpha^*}}\\ &\quad+\big\|\varepsilon(\overline{v})\|^{\alpha-1}_{L^\alpha}+\Big[\big\|\varepsilon(\overline{v})\big\|_{L^\alpha}+\rho\,(\overline{v})\Big]^{\alpha-1}\Big\}\;\rho\,(\overline{v})\\ &+\frac{\beta}{2}\,\Big(\big\|\,|\tau|^{\alpha^*-2}\tau-\varepsilon(\overline{v})\big\|^2_{L^\alpha}+2\rho\,(\overline{v})\Big),\end{aligned}$$

the quantities $\mathcal{N}_1$ and $\mathcal{N}_2$ being the same as in Theorem 3.1.

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