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Abstract

We consider the strong solution of an initial boundary value problem for a system of evolution equations describing the flow of a generalized Newtonian fluid of power law type. For a rather large scale of growth rates we prove local initial regularity results such as higher integrability of the pressure function or the existence of the second spatial derivatives of the velocity field.

1 Introduction and statement of the results

In the present note we investigate some basic local regularity properties of solutions of an initial boundary value problem for a system of nonlinear evolution equations describing the flow of certain generalized Newtonian fluids. These equations can be seen as a modification and an extension of the classical Navier–Stokes system, and they might be also used for a deterministic description of flows of standard viscous incompressible fluids. That was Ladyzhenskaya's point of view which she explained in the works [L1] and [L2]. A further discussion of this issue can be found in [MNR] and [MNRR]. To be precise, let us fix our setting: given a domain $\Omega \subset \mathbb{R}^N, N \geq 2$, and a number $T > 0$ we look at solutions $v : Q_T := \Omega \times (0,T) \to \mathbb{R}^N$ (to be defined in a suitable sense) of the problem

$$\text{(1.1)} \qquad \begin{cases} \partial_t v + v \cdot \nabla v - \operatorname{div} \sigma = -\nabla p + f, \\ \operatorname{div} v = 0 \qquad \text{in } Q_T. \end{cases}$$

Here p stands for the apriori unknown pressure function, and f denotes a given system of forces. The tensor σ represents the viscous part of the Cauchy stress tensor, and we assume that σ is the gradient of some smooth potential $\Phi : \mathbb{S}^N \to \mathbb{R}$ acting on the space $\mathbb{S}^N$ of symmetric $(N \times N)$–matrices, more precisely, we require the relation

$$\sigma = \frac{\partial}{\partial \varepsilon} \, \Phi\big(\varepsilon(v)\big), \varepsilon(v) := \, \frac{1}{2} \, (\nabla v + \nabla^T v),$$

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where here and in what follows "∇" has to be understood just w.r.t. the spatial variables, and the same is true for the operator "div". Finally, in (1.1) the symbol $v \cdot \nabla v$ has the usual meaning of the convective term, i.e. $v \cdot \nabla v := v^k \frac{\partial}{\partial x_k} v$ (using summation w.r.t. $k = 1, \dots, N$). We now address the following problem: suppose that we are given "reasonable" solutions v and p of the problem (1.1) which means that v and p are located in such function spaces for which an existence theory can be established under suitable restriction on the potential Φ. It is well known from the general theory of partial differential equations that these spaces consist of generalized functions, and so we ask if our solutions possess some additional degree of smoothness. To make our formulations and arguments more transparent, we restrict ourselves to a rather simple model of a class of generalized Newtonian fluids, i.e. we assume that Φ is the power growth potential

$$\Phi(\varepsilon) = \left(1 + |\varepsilon|^2\right)^{m/2},\ \varepsilon \in \mathbb{S}^N, \tag{1.2}$$

with exponent $m \in (1, \infty)$, where of course it would be possible to replace Φ form (1.2) by a more general function with appropriate estimates for the derivatives. The physical relevance of power growth potentials is explained for example in the monographs [AM] and [BAH], and the question of regularity of weak solutions is well investigated for stationary flows where under reasonable assumptions interior $C^{1,\alpha}$–regularity is proved in the two–dimensional case, whereas in higher dimensions interior partial regularity is established. Without being complete we mention the paper [KMS] discussing the case of planar stationary flows, the monograph [FS1], where stationary and also slow flows are investigated in higher dimensions with the help of variational methods, and the paper [ABF], where stationary flows with non–vanishing convective term are analyzed for domains $\Omega \subset \mathbb{R}^N, N = 2, 3$. It should be noted that in the presence of the convective term the above mentioned regularity results require the lower bound $m > 6/5$, if $N = 2$, whereas $m > 9/5$ is sufficient for partial regularity, if $N = 3$. For slow flows just $m > 1$ has to be required.
Let us now turn to the evolution problem (1.1) with a potential Φ as given in (1.2). As we shall see below we are then confronted with more serious restrictions on the exponent m. As a matter of fact, as in the stationary case, the presence of the convective term in (1.1) makes it necessary to bound m from below. An additional upper bound for m comes from the fact that our problem inherits a certain anisotropy: the tensor σ is of growth order $m - 1$ w.r.t. to the symmetric gradient of the velocity field v, whereas the pressure function p enters (1.1) in a linear way. Clearly this hidden anisotropy also occurs in the stationary case but it is of no effect if one for example likes to prove partial regularity for $N \geq 3$ via blow–up, we refer to [ABF] or [FS1]. As outlined in [S] this anisotropy immediately leads to severe restrictions on m, if one carries out the parabolic blow–up procedure: in [S] partial regularity is shown to be true in three dimensions for exponents m such that $\frac{12}{5} < m < \frac{10}{3}$.

In our paper we now like to investigate the influence of this anisotropy in a more careful way, i.e. we like to improve the upper bound for the exponent m, where for technical simplicity we assume that the convective term vanishes. Moreover, we concentrate on proving some initial regularity from which we hope that with some work but with no additional bound on m, partially regularity can be deduced, i.e. we like to show that $m < 6$ (in case $N = 3$) is sufficient for proving the existence of the second spatial derivatives of the velocity field v. So we are going to consider the simplified evolution problem

$$\left.\begin{aligned} &\partial_t v - \operatorname{div} \sigma = f - \nabla p, \\ &\operatorname{div} v = 0,\ \sigma = \tfrac{\partial \Phi}{\partial \varepsilon}\ \big(\varepsilon(v)\big) \end{aligned}\right\} \text{in } Q_T \tag{1.3}$$

with Φ from (1.2), but the reader should note that results for solutions of (1.3) obtained for "large" m clearly extend to solutions to (1.1) since in this case the convective term $v \cdot \nabla v$ can be included into the forces f. To (1.3) we add the following initial boundary conditions

$$v|_{\partial\Omega\times[0,T]} = 0, \tag{1.4}$$

$$v|_{t=0} = a \tag{1.5}$$

with a given function $a : \Omega \to \mathbb{R}^N$ such that $a|_{\partial\Omega} = 0$ and $\operatorname{div} a = 0$. Assuming that Ω is a bounded Lipschitz domain we require that

$$f \in L^2(Q_T) \tag{1.6}$$

and

$$a \in V_m := \text{ closure of } \overset{\circ}{C}{}_0^\infty(\Omega) \text{ w.r.t. } W_m^1(\Omega), \tag{1.7}$$

where $\overset{\circ}{C}{}_0^\infty(\Omega) := \{v \in C_0^\infty(\Omega) : \operatorname{div} v = 0\}$ and $W_m^1(\Omega)$ is the standard Sobolev space. Note that in all cases the function spaces consist of vector–functions with values in $\mathbb{R}^N$. Then we can show the existence of a (unique) so–called strong solution to (1.3) – (1.5) which means that there exists a velocity field $v : Q_T \to \mathbb{R}^N$ and a pressure function $p : Q_T \to \mathbb{R}$ satisfying

$$\nabla v \in L^\infty(0,T; V_m),\ \partial_t\, v \in L^2(Q_T),\ p \in L^{m'}(Q_T), \tag{1.8}$$

where $m' := m/(m-1)$, and moreover

$$v \in C^0\big([0,T]; L^2(\Omega)\big),\ v(\cdot, 0) = a \tag{1.9}$$

such that we have the following weak form of (1.3)

$$\begin{aligned} &\int_\Omega \partial_t v(x,t) \cdot w(x)\, dx + \int_\Omega \sigma(x,t) : \varepsilon(w)(x)\, dx \\ &\quad = \int_\Omega p(x,t) \operatorname{div} w(x)\, dx + \int_\Omega f(x,t) \cdot w(x)\, dx \end{aligned} \tag{1.10}$$

valid for all $w \in C_0^\infty(\Omega)$ and almost all $t \in [0,T]$. The existence proof can be carried out in a rather elementary and classical way, we refer to [MNR], [MNRR], [S] and [FS2].

Our main result now reads as follows

THEOREM 1.1. *Let (1.6) and (1.7) hold and consider the strong solution v, p to the initial boundary value problem (1.3) –(1.5) with tensor σ defined according to $\sigma = \frac{\partial \Phi}{\partial \varepsilon}\,\big(\varepsilon(v)\big)$ and potential Φ as in (1.2). Suppose further that*

$$2 < m < \frac{2N}{N-2}\,.$$

Then, for any $\delta \in]0,T[$ and for any subdomain $\Omega' \Subset \Omega$ we have that

$$\sigma,\ p \in L^2(Q'_{\delta,T}), \tag{1.11}$$

$$\int_{Q'_{\delta,T}} \left(1 + |\varepsilon(v)|^2\right)^{\frac{m-2}{2}} \; |\nabla\varepsilon(v)|^2 \, dx\, dt \le c\,(a, f, \delta, \Omega', N, m) < \infty, \tag{1.12}$$

where $Q'_{\delta,T} := \Omega' \times]\delta, T[$. If in addition we assume that

$$\partial_t\, f \in L^2(Q_T), \tag{1.13}$$

then

$$p \in L^\gamma(Q'_{\delta,T}),\ \ \gamma := \frac{m}{m-1}\ \frac{N+1}{N} > 2, \tag{1.14}$$

and

$$\partial_t\ v \in L^{2,\infty}(Q_{\delta,T}), \nabla\partial_t\ v \in L^2(Q_{\delta,T}), \tag{1.15}$$

where $Q_{\delta,T} := \Omega \times]\delta, T[$.

REMARK 1.2. *In (1.15) the first statement means that*

$$\sup_{\delta \le t \le T} \int_\Omega |\partial_t v(t,x)|^2 \, dx < \infty.$$

REMARK 1.3. *Our results are formulated as local initial regularity results for the strong solution of an initial boundary value problem so that one may hope for similar statements in case of local solutions. Unfortunately our proof uses the fact that we deal with a global solution.*

REMARK 1.4. *In [MNR] Theorem 1.1 is proved even including the convective term but under the restriction that $N = 3$ together with $\frac{9}{4} \leq m < 3$.*

REMARK 1.5. *The properties (1.11) and (1.14) are the starting points for the further investigation of the regularity properties of strong solutions in the spirit of the paper [S]. Since the details are rather involved, they will be presented in a separate paper.*

Our paper is organized as follows: in Section 2 we introduce a suitable approximation for our initial boundary value problem. This is done in such a way that the hidden anisotropy discussed above dissappears. More precisely, we replace the potential Φ by a sequence Φ_M of quadratic potentials approximating Φ from below and prove appropriate apriori estimates for the corresponding strong solutions. Section 3 then is devoted to the limiting procedure leading to the proof of the first part of Theorem 1.1. The second part is established in the final Section 4.

2 Approximation of the initial boundary value problem and apriori estimates

We let for $M > 0$

$$\begin{aligned} d_M(s) \quad &:= \quad \begin{cases} d(s),\ 0 \leq s \leq M \\ d(M) + d'(M)(s-M) + \frac{1}{2}\, d''(M)(s-M)^2,\ s \geq M, \end{cases} \\ \Phi_M(\varepsilon) \quad &:= \qquad d_M(|\varepsilon|),\ \varepsilon \in \mathbb{S}^N, \end{aligned}$$

where $d(s) := (1+s^2)^{\frac{m}{2}}$. The potentials Φ_M are of quadratic growth satisfying

$$\frac{\partial^2}{\partial \varepsilon^2}\, \Phi_M(\varepsilon)(\tau,\tau) = d''_M\big(|\varepsilon|\big)\, \frac{|\varepsilon : \tau|^2}{|\varepsilon|^2} + \frac{d'_M(|\varepsilon|)}{|\varepsilon|}\, \Big[|\tau|^2 - \frac{|\varepsilon : \tau|^2}{|\varepsilon|^2}\Big] \tag{2.1}$$

for all tensors $\varepsilon, \tau \in \mathbb{S}^N$. Let us fix $\varepsilon \in \mathbb{S}^N$ such that $|\varepsilon| \geq 2M$ and consider some $\tau \in \mathbb{S}^N$. If $\frac{|\varepsilon:\tau|^2}{|\varepsilon|^2} \geq \frac{1}{2}\, |\tau|^2$, then (2.1) implies $\frac{\partial^2 \Phi_M}{\partial \varepsilon^2}\, (\varepsilon)(\tau,\tau) \geq d''_M(|\varepsilon|)\frac{1}{2}\, |\tau|^2 = d''(M)\frac{1}{2}\, |\tau|^2$, whereas for $\frac{|\varepsilon:\tau|^2}{|\varepsilon|^2} \leq \frac{1}{2}\, |\tau|^2$ we see that

$$\begin{aligned} \frac{\partial^2 \Phi_M}{\partial \varepsilon^2}\, (\varepsilon)(\tau,\tau) \quad &\geq \quad \frac{1}{2}\, \frac{d'_M\big(|\varepsilon|\big)}{|\varepsilon|}\, |\tau|^2 \\ &= \quad \frac{1}{2}\, |\tau|^2 \left\{ \frac{d'(M)}{|\varepsilon|} + d''(M) \frac{|\varepsilon| - M}{|\varepsilon|} \right\} \\ &\geq \quad \frac{1}{2}\, |\tau|^2 d''(M) \frac{|\varepsilon| - M}{|\varepsilon|} \geq \frac{1}{4}\, |\tau|^2 d''(M), \end{aligned}$$

where the last inequality follows from the choice of ε. From these calculations we easily deduce the existence of constants $\lambda, \Lambda > 0$ such that

$$\lambda(1+M^2)^{\frac{m-2}{2}}\ |\tau|^2 \leq \frac{\partial^2\Phi_M}{\partial\varepsilon^2}\ (\varepsilon)(\tau,\tau) \leq \Lambda(1+M^2)^{\frac{m-2}{2}}\ |\tau|^2 \tag{2.2}$$

for all $\varepsilon, \tau \in \mathbb{S}^N, |\varepsilon| \geq M$. For tensors ε such that $|\varepsilon| \leq M$ we obviously have $\frac{\partial^2\Phi_M}{\partial\varepsilon^2}\ (\varepsilon) = \frac{\partial^2\Phi}{\partial\varepsilon^2}\ (\varepsilon)$. (The reader should note that for $M \leq |\varepsilon| \leq 2M$ the inequality (2.2) follows from (2.1) in more or less the same way as in case $|\varepsilon| \geq 2M$.) We now consider the initial boundary value problem (1.3) – (1.5) with Φ replaced by Φ_M. Let v^M and p^M denote the corresponding velocity field and pressure function, moreover, we abbreviate $\sigma^M = \frac{\partial\Phi_M}{\partial\varepsilon}\ (\varepsilon(v^M))$. Here of course v^M, p^M have the meaning of the strong solution discussed in Section 1. Since Φ_M is of quadratic growth, we have the following additional information concerning the solution

$$\partial_t v^M \in L^2(Q_T), \nabla^2 v^M \in L^2(Q'_{\delta,T}), \nabla p^M \in L^2(Q'_{\delta,T}), \tag{2.3}$$

where $Q'_{\delta,T}$ is defined as in Theorem 1.1. If we let $\omega(s) := (1+s^2)^{\frac{m-2}{2}}, \omega_M(s) := \omega(s)$, if $|s| \leq M, \omega_M(s) := (1+M^2)^{\frac{m-2}{2}}$, if $|s| \geq M, s \in \mathbb{R}$, then it is easy to check that (2.2) implies the following estimate with positive constants c_1, c_2 being independent of M:

$$c_1\ \omega_M\big(|\varepsilon(w)|\big)\ |\nabla\varepsilon(w)|^2 \leq \tau_{,\,k} :\ \varepsilon(w_{,\,k}) \leq c_2\ \omega_M\big(|\varepsilon(w)|\big)\ |\nabla\varepsilon(w)|^2. \tag{2.4}$$

Here and in what follows we use the symbols $w :=\ v^M,\ \tau :=\ \sigma^M,\ q :=\ p^M$, and denote by $w_{,\,k}$ etc. the partial derivative w.r.t. the k th spatial variable. Moreover, we always take the sum w.r.t. to indices repeated twice.

Consider a smooth, non-negative cut–off function φ vanishing in a neighborhood of the parabolic boundary $\partial' Q_T :=\ \big(\overline{\Omega}\times\{0\}\big)\cup\big(\partial\Omega\times[0,T]\big)$ of the cylinder Q_T. From the equation satisfied by w, τ and q we deduce (by multiplying with $(\varphi w_{,\,k})_{,\,k}$ and integrating over Q_T)

$$I_1 = I_2 + I_3 + I_4 + I_5, \tag{2.5}$$

where

$$\begin{aligned}
I_1 &:= \int_{Q_T} \varphi\, \tau_{,\,k} : \varepsilon(w_{,\,k})\ dx\ dt,\\
I_2 &:= -\int_{Q_T} \tau_{,\,k} : w_{,\,k} \otimes \nabla\varphi\ dx\ dt,\\
I_3 &:= -\int_{Q_T} \overline{q}\ (w_{,\,k}\cdot\nabla\varphi)_{,\,k}\ dx\ dt,\\
I_4 &:= -\int_{Q_T} f\cdot(w\,\varphi_{,\,k})_{,\,k}\ dx\ dt,\\
I_5 &:= \tfrac{1}{2}\int_{Q_T} |\nabla w|^2\,\partial_t\,\varphi\ dx\ dt.
\end{aligned}$$

In I_3 we have set $\overline{q} := q - c\,(t), c\,(t)$ being a function just depending on t. To I_1 we can apply the l.h.s. of (2.4) to get a lower bound for this integral We split

$$\begin{aligned} I_2 &= I_2' + I_2'', \\ I_2' &:= -2\int_{Q_T} \tau_{ij,k}\, \varepsilon_{ik}(w)\varphi_{,j}\; dx\; dt, \\ I_2'' &:= -\int_{Q_T} \tau_{ij,k}\, w_{i,k}\varphi_{,j}\; dx\; dt. \end{aligned}$$

If we replace φ by Ψ^2, then the Cauchy-Schwarz inequality $\big($for the bilinear form $\frac{\partial^2 \Phi_M}{\partial \varepsilon^2}(\varepsilon)\ \big)$ together with (2.4) implies

$$|I_2'| \;\leq\; c\, I_1^{1/2} \,\Big(\int_{Q_T} |\nabla \Psi|^2 \omega_M\big(|\varepsilon(w)|\big)|\varepsilon(w)|^2\; dx\; dt\Big)^{1/2} \tag{2.6}$$

with $0 < c < \infty$ independent of M. We transform I_2'' using integration by parts together with the equation for w, τ and q and get

$$\begin{aligned} I_2'' &= \int_{Q_T} \tau_{ij}\; w_{k,\,i}\; \varphi_{,\,jk}\; dx\; dt \\ &= -\int_{Q_T} \tau_{ij}\; w_k\; \varphi_{,\,ijk}\; dx\; dt - \int_{Q_T} \tau_{ij,\,i}\; w_k\; \varphi_{,\,jk}\; dx\; dt \\ &= -\int_{Q_T} \tau_{ij}\; w_k\; \varphi_{,\,ijk}\; dx\; dt - \int_{Q_T} \big(\partial_t w_j + q_{,j} - f_j\big) w_k\; \varphi_{,jk}\; dx\; dt \\ &= -\int_{Q_T} \tau_{ij} w_k\; \varphi_{,\,ijk}\; dx\; dt + \frac{1}{2}\int_{Q_T} w_j\; w_k\; \partial_t\; \varphi_{,\,jk}\; dx\; dt \\ &\quad + \int_{Q_T} f_j\; w_k\; \varphi_{,\,jk}\; dx\; dt + \int_{Q_T} \overline{q}\; w_k\; \Delta\, \varphi_{,\,k}\; dx\; dt \\ &\quad + \int_{Q_T} \overline{q}\; w_{k,\,j}\; \varphi_{,\,jk}\; dx\; dt, \end{aligned}$$

hence we can estimate I_2'' in the following way:

$$\begin{aligned} |I_2''| \;&\leq\; \Big(\int_{spt\varphi} \big(|\tau|^2 + |\overline{q}|^2\big)\; dx\; dt\Big)^{1/2} \Big(\int_{Q_T} \big(|w|^2|\nabla^3\varphi|^2 + |\nabla w|^2 |\nabla^2 \varphi|^2\big)\; dx\; dt\Big)^{1/2} \\ &+ \int_{Q_T} \big(|f||w||\nabla^2\varphi| + |w|^2 |\partial_t \nabla^2 \varphi|\big)\; dx\; dt. \end{aligned}$$

This implies the bound (recall (2.6)

$$
\begin{aligned}
|I_2| \ \le\ & c\, I_1^{1/2}\Big(\int_{Q_T} |\nabla\Psi|^2 \omega_M\big(|\varepsilon(w)|\big)|\varepsilon(w)|^2\,dx\,dt\Big)^{1/2} \\
+ & \int_{Q_T} \big(|f||w||\nabla^2\varphi| + |w|^2|\partial_t\nabla^2\varphi|\big)\,dx\,dt \\
+ & \int_{spt\varphi} \big(|\tau|^2 + |\overline{q}|^2\big)\,dx\,dt\Big)^{1/2} \\
\cdot & \Big(\int_{Q_T} \big(|w|^2|\nabla^3\varphi|^2 + |\nabla w|^2|\nabla^2\varphi|^2\big)\,dx\,dt\Big)^{1/2}.
\end{aligned} \tag{2.7}
$$

For I_3 we use the decomposition

$$
\begin{aligned}
I_3 \ &=\ I_3' + I_3'', \\
I_3' \ &:=\ -\int_{Q_T} \overline{q}\,\Delta w \cdot \nabla\varphi\,dx\,dt, \\
I_3'' \ &:=\ -\int_{Q_T} \overline{q}\,w_{,\,k} \cdot \nabla\varphi_{,\,k}\,dx\,dt,
\end{aligned}
$$

hence

$$
|I_3'| \ \le\ c\int_{Q_T} |\overline{q}|\Psi|\nabla\Psi||\nabla\varepsilon(w)|\,dx\,dt \le c\, I_1^{1/2}\Big(\int_{Q_T} |\overline{q}|^2|\nabla\Psi|^2\,dx\,dt\Big)^{1/2}, \tag{2.8}
$$

$$
|I_3''| \ \le\ \Big(\int_{spt\varphi} |\overline{q}|^2\,dx\,dt\Big)^{1/2}\Big(\int_{Q_T} |\nabla w|^2|\nabla^2\varphi|^2\,dx\,dt\Big)^{1/2}. \tag{2.9}
$$

In order to transform (2.7) – (2.9) into more suitable estimates, we have to control the integrals $\int_{spt\varphi}|\tau|^2\,dx\,dt$ and $\int_{spt\varphi}|\overline{q}|^2\,dx\,dt$. Of course it is sufficient to discuss the first one, then we can use the equation to bound the second integral. To this purpose let

$$
f_M := \Big(\omega_M\big(|\varepsilon(w)|\big)|\varepsilon(w)|^2\Big)^{\frac{1}{m}}
$$

and observe

$$
\Big|\nabla\big(f_M^{\frac{m}{2}}\big)\Big|^2 \le c\,\omega_M\big(|\varepsilon(w)|\big)|\nabla\varepsilon(w)|^2. \tag{2.10}
$$

Let us define $\mu := \frac{m-2}{2m}\,(N-2), \kappa := \frac{\mu N}{N-2}$. Then

$$
\kappa < 1 \tag{2.11}
$$

which follows from our assumption that $2 < m < \frac{2N}{N-2}$ stated in Theorem 1.1. For any ball $B_\rho(x_0) \Subset \Omega$ we get by Hölder's inequality and the Gagliardo–Nirenberg estimate (note by (2.11) $\mu \frac{N}{N-2} < 1$, hence $\mu < 1$)

$$
\begin{aligned}
\int_{B_\rho(x_0)} f_M^{2(m-1)}\, dx &\leq \Big(\int_{B_\rho(x_0)} f_M^m\, dx\Big)^{1-\mu} \\
&\quad \cdot \Big(\int_{B_\rho(x_0)} f_M^{\frac{m}{2}\frac{2N}{N-2}}\, dx\Big)^{\mu} \\
&\leq c\Big(\int_{B_\rho(x_0)} f_M^m\, dx\Big)^{1-\mu}\Big(\int_{B_\rho(x_0)} |\nabla\big(f_M^{\frac{m}{2}}\big)|^2\, dx \\
&\quad + \rho^{-2}\int_{B_\rho(x_0)} f_M^m\, dx\Big)^{\kappa}.
\end{aligned}
\tag{2.12}
$$

Combining (2.10) and (2.12) we find that

$$
\begin{aligned}
\int_{B_\rho(x_0)} |\tau|^2\, dx &\leq c\Big(\int_{B_\rho(x_0)} \omega_M\big(|\varepsilon(w)|\big)\, |\varepsilon(w)|^2\, dx\Big)^{1-\mu} \\
&\quad \cdot \Big(\int_{B_\rho(x_0)} \omega_M\big(|\varepsilon(w)|\big)\, |\nabla\varepsilon(w)|^2\, dx \\
&\quad + \rho^{-2}\int_{B_\rho(x_0)} \omega_M\big(|\varepsilon(w)|\big)\, |\varepsilon(w)|^2\, dx\Big)^{\kappa}.
\end{aligned}
\tag{2.13}
$$

Let $z_0 = (x_0, t_0) \in \Omega \times (0,T)$ and $Q(z_0,\rho) := B_\rho(x_0) \times]t_0 - \rho^2, t_0[$. The integration of (2.13) w.r.t. time yields

$$
\begin{aligned}
\int_{Q(z_0,\rho)} |\tau|^2\, dx\, dt &\leq c\, \rho^{2(1-\kappa)}\Big(\sup_{t_0-\rho^2<t<t_0} \int_{B_\rho(x_0)} \omega_M\big(|\varepsilon(w)|\big)\, |\varepsilon(w)|^2\, dx\Big)^{1-\mu} \\
&\quad \cdot \Big(\int_{Q(z_0,\rho)} \omega_M\big(|\varepsilon(w)|\big)\, |\nabla\varepsilon(w)|^2\, dx\, dt \\
&\quad + \rho^{-2}\int_{Q(z_0,\rho)} \omega_M\big(|\varepsilon(w)|\big)\, |\varepsilon(w)|^2\, dx\, dt\Big)^{\kappa}.
\end{aligned}
\tag{2.14}
$$

(2.14) is the desired bound for the integral of $|\tau|^2$, and as explained above this gives the following estimate for the pressure ($(\cdot)_{x_0,\rho} := \fint \cdot\, dx$)

$B_\rho(x_0)$

$$
\begin{aligned}
&\int_{Q(z_0,\rho)} |q-(q)_{x_0,\rho}|^2 \quad dx\,dt \\
&\qquad \le c\,\rho^{2(1-\kappa)}\Big(\sup_{t_0-\rho^2<t<t_0}\int_{B_\rho(x_0)}\omega_M\big(|\varepsilon(w)|\big)\ |\varepsilon(w)|^2\,dx\Big)^{1-\lambda} \\
&\qquad\quad \cdot\Big(\int_{Q(z_0,\rho)}\omega_M\big(|\varepsilon(w)|\big)\ |\nabla\varepsilon(w)|^2\,dx\,dt \qquad (2.15)\\
&\qquad\quad + \rho^{-2}\int_{Q(z_0,\rho)}\omega_M\big(|\varepsilon(w)|\big)\ |\varepsilon(w)|^2\,dx\,dt\Big)^{\kappa} \\
&\qquad\quad + c\,\rho^2\int_{Q(z_0,\rho)}\big(|f|^2+|\partial_t w|^2\big)\,dx\,dt.
\end{aligned}
$$

Having established (2.15) we return to (2.5) and estimate I_4 in an obvious way:

$$
|I_4| \le c\,\Big(\int_{spt\varphi}|f|^2\,dx\,dt\Big)\Big(\int_{Q_T}|w|^2|\nabla\varphi|^2\,dx\,dt + I_1\Big)^{1/2}. \tag{2.16}
$$

For discussing I_5 we specify our function Ψ (recall $\varphi=\Psi^2$). Let $\Psi(x,t)=\eta(x)\sqrt{\chi(t)}$ with $\eta=1$ on $B_r(x_0), \eta=0$ outside of $B_\rho(x_0)$ for balls $B_r(x_0)\subset B_\rho(x_0)\Subset\Omega$ and in addition assume that $|\nabla^k\eta|\le c\,(\rho-r)^{-k}, k=1,2,3$. Let further $\frac{R}{2}\le r<\rho\le R\le 1$ with $B_R(x_0)\Subset\Omega$. The function $\chi(t)$ is defined as follows:

$$
\chi(t)=\begin{cases}
0, & 0\le t\le t_0-\rho^2\\
\frac{t-t_0+\rho^2}{\rho^2-r^2}, & t_0-\rho^2\le t\le t_0-r^2\\
1, & t_0-r^2\le t\le t_0\\
\frac{t_0+\varepsilon-t}{\varepsilon}, & t_0\le t\le t_0+\varepsilon\\
0, & t_0+\varepsilon\le t\le T.
\end{cases}
$$

With this choice of Ψ we get

$$
\begin{aligned}
I_5 &= -\tfrac{1}{2\varepsilon}\int_{t_0}^{t_0+\varepsilon}\int_{B_r(x_0)}\eta^2|\nabla w|^2\,dx\,dt \qquad (2.17)\\
&\quad +\tfrac{1}{2(\rho^2-r^2)}\int_{t_0-\rho^2}^{t_0-r^2}\int_{B_R(x_0)}\eta^2|\nabla w|^2\,dx\,dt.
\end{aligned}
$$

Putting together (2.7), (2.8), (2.9), (2.16) and (2.17) we finally arrive at

$$\begin{aligned}
&\frac{1}{\varepsilon}\int_{t_0}^{t_0+\varepsilon}\int_{B_r(x_0)}|\nabla w|^2\,dx\,dt + I_1\\
&\leq\; c\Big\{\int_{Q_T}|\nabla\Psi|^2\,\omega_M\big(|\varepsilon(w)|\big)\,|\varepsilon(w)|^2\,dx\,dt\\
&\quad+\int_{Q(z_0,R)}|f|^2\,dx\,dt\,(\rho-r)^{-2}+\int_{Q(z_0,R)}|w|^2\,dx\,dt\,(\rho-r)^{-6}\\
&\quad+\int_{Q(z_0,R)}|\nabla w|^2\,dx\,dt\,(\rho-r)^{-4}+\int_{Q(z_0,R)}\big(|f|^2+|\partial_t w|^2\big)\,dx\,dt\,R^2\\
&\quad+\frac{\rho^{2(1-\kappa)}}{(\rho-r)^2}\,\Big(\sup_{t_0-\rho^2<t<t_0}\int_{B_\rho(x_0)}\omega_M\big(|\varepsilon(w)|\big)\,|\varepsilon(w)|^2\,dx\Big)^{1-\mu}\\
&\quad\cdot\Big[\int_{Q(z_0,\rho)}\omega_M\big(|\varepsilon(w)|\big)\,|\nabla\varepsilon(w)|^2\,dx\,dt\\
&\quad+R^{-2}\int_{Q(z_0,\rho)}\omega_M\big(|\varepsilon(w)|\big)\,|\varepsilon(w)|^2\,dx\,dt\Big]^{\kappa}\Big\}.
\end{aligned}$$

Since $\kappa<1$ we deduce from the above inequality the estimate

$$\begin{aligned}
&\int_{Q(z_0,r)}\omega_M\big(|\nabla w|\big)\,|\nabla\varepsilon(w)|^2\,dx\,dt\\
&\leq\;\frac{1}{2}\int_{Q(z_0,\rho)}\omega_M\big(|\nabla w|\big)\,|\nabla\varepsilon(w)|^2\,dx\,dt\\
&\quad+c\Big\{R^2\int_{Q(z_0,R)}|\partial_t w|^2\,dx\,dt+(\rho-r)^{-2}\int_{Q(z_0,R)}|f|^2\,dx\,dt\\
&\quad+(\rho-r)^{-4}\int_{Q(z_0,R)}|\varepsilon(w)|^2\,dx\,dt+(\rho-r)^{-6}\int_{Q(z_0,R)}|w|^2\,dx\,dt\\
&\quad+R^{-2}\int_{Q(z_0,R)}\omega_M\big(|\varepsilon(w)|\big)\,|\varepsilon(w)|^2\,dx\,dt\\
&\quad+\frac{R^{4(1-\kappa)}}{(\rho-r)^{2(1-\kappa)}}\,\Big(\sup_{t_0-R^2<t<t_0}\int_{B_R(x_0)}\omega_M\big(|\varepsilon(w)|\big)\,|\varepsilon(w)|^2\,dx\,dt\Big)^{(1-\mu)(1-\kappa)}\Big\}.
\end{aligned}$$

Thus we may apply a wellknown reasoning to get

$$
\begin{aligned}
&\int_{Q(z_0,R/2)} \omega_M\big(|\varepsilon(w)|\big)\, |\nabla\varepsilon(w)|^2\, dx\, dt \\
&\qquad \le\ c\,\Big\{R^2\int_{Q(z_0,R)} |\partial_t w|^2\, dx\, dt + R^{-2}\int_{Q(z_0,R)} |f|^2\, dx\, dt \\
&\qquad\quad + R^{-4}\int_{Q(z_0,R)} |\varepsilon(w)|^2\, dx\, dt + R^{-6}\int_{Q(z_0,R)} |w|^2\, dx\, dt \\
&\qquad\quad + R^{-2}\int_{Q(z_0,R)} \omega_M\big(|\varepsilon(w)|\big)\, |\varepsilon(w)|^2\, dx\, dt \\
&\qquad\quad + R^{2(1-\kappa)}\Big(\sup_{t_0-R^2<t<t_0}\int_{B_R(x_0)} \omega_M\big(|\varepsilon(w)|\big)\, |\varepsilon(w)|^2\, dx\, dt\Big)^{(1-\mu)(1-\kappa)}\Big\}.
\end{aligned}
\tag{2.18}
$$

Here the constant c is independent of the parameter M and also independent of the cylinder $Q(z_0,R)$. With (2.18) we have established an apriori estimate for the approximation which will be of central importance in the next section.

3 Limiting procedure and proof of the first part of Theorem 1.1

We use the same notation as in Section 2, in particular we recall the definitions of v^M, σ^M and p^M. Testing the " M–version" of (1.1) with v^M and $\frac{\partial}{\partial t}\, v^M$, respectively, we get the apriori estimates (valid for a.a.t)

$$
\frac{1}{2}\,\frac{d}{dt}\int_\Omega |v^M|^2\, dx + \int_\Omega \sigma^M : \varepsilon(v^M)\, dx \ = \ \int_\Omega f\cdot v^M\, dx, \tag{3.1}
$$

$$
\int_\Omega |\partial_t v^M|^2\, dx + \frac{d}{dt}\int \Phi_M\big(\varepsilon(v^M)\big)\, dx \ = \ \int_\Omega f\cdot \partial_t v^M\, dx. \tag{3.2}
$$

Integrating (3.1), (3.2) w.r.t. time we deduce the global bound

$$
\begin{aligned}
&\int_{Q_T} |\partial_t v^M|^2\, dx\, dt + \int_{Q_T} \omega_M\big(|\varepsilon(v^M)|\big)\, |\varepsilon(v^M)|^2\, dx\, dt \\
&\qquad + \int_{Q_T} |p^M|^{m'}\, dx\, dt \\
&\qquad + \sup_{0<t<T}\int_\Omega \Big(\omega_M\big(|\varepsilon(v^M)|\big)\, |\varepsilon(v^M)|^2 + |v^M|^2\Big)\, dx \\
&\qquad \le C\,\Big(m, N, Q_T, \|f\|_{L^2(Q_T)}, \|a\|_{V_m}\Big) =:\ c\,(a,f) < \infty,
\end{aligned}
\tag{3.3}
$$

the constant $c(a,f)$ being independent of M. In fact, it is immediate, how to estimate the first two integrals and the last term on the l.h.s. of (3.3) with the help of (3.1), (3.2) and Young's inequality. The pressure term ist discussed in the standard way, i.e. by using the equation and the foregoing estimates.

So if we combine (2.13), the pressure estimate (2.15), (2.18) and (3.3) we find the local inequality

$$\int_{Q(z_0,R)} \Big(|p^M|^2 + |\sigma^M|^2 + \omega_M\big(|\varepsilon(v^M)|\big)\, |\nabla\varepsilon(v^M)|^2\Big)\, dx\, dt \leq c\,(z_0,R,f,a) \tag{3.4}$$

for any $B_R(x_0) \Subset \Omega, t_0 < T, t_0 - R^2 > 0, z_0 = (x_0,t_0)$.
We note two obvious consequences of (3.1) – (3.4):

$$\int_{Q_T} \Big(|\partial_t v^M|^2 + |\nabla v^M|^2\Big)\, dx\, dt \;\leq\; c\,(a,f), \tag{3.5}$$

$$\int_{Q(z_0,R)} |\nabla^2 v^M|^2\, dx\, dt \;\leq\; c\,(z_0,R,a,f). \tag{3.6}$$

Here of course we use $m > 2$ together with the pointwise inequality $|\nabla^2 v^M| \leq c\,|\nabla\varepsilon(v^M)|$.
From (3.5) and (3.6) we deduce the existence of suitable subsequences such that as $M \to \infty$

$$\begin{cases} v^M & \rightarrow: & v^* & \text{in} & L^2(Q_T), \\ \partial_t v^M & \rightharpoondown & \partial_t v^* & \text{in} & L^2_{\text{loc}}(Q_T), \\ p^M & \rightharpoondown: & p^* & \text{in} & L^2_{\text{loc}}(Q_T), \\ \nabla v^M & \rightharpoondown & \nabla v^* & \text{in} & L^2(Q_T), \\ \nabla^2 v^M & \rightharpoondown & \nabla^2 v^* & \text{in} & L^2_{\text{loc}}(Q_T). \end{cases} \tag{3.7}$$

Using compactness arguments, (3.7) implies

$$\begin{cases} \nabla v^M & \rightarrow & \nabla v^* & \text{in} & L^2(Q_T), \\ \varepsilon(v^M) & \rightarrow & \varepsilon(v^*) & \text{in} & L^2(Q_T), \\ \nabla v^M & \rightarrow & \nabla v^* & \text{a.e.in} & Q_T, \\ \varepsilon(v^M) & \rightarrow & \varepsilon(v^*) & \text{a.e.in} & Q_T, \\ \Phi_M\big(\varepsilon(v^M)\big) & \rightarrow & \Phi\big(\varepsilon(v^*)\big) & \text{a.e.in} & Q_T, \\ \sigma^M = \frac{\partial}{\partial\varepsilon}\,\Phi_M\big(\varepsilon(v^M)\big) & \rightarrow & \sigma^* := \frac{\partial}{\partial\varepsilon}\,\Phi\big(\varepsilon(v^*)\big) & \text{a.e.in} & Q_T. \end{cases} \tag{3.8}$$

By(3.4) and (3.8) and with the help of Fatou's lemma we see

$$\sigma^* \in L^2_{\text{loc}}(Q_T). \tag{3.9}$$

If we fix a number $L > 0$, then for $M \geq L$ we deduce from (3.4) that

$$\int_{Q(z_0,R)} \omega_L\big(|\varepsilon(v^M)|\big)\, |\nabla\varepsilon(v^M)|^2\, dx\, dt \;\leq\; c\,(z_0, R, a, f).$$

Since $\omega_L\big(|\varepsilon(v^M)|\big)^{1/2}$ is bounded and converging to $\omega_L\big(|\varepsilon(v^*)|\big)^{1/2}$ a.e. as $M \to \infty$, we see that (recall (3.7))

$$\omega_L\big(|\varepsilon(v^M)|\big)^{1/2}\, \nabla\varepsilon(v^M) \rightharpoonup \omega_L\big(|\varepsilon(v^*)|\big)^{1/2}\, \nabla\varepsilon(v^*)$$

in $L^2_{\text{loc}}(Q_T)$ as $M \to \infty$, thus by lower semicontinuity

$$\int_{Q(z_0,R)} \omega_L\big(|\varepsilon(v^*)|\big)\, |\nabla\varepsilon(v^*)|^2\, dx\, dt \;\leq\; c\,(z_0, R, a, f),$$

more precisely

$$\begin{aligned}\int_{Q(z_0,R)} &\omega_L\big(|\varepsilon(v^*)|\big)\, |\nabla\varepsilon(v^*)|^2\, dx\, dt\\ &\leq \liminf_{M\to\infty} \int_{Q(z_0,R)} \omega_L\big(|\varepsilon(v^M)|\big)\, |\nabla\varepsilon(v^M)|^2\, dx\, dt\\ &\leq \liminf_{M\to\infty} \int_{Q(z_0,R)} \omega_M\big(|\varepsilon(v^M)|\big)\, |\nabla\varepsilon(v^M)|^2\, dx\, dt.\end{aligned}$$

If we let $L \to \infty$ on the l.h.s. using Fatou's lemma, we end up with

$$\begin{aligned}\int_{Q(z_0,R)} &\omega\big(|\varepsilon(v^*)|\big)\, |\nabla\varepsilon(v^*)|^2\, dx\, dt\\ &\leq \liminf_{M\to\infty} \int_{Q(z_0,R)} \omega_M\big(|\varepsilon(v^M)|\big)\, |\nabla\varepsilon(v^M)|^2\, dx\, dt\\ &\leq c\,(z_0, R, a, f).\end{aligned} \tag{3.10}$$

Note that from (3.3) together with the convergences from above it follows that

$$\sup_{0<t<T} \int_\Omega \Phi\big(\varepsilon(v^*)\big)\, dx < \infty. \tag{3.11}$$

Clearly (3.7), (3.9) and (3.10) imply the first part of Theorem 1.1, i.e. the statements (1.11) and (1.12), as soon as we can show that $v^* = v, \sigma^* = \sigma$ and $p^* = p$. In order to do so, we first claim that

$$\sigma^M \to \sigma^* \text{ in } L^1_{\text{loc}}(Q_T). \tag{3.12}$$

But this follows from the pointwise convergence together with the equi-integrability of the sequence $\{\sigma^M\}$: for sets $Q_0 \subset Q_T$ we have

$$\begin{aligned}\int_{Q_0} |\sigma^M|\, dx\, dt &\leq c \int_{Q_0} \omega\big(|\varepsilon(v^M)|\big)\, |\varepsilon(v_M)|\, dx\, dt \\ &\leq c \int_{Q_0} \Big(1+|\varepsilon(v^M)|^2\Big)^{\frac{m-1}{2}}\, dx\, dt \\ &\leq c \Big(\int_{Q_0} \Big(1+|\varepsilon(v^M)|^2\Big)^{\frac{m}{2}}\, dx\, dt\Big)^{\frac{m-1}{m}} \mathcal{L}^{N+1}(Q_0)^{1/m}\,.\end{aligned}$$

Clearly the integral on the r.h.s. stays locally bounded independent of M (if Q_0 has positive distance to the parabolic boundary), hence $\int_{Q_0} |\sigma^M|\, dx\, dt \to 0$ as $\mathcal{L}^{N+1}(Q_0) \to 0$ uniformly in M. Now, with (3.12) and the other convergences, it is easy to show that $v^*, \sigma^* = \frac{\partial \Phi}{\partial \varepsilon}\big(\varepsilon(v^*)\big)$ and p^* strongly solve (1.3) – (1.5), uniqueness then implies the first part of Theorem 1.1. □

4 Steps toward partial regularity: proof of the second part of Theorem 1.1

As it is shown in the paper [S] the partial regularity theory makes essential use of the higher integrability of the pressure function p. Such a property is formulated in (1.14). In order to get this result we recall the definition of f_M stated before (2.10) and define the numbers $\gamma := \frac{m}{m-1}\,\frac{N+1}{N}$ (> 2 on account of our assumption that $2 < m < \frac{2N}{N-2}$), $\overline{\kappa} := \overline{\mu}\,\frac{N}{N-2}$, where $\overline{\mu}$ is fixed through the requirement that

$$\gamma = \frac{1}{m-1}\,\Big(m\big(1-\overline{\mu}\big) + \overline{\mu}\,\frac{mN}{N-2}\Big)$$

This implies that $\overline{\mu} = \frac{N-2}{2N}$, hence $\overline{\kappa} = 1/2$. Proceeding as in (2.12) we get

$$\begin{aligned}\int_{B_\rho(x_0)} f_M^{\gamma(m-1)}\, dx &\leq \Big(\int_{B_\rho(x_0)} f_M^m\, dx\Big)^{1-\overline{\mu}} \cdot \Big(\int_{B_\rho(x_0)} f_M^{\frac{m}{2}\frac{2N}{N-2}}\, dx\Big)^{\overline{\mu}} \\ &\leq c\,\Big(\int_{B_\rho(x_0)} f_M^m\, dx\Big)^{1-\overline{\mu}}\Big(\int_{B_\rho(x_0)} |\nabla f_M^{\frac{m}{2}}|^2\, dx + \rho^{-2}\int_{B_\rho(x_0)} f_M^m\, dx\Big)^{\overline{\kappa}}.\end{aligned}$$

This implies

$$\int_{Q(z_0,R)} |\sigma_M|^\gamma\, dx\, dt \leq c\,(z_0, R, a, f)$$

and in conclusion

$$\int_{Q(z_0,R)} |\sigma|^\gamma\, dx\, dt \leq c\,(z_0, R, a, f).$$

From the equation we then deduce the pressure bound

$$\int_{t_0-R^2}^{t_0} \int_{B_R(x_0)} |p-(p)_{x_0,R}|^\gamma \, dx \, dt$$

$$\leq c \Big\{ \int_{Q(z_0,R)} |\sigma|^\gamma \, dx \, dt + \int_{t_0-R^2}^{t_0} \Big(\int_{B_{\mathbb{R}}(x_0)} \Big(|\partial_t v|^2 + |f|^2 \Big) \, dx \Big)^{\gamma/2} dt \Big\},$$

which gives the result (1.14) provided the second integral on the r.h.s. is finite which clearly is the case if we know that $\partial_t v, f \in L^{2,\gamma}_{\text{loc}}(Q_T)$.

Let us now look at our assumption (1.13). Then we have from the equation the identity $\frac{1}{2}\frac{d}{dt}\int_\Omega |\partial_t v|^2 \, dx + \int_\Omega \partial_t \sigma : \varepsilon(\partial_t v) \, dx = \int_\Omega \partial_t f \cdot \partial_t v \, dx$ which implies $\partial_t v \in L^{2,\infty}(Q_{\delta,T})$ (see Remark 1.2) for any $0 < \delta < T$. This completes the proof of (1.14), Theorem 1.1 is established. □

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