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Abstract

In this note we consider an initial-boundary value problem describing a nonlinear variant of the nonstationary Stokes equation. We prove the existence of a (unique) global solution with Galerkin-type arguments.

1 Notation and results

In this note we prove the existence of a global solution $(u, \pi) : (0, T) \times \Omega \to \mathbb{R}^n \times \mathbb{R}$ for the following nonstationary and nonlinear variant of the Stokes-type problem:

$$(1.1) \qquad \begin{cases} u(0, \cdot) \;=\; u_0 \;\text{ on } \Omega, u \;=\; 0 \;\text{ on } (0, T) \times \partial\Omega, \\ \operatorname{div}\; u \;=\; 0 \quad \text{and} \\ \frac{\partial u}{\partial t} - \operatorname{div}\; \big(Df(\varepsilon(u))\big) \;=\; -\nabla\pi \;\text{ in } (0, T) \times \Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain, T denotes a positive number, and differential operators like div or the symmetric gradient $\varepsilon(u)$ act w.r.t. the spatial variable $x \in \Omega$. We assume that $f : \mathbb{S}^n \to [0, \infty)$ is a potential of class C^2 defined on the space $\mathbb{S}^n$ of symmetric matrices satisfying for some $p \in [2, \infty)$ the ellipticity estimate

$$(1.2) \qquad \lambda(1 + |\varepsilon|^2)^{\frac{p-2}{2}} |\sigma|^2 \;\le\; D^2 f(\varepsilon)(\sigma, \sigma) \;\le\; \Lambda(1 + |\varepsilon|^2)^{\frac{p-2}{2}} |\sigma|^2$$

for all $\varepsilon, \sigma \in \mathbb{S}^n$ with positive constants λ, Λ. As usual u stands for the velocity field, and π denotes the apriori unknown pressure.

We let

$$\begin{aligned} V \;&:=\; \overset{\circ}{W}{}^1_p(\Omega; \mathbb{R}^n), \\ V_0 \;&:=\; \{v \in V \;:\; \operatorname{div} v \;=\; 0\} \end{aligned}$$

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equipped with the norm

$$\|v\|_V \ := \ \|\varepsilon(v)\|_{L^p}$$

which by Korn's and Poincaré's inequality is equivalent to the usual norm. Finally, we consider a function $u_0 \in V_0$. Note that for technical simplicity we restrict ourselves to functions vanishing on $(0,T) \times \partial\Omega$. Nonhomogeneous boundary conditions can be handled in a similar way. Potentials f satisfying (1.2) are of power growth type since they can be bounded from above and from below (up to irrelevant terms) by the standard model $\Phi(\varepsilon) = (1+|\varepsilon|^2)^{p/2}, \varepsilon \in \mathbb{S}^n$. The physical relevance of power growth potentials is explained in the monographs [AM] and [BAH], the mathematical background is discussed in the works [L1] and [L2], we also refer to [MNR] and [MNRR].

Our main concern is to give an elementary existence proof for problem (1.1), more precisely, we are going to show

THEOREM 1.1. *There exists a unique function $u \in L^p(0,T;V)$ with distributional time derivative $\frac{d}{dt}\, u \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ such that $u(0) = u_0, u(t) \in V_0$ and*

$$\int_\Omega \frac{d}{dt}\, u \cdot \varphi \, dx + \int_\Omega Df(\varepsilon(u)) : \varepsilon(\varphi)\, dx \ = \ 0 \tag{1.3}$$

for all $\varphi \in V_0$ and almost all $t \in (0,T)$.

In (1.3) " $\cdot$ " denotes the scalar product in $\mathbb{R}^n$, " $:$ " stands for the scalar product of matrices. We recall that "$\frac{d}{dt}\, u \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$" means that there exists a function v in this space such that

$$\frac{d}{dt}\int_\Omega u \cdot w \, dx = \int_\Omega v \cdot w \, dx \quad \forall w \in L^2(\Omega;\mathbb{R}^n)$$

holds in the scalar distributional sense on $(0,T)$. This is equivalent to

$$\int_0^T u(t)\eta'(t)\, dt \ = \ -\int_0^T v(t)\eta(t)\, dt$$

for all $\eta \in C_0^\infty(0,T)$, where $\int_0^T$ ...... are Bochner integrals.

REMARK 1.1. *The solution described in Theorem 1.1 usually is called the strong solution to problem (1.1).*

2 A lemma on finite dimensional nonstationary variational inequalities

In this section we consider the space $\mathbb{R}^m$ equipped with some scalar product $[\cdot,\cdot]$ and associated norm $\|x\| = \sqrt{[x,x]}$.

LEMMA 2.1. *Let K denote a compact and convex set in $\mathbb{R}^m$. Suppose further that we are given a point $x_0 \in K$ and a vectorfield $F : \mathbb{R}^m \to \mathbb{R}^m$ of class C^1 satisfying the monotonicity inequality*

$$[F(x) - F(y), x - y] \ \geq \ \nu(\|x - y\|) \quad \forall x, y \in \mathbb{R}^m \tag{2.1}$$

for a function $\nu : [0, \infty) \longrightarrow [0, \infty)$ being continuous and such that $\nu(0) = 0, \nu(t) > 0$ for $t > 0$. Then there exists a unique Lipschitz curve $x : [0, T] \longrightarrow \mathbb{R}^m$ satisfying

$$\begin{cases} x(0) = x_0, \ x(t) \in K \ \ \textit{for all} \ \ t \in [0, T] \qquad \textit{and} \\ \big[\dot{x}(t), y - x(t)\big] + \big[F(x(t)), y - x(t)\big] \ \geq \ 0 \ \ \textit{a.e.} \qquad \textit{for all } y \in K. \end{cases} \tag{2.2}$$

Proof: **Step 1** "Uniqueness"

Let $x(t), y(t)$ satisfy (2.2). Then we get

$$\big[\dot{x}(t) - \dot{y}(t), x(t) - y(t)\big] + \big[F(x(t)) - F(y(t)), x(t) - y(t)\big] \ \leq \ 0,$$

thus (see (2.1)) $\frac{d}{dt} \|x(t) - y(t)\|^2 \ \leq \ 0$ which gives $x(t) = y(t)$ on account of the initial condition.

Step 2 "Existence for special sets K"

Let $K = \overline{G}$ for an open, bounded and convex set G with ∂G of class C^2. For ρ sufficiently small let $U = \{x \in \mathbb{R}^m : \operatorname{dist}(x, \partial G) < \rho\}$ and define $d : U \to \mathbb{R}$ as

$$d(x) = \begin{cases} \operatorname{dist}(z, \partial G), & \text{if} \quad z \in U \cap K \\ -\operatorname{dist}(z, \partial G), & \text{if} \quad z \in U - K. \end{cases}$$

Here of course "dist" is measured with respect to $\| \cdot \|$. Finally, let $N : U \to \mathbb{R}^m$, $N = \operatorname{grad} d$ (calculated w.r.t. $[\cdot, \cdot]$) and observe that on ∂K N is just the interior normal vectorfield of ∂K. Let a denote a number such that

$$a > \sup\big\{\|F(y)\| : \ y \in K\big\} \tag{2.3}$$

and fix $\varepsilon \in (0, \rho)$. We choose a smooth function $h_\varepsilon : \mathbb{R} \to [0, \infty)$ with the properties

$$h_\varepsilon(s) = a \quad \text{for } |s| \leq \frac{\varepsilon}{2}, \ h_\varepsilon(s) = 0 \quad \text{for } |s| \geq \frac{3\varepsilon}{4}. \tag{2.4}$$

With $\eta \in C_0^1\big(\mathbb{R}^m; [0, 1]\big), \eta = 1$ on $K \cup U$, we finally let

$$F_\varepsilon(x) = \eta(x) F(x) - h_\varepsilon\big(d(x)\big) \ N(x).$$

Note that F_ε is of class $C^1(\mathbb{R}^m;\mathbb{R}^m)$ with compact support, hence the initial value problem

$$\dot{x}(t) + F_\varepsilon\bigl(x(t)\bigr) = 0 \quad \text{on } [0,T],\ x(0) = x_0, \tag{2.5}$$

admits a unique solution $x_\varepsilon \in C^1\bigl([0,T];\mathbb{R}^m\bigr)$.

We claim that

$$x_\varepsilon(t) \in K \quad \forall\, 0 \le t \le T. \tag{2.6}$$

For proving (2.6) let us assume that $x_\varepsilon(t) \notin K$ for some time $t > 0$. Since $x_\varepsilon(0) \in K$ we can find $t_1 \in [0,t)$ such that $x_\varepsilon(t_1) \in \partial K$ and $x(s) \notin K$ for $s > t_1$ close to t_1. From (2.5) we get

$$\begin{aligned} 0 &= \bigl[\dot{x}_\varepsilon(t_1), N\bigl(x_\varepsilon(t_1)\bigr)\bigr] + \bigl[F_\varepsilon\bigl(x_\varepsilon(t_1)\bigr), N\bigl(x_\varepsilon(t_1)\bigr)\bigr] \\ &\underset{(2.4)}{=} \frac{d}{dt|_{t_1}} d\bigl(x_\varepsilon(t)\bigr) + \bigl[F\bigl(x_\varepsilon(t_1)\bigr), N\bigl(x_\varepsilon(t_1)\bigr)\bigr] - a \\ &\le \frac{d}{dt|_{t_1}} d\bigl(x_\varepsilon(t)\bigr) + \sup\bigl\{\|F(y)\| :\ y \in K\bigr\} - a, \end{aligned}$$

and we see by (2.3) that $\frac{d}{dt|_{t_1}} d\bigl(x_\varepsilon(t)\bigr) > 0$. This gives $d\bigl(x_\varepsilon(s)\bigr) > 0$ for $s > t_1$ sufficiently close to t_1, hence the solution curve stays inside K which is a contradiction.

Equation (2.5) together with (2.6) implies

$$\bigl\|\dot{x}_\varepsilon(t)\bigr\| \le \sup\bigl\{\bigl\|F(y)\bigr\| :\ y \in K\bigr\} + a \tag{2.7}$$

for all $t \in [0,T]$ and all $\varepsilon \in (0,\rho)$, thus we find a Lipschitz curve $x : [0,T] \longrightarrow \mathbb{R}^m$ such that $x_\varepsilon \underset{\varepsilon\downarrow 0}{\longrightarrow} x$ uniformly (at least for a subsequence). Clearly $x(0) = x_0$, $x(t) \in K$ for all $t \in [0,T]$, and (2.7) continues to hold for $\dot{x}(t)$ and almost all $t \in [0,T]$. We claim that x is the solution of (2.2). To this purpose we assume w.l.o.g. that $\dot{x}_\varepsilon \rightharpoonup \dot{x}$ as $\varepsilon \searrow 0$ weakly in $L^2(0,T;\mathbb{R}^m)$. From (2.5) we obtain for any $y \in L^2(0,T;\mathbb{R}^m)$ with the property $y(t) \in K$ a.e.

$$\begin{aligned} 0 &= \int_0^T \bigl[\dot{x}_\varepsilon, y - x\bigr]\,dt + \int_0^T \bigl[F_\varepsilon(x_\varepsilon), y - x\bigr]\,dt \\ &= \int_0^T \bigl[\dot{x}_\varepsilon, y - x\bigr]\,dx + \int_0^T \bigl[F(x_\varepsilon), y - x\bigr]\,dt \\ &\quad - \int_0^T h_\varepsilon\bigl(d(x_\varepsilon)\bigr)\bigl[N(x_\varepsilon), y - x\bigr]\,dt \\ &=: I_1 + I_2 - I_3. \end{aligned}$$

Here we have used that $x_\varepsilon(t) \in K$, hence $\eta\big(x_\varepsilon(t)\big) = 1$. Obviously

$$\begin{aligned} I_1 &\longrightarrow \int_0^T [\dot{x}, y - x]\, dt, \\ I_2 &\longrightarrow \int_0^T \big[F(x), y - x\big]\, dt \end{aligned}$$

as $\varepsilon \downarrow 0$. Let us look at I_3:

$$\begin{aligned} I_3 &= \int_0^T h_\varepsilon\big(d(x_\varepsilon)\big)\big[N(x), y - x\big]\, dt \\ &\quad + \int_0^T h_\varepsilon\big(d(x_\varepsilon)\big)\big[N(x_\varepsilon) - N(x), y - x\big]\, dt \\ &=: J_1 + J_2, \end{aligned}$$

$|J_2| \;\leq\;$ diam $(K) \int_0^T h_\varepsilon\big(d(x_\varepsilon)\big)\, \big\|N(x_\varepsilon) - N(x)\big\|\, dt \underset{\varepsilon\downarrow 0}{\longrightarrow} 0$ (by the boundedness of h_ε),

$$\begin{aligned} J_1 &= \int_{\{t\in[0,T]:\ x(t)\,\in\,\partial K\}} h_\varepsilon\big(d(x_\varepsilon)\big)\big[N(x), y - x\big]\, dt \\ &\quad + \int_{\{t\in[0,T]:\ x(t)\,\notin\,\partial K\}} h_\varepsilon\big(d(x_\varepsilon)\big)\big[N(x), y - x\big]\, dt \\ &=: \alpha + \beta. \end{aligned}$$

Clearly $h_\varepsilon\big(d(x_\varepsilon)\big) \underset{\varepsilon\downarrow 0}{\longrightarrow} 0$ on the set $\big\{t \in [0,T] :\ x(t) \notin \partial K\big\}$, hence $\beta \longrightarrow 0$ as $\varepsilon \downarrow 0$ by dominated convergence. By convexity of K we see that $\alpha \geq 0$, hence we finally get

$$\int_0^T [\dot{x}, y - x]\, dt + \int_0^T \big[F(x), y - x\big]\, dt \geq 0$$

for all L^2– curves $y : [0,T] \longrightarrow K$, in particular we may choose $y(t) = x(t) + \varphi(t)\big(y - x(t)\big)$ for $y \in K$ and $\varphi \in C_0^0(0,T), 0 \leq \varphi \leq 1$.

This gives

$$0 \leq \int_0^T \varphi(t)\big\{\big[\dot{x}(t), y - x(t)\big] + \big[F\big(x(t)\big), y - x(t)\big]\big\}\, dt,$$

and by the arbitrariness of φ inequality (2.2) follows.

Step 3 “Existence for general sets K”

Let now $K \subset \mathbb{R}^m$ denote an arbitrary compact and convex set. If φ is a symmetric mollifier, we let

$$g_\varepsilon(z) = \int_{\mathbb{R}^m} \varepsilon^{-m} \varphi\,(\frac{y-z}{\varepsilon})\ \mathrm{dist}\,(y,K)\,dy,$$

i.e. g_ε denotes the mollification of the distance to K. For any $\rho > 0$ the sets $[g_\varepsilon < \rho]$ are open, bounded and convex, moreover, Sard's theorem shows that $\partial\,[g_\varepsilon < \rho]$ is a smooth hypersurface for almost all values of ρ. For $z \in K$ we have

$$\begin{aligned} g_\varepsilon(z) &= \int_{\mathbb{R}^m} \varepsilon^{-m} \varphi\,(\frac{y}{\varepsilon})\ \mathrm{dist}\,(y+z,K)\,dy \\ &= \int_{B_\varepsilon(0)} \varepsilon^{-m} \varphi\,(\frac{y}{\varepsilon})\ \mathrm{dist}\,(y+z,K)\,dy\ \le \varepsilon, \end{aligned}$$

hence

$$K \subset [g_\varepsilon \le \varepsilon] \subset [g_\varepsilon < \rho]$$

if $\rho > \varepsilon$.

Let $\varepsilon_k \searrow 0, k \to \infty$, and choose $\rho_k \in (\varepsilon_k, 2\varepsilon_k)$ such that $\partial\,[g_{\varepsilon_k} < \rho_k]$ is smooth. We apply Step 2 to the sets $K_k = [g_{\varepsilon_k} \le \rho_k]$, i.e. we get for each k a solution of (2.2) with K replaced by K_k. Since obviously $K_k \subset \overline{B}_R(x_0)$ for a sufficiently large ball $\overline{B}_R(x_0)$, we have that $\sup_{k\in\mathbb{N}} \|x_k\|_{L^\infty(0,T)} < \infty$.

Note that in Step 2 we proved that

$$\|\dot{x}_k\|_{L^\infty(0,T)} \le \sup\big\{\|F(y)\|:\ y \in K_k\big\} + a_k,$$

where a_k has to be chosen according to (see (2.3))

$$a_k > \sup\big\{\|F(y)\|:\ y \in K_k\big\} =: \xi_k.$$

Let us define $a_k = \xi_k + 1$. Since $\xi_k \le \sup\big\{\|F(y)\|:\ y \in \overline{B}_R(x_0)\big\} =:\ \xi$ we get

$$\|\dot{x}_k\|_{L^\infty(0,T)} \le 2\xi + 1,$$

hence (after passing to a subsequence) there is a Lipschitz curve $x : [0,T] \to \mathbb{R}^m$ with the properties $x(0) = x_0, x_k \to x$ uniformly on $[0,T]$ and $\dot{x}_k \rightharpoonup \dot{x}$ weakly in $L^2(0,T;\mathbb{R}^m)$. We have

$$\begin{aligned} \mathrm{dist}\,\big(x(t),K\big) &\le \mathrm{dist}\,\big(x_k(t),K\big) + \big\|x_k(t) - x(t)\big\| \\ &\le 3\varepsilon_k + \|x_k(t) - x(t)\| \underset{k\to\infty}{\longrightarrow} 0, \end{aligned}$$

hence $x(t) \in K$. Note that $\mathrm{dist}\,\big(x_k(t),K\big) > 3\varepsilon_k$ would give $\mathrm{dist}\,(y,K) \ge 2\varepsilon_k$ on $B_{\varepsilon_k}(x_k(t))$, hence $g_{\varepsilon_k}\,\big(x_k(t)\big) \ge 2\varepsilon_k$ which contradicts the fact that $x_k(t) \in \big[g_{\varepsilon_k} \le \rho_k\big]$

and $\rho_k < 2\varepsilon_k$.

Consider $y \in L^2(0,T;\mathbb{R}^m), y(t) \in K$ a.e., and observe $K \subset K_k$, thus

$$\int_0^T \big[\dot{x}_k(t), y(t) - x_k(t)\big]\, dt + \int_0^T \big[F\big(x_k(t)\big), y(t) - x_k(t)\big]\, dt \geq 0.$$

By uniform convergence the second integral converges to $\int_0^T \big[F\big(x(t)\big), y(t) - x(t)\big]\, dt$, the first integral equals

$$\int_0^T \big[\dot{x}_k(t), y(t) - x(t)\big]\, dt + \int_0^T \big[\dot{x}_k(t), x(t) - x_k(t)\big]\, dt \underset{k\to\infty}{\longrightarrow} \int_0^T \big[\dot{x}(t), y(t) - x(t)\big]\, dt,$$

which follows from $\dot{x}_k \rightharpoonup \dot{x}$ weakly in $L^2(0,T;\mathbb{R}^m)$ together with $x_k \longrightarrow x$ uniformly on $[0,T]$.

This implies

$$\int_0^T \big[\dot{x}(t), y(t) - x(t)\big]\, dt + \int_0^T \big[F\big(x(t)\big), y(t) - x(t)\big]\, dt \geq 0$$

for all L^2 – curves $y:\ [0,T] \longrightarrow \mathbb{R}^m$ such that $y(t) \in K$, and the final claim follows as at the end of Step 2. This completes the proof of Lemma 2.1. $\square$

3 Proof of Theorem 1.1 with a Galerkin-type argument

Let $\{v_i:\ i \in \mathbb{N}\}$ denote a dense subset of V_0 and let V_m the subspace of V_0 generated by the functions $u_0, v_1, \ldots, v_m, m \in \mathbb{N}$. We define K_m as the convex hull of $\{u_0, v_1, \ldots, v_m\}$. By Lemma 2.1 there exists a unique Lipschitz function $u_m:\ [0,T] \to V_m$ s.t.

$$(3.1)\quad \begin{cases} u_m(0) = u_0, u_m(t) \in K_m,\ 0 \leq t \leq T, \\ \displaystyle\int_\Omega \frac{d}{dt} u_m(t) \cdot \big(v - u_m(t)\big)\, dx + \int_\Omega Df\big(\varepsilon(u_m(t))\big) :\ \big(\varepsilon(v) - \varepsilon(u_m(t)\big)\, dx \geq 0 \\ \text{for almost all } t \in [0,T] \quad \text{and all } v \in K_m. \end{cases}$$

To justify this suppose that $w_1, \ldots, w_M$ is some basis in V_m and consider the linear isomorphism

$$\Phi_m:\ \mathbb{R}^M \longrightarrow V_m, x \longmapsto \sum_{i=1}^M x_i w_i, x = (x_i)_{1\leq i\leq M}.$$

We define the scalar product (" $\cdot$ " denoting the product in $\mathbb{R}^n$)
$[x,y] = \sum_{i,j=1}^{M} x_i y_j \int_\Omega w_i \cdot w_j \, dz, x, y \in \mathbb{R}^M$, and the vectorfield $F : \mathbb{R}^M \to \mathbb{R}^M$ through the relation

$$\big[F(x), y\big] = \int_\Omega Df\big(\varepsilon(\Phi_m(x))\big) : \varepsilon\big(\Phi_m(y)\big) \, dz \quad \forall\, y \in \mathbb{R}^M .$$

Note that for fixed $x \in \mathbb{R}^M$ the r.h.s. is a linear functional w.r.t. $y \in \mathbb{R}^M$, thus can be represented by a unique vector $F(x)$ via the scalar product $[\cdot,\cdot]$. Let $x_0 = \Phi_m^{-1}(u_0)$ and $K = \Phi_m^{-1}(K_m)$ which is a compact and convex subset of $\mathbb{R}^M$. We show that F satisfies (2.1): for any $x, \tilde{x} \in \mathbb{R}^M$ we have

$$\begin{aligned} \big[F(x) - F(\tilde{x}), x - \tilde{x}\big] &= \int_\Omega \big(Df\big(\varepsilon(u)\big) - Df\big(\varepsilon(\tilde{u})\big)\big) : \big(\varepsilon(u) - \varepsilon(\tilde{u})\big) \, dz \\ &= \int_0^1 \int_\Omega D^2 f\big(\varepsilon(\tilde{u}) + t\big(\varepsilon(u) - \varepsilon(\tilde{u})\big)\big)\big(\varepsilon(u) - \varepsilon(\tilde{u}), \varepsilon(u) - \varepsilon(\tilde{u})\big) \, dz \, dt, \end{aligned}$$

where $u = \Phi_m(x), \tilde{u} = \Phi_m(\tilde{x})$. Since $p \geq 2$ we can bound the double integral from below by

$$c \int_\Omega \big|\varepsilon(u) - \varepsilon(\tilde{u})\big|^2 \, dz,$$

c denoting a positive constant independent of m. Korn's and Poincaré's inequality give

$$\begin{aligned} \int_\Omega \big|\varepsilon(u) - \varepsilon(\tilde{u})\big|^2 \, dz \;&\geq\; c \int_\Omega |u - \tilde{u}|^2 \, dz \\ &=\; c \int_\Omega |\Phi_m(x) - \Phi_m(\tilde{x})|^2 \, dz \\ &=\; c \int_\Omega \Big| \sum_{i=1}^{M} \big(x_i - \tilde{x}_i\big) w_i \Big|^2 \, dz \\ &=\; c \sum_{i,j=1}^{M} \big(x_i - \tilde{x}_i\big)\big(x_j - \tilde{x}_j\big) \int_\Omega w_i \cdot x_j \, dz \;=\; c\|x - \tilde{x}\|^2, \end{aligned}$$

thus $\big[F(x) - F(\tilde{x}), x - \tilde{x}\big] \geq c\|x - \tilde{x}\|^2$, and we can apply Lemma 2.1 to our choices of x_0, K and F. Let x_m denote the corresponding solution. By construction it is now immediate that $u_m = \Phi_m(x_m)$ is the solution of (3.1).

Next we derive suitable apriori bounds for the sequence $\{u_m\}$. By definition $u_0 \in K_m$, thus

$$\int_\Omega \frac{d}{dt}\, u_m \cdot (u_0 - u_m)\, dx \geq \int_\Omega Df\big(\varepsilon(u_m)\big) : \big(\varepsilon(u_m) - \varepsilon(u_0)\big)\, dx$$

for almost all t, and we get:

$$\begin{aligned} &\int_\Omega Df\big(\varepsilon(u_m)\big) : \varepsilon(u_m)\, dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |u_m|^2\, dx \\ &\leq \int_\Omega Df\big(\varepsilon(u_m)\big) : \varepsilon(u_0)\, dx + \int_\Omega \frac{d}{dt}\, u_m \cdot u_0\, dx. \end{aligned} \tag{3.2}$$

W.l.o.g. we may assume that $Df(0) = 0$. Then we obtain (all constants are independent of m)

$$c_1\|u_m(t)\|_V^p + \frac{d}{dt}\int_\Omega \Big(\frac{1}{2}|u_m(t)|^2 - u_m(t)\cdot u_0\Big)\, dx \leq c_2\|u_m(t)\|_V^{p-1}\|u_0\|_V$$

and with Young's inequality this implies

$$c_3\|u_m(t)\|_V^p + \frac{d}{dt}\int_\Omega \Big(\frac{1}{2}|u_m(t)|^2 - u_m(t)\cdot u_0\Big)\, dx \leq c_4\|u_0\|_V^p$$

being valid for almost all t. Integrating this inequality from 0 to T gives the bound

$$c_3\int_0^T \|u_m(t)\|_V^p\, dt + \int_\Omega \Big(\frac{1}{2}\,|u_m(T)|^2 - u_m(T)\cdot u_0\Big)\, dx \leq c_4 T\|u_0\|_V^p - \frac{1}{2}\int_\Omega |u_0|^2\, dx,$$

hence

$$c_3\int_0^T \|u_m(t)\|_V^p\, dt + \int_\Omega \frac{1}{4}\,|u_m(T)|^2\, dx \leq c_5\Big(T\|u_0\|_V^p + \|u_0\|^2_{L^2(\Omega)}\Big),$$

and if we neglect the second term on the l.h.s. we arrive at the apriori bound

$$\begin{aligned} &\int_0^T \|u_m(t)\|_V^p\, dt \leq c_6, \\ &c_6 = c_6\big(n, p, \lambda, \Lambda, \Omega, T, \|u_0\|_V\big), \end{aligned} \tag{3.3}$$

which means that $\{u_m\}$ is a bounded sequence in the space $L^p(0,T;V)$.

Next consider $t \in (0,T)$ s.t. $\frac{d}{dt}\, u_m(t)$ exists and observe that for $h > 0$ the function $u_m(t-h)$ belong to the set K_m (since $u_m(s) \in K_m$ for all $s \in [0,T]$), hence by (3.1)

$$\begin{aligned} &\int_\Omega \frac{d}{dt}\, u_m(t)\cdot\big(u_m(t) - u_m(t-h)\big)\frac{1}{h}\, dx \\ &+ \int_\Omega Df\big(\varepsilon(u_m(t))\big) : \Big(\varepsilon\big(u_m(t)\big) - \varepsilon\big(u_m(t-h)\big)\Big)\frac{1}{h}\, dx \leq 0 \end{aligned}$$

and we get by passing to the limit $h \downarrow 0$

$$\int_\Omega \big|\frac{d}{dt}\, u_m(t)\big|^2\, dx + \int_\Omega Df\big(\varepsilon(u_m(t))\big) :\, \frac{d}{dt}\, \varepsilon(u_m(t))\, dx \le 0.$$

In the same way we use $v = u_m(t+h), h > 0$, in (3.1) with the result

$$\int_\Omega \big|\frac{d}{dt}\, u_m(t)\big|^2\, dx + \int_\Omega Df\big(\varepsilon(u_m(t))\big) :\, \frac{d}{dt}\, \varepsilon(u_m(t))\, dx \ge 0.$$

Observing $\frac{d}{dt} f\big(\varepsilon(u_m(t))\big) = Df\big(\varepsilon(u_m(t))\big) :\, \frac{d}{dt}\, \varepsilon(u_m(t))$ we get

$$0 = \|\frac{d}{dt}\, u_m(t)\|^2_{L^2(\Omega)} + \frac{d}{dt}\int_\Omega f\big(\varepsilon(u_m(t))\big)\, dx$$

which yields upon integration

$$\begin{aligned}
&\int_0^T \big\|\frac{d}{dt}\, u_m(t)\big\|^2_{L^2(\Omega)}\, dt \\
&= \; -\int_0^T \frac{d}{dt} \int_\Omega f\big(\varepsilon(u_m(t))\big)\, dx\, dt \\
&= \; \int_\Omega f\big(\varepsilon(u_m(0))\big)\, dx - \int_\Omega f\big(\varepsilon(u_m(T))\big)\, dx \le \int_\Omega f\big(\varepsilon(u_0)\big)\, dx.
\end{aligned}$$

Taking into account (3.3), we have shown

LEMMA 3.1. *The sequence $\{u_m\}$ is bounded in the space $\big\{w \in L^p(0,T;V) :\, \frac{d}{dt}\, w \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))\big\}$ equipped with the norm*

$$\Big(\int_0^T \|w\|^p_V\, dt\Big)^{1/p} + \Big(\int_0^T \|\frac{d}{dt}\, w\|^2_{L^2}\, dt\Big)^{1/2}.$$

With Lemma 3.1 we may pass to a suitable subsequence and find a function $u \in L^p(0,T;V)$ with $\frac{du}{dt} \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ such that

$$\begin{aligned}
u_m \quad &\rightharpoonup \quad u \quad \text{weakly in} \quad L^p(0,T;V), \\
\tfrac{d}{dt}\, u_m \quad &\rightharpoonup \quad \tfrac{d}{dt}\, u \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega;\mathbb{R}^n))
\end{aligned}$$

and $u_m \to u$ strongly in $L^2\big(0,T;L^2(\Omega;\mathbb{R}^n)\big)$. The last statement follows from the compactness of the embedding (see, e.g. [Li])

$$\big\{w \in L^2\big(0,T;\overset{\circ}{W}{}^1_2(\Omega;\mathbb{R}^n)\big) :\, \tfrac{d}{dt}\, w \in L^2\big(0,T;L^2(\Omega;\mathbb{R}^n)\big)\big\} \hookrightarrow L^2\big(0,T;L^2(\Omega;\mathbb{R}^n)\big).$$

Next we fix $\eta \in C_0^0(0,T), \eta \geq 0$. Multiplying (3.1) with η and integrating over $[0,T]$ gives

$$\begin{aligned} 0 \;&\leq \int_0^T \int_\Omega \eta(t) \frac{d}{dt}\, u_m(t) \cdot (v - u_m(t))\, dx\, dt \\ &+ \int_0^T \int_\Omega \eta(t) Df\big(\varepsilon(u_m(t))\big) : \big(\varepsilon(v) - \varepsilon(u_m(t))\big)\, dx\, dt \end{aligned} \tag{3.4}$$

for any function $v \in K_k$ and all $m \geq k$. Passing to the limit $m \to \infty$ we see that the first integral converges to

$$\int_0^T \int_\Omega \eta(t) \frac{d}{dt}\, u(t) \cdot \big(v - u(t)\big)\, dx\, dt.$$

For the second one we observe that it is equal to

$$\begin{aligned} &\int_0^T \int_\Omega \eta(t) Df\big(\varepsilon(v)\big) : \big(\varepsilon(v) - \varepsilon(u_m(t))\big)\, dx\, dt \\ &\quad + \int_0^T \int_\Omega \eta(t) \big\{ Df\big(\varepsilon(u_m(t))\big) - Df\big(\varepsilon(v)\big)\big\} : \big(\varepsilon(v) - \varepsilon(u_m(t))\big)\, dx\, dt \\ &\quad \leq \int_0^T \int_\Omega \eta(t) Df\big(\varepsilon(v)\big) : \big(\varepsilon(v) - \varepsilon(u_m(t))\big)\, dx\, dt \\ &\underset{m\to\infty}{\longrightarrow} \int_0^T \int_\Omega \eta(t) Df\big(\varepsilon(v)\big) : \big(\varepsilon(v) - \varepsilon(u(t))\big)\, dx\, dt. \end{aligned}$$

Here we used the fact that

$$L^p(0,T;V) \ni \varphi \mapsto \int_0^T \eta(t) \int_\Omega Df\big(\varepsilon(v)\big) : \varepsilon\big(\varphi(t)\big)\, dx\, dt$$

is a continuous linear functional and $u_m \rightharpoonup u$ weakly in $L^p(0,T;V)$.

Our calculations now show that

$$\begin{aligned} &\int_0^T \int_\Omega \eta(t) \frac{d}{dt}\, u(t) \cdot \big(v - u(t)\big)\, dx\, dt \\ &\quad + \int_0^T \int_\Omega \eta(t) Df\big(\varepsilon(v)\big) : \big(\varepsilon(v) - \varepsilon(u(t))\big)\, dx\, dt \geq 0 \end{aligned}$$

for any $v \in K_k, k \in \mathbb{N}$.

Recalling that $\{v_i : i \in \mathbb{N}\}$ is dense in V_0 we see that the last inequality holds for any $v \in V_0$. Let us also remark that actually u belongs to $L^p(0,T;V_0)$ since the u_m are in this closed subspace of $L^p(0,T;V)$. By the arbitrariness of η we deduce that for almost all t we have

$$\int_\Omega \frac{d}{dt}\, u(t)\cdot\big(v-u(t)\big)\,dx + \int_\Omega Df\big(\varepsilon(v)\big) : \,\big(\varepsilon(v)-\varepsilon(u(t))\big)\,dx \geq 0.$$

Finally, we replace v by $u(t)+\varepsilon w, w\in V_0, \varepsilon>0$, and end up with (after passing to the limit $\varepsilon\downarrow 0$)

$$0=\int_\Omega \frac{d}{dt}\, u(t)\cdot w\,dx + \int_\Omega Df\big(\varepsilon(u(t))\big) : \,\varepsilon(w)dx$$

which proves Theorem 1.1 by remarking that the uniqueness follows in a standard way.

□

4 Some extensions

We assume first that f can be written as

$$f(\varepsilon)=A(|\varepsilon|), \tag{4.1}$$

where $A:\ [0,\infty)\to[0,\infty)$ is a smooth, i.e. C^2, N-function for which the Δ_2–property holds and which satisfies for some $p\geq 2$

$$\begin{aligned} D^2f(\varepsilon)(\sigma,\sigma) &= \tfrac{1}{|\varepsilon|}\,A'(|\varepsilon|)\big[|\sigma|^2-\tfrac{1}{|\varepsilon|^2}\,(\varepsilon:\sigma)^2\big] \\ &\quad +A''(|\varepsilon|)\tfrac{1}{|\varepsilon|^2}\,(\varepsilon:\sigma)^2 \\ &\geq \lambda\big(1+|\varepsilon|^2\big)^{\frac{p-2}{2}}\,|\sigma|^2 \end{aligned} \tag{4.2}$$

for all $\varepsilon,\sigma\in\mathbb{S}^n$. Note that (4.1), (4.2) imply that f grows at least as $|\varepsilon|^p$. The space V has to be replaced by $\overset{\circ}{W}{}^1_A(\Omega;\mathbb{R}^n)$, and we let

$$V_0=\{v\in V:\ \operatorname{div} v=0\}$$

equipped with the norm

$$\|v\|_V=\|\varepsilon(v)\|_{L_A}.$$

Of course, now u_0 is assumed to be an element of V_0. By choosing a countable dense subset of V_0 and by the appropriate use of Young's inequality it is easy to modify the arguments of Section 3 and to prove Theorem 1.1 in this more general setting.

Next we consider the case that f satisfies an anisotropic (p,q)–growth condition, i.e.

$$\lambda\big(1+|\varepsilon|^2\big)^{\frac{p-2}{2}}\leq D^2f(\varepsilon)\leq \Lambda\big(1+|\varepsilon|^2\big)^{\frac{q-2}{2}} \tag{4.3}$$

with exponents $2 \le p < q < \infty$. Let V, V_0 denote the spaces introduced in Section 1 (with exponent p !), and we try to find a solution of the evolution problem in the class

$$X = \Big\{ u \in L^p(0,T;V) : \ u(t) \in V_0, \int_0^T \int_\Omega f\big(\varepsilon(u(t))\big)\, dx\, dt < \infty \Big\}.$$

Of course we assume that u_0 is a given function from V_0 s.t. $\int_\Omega f\big(\varepsilon(u_0)\big)\, dx < \infty$. This clearly will follow if $\varepsilon(u_0) \in L^q(\Omega;\mathbb{S}^n)$ is assumed (see (4.3)). Let $\{v_m\}_{m\in\mathbb{N}}$ a set of functions in $C_0^\infty(\Omega;\mathbb{R}^n) \cap$ Kern (div) being dense in $\overset{\circ}{W}{}^1_q(\Omega;\mathbb{R}^n) \cap$ Kern (div) w.r.t. the norm $\|\varepsilon(v)\|_{L^q}$ and let

$$\begin{aligned} V_m :=& \quad \text{space generated by } u_0, v_1, \dots, v_m, \\ K_m :=& \quad \text{convex hull of } u_0, v_1, \dots, v_m. \end{aligned}$$

As before we can solve problem (3.1) with unique solution $u_m : [0,T] \to V_m$.

In order to get apriori bounds for the sequence $\{u_m\}$, we assume

$$\varepsilon(u_0) \in L^\infty(\Omega;\mathbb{S}^n).$$

If this is not the case, we may insert $v := v_1$ in inequality (3.1). As a result we get estimate (3.2). Assuming $f(0) = 0$, the convexity of f implies $\int_\Omega Df\big(\varepsilon(u_m)\big) : \varepsilon(u_m)\, dx \ge \int_\Omega f\big(\varepsilon(u_m)\big)\, dx$, whereas from (4.3) it follows that

$$\Big| \int_\Omega Df\big(\varepsilon(u_m)\big) : \ \varepsilon(u_0)\, dx \Big| \le c \|\varepsilon(u_0)\|_{L^\infty} \int_\Omega \big|\varepsilon(u_m)\big|^{q-1}\, dx.$$

In order to continue we need the restriction that

$$q < p + 1. \tag{4.4}$$

Then $\int_\Omega \big|\varepsilon(u_m)\big|^{q-1}\, dx$ can be absorbed in $\int_\Omega f\big(\varepsilon(u_m)\big)\, dx$ (which is bounded from below by $\int_\Omega \big|\varepsilon(u_m)\big|^p\, dx$ on account of (4.3)), and we find that

$$c_1 \int_\Omega f\big(\varepsilon(u_m)\big)\, dx + \frac{d}{dt} \int_\Omega \Big(\frac{1}{2}\, |u_m(t)|^2 - u_m(t) \cdot u_0 \Big)\, dx \le c_2$$

with c_2 depending on u_0 but both constants c_1, c_2 being independent of m. Following the arguments presented in Section 3 we see that (3.3) has to be replaced by

$$\int_0^T \int_\Omega f\big(\varepsilon(u_m(t))\big)\, dx\, dt \le c_3, \tag{4.5}$$

i.e. $\{u_m\}$ "stays bounded" in the space X, in particular we have boundedness in $L^p(0,T;V)$. The calculations carried out before Lemma 3.1 also show that

$$\int_0^T \Big\| \frac{d}{dt}\, u_m(t) \Big\|^2_{L^2(\Omega)}\, dt \le c_4,$$

thus we find $u \in L^p(0,T;V)$ with $\frac{d}{dt} u \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ and such that the convergences stated after Lemma 3.1 hold. By l.s.c. and (4.5) we see that $u \in X$. Returning to (3.4) and estimating the second integral on the r.h.s. of (3.4) in an obvious way, we deduce the inequality

$$
\begin{aligned}
0 \le & \int_0^T \eta(t) \int_\Omega \frac{d}{dt} u_m(t) \cdot (v - u_m(t))\, dx\, dt \\
& + \int_0^T \eta(t) \int_\Omega Df\big(\varepsilon(v)\big) : \big(\varepsilon(v) - \varepsilon(u_m(t))\big)\, dx\, dt
\end{aligned}
$$

for any $v \in K_k$ and all $m \ge k$. Since the functions v_k are smooth, we see that we may pass to the limit $m \to \infty$, thus

$$
\begin{aligned}
0 \le & \int_0^T \eta(t) \int_\Omega \frac{d}{dt} u(t) \cdot (v_k - u(t))\, dx\, dt \\
& + \int_0^T \eta(t) \int_\Omega Df\big(\varepsilon(v_k)\big) : \big(\varepsilon(v_k) - \varepsilon(u(t))\big)\, dx\, dt.
\end{aligned}
$$

Now, if $v \in \overset{\circ}{W}{}^1_q(\Omega;\mathbb{R}^n) \cap$ Kern (div) is arbitrary, we have that $\varepsilon(v'_k) \to \varepsilon(v)$ in $L^q(\Omega;\mathbb{S}^n)$ for a subsequence, but this is <u>not</u> enough to get

$$
* \qquad \int_\Omega Df\big(\varepsilon(v'_k)\big) : \varepsilon(u(t))\, dx \underset{k\to\infty}{\longrightarrow} \int_\Omega Df\big(\varepsilon(v)\big) : \varepsilon(u(t))\, dx.
$$

$*$ is correct if $\varepsilon(v) \in L^{\frac{p}{p-1}(q-1)}(\Omega;\mathbb{S}^n)$ and $\varepsilon(v'_k) \to \varepsilon(v)$ strongly in this space. So if we replace in the beginning $\overset{\circ}{W}{}^1_q \cap$ Kern (div) by $\overset{\circ}{W}{}^1_{\frac{p}{p-1}(q-1)} \cap$ Kern (div) and choose $\{v_k\}$ as a dense subset, then we have shown:

THEOREM 4.1. *Let f satisfy (4.3) together with (4.4). Then there exists a function $u : [0,T] \to \overset{\circ}{W}{}^1_p \cap$ Kern (div), $u(0) = u_0$, such that $u \in L^p(0,T;W^1_p)$, $\frac{d}{dt} u \in L^2(0,T;L^2)$ and $0 \le \int_\Omega \frac{d}{dt} u(t) \cdot (v - u(t))\, dx + \int_\Omega Df\big(\varepsilon(v)\big) : \big(\varepsilon(v) - \varepsilon(u(t))\big)\, dx$ for a.a. t and all $v \in \overset{\circ}{W}{}^1_{\frac{p}{p-1}(q-1)} \cap$ Kern (div).*

REMARK 4.1. *If one can prove higher integrability of $\varepsilon(u(t))$, then the argument from Section 3 turns this evolution variational inequality into the evolution equation. We therefore have produced some kind of "weak" solution.*

Finally we discuss how to remove (4.4) in the anisotropic case (4.3).

If we consider the example

$$f(\varepsilon) = \left(1 + |\varepsilon_1|^2\right)^{p/2} + \left(1 + |\varepsilon_2|^2\right)^{q/2}$$

for some decomposition $\varepsilon = (\varepsilon_1 \varepsilon_2)$, then it is easy to see that for each $\delta > 0$ there is a constant $c\,(\delta)$ such that

$$\left|Df(\varepsilon)\right| \leq \delta f(\varepsilon) + c\,(\delta) \quad \forall\, \varepsilon \in \mathbb{S}^n \tag{4.6}$$

holds. Thus we may replace the inequality stated before (4.4) by (recall $\varepsilon(u_0) \in L^\infty$)

$$\left| \int_\Omega Df\big(\varepsilon(u_m)\big) : \varepsilon(u_0)\, dx \right| \leq \delta \int_\Omega f\big(\varepsilon(u_m)\big)\, dx + c\,(\delta),$$

and for δ small enough we arrive at (4.5). We therefore can replace (4.4) in Theorem 4.1 by the condition (4.6) with the same result. Note that under the assumption (4.4) we clearly have (4.6), thus (4.6) is less restrictive than (4.4).

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