On the periodic
Wigner-Poisson-Fokker-Planck system

Anton Arnold, Jose A. Carrillo and Elidon Dhamo
On the periodic Wigner-Poisson-Fokker-Planck system

Anton Arnold
Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken
Germany
E-Mail: arnold@num.uni-sb.de

Jose A. Carrillo
Departamento de Matemática Aplicada
Universidad de Granada
E-18071 Granada
Spain
E-Mail: carrillo@goliat.ugr.es

Elidon Dhamo
Saarland University
Department of Mathematics
Postfach 15 11 50
D–66041 Saarbrücken
Germany
E-Mail: dhamo@num.uni-sb.de

submitted: February 4, 2002

Preprint No. 50
Saarbrücken 2002
On the periodic Wigner-Poisson-Fokker-Planck system

February 4, 2002

AMS 2000 Subject Classification: 35Q40, 35K55, 35S10, 81S30
Key words: Wigner equation, Fokker-Planck equation, diffusive quantum systems, semigroup theory

Last update February 4, 2002

Abstract

This paper is concerned with the existence and uniqueness analysis of global classical solutions of a diffusive quantum evolution equation with non-linear coupling to the Poisson equation. The main technical difficulty in the existence proof is to show that the quantum Fokker-Planck term is a semigroup-generator in a weighted $L^2$-space. The potential term is then a Lipschitz-perturbation of it.

1 Introduction

The object of this paper is the analysis of the coupled Wigner-Poisson-Fokker-Planck (WPFP) system in one dimension with periodic boundary conditions. We focus on the existence and uniqueness of global-in-time solutions to this system.

Wigner functions provide a kinetic description of quantum mechanics (cf. [19]) and have recently become a valuable modeling and simulation tool in fields like semiconductor device modeling (cf. [12] and references therein), quantum Brownian motion, and quantum optics ([6, 8]). The real-valued Wigner function $w(x,v,t)$ is a probabilistic quasi-distribution function in the position-velocity $(x,v)$ phase space for the considered quantum system at time $t$.

Its temporal evolution is governed by the Wigner-Fokker-Planck (WFP) equation:

$$ w_t + vw_x + \Theta[V]w = \beta(vw)_v + \sigma w_{vv} + 2\gamma w_{xx} + \alpha w_{xx}, \quad t > 0, \quad (1.1) $$

on the phase space slab $x \in (0,2\pi)$, $v \in \mathbb{R}$ with periodic boundary conditions in $x$:

$$ w(0,v,t) = w(2\pi,v,t), $$

and the initial condition

$$ w(x,v,t = 0) = w^f(x,v). $$

With a vanishing right hand side Equation (1.1) would be the (diffusion-free) Wigner equation. It describes the reversible evolution of a quantum system under the action
of a (possibly time-dependent) electrostatic potential $V = V(x, t)$. Its effect enters in the equation via the pseudo-differential operator $\Theta[V]$:

\[
(\Theta[V]w)(x, v, t) = i[V(x + \frac{1}{2i}\nabla_v, t) - V(x - \frac{1}{2i}\nabla_v, t)]w(x, v, t)
\]

\[
= \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \delta V(x, \eta, t)\mathcal{F}_v w(x, \eta, t)e^{i\eta v} \, d\eta
\]

\[
= \frac{i}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \delta V(x, \eta, t)w(x, v', t)e^{i(v-v')\eta} \, dv' \, d\eta,
\]

where $\delta V(x, \eta, t) = V(x + \frac{\eta}{2}, t) - V(x - \frac{\eta}{2}, t)$ and $\mathcal{F}_v w$ denotes the Fourier transform of $w$ with respect to $v$:

\[
\mathcal{F}_v w(x, \eta, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} w(x, v', t)e^{-i\eta v'} \, dv'.
\]

For simplicity of the notation we have here set the Planck constant, particle mass and charge equal to unity.

The right hand side of (1.1) is a Fokker-Planck type model for the non-reversible interaction of this quantum system with an environment, e.g. the interaction of electrons with a phonon bath (cf. [9, 18]). In (1.1) $\beta \geq 0$ is the friction parameter and the parameters $\alpha, \gamma \geq 0, \sigma > 0$ constitute the phase-space diffusion matrix of the system. In the kinetic Fokker-Planck equation of classical mechanics (cf. [17, 7]) one would have $\alpha = \gamma = 0$. For the WFP equation (1.1) we have to assume

\[
\begin{pmatrix}
\alpha & \gamma + \frac{i}{4}\beta \\
\gamma - \frac{i}{4}\beta & \sigma
\end{pmatrix} \geq 0,
\]

which guarantees that the system is quantum mechanically correct. More precisely, it guarantees that the corresponding von Neumann equation is in Lindblad form and that the density matrix of the quantum system stays a positive operator under temporal evolution (see [4] for details).

In the sequel we shall hence assume

\[
\alpha \sigma \geq \gamma^2 + \frac{\beta^2}{16}, \quad (1.3)
\]

However, the subsequent mathematical analysis will even hold for

\[
\alpha \sigma \geq \gamma^2.
\]

The WFP equation (1.1) is self-consistently coupled with the Poisson equation for the (real-valued) potential $V[w](x, t)$:

\[
V_{xx} = n[w] - D, \quad x \in (x, 2\pi), \ t > 0,
\]

\[
V(0, t) = V(2\pi, t),
\]

with the particle density

\[
n[w](x, t) = \int_{\mathbb{R}} w(x, v, t) \, dv. \quad (1.4)
\]

$D = D(x)$ denotes the density of some fixed charges ("doping profile" in the context of semiconductor modeling), which is assumed to be given.

Mathematical properties of the Wigner-Poisson equation and dissipative Wigner systems have been intensively studied in the last decade (see [12, 3] and references therein). The (friction-free) WFPFP equation in 3 dimensions was first analyzed
in [4], where unique local-in-time solutions were constructed. The main analytical challenge of Wigner-Poisson systems lies in controlling the particle density \(1.4\) in appropriate \(L^p\) spaces. Usually this is achieved by either reformulating the Wigner equation as a Schrödinger system or a von Neumann equation ([12, 3]) or by exploiting the dissipative structure of the system ([4]). The 1-dimensional Wigner-Poisson equation, however, allows for a ‘direct’ analysis (cf. [5], §5). Hence our interest in this analytical framework.

2 Existence and Uniqueness of Global-in-Time Solution

In this section we shall establish existence and uniqueness of global mild and classical solutions to the WPFP system (1.1)-(1.4). This system will be considered as an evolution problem in the weighted (real-valued) \(L^2\)-space

\[ X = L^2((0,2\pi) \times \mathbb{R}; (1 + v^2)dxdv), \]

endowed with the scalar product

\[ < u, w >_X = \int_0^{2\pi} \int_{\mathbb{R}} uw(1 + v^2)dvdx. \]

This choice of the space \(X\) allows to define the particle density \(n[w]\) of a Wigner function \(w \in X\): a simple estimate (using Cauchy-Schwartz) yields

\[ \|n[w]\|_{L^2(0,2\pi)} \leq C\|w\|_X. \tag{2.1} \]

Here and in the sequel \(C\) denotes generic, but not necessarily equal constants.

We shall use semigroup techniques to prove existence and uniqueness of a solution to the semi-linear WPFP system (1.1)-(1.4). To this end the quadratically nonlinear potential term \(\Theta[V]w\) will be considered as a bounded perturbation in the kinetic Fokker-Planck equation \(w_t + vw_x = \beta(vw)_x + \sigma w_{xx} + 2\gamma w_{xy} + \alpha w_{xx}\).

We first consider the unbounded linear operator \(A : D(A) \rightarrow X\),

\[ Au = -v\partial_v u + \beta \partial_v(vu) + \sigma \partial_x^2 u + 2\gamma \partial_x \partial_x^2 u + \alpha \partial_x^2 u, \tag{2.2} \]

developed on

\[ D(A) = \{ u \in X | vu_x, vu_v, \partial_v u, \partial_x u, \partial_x^2 u, \partial_x \partial_x^2 u \in X; u(0,v) = u(2\pi,v), \]

\[ u_x(0,v) = u_x(2\pi,v) \forall v \in \mathbb{R} \}. \]

Clearly, the restriction (to \((0,2\pi) \times \mathbb{R}\)) of \(C^\infty(\mathbb{R}^2)\)-functions that are 2\(\pi\)-periodic in \(x\) and have a compact support in \(v\) are included in \(D(A)\). Hence, \(D(A)\) is dense in \(X\). A simple calculation shows that for \(u \in D(A), u_x\) is also in \(X\).

A straightforward calculation using the periodicity in \(x\) and integrations by part yields

\[ < A_u w >_X = < u, A_1^* w >_X + < u, A_2^* w >_X, \forall u, w \in D(A), \]

with

\[ A_1^* w = v \partial_v w - \beta v \partial_v w + \sigma \partial_x^2 w + 2\gamma \partial_x \partial_x^2 w + \alpha \partial_x^2 w, \]

\[ A_2^* w = \frac{1}{1 + v^2} [2\sigma (v^2 + \partial_v w) - 2\beta v^2 w + 4\gamma vw_x]. \]

Hence, \(A^* |_{D(A)}\) - the restriction of the adjoint of the operator \(A\) to \(D(A)\) - is given by \(A^* w = A_1^* w + A_2^* w, w \in D(A)\). \(A^*\) is densely defined on \(D(A^*) \supset D(A)\), and
hence $A$ is a closable operator (cf. Theorem VIII.1.b of [15]). Its closure $\overline{A}$ satisfies $(\overline{A})^* = A^*$ (cf. [15], Theorem VIII.1.c).

Next we study the dissipation property of the operator $A$, which is defined on the Hilbert space $X$ (over $\mathbb{R}$) by:

$$< Au, u > _X \leq 0, \ \forall u \in D(A).$$

Lemma 2.1 Let the coefficients of the operator $A$ satisfy $\alpha \sigma \geq \gamma^2$. Then $A - (\sigma + \frac{\beta}{2})I$ and its closure is dissipative.

Proof.- Using integrations by part we have for $u \in D(A)$:

$$< Au, u > _X = - \int \int \int (vu)_x u + \alpha \int \int \int (vu)_x u + 2\gamma \int \int \int u u_x u$$

$$+ \beta \int \int \int (vu)_x u + \alpha \int \int \int (vu)_x u + \beta \int \int \int (vu)_x u + \sigma \int \int \int (vu)_x u$$

$$+ 2\gamma \int \int \int (vu)_x u + \alpha \int \int \int (vu)_x u$$

$$= - \beta \int \int \int (wu)_v u - \alpha \int \int \int (wu)_v u - 2\gamma \int \int \int (wu)_v u - \alpha \int \int \int (wu)_v u - \beta \int \int \int (wu)_v u$$

where $\int \int \int$ denotes the integral $\int_0^2 \int_0^2 f(x, v) dv dx$.

For the two integrals of the right side that involve mixed $x$-$v$-derivatives we shall now use the interpolation inequality

$$\int \int \int u_x u \leq \epsilon \frac{\epsilon}{2} \| u_x \|^2 + \epsilon \frac{\epsilon}{2} \| u_v \|^2, \quad \epsilon > 0,$$

which is immediately obtained by an integration by parts (in $v$) and Young's inequality. With $\epsilon = \frac{\sigma}{2}$ we then obtain

$$< Au, u >_X \leq \epsilon \frac{\epsilon}{2} \| u \|^2 - \sigma \| u_v \|^2 + \epsilon \gamma \| u_x \|^2 + \frac{1}{\epsilon} \gamma \| u_x \|^2$$

$$- \alpha \| u_x \|^2 - 2\beta \| u_v \|^2 - \beta \int \int \int (wu)_v u - 2\gamma \int \int \int (wu)_v u - \alpha \int \int \int (wu)_v u$$

$$- \sigma \| u_v \|^2 + \epsilon \gamma \| u_x \|^2 + \frac{1}{\epsilon} \gamma \| (vu)_v \|^2 - \alpha \| u_x \|^2$$

$$= \frac{\beta}{2} \| u \|^2 + \frac{\gamma^2}{\sigma} \| u_x \|^2 - \alpha \| u_x \|^2 - 2\beta \| u_v \|^2 + \frac{3}{2} \beta \| u_v \|^2$$

$$+ \sigma \| u \|^2 + \frac{\gamma^2}{\sigma} \| u_x \|^2 - \alpha \| u_x \|^2$$

$$\leq (\sigma + \frac{\beta}{2}) \| u \|^2,$$

Thus

$$< [A - (\sigma + \frac{\beta}{2})I] u, u > _X \leq - \sigma \| (wu)_v \|^2 - \frac{\beta}{2} \| (wu)_v \|^2 \leq 0 \quad (2.4)$$

and the operator $A - (\sigma + \frac{\beta}{2})I$ is dissipative. By Theorem 1.4.5b of [14] its closure, $A - (\sigma + \frac{\beta}{2})I = A - (\sigma + \frac{\beta}{2})I$ is also dissipative. \[\Box\]

It is easy to see that the operator $A - \frac{\beta}{2}I$ defined on $\overline{D(A)} = \{ u \in L^2((0, 2\pi) \times \mathbb{R}) | vu_x, vu_v, vu_v, vu_v, vx_{x} \in L^2((0, 2\pi) \times \mathbb{R}); u(0, v) = u(2\pi, v), v(0, v) = u(x, 2\pi, v), \forall v \in \mathbb{R} \}$ is dissipative in $L^2((0, 2\pi) \times \mathbb{R})$ and the $L^2$-adjoint of $A$ is $A^* = A_{\ast}$ on $D(A)$. 

4
Let us now study the dissipativity of the operator \( A^* \) restricted to \( D(A) \). Analogously to Lemma 2.1 we have:

\[
< A^* u, u >_X \leq (\sigma + \frac{\beta}{2}) \| u \|_2^2, \quad \forall u \in D(A).
\]

Hence the restriction of the operator \( A^* - (\sigma + \frac{\beta}{2})I = [A - (\sigma + \frac{\beta}{2})I]^* \) to \( D(A) \) is dissipative.

Next we consider the dissipativity of this operator on its proper domain \( D(A^*) \), which, however, is not known explicitly. To this end we shall use the following technical lemma whose proof is deferred to the appendix. Here we shall denote by \( \tilde{u} \) the (in \( x \)) 2\( \pi \)-periodic extension of a function \( u \in X \) to \( \mathbb{R}^2 \).

**Lemma 2.2** Let \( P := p(\nu, \partial_x, \partial_v) \) be a linear operator in \( X \), where \( p \) is a quadratic polynomial and

\[
D(P) := \{ u \in X / \tilde{u} \in C^\infty(\mathbb{R}^2) \text{ with compact support in } v \} \subset X.
\]

Then \( \overline{P} \) is the maximum extension of \( P \) in the sense that

\[
D(\overline{P}) := \{ u \in X / \text{ the distribution } P u \in X \}.
\]

We now apply Lemma 2.2 to \( P = A^* - (\sigma + \frac{\beta}{2})I \), which is dissipative on \( D(P) \subset D(A) \). Since \( A^* \) is closed, we have \( D(A^*) = D(\overline{P}) = \{ u \in X / A^* u \in X \} \) and \( A^* - (\sigma + \frac{\beta}{2})I \) is dissipative on all of \( D(A^*) \).

Applying Corollary 1.4.4 of [14] to \( \overline{A} - (\sigma + \frac{\beta}{2})I \) (with \( \overline{A}^* = A^* \)), then implies that \( \overline{A} - (\sigma + \frac{\beta}{2})I \) generates a \( C_0 \) semigroup of contractions on \( X \), and the \( C_0 \) semigroup generated by \( \overline{A} \) satisfies

\[
\| e^{\overline{A} t} u \|_X \leq e^{(\sigma + \frac{\beta}{2}) t} \| u \|_X, \ u \in X, t \geq 0.
\]

By the same arguments \( \overline{A} - \frac{\beta}{2}I \) generates a \( C_0 \) semigroup of contractions on the space \( L^2((0, 2\pi) \times \mathbb{R}) \).

Next we shall analyze the properties of the quadratically nonlinear term \( \Theta[V]w \), which will later be considered as a perturbation of the generator \( \overline{A} \).

For \( V \in L^\infty(\mathbb{R}) \) the pseudo-differential operator \( \Theta[V] \) from (1.2) is defined by

\[
(\mathcal{F}_\nu \Theta[V] u)(x, \eta) = i \delta V(x, \eta) \mathcal{F}_\nu u(x, \eta), \ u \in L^2((0, 2\pi) \times \mathbb{R}_v).
\]

Since \( \delta V(x, \eta) \in \mathbb{R} \), the operator \( \Theta[V] \) is skew-symmetric on \( L^2((0, 2\pi) \times \mathbb{R}_v) \) and it satisfies (cf. [11], [5]):

\[
\| \Theta[V] \|_{B(L^2((0, 2\pi) \times \mathbb{R}_v))} \leq 2 \| V \|_\infty.
\]

For \( V \in L^\infty(\mathbb{R}) \) we define the pseudo-differential operator \( \Omega[V] \) on \( L^2((0, 2\pi) \times \mathbb{R}_v) \) by

\[
(\Omega[V] u)(x, v) = \frac{1}{2} [V(x + \frac{\eta}{2}) \nabla_v + V(x - \frac{\eta}{2}) \nabla_v] u(x, v)
= \frac{1}{2 \sqrt{2\pi}} \int_{\mathbb{R}_v} [V(x + \frac{\eta}{2}) \nabla_v + V(x - \frac{\eta}{2}) \nabla_v] \mathcal{F}_\nu u(x, \eta) e^{i\eta \eta} \, d\eta. \tag{2.5}
\]

As for the operator \( \Theta[V] \) we obtain

\[
\| \Omega[V] \|_{B(L^2((0, 2\pi) \times \mathbb{R}_v))} \leq \| V \|_\infty. \tag{2.6}
\]
Proposition 2.3 Let \( V \in W^{1,\infty}(\mathbb{R}) \). Then,
\[
\Theta[V](vw) = v\Theta[V]w + \Omega[V_x]w
\]
(2.7)
holds for \( w \in X \).

Proof. By partial integration we obtain
\[
\Theta[V](vw) = \frac{i}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} (V(x + \frac{\eta}{2}) - V(x - \frac{\eta}{2})) w(x, v) e^{i(v-v')\eta} dw d\eta \\
= \frac{i}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ (V(x + \frac{\eta}{2}) - V(x - \frac{\eta}{2})) w(x, v) e^{i\eta} \right] [v' e^{-i\eta}] d\eta dv \\
= \frac{1}{4\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \left[ V_x(x + \frac{\eta}{2}) + V_x(x - \frac{\eta}{2}) \right] w(x, v) e^{i(v-v')\eta} d\eta dv \\
+ \frac{i}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} v(V(x + \frac{\eta}{2}) - V(x - \frac{\eta}{2})) w(x, v) e^{i(v-v')\eta} d\eta dv \\
= \Omega[V_x]w + v\Theta[V]w.
\]

Now, let us consider the nonlinear operator \( B \) defined on \( X \) by
\[
u \rightarrow Bu := -\Theta[V]\|u\|u,
\]
where \( V[u] \) is the \( 2\pi \)-periodically extended solution of the Poisson equation
\[
\begin{align*}
V_{xx} &= n[u] - D, \quad x \in (0, 2\pi), \\
V(0) &= V(2\pi),
\end{align*}
\]
(2.8)
with \( n[u](x) = \int_{\mathbb{R}} u(x, v) dv \).

Lemma 2.4 Let \( D \in L^1(0, 2\pi) \). Then
(a) \( B \) maps \( X \) into itself.
(b) Moreover, the operator \( B \) is of class \( C^\infty \) in \( X \), and satisfies
\[
\|Bu_1 - Bu_2\|_X \leq C(\|u_1\|_X + \|u_2\|_X + \|D\|_{L^1(0,2\pi)})\|u_1 - u_2\|_X,
\]
for \( u_1, u_2 \in X \).

For the simple proof we refer the reader to [5].

Remark 2.5 In the proof of Lemma 2.4 it is essential that \( \|u\|_X \) controls \( n[u] \) in \( L^1(0, 2\pi) \) (see (2.1)). Hence the solution of the Poisson equation (2.8) satisfies \( V[u] \in W^{1,\infty}(0, 2\pi) \), and \( \|\Theta[V[u]\|_{\mathcal{B}(X)} \leq C\|V[u]\|_{W^{1,\infty}(\mathbb{R})} \).

We rewrite the WFPF system as
\[
\begin{align*}
w_t &= \overline{A}w + Bw, \quad t > 0, \\
w(t = 0) &= w'_t \in X.
\end{align*}
\]
(2.9)

The main result of this paper is:

Theorem 2.6 Let \( D \in L^1(0, 2\pi) \).
(a) For every \( w \in X \), the WFPF problem (2.9) has a unique mild solution \( w \in C([0, \infty), X) \).
(b) If \( w \in D(\overline{A}) \), \( w \) is a classical solution, i.e. \( w \in C^1([0, \infty), X) \), and \( w(t) \in D(\overline{A}) \) for \( t \geq 0 \).
**Proof.** - We consider $B$ as a bounded perturbation of the generator $\overline{A}$. Since $B$ is locally Lipschitz continuous, Theorem 6.1.4 of [14] shows that (2.9) has a unique mild solution for every $w' \in X$ on some time interval $[0, t_{max})$. Moreover, if $t_{max} = t_{max}(w') < \infty$ then $\lim_{t \to t_{max}} \|w\|_X = \infty$. Since $B$ is of class $C^\infty$ in $X$, Theorem 6.1.5 in [14] proves that $w$ is a classical solution on $[0, t_{max})$ for $w' \in D(\overline{A})$.

To prove $t_{max} = \infty$ we shall now derive an a-priori estimate for $\|w(t)\|_X$. By computing its time derivative and taking into account (2.9), we deduce

$$\frac{1}{2} \frac{d}{dt} \|w\|^2_X = < \overline{A}w, w >_X + < Bw, w >_X.$$ 

Using the dissipativity of $\overline{A} - (\sigma + \frac{\beta}{2})I$ (cf. (2.4)) we conclude that

$$\frac{1}{2} \frac{d}{dt} \|w\|^2_X \leq \left( \sigma + \frac{\beta}{2} \right) \|w\|^2_X + < Bw, w >_X.$$ 

The skew-symmetry of the operator $\Theta[V]$ implies finally that

$$\frac{1}{2} \frac{d}{dt} \|w\|^2_X \leq \left( \sigma + \frac{\beta}{2} \right) \|w\|^2_X + \int \|Vw\| \|V_x(t)\|w.$$ 

On the other hand, since $\overline{A} - \frac{\beta}{2}I$ is dissipative on the space $L^2((0, 2\pi) \times \mathbb{R})$, the estimates

$$\frac{d}{dt} \|w\|^2_X \leq \beta \|w\|^2_X \quad \text{and} \quad \|w\|^2_X \leq \|w'\|^2_X e^{\beta t}$$

follow. From the proof of Lemma 2.4 in [5] we have for the solution of (2.8):

$$\|V[w]\|_{W^{1, \infty}(0, 2\pi)} \leq C(\|w\|_X + \|D\|_{L^1(0, 2\pi)}).$$

Using (2.6), (2.10) and (2.11) we hence obtain

$$\frac{1}{2} \frac{d}{dt} \|w\|^2_X - \left( \sigma + \frac{\beta}{2} \right) \|w\|^2_X \leq \int \|Vw\| \|V_x(t)\|w \\
\leq \|V_x(t)\|_{\infty} \|Vw\|_2 \|w\|_2 \\
\leq C(\|w\|_X + \|D\|_1) \|Vw\|_2 \|w'\|_2 e^{\frac{\beta t}{2}} \\
\leq C\|w'\|_2 e^{\frac{\beta t}{2}} (\|w\|^2_X + \|w\|_X \|D\|_1) \\
\leq C\|w'\|_2 e^{\frac{\beta t}{2}} (\|w\|^2_X + \|D\|^2_1).$$

Thus

$$\frac{d}{dt} \|w\|^2_X \leq a(t) \|w\|^2_X + b(t),$$

where

$$a(t) = C\|w'\|_2 e^{\frac{\beta t}{2}} + \beta + 2\sigma,$$

$$b(t) = C\|w'\|_2 e^{\frac{\beta t}{2}} \|D\|^2_1.$$ 

Finally, applying Gronwall’s inequality yields:

$$\|w(t)\|^2_X \leq \|w'\|^2_X e^{\int_0^t a(s) \, ds} + \int_0^t b(s) e^{\int_s^t a(\tau) \, d\tau} \, ds, \quad t \geq 0. \quad (2.12)$$

Hence $t_{max} = \infty$ holds.
Step 2. Since (2.12) only involves \( \|w^t\|_X \) this result carries over to \( w^t \) only in \( X \) by the following density argument. 
For \( w^t \in X \) let \( (w^t_n) \) be a sequence in \( D(\bar{X}) \) such that \( w^t_n \to w^t \) in \( X \). Using (2.12) we have for every \( w^t_n \) an a-priori estimate for the corresponding classical solution:

\[
\|w^t_n(t)\|_X \leq h(t), \quad \forall t \geq 0, \ n \in \mathbb{N}
\]

with \( h \in C[0, \infty) \) independent of \( n \). 
Let \( w \in C([0, t_{\max}(w^t)), X) \) be the unique mild solution for \( w^t \), which exists according to the first part of this theorem. 
Next we assume \( t_{\max}(w^t) < \infty \). Thus \( \lim_{t \uparrow t_{\max}(w^t)} \|w(t)\|_X = \infty \). For the continuous, monotonously increasing function \( g(t) := \max \{\|w(\tau)\|_X, 0 \leq \tau \leq t \} \) we also have \( \lim_{t \uparrow t_{\max}(w^t)} g(t) = \infty \). 
Choose \( N \in \mathbb{N} \) with \( N \geq 2 \max \{h(t), \ t \in [0, t_{\max}(w^t)]\} \). Then there exists a \( t_N < t_{\max}(w^t) \) such that

\[
g(t_N) = N, \
g(t) \leq N, \quad t \leq t_N, \
g(t) \geq N, \quad t_N \leq t < t_{\max}(w^t). 
\]

We denote by \( L_N \) the Lipschitz constant of the operator \( B \) on

\[
B_N := \{u \in X, \|u\|_X < N\}.
\]

Let \( \hat{B} \) be a (globally) Lipschitz extension of \( B \) outside of \( B_N \). Thus, applying Theorem 6.1.2 in [14] on \([0, t_N]\) we obtain a Lipschitz dependence of the solutions on their initial values,

\[
\|w - w^t_n\|_{C([0, t_N], X)} \leq C(L_N) \|w^t - w^t_n\|_X.
\]

Thus, \( w_n \to w \) in \( C([0, t_N], X) \), and \( \|w(t)\|_X \leq h(t) \leq \frac{N}{2} \), for \( 0 \leq t \leq t_N \) follows. 
This contradicts the assumption (2.13). \( \Box \)

3 Appendix: Proof of Lemma 2.2

To prove the assertion we shall construct for each \( f \in D(P) \subset L^2((0, 2\pi) \times \mathbb{R}) \) a sequence \( \{f_n\} \subset D(P) \) such that \( f_n \to f \) in the graph norm \( \|f\|_P = \|f\|_{L^2} + \|vf\|_{L^2} + \|Pf\|_{L^2} + \|v Pf\|_{L^2} \).

To shorten the proof we shall consider here only the case

\[
P = \mu + vv \partial_x + \beta v \partial_v + \sigma \partial_v^2 + 2\gamma \partial_v \partial_x + \alpha \partial_x^2
\]

(cf. the definition of the operator \( A \) in (2.2)), but exactly the same strategy extends to the general case.

First we define the mollifying delta sequence

\[
\varphi_n(x, v) := n^2 \varphi(nx, nv); \quad n \in \mathbb{N}; \ x, v \in \mathbb{R},
\]

with the properties:

\[
\varphi \in C_0^{\infty}(\mathbb{R}^2), \\
\varphi(x, v) \geq 0, \\
\int \int \varphi(x, v) dx dv = 1, \\
\text{supp} \varphi \subset \{\|x\|^2 + |v|^2 \leq 1\}.
\]
The velocity-cutoff function

\[ \psi_n(v) := \psi\left(\frac{v}{n}\right); \quad n \in \mathbb{N}; \quad v \in \mathbb{R} \]

is assumed to have the properties

\[ \psi \in C_0^\infty(\mathbb{R}), \]
\[ 0 \leq \psi(v) \leq 1, \]
\[ |\psi^{(j)}(v)| \leq C_j \quad \forall v \in \mathbb{R}; \quad j = 1, 2, \]
\[ \text{supp} \psi \subseteq [-1, 1], \]
\[ \psi\big|_{[-\frac{1}{2}, \frac{1}{2}]} = 1. \]

We now define the approximating sequence

\[ \tilde{f}_n(x, v) := (\tilde{f} \ast \varphi_n)(x, v) \cdot \psi_n(v), \quad n \in \mathbb{N}, \]

where ‘\( \ast \)' denotes the convolution in \( x \) and \( v \). Remember that \( \tilde{f} \) denotes the (in \( x \)) \( 2\pi \)-periodic extension of the function \( f \in X \) to \( \mathbb{R}^2 \). By construction we have \( \tilde{f}_n \in C^\infty(\mathbb{R}^2) \) and \( f_n \) is \( 2\pi \)-periodic in \( x \) with compact support in \( v \). Now, let \( R \) denote the restriction operator of (in \( x \)) \( 2\pi \)-periodic functions to \( (0, 2\pi) \times \mathbb{R} \). Then, \( f_n := R \tilde{f}_n \in D(P) \). According to the 4 terms of the graph norm we split the proof into 4 steps:

**Step 1:** Since \( \varphi_n \to \delta \) in \( D'(\mathbb{R}^2) \) and \( \psi_n(v) \to 1 \) pointwise, we have \( \tilde{f}_n \to \tilde{f} \) in \( L^2_{loc}(\mathbb{R}^2) \times L^2(\mathbb{R}) \) and

\[ f_n \to f \quad \text{in} \quad L^2((0, 2\pi) \times \mathbb{R}). \]

**Step 2:** For the second term of the graph norm we write

\[ v \tilde{f}_n = (v \tilde{f} \ast \varphi_n) \psi_n + (\tilde{f} \ast v \varphi_n) \psi_n. \]

The restriction of the first summand converges to \( vf \) in \( L^2((0, 2\pi) \times \mathbb{R}) \) and the second term converges to 0 since \( v \varphi_n \to 0 \) in \( D'(\mathbb{R}^2) \). Hence we have

\[ f_n \to f \quad \text{in} \quad X. \]

**Step 3:** To prove that \( Pf_n \to Pf \) in \( L^2((0, 2\pi) \times \mathbb{R}) \) we write:

\[ Pf_n = \mu(\tilde{f} \ast \varphi_n) \psi_n + \nu(\tilde{f}_x \ast \varphi_n) \psi_n + \beta(\tilde{f}_0 \ast \varphi_n) \psi_n \]
\[ + \sigma(\tilde{f}_v \ast \varphi_n) \psi_n + 2\gamma(\tilde{f}_x \ast \varphi_n) \psi_n + \alpha(\tilde{f}_x \ast \varphi_n) \psi_n \]
\[ + r^1_n(x, v) = (Pf \ast \varphi_n) \psi_n + r^1_n(x, v). \]

As we shall show, the restriction of all six terms of the remainder

\[ r^1_n = \nu(\tilde{f} \ast v \partial_x \varphi_n) \psi_n + \beta(\tilde{f} \ast \varphi_n) (v \partial_v \psi_n) \]
\[ + \beta(\tilde{f} \ast \partial_v (v \varphi_n)) \psi_n + 2\beta(\tilde{f} \ast (\frac{1}{n} \partial_v \varphi_n))(\partial_v \psi_n) \]
\[ + \sigma(\tilde{f} \ast \varphi_n) \partial_v^2 \psi_n + 2\gamma(\tilde{f} \ast (\frac{1}{n} \partial_x \varphi_n))(\partial_v \psi_n) \]
converge to 0 in $L^2((0,2\pi) \times \mathbb{R})$:

In the first term $v \partial_x \varphi_n \to 0$ in $\mathcal{D}'(\mathbb{R}^2)$. Hence we have

$$R \left( \tilde{f} * v \partial_x \varphi_n \right) \to 0 \text{ in } L^2((0,2\pi) \times \mathbb{R}),$$

and the same argument holds for the third term.

For the second term we have

$$v \partial_v \psi_n = \frac{v}{n} \psi'\left( \frac{n}{v} \right),$$

which is in $L^\infty(\mathbb{R})$, uniformly for $n \in \mathbb{N}$ and with support in $[-n,-\frac{n}{2}] \cup [\frac{n}{2},n]$. Hence, the second term converges to 0 in $L^2((0,2\pi) \times \mathbb{R})$.

In the fourth term $\frac{1}{n} \partial_v \varphi_n \to 0$ in $\mathcal{D}'(\mathbb{R}^2)$, and hence

$$R \left( \tilde{f} * \left( \frac{1}{n} \partial_v \varphi_n \right) \right) \to 0 \text{ in } L^2((0,2\pi) \times \mathbb{R}).$$

Furthermore, $n \partial_v \psi_n = \psi'\left( \frac{n}{v} \right)$ with $|\psi'| \leq C_1$. By the same argument also the sixth term converges to 0 in $L^2((0,2\pi) \times \mathbb{R})$.

Finally, the fifth term converges to 0 since $\partial_v^2 \psi_n = \frac{1}{n} \psi''\left( \frac{n}{v} \right)$ with $|\psi''| \leq C_2$.

**Step 4:** To prove that $v P f_n \to v Pf$ in $L^2((0,2\pi) \times \mathbb{R})$ we write:

$$v P f_n = \mu(v \tilde{f} \ast \varphi_n) \psi_n + \nu(v^2 \tilde{f} \ast \varphi_n) \psi_n + \beta(v^2 \tilde{f} \ast \varphi_n) \psi_n + \sigma(v \tilde{f} \ast \varphi_n) \psi_n + 2\gamma(v \tilde{f} \ast \varphi_n) \psi_n + \alpha(v \tilde{f} \ast \varphi_n) \psi_n + r_n^2(x,v),$$

with the remainder

$$r_n^2 = \mu(\tilde{f} * v \varphi_n) \psi_n + \nu(\tilde{f} * v \partial_x \varphi_n) \psi_n + \frac{v}{n} \psi'\left( \frac{n}{v} \right) + \beta(\tilde{f} * v^2 \partial_x \varphi_n) \psi_n + 2\gamma(\tilde{f} * v \partial_x (v \varphi_n)) \psi_n + 2\gamma(\tilde{f} * v \partial_x \varphi_n) \psi_n + \alpha(\tilde{f} * v \partial_x (v \varphi_n)) \psi_n.$$

For proving that the restriction of all terms of $r_n^2$ converge to 0 in $L^2((0,2\pi) \times \mathbb{R})$ we recall that both $f, v \in L^2((0,2\pi) \times \mathbb{R})$. Since the strategy of the proof is the same as in Step 3 we shall only give the key points:

The distributions $v \varphi_n, v \partial_x \varphi_n, v^2 \partial_x \varphi_n, \partial_t (v \varphi_n), v^2 \partial_t \varphi_n, \partial_t (v \varphi_n), v \partial_x (v \varphi_n), v \partial_x^2 \varphi_n, v \partial_{xx} (v \varphi_n)$, and $v \partial_{xx} \varphi_n$ all converge to 0 in $\mathcal{D}'(\mathbb{R}^2)$. Further, $v \partial_v^2 \psi_n \to 0$ in $L^\infty(\mathbb{R})$ and the term $v \partial_v \psi_n$ was already discussed in Step 3.

**Acknowledgements.-** Partially supported by the grants ERBFMRXCT970157 (TMR-Network) from the EU, the bilateral DAAD-Acciones Integradas Program, and DGES (Spain) PB98-1281. The third author acknowledges support from the DFG under Grant-No. AR 277/3-1.
References