Apriori Gradient Estimates for Bounded Generalized Solutions of a Class of Variational Problems with Linear Growth

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Abstract

Given an integrand \( f \) of linear growth and assuming an ellipticity condition of the form

\[
D^2 f(Z)(Y, Y) \geq c (1 + |Z|^2)^{-\frac{\mu}{2}} |Y|^2, \quad 1 < \mu \leq 3,
\]

we consider the variational problem \( J(w) = \int_\Omega f(\nabla w) \, dx \to \min \) among mappings \( w: \mathbb{R}^n \supset \Omega \to \mathbb{R}^N \) with prescribed Dirichlet boundary data. If we impose some boundedness condition, then the existence of a generalized minimizer \( u^* \) is proved such that \( \int_{\Omega'} |\nabla u^*|^2 \log(1 + |\nabla u^*|^2) \, dx \leq c(\Omega') \) for any \( \Omega' \subset \Omega \). Here the limit case \( \mu = 3 \) is included. Moreover, if \( \mu < 3 \) and if \( f(Z) = g(|Z|^2) \) is assumed in the vector-valued case, then we show local \( C^{1,\alpha} \)-regularity and uniqueness up to a constant of generalized minimizers. These results substantially improve earlier contributions of [BF3] where only the case of exponents \( 1 < \mu < 1 + 2/n \) could be considered.

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Contents

1 Introduction \hfill 2
2 Assumptions and main results \hfill 6
3 Regularization \hfill 10
4 Higher integrability in the limit case \( \mu = 3 \) \hfill 14
5 \( L^p \)-estimates in the case \( \mu < 3 \) \hfill 16
6 Apriori gradient bounds \hfill 19
7 Proof of the main theorem \hfill 23
1 Introduction

Suppose we are given a smooth, strictly convex (in the sense of definitions) integrand \( f : \mathbb{R}^n \to \mathbb{R} \) of linear growth (compare Assumption 2.1 for details). Then we consider the variational problem

\[
(P) \quad J[w] = \int_{\Omega} f(\nabla w) \, dx \quad \rightarrow \quad \min
\]

among mappings \( w \in u_0 + \overset{\circ}{W}_1^1(\Omega; \mathbb{R}^N) \) where \( \Omega \subset \mathbb{R}^n \) is a bounded Lipschitz domain and \( u_0 \) is of class \( W_1^1(\Omega; \mathbb{R}^N) \).

One prominent (scalar) example is the minimal surface case \( f(Z) = \sqrt{1 + |Z|^2} \).

A variety of references is available for the study of this variational integrand. With regard to the following discussion we just want to mention the monographs [Gl], [GMS2] and the apriori estimates given in [LU2] and [GMS1]. The theory of perfect plasticity provides another famous variational integrand of linear growth (the assumptions of smoothness and strict convexity however are not satisfied in this case). Here we like to refer to the studies of Seregin (see, for instance, [SE1]–[SE4]) and to the recent monograph [FS2].

In any case, on account of the lack of compactness in the non-reflexive Sobolev space \( W_1^1(\Omega; \mathbb{R}^N) \), problem (P) in general fails to have solutions. Thus one either has to study suitable relaxations (possibility \( i \)) or we may pass to the dual variational problem (possibility \( ii \)).

ad \( i \): since the integrand \( f \) under consideration is of linear growth, any \( J \)-minimizing sequence \( \{u_m\} \), \( u_m \in u_0 + \overset{\circ}{W}_1^1(\Omega; \mathbb{R}^N) \), is uniformly bounded in the space \( BV(\Omega; \mathbb{R}^N) \). This ensures the existence of a subsequence (not relabelled) and a function \( u \) in \( BV(\Omega; \mathbb{R}^N) \) such that \( u_m \to u \) in \( L^1(\Omega; \mathbb{R}^N) \).

So we define the set \( \mathcal{M} \) of all generalized minimizers of problem (P) via

\[
\mathcal{M} = \{ u \in BV(\Omega; \mathbb{R}^N) : u \text{ is the } L^1-\text{limit of a } J-\text{minimizing sequence from } u_0 + \overset{\circ}{W}_1^1(\Omega; \mathbb{R}^N) \}.
\]

ad \( ii \): following [ET] it is possible to write

\[
J[w] = \sup_{\tau \in L^\infty(\Omega; \mathbb{R}^n)} l(w, \tau), \quad w \in u_0 + \overset{\circ}{W}_1^1(\Omega; \mathbb{R}^N),
\]

where the Lagrangian \( l(w, \tau) \) for \( (w, \tau) = (u_0 + \varphi, \tau) \in (u_0 + \overset{\circ}{W}_1^1(\Omega; \mathbb{R}^N)) \times L^\infty(\Omega; \mathbb{R}^n) \) is defined through the formula

\[
l(w, \tau) := \int_{\Omega} \tau : \nabla w \, dx - \int_{\Omega} f^*(\tau) \, dx = l(u_0, \tau) + \int_{\Omega} \tau : \nabla \varphi \, dx,
\]

2
and where \( f^* \) denotes the conjugate function of \( f \). If we let

\[
R : L^\infty(\Omega; \mathbb{R}^N) \to \mathbb{R},
\]

\[
R(\tau) := \inf_{u \in u_0 + \dot{W}_1^1(\Omega; \mathbb{R}^N)} l(u, \tau) = \begin{cases} -\infty, & \text{if } \text{div } \tau \neq 0, \\
 l(u_0, \tau), & \text{if } \text{div } \tau = 0, \end{cases}
\]

then the dual problem reads as

(\( \mathcal{P}^* \)) to maximize \( R \) among all functions in \( L^\infty(\Omega; \mathbb{R}^N) \).

Although the set \( \mathcal{M} \) of generalized minimizers of problem (\( \mathcal{P} \)) may be very “large”, the solution of the dual problem is unique. This is a well known fact from duality theory (compare [ET]), a generalization (without imposing any conditions on the conjugate function) is given in [BI]. Moreover, the dual solution \( \sigma \) admits clear physical or geometrical interpretation: in the minimal surface case the dual solution corresponds to the normal of the surface, in the theory of perfect plasticity we obtain the stress tensor. Let us finally mention that (see again [ET])

\[
\inf_{u \in u_0 + \dot{W}_1^1(\Omega; \mathbb{R}^N)} J(u) = \sup_{\tau \in L^\infty(\Omega; \mathbb{R}^N)} R(\tau).
\]

Next, some known results are briefly summarized.

i) In the minimal surface case it is possible to benefit from the geometric structure of the problem (compare Remark 2.3). A class of integrands with this structure is studied, for instance, in [GMS1] following the apriori gradient bounds given in [LU2]. It turns out that generalized minimizers have (locally) Hölder continuous derivatives, they are unique up to a constant and the dual solution \( \sigma \) is of class \( C^{0,\alpha}_{loc} \) for any \( 0 < \alpha < 1 \).

ii) In the theory of perfect plasticity only partial regularity of the stress tensor is known (compare [SE3]). Even in the two-dimensional setting \( n = 2 \) we just have some additional information on the singular set (see [SE4]). As an approximation, plastic materials with logarithmic hardening are studied, i.e. the integrand under consideration is given by

\[
f(Z) := |Z| \log(1 + |Z|).
\]

This integrand is of nearly linear growth, and, as a consequence, a unique solution of Problem (\( \mathcal{P} \)) exists. The solution is known to be of class \( C^{1,\alpha}_{loc} \) implying the stress tensor to be (locally) continuous (see [FRS], [FS1], [MS] — generalizations are given in [FM], [BFM], [BF2]).
iii) Considering the general situation of strictly convex, smooth integrands with linear growth, singularities have to be expected (see [GMS1] for examples). However, partial $C^1,\alpha$-regularity of one particular generalized minimizer $u^*$ is established in [BF1]. This is done by proving $u^*$ to be a local minimizer of a relaxed problem studied in [AG]. Finally, the duality relation $\sigma = \nabla f(\nabla u^*)$ ensures partial $C^0,\alpha$-regularity of the stress tensor.

iv) In contrast to the nearly linear logarithmic hardening, the idea in [BF3] is to study a regular class of variational integrals with linear growth. Here, on one hand, existence and regularity results are comparable to the minimal surface situation. On the other hand no geometric structure conditions are imposed.

As an example one may think of

$$f(Z) = \int_0^{[Z]} \int_0^s (1 + t^2)^{-\frac{p}{2}} dt ds,$$

where $\mu > 1$ is some fixed real number. If, as a substitute for the geometric structure, ellipticity is assumed to be “good enough”, i.e. if $\mu < 1 + 2/n$ is assumed, then $C^1,\alpha$-regularity of generalized minimizers (which again are unique up to a constant) and local Hölder continuity of the stress tensor are valid.

Let us shortly discuss the limitation $\mu < 1 + 2/n$. Given a suitable regularization $u_\delta$, it is shown that

$$\omega_\delta := (1 + |\nabla u_\delta|^2)^{\frac{1-\mu}{2}}$$

is uniformly bounded in the class $W^{1,\text{loc}}_2(\Omega)$. This provides no information at all if the exponent is negative, i.e. if $\mu > 2$. An application of Sobolev’s inequality, which needs the bound $\mu < 1 + 2/n$, proves local uniform higher integrability of the gradients.

In a similar way, the DeGiorgi type reasoning of [BF3] leads to the same limitation on the ellipticity exponent $\mu$.

The purpose of our paper is to cover the whole scale of $\mu$-elliptic integrands with linear growth (as introduced in [BF3]) up to $\mu = 3$. This is the limit induced by the minimal surface example (see Remark 2.3).

As an additional assumption, the boundary values $u_0$ are supposed to be of class $L^\infty(\Omega; \mathbb{R}^N)$ (with the approximation arguments of [BF1] w.l.o.g $u_0 \in L^\infty \cap W^1_2(\Omega; \mathbb{R}^N)$). Moreover, a maximum principle is imposed:

**ASSUMPTION 1.1** Let $u_\delta$ denote the unique minimizer of

$$J_\delta[w] := \frac{\delta}{2} \int_\Omega |\nabla w|^2 dx + J[w], \quad w \in u_0 + W^1_2(\Omega; \mathbb{R}^N),$$
\( \delta \in (0,1) \). Then there is a real number \( M \), independent of \( \delta \), such that
\[
\|u_\delta\|_{L^\infty(\Omega;\mathbb{R}^N)} \leq M \|u_0\|_{L^\infty(\Omega;\mathbb{R}^N)}.
\]

**Remark 1.2** Alternatively Assumption 1.1 may be replaced by
\[
\|u_\delta\|_{L^\infty_{\text{loc}}(\Omega;\mathbb{R}^N)} \leq K
\]
for some real number \( K \) not depending on \( \delta \). In this case no restriction on
the boundary values is needed.

**Remark 1.3** Of course there are a lot of contributions on the boundedness
of solutions of variational problems. Let us mention [TA] in the scalar case,
a maximum principle for \( N > 1 \) is given in [DLM]. Let us also remark that in
the case of nonstandard growth conditions, a boundedness assumption
serves as an important tool in [CH] and [ELM].

Given some preliminary results on the regularization (see Section 3), we
exploit these hypotheses in the main sections 4–6 to obtain uniform apriori
gradient estimates for the sequence \( \{u_\delta\} \).

In contrast to [BF3] we do not differentiate the Euler equation in Sections 4
and 5 by the way avoiding Sobolev’s inequality.

As outlined in Section 4, a generalized minimizer \( u^* \in W_{1,\text{loc}}^1(\Omega; \mathbb{R}^N) \)
is found in the first step (in fact, integrability is slightly better, compare Theorem
2.5). Thus, following [BF1], \( u^* \) is a local minimizer of \( \int \nabla f(\nabla w) \, dx \) (see
Corollary 2.6).

It turns out that in the limit case \( \mu = 3 \) we have to stop at this point, i.e. full
regularity in the minimal surface case depends on the geometric structure of
the problem (again compare Remark 2.3).

However, if \( \mu < 3 \) and if some additional assumptions are imposed in the
vectorial setting, then Section 5 proves uniform local \( L_\mu \)-integrability of the
gradients for any \( 1 < p < \infty \) (see Theorem 5.1).

Once this is established, uniform local apriori gradient bounds for the sequence \( \{u_\delta\} \) are shown in Theorem 6.1. Here DeGiorgi’s technique is mod-
ified: since on one hand we benefit from Hölder’s inequality, on the other
hand we have to check carefully that the iteration works (see the definition
of \( \beta \)).

Finally, in Section 7, the proof of the main Theorem 2.7 is completed.
2 Assumptions and main results

The boundary values $u_0$ are supposed to be of class $L^\infty \cap W^{1,1}_0(\Omega; \mathbb{R}^N)$. As mentioned above, the case $u_0 \in L^\infty \cap W^{1,1}_0(\Omega; \mathbb{R}^N)$ is covered with the help of the approximation arguments given in [BF1]. The class of integrands under consideration is defined by

**ASSUMPTION 2.1** There exist positive constants $\nu_1$, $\nu_2$, $\nu_3$ and a real number $1 < \mu \leq 3$ such that for any $Z \in \mathbb{R}^N$

i) $f \in C^2(\mathbb{R}^N)$;

ii) $|\nabla f(Z)| \leq \nu_1$;

iii) for any $Y \in \mathbb{R}^N$ we have

$$
\nu_2 (1 + |Z|^2)^{-\frac{\mu}{2}} |Y|^2 \leq D^2f(Z)(Y, Y) \leq \nu_3 (1 + |Z|^2)^{-\frac{\mu}{2}} |Y|^2.
$$

**REMARK 2.2** Assumption 2.1 implies the following structure conditions.

i) There are real numbers $\nu_4 > 0$ and $\nu_5$ such that for any $Z \in \mathbb{R}^N$

$$
\nabla f(Z) : Z \geq \nu_4 (1 + |Z|^2)^{\frac{1}{2}} - \nu_5,
$$

where we use the symbol $Y : Z$ to denote the standard scalar product in $\mathbb{R}^N$.

ii) The integrand $f$ is of linear growth in the sense that for real numbers $\nu_6 > 0$, $\nu_7$, $\nu_8 > 0$, $\nu_9$ and for any $Z \in \mathbb{R}^N$

$$
\nu_6 |Z| - \nu_7 \leq f(Z) \leq \nu_8 |Z| + \nu_9.
$$

iii) The integrand satisfies a balancing condition: there is a positive number $\nu_{10}$ such that

$$
|D^2f(Z)||Z|^2 \leq \nu_{10} (1 + f(Z)) \quad \text{holds for any } Z \in \mathbb{R}^N.
$$

**Proof.** ad i): replace $f$ by $\tilde{f} : \mathbb{R}^N \to \mathbb{R}$,

$$
\tilde{f}(Z) := f(Z) - \nabla f(0) : Z \quad \text{for all } Z \in \mathbb{R}^N.
$$

A partial integration gives a real number $c$ such that we have for all $w \in u_0 + W^{1,1}_0(\Omega; \mathbb{R}^N)$

$$
\tilde{J}[w] := \int_{\Omega} f(\nabla w) \, dx - \int_{\Omega} \nabla f(0) : \nabla w \, dx = J[w] + c.
$$
Thus minimizing sequences and generalized minimizers of $J$ and $\bar{J}$ respectively coincide, and w.l.o.g. $\nabla f(0) = 0$ may be assumed. This implies by Assumption 2.1 $i\text{ii}$)

$$\nabla f(Z) : Z = \int_0^1 \frac{d}{d\theta} \nabla f(\theta Z) : Z \, d\theta$$

$$= \int_0^1 D^2 f(\theta Z)(Z,Z) \, d\theta$$

$$\geq \nu_2 \int_0^1 (1 + \theta^2 |Z|^2)^{-\frac{\nu}{2}} |Z|^2 \, d\theta$$

$$= \nu_2 |Z| \int_0^{|Z|} (1 + \rho^2)^{-\frac{\nu}{2}} \, d\rho,$$

i.e. $\nabla f(Z) : Z$ is at least of linear growth and $i$) follows.

ad $ii$): the upper bound is immediate by Assumption 2.1 $i\text{ii}$). Proving the left-hand inequality we observe that (2.1) gives $\nabla f(Z) : Z \geq 0$ for any $Z \in \mathbb{R}^{n,N}$. W.l.o.g. we additionally assume $f(0) = 0$ to write (using $i$))

$$f(Z) = \int_0^1 \frac{d}{d\theta} f(\theta Z) \, d\theta$$

$$\geq \int_{1/2}^1 \nabla f(\theta Z) : \theta Z \, d\theta$$

$$\geq \frac{1}{2} \left[ \nu_5 \left( 1 + \frac{|Z|^2}{4} \right)^{\frac{1}{2}} - \nu_5 \right],$$

hence $ii$) is clear as well.

ad $iii$): this assertion follows from $ii$) and the right-hand side of Assumption 2.1 $i\text{iii}$).

A comparison of the minimal surface integrand with the above definition provides the following

**Remark 2.3** The minimal surface example $f(Z) = \sqrt{1 + |Z|^2}$ satisfies Assumption 2.1 with the limit exponent $\mu = 3$. On the other hand, there is much better information on account of the geometric structure of this exam-
ple, in particular we have

$$\frac{c_1}{\sqrt{1 + |Z|^2}} \left[ |Y|^2 - \frac{(Y : Z)^2}{1 + |Z|^2} \right] \leq D^2 f(Z)(Y, Y)$$

$$\leq \frac{c_2}{\sqrt{1 + |Z|^2}} \left[ |Y|^2 - \frac{(Y : Z)^2}{1 + |Z|^2} \right]$$

for all $Z, Y \in \mathbb{R}^n$ with some real numbers $c_1, c_2$.

Given an integrand satisfying this condition, Ladyzhenskaya/Ural’teeva ([LU2]) and Giaquinta/Modica/Souček ([GMS1]) then use Sobolev’s inequality for functions defined on minimal hypersurfaces (compare [MI] and [BGM]) as an essential tool for proving their regularity results.

Finally, the vectorial setting $N > 1$ needs some additional

**REMARK 2.4** Suppose that $N > 1$ and that Assumptions 1.1 and 2.1 hold. This is sufficient to prove a higher integrability result for generalized minimizers — even in the limit case $\mu = 3$ (see Theorem 2.5 and Corollary 2.6).

The main theorem in the vectorial setting however is obtained for integrands $f$ with some additional “special structure” in the sense that

$$f(Z) = g(|Z|^2) \quad \text{for all } Z \in \mathbb{R}^{nN}$$

with $g: [0, \infty) \to [0, \infty)$ of class $C^2(\mathbb{R})$.

Note that (2.2) is not needed to prove a maximum principle (compare [DLM]).

As an immediate consequence, (2.2) gives

$$\frac{\partial^2 f}{\partial z^i_\alpha \partial z^j_\beta}(Z) = 4g''(|Z|^2) z^i_\alpha z^j_\beta + 2g'(|Z|^2) \delta^i_\alpha \delta^j_\beta .$$

In addition to (2.2) we impose some Hölder condition on the second derivatives: there are real numbers $\alpha \in (0, 1)$, $K > 0$ such that

$$|D^2 f(Z) - D^2 f(\bar{Z})| \leq K |Z - \bar{Z}|^\alpha .$$

Now let us give a precise formulation of the results.

**THEOREM 2.5** If $N \geq 1$ and if Assumptions 1.1 and 2.1 are supposed to be true in the limit case $\mu = 3$, then there is a generalized minimizer $u^* \in \mathcal{M}$ such that
i) $\nabla^a u^* \equiv 0$, i.e. $\nabla u^* \equiv \nabla^a u^*$, where the absolutely continuous part of $\nabla u^*$ w.r.t. the Lebesgue measure is denoted by $\nabla^a u^*$, the singular part by $\nabla^s u^*$.

ii) For any $\Omega \subset \subset \Omega$ there is a constant $c(\Omega)$ satisfying

$$
\int_{\Omega} |\nabla u^*| \log^2 \left( 1 + |\nabla u^*|^2 \right) \, dx \leq c(\Omega) < \infty.
$$

In [BF1] the behaviour of generalized solutions is studied on suitable balls, which means that no mass of $\nabla u^*$ is concentrated on the boundary of these balls. As a result the following Corollary immediately is verified.

**COROLLARY 2.6** The generalized minimizer $u^*$ given in Theorem 2.5 is a local minimizer of the functional $\int_{\Omega} f(\nabla w) \, dx$.

A slight improvement of the ellipticity condition yields:

**THEOREM 2.7** Suppose Assumptions 1.1 and 2.1 to be true with $\mu < 3$. In the case $N > 1$ we additionally impose (2.2) and (2.3).

i) Each generalized minimizer $u \in \mathcal{M}$ is in the space $C^{1,\alpha}(\Omega; \mathbb{R}^N)$ for any $0 < \alpha < 1$.

ii) The dual solution $\sigma$ is of class $C^{0,\alpha}(\Omega; \mathbb{R}^n)$ for any $0 < \alpha < 1$. Moreover, $\sigma$ has weak derivatives in the space $L^2_{\text{loc}}(\Omega; \mathbb{R}^N)$.

iii) For $u, v \in \mathcal{M}$ we have $\nabla u = \nabla v$, i.e. up to a constant uniqueness of generalized minimizers holds true.

**REMARK 2.8** Although we concentrate on generalized minimizers, the (local) continuity of the dual solution is needed to obtain iii) from i). The fact $\sigma \in W^{1,2}_{2,\text{loc}}(\Omega; \mathbb{R}^N)$ is well known (compare [SE1], [BF1]) and just mentioned for the sake of completeness.

To finish this section we fix some notation.

i) With a slight abuse of notation, constants are denoted by $c$ without being relabelled.

ii) We take the sum w.r.t. repeated Greek indices $\alpha = 1, \ldots, n$ and w.r.t. repeated Latin indices $i = 1, \ldots, N$.

iii) We always assume that $x_0 \in \Omega$ and that $B_r(x_0) \subset \Omega$ is satisfied for each ball under consideration.
3 Regularization

As mentioned above, Problem (\( \mathcal{P} \)) is approximated in the following way: consider for any \( \delta \in (0,1) \) the functional

\[
J_\delta[w] := \frac{\delta}{2} \int_\Omega |\nabla w|^2 \, dx + J[w], \quad w \in u_0 + W^{1,0}_2(\Omega; \mathbb{R}^N),
\]

and denote by \( u_\delta \) the unique solution of

\[
(\mathcal{P}_\delta) \quad J_\delta[w] \to \min, \quad w \in u_0 + W^{1,0}_2(\Omega; \mathbb{R}^N).
\]

Letting \( f_\delta := \frac{\delta}{2} |\cdot|^2 + f \), observe that the minimality of \( u_\delta \) implies \( J_\delta(u_\delta) \leq J_\delta(u_0) \leq J_1(u_0) \), hence

\[
(3.1) \quad \int_\Omega f_\delta(\nabla u_\delta) \, dx \leq c
\]

follows for some real number \( c \). Moreover, by the definition of \( u_\delta \),

\[
(3.2) \quad \int_\Omega \nabla f_\delta(\nabla u_\delta) : \nabla \varphi \, dx = 0 \quad \text{for all } \varphi \in W^{1,0}_2(\Omega; \mathbb{R}^N).
\]

With the notation \( \sigma_\delta = \nabla f_\delta(\nabla u_\delta) \) we may assume on account of (3.1) that \( \sigma_\delta \rightharpoonup \sigma \) in \( L^2(\Omega; \mathbb{R}^{nN}) \) as \( \delta \to 0 \), and following [BF1] it is easily seen that

**Lemma 3.1** i) The sequence \( \{u_\delta\} \) is a \( J \)-minimizing sequence. Hence, the \( L^1 \)-cluster points of \( \{u_\delta\} \) provide generalized minimizers in the above sense.

ii) The limit \( \sigma \) of the sequence \( \{\sigma_\delta\} \) maximizes the dual problem \( \mathcal{P}^* \).

Next, a preliminary lemma (nevertheless an essential tool) is proved.

**Lemma 3.2** Suppose that Assumption 2.1 is true and that we have (2.2) in the case \( N > 1 \).

i) There is a real number \( c > 0 \) such that for any \( s \geq 0 \), for any \( \eta \in C_0^\infty(\Omega) \), \( 0 \leq \eta \leq 1 \) and for any \( \delta \in (0,1) \)

\[
\int_\Omega D^2f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \Gamma_\delta^s \eta^2 \, dx
\leq c \int_\Omega D^2f_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \otimes \nabla \eta, \partial_\gamma u_\delta \otimes \nabla \eta) \Gamma_\delta^s \, dx,
\]

where we have abbreviated \( \Gamma_\delta = 1 + |\nabla u_\delta|^2 \).
ii) Letting $A(k, r) = A_\delta(k, r) = \{ x \in B_r(x_0) : \Gamma_{\delta} > k \}$, $k > 0$, there is a real number $c > 0$, independent of $\delta$, such that for any $\eta \in C_0^\infty(B_r(x_0))$, $0 \leq \eta \leq 1$ and for any $\delta \in (0, 1)$

$$
\int_{A(k, r)} \Gamma_{\delta}^\frac{\eta}{k} |\nabla \Gamma_{\delta}|^2 \eta^2 \, dx \\
\leq c \int_{A(k, r)} D^2 f_\delta(\nabla u_\delta)(e_j \otimes \nabla \eta, e_j \otimes \nabla \eta) (\Gamma_{\delta} - k)^2 \, dx .
$$

Here $e_j$ denotes the $j^{th}$ coordinate vector.

**Remark 3.3** Following the proof of Lemma 3.2 we see that (2.2) is not needed for assertion i) if $s = 0$.

**Proof of Lemma 3.2.** ad i): Using the standard difference quotient technique it is easily seen that $u_\delta$ is of class $W^2_{2, \text{loc}}(\Omega; \mathbb{R}^N)$. Moreover, since $|D^2 f_\delta|$ is bounded, $\nabla f_\delta(\nabla u_\delta)$ is of class $W^1_{2, \text{loc}}(\Omega; \mathbb{R}^N)$ with partial derivatives (almost everywhere)

$$
\partial_\gamma (\nabla f_\delta(\nabla u_\delta)) = D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \cdot), \gamma = 1, \ldots, n.
$$

Now, given $\varphi \in C_0^\infty(\Omega; \mathbb{R}^N)$, we take $\partial_\gamma \varphi$, $\gamma = 1, \ldots, n$, as an admissible choice in the Euler equation (3.2). A partial integration implies by the above remarks

$$
(3.3) \quad \int_{\Omega} D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla \varphi) \, dx = 0 \quad \text{for all } \varphi \in C_0^\infty(\Omega; \mathbb{R}^N),
$$

and, using standard approximation arguments, (3.3) is seen to be true for all $\varphi \in W^1_2(\Omega; \mathbb{R}^N)$ which are compactly supported in $\Omega$. Next we cite [LU1], Chapter 4, Theorem 5.2, in the scalar case and the Uhlenbeck/Ural’tseva estimates (see [UH], [UR], we refer to [GM], Theorem 3.1) if $N > 1$ to see that $u_\delta \in W^1_{2, \text{loc}}(\Omega; \mathbb{R}^N)$. As a consequence, $\varphi = \eta^2 \partial_\gamma u_\delta \Gamma_{\delta}^s$ with $\eta$ given above is admissible in (3.3) (recall the product and chain rules for Sobolev functions). Summarizing the results we arrive at

$$
(3.4) \quad \int_{\Omega} D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \Gamma_{\delta}^s \eta^2 \, dx \\
+ s \int_{\Omega} D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \Gamma_{\delta}) \Gamma_{\delta}^{s-1} \eta^2 \, dx \\
= -2 \int_{\Omega} D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \eta) \eta \Gamma_{\delta}^s \, dx .
$$

11
In the scalar case \( N = 1 \) the second integral on the left-hand side can be neglected on account of

\[
D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \Gamma_\delta) = \frac{1}{2} D^2 f_\delta(\nabla u_\delta)(\nabla \Gamma_\delta, \nabla \Gamma_\delta) \geq 0 \text{ a.e.}
\]

In the vectorial setting \( N > 1 \) we first consider the case \( s = 0 \). Then the second term on the left-hand side trivially vanishes without any additional assumption (compare Remark 3.3). If \( s > 0 \), then (2.2) is needed: given a weakly differentiable function \( \psi : \Omega \rightarrow \mathbb{R} \), and letting \( f_\delta(Z) = g_\delta(|Z|^2) \) we obtain almost everywhere

\[
D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \psi)
= 4 g_\delta \partial_\alpha u_\delta^\beta \partial_\gamma \partial_\alpha u_\delta^\beta \partial_\gamma \partial_\beta \psi + 2 g_\delta \partial_\gamma \partial_\alpha u_\delta^\beta \partial_\gamma u_\delta^\beta \partial_\alpha \psi
= 2 g_\delta \partial_\gamma |\nabla u_\delta|^2 \partial_\beta \psi \partial_\beta u_\delta^\beta + g_\delta \partial_\alpha \nabla u_\delta \partial_\alpha \psi
= \frac{1}{2} \frac{\partial^2 f_\delta}{\partial z_\beta^2 \partial z_\gamma^2}(\nabla u_\delta) \partial_\beta \psi \partial_\gamma \Gamma_\delta
= \frac{1}{2} D^2 f_\delta(\nabla u_\delta)(e_j \otimes \nabla \psi, e_j \otimes \nabla \Gamma_\delta).
\]

(3.5)

Choosing \( \psi = \Gamma_\delta \) we see that in the vectorial setting the second integral on the left-hand side of (3.4) is non-negative as well. In any case we obtain for any \( \varepsilon > 0 \)

\[
\int_\Omega D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \Gamma_\delta^{s} \eta^2 \, dx
\leq c \int_\Omega \left[ D^2 f_\delta(\nabla u_\delta)(\partial_\gamma u_\delta, \partial_\gamma \nabla \eta) \right] \frac{1}{2} \Gamma_\delta^{s} \, dx
\leq c \varepsilon \int_\Omega D^2 f_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \otimes \nabla \eta, \partial_\gamma \nabla u_\delta \otimes \nabla \eta) \Gamma_\delta^{s} \eta^2 \, dx
+ \varepsilon^{-1} \int_\Omega D^2 f_\delta(\nabla u_\delta)(\partial_\gamma u_\delta \otimes \nabla \eta, \partial_\gamma \nabla u_\delta \otimes \nabla \eta) \Gamma_\delta^{s} \, dx.
\]

If \( \varepsilon \) is sufficiently small, then we may absorb the first integral on the right-hand side and i) is proved.
ad \(ii\)): This time we choose \( \varphi = \eta^2 \, \partial_\gamma u_\delta \max \{ \Gamma_\delta - k, 0 \} \). Moreover, given a measurable function \( w: \Omega \to \mathbb{R} \) and writing \( w^+ = \max \{ w, 0 \} \), we recall (see, for instance, [GT], Lemma 7.6, p. 152) that for \( w \in W_1^1(\Omega) \)

\[
\nabla w^+ = \begin{cases} 
\nabla w & \text{if } w > 0, \\
0 & \text{if } w \leq 0.
\end{cases}
\]

Then the same arguments as before prove \( \varphi \) to be admissible in (3.3):

\[
\int_{A(k,r)} D^2 f_\delta(\nabla u_\delta) \left( \partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta \right) (\Gamma_\delta - k) \eta^2 \, dx
\]

\[
+ \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta) \left( \partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \Gamma_\delta \right) \eta^2 \, dx
\]

\[
= -2 \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta) \left( \partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \eta \right) \eta (\Gamma_\delta - k) \, dx.
\]

Here the non-negative first integral on the left-hand side is neglected and the second integral is estimated as above:

\[
\frac{1}{2} \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta) \left( e_j \otimes \nabla \Gamma_\delta, e_j \otimes \nabla \Gamma_\delta \right) \eta^2 \, dx
\]

\[
\leq \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta) \left( \partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \Gamma_\delta \right) \eta^2 \, dx.
\]

According to (3.5), the right-hand side of (3.6) satisfies almost everywhere

\[
D^2 f_\delta(\nabla u_\delta) \left( \partial_\gamma \nabla u_\delta, \partial_\gamma u_\delta \otimes \nabla \eta \right) = \frac{1}{2} D^2 f_\delta(\nabla u_\delta) \left( e_j \otimes \nabla \eta, e_j \otimes \nabla \Gamma_\delta \right).
\]

(3.6)–(3.8) imply with the Cauchy-Schwarz inequality

\[
\int_{A(k,r)} D^2 f_\delta(\nabla u_\delta) \left( e_j \otimes \nabla \Gamma_\delta, e_j \otimes \nabla \Gamma_\delta \right) \eta^2 \, dx
\]

\[
\leq c \left\{ \varepsilon \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta) \left( e_j \otimes \nabla \Gamma_\delta, e_j \otimes \nabla \Gamma_\delta \right) \eta^2 \, dx \\
+ \varepsilon^{-1} \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta) \left( e_j \otimes \nabla \eta, e_j \otimes \nabla \eta \right) (\Gamma_\delta - k)^2 \, dx \right\},
\]

hence \( iii \) is proved recalling Assumption 2.1 \( iii \) and choosing \( \varepsilon > 0 \) sufficiently small. \( \blacksquare \)
4 Higher integrability in the limit case $\mu = 3$

In this section we consider the limit case $\mu = 3$ and prove local uniform integrability of $|\nabla u_\delta| \log^2(1 + |\nabla u_\delta|^2)$ by the way establishing Theorem 2.5. Here the discussion of the vectorial setting does not depend on additional conditions (compare Remark 2.4).

**THEOREM 4.1** Suppose we are given Assumptions 1.1 and 2.1 in the limit case $\mu = 3$. Then for any $\Omega' \Subset \Omega$ there is a real number $c(\Omega')$ independent of $\delta$ satisfying

$$
\int_{\Omega'} |\nabla u_\delta| \log^2(1 + |\nabla u_\delta|^2) \, dx \leq c(\Omega') < \infty.
$$

**Proof.** This time we have to show that $\varphi = u_\delta \omega_\delta^2 \eta^2$, $\omega_\delta = \log(\Gamma_\delta)$, $\eta \in C^\infty_0(B_{2r}(x_0))$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ on $B_r(x_0)$, is admissible in the Euler equation (3.2). Since condition (2.2) is dropped in this section, we now refer to the discussion of asymptotically regular integrands given in [CE]; a generalization is proved in [GM, Theorem 5.1]. As a result, $u_\delta$ is seen to be of class $W^2_2, \cap W^1_\infty, \cap (\Omega; \mathbb{R}^N)$ which proves $\varphi$ to be admissible. Alternatively, we could replace $\omega_\delta$ by a suitable truncation $\omega_{\delta,M}$ and prove Theorem 4.1 passing to the limit $M \to \infty$.

With the above choice, the Euler equation reads as

$$
\int_{B_{2r}(x_0)} \nabla f(\nabla u_\delta) : \nabla u_\delta \omega_\delta^2 \eta^2 \, dx + \delta \int_{B_{2r}(x_0)} |\nabla u_\delta|^2 \omega_\delta^2 \eta^2 \, dx
$$

$$
= -\int_{B_{2r}(x_0)} \nabla f(\nabla u_\delta) : u_\delta \otimes [\nabla \omega_\delta^2 \eta^2 + \nabla \eta^2 \omega_\delta^2] \, dx
$$

$$
- \delta \int_{B_{2r}(x_0)} \nabla u_\delta : u_\delta \otimes [\nabla \omega_\delta^2 \eta^2 + \nabla \eta^2 \omega_\delta^2] \, dx.
$$

(4.1)

Remark 2.2 i) proves the left-hand side of (4.1) to be greater than or equal to

$$
\int_{B_{2r}(x_0)} [\nu_1 \Gamma_\delta^\frac{1}{\nu_1} \omega_\delta^2 \eta^2 - \nu_2 \omega_\delta^2 \eta^2] \, dx + \delta \int_{B_{2r}(x_0)} |\nabla u_\delta|^2 \omega_\delta^2 \eta^2 \, dx.
$$

(4.2)

14
Since $|\nabla f|$ and $|u_\delta|$ are bounded, we find an upper bound for the right-hand side of (4.1) (using Young’s inequality with $\varepsilon > 0$ fixed)

\[
\text{r.h.s.} \leq c \int_{B_{2r}(x_0)} \eta^2 \left[ \varepsilon \Gamma_{\delta}^{-\frac{1}{2}} \omega_\delta^2 + \varepsilon^{-1} \Gamma_{\delta}^{-\frac{1}{2}} |\nabla \omega_\delta|^2 \right] dx \\
+ c(r) \int_{B_{2r}(x_0)} \omega_\delta^2 dx \\
+ c \delta \int_{B_{2r}(x_0)} \eta^2 \left[ \varepsilon |\nabla u_\delta|^2 \omega_\delta^2 + \varepsilon^{-1} |\nabla \omega_\delta|^2 \right] dx \\
+ c(r) \delta \int_{B_{2r}(x_0)} |\nabla u_\delta| \omega_\delta^2 dx.
\]

(4.3)

Clearly $\int_{B_{2r}(x_0)} \omega_\delta^2 dx$ and $\delta \int_{B_{2r}(x_0)} |\nabla u_\delta| \omega_\delta^2 dx$ are uniformly bounded w.r.t. $\delta$ (compare (3.1)). Hence (4.1)–(4.3) imply after absorbing terms (for $\varepsilon$ sufficiently small)

\[
\int_{B_r(x_0)} \Gamma_{\delta}^\frac{1}{2} \omega_\delta^2 dx \leq c \left[ 1 + \int_{B_{2r}(x_0)} \Gamma_{\delta}^{-\frac{1}{2}} |\nabla \omega_\delta|^2 \eta^2 dx \\
+ \delta \int_{B_{2r}(x_0)} |\nabla \omega_\delta|^2 \eta^2 dx \right].
\]

(4.4)

Given (4.4) observe that a.e.

\[
|\nabla \omega_\delta|^2 \leq c \frac{1}{1 + |\nabla u_\delta|^2 |D^2 u_\delta|^2},
\]

thus we may use Assumption 2.1, $iii)$, with $\mu = 3$, Lemma 3.2 (letting $s = 0$ and recalling Remark 3.3) as well as Remark 2.2 $iii)$ and (3.1) to obtain the final result

\[
\int_{B_r(x_0)} \Gamma_{\delta}^\frac{1}{2} \omega_\delta^2 dx \leq c \left[ 1 + c \int_{B_{2r}(x_0)} \left( \Gamma_{\delta}^{-\frac{1}{2}} + \delta \right) \Gamma_{\delta}^{-1} |D^2 u_\delta|^2 \eta^2 dx \right] \\
\leq c \left[ 1 + c \int_{B_{2r}(x_0)} D^2 f_\delta(\nabla u_\delta) \left( \partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta \right) \eta^2 dx \right] \\
\leq c \left[ 1 + c(r) \int_{B_{2r}(x_0)} |D^2 f_\delta(\nabla u_\delta)| |\nabla u_\delta|^2 dx \right] \leq c.
\]
5 \( L^p \)-estimates in the case \( \mu < 3 \)

From now on we concentrate on the case \( \mu < 3 \). Moreover, we impose some additional structure in the vectorial setting \( N > 1 \) (compare Remark 2.4). Then it is possible to modify the arguments of Section 4 such that the results obtained there may be iterated. This gives uniform \( L^p \)-estimates in the sense of

**THEOREM 5.1** Suppose that \( \mu < 3 \), that we have Assumptions 1.1, 2.1 and that (2.2) is satisfied. Then for any \( 1 < p < \infty \) and for any \( \Omega' \subseteq \Omega \) there is a constant \( c(p, \Omega') \), which does not depend on \( \delta \), such that

\[
\| \nabla u_\delta \|_{L^p(\Omega'; \mathbb{R}^N)} \leq c(p, \Omega') < \infty.
\]

**REMARK 5.2** As an immediate consequence a generalized minimizer \( u^* \in \mathcal{M} \) is found which is of class \( W^{1, p}_{\text{loc}}(\Omega; \mathbb{R}^N) \) for any \( 1 < p < \infty \).

**Proof.** Fix a ball \( B_{r_0}(x_0) \subseteq \Omega \) and assume that there is real number \( \alpha_0 \geq 0 \) such that (uniformly w.r.t. \( \delta \))

\[
\int_{B_{r_0}(x_0)} \Gamma_{\delta}^{\frac{1}{\alpha_0}} dx + \delta \int_{B_{r_0}(x_0)} \Gamma_{\delta}^{\frac{\alpha_0}{\alpha_0 + 1}} dx \leq c.
\]

Note that by (3.1) this assumption is true for \( \alpha_0 = 0 \). Next define \( \alpha = \alpha_0 + 3 - \mu \) and choose \( \varphi = u_\delta \Gamma_\delta^\alpha \eta^2 \), \( \eta \in C_0^\infty(B_{r_0}(x_0)) \), \( 0 \leq \eta \leq 1 \), \( \eta \equiv 1 \) on \( B_{r_0/2}(x_0) \), \( \| \nabla \eta \| \leq c/r_0 \). As outlined in the proof of Lemma 3.2 \( u_\delta \) is of class \( W^{2, \infty}_{\text{loc}}(\Omega; \mathbb{R}^N) \), hence \( \varphi \) is admissible in (3.2) with the result

\[
\int_{B_{r_0}(x_0)} \nabla f(\nabla u_\delta) : \nabla u_\delta \Gamma_\delta^\alpha \eta^2 dx + \delta \int_{B_{r_0}(x_0)} |\nabla u_\delta|^2 \Gamma_\delta^\alpha \eta^2 dx
\]

\[
\leq c(\alpha) \int_{B_{r_0}(x_0)} \Gamma_\delta^{\alpha - 1} |D^2 u_\delta| \eta^2 dx + c(\alpha) \delta \int_{B_{r_0}(x_0)} \Gamma_\delta^\alpha |D^2 u_\delta| \eta^2 dx
\]

\[
+ c \int_{B_{r_0}(x_0)} \Gamma_\delta^\alpha |\nabla \eta^2| dx + c \delta \int_{B_{r_0}(x_0)} \Gamma_\delta^{\alpha + 1} |\nabla \eta^2| dx.
\]

Here Assumption 1.1 and the boundedness of \( \nabla f \) (compare Assumption 2.1 ii)) are already used. Analogous to the previous section, the left-hand side of (5.2) is estimated with the help of Remark 2.2, i):

\[
l.h.s. \geq \nu_4 \int_{B_{r_0}(x_0)} \Gamma_\delta^{\frac{1}{\alpha_0}} \eta^2 dx - \nu_5 \int_{B_{r_0}(x_0)} \Gamma_\delta^\alpha \eta^2 dx
\]

\[
+ \delta \int_{B_{r_0}(x_0)} \Gamma_\delta^{\frac{\alpha + 1}{\alpha_0}} \eta^2 dx - \delta \int_{B_{r_0}(x_0)} \Gamma_\delta^\alpha \eta^2 dx.
\]
The right-hand side of (5.2) is handled via (fix \( \varepsilon > 0 \) and use Young’s inequality)

\[
\text{r.h.s.} \leq c \int_{B_{\varepsilon}(x_0)} \eta^2 \left[ \varepsilon \Gamma_{\frac{1+\alpha}{\delta}} + \varepsilon^{-1} \Gamma_{\frac{1}{\delta}} - \frac{1+\alpha}{\delta} \Gamma_{\delta}^{-1} |D^2u_{\delta}|^2 \right] dx \\
+ c \int_{B_{\varepsilon}(x_0)} \left[ \varepsilon \Gamma_{\frac{1+\alpha}{\delta}} \eta^2 + \varepsilon^{-1} \Gamma_{\frac{1}{\delta}} - \frac{1+\alpha}{\delta} \Gamma_{\delta}^{-1} |\nabla \eta|^2 \right] dx \\
+ c \delta \int_{B_{\varepsilon}(x_0)} \eta^2 \left[ \varepsilon \Gamma_{\frac{1+\alpha}{\delta}} + \varepsilon^{-1} \Gamma_{\frac{1}{\delta}} - \frac{1+\alpha}{\delta} \Gamma_{\delta}^{-1} |D^2u_{\delta}|^2 \right] dx \\
+ c \delta \int_{B_{\varepsilon}(x_0)} \left[ \varepsilon \Gamma_{\frac{1+\alpha}{\delta}} \eta^2 + \varepsilon^{-1} \Gamma_{\frac{1}{\delta}} - \frac{1+\alpha}{\delta} \Gamma_{\delta}^{-1} |\nabla \eta|^2 \right] dx.
\]

Hence, absorbing terms, (5.2) yields

\[
\int_{B_{\varepsilon}(x_0)} \Gamma_{\frac{1+\alpha}{\delta}} dx + \delta \int_{B_{\varepsilon}(x_0)} \Gamma_{\frac{1}{\delta}} dx \\
\leq c \left[ \int_{B_{\varepsilon}(x_0)} \eta^2 \Gamma_{\frac{1+\alpha}{\delta}} |D^2u_{\delta}|^2 dx \\
+ \int_{B_{\varepsilon}(x_0)} \Gamma_{\frac{1}{\delta}} \eta^2 |\nabla \eta|^2 dx + \int_{B_{\varepsilon}(x_0)} \Gamma_{\frac{1+\alpha}{\delta}} \eta^2 dx \right] \\
+ c \delta \left[ \int_{B_{\varepsilon}(x_0)} \eta^2 \Gamma_{\frac{1}{\delta}} |D^2u_{\delta}|^2 dx \\
+ \int_{B_{\varepsilon}(x_0)} \Gamma_{\frac{1}{\delta}} |\nabla \eta|^2 dx + \int_{B_{\varepsilon}(x_0)} \Gamma_{\frac{1+\alpha}{\delta}} \eta^2 dx \right] \\
= c \sum_{i=1}^{6} I_i + c \delta \sum_{i=4}^{6} I_i.
\]

Starting with \( I_1 \), we recall that by definition \( \mu + \alpha - 3 = 0 \geq 0 \), thus Assumption 2.1 iii) and Lemma 3.2 i) give

\[
I_1 = \int_{B_{\varepsilon}(x_0)} \eta^2 \Gamma_{\frac{1+\alpha}{\delta}} |D^2u_{\delta}|^2 \Gamma_{\frac{1+\alpha}{\delta}} \eta^2 dx \\
\leq c \int_{B_{\varepsilon}(x_0)} D^2 f_{\delta}(\nabla u_{\delta}) (\partial_{r} \nabla u_{\delta}, \partial_{\gamma} \nabla u_{\delta}) \Gamma_{\delta}^{-\alpha} \eta^2 dx \\
\leq c(r_0) \int_{B_{\varepsilon}(x_0)} \left[ \Gamma_{\delta}^{-\frac{1}{2}} + \delta \right]^{1+\alpha} \eta^2 dx \leq c,
\]

where the last inequality is due to Assumption (5.1). An upper bound (not depending on \( \delta \)) for \( I_5 \) is found since we may assume w.l.o.g. that \( \mu \geq 2. \)
This clearly proves $I_2$ to be bounded independent of $\delta$ as well. Studying $I_4$ let us first assume that $\alpha \leq 2$. Then, again by Lemma 3.2 i)

$$
\delta I_4 \leq \int_{B_{r_0}(x_0)} D^2 f_{\delta}(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \eta^2 \, dx
\leq c(r_0) \int_{B_{r_0}(x_0)} \left[ \Gamma_\delta^{-\frac{1}{2}} + \delta \right] \Gamma_\delta \, dx \leq c.
$$

In the case $\alpha > 2$, Lemma 3.2 i) gives

$$
\delta I_4 \leq \int_{B_{r_0}(x_0)} D^2 f_{\delta}(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \Gamma_\delta^{\frac{\alpha}{2} - 1} \eta^2 \, dx
\leq c(r_0) \int_{B_{r_0}(x_0)} \left[ \Gamma_\delta^{-\frac{1}{2}} + \delta \right] \Gamma_\delta^{1 + \frac{\alpha}{2} - 1} \, dx \leq c,
$$

where we once more recall (5.1) and observe that $\frac{\alpha}{2} - 1 = (\alpha_0 + 1 - \mu)/2 \leq \alpha_0/2$. This condition trivially bounds $\delta I_5$ and $\delta I_6$ independent of $\delta$, and we have proved with (5.3): suppose that (5.1) holds for some given $r_0 > 0$ and $\alpha_0 > 0$. Then there is a constant, independent of $\delta$ such that

$$(5.4) \int_{B_{r_0/3}(x_0)} \frac{1 + \alpha_0 + 3 - \mu}{2} \Gamma_\delta^2 \, dx + \delta \int_{B_{r_0/2}(x_0)} \frac{1 + \alpha_0 + 3 - \mu}{2} \Gamma_\delta^{1 + \frac{\alpha_0 + 3 - \mu}{2}} \, dx \leq c.$$ 

We now claim that for any $n \in \mathbb{N}$ there is a constant $c(n)$, independent of $\delta$, such that

$$(5.5) \int_{B_{r_0/n}(x_0)} \frac{1 + n(3 - \mu)}{2} \Gamma_\delta^2 \, dx + \delta \int_{B_{r_0/2n}(x_0)} \frac{1 + n(3 - \mu)}{2} \Gamma_\delta^{1 + \frac{n(3 - \mu)}{2}} \, dx \leq c.$$ 

In fact, as mentioned above, $\alpha_0 = 0$ is an admissible choice to obtain (5.5) from 5.4 in the case $n = 1$. Next assume by induction that (5.5) is true for some $n \in \mathbb{N}$. Then we may take $\alpha_0 = n(3 - \mu)$ in (5.1) and (5.4) establishes (5.5) with $n$ replaced by $n + 1$, thus the claim is proved. Obviously this implies Theorem 5.1. 

**Remark 5.3** If we omit condition (2.2) in the vectorial setting, then analogous arguments prove higher integrability up to a finite number $1 < p(\mu)$.
6 Apriori gradient bounds

In this section the DeGiorgi type arguments, outlined for example in [BF3], are modified: on one hand, given Theorem 5.1, we benefit from Hölder’s inequality. This decreases on the other hand the exponent of iteration (see definition of $\beta$). Nevertheless it turns out that Lemma 6.3 still is applicable to obtain

**THEOREM 6.1** Consider a ball $B_{R_0}(x_0) \subseteq \Omega$. With the assumptions of Theorem 5.1 there is a local constant $c > 0$ such that for any $\delta \in (0, 1)$

$$\|\nabla u_\delta\|_{L^\infty(B_{R_0}, \mathbb{R}^N)} \leq c.$$

Before proving Theorem 6.1 we recall the definition

$$A(k, r) = \{ x \in B_r(x_0) : \Gamma_\delta > k \}, \quad B_r(x_0) \subseteq \Omega, \quad k > 0,$$

and establish the following result.

**LEMMA 6.2** Fix some $x_0 \in \Omega$ and suppose $0 < r < R < R_0$ such that $B_{R_0}(x_0) \subseteq \Omega$. Then there is a real number $c$, independent of $r, R, R_0, k$ and $\delta$, satisfying

$$\int_{A(k, r)} (\Gamma_\delta - k)^{\frac{n}{n-1}} dx \leq \frac{c}{(R - r)^{\frac{n}{n-1}}} \left[ \int_{A(k, R)} (\Gamma_\delta - k)^2 dx \right]^{\frac{1}{2} - \frac{1}{n}} \left[ \int_{A(k, R)} \Gamma_\delta^{\frac{p}{2}} dx \right]^{\frac{1}{2} - \frac{1}{n}}. \tag{6.1}$$

**Proof of Lemma 6.2.** Recalling the notion $w^+$ of Section 3, Sobolev’s inequality yields with $\eta \in C_0^\infty(B_R(x_0)), 0 \leq \eta \leq 1, \eta \equiv 1$ on $B_r(x_0)$, $|\nabla \eta| \leq c/(R - r)$,

$$\int_{A(k, r)} (\Gamma_\delta - k)^{\frac{n}{n-1}} dx \leq \int_{B_R(x_0)} \left[ \eta(\Gamma_\delta - k)^+ \right]^{\frac{n}{n-1}} dx \leq c \left[ \int_{B_R(x_0)} |\nabla \left[ \eta(\Gamma_\delta - k)\right]^+ \right| dx \right]^{\frac{n}{n-1}} \leq c \left[ \int_{A(k, r)} \left| \nabla \left[ \eta(\Gamma_\delta - k)\right] \right| dx \right]^{\frac{n}{n-1}} \leq c \left[ I_1^{\frac{n}{n-1}} + I_2^{\frac{n}{n-1}} \right]. \tag{6.2}$$
Here we have
\[
I_1^{\frac{n}{n-1}} = \left[ \int_{A(k,R)} |\nabla \eta| (\Gamma_\delta - k) \, dx \right]^{\frac{n}{n-1}}
\]
\[
\leq \left[ \int_{A(k,R)} |\nabla \eta|^2 (\Gamma_\delta - k)^2 \, dx \right]^{\frac{1}{2}} \left[ \int_{A(k,R)} 1 \, dx \right]^{\frac{n}{2(n-1)}}
\]
\[
\leq \frac{c}{(R - r)^{\frac{n}{n-1}}} \left[ \int_{A(k,R)} (\Gamma_\delta - k)^2 \, dx \right]^{\frac{1}{2}} \left[ \int_{A(k,R)} 1 \, dx \right]^{\frac{n}{2(n-1)}},
\]
thus \(I_1^{\frac{n}{n-1}}\) is seen to be bounded from above by the right-hand side of 6.1. Estimating \(I_2\), Lemma 3.2 ii) is needed with the result
\[
I_2^{\frac{n}{n-1}} = \left[ \int_{A(k,R)} \eta |\nabla \Gamma_\delta| \, dx \right]^{\frac{n}{n-1}}
\]
\[
\leq \left[ \int_{A(k,R)} \eta^2 |\nabla \Gamma_\delta|^2 \Gamma_\delta^{-\frac{\rho}{2}} \, dx \right]^{\frac{1}{2}} \left[ \int_{A(k,R)} \Gamma_\delta^{\frac{\rho}{2}} \, dx \right]^{\frac{n}{2(n-1)}}
\]
\[
\leq c \left[ \int_{A(k,R)} D^2 f_\delta(\nabla u_\delta)(e_j \otimes \nabla \eta, e_j \otimes \nabla \eta) (\Gamma_\delta - k)^2 \, dx \right]^{\frac{1}{2}} \left[ \int_{A(k,R)} \Gamma_\delta^{\frac{\rho}{2}} \, dx \right]^{\frac{n}{2(n-1)}}
\]
\[
\times \left[ \int_{A(k,R)} \Gamma_\delta^{\frac{\rho}{2}} \, dx \right]^{\frac{n}{2(n-1)}},
\]
hence (6.2) proves the lemma.

We now come to the **Proof of Theorem 6.1**. Consider the left-hand side of (6.1): for any real number \(s > 1\) Hölder’s inequality implies
\[
\int_{A(k,r)} (\Gamma_\delta - k)^2 \, dx = \int_{A(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1} \frac{s}{2}} (\Gamma_\delta - k)^{2 - \frac{n}{n-1} \frac{s}{2}} \, dx
\]
\[
\leq \left[ \int_{A(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1} \frac{s}{2}} \, dx \right]^\frac{1}{s}
\]
\[
\times \left[ \int_{A(k,r)} (\Gamma_\delta - k)^{2 - \frac{n}{n-1} \frac{s}{2}} \, dx \right]^\frac{n-1}{2s}.
\]

20
Hence, on account of Theorem 5.1 there is a real number \( c_1(s, n, B_{R_0}(x_0)) \), independent of \( \delta \),

\[
c_1(s, n, B_{R_0}(x_0)) := \sup_{\delta > 0} \left[ \int_{B_{R_0}(x_0)} \frac{1}{2 \pi} \left( \frac{2s - n}{\pi r} \right)^{\frac{\delta}{2}} \frac{r^{n-1}}{r^{1/2}} \, dx \right]^{\frac{2}{1-n}} < \infty,
\]

such that

\[
(6.3) \quad \int_{A(k, r)} (\Gamma_\delta - k)^2 \, dx \leq c_1(s, n, B_{R_0}(x_0)) \left[ \int_{A(k, r)} (\Gamma_\delta - k)^{\frac{n}{n-1}} \, dx \right]^{\frac{1}{n-1}}.
\]

Studying the right-hand side of (6.1), we fix a second real number \( t > 1 \) and applying Hölder’s inequality once more it is seen that

\[
\int_{A(k, R)} \Gamma_\delta^t \, dx \leq |A(k, R)|^{\frac{1}{t}} \left[ \int_{A(k, R)} \Gamma_\delta^{\frac{t}{n-1}} \, dx \right]^{\frac{n-1}{t}}.
\]

Similar as above, one defines

\[
c_2(t, \mu, B_{R_0}(x_0)) := \sup_{\delta > 0} \left[ \int_{B_{R_0}(x_0)} \Gamma_\delta^{\frac{t}{n-1}} \, dx \right]^{\frac{n-1}{t}} < \infty
\]

to obtain

\[
(6.4) \quad \int_{A(k, R)} \Gamma_\delta^t \, dx \leq c_2(t, \mu, B_{R_0}(x_0)) |A(k, R)|^{\frac{1}{t}}.
\]

Summarizing the results we arrive at

\[
\int_{A(k, r)} (\Gamma_\delta - k)^2 \, dx \overset{[6.3]}{\leq} c \left[ \int_{A(k, r)} (\Gamma_\delta - k)^{\frac{n}{n-1}} \, dx \right]^{\frac{1}{n-1}} \overset{[6.1]}{\leq} \frac{c}{(R - r)^{\frac{n-1}{2}}} \left[ \int_{A(k, r)} (\Gamma_\delta - k)^2 \, dx \right]^{\frac{1}{2}} \times \left[ \int_{A(k, R)} \Gamma_\delta^t \, dx \right]^{\frac{n-1}{2t}} \overset{[6.4]}{\leq} \frac{c}{(R - r)^{\frac{n-1}{2}}} \left[ \int_{A(k, r)} (\Gamma_\delta - k)^2 \, dx \right]^{\frac{1}{2}} \times \left| A(k, R) \right|^{\frac{1}{2}} \frac{n-1}{2t} \times \frac{1}{2}.
\]
As the next step, define for \( k \) and \( r < R \) as above

\[
\tau(k, r) := \int_{A(k,r)} (\Gamma_\delta - k)^2 \, dx , \\
a(k, r) := |A(k,r)| .
\]

With this notation (6.5) reads as

\[
(6.6) \quad \tau(k, r) \leq \frac{c}{(R-r)^{\frac{n+1}{2}}} \left[ \tau(k, R) \right]^{\frac{1}{2}} \left[ \frac{n}{n-1} \right]^{\frac{1}{2}} \left[ a(k, R) \right]^{\frac{1}{2}} \left[ \frac{1}{(h-k)^{1+\frac{1}{n}}} \right] .
\]

Given two real numbers \( h > k > 0 \), we now observe

\[
a(h, R) = \int_{A(h,r)} dx \leq \int_{A(h,r)} (\Gamma_\delta - k)^2 (h-k)^{-2} \, dx ,
\]

thus \( h > k > 0 \) implies

\[
(6.7) \quad a(h, R) \leq \frac{1}{(h-k)^2} \tau(k, R) .
\]

With (6.6) and (6.7) it is proved that for \( h > k > 0 \)

\[
\tau(h, r) \leq \frac{c}{(R-r)^{\frac{n+1}{2}}} \left[ \tau(h, R) \right]^{\frac{1}{2}} \left[ \frac{n}{n-1} \right]^{\frac{1}{2}} \left[ \frac{1}{(h-k)^{1+\frac{1}{n}}} \right] \left[ \tau(k, R) \right]^{\frac{1}{2}} \left[ \frac{1}{(h-k)^{1+\frac{1}{n}}} \right] \left[ \frac{n}{n-1} \right]^{\frac{1}{2}} .
\]

If \( s \) and \( t \) are chosen sufficiently close to 1 (depending on \( n \)), then it is achieved that

\[
\frac{1}{2} \frac{n}{n-1} s \left[ 1 + \frac{1}{t} \right] =: \beta > 1 .
\]

With this choice of \( s \) and \( t \) we additionally let

\[
\alpha := \frac{n}{n-1} s t > 0 , \quad \gamma := \frac{n}{n-1} s t > 0 ,
\]

hence the following lemma, stated for example in [ST], Lemma 5.1, p.219, may be applied.
Lemma 6.3 Assume that \( \varphi(h, \rho) \) is a non-negative real valued function defined for \( h > k_0 \) and \( \rho < R_0 \). Suppose further that for fixed \( \rho \) the function is non-increasing in \( h \) and that is non-decreasing in \( \rho \) if \( h \) is fixed. Then

\[
\varphi(h, \rho) \leq \frac{C}{(h - k)^{\alpha}(R - \rho)^{\gamma}} \left[ \varphi(k, R) \right]^{\beta}, \quad h > k > k_0, \quad \rho < R < R_0,
\]

with some positive constants \( C, \alpha, \beta > 1, \gamma \), implies for all \( 0 < \sigma < 1 \)

\[
\varphi(k_0 + d, R_0 - \sigma R_0) = 0,
\]

where the quantity \( d \) is given by

\[
d^\alpha = \frac{2^{(\alpha + \beta)\beta/(\beta - 1)} C \left[ \varphi(k_0, R_0) \right]^{\beta - 1}}{\sigma^{\gamma} R_0^\beta}.
\]

This lemma yields

\[
\tau(d, R_0/2) = \int_{A(d, R_0/2)} (\Gamma_\delta - d)^2 \, dx = 0,
\]

and, as a consequence,

(6.8) \( \Gamma_\delta \leq d \) on \( B_{R_0/2}(x_0) \).

Here the quantity \( d \) is uniformly bounded w.r.t. \( \delta \) if and only if there is a constant (independent of \( \delta \)) such that

\[
\tau(0, R_0) = \int_{B_{R_0}(x_0)} \Gamma_\delta^2 dx \leq c.
\]

This fact is proved in Theorem 5.1 and the apriori estimate Theorem 6.1 follows from (6.8).

\[\square\]

7 Proof of the main theorem

Once Theorem 6.1 is established, Theorem 2.7 follows exactly as outlined in [BF3]. Let us first sketch the main arguments to obtain local \( C^{1,\alpha} \)-regularity for weak \( \{u_\delta\} \)-cluster points \( u^* \): recall Corollary 2.6 which implies the Euler equation

\[
\int_{\Omega} \nabla f(\nabla u^*) : \nabla \varphi \, dx = 0 \quad \text{for all} \quad \varphi \in C^j_0(\Omega; \mathbb{R}^N).
\]
In the scalar case $N = 1$ we argue with the standard difference quotient technique and, since $u^*$ is Lipschitz, it follows that $u^*$ is of class $W^{2,\infty}_{2,\text{loc}}(\Omega; \mathbb{R}^N)$. Then, letting $v = \partial_\alpha u^*$, one arrives at

\[ \int_\Omega D^2 f(\nabla u^*) (\nabla v, \nabla \varphi) \, dx = 0 \quad \text{for all } \varphi \in C^1_0(\Omega; \mathbb{R}^N) \]

(compare (3.3)), where the coefficients $\frac{\partial f}{\partial x_\alpha x_\beta}(\nabla u^*)$ are uniformly elliptic on $\Omega' \subset \Omega$. Theorem 8.22 of [GT] finally proves Hölder continuity of $v$.

In the vectorvalued case an auxiliary integrand $\tilde{f}$ is constructed following the lines of [MS]. As a result, Theorem 3.1 of [GM] may be applied since we also have imposed the Hölder condition (2.3). Thus $C^{1,\alpha}$-regularity is proved in the vectorial setting as well.

Next consider the dual solution $\sigma$. As it is proved in [BF1] or [BF3], the duality relation

$$ \sigma = \nabla \tilde{f}(\nabla u^*) \quad \text{for a.a. } x \in \Omega $$

holds true for any weak cluster point $u^*$ of the sequence $\{u_k\}$. Since these cluster points are known to have (locally) Hölder continuous first derivatives, $\sigma \in C^{0,\alpha}(\Omega; \mathbb{R}^N)$ is immediate.

Uniqueness of generalized minimizers up to a constant is shown in Section 5 of [BF3] — the idea is due to [SE4]: a suitable variation of $\sigma$ is seen to be admissible on account of $\sigma \in C^{0,\alpha}(\Omega; \mathbb{R}^N)$. This, together with the uniqueness of $\sigma$, yields

$$ \nabla \tilde{u} = \nabla \tilde{f}^*(\sigma) $$

for any generalized minimizer $\tilde{u} \in \mathcal{M}$, and Theorem 2.7 is proved.

\[ \blacksquare \]

References


[BF2] Bildhauer, M., Fuchs, M., Partial regularity for variational integrals with $(s, \mu, q)$-growth. To appear in Calc. Var.


