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Convex Variational Integrals with a Wide Range of Anisotropy. Part I: Regularity Results

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Abstract

We consider strictly convex energy densities $f\colon \mathbb{R}^{nN} \to \mathbb{R}$, $f(Z) = g(|Z_1|, \dots, |Z_n|)$ if $N > 1$, under non-standard growth conditions. More precisely we assume that for some constants λ, Λ and for all Z, $Y \in \mathbb{R}^{nN}$

$$\lambda\big(1+|Z|^2\big)^{-\frac{\mu}{2}}|Y|^2 \le D^2 f(Z)(Y,Y) \le \Lambda\big(1+|Z|^2\big)^{\frac{q-2}{2}}|Y|^2$$

holds with exponents $\mu \in \mathbb{R}$ and $q > 1$. If u denotes a local minimizer w.r.t. the energy $\int f(\nabla w)\, dx$, then we prove $L^{q+\varepsilon}$-integrability of $|\nabla u|$ provided that u is locally bounded and $q < 4 - \mu$. In particular this is true in the vectorvalued setting and implies partial $C^{1,\alpha}$-regularity of u together with the additional assumption $q < (2-\mu)n/(n-2)$. In the scalar case we derive local $C^{1,\alpha}$-regularity from the condition $q < 4 - \mu$, again if u is locally bounded. Both results substantially improve what is known up to now (see, for instance, [ELM], [CH], [BF1], [BF2] and the references quoted therein).

1 Introduction

Given a smooth, strictly convex integrand $f\colon \mathbb{R}^{nN} \to \mathbb{R}$ and a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ we are interested in smoothness properties of local minimizers of the energy

$$J[w] := \int_\Omega f(\nabla w)\, dx \tag{1.1}$$

in a suitable energy class, where – for $N = 1$ – we also include obstacle problems in our considerations.

Given a power growth integrand f, it is well known that we obtain local $C^{1,\alpha}$-regularity in the scalar case (we just mention the names of DeGiorgi, Moser, Nash, Ladyzhenskaya, Ural'tseva where of course many others could be mentioned). With the work of Uhlenbeck ([UH]) this is also seen to be true for vectorial problems with some additional structure. General vectorvalued problems only provide partially regular minimizers, i.e. $C^{1,\alpha}$-regularity holds on an open set of full measure — details and references are found, for instance, in [Gi1].

In recent years variational problems with non-standard growth conditions became more and more popular. Here, on one hand we may assume that f (together with the corresponding estimates for the second derivatives) is bounded from above and below by different growth rates $q > p > 1$, i.e. typical examples are given by anisotropic integrands (see Example 1.2, i), below). Studies in this direction were forced in particular by Marcellini starting with [Ma1]–[Ma3]. Here the most important tool is to find an appropriate condition relating the exponents p and q. In fact, the existence of irregular solutions was already observed by Giaquinta ([Gi2]) if p and q differ too much.

As a model case for the second class of non-standard integrands one may think of $f(Z) = |Z| \ln(1 + |Z|)$, a function of nearly linear growth which is studied in the theory of Prandtl-Eyring fluids and of plastic materials with logarithmic hardening (an exhaustive overview is found in [FS]). Motivated by this logarithmic example, the consideration of variational problems in general Orlicz-Sobolev energy classes is continued in the papers [FO], [FM].

Finally, a unified approach to anisotropic energy densities together with an Orlicz-Sobolev growth condition is given in [BFM] and [BF1] by introducing the notion of (s, μ, q)-growth. Here the variational integrand f is bounded from below by some Orlicz-Sobolev function F (which in turn has at least the growth rate $s \geq 1$) and the second derivatives are supposed to satisfy

$$\lambda \left(1 + |Z|^2\right)^{-\frac{\mu}{2}} |Y|^2 \;\leq\; D^2 f(Z)(Y, Y) \;\leq\; \Lambda \left(1 + |Z|^2\right)^{\frac{q-2}{2}} |Y|^2 \tag{1.2}$$

for all Z, $Y \in \mathbb{R}^{nN}$, for some positive constants λ, Λ, for $\mu \in \mathbb{R}$ and with the choice $q > 1$. Hence, anisotropic power growth is covered by letting $2 - \mu = p = s > 1$, the logarithmic integrand of above satisfies (1.2) choosing $s = 1$, $\mu = 1$ and $q = 1 + \varepsilon$. Given the notion of (s, μ, q)-growth, both smoothness in the scalar case and partial regularity for $N > 1$ are established under the so called (s, μ, q)-condition relating these exponents such that a variety of known results is included and extended (see also [Bi2] for a detailed discussion). Note that, from the technical point of view, the (first) restriction on the exponents enters through an application of Sobolev's inequality which gives uniform higher local integrability of the gradients of some regularization.

Studying a similar linear growth situation it turns out in [Bi1] that much better results are obtained if the solution (and its regularization) is uniformly bounded (this assumption is quite natural for Dirichlet problems with L^∞-traces and admitting some maximum principle). In this case, Sobolev's inequality may be replaced by an additional application of the (non-differentiated) Euler equation. This method enables us to reach (up to a certain extend) the limit case $\mu = 3$ and $q = 1$ in (1.2). Moreover, as outlined in [Bi2], we do not expect regular solutions if ellipticity is given in terms of $\mu > 3$.

Let us turn our attention back to variational problems with non-standard, superlinear growth. If we impose an analogous boundedness condition, then, as a formal correspondence to the results given in [Bi1], the relation $1 < q < 4 - \mu$ (for anisotropic power growth integrands $1 < q < 2 + p$) is expected to be the best possible one inducing (partially) regular solutions. Note that the relevance of the restriction $q < 2 + p$ was already discovered in [ELM]: given a L^∞-solution u, uniform local higher (up to a certain extend) integrability of $|\nabla u|$ is established in the vectorvalued setting (choosing $2 \le p$).

Nevertheless, the full strength of the above stated correspondence could not be established in the paper [BF2] on anisotropic variational integrals with convex hull property: instead of $1 < q < 2 + p$ (plus some natural condition improving higher integrability to partial regularity) the exponents have to be related via $1 < q < p + 2/3$. This is caused by an essential difference to the linear growth situation: in [Bi1] we benefit from the growth rate $1 = q$ of the main quantity $\nabla f(Z) : Z$ under consideration. Given an anisotropic power growth integrand, we just have the lower bound $p < q$ of this quantity.

As a consequence, the techniques again have to be changed such that we do not have to rely on the quantity $\nabla f(Z) : Z$. This leads to the study of Choe's article [CH], where bounded solutions w.r.t. anisotropic integrands $f(Z) = g(|Z|^2)$ (an assumption both for $N > 1$ and the scalar situation) are handled up to $1 < q < p + 1$. As a third approach, his results depend on a partial integration combined with a Caccioppoli-type inequality (of course this type of inequality also enters the two other techniques mentioned above).

In our paper we are interested in the question whether Choe's Ansatz can be improved. Following the above listed references ([FM], [BFM], [BF1], [BF2], [Bi1]) we introduce a regularization satisfying a Caccioppoli-type inequality which is slightly different from the one given in [CH]. This enables us to refine Choe's reasoning in various directions with surprisingly strong results. Roughly speaking we otain

THEOREM. *Assume that f admits some maximum principle and consider a local minimizer u of the energy J given in (1.1). If u is of class L^∞_{loc} and if (1.2) is supposed, then*

i) in the vectorvalued setting local L^t-integrability of ∇u follows whenever $1 < q < t < 4 - \mu$ ($1 < q < t < 2 + p$);

ii) in the scalar case (here we may also consider obstacle problems) the same condition ensures full regularity of the solution.

REMARK 1.1 i) *The higher integrability stated in* i) *is the starting point to prove the Corollary 2.4 on partial regularity.*

ii) *It is well known that vectorvalued problems with the additional structure $f(Z) = g(|Z|^2)$ can be handled more or less in the same way as scalar problems. The slight changes which are needed in our setting are outlined, for instance, in [FM], [Bi2].*

iii) *In contrast to [BFM] and [BF1] the growth rate s of the variational integrand is not specified. We just have the bounds induced by (1.2).*

iv) *In terms of anisotropic integrands with (p,q)-growth the above assertions are established in [BFM] and [BF1] whenever*

$$q \;<\; p\frac{n+2}{n}.$$

These results do not rely on L^∞-bounds for the solution. Hence, at the first glance one may wonder about the case

$$2+p \;\le\; p\frac{n+2}{n}\,.$$

However, $p = n$ is the point of bifurcation, hence, by Sobolev's embedding theorem, boundedness becomes no restriction at all.

v) *On account of the few counterexamples we do not dare to state a conjecture on sharpness. Nevertheless, the above remark and the detailed discussion of admissible exponents given in [Bi2] at least shows our results to be reasonable and consistent.*

EXAMPLES 1.2

i) *As a first example let us have a look at the anisotropic energy density*

$$f(Z) \;=\; \big(1+|Z|^2\big)^{\frac{p}{2}} + \big(1+|Z_n|^2\big)^{\frac{q}{2}}\,, \qquad Z=(Z_1,\dots,Z_n)\in\mathbb{R}^{nN}\,,$$

with exponents $2 \le p < q$. This structure is imposed in [AF] to obtain partial regularity under a rather weak condition relating p and q (see [BF2] for a detailed comparison with the reults of [PS], [BF1] and [BF2]). In the scalar situation, $q < p(n+2)/n$ gives regular solutions (see [Ma1], [Ma2], [BFM]), a relation which can be improved by some additional assumptions (see [UU]).

ii) *If we do not have the above decomposition, for instance* $(2 \le p < q)$

$$f(Z) \;=\; \Big[1 + |Z|^2 + \big(1 + |Z_n|^2\big)^{\frac{q}{2}}\Big]^{\frac{p}{2}}\,, \qquad Z = (Z_1, \dots, Z_n) \in \mathbb{R}^{nN}\,,$$

or if the energy density is completely anisotropic in the sense that

$$f(Z) \;=\; \sum_{i=1}^{n} \phi_i(Z_i)\,, \qquad \phi_i = \big(1 + |Z_i|^2\big)^{\frac{q_i}{2}}\,, \qquad Z_i \in \mathbb{R}^N\,,$$

with exponents $q_i \ge 2$*, then the results of [AF] do not apply any more and, with the notation introduced in (1.2), partial regularity follows if* $q < pn/(n-2)$ *and* $q < \widehat{q} := \max\{p + 2/3; p(n+2)/n\}$ *(see [BF2]). In the following we will improve* $p + 2/3$ *to* $p + 2$*. If* $N = 1$*, then the condition* $q < p(n+2)/n$ *will be replaced by* $q < \max\{2+p, p(n+2)/n\}$ *(compare Remark 1.1, iv)).*

iii) *Let us finally discuss an example which is the most interesting one from our point of view. Consider the scalar case* $N = 1$ *and let* $Z = (Z_1, Z_2) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ *for some* $1 \le k < n$*. Moreover, suppose we are given exponents* $1 < p < q < 2$ *and*

$$f(Z) \;=\; \big(1 + |Z_1|^2\big)^{\frac{p}{2}} + \big(1 + |Z_2|^2\big)^{\frac{q}{2}}\,.$$

In this subquadratic situation (by elementary calculations) the estimate

$$\lambda\big(1 + |Z|^2\big)^{\frac{p-2}{2}} |Y|^2 \;\le\; D^2 f(Z)(Y,Y) \;\le\; \Lambda |Y|^2$$

is the best possible one. As a consequence, no regularity results are available up to now if p *is close to* 1 *— even if* $(q-p)$ *becomes very small. Hence, with the trivial inequality* $2 < p + 2$*, our theorem really covers a new class of variational integrals.*

Our paper is organized as follows: the precise setting together with a statement of the main results is given in the next section. In Section 3 we introduce a suitable regularization and proof some Caccioppoli-type inequalities. The vectorvalued case is handled in Section 4, whereas scalar obstacle problems are studied in Section 5.

2 Assumptions and main results

Starting with the vectorvalued setting we suppose the variational integrand f to satisfy

ASSUMPTION 2.1 *The energy density $f\colon \mathbb{R}^{nN} \to \mathbb{R}$ is a function of class $C^2(\mathbb{R}^{nN})$ and its second derivative is estimated for all Z, $Y \in \mathbb{R}^{nN}$ via*

$$(2.1)\qquad \lambda\big(1+|Z|^2\big)^{-\frac{\mu}{2}}|Y|^2 \;\le\; D^2f(Z)(Y,Y) \;\le\; \Lambda\big(1+|Z|^2\big)^{\frac{q-2}{2}}|Y|^2 \,.$$

Here λ, Λ denote some positive constants and the exponents $\mu \in \mathbb{R}$, $q > 1$ are related by

$$(2.2)\qquad q \;<\; 4-\mu \,.$$

Moreover, the representation formula

$$(2.3)\qquad f(Z) \;=\; g\big(|Z_1|,\dots,|Z_n|\big)\,, \qquad Z = \big(Z_1,\dots,Z_n\big) \in \mathbb{R}^{nN}\,,$$

is supposed to be valid for some function g which is increasing w.r.t. each argument. Let us finally assume that there is some N-function $F\colon [0,\infty) \to [0,\infty)$ having the Δ_2-property (see, e.g. [Ad] for details) such that

$$(2.4)\qquad c_1\,F(|Z|) - c_2 \;\le\; f(Z) \quad \text{for all } Z \in \mathbb{R}^{nN}$$

and for some positive numbers c_1, c_2.

REMARK 2.2 i) *Condition (2.4) gives existence and uniqueness results for Dirichlet problems in Orlicz-Sobolev spaces $W^1_F(\Omega;\mathbb{R}^{nN})$ (see [FO]). For the consideration of local minimizers it is sufficient to suppose that F is a continuous function of superlinear growth (see [BFM]).*

ii) *If $\mu < 1$, then ellipticity is good enough to improve (2.4) to the power growth estimate (with suitable constants c_1, c_2)*

$$c_1\,|Z|^{2-\mu} - c_2 \;\le\; f(Z) \quad \text{for all } Z \in \mathbb{R}^{nN}\,.$$

In fact, we may assume w.l.o.g. that $f \ge 0$ and combine convexity of f, in particular

$$f(Z) \;\ge\; f(Z/2) + \nabla f(Z/2) : Z/2\,,$$

with the inequality

$$\nabla f(Z) : Z \;=\; \int_0^1 D^2 f(\theta Z)(Z,Z)\,d\theta - \nabla f(0) : Z\,.$$

iii) *The structure (2.3) can be replaced by any condition ensuring an appropriate maximum principle (compare [DLM]). We prefer the above formulation which in particular is is a natural approach to anisotropic variational problems. Moreover, note that (2.3) even gives the convex hull property for local minimizers of (1.1). This is proved in [BF2].*

iv) *Note that (2.2) – in complete accordance with [Bi1] – implies*

$$\mu \;<\; 3\,.$$

v) *It should be kept in mind that anisotropic power growth examples as discussed in the introduction satisfy (2.2) whenever*

$$q \;<\; 2+p\,.$$

vi) *Following Remark 1.1, iv), the results extend to the case*

$$q \;<\; \max\left\{4-\mu, (2-\mu)\,\frac{n+2}{n}\right\}.$$

Our first theorem addresses the general vectorvalued setting

THEOREM 2.3 *Let f be given as above and denote by $u \in W^1_{1,loc}(\Omega;\mathbb{R}^N)$ a local minimizer of (1.1), i.e.*

$$\int_{\widehat{\Omega}} f(\nabla u)\,dx \;<\; \infty \quad \textit{for any } \widehat{\Omega} \Subset \Omega\,, \tag{2.5}$$

$$\int_{\mathrm{spt}\,(u-v)} f(\nabla u)\,dx \;\le\; \int_{\mathrm{spt}\,(u-v)} f(\nabla v)\,dx \tag{2.6}$$

for any $v \in W^1_{1,loc}(\Omega;\mathbb{R}^N)$, $\mathrm{spt}\,(u-v) \Subset \Omega$. If u is of class $L^\infty_{loc}(\Omega;\mathbb{R}^N)$, then for any $q < s < 4-\mu$ and for any $\widehat{\Omega} \Subset \Omega$ there is a positive number c such that

$$\int_{\widehat{\Omega}} |\nabla u|^s\,dx \;\le\; c \;<\; \infty\,.$$

Given Theorem 2.3 we obtain the following corollary on partial regularity. To this purpose we follow the blow-up arguments of [BF1] which remain unchanged once a Caccioppoli-type inequality and higher local integrability of the gradient are verified. Some more details are outlined in [BF2], here we just note (see Remark 3.1) that the way of regularizing the problem is irrelevant since these ingredients are formulated in terms of the solution u.

The restriction

$$q \;<\; (2-\mu)\,\frac{n}{n-2} \quad \text{if } n \ge 3$$

is due to the properties of some auxiliary functions ψ_m as introduced in [BF1] (compare [FO]). Since our boundedness condition does not improve the Caccioppoli-type inequality, which in turn is the basis of the discussion of ψ_m, we can not expect to get rid of this assumption.

COROLLARY 2.4 *The hypotheses of Theorem 2.3 together with the condition*

$$q \ < \ \min\left\{4-\mu, (2-\mu)\frac{n}{n-2}\right\} \quad \text{if } n \geq 3$$

yield an open set $\Omega_0 \subset \Omega$ *of full measure,* $|\Omega - \Omega_0| = 0$*, such that* $u \in C^{1,\alpha}(\Omega_0; \mathbb{R}^N)$ *for any* $0 < \alpha < 1$*.*

Let us now turn our attention to scalar obstacle problems, where the general hypothesis under consideration is slightly different.

ASSUMPTION 2.5 *In the scalar case* $N = 1$ *we suppose that the energy density* $f\colon \mathbb{R}^n \to \mathbb{R}$ *is a strictly convex function of class* $C^2(\mathbb{R}^n)$ *with properties (2.2) and (2.4), where (2.1) is just assumed to be true whenever* $|Z| > 1$*.*

REMARK 2.6 *Clearly the setting is much more general in comparison to the one considered in [CH]: we do not suppose* $f(Z) = g(|Z|^2)$ *and we just assume (2.2). As a formal difference, Choe studies energy densities admitting some kind of degeneracy as* $|Z| \to 0$*. This behaviour of the second derivative is covered by the Assumption 2.5 since the validity of (2.1) is not supposed in the case* $|Z| < 1$*. We already like to remark that this causes no additional technical difficulties since in any way we have to rely on a cut off function in order to study obstacle problems.*

Now the precise formulation of our second main result reads as

THEOREM 2.7 *Let the energy density* f *be given according to Assumption 2.5 and suppose we are given a function* ψ *of class* $W^1_{\infty,loc}(\Omega)$*. Moreover, let* $u \in W^1_{1,loc}(\Omega)$*,* $u \geq \psi$ *almost everywhere, denote a local minimizer of the energy (1.1) in the sense of (2.5), (2.6), where the comparison functions are supposed to respect the obstacle as well. If* u *is of class* $L^\infty_{loc}(\Omega)$*, then we have:*

i) u *is of class* $W^1_{\infty,loc}(\Omega)$*.*

ii) *If (2.1) holds for any* $Z \in \mathbb{R}^n$ *then* u *is of class* $C^{1,\alpha}(\Omega)$ *if so is the obstacle.*

REMARK 2.8 *If we consider degenerate energy densities with* (p,q)*-growth, then local Lipschitz continuity is improved to* $C^{1,\alpha}_{loc}$*-regularity following [BFM] (compare [MUZ]). Here an additional hypothesis is needed to control the kind of degeneration of* $D^2 f$*.*

3 Regularization and Caccioppoli-type inequalities

I. Vectorvalued problems. Given Assumption 2.1 we denote by $(u)^\varepsilon$ the ε-mollification of the local minimizer u under consideration through a family of smooth mollifiers, we fix $B := B_R(x_0) \Subset \Omega$ and assume that $B \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$ for any small $\varepsilon > 0$ as above. Moreover, fix some exponent $t > \max\{2, q\}$ and let for any $\delta \in (0,1)$

$$f_\delta(Z) \;:=\; f(Z) + \delta\left(1 + |Z|^2\right)^{\frac{t}{2}}.$$

Then we define u_δ $(= u_\delta^\varepsilon)$ as the unique solution of the Dirichlet problem

$$(\mathcal{P}_\delta) \qquad J_\delta[w, B] \;:=\; \int_B f_\delta(\nabla w)\,dx \;\to\; \min\,, \qquad w \in (u)^\varepsilon_{|B} + \overset{\circ}{W}{}^1_t\left(B; \mathbb{R}^N\right).$$

REMARK 3.1 *In [BF1] the regularization was done w.r.t. $t = q$. On one hand, this choice gives some difficulties concerning starting integrability of the regularization. On the other hand, once a Caccioppoli-type inequality is established, it is not necessary to care about additional δ-terms in the case $t = q$.*

The technique outlined below relies on the condition $t > \max\{2, q\}$: the discussion of asymptotic regular integrands (compare [CE] or [GM], Theorem 5.1) includes the vectorial case and yields

$$u_\delta \in W^1_{\infty,loc}\left(B; \mathbb{R}^N\right) \cap W^2_{2,loc}\left(B; \mathbb{R}^N\right). \tag{3.1}$$

If $\delta = \delta(\varepsilon)$ is chosen sufficiently small, then we obtain using Jensen's inequality (compare [MS], [FM], [BFM], [BF1], [BF2] for detailed arguments)

$$\begin{aligned} \int_B f(\nabla u_{\delta(\varepsilon)})\,dx \;&\le\; \int_B f_{\delta(\varepsilon)}(\nabla u_{\delta(\varepsilon)})\,dx \;\le\; \int_B f_{\delta(\varepsilon)}(\nabla (u)^\varepsilon)\,dx \\ &\le\; \int_B f(\nabla u)\,dx + O(\varepsilon)\,. \end{aligned} \tag{3.2}$$

Hence, (3.2) gives together with assumptions (2.3), (2.4), DeGiorgi's lower semicontinuity theorem and strict convexity

LEMMA 3.2 *With the above notation we have*

i) $\|u_{\delta(\varepsilon)}\|_{W^1_F(B;\mathbb{R}^N)} \le$ *const* $< \infty$;

ii) $u_{\delta(\varepsilon)} \rightharpoondown u$ *in* $W^1_1(B; \mathbb{R}^N)$ *and almost everywhere as* $\varepsilon \to 0$;

iii) $\sup_B |u_\delta(\varepsilon)| \leq \sup_{B_{R+\varepsilon}(x_0)} |u| < \infty$;

iv) $\delta(\varepsilon) \int_B \big(1 + |\nabla u_{\delta(\varepsilon)}|^2\big)^{t/2}\, dx \to 0$ *as* $\varepsilon \to 0$;

v) $\int_B f(\nabla u_{\delta(\varepsilon)})\, dx \to \int_B f(\nabla u)\, dx$ *as* $\varepsilon \to 0$;

vi) $\int_B f_{\delta(\varepsilon)}(\nabla u_{\delta(\varepsilon)})\, dx \to \int_B f(\nabla u)\, dx.$

Keeping Lemma 3.2 in mind we abbreviate $\delta = \delta(\varepsilon)$ in the following. Then the first Caccioppoli-type inequality reads as (again compare, for instance, [FM], [BFM], [BF1], [BF2])

LEMMA 3.3 *Suppose we are given Assumption 2.1 in the vectorvalued setting. Then there is a real number* $c > 0$*, independent of* δ*, such that for any* $\eta \in C_0^\infty(B)$*,* $0 \leq \eta \leq 1$*,*

$$\int_B D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\, \eta^2\, dx \leq c \int_B \big|D^2 f_\delta(\nabla u_\delta)\big|\, |\nabla u_\delta|^2\, |\nabla \eta|^2\, dx\, .$$

Note that we always take the sum w.r.t. repeated Greek indices $\gamma = 1, \dots, n$ *and w.r.t. repeated Latin indices* $i = 1, \dots, N$.

Proof. Following the above references the proof is standard: on account of (3.1) we may differentiate the Euler equation and take $\varphi = \eta^2 \partial_\gamma u_\delta$ as test-function. ■

II. Scalar obstacle problems. In the setting of Assumption 2.5 and with the notation introduced above we now regularize w.r.t. $t = q$.

REMARK 3.4 *(compare Remark 3.1) If* $t = q$*, then it is much easier to handle the iteration arguments and the succeeding DeGiorgi-type reasoning — the paper [BFM] also benefits from this choice.*

Note that the problem of starting integrability disappears in the scalar case (see Lemma 3.6 below).

If $(\psi)^\varepsilon$ denotes the ε-mollification of the obstacle, then u_δ denotes the unique solution of problem

$$(\mathcal{P}_\delta^{(\psi)^\varepsilon}) \qquad J_\delta[w, B] := \int_B f_\delta(\nabla w)\, dx \to \min\, , \quad w \in \mathbb{K}_\varepsilon\, ,$$

where $\mathbb{K}_\varepsilon = \{w \in (u)^\varepsilon_{|B} + \overset{\circ}{W}{}^1_q(B) \, : \, (\psi)^\varepsilon \le w \text{ a.e.}\}$

REMARK 3.5 *In the situation at hand Lemma 3.2 remains unchanged. Moreover, on account of a.e. convergence, the limit u is seen to respect the obstacle ψ.*

The study of obstacle problems needs an additional linearization. This procedure is well known, a detailed proof of the following lemma is given in [BFM] together with a list of further references.

LEMMA 3.6 *If we consider the setting as introduced in Assumption 2.5, then u_δ is of class $W^2_t(B)$ for any $t < \infty$ and*

$$\nabla f(\nabla u_\delta) \in W^1_{t,loc}(B) \ .$$

Moreover, the equation

$$\int_B \nabla f(\nabla u_\delta) \cdot \nabla \varphi \, dx \ = \ \int_B \varphi \, g \, dx \tag{3.3}$$

is valid for any $\varphi \in C^1_0(B)$, where

$$g \ := \ \mathbf{1}_{\{x \in B : \, u_\delta = (\psi)^\varepsilon\}} \Big(- \operatorname{div} \big[\nabla f (\nabla (\psi)^\varepsilon) \big] \Big) \, .$$

Given Lemma 3.6 we end up with

LEMMA 3.7 *Suppose Assumption 2.5 to be true and fix $L > 1$ such that for any ε as above*

$$L \ > \ 1 + \|\nabla (\psi)^\varepsilon\|^2_{L^\infty(B)} \, .$$

i) *If $B_\varkappa := \{x \in B \, : \, \Gamma_\delta := 1 + |\nabla u_\delta|^2 > \varkappa\}$, $\varkappa > 1$, then there is a constant c, independent of δ, such that for any $\varkappa > L$, for any real number $s \ge 0$ and for any $\eta \in C^\infty_0(B)$, $0 \le \eta \le 1$,*

$$\begin{aligned} &\int_{B_{2\varkappa}} D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \, \Gamma^s_\delta \, \eta^2 \, dx \\ &\qquad \le \ c \int_{B_\varkappa} \big| D^2 f_\delta(\nabla u_\delta) \big| \, \Gamma^{1+s}_\delta \, |\nabla \eta|^2 \, dx \, . \end{aligned}$$

ii) *Recall that $\Gamma_\delta := 1 + |\nabla u_\delta|^2$ and denote for $0 < r < R$*

$$A(k,r) \ = \ A_\delta(k,r) \ = \ \big\{ x \in B_r(x_0) \, : \, \Gamma_\delta > k \big\} \, , \qquad k > 1 + L \, .$$

Then there is a real number $c > 0$, independent of δ, such that for any $\eta \in C_0^\infty(B_r(x_0))$, $0 \le \eta \le 1$ and for any $\delta \in (0,1)$

$$\int_{A(k,r)} \Gamma_\delta^{-\frac{\mu}{2}} \, |\nabla \Gamma_\delta|^2 \, \eta^2 \, dx \le c \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)(\nabla \eta, \nabla \eta) \, (\Gamma_\delta - k)^2 \, dx \, .$$

Proof. ad i). This time we shortly sketch the proof following the idea of [BFM], Lemma 2.3: fix $\varkappa > L$ and let for all $t \in \mathbb{R}$

$$\tilde{h}(t) \; := \; \min\big\{ \max[t-1, 0], 1 \big\} \, , \qquad h(t) \; = \; h_\varkappa(t) \; = \; \tilde{h}(\varkappa^{-1} t) \, , \tag{3.4}$$

i.e. $h(t) \equiv 0$ if $t < \varkappa$ and $h(t) \equiv 1$ if $t > 2\varkappa$. Again integrability is good enough (see Lemma 3.6) to differentiate the Euler equation (3.3) with the result

$$\int_B D^2 f_\delta(\nabla u_\delta) \Big(\partial_\gamma \nabla u_\delta, \nabla \big(\eta^2 \, \partial_\gamma u_\delta \, h(\Gamma_\delta) \, \Gamma_\delta^s \big) \Big) \, dx = - \int_B g \, \partial_\gamma \big(\eta^2 \, \partial_\gamma u_\delta \, h(\Gamma_\delta) \, \Gamma_\delta^s \big) \, dx \, .$$

On the set of coincidence we have almost everywhere $\nabla u_\delta = \nabla(\psi)^\varepsilon$, (see [GT], Lemma 7.7, p. 152) hence the auxiliary function $h(\Gamma_\delta)$ vanishes on account of $\varkappa > L$. This, together with

$$\int_B D^2 f_\delta(\nabla u_\delta) \big(\partial_\gamma u_\delta \, \partial_\gamma \nabla u_\delta, \nabla \Gamma_\delta \big) \, h'(\Gamma_\delta) \, \Gamma_\delta^s \, \eta^2 \, dx \; \ge \; 0 \, ,$$

$$s \int_B D^2 f_\delta(\nabla u_\delta) \big(\partial_\gamma u_\delta \, \partial_\gamma \nabla u_\delta, \nabla \Gamma_\delta \big) \, \Gamma_\delta^{s-1} \, h(\Gamma_\delta) \, \eta^2 \, dx \; \ge \; 0$$

(which follows from $h' \ge 0$ and $2 \partial_\gamma u_\delta \, \partial_\gamma \nabla u_\delta = \nabla \Gamma_\delta$) yields

$$\int_B D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \, h(\Gamma_\delta) \, \Gamma_\delta^s \, \eta^2 \, dx \le - \int_B D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla \eta) \, \eta \, \partial_\gamma u_\delta \, h(\Gamma_\delta) \, \Gamma_\delta^s \, dx \, .$$

Finally Young's inequality proves the claim after absorbing terms.

ad ii). Following the reasoning of [Bi1], Lemma 3.2, ii), we now have to include the obstacle condition. If we are given $k > 1 + L$ then we choose $\varphi = \eta^2 \, \partial_\gamma u_\delta \, \max\big[\Gamma_\delta - k, 0 \big]$, η as above. Again Lemma 3.6 shows the validity of the Euler equation (3.3) and its differentiated version. As before the right-hand side vanishes since k is large enough, thus

$$
\begin{aligned}
&\int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta\big)\,(\Gamma_\delta - k)\,\eta^2\,dx \\
(3.5)\qquad &\quad + \int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \nabla\Gamma_\delta\big)\,\partial_\gamma u_\delta\,\eta^2\,dx \\
&\quad = \; -2\int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \nabla\eta\big)\,\eta\,\partial_\gamma u_\delta\,(\Gamma_\delta - k)\,dx\,.
\end{aligned}
$$

Here the non-negative first integral on the left-hand side is neglected, the second one satisfies:

$$
\begin{aligned}
&\int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)\big(\partial_\gamma \nabla u_\delta, \nabla\Gamma_\delta\big)\,\partial_\gamma u_\delta\,\eta^2\,dx \\
(3.6)\qquad &\quad = \; \frac{1}{2}\int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)\big(\nabla\Gamma_\delta, \nabla\Gamma_\delta\big)\,\eta^2\,dx\,.
\end{aligned}
$$

The right-hand side of (3.5) is estimated from above via

$$
\begin{aligned}
&c\varepsilon\int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)\big(\nabla\Gamma_\delta, \nabla\Gamma_\delta\big)\,\eta^2\,dx \\
(3.7)\qquad &\quad + c\varepsilon^{-1}\int_{A(k,r)} D^2 f_\delta(\nabla u_\delta)(\nabla\eta, \nabla\eta)\,(\Gamma_\delta - k)^2\,dx\,,
\end{aligned}
$$

where we made use of Young's inequality for $\varepsilon > 0$ sufficiently small. Absorbing terms the lemma is proved by (3.5) – (3.7) and the ellipticity condition (2.1), which can be applied on account of $k > 1 + L$. ■

4 Proof of Theorem 2.3

Recalling Lemma 3.2 we obviously have established our first main result once uniform local higher integrability of the regularization is proved in the sense of

THEOREM 4.1 *Consider the vectorvalued setting and assume that f satisfies Assumption 2.1. Then for any $q < s < 4 - \mu$ and for any ball $B_r(x_0)$, $r < R$, there is a constant c just depending on the data, $sup_B|(u)^\varepsilon|$, r and s, such that*

$$
\int_{B_r(x_0)} |\nabla u_\delta|^s\,dx \;\le\; c \;<\; \infty\,.
$$

Proof. With s fixed as above it is possible to define

$$q+\mu-4 \;<\; \alpha \;:=\; s+\mu-4 \;<\; 0\,, \tag{4.1}$$

where the negative sign of α ensures that

$$\sigma \;:=\; 2+\alpha-\frac{\mu}{2} \;<\; 2+\frac{\alpha-\mu}{2} \;=:\; \sigma'\,. \tag{4.2}$$

Hence we may choose in addition $k \in \mathbb{N}$ sufficiently large satisfying

$$2\,k\,\frac{\sigma}{\sigma'} \;<\; 2k-2\,.$$

Now, given $\eta \in C_0^\infty(B)$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(x_0)$, $|\nabla \eta| \le c/(R-r)$, we recall the abbreviation $\Gamma_\delta = 1+|\nabla u_\delta|^2$ and (3.1) to justify that u_δ is smooth enough to perform the following partial integration

$$\begin{aligned}
\int_B |\nabla u_\delta|^2\,\Gamma_\delta^{1+\frac{\alpha-\mu}{2}}\,\eta^{2k}\,dx \;&=\; -\int_B u_\delta^i \cdot \nabla\Big[\nabla u_\delta^i\,\Gamma_\delta^{1+\frac{\alpha-\mu}{2}}\,\eta^{2k}\Big]\,dx\\
&\le\; c\int_B |\nabla^2 u_\delta|\,\Gamma_\delta^{1+\frac{\alpha-\mu}{2}}\,\eta^{2k}\,dx\\
&\qquad +c\int_B \Gamma_\delta^{\frac{3+\alpha-\mu}{2}}\,\eta^{2k-1}\,|\nabla\eta|\,dx\,.
\end{aligned}$$

Here we already made use of the fact that u_δ is uniformly bounded. If a positive constant M is fixed, then the left-hand side is immediately estimated via

$$\begin{aligned}
\int_B |\nabla u_\delta|^2\,\Gamma_\delta^{1+\frac{\alpha-\mu}{2}}\,\eta^{2k}\,dx \;&\ge\; c\int_{B\cap[|\nabla u_\delta|\ge M]} \Gamma_\delta^{2+\frac{\alpha-\mu}{2}}\,\eta^{2k}\,dx\\
&\ge\; c\int_B \Gamma_\delta^{2+\frac{\alpha-\mu}{2}}\,\eta^{2k}\,dx - c(M)\,,
\end{aligned}$$

therefore the starting inequality reads as

$$\begin{aligned}
\int_B \Gamma_\delta^{2+\frac{\alpha-\mu}{2}}\,\eta^{2k}\,dx \;&\le\; c\Big\{1+\int_B |\nabla^2 u_\delta|\,\Gamma_\delta^{1+\frac{\alpha-\mu}{2}}\,\eta^{2k}\,dx\\
&\qquad +\int_B \Gamma_\delta^{\frac{3+\alpha-\mu}{2}}\,\eta^{2k-1}\,|\nabla\eta|\,dx\Big\}\\
&=:\; c\big\{1+I+II\big\}\,.
\end{aligned} \tag{4.3}$$

At this point we like to emphasize that the choice (4.1) of α gives

$$2+\frac{\alpha-\mu}{2} \;=\; \frac{s}{2} \;>\; \frac{q}{2}\,. \tag{4.4}$$

Now for $\varepsilon > 0$ sufficiently small Young's inequality yields a bound for II

$$
\begin{aligned}
II \;&\le\; \varepsilon \int_B \Gamma_\delta^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx + \varepsilon^{-1} \int_B \Gamma_\delta^{-2-\frac{\alpha-\mu}{2}} \Gamma_\delta^{3+\alpha-\mu} \eta^{2k-2} |\nabla \eta|^2\, dx \\
&\le\; \varepsilon \int_B \Gamma_\delta^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx + \frac{c\varepsilon^{-1}}{(R-r)^2} \int_B \Gamma_\delta^{1+\frac{\alpha-\mu}{2}} \eta^{2k-2}\, dx\,.
\end{aligned} \tag{4.5}
$$

Note that the first integral on the right-hand side of (4.5) may be absorbed on the left-hand side of (4.3) whereas the second one remains uniformly bounded on account of Remark 2.2, ii), the uniform bound (3.2) and $\alpha < 0$. Hence, the theorem is proved if an appropriate estimate for I is found. To this purpose we derive (again $\varepsilon > 0$ is sufficiently small and Young's inequality is applied)

$$
\begin{aligned}
I \;&\le\; \varepsilon \int_B \Gamma_\delta^{-\frac{\mu}{2}} |\nabla^2 u_\delta|^2 \eta^{2k+2}\, dx + \varepsilon^{-1} \int_B \Gamma_\delta^{2+\alpha-\frac{\mu}{2}} \eta^{2k-2}\, dx \\
&=:\; \varepsilon I_1 + \varepsilon^{-1} I_2\,.
\end{aligned} \tag{4.6}
$$

Using Lemma 3.2, iv), as well as Lemma 3.3 one obtains

$$
\begin{aligned}
I_1 \;&\le\; \int_B D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta) \left(\eta^{k+1}\right)^2 dx \\
&\le\; c \int_B \left|D^2 f_\delta(\nabla u_\delta)\right| |\nabla u_\delta|^2 \eta^{2k} |\nabla \eta|^2\, dx \\
&\le\; \frac{c}{(R-r)^2} \left\{ \int_B \Gamma_\delta^{\frac{q}{2}} \eta^{2k}\, dx + \delta \int_B \Gamma_\delta^{\frac{t}{2}} \eta^{2k}\, dx \right\} \\
&\le\; \frac{c}{(R-r)^2} \left\{ 1 + \int_B \Gamma_\delta^{\frac{q}{2}} \eta^{2k}\, dx \right\}.
\end{aligned}
$$

As the result, (4.6) yields (using (4.4))

$$
I \;\le\; \frac{c\varepsilon}{(R-r)^2} \left\{ 1 + \int_B \Gamma_\delta^{2+\frac{\alpha-\mu}{2}} \eta^{2k}\, dx \right\} + \varepsilon^{-1} \int_B \Gamma_\delta^{2+\alpha-\frac{\mu}{2}} \eta^{2k-2}\, dx\,. \tag{4.7}
$$

Choosing $\varepsilon = \hat{\varepsilon}(R-r)^2$ with $\hat{\varepsilon} > 0$ sufficiently small, the first integral on the right-hand side of (4.7) may also be absorbed on the left-hand side of (4.3), hence it remains to bound the second one. Here the negative sign of α and, as a consequence, (4.2) and our choice of k come into the play. For $\tilde{\varepsilon} > 0$ sufficiently small we get with a final application of Young's inequality

$$\begin{aligned} &\hat{\varepsilon}^{-1}(R-r)^{-2}\int_B \Gamma_\delta^{2+\alpha-\frac{\mu}{2}}\,\eta^{2k-2}\,dx \\ &\quad \leq\; c\hat{\varepsilon}^{-1}(R-r)^{-2}\Big\{\tilde{\varepsilon}\int_B \Gamma_\delta^{2+\frac{\alpha-\mu}{2}}\,\eta^{2k}\,dx + \tilde{\varepsilon}^{-\frac{\sigma}{\sigma'-\sigma}}\,|B|\Big\}\,. \end{aligned}$$

Absorbing terms for $\tilde{\varepsilon} = \varepsilon'\,\hat{\varepsilon}(R-r)^2$, $1 \gg \varepsilon' > 0$, Theorem 4.1 is proved implying the validity of Theorem 2.3 as well. ∎

5 Proof of Theorem 2.7

In the case of scalar obstacle problems higher integrability is improved by an iteration argument to

THEOREM 5.1 *Assume that f satisfies Assumption 2.5 in the case $N = 1$. Then for any $1 < s < \infty$ and for any ball $B_r(x_0)$, $r < R$, there is a constant c just depending on the data, $sup_B|(u)^\varepsilon|$, r and s, such that*

$$\int_{B_r(x_0)} |\nabla u_\delta|^s\,dx \;\leq\; c \;<\; \infty\,.$$

Proof. We now fix some non-negative number $\alpha \geq 0$ and let

$$\beta \;=\; 4-\mu-q \;>\; 0\,,$$

where the positive sign follows from Assumption (2.2). As a consequence, the counterpart of (4.2) reads as

$$\sigma \;:=\; 2+\frac{\alpha-\beta-\mu}{2} \;<\; 2+\frac{\alpha-\mu}{2} \;=:\; \sigma'\,.$$

Again we may choose $k \in \mathbb{N}$ sufficiently large such that

$$2k\,\frac{\sigma}{\sigma'} \;<\; 2k-2\,.$$

Next we have to rely on the auxiliary function h as defined in (3.4). Here we define h w.r.t. $2\varkappa$, $\varkappa > L+1$, L given in Lemma 3.7. Once more, the starting inequality is derived performing a partial integration which is admissible on account of Lemma 3.6

$$\begin{aligned}\int_B |\nabla u_\delta|^2\,\Gamma_\delta^{1+\frac{\alpha-\mu}{2}}\,h(\Gamma_\delta)\,\eta^{2k}\,dx &= -\int_B u_\delta\,\nabla\Big[\nabla u_\delta\,\Gamma_\delta^{1+\frac{\alpha-\mu}{2}}\,h(\Gamma_\delta)\,\eta^{2k}\Big]\,dx\\ &\leq c\int_B |\nabla^2 u_\delta|\,\Gamma_\delta^{1+\frac{\alpha-\mu}{2}}\,h(\Gamma_\delta)\,\eta^{2k}\,dx\\ &\quad +c\int_B \Gamma_\delta^{\frac{3+\alpha-\mu}{2}}\,h(\Gamma_\delta)\,\eta^{2k-1}\,|\nabla\eta|\,dx\\ &\quad +c\int_B \Gamma_\delta^{\frac{3+\alpha-\mu}{2}}\,h'(\Gamma_\delta)\,|\nabla u_\delta|\,|\nabla^2 u_\delta|\,\eta^{2k}\,dx\,.\end{aligned}$$

This time it is supposed that $r<\rho<\rho'<R$, $\eta\in C_0^\infty(B_{\rho'}(x_0))$, $\eta\equiv 1$ on $B_\rho(x_0)$, $\nabla\eta\leq c(\rho'-\rho)^{-1}$. A lower bound for the left-hand side of this inequality is given by (compare Section 4)

$$\int_B \Gamma_\delta^{2+\frac{\alpha-\mu}{2}}\,\eta^{2k}\,dx - c(\varkappa)\,,$$

on the right-hand side we observe that $h'(\Gamma_\delta)$ identically vanishes outside the set $[2\varkappa\leq\Gamma_\delta\leq 4\varkappa]$, i.e. as an immediate consequence

$$\int_B \Gamma_\delta^{\frac{3+\alpha-\mu}{2}}\,h'(\Gamma_\delta)\,|\nabla u_\delta|\,|\nabla^2 u_\delta|\,\eta^{2k}\,dx \;\leq\; c(\varkappa)\int_{B_{2\varkappa}} |\nabla^2 u_\delta|\,\Gamma_\delta^{1+\frac{\alpha-\mu}{2}}\,\eta^{2k}\,dx\,,$$

where the definition of $B_{2\varkappa}$ is the same as introduced in Lemma 3.7. Since it is also obvious that

$$\int_B |\nabla^2 u_\delta|\,\Gamma_\delta^{1+\frac{\alpha-\mu}{2}}\,h(\Gamma_\delta)\,\eta^{2k}\,dx \;\leq\; \int_{B_{2\varkappa}} |\nabla^2 u_\delta|\,\Gamma_\delta^{1+\frac{\alpha-\mu}{2}}\,\eta^{2k}\,dx\,,$$

an analogous estimate holding for the remaining integral, we arrive at

$$\tag{5.1}\begin{aligned}\int_B \Gamma_\delta^{2+\frac{\alpha-\mu}{2}}\,\eta^{2k}\,dx &\leq c\Big\{1+\int_{B_{2\varkappa}} |\nabla^2 u_\delta|\,\Gamma_\delta^{1+\frac{\alpha-\mu}{2}}\,\eta^{2k}\,dx\\ &\qquad +\int_{B_{2\varkappa}} \Gamma_\delta^{\frac{3+\alpha-\mu}{2}}\,\eta^{2k-1}\,|\nabla\eta|\,dx\Big\}\\ &=: c\big\{1+I+II\big\}\,.\end{aligned}$$

Now, given $\varepsilon>0$ sufficiently small, II is handled in the same manner as in Section 4

$$\tag{5.2} II \;\leq\; \varepsilon\int_B \Gamma_\delta^{2+\frac{\alpha-\mu}{2}}\,\eta^{2k}\,dx + \frac{c\varepsilon^{-1}}{(\rho'-\rho)^2}\int_B \Gamma_\delta^{1+\frac{\alpha-\mu}{2}}\,\eta^{2k-2}\,dx\,,$$

where the first integral can be absorbed on the left-hand side of (5.1). For the discussion of I we first observe that

$$
\begin{aligned}
I \;\le\; & \varepsilon \int_{B_{2\varkappa}} \Gamma_\delta^{\frac{\alpha+\beta}{2}}\, \Gamma_\delta^{-\frac{\mu}{2}}\, |\nabla^2 u_\delta|^2\, \eta^{2k+2}\, dx \\
& +\varepsilon^{-1} \int_{B_{2\varkappa}} \Gamma_\delta^{2+\frac{\alpha-\beta-\mu}{2}}\, \eta^{2k-2}\, dx \;=:\; \varepsilon I_1 + \varepsilon^{-1} I_2\, .
\end{aligned}
$$

Then we have to check that I_1 can be handled via Lemma 3.7, i): by definition it is clear that $\alpha+\beta \ge 0$. Moreover, the choice of $\varkappa$ verifies Assumption (2.1) on the set $B_\varkappa$ (recall that in the situation at hand we only have (2.1) whenever $|Z|>1$), hence one gets

$$
\begin{aligned}
I_1 \;\le\; & c \int_{B_{2\varkappa}} D^2 f_\delta(\nabla u_\delta)(\partial_\gamma \nabla u_\delta, \partial_\gamma \nabla u_\delta)\, \Gamma_\delta^{\frac{\alpha+\beta}{2}} \left(\eta^{k+1}\right)^2 dx \\
\le\; & c \int_{B_\varkappa} \left|D^2 f_\delta(\nabla u_\delta)\right| \Gamma_\delta^{1+\frac{\alpha+\beta}{2}}\, \eta^{2k}\, |\nabla \eta|^2\, dx \\
\le\; & \frac{c}{(\rho'-\rho)^2} \int_B \Gamma_\delta^{\frac{\alpha+\beta}{2}}\, \Gamma_\delta^{\frac{q}{2}}\, \eta^{2k}\, dx\, .
\end{aligned}
$$

For the last inequality we like to recall that the regularization was done w.r.t. $t=q$. Finally, the choice of β implies

$$
I \;\le\; \frac{c\,\varepsilon}{(\rho'-\rho)^2} \int_B \Gamma_\delta^{2+\frac{\alpha-\mu}{2}}\, \eta^{2k}\, dx + \varepsilon^{-1} \int_B \Gamma_\delta^{2+\frac{\alpha-\beta-\mu}{2}}\, \eta^{2k-2}\, dx\, . \tag{5.3}
$$

If again $\varepsilon = \hat{\varepsilon}(\rho'-\rho)^2$ and if $\hat{\varepsilon}>0$ is sufficiently small, then we argue exactly as in Section 4, i.e. the first integral on the right-hand side of (5.3) is absorbed on the left-hand side of (5.1) whereas

$$
\begin{aligned}
& \hat{\varepsilon}^{-1} (\rho'-\rho)^{-2} \int_B \Gamma_\delta^{2+\frac{\alpha-\beta-\mu}{2}}\, \eta^{2k-2}\, dx \\
& \quad \le\; c\,\hat{\varepsilon}^{-1} (\rho'-\rho)^{-2} \left\{ \tilde{\varepsilon} \int_B \Gamma_\delta^{2+\frac{\alpha-\mu}{2}}\, \eta^{2k}\, dx + \tilde{\varepsilon}^{-\frac{\sigma}{\sigma'-\sigma}}\, |B| \right\}.
\end{aligned} \tag{5.4}
$$

Following (5.1)–(5.4), letting $\tilde{\varepsilon} = \varepsilon' \hat{\varepsilon} (\rho'-\rho)^2$, $1 \gg \varepsilon' > 0$ and absorbing terms for a last time we have found a real number $c = c(\varkappa, \alpha, \rho'-\rho, \sup_B |(u)^\varepsilon|)$, independent of δ, such that

$$
\int_B \Gamma_\delta^{2+\frac{\alpha-\mu}{2}}\, \eta^{2k}\, dx \;\le\; c \left\{ 1 + \int_B \Gamma_\delta^{1+\frac{\alpha-\mu}{2}}\, \eta^{2k-2}\, dx \right\}. \tag{5.5}
$$

To start an iteration of (5.5) let

$$\rho_m \;=\; r + (R-r)\,2^{-m}\,, \qquad m = 0,1,2,\dots\,,$$

as well as

$$\alpha_m \;=\; 2m\,, \qquad \text{i.e.} \qquad \alpha_{m+1} \;=\; 2+\alpha_m\,, \qquad m = 0,1,2,\dots\,,$$

where for any m as above α_m is non-negative, hence admissible in the above calculations. Then we obtain (5.5) for any $m = 0,1,2,\dots$, with the choice $\rho = \rho_{m+1}$, $\rho' = \rho_m$, $\alpha = \alpha_m$, i.e.

$$\int_{B_{\rho_{m+1}}(x_0)} \Gamma_\delta^{1+\frac{\alpha_{m+1}-\mu}{2}}\, dx \;\le\; c\left\{1 + \int_{B_{\rho_m}(x_0)} \Gamma_\delta^{1+\frac{\alpha_m-\mu}{2}}\, dx\right\}.$$

Iteration completes the proof since $\alpha_0 = 0$ gives a uniformly bounded right-hand side (once more compare Remark 2.2, ii). ■

Given Theorem 5.1 one may apply a Moser-type iteration (as done in [CH]) to obtain uniform local apriori gradient bounds. We pefer DeGiorgi-type arguments (similar to [Bi1]) which seem to be more convenient in the setting of "bad" ellipticity, moreover, the side condition is easily eliminated.

THEOREM 5.2 *Consider a ball $B_{R_0}(x_0) \Subset B$ and an energy density as classified in Assumption 2.5. Then there is a local constant $c > 0$ such that for any $\delta \in (0,1)$*

$$\|\nabla u_\delta\|_{L^\infty(B_{R_0/2},\mathbb{R}^n)} \;\le\; c\,.$$

Before proving Theorem 5.2 we have to establish an auxiliary Lemma which is shown in [Bi1], Lemma 6.2, in the case $q = 2$.

LEMMA 5.3 *Suppose $0 < r < \hat{r} < R_0$ such that $B_{R_0}(x_0) \Subset B$. Then there is a real number c, independent of r, $\hat{r}$, R_0, k and δ, satisfying for any $k > 1 + L$ (L as above)*

$$\begin{aligned}
&\int_{A(k,r)} (\Gamma_\delta - k)^{\frac{n}{n-1}}\, dx \\
(5.6)\qquad &\le\; \frac{c}{(\hat{r}-r)^{\frac{n}{n-1}}}\left[\int_{A(k,\hat{r})} \Gamma_\delta^{\frac{q-2}{2}}\,(\Gamma_\delta - k)^2\, dx\right]^{\frac{1}{2}\frac{n}{n-1}} \left[\int_{A(k,\hat{r})} \Gamma_\delta^{\frac{\mu}{2}}\, dx\right]^{\frac{1}{2}\frac{n}{n-1}},
\end{aligned}$$

where the sets $A(k,r) = \{x \in B_r(x_0) : \Gamma_\delta > k\}$ are introduced in Lemma 3.7.

Proof of Lemma 5.3. With the notion $w^+ = \max[w, 0]$, Sobolev's inequality yields for $\eta \in C_0^\infty(B_{\hat r}(x_0))$, $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_r(x_0)$, $|\nabla\eta| \le c/(\hat r - r)$,

$$\begin{aligned}\int_{A(k,r)} \left(\Gamma_\delta - k\right)^{\frac{n}{n-1}} dx &\le \int_{B_{\hat r}(x_0)} \left[\eta\left(\Gamma_\delta - k\right)^+\right]^{\frac{n}{n-1}} dx\\ &\le c\left[\int_{B_{\hat r}(x_0)} \left|\nabla\left[\eta\left(\Gamma_\delta - k\right)^+\right]\right| dx\right]^{\frac{n}{n-1}}\\ &\le c\left[\int_{A(k,\hat r)} \left|\nabla\left[\eta\left(\Gamma_\delta - k\right)\right]\right| dx\right]^{\frac{n}{n-1}}\\ &\le c\left[I_1^{\frac{n}{n-1}} + I_2^{\frac{n}{n-1}}\right].\end{aligned}\tag{5.7}$$

The first integral on the right-hand side is handled via (recall $2 - \mu < q$)

$$\begin{aligned}I_1^{\frac{n}{n-1}} &= \left[\int_{A(k,\hat r)} |\nabla\eta|\left(\Gamma_\delta - k\right) dx\right]^{\frac{n}{n-1}}\\ &\le \left[\int_{A(k,\hat r)} |\nabla\eta|^2\,\Gamma_\delta^{\frac{q-2}{2}}\left(\Gamma_\delta - k\right)^2 dx\right]^{\frac12\frac{n}{n-1}} \left[\int_{A(k,\hat r)} \Gamma_\delta^{\frac{2-q}{2}}\, dx\right]^{\frac12\frac{n}{n-1}}\\ &\le \frac{c}{(\hat r - r)^{\frac{n}{n-1}}}\left[\int_{A(k,\hat r)} \Gamma_\delta^{\frac{q-2}{2}}\left(\Gamma_\delta - k\right)^2 dx\right]^{\frac12\frac{n}{n-1}} \left[\int_{A(k,\hat r)} \Gamma_\delta^{\frac{\mu}{2}}\, dx\right]^{\frac12\frac{n}{n-1}},\end{aligned}$$

thus $I_1^{\frac{n}{n-1}}$ is seen to be bounded from above by the right-hand side of (5.6). Estimating I_2 we recall the choice $k > 1 + L$, hence it is possible to refer to Lemma 3.7, ii), with the result

$$\begin{aligned}I_2^{\frac{n}{n-1}} &= \left[\int_{A(k,\hat r)} \eta\,|\nabla\Gamma_\delta|\, dx\right]^{\frac{n}{n-1}}\\ &\le \left[\int_{A(k,\hat r)} \eta^2\,|\nabla\Gamma_\delta|^2\,\Gamma_\delta^{-\frac{\mu}{2}}\, dx\right]^{\frac12\frac{n}{n-1}} \left[\int_{A(k,\hat r)} \Gamma_\delta^{\frac{\mu}{2}}\, dx\right]^{\frac12\frac{n}{n-1}}\\ &\le c\left[\int_{A(k,\hat r)} D^2 f_\delta(\nabla u_\delta)\big(\nabla\eta,\nabla\eta\big)\left(\Gamma_\delta - k\right)^2\right]^{\frac12\frac{n}{n-1}} \left[\int_{A(k,\hat r)} \Gamma_\delta^{\frac{\mu}{2}}\, dx\right]^{\frac12\frac{n}{n-1}}\\ &\le \frac{c}{(\hat r - r)^{\frac{n}{n-1}}}\left[\int_{A(k,\hat r)} \Gamma_\delta^{\frac{q-2}{2}}\left(\Gamma_\delta - k\right)^2 dx\right]^{\frac12\frac{n}{n-1}} \left[\int_{A(k,\hat r)} \Gamma_\delta^{\frac{\mu}{2}}\, dx\right]^{\frac12\frac{n}{n-1}}.\end{aligned}$$

and the lemma follows from (5.7). ■

Proof of Theorem 5.2. Again we have to modify the reasoning of [Bi1]. Starting with the left-hand side of (5.6) we fix a real number $s > 1$ and observe that Hölder's inequality implies

$$\begin{aligned}\int_{A(k,r)} \Gamma_\delta^{\frac{q-2}{2}} \big(\Gamma_\delta - k\big)^2\, dx &= \int_{A(k,r)} \big(\Gamma_\delta - k\big)^{\frac{n}{n-1}\frac{1}{s}} \Gamma_\delta^{\frac{q-2}{2}} \big(\Gamma_\delta - k\big)^{2-\frac{n}{n-1}\frac{1}{s}}\, dx \\ &\le \left[\int_{A(k,r)} \big(\Gamma_\delta - k\big)^{\frac{n}{n-1}}\, dx\right]^{\frac{1}{s}} \\ &\quad \times \left[\int_{A(k,r)} \Gamma_\delta^{\frac{q-2}{2}\frac{s}{s-1}} \big(\Gamma_\delta - k\big)^{\left(2-\frac{n}{n-1}\frac{1}{s}\right)\frac{s}{s-1}}\, dx\right]^{\frac{s-1}{s}} .\end{aligned}$$

Theorem 5.1 ensures the existence of a real number $c_1(s, n, B_{R_0}(x_0))$, independent of δ,

$$c_1(s, n, B_{R_0}(x_0)) := \sup_{\delta > 0} \left[\int_{B_{R_0}(x_0)} \Gamma_\delta^{\frac{s}{s-1}\left(\frac{q-2}{2}+2-\frac{n}{n-1}\frac{1}{s}\right)}\, dx\right]^{\frac{s-1}{s}} < \infty ,$$

such that

$$\int_{A(k,r)} \Gamma_\delta^{\frac{q-2}{2}} \big(\Gamma_\delta - k\big)^2\, dx \le c_1 \left[\int_{A(k,r)} \big(\Gamma_\delta - k\big)^{\frac{n}{n-1}}\, dx\right]^{\frac{1}{s}} . \tag{5.8}$$

In a similar way one obtains

$$\int_{A(k,\hat{r})} \Gamma_\delta^{\frac{\mu}{2}}\, dx \le c_2(t, \mu, B_{R_0}(x_0)) \left[\int_{A(k,\hat{r})} \Gamma_\delta^{\frac{q-2}{2}}\, dx\right]^{\frac{1}{t}} , \tag{5.9}$$

where $t > 1$ is a fixed second parameter. Combining (5.6), (5.8) and (5.9) it is proved that

$$\begin{aligned}\int_{A(k,r)} \Gamma_\delta^{\frac{q-2}{2}} \big(\Gamma_\delta - k\big)^2\, dx &\le \frac{c}{(\hat{r} - r)^{\frac{n}{n-1}\frac{1}{s}}} \left[\int_{A(k,\hat{r})} \Gamma_\delta^{\frac{q-2}{2}} \big(\Gamma_\delta - k\big)^2\, dx\right]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}} \\ &\quad \times \left[\int_{A(k,\hat{r})} \Gamma_\delta^{\frac{q-2}{2}}\, dx\right]^{\frac{1}{2}\frac{n}{n-1}\frac{1}{s}\frac{1}{t}} .\end{aligned} \tag{5.10}$$

For $k > 1 + L$ and $r < \hat{r}$ as above let

$$\tau(k,r) := \int_{A(k,r)} \Gamma_\delta^{\frac{q-2}{2}} \big(\Gamma_\delta - k\big)^2\, dx , \quad a(k,r) := \int_{A(k,r)} \Gamma_\delta^{\frac{q-2}{2}}\, dx ,$$

hence (5.10) can be rewritten as

$$\tag{5.11} \tau(k,r) \;\le\; \frac{c}{(\hat r - r)^{\frac{n}{n-1}\frac{1}{s}}}\,\big[\tau(k,\hat r)\big]^{\frac12\frac{n}{n-1}\frac1s}\,\big[a(k,\hat r)\big]^{\frac12\frac{n}{n-1}\frac1s\frac1t}\,.$$

Given two real numbers $h > k > 1 + L$ we have the immediate estimate

$$a(h,\hat r) \;\le\; \frac{1}{(h-k)^2}\,\tau(k,\hat r)\,,$$

hence together with (5.11)

$$\begin{aligned}\tau(h,r) \;&\le\; \frac{c}{(\hat r - r)^{\frac{n}{n-1}\frac1s}}\,\big[\tau(h,\hat r)\big]^{\frac12\frac{n}{n-1}\frac1s}\,\frac{1}{(h-k)^{\frac{n}{n-1}\frac1s\frac1t}}\,\big[\tau(k,\hat r)\big]^{\frac12\frac{n}{n-1}\frac1s\frac1t}\\ &\le\; \frac{c}{(\hat r - r)^{\frac{n}{n-1}\frac1s}}\,\frac{1}{(h-k)^{\frac{n}{n-1}\frac1s\frac1t}}\,\big[\tau(k,\hat r)\big]^{\frac12\frac{n}{n-1}\frac1s\left(1+\frac1t\right)}\,.\end{aligned}$$

Finally s and t are chosen sufficiently close to 1 (depending on n) such that

$$\frac12\,\frac{n}{n-1}\,\frac1s\left[1+\frac1t\right] \;=:\; \beta \;>\; 1\,,$$

moreover we let

$$\alpha \;:=\; \frac{n}{n-1}\,\frac1s\,\frac1t \;>\; 0\,,\qquad \gamma \;:=\; \frac{n}{n-1}\,\frac1s \;>\; 0\,.$$

Then Theorem 5.2 is an immediate application of the following well known lemma (compare, for instance, [ST], Lemma 5.1, p. 219) to the function $\tau(h,r)$, where we once more benefit from Theorem 5.1. ∎

LEMMA 5.4 *Assume that $\varphi(h,\rho)$ is a non-negative real valued function defined for $h > k_0$ and $\rho < R_0$. Suppose further that for fixed ρ the function is non-increasing in h and that is non-decreasing in ρ if h is fixed. Then*

$$\varphi(h,\rho) \;\le\; \frac{C}{(h-k)^\alpha (R-\rho)^\gamma}\,\big[\varphi(k,R)\big]^\beta\,,\quad h > k > k_0\,,\quad \rho < R < R_0\,,$$

with some positive constants C, α, $\beta > 1$, γ, implies for all $0 < \sigma < 1$

$$\varphi\big(k_0 + d, R_0 - \sigma R_0\big) \;=\; 0\,,$$

where the quantity d is given by

$$d^\alpha \;=\; \frac{2^{(\alpha+\beta)\beta/(\beta-1)}\,C\,\big[\varphi(k_0,R_0)\big]^{\beta-1}}{\sigma^\gamma R_0^\gamma}\,.$$

Once it is noticed that the data of the obstacle just enter through the constant L, the **Proof of Theorem 2.7**, i), is an immediate consequence of Lemma 3.2 and Remark 3.5. Now that i) is established, the second assertion follows from the well known paper [MUZ] (compare also [FM] for details). ■

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