

On the Construction of Abstract Voronoi Diagrams¹

by

K. Mehlhorn²

St. Meiser²

C. Ó'Dúnlaing³

A 01/89

Abstract: We show that the abstract Voronoi diagram of n sites in the plane can be constructed in time $O(n \log n)$ by a randomized algorithm. This yields an alternative, but simpler, $O(n \log n)$ algorithm in many previously considered cases and the first $O(n \log n)$ algorithm in some cases, e.g., disjoint convex sites with the Euclidean distance function. Abstract Voronoi diagrams are given by a family of bisecting curves and were recently introduced by Klein [Kl88a]. Our algorithm is based on Clarkson and Shor's randomized incremental construction technique [CS].

Key words: Voronoi diagrams, randomized algorithms

¹ This work was supported partially by the DFG, grant SPP Me 620/6, and partially by the ESPRIT II Basic Research Actions Program of the EC under contract No. 3075 (project ALCOM)

² FB Informatik, Universität des Saarlandes, D-6600 Saarbrücken, West Germany

³ Department of Mathematics, Trinity College Dublin, Dublin, Ireland

1. Introduction

The Voronoi diagram of a set of sites in the plane partitions the plane into regions, called Voronoi regions, one to a site. The Voronoi region of a site s is the set of points in the plane for which s is the closest site among all the sites.

The Voronoi diagram has many applications in diverse fields, cf. Leven/Sharir [LS86] or Aurenhammer [A88b] for a list of applications and a history of Voronoi diagrams. Different types of diagrams result from considering different notions of distance, e.g., Euclidean or L_p -norm or convex distance functions, and different sorts of sites, e.g., points, line segments, or circles; cf. also section IV. For many types of diagrams efficient construction algorithms have been found; these are either based on the divide-and-conquer technique due to Shamos/Hoey [SH], the sweepline technique due to Fortune [F87] or geometric transforms due to Brown [Br] and Edelsbrunner/Seidel [ES].

A unifying approach to Voronoi diagrams was recently proposed by Klein [Kl88a]. He does not use the concept of distance as the basic notion but rather the concept of bisecting curves, i.e., he assumes for each pair $\{p, q\}$ of sites the existence of a bisecting curve $J(p, q)$ which divides the plane into a p -region and a q -region. The intersection of all p -regions for different q 's is then the Voronoi-region of site p . He also postulates that Voronoi-regions are simply-connected and partition the plane. He shows that abstract Voronoi diagrams have already many of the properties of concrete Voronoi diagrams, cf. section II. He also shows that the divide-and-conquer technique can be used to construct abstract diagrams efficiently. More precisely, if the basic geometric operations on bisecting curves take time $O(1)$ and if any set S of sites can be split in time $O(|S|)$ into about equal sized subsets L and R such that the bisector between L and R (= the common boundary of regions in L with regions in R) is acyclic then the Voronoi diagrams of L and R can be merged in time $O(|S|)$ and hence the diagram of n sites can be constructed in time $O(n \log n)$. Klein's result subsumes many of the previous results and goes far beyond them. There are, however, situations, e.g., circle sites under Euclidean distance, where it is not known how to determine L and R in the divide-and-conquer algorithm such that their bisector is acyclic; cf. Sharir [S].

The purpose of this paper is to show that there is an $O(n \log n)$ randomized algorithm for constructing (a subset of Klein's) abstract Voronoi diagrams even without the acyclicity assumption. The subset is defined by the following two general position assumptions: We do not allow bisecting curves to touch but require that all intersections are crossings and that no four bisecting curves go through a common point.

The algorithm is given in section III and applications can be found in section IV. In many concrete situations, e.g., point sites with Euclidean distance function, our algorithm is just another $O(n \log n)$ algorithm, albeit simpler. There are however at least two cases where we achieve $O(n \log n)$ for the first time: For disjoint convex sites the best deterministic algorithm runs in time $O(n(\log n)^2)$ [LS86] and for line segments under the Hausdorff metric, i.e., a point x and a line segment $s = \overline{s_1 s_2}$ have distance $\max(|x - s_1|, |x - s_2|)$, an $O(n \log n)$ algorithm was only known in the special case of so-called α -disjoint segments [A88b]. We also want to stress that the new algorithm is uniform in the sense that only a small number of primitives, cf. section II, are problem specific.

Our algorithm is based on Clarkson and Shor's randomized incremental construction technique [CS]. The idea is to construct the abstract Voronoi diagram of a set S of sites incrementally by adding site after site in random order. When $R \subseteq S$ is the current set of sites, the

Voronoi diagram $V(R)$ and a conflict graph $G(R)$ is maintained. The conflict graph contains all pairs $\{e, t\}$, where e is an edge of $V(R)$ and $t \in S - R$ is a site still to be considered, such that addition of site t causes the edge e to be removed (either completely or partially) from the diagram. In order to make Clarkson and Shor's method applicable one has to show that for a site $s \in S - R$ the diagram $V(R \cup \{s\})$ and the conflict graph $G(R \cup \{s\})$ can be constructed from $V(R)$ and $G(R)$ in time

$$O\left(\sum_{\{e, s\} \in G(R)} \deg_{G(R)}(e)\right)$$

where $\deg_{G(R)}(e)$ is the degree of e in $G(R)$ and the summation is over all edges e of $V(R)$ which conflict with the new site s . This is the content of Theorem 1 of section III. If the method is applicable the expected running time is

$$O(n + m(n) + n \cdot \sum_{1 \leq r \leq n/2} m(r)/r^2)$$

where $m(r)$ is the expected number of edges in $V(R)$. For abstract diagrams $m(r) \leq 3r$ and hence the algorithm runs in time $O(n \log n)$.

Throughout we use the following notation:

For a subset $X \subseteq \mathbb{R}^2$ the closure, boundary and interior of X are denoted by $cl X$, $bd X$ and $int X$ respectively.

II. Abstract Voronoi Diagrams

Let $n \in \mathbb{N}$, and for each pair of integers p, q such that $1 \leq p \neq q \leq n$ let $D(p, q)$ be either empty or an open unbounded subset of \mathbb{R}^2 and let $J(p, q)$ be the boundary of $D(p, q)$. We postulate:

- 1) $J(p, q) = J(q, p)$ and for each p, q such that $p \neq q$ the regions $D(p, q)$, $J(p, q)$ and $D(q, p)$ form a partition of \mathbb{R}^2 into three disjoint sets.
- 2) If $\emptyset \neq D(p, q) \neq \mathbb{R}^2$ then $J(p, q)$ is homeomorphic to the open interval $(0, 1)$.

We call $J(p, q)$ the bisecting curve for sites p and q . The abstract Voronoi diagram is now defined as follows:

Definition (R. Klein [Kl88a]):

- a) Let $S = \{1, \dots, n-1\}$ and

$$R(p, q) := \begin{cases} D(p, q) \cup J(p, q) & \text{if } p < q \\ D(p, q) & \text{if } p > q \end{cases}$$

$$VR(p, S) := \bigcap_{\substack{q \in S \\ q \neq p}} R(p, q)$$

$$V(S) := \bigcup_{p \in S} bd VR(p, S)$$

$VR(p, S)$ is called the Voronoi region of p w.r.t. S and $V(S)$ is called the Voronoi diagram of S .

b) We postulate that the Voronoi regions and the bisecting curves satisfy the following two conditions:

- 1) Any two bisecting curves have only a finite number of points in common. Any point in common to two bisecting curves is a proper crossing between the two curves, cf. Figure 1.
- 2) For any non-empty subset S' of S
 - A) if $VR(p, S')$ is non-empty then $VR(p, S')$ is path-connected and has non-empty interior for each $p \in S'$,
 - B) $\mathbb{R}^2 = \bigcup_{p \in S'} VR(p, S')$ (disjoint) □

Remark 1: Klein's definition is actually more liberal. He allows that bisecting curves may touch and only requires that their intersection consists of finitely many connected components. In 2A) he postulates that each $VR(p, S')$ is non-empty. The weaker assumption made here does not harm his theory. □

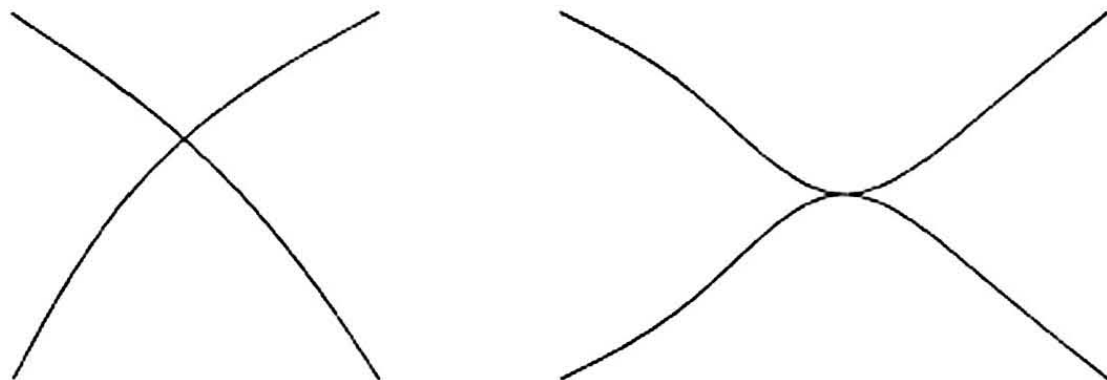


Figure 1. A crossing and touching point

Fact 1 (R. Klein [Kl88c]):

- a) Voronoi regions are simply connected.
- b) The following holds for each point $v \in V(S)$: There are arbitrarily small neighborhoods U of v that have the following properties. Let $VR(p_1, S), VR(p_2, S), \dots, VR(p_k, S)$ be the sequence of Voronoi regions traversed on a counterclockwise march around the boundary of U and let I_1, I_2, \dots, I_k denote the corresponding intervals of ∂U , where $I_j = (w_j, w_{j+1}) \subseteq VR(p_j, S)$ for $1 \leq j \leq k$ (indices must be read mod k). The intervals may be open, half-open or closed. We have $w_j \neq w_{j+1}$ for $1 \leq j \leq k$. The common boundary of $VR(p_{j-1}, S)$ and $VR(p_j, S)$ defines a curve segment $\beta_j \subseteq J(p_{j-1}, p_j)$ connecting v and w_j . $V(S) \cap U$ is the union of the curve segments β_j together with the point v . Each β_j is contained in the Voronoi region of $\min\{p_{j-1}, p_j\}$. The open "piece of pie" bordered by β_j, β_{j+1} and I_j belongs to $VR(p_j, S)$. The point v belongs to the region of $\min\{p_1, \dots, p_k\}$. Finally, $p_i \neq p_j$ for $i \neq j$. □

For the sequel, it is helpful to restrict attention to the "finite part" of $V(S)$. Let Γ be a simple closed curve such that all intersections between bisecting curves lie in the inner domain of Γ . We add a site ∞ to S , define $J(p, \infty) = J(\infty, p) = \Gamma$ for all p , $1 \leq p < n$, and $D(\infty, p)$ to be the outer domain of Γ for each p , $1 \leq p < n$.

Fact 2 (R. Klein [Kl88c]):

The boundary of each non-empty Voronoi region is a simple closed curve. Moreover, the closure of each non-empty Voronoi region $VR(p, S)$, $p \neq \infty$, is homeomorphic to a closed disc. A Voronoi diagram can be represented as a planar graph in a natural way. The vertices of the graph are the points of $V(S)$ which belong to the boundary of three or more Voronoi regions and the edges of the graph correspond to the maximal connected subsets of $V(S)$ belonging to the boundary of exactly two Voronoi regions. The faces of the graph correspond to the non-empty Voronoi regions. We use $V(S)$ to also denote this graph. For the algorithmic treatment of Voronoi diagrams we also need to make a feasibility assumption about the bisecting curves.

Definition (R. Klein):

The following operations on bisecting curves are assumed to take time $O(1)$.

- 1) Given $J(p, q)$ and a point v , determine if $v \in D(p, q)$ holds.
- 2) Given a point v in common to three bisecting curves, determine the clockwise order of the curves around v .
- 3) Given points $v \in J(p, q)$ and $w \in J(p, r)$ and orientations of these curves, determine the first point of $J(p, r)|_{(w, \infty]}$ crossed by $J(p, q)|_{(v, \infty]}$.
- 4) Given $J(p, q)$ with an orientation, and points v, w, x on $J(p, q)$, determine if v comes before w on $J(p, q)|_{(x, \infty]}$. \square

For simplicity we also make the following general position assumption.

General Position Assumption: No four bisecting curves have a point in common. \square

The general position assumption and Fact 1 imply that each vertex of the Voronoi diagram has degree three. It lies at the intersection of three bisecting curves as shown in Figure 2.

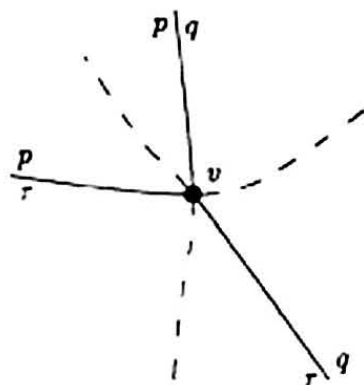


Figure 2. The bisecting curves $J(p, q)$, $J(p, r)$, $J(r, q)$ intersect at v . The domains $D(p, q)$ and $D(q, p)$ are indicated by the letters p and q on the two sides of the bisecting curve $J(p, q)$. The parts of the bisecting curves which define region boundaries are shown solid.

Remark 2: The requirement that the Voronoi regions partition the plane is a severe restriction on the family of bisecting curves. Consider a crossing of $J(r, p)$ and $J(r, q)$ as in Figure 3. Then $J(p, q)$ must also pass through v with $D(q, p)$ on its right.

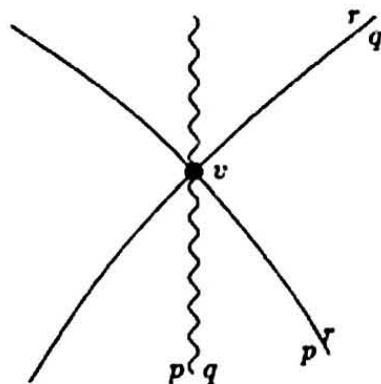


Figure 3.

We close this section with a simple but important property of Voronoi edges:

Lemma 1: Let $R \subseteq S$ and $t \in S - R$. Let e be an edge of $V(R)$ which separates the regions $VR(p, R)$ and $VR(q, R)$ of the two sites $p, q \in R$. Then $e \cap VR(t, R \cup \{t\}) = e \cap VR(t, \{p, q, t\})$.

Proof: \subseteq : This follows immediately from $VR(t, R \cup \{t\}) \subseteq VR(t, \{p, q, t\})$.

\supseteq : Let $x \in e \cap VR(t, \{p, q, t\})$. From $x \in e$ we conclude $x \in VR(p, R) \cup VR(q, R)$ and hence $x \notin VR(r, R) \supseteq VR(r, R \cup \{t\})$ for any $r \in R - \{p, q\}$. From $x \in VR(t, \{p, q, t\})$ we conclude $x \notin VR(p, \{p, q, t\}) \cup VR(q, \{p, q, t\}) \supseteq VR(p, R \cup \{t\}) \cup VR(q, R \cup \{t\})$. Thus $x \notin VR(r, R \cup \{t\})$ for any $r \in R$ and hence $x \in VR(t, R \cup \{t\})$. \square

Informally, Lemma 1 states that the influence of a site on a given edge depends only on the sites defining this particular edge.

III. Incremental Construction of Abstract Voronoi Diagrams

In this section we describe the incremental construction algorithm. We start with three sites ∞, p, q where p and q are chosen at random and then add the remaining sites in random order. At the general step we have to consider a set $R \subseteq S$ of sites with $\infty \in R$ and $|R| \geq 3$. We maintain the following data structures.

- 1) The Voronoi diagram $V(R)$: It is stored as a planar graph as described in the previous section.
- 2) The conflict graph $G(R)$: The vertices of the conflict graph $G(R)$ are the edges of $V(R)$ and the sites in $S - R$. There is an edge (read: conflict) between the edge e of $V(R)$ and the site $s \in S - R$ iff $e \cap VR(s, R \cup \{s\}) \neq \emptyset$.

Remark: Recall that an edge of a Voronoi diagram is an open set and that a Voronoi region may contain part of its boundary. For the definition of conflict graph it is however immaterial whether we intersect open sets or their closures.

Lemma 2: $cl\ e \cap cl\ VR(s, R \cup \{s\}) \neq \emptyset$ implies $e \cap VR(s, R \cup \{s\}) \neq \emptyset$.

Proof: Let $x \in cl\ e \cap cl\ VR(s, R \cup \{s\})$. Assume first that x is an endpoint of e . Then x lies at the intersection of three bisecting curves of sites in R . Hence no bisecting curve $J(s, r)$, $r \in R$, can go through x and therefore an entire neighborhood of x must belong to $VR(s, R \cup \{s\})$. Thus $e \cap VR(s, R \cup \{s\}) \neq \emptyset$.

Assume next that $x \in e \cap bd\ VR(s, R \cup \{s\})$. Then $x \in J(p, q) \cap J(s, r)$ for some sites $p, q, r \in R$. The bisecting curves $J(p, q)$ and $J(s, r)$ cross at point x and hence there is a point $y \in e$ in the neighborhood of x such that $y \in VR(s, R \cup \{s\})$. \square

We next discuss how to update the data structures after the addition of a site $s \in S - R$ to R . We first concentrate on the construction of the Voronoi diagram $V(R \cup \{s\})$ from $V(R)$ and $G(R)$.

Let $S = VR(s, R \cup \{s\})$. We proceed in several steps. Lemma 3 deals with the case $S = \emptyset$. The case $S \neq \emptyset$ is dealt with in Lemmas 4 and 5. We show that the intersection of the current diagram $V(R)$ with the region S is a connected set (Lemma 4) and that the intersection $e \cap S$ for an edge e of $V(R)$ consists of at most two components (Lemma 5). From Lemmas 4 and 5 we derive the update algorithm.

Lemma 3: $S = \emptyset$ iff $deg_{G(R)}(s) = 0$.

Proof: If $S = \emptyset$ then clearly $deg_{G(R)}(s) = 0$. So let us assume $S \neq \emptyset$. If $deg_{G(R)}(s) = 0$ then $cl\ S \subseteq int\ VR(r, R)$ for some $r \in R$. Next observe that $VR(r, R \cup \{s\}) = VR(r, R) - S$. Also $r \neq \infty$ since $VR(\infty, P)$ is the outer domain of the closed curve Γ for all P , $\infty \in P \subseteq S$. Thus $VR(r, R \cup \{s\})$ is bounded but not simply connected. This contradicts Fact 1a. \square

If $S = \emptyset$ then $V(R \cup \{s\}) = V(R)$. So let us assume $S \neq \emptyset$ and hence $deg_{G(R)}(s) \neq 0$. Let $I = V(R) \cap cl\ S$.

Lemma 4: I is a connected set which intersects $bd\ S$ in at least two points.

Proof: The boundary $bd\ S$ is a simple closed curve which does not go through any vertex of $V(R)$. This follows from Fact 2 and the general position assumption. Also $I \neq \emptyset$ by Lemma 2. Let I_1, I_2, \dots, I_k be the connected components of I .

Claim: Each I_j , $1 \leq j \leq k$, contains two points of $bd\ S$.

Proof: Assume first that I_j contains no points of $bd\ S$, i.e., $I_j \subseteq int\ S$. Then there is a simple closed curve $C \subseteq int\ S$ such that I_j is contained in the inner domain of C and C does not intersect $V(R)$. Thus $C \subseteq int\ VR(r, R)$ for some $r \in R$. Since Voronoi regions are simply connected, C and its interior must belong to $VR(r, R)$ and hence C cannot contain a component I_j in its interior.

Assume next that I_j intersects $bd\ S$ in exactly one point, say x . Then there is a simple closed curve C containing I_j in its inner domain such that $x \in C$, $C - \{x\} \subseteq int\ S$ and $C - \{x\}$ does not intersect $V(R)$. Thus $C - \{x\} \subseteq VR(r, R)$ for some $r \in R$ and hence x is a point on an edge of $V(R)$ such that both sides of the edge belong to the same Voronoi region. This contradicts Fact 1. \square

Assume now that $k \geq 2$. Then there is a path $P \subseteq cl S - (I_1 \cup \dots \cup I_k)$ connecting two points on the boundary $bd S$ such that one component of $S - P$ contains I_1 and the other component contains I_2 . Let x and y be the endpoints of P and let $r \in R$ be such that $P \subseteq VR(r, R)$. Since $x, y \notin V(R)$ we conclude that $VR(r, R \cup \{s\}) = VR(r, R) - S \neq \emptyset$. Thus $x, y \in cl VR(r, R \cup \{s\})$ and hence there is a simple path $Q \subseteq cl VR(r, R \cup \{s\})$ with endpoints x and y . The cycle $P \circ Q$ is then contained in $cl VR(r, R)$ and contains either I_1 or I_2 in its interior. Thus $VR(r, R)$ is not simply connected, a contradiction to Fact 2. \square

Lemma 5: Let e be an edge of $V(R)$. If $e \cap S \neq \emptyset$ then either $e \cap S = V(R) \cap S$ and $e \cap S$ is a single component or $e - S$ is a single component; cf. Figure 4.

Proof: Assume first that $e \cap S = V(R) \cap S$. Since $V(R) \cap S$ is connected by Lemma 4 we conclude that $e \cap S$ is connected. Assume next that $e \cap S \neq V(R) \cap S$. Then with every point $x \in e \cap S$ one of the subpaths of e connecting x to an endpoint of e must be contained in S . Hence $e - S$ is a single component. \square

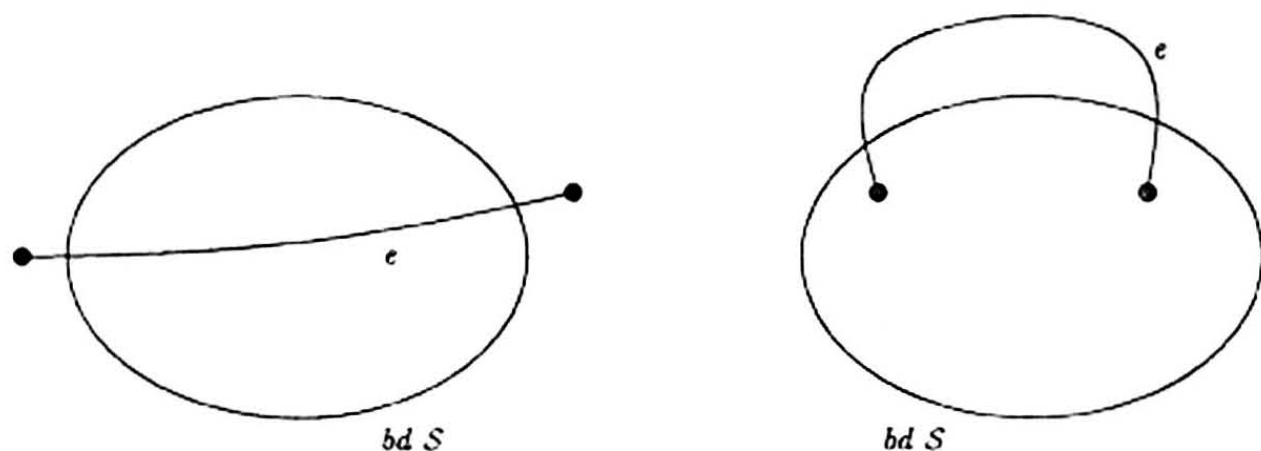


Figure 4. Two cases of Lemma 5

Let $L = \{e \text{ edge of } V(R); \{e, s\} \in G(R)\}$. For $e \in L$ let $e' = e \cap S$. Note that $e' = e \cap VR(s, \{p, q, s\})$ by Lemma 1 where e separates the regions of sites p and q ; hence e' can be computed from e in time $O(1)$. We have shown above that the set $\bigcup_{e \in L} cl e' = V(R) \cap cl S$ is connected. Let $B = \{x; x \text{ is an endpoint of } e' \text{ which is not an endpoint of } e \text{ for some } e \in L\} = V(R) \cap bd S$. Since $bd S$ is a simple closed curve by Fact 2, $bd S$ induces a cyclic ordering on the points in B . Since $V(R) \cap cl S$ is connected this cyclic ordering can be determined by a traversal of the planar graph $V(R) \cap cl S$. It is now easy to update the Voronoi diagram as follows:

Step 1: Compute e' for each $e \in L$. Remove e' from $V(R)$ for each $e \in L$.

Step 2: Compute B and the cyclic ordering on B induced by $bd S$.

Step 3: Let x_1, \dots, x_k be the set B in its cyclic ordering and

let $r_i \in R$ be such that $\{x_i, x_{i+1}\} \subseteq bd VR(r_i, R)$.

(1) for i from 1 to k

(2) do add the part of $J(r_i, s)$ with endpoints x_i and x_{i+1} to the Voronoi diagram

(3) od

For the time bound we only have to observe that steps 1 and 2 take time $O(|L|)$ and that step 3 takes time $O(k) = O(|L|)$. This proves the following

Lemma 6: Let $s \in S - R$. Then $V(R \cup \{s\})$ can be constructed from $V(R)$ and $G(R)$ in time $O(\deg_{G(R)}(s) + 1)$.

We now turn to the update of the conflict graph.

Lemma 7: Let $s \in S - R$. Then $G(R \cup \{s\})$ can be constructed from $V(R)$ and $G(R)$ in time

$$O\left(\sum_{\{e,s\} \in G(R)} \deg_{G(R)}(e)\right).$$

Proof: In this proof we distinguish three cases: edges of $V(R \cup \{s\})$ which already were edges of $V(R)$, edges which are part of edges of $V(R)$, and edges which are completely new. The only difficult case is the third one; it is dealt with in Lemma 8.

As above let $L = \{e; e \text{ is an edge of } V(R) \text{ and } e \cap S \neq \emptyset\}$ where $S = VR(s, R \cup \{s\})$. For $e \notin L$ the conflict information does not change. This follows from $e \cap VR(t, R \cup \{s, t\}) = e \cap (VR(t, R \cup \{t\}) - VR(s, R \cup \{s, t\}))$ and $(e \cap VR(t, R \cup \{t\})) - (e \cap VR(s, R \cup \{s, t\})) = e \cap VR(t, R \cup \{t\})$.

Let us next consider an edge $e \in L$. If $e \subseteq S$ then e has to be deleted from the conflict graph. This certainly takes time $O(\deg_{G(R)}(e))$. If $e \not\subseteq S$ then $e - S$ consists of at most two subsegments by Lemma 5. Let e' be one of those subsegments and let $t \in S - R - \{s\}$. Then $e' \cap VR(t, R \cup \{s\} \cup \{t\}) = e' \cap \bigcap_{r \in R} R(t, r) \cap R(t, s) = e' \cap VR(t, R \cup \{t\}) \cap R(t, s) \subseteq e \cap VR(t, R \cup \{t\})$ and hence any site t in conflict with e' must be in conflict with e .

It remains to consider those edges of $V(R \cup \{s\})$ which are not fragments of edges of $V(R)$. Let e_{12} be one of those edges. The endpoints x_1 and x_2 of e_{12} lie in the interior of edges e_1 and e_2 on $bd VR(p, R)$ for some $p \in R$. Also e_{12} is part of the bisecting curve $J(p, s)$. Note that $p \neq \infty$ since $J(\infty, s) = \Gamma \subseteq V(R)$. Let P be that part of $bd VR(p, R)$ which connects x_1 and x_2 and is contained in S in all sufficiently small neighborhoods of x_1 and x_2 .

Claim: $P \subseteq S$.

Proof: $bd VR(p, R)$ is a simple closed curve and $int VR(p, R)$ is the bounded domain defined by this curve. Assume now that P crosses $bd S$. Then $VR(p, R \cup \{s\}) = VR(p, R) - S$ is not connected, a contradiction. \square

Lemma 8: Let $t \in S - R - \{s\}$, and let t conflict with e_{12} in $V(R \cup \{s\})$. Then t conflicts in $V(R)$ with either e_1 or e_2 or one of the edges of P .

Proof: Consider $VR(p, R)$.

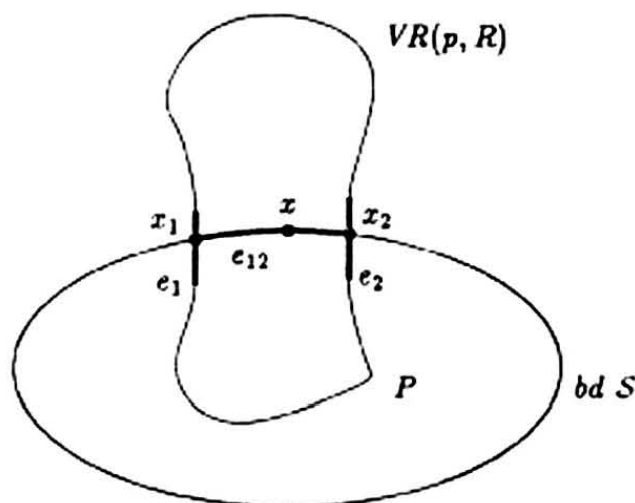


Figure 5.

By the definition of conflict a point $x \in e_{12}$ exists such that $x \in VR(t, R \cup \{s, t\}) \subseteq VR(t, R \cup \{t\})$. Since we claim a contradiction we assume that t is not in conflict with P , e_1 or e_2 in $V(R)$. Thus, $VR(t, R \cup \{s, t\}) \cap U(x_1) \subseteq VR(t, R \cup \{t\}) \cap U(x_1) = \emptyset$ for any sufficiently small neighborhood $U(x_1)$ of x_1 . Now consider in any such neighborhood of x_1 the wedge spanned by e_{12} and the part of e_1 outside S . The points in this wedge all belong to $VR(p, R \cup \{s, t\})$. The same is true for any sufficiently small neighborhood of x_2 with e_2 instead of e_1 . Since $VR(p, R \cup \{s, t\})$ is connected, there is a path Q from x_1 to x_2 running completely inside $VR(p, R \cup \{s, t\}) \subseteq VR(p, R \cup \{t\})$ except at the endpoints.

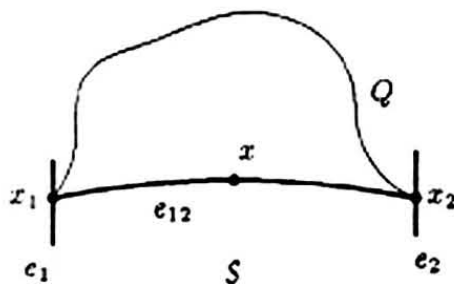


Figure 6.

By definition of P and Q the Voronoi region $VR(t, R \cup \{t\})$ cannot intersect these two paths. Moreover, x lies in the interior of the cycle $x_1 \circ P \circ x_2 \circ Q$; otherwise $VR(p, R)$ would not be simply connected. From $x_1, x_2 \notin VR(t, R \cup \{t\})$ and $x \in VR(t, R \cup \{t\})$ we conclude that $VR(t, R \cup \{t\})$ lies in the interior of the cycle. This is a contradiction to the fact that $VR(p, R \cup \{t\})$ is simply connected. \square

Lemmas 8 and 1 together allow us to compute the conflict information for the new edges. Let $e_{12} \subseteq J(p, s)$ be any new edge. A site t in conflict with edge e_{12} must have conflicted in $G(R)$ with either e_1 , e_2 or one of the edges on the path P by Lemma 8. Also for any such site t we can compute the conflict information $e_{12} \cap VR(t, R \cup \{s, t\})$ in time $O(1)$ by Lemma 1. Thus the set of neighbors of edge e_{12} in $G(R \cup \{s\})$ can be computed in time

$$O\left(\sum_{e \in P \cup \{e_1, e_2\}} \deg_{G(R)}(e)\right)$$

where the sum is over all edges in $P \cup \{e_1, e_2\}$. Next observe that every edge $e \in V(R)$ with $e \cap VR(s, R \cup \{s\}) \neq \emptyset$ can belong at most two times to a path P for some new edge by planarity. Thus $G(R \cup \{s\})$ can be obtained from $G(R)$ in time

$$O\left(\sum_{\{e,s\} \in G(R)} \deg_{G(R)}(e)\right).$$

This proves Lemma 7. □

Theorem 1: a) Let $s \in S - R$. Then the data structures $G(R \cup \{s\})$ and $V(R \cup \{s\})$ can be obtained from $G(R)$ and $V(R)$ in time

$$O\left(\sum_{\{e,s\} \in G(R)} \deg_{G(R)}(e)\right).$$

b) For $R \subseteq S$, $|R| = 3$ and $\infty \in R$ the data structures $V(R)$ and $G(R)$ can be set up in time $O(n)$ where $n = |S|$.

Proof: a) This point summarizes Lemma 6 and 7.

b) The Voronoi diagram $V(R)$ for three sites ∞, p and q has the structure shown in Figure 7 and can certainly be set up in time $O(1)$. Also for each of the edges e of $V(R)$ and each of the $n - 3$ sites in $S - R$ one can test $e \cap VR(t, R \cup \{t\}) \neq \emptyset$ in $O(1)$ by Lemma 1. This proves b). □

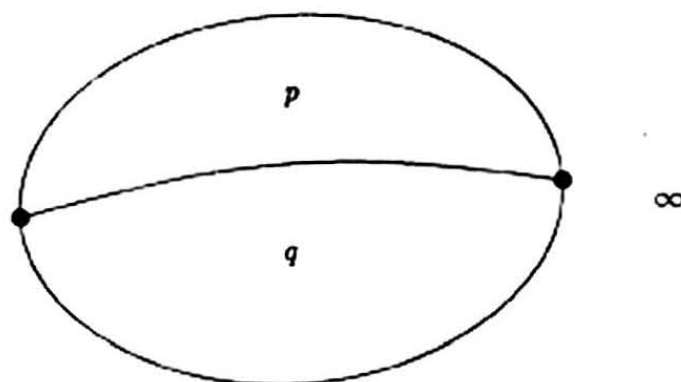


Figure 7. The Voronoi diagram for sites ∞, p and q .

Lemma 9: The number of edges of $V(R)$ is at most $3|R|$.

Proof: $V(R)$ is a planar graph with at most $|R|$ regions. Also, each vertex has degree three. The number of edges is therefore at most $3|R|$ by Euler's Formula. □

Theorem 2: The abstract Voronoi diagram $V(S)$ of n sites can be constructed by a randomized algorithm in time $O(n \log n)$.

Proof: In [CS], Clarkson and Shor show that randomized incremental construction has expected running time

$$O(m(n) + n \cdot \sum_{1 \leq r \leq n/2} m(r)/r^2 + n)$$

provided that initialization takes time $O(n)$ and addition of an object (here site) s to the set R takes time proportional to

$$\sum_{\{e,s\} \in G(R)} \deg_{G(R)}(e),$$

where the summation is over all regions (here edges) of the current structure (here Voronoi diagram $V(R)$) which conflict with site s . Also $m(r)$ is the expected size of the structure for a random subset $R \subseteq S$ of r elements. In our case we have $m(r) \leq 3r$ by Lemma 9. Finally, the assumptions of Clarkson's theorem are satisfied by Theorem 1. The time bound follows. \square

Remark: In our algorithm $\infty \in R$ always. An inspection of Clarkson's argument shows that this minor deviation from randomness does not change the time bound.

IV. Applications

Many previously considered types of Voronoi diagrams fall under the framework described above.

1. Point Sites: In their pioneering paper Shamos/Hoey [SH] showed how to construct the Voronoi diagram for point sites under the Euclidean metric in time $O(n \log n)$. This was later extended to arbitrary L_p -metrics, $1 \leq p \leq \infty$, by Lee [L], to the L_2 -metric with additive weights by Sharir [S] and Fortune [F87], to the so-called Moscow-metric by Klein [Kl88b], to convex distance functions by Chew/Drysdale [CD] and Fortune [F85] and to abstract Voronoi diagrams by Klein/Wood [KW] and Klein [Kl88a]. The previous algorithms for abstract diagrams had to assume, as they were based on the divide-and-conquer approach, that the set of sites S can be partitioned into about equal sized subsets L and R such that the bisector between L and R is acyclic. This assumption is crucial for the efficiency of the merging step. For all cases mentioned our algorithm gives an alternative $O(n \log n)$ solution, albeit randomized. For abstract diagrams ([Kl88a]) we do not need the acyclicity assumption, however, and for the L_p -norm we may also add additive weights.

2. Beyond Point Sites: Point and line sites were considered by Kirkpatrick [Ki] and Fortune [F87], and disjoint convex objects were considered by Leven/Sharir [LS86]. In the latter case, the running time is $O(n(\log n)^2)$ since the Leven/Sharir algorithm uses divide-and-conquer and the bisector between the subsets L and R of S mentioned above is not necessarily acyclic. Our algorithm runs in time $O(n \log n)$. Other applications are the Voronoi diagrams for circles under the Laguerre distance (Imai/Iri/Murota [IIM], Aurenhammer [A87], [A88a]) and for disjoint convex polygons under a convex distance function (Leven/Sharir [LS87]).

Of course, there are also types of Voronoi diagrams which do not fall under the framework, e.g., the diagram for points sites under the Euclidean metric with multiplicative weight (Aurenhammer/Edelsbrunner [AE]), the diagram for points and circular arcs, and the diagram for points under metrics which arise from weighted partitions of the plane (Mitchell/Papadimitriou [MP]). In all three cases the bisector $J(p, q)$ of two sites may be a closed curve, cf. Figure 8.

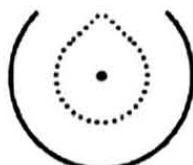


Figure 8. The bisector for a point and a circular arc.

V. Conclusions and Open Problems

We showed that Clarkson and Shor's randomized incremental construction method works for (a subset of) Klein's abstract Voronoi diagrams. Many previously considered types of Voronoi diagrams can thus be handled by the same simple algorithm. In [KMM] the results of this paper are extended in two ways. We show that the algorithm can be programmed on a schema level such that specific Voronoi diagram algorithms can be derived in a simple way; we also drop the general position assumption and the assumption that bisecting curves may not touch. Nevertheless, many open problems remain:

- 1) Can the concept of abstract Voronoi diagram be generalized to higher dimensions?
- 2) What can be done in two dimensions without the assumption that bisectors are non-closed curves?
- 3) Can the algorithm be modified in order to handle higher-order Voronoi diagrams?

VI. References

- [A87] Aurenhammer, F.: *Power diagrams: properties, algorithms and applications*, SIAM J. of Computing 16 (1987), pp. 78–96.
- [A88a] Aurenhammer, F.: *Improved algorithms for discs and balls using power diagrams*, J. Algorithms 9 (1988), pp. 151–161.
- [A88b] Aurenhammer, F.: *Voronoi Diagrams — A survey*, Tech. Report 263, Institutes for Information Processing, Graz Technical University, Austria (1988)
- [AE] Aurenhammer, F., and Edelsbrunner, H.: *An optimal algorithm for constructing the weighted Voronoi diagram in the plane*, Pattern Recognition 17 (2) (1984), pp. 251–257.
- [Br] Brown, K.Q.: *Voronoi diagrams from convex hulls*, IPL 9 (1979), pp. 223–228.

- [CD] Chew, L.P., and Drysdale III, R.L.: *Voronoi diagrams based on convex distance functions*, Proc. 1st ACM Symp. on Computational Geometry (1985), pp. 235–244.
- [CS] Clarkson, K.L., and Shor, P.W.: *Algorithms for diametral pairs and convex hulls that are optimal, randomized and incremental*, Proc. 4th ACM Symp. on Computational Geometry (1988), pp. 12–17.
- [ES] Edelsbrunner, H. and Seidel, R.: *Voronoi diagrams and arrangements*, Discrete & Computational Geometry 1 (1986), pp. 25–44.
- [F85] Fortune, S.: *A fast algorithm for polygon containment by translation (extended abstract)*, Proc. 12th Int. Colloq. Automata, Languages and Programming (1985), LNCS 194, Springer-Verlag, New York, pp. 189–198.
- [F87] Fortune, S.: *A sweepline algorithm for Voronoi diagrams*, Algorithmica 2 (1987), pp. 153–174.
- [IIM] Imai, H., Iri, M., and Murota, K.: *Voronoi diagram in the Laguerre geometry and its applications*, SIAM J. of Computing 14 (1985), pp. 93–105.
- [Ki] Kirkpatrick, D.G.: *Efficient computation of continuous skeletons*, Proc. 20th Symp. on Foundations of Computer Science (1979), pp. 18–27.
- [K188a] Klein, R.: *Abstract Voronoi diagrams and their applications (extended abstract)*, in: H. Noltemeier (ed.), Computational Geometry and its Applications (CG '88), Würzburg (1988), LNCS 333, pp. 148–157.
- [K188b] Klein, R.: *Voronoi diagrams in the Moscow metric (extended abstract)*, in: Graph-theoretic Concepts in Computer Science (WG '88), Amsterdam (1988), to appear in LNCS.
- [K188c] Klein, R.: *On a generalization of planar Voronoi diagrams (Habilitationsschrift)*, Mathematische Fakultät der Universität Freiburg i. Br. (1988)
- [KMM] Klein, R., Mehlhorn, K., and Meiser, S.: *On the Construction of Abstract Voronoi Diagrams, II*, Tech. Report, FB Informatik, Universität des Saarlandes (1989)
- [KW] Klein, R., and Wood, D.: *Voronoi diagrams based on general metrics in the plane*, in: R. Cori and M. Wirsing (eds.), Proc. 5th Annual Symp. on Theoretical Aspects of Computer Science (STACS), Bordeaux (1988), LNCS 294, pp. 281–291.
- [L] Lee, D.T.: *Two-dimensional Voronoi diagrams in the L_p -metric*, J. ACM 27 (1980), pp. 604–618.
- [LS86] Leven, D., and Sharir, M.: *Intersection and proximity problems and Voronoi diagrams*, in: J. Schwartz and C.K. Yap (eds.), Advances in Robotics, Vol. 1 (1986), Lawrence Erlbaum.
- [LS87] Leven, D., and Sharir, M.: *Planning a purely translational motion for a convex object in two-dimensional space using generalized Voronoi diagrams*, Discrete & Computational Geometry 2 (1987), pp. 9–31.
- [MP] Mitchell, J.S.B., and Papadimitriou, C.H.: *The weighted region problem*, in: Proc. 3rd ACM Symp. on Computational Geometry, Waterloo (1987), pp. 30–38.
- [S] Sharir, M.: *Intersection and closest-pair problems for a set of planar discs*, SIAM J. of Computing 14 (1985), pp. 448–468.
- [SH] Shamos, M.I., and Hoey, D.: *Closest point problems*, Proc. 16th IEEE Symp. on Foundations of Computer Science (1975), pp. 151–162.