Picture Reconstruction from
Projections in Restricted
Range

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Abstract

In order to reduce scanning time modern x-ray scanners
provide projections only in a restricted range $[0, \Phi]$
with $\Phi < \pi$. We consider the reconstruction of pictures
from $p+1$ complete projections in $[0, \Phi]$. An extrapolation
procedure is given to achieve approximations $g_p$ of the
data in the whole range. We show that the $L_2$-error of the
corresponding picture is of order $p^{-\alpha}$ if the original
belongs to the SOBOLEV space $H_0^{\alpha}$. The validity of our
error estimate is investigeted by numerical experiments.

# 1   Introduction

Scanning with x-rays the density f of a tissue lying in a domain $\Omega \subset \mathbb{R}^2$ we get the integrals of f along the paths of the rays, see figure 1,

$$(1.1) \quad g(s,\omega) = Rf(s,\omega) = \int_{-\infty}^{\infty} f(s\omega + t\omega^{\perp}) \, dt$$

with $\omega = (\cos \varphi, \sin \varphi)$, $\omega^{\perp} = (-\sin \varphi, \cos \varphi)$. The function g = Rf is called the RADON transform of f.
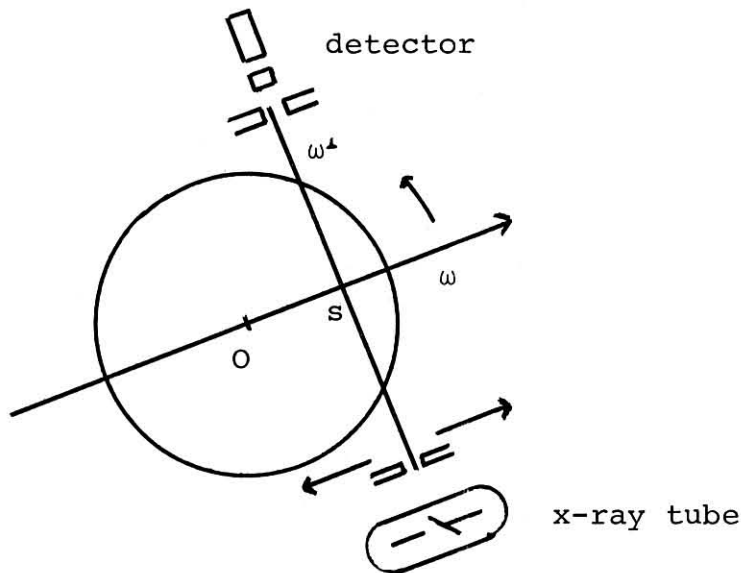


Figure 1.

The reconstruction of the density f from its integrals along all lines, i.e. the analytical inversion of the RADON trans - form (1.1) requires the values of g for all $s \in \mathbb{R}$ and all $\varphi \in [0,\Phi]$ for some $\Phi \in \, ]0,\pi[$, but the fast reconstruction algorithms need projections in the whole range $[0,\pi[$. In order to apply such a discrete inversion formula projections have to be provided for angles equally distributed over the whole interval $[0,\pi[$. Conventional scanners need several minutes for x-raying, errors occur by motions of the patient, and scanning of the heart for example is impossible. Modern scanners with modified scan geometry need only a few seconds but in general they cannot cover the whole interval $[0,\pi[$.

The discretization of the integral equation (1.1) and appli -
cation of ART methods, see GORDON-HERMAN, [2], takes the
restricted range into consideration, but the computing time
is to large. In order to apply the fast algorithms we have
to provide approximations of the data in the missing range.
The interpolation of g, as proposed by WAGNER, [12], has
useful results only if the missing range is sufficiently
small.

In §3 we give an extrapolation algorithm which is applicable
without this strong restriction and which achieves better
results. The number of arithmetic operations is of the same
order as that of the fast reconstruction algorithms and so
the procedure is practicable.

In §4 we show that the approximated data $g_p$ lead to a
reconstruction $f_p$ with $L_2$-error of the order $p^{-\alpha}$ if the
original picture belongs to the SOBOLEV space $H_o^\alpha$. This
coincides with the error in using filtered backprojection
with p angles distributed over the whole range as shown in
NATTERER, [7].

The numerical examples in §5 verify the error bounds of §4
and show that the procedure gives useful results even if
the range is only $[0,\Phi]$ with $\Phi = \frac{2}{3}\pi$.

## 2 The RADON Transform in SOBOLEV Spaces

In this chapter we first introduce the SOBOLEV spaces used in the following. Let k be an integer and $G \subset \mathbb{R}^k$ be an open and bounded domain. $C_o^\infty(G)$ is the set of all real-valued infinitely differentiable functions with compact support in G. For real $\alpha$ we denote by $H^\alpha(\mathbb{R}^k)$ the SOBOLEV space of order $\alpha$, i.e. the set of all tempered distributions with

$$(2.1) \qquad ||f||^2_{H^\alpha(\mathbb{R}^k)} = \int_{\mathbb{R}^k} (1+|\xi|^2)^\alpha |\hat{f}(\xi)|^2 \, d\xi < \infty$$

where $\hat{f}$ is the FOURIER transform of f :

$$(2.2) \qquad \hat{f}(\xi) = \int f(x) \, e^{-2\pi i \, x \cdot \xi} \, dx.$$

For domains G with sufficiently regular boundary we put

$$(2.3) \qquad H_o^\alpha(G) = \{ \, f \in H^\alpha(\mathbb{R}^k) \, : \, \text{supp}(f) \subset \overline{G} \, \},$$

$$(2.4) \qquad ||f||_{H_o^\alpha(G)} = ||f||_{H^\alpha(\mathbb{R}^k)} \qquad \text{for } f \in H_o^\alpha(G).$$

These spaces are the $\tilde{H}_2^\alpha(G)$ spaces of TRIEBEL, [11], chapt. 4.3.2 and for $\alpha \neq k + 1/2$, k integer, they coincide with the closure of $C_o^\infty(G)$ in the $H^\alpha$ - norm. For all $\alpha$, $\beta \in \mathbb{R}$ and $0 \leq \theta \leq 1$ they fulfill the interpolation property

$$(2.5) \qquad [H_o^\alpha(G), H_o^\beta(G)]_\theta = H_o^{\alpha(1-\theta)+\beta\theta}(G)$$

where $[X,Y]_\theta$ denotes the interpolation space of definition 2.1 in LIONS-MAGENES, [4].

For a real interval I and a weight function p we define the SOBOLEV space with weight

$$W_2^s(I,p,1), \ s \geq 0,$$

as the set of all $L_2$ functions with $||f||_{W_2^s(I,p,1)} < \infty$

If s is an integer this norm is defined as

$$(2.6) \qquad ||f||_{W_2^s(I,p,1)} = [ \, \int_I ( \, p(x)|f^{(s)}(x)|^2 + |f(x)|^2 \, ) \, dx]^{1/2}$$

and for $s = k + \sigma$, $k \in \mathbb{N}_o$ one sets

$$(2.7) \quad ||f||_{W_2^s(I,p,1)} = [\int_I \int_I \frac{|p^{1/2}(x) f^{(k)}(x) - p^{1/2}(y) f^{(k)}(y)|^2}{|x-y|^{1+2\sigma}} dxdy]^{1/}$$

$$+ \, ||f||_{W_2^k(I,p,1)} \quad ,$$

see TRIEBEL, [10], chapt. 3.2.6.

Finally let S be the boundary of the unit circle in $\mathbb{R}^2$ and $Z = \mathbb{R} \times S \subset \mathbb{R}^3$. On the manifold Z we use the SOBOLEV space $H^\alpha(Z)$ with the norm

$$(2.8) \quad ||g||_{H^\alpha(Z)} = [\int_0^{2\pi} ||g(\cdot,\omega)||^2_{H^\alpha(\mathbb{R})} d\varphi]^{1/2}$$

for $\alpha \geq 0$.

Next we consider the RADON transform in the SOBOLEV space $H_o^\alpha(\Omega)$ where $\Omega \subset \mathbb{R}^2$ is bounded. The line integrals in (1.1) are defined for $f \in H_o^\alpha(\Omega)$ for $\alpha > 1/2$. For $f \in C_o^\infty(\Omega)$ the norms $||f||_{H_o^{-1/2}(\Omega)}$ and $||Rf||_{L_2(Z)}$ are equivalent, see NATTERER, [7], theorem 3.1. So we can extend the RADON transform to all of $H_o^{-1/2}(\Omega)$ by continuity because $C_o^\infty(\Omega)$ is dense in $H_o^\alpha(\Omega)$; the extension is again denoted by R.

THEOREM 2.1

For real $\alpha \geq -\frac{1}{2}$ the RADON transform is a bounded operator from $H_o^\alpha(\Omega)$ into $H^{\alpha+1/2}(Z)$.

PROOF

For $\alpha > 1/2$ this is theorem 3.2 in NATTERER, [7]. Theorem 3.1 in [7] states that $||Rf||_{L_2(Z)} \leq c ||f||_{H_o^{-1/2}(\Omega)}$ for $f \in C_o^\infty(\Omega)$. This is also true for the extension of R on $H_o^{-1/2}(\Omega)$. Now the result follows from the interpolation theorem 5.1, chapt. 1 of LIONS-MAGENES, [4].

## 3   The Extrapolation Procedure

The extrapolation procedure given here is based upon the
PALEY-WIENER theorem for the RADON transform. It characterizes
the small range of that ill-posed problem.

THEOREM 3.1

A function g $\in H^{1/2}(Z)$ with compact support is the RADON trans -
form of a function f $\in L_2(\mathbb{R}^2)$ with compact support if and
only if

(3.1)    i) $g(-s,-\omega) = g(s,\omega)$ on Z,

    ii) for all m $\in \mathbb{N}_0$

(3.2)    $\int\limits_{\mathbb{R}} s^m g(s,\omega)\, ds$

    is a homogeneous polynomial of degree $\leq$ m in $\omega$.

For a proof see LAX-PHILLIPS, [3]. This result suggests a series
expansion of g in terms of polynomials, and especially the
LEGENDRE polynomials are well suited.

Let us assume that the tissue to be scanned is lying in a
circle $\Omega$ around O with radius 1/2. That means that the support
of the density f is lying in $\Omega$ and the support of the functions
$g(\cdot,\omega)$ is in [-1/2,1/2] for all $\varphi$. Further we assume that we
have p+1 complete projections in the interval [O,$\Phi$] with $\Phi < \pi$,
i.e. the functions

   $g(\cdot,\omega_j)$, j = O,...,p

with $\omega_j$ = (cos $\varphi_j$, sin $\varphi_j$), $\varphi_j \in [0,\Phi]$, $\varphi_j \neq \varphi_k$ for j $\neq$ k
are known. In practice this means that at sufficiently many
positions s for each $\omega_j$ the tissue is scanned.

Let us denote with $Q_m$ the normalized LEGENDRE polynomials of
degree m. From theorem 3.1 we know that the coefficients of
the expansion in terms of LEGENDRE polynomials

(3.3)    $q_m(\varphi) = \int\limits_{-1}^{1} Q_m(s)\, g(s,\omega)\, ds$

are polynomials of degree less or equal m in $\omega$. Because of

the periodicity of g the $q_m$ have the representation

(3.4)    $q_m(\varphi) = \sum\limits_{k=0}^{[m/2]} (\, a_k^m \cos c_k^m \varphi + b_k^m \sin c_k^m \varphi\, )$,

where $[x]$ denotes the integral part of x and the $a_k^m$, $b_k^m$ are coefficients and

(3.5)    $c_k^m = \begin{cases} 2k & \text{for m even} \\ 2k+1 & \text{for m odd.} \end{cases}$

The $q_m$ defined in (3.3) can be calculated for the angles $\varphi_j$, $j = 0,\ldots,p$ and then (3.4) is a system of linear equations for the coefficients $a_k^m$, $b_k^m$. This system is overdetermined for $m \leq p$ and can be solved for example by singular value decom - position. Now for those m the coefficients $q_m$ of the series expansion are detemined and we have the following approxi - mation of the function g :

(3.6)    $g_p(s,\omega) = \begin{cases} \sum\limits_{m=0}^{p} Q_m(s)\, q_m(\varphi) & \text{for } |s| < 1 \\ 0 & \text{for } |s| \geq 1 \end{cases}$

With this function, approximations for the data in the missing range can be calculated. If the angles in $[0,\Phi]$ are not equally distributed it is possible to compute the data for equally distributed angles in the whole range. With $\theta$ we denote

$\theta = [\pi/\Phi]$.

This leads to the following algorithm

STEP 1 : Compute $q_m(\varphi_j)$ according to (3.3) for $j = 0,\ldots,p$, $m = 0,\ldots,p$.

STEP 2 : Solve the system of equations (3.4) with $\varphi = \varphi_j$ for $j,m = 0,\ldots,p$.

STEP 3 : Compute $q_m(\varphi_j)$ according to (3.4) for $m = 0,\ldots,p$ and $j = p+1,\ldots,\theta p-1$, or $q_m(\psi_j)$ for $m = 0,\ldots,p$ and $j = 0,\ldots,\theta p-1$ with $\psi_j$ equally distributed over $[0,\pi[$.

STEP 4 : Compute $g_p(s_i,\omega_j)$ , $j = p+1,\ldots,\theta p-1$ or $g_p(s_i,\tilde{\omega}_j)$ , $j = 0,\ldots,\theta p-1$, $\tilde{\omega}_j = (\, \cos \psi_j,\ \sin \psi_j)$ for $i = -q,\ldots,q-1$.

The count of operations shows that we have a procedure with $O(p^s q^t)$ arithmetic operations where $s+t = 3$. This is the same order as for the fast reconstruction algorithms. If we calculate the integrals in step 1 with the trapezoidal rule at the points $s_i$, $i = -q,\ldots,q-1$, we need $2q(p+1)^2$ operations. The singular value decomposition in step 2 takes more effort, but for each scanner the directions $\omega_j$ are known a priori, so the matrices in (3.4) can be calculated and be decomposed independently of the data once and for all. In the actual computation we then have to compute only products of matrices and vectors, which requires $cp^3$ operations. In step 3 there are at most $\frac{1}{2}\theta p(p+1)^2$ operations and finally $2\theta p^2 q$ operations in step 4.

## 4   Error Estimates

In this chapter we prove error estimates in SOBOLEV norms for the picture reconstructed from the data $g_p$ instead of $g$. First we show that $g_p$ lies in the range of the RADON transform.

LEMMA 4.1

Let $g_p$ be defined as in (3.6).
Then there exists $f_p \in L_2(\mathbb{R}^2)$ with compact support and

$$(4.1) \qquad g_p(s,\omega) = \int f_p(s\omega + t\omega^\perp)\, dt.$$

PROOF

The function $g_p$ is in $C^\infty([-1,1] \times S)$ and has compact support. So we can use theorem 3.1 to prove this lemma. First we show the symmetry of $g_p$

$$(4.2) \qquad g_p(-s,-\omega) = g_p(s,\omega) \text{ on } Z.$$

Because of $-\omega = (-\cos\varphi, -\sin\varphi) = (\cos(\varphi+\pi), \sin(\varphi+\pi))$ we get with the addition formula for sin and cos that
$\cos c_k^m(\varphi+\pi) = (-1)^m \cos c_k^m\varphi$ and $\sin c_k^m(\varphi+\pi) = (-1)^m \sin c_k^m\varphi$
and so

$$q_m(\varphi+\pi) = (-1)^m q_m(\varphi).$$

Now (4.2) follows from the symmetry of the LEGENDRE polynomials, i.e. $Q_m(-s) = (-1)^m Q_m(s)$.

The second condition in theorem 3.1 for $g_p$ is a consequence of the orthogonality of the LEGENDRE polynomials. Each monomial $s^m$ can be represented in the form

$$s^m = \sum_{j=0}^{m} d_j^m Q_j(s) \text{ with } d_j^m = 0 \text{ for } j+m \text{ odd.}$$

So we get

$$\int s^m g_p(s,\omega)\, ds = \sum_{k=0}^{p} q_k(\varphi) \int s^m Q_k(s)\, ds$$

$$= \sum_{k=0}^{p} q_k(\varphi) \sum_{j=0}^{m} d_j^m \int Q_j(s) Q_k(s)\, ds$$

$$= \sum_{k=0}^{t} q_k(\varphi) d_k^m$$

where $t = \min(p,k)$.

The $q_m$ defined in (3.3) can be represented by a homogeneous polynomial of degree $\leq m$ because of (3.4), (3.5), which completes the proof.

Next we give an error estimate for the partial sum of an expansion of a function in terms of LEGENDRE polynomials. Let I be the interval $]-1,1[$ and p be the weight function $p(x) = 1-x^2$. With $K^s$ we denote

$$K^s = W_2^s(I,p^s,1) \quad \text{for } s \neq \frac{1}{2} + k, \text{ where } k \in \mathbb{N}_o$$

and for $s = \frac{1}{2} + k$, $k \in \mathbb{N}_o$, the space $K^s$ is the completion of $C^\infty(\bar{I})$ in the norm

$$(4.3) \quad ||u||_{K^s} = [\ ||u||^2_{W_2^s(I,p^s,1)} + \int_{-1}^{1} p^{s-1}(x)\ |u^{(k)}(x)|^2 dx\ ]^{1/2}.$$

LEMMA 4.2

Let $u \in K^t$ and $u_p$ be the partial sum of the expansion of u in terms of LEGENDRE polynomials

$$u_p(x) = \sum_{k=0}^{p} c_k\ Q_k(x)$$

with

$$(4.4) \quad c_k = \int_{-1}^{1} u(x)\ Q_k(x)\ dx.$$

Then there is a constant $c > 0$ independent of u and p such that

$$(4.5) \quad ||u-u_p||_{K^s} \leq c\ p^{-(t-s)}\ ||u||_{K^t}$$

for $0 \leq s \leq t$.

COROLLARY 4.3

Let $\gamma \in ]0,1[$ and $u \in H_o^t(]-\gamma,\gamma[)$, then

$$(4.6) \quad ||u-u_p||_{H^s(]-\gamma,\gamma[)} \leq c_1\ p^{-(t-s)}\ ||u||_{H^t(]-\gamma,\gamma[)}$$

for $0 \leq s \leq t$.

PROOF

Let A be the LEGENDRE differential operator $-\frac{d}{dx}((1-x^2)\frac{d}{dx})$ and $\bar{A}$ be the closure of A. Then it is shown in TRIEBEL, [11], chapt. 7.7.1 that

$$D(\bar{A}^s) = K^{2s} \text{ if } s \geq 0.$$

$\bar{A}^s$ is a selfadjoint operator with the eigenfunctions $Q_k$ and the eigenvalues $\lambda_k^s$ with

$$\lambda_k = k(k+1).$$

Following LIONS-MAGENES, [4], chapt. 1, formula (16.12) we define the norm

(4.7) $$\|u\|'_{K^s} = (u, \bar{A}^s u)_{L_2}^{1/2} = [\sum_{k=0}^{\infty} \lambda_k^s c_k^2]^{1/2}$$

with $c_k$ as defined in (4.4). This norm is equivalent with the $K^s$ norm defined above. Then we have

$$\|u-u_p\|_{K^s}^2 \leq c' \|u-u_p\|_{K^s}'^2$$

$$= c' \sum_{k=p+1}^{\infty} [k(k+1)]^s c_k^2$$

$$= c' \sum_{k=p+1}^{\infty} [k(k+1)]^{s-t} [k(k+1)]^t c_k^2$$

$$\leq c' [(p+1)(p+2)]^{s-t} \sum_{k=p+1}^{\infty} [k(k+1)]^t c_k^2$$

$$\leq c'' [(p+1)(p+2)]^{s-t} \|u\|_{K^t}^2$$

which completes the proof of lemma 4.2. Now the result of the corollary follows from the fact that the weight function p is bounded away from zero on the interval $[-\gamma,\gamma]$ for $0 < \gamma < 1$. So the norms $\|\cdot\|_{K^s}$ and $\|\cdot\|_{H^s(]-\gamma,\gamma[)}$ are equivalent for functions with support in $[-\gamma,\gamma]$.

Now we are able to give an error estimate for the function $f_p$ defined in (4.1).

THEOREM 4.4

Let $f \in H_o^\alpha(\Omega)$, $\alpha > 0$, and $f_p$ be defined as in (4.1). Then we have

$$(4.10) \qquad \| f - f_p \|_{H_o^\beta(\Omega)} \le c \, p^{-(\alpha - \beta)} \| f \|_{H_o^\alpha(\Omega)} \; .$$

for $-\frac{1}{2} \le \beta < \alpha$.

PROOF

The proof is based upon the continuity of $R^{-1}$ from a subspace of $H^{\alpha + 1/2}(Z)$ into $H_o^\alpha(\Omega)$, see SMITH-SOLMON-WAGNER, [10], theorem 12.6.

For $\beta \ge -\frac{1}{2}$ we have

$$
\begin{aligned}
\| f - f_p \|_{H_o^\beta(\Omega)}^2 \; &\le \; c \, \| g - g_p \|_{H^{\beta + 1/2}(Z)}^2 \\
&= \; c \int_0^{2\pi} \| (g - g_p)(\cdot, \omega) \|_{H^{\beta + 1/2}(I)}^2 \, d\varphi \\
&\le \; \bar{c} \, p^{-2(\alpha - \beta)} \int_0^{2\pi} \| g(\cdot, \omega) \|_{H^{\alpha + 1/2}(I)}^2 \, d\varphi \\
&= \; \bar{c} \, p^{-2(\alpha - \beta)} \| g \|_{H^{\alpha + 1/2}(Z)}^2 \\
&\le \; \bar{\bar{c}} \, p^{-2(\alpha - \beta)} \| f \|_{H_o^\alpha(\Omega)}^2
\end{aligned}
$$

where we have used corollary 4.3 with $I = \,]-\frac{1}{2}, \frac{1}{2}[$ and the regularity theorem 2.1.

REMARKS

1. In the important case $f \in H_o^{1/2}(\Omega)$ we have a $L_2$ error of order $p^{-1/2}$.

2. If one uses the filtered backprojection algorithm as described in RAMACHANDRAN-LAKSHMINARAYANAN, [9], with $p_1 = \theta p$, $\theta \sim \pi/\Phi$, directions with the ideal low pass filter and cut off frequency $\sigma_o = cp$ and the approximated data $g_p$, then the total error in $L_2$ is also of the order $p^{-\alpha}$, see NATTERER, [7].

## 5   Numerical Experiments

In order to investigate the error estimate of theorem 4.4 we consider first a rather simple example.
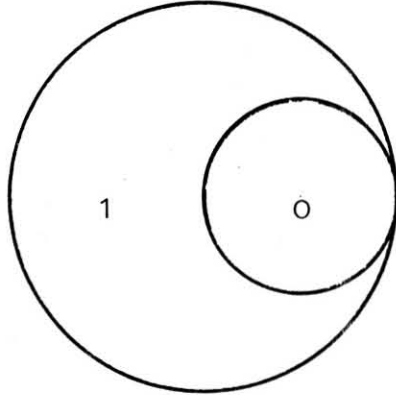


Figure 2.   First test object

The function $f \in H_o^s(\Omega)$, $s < 1/2$, as shown in figure 2 is defined as 1 in $\{ x \in \mathbb{R}^2 : |x| < 0.3 \wedge |x-(0.15,0)| > 0.15 \}$ and 0 otherwise. To simulate the fact of complete projections we have analytically computed the function $g(s,\omega_j)$ with a stepsize 1/256 in the s - direction and for p+1 angles in the range $[0,2/3\pi]$, i.e. within a range of $120^\circ$. To reconstruct f, we have used the filtered backprojection with 3p/2 angles equally distributed over $[0,\pi[$. The results are given in table 1 and 2 and show the order $p^{-1/2}$ for the $L_2$-error. The function $f_p^1$ is reconstructed using $g_p(s,\omega_j)$ for all $j = 0,\dots,\frac{3}{2}p-1$, see table 1, in the second case the data are approximated only in the interval $]\frac{2}{3}\pi,\pi[$ and in $[0,\frac{2}{3}\pi]$ the given exact data are used, see table 2. Finally table 3 shows the result when only exact data are used.

Table 1

| Projections | $\|f-f_p^1\|_{L_2}$ | $p^{1/2}\|f-f_p^1\|_{L_2}$ |
|---|---|---|
| 20 | $1.8026\cdot10^{-1}$ | 0.81 |
| 30 | $1.5635\cdot10^{-1}$ | 0.86 |
| 40 | $1.3255\cdot10^{-1}$ | 0.84 |
| 60 | $1.0842\cdot10^{-1}$ | 0.84 |

$L_2$-error of the reproduction $f_p^1$ if all the data are approximated by $g_p$.

Table 2

| Projections | $\|f-f_p^2\|_{L_2}$ | $p^{1/2}\|f-f_p^2\|_{L_2}$ |
|---|---|---|
| 20 | $1.1378\cdot10^{-1}$ | 0.51 |
| 30 | $1.0031\cdot10^{-1}$ | 0.55 |
| 40 | $8.4308\cdot10^{-2}$ | 0.53 |
| 60 | $6.8286\cdot10^{-2}$ | 0.53 |

$L_2$-error of the reproduction $f_p^2$ if the data are approximated by $g_p$ only in the missing range.

Table 3

| Projections | $\|f-f_p^3\|_{L_2}$ | $p^{1/2}\|f-f_p^3\|_{L_2}$ |
|---|---|---|
| 20 | $8.9923\cdot10^{-2}$ | 0.40 |
| 30 | $7.0715\cdot10^{-2}$ | 0.39 |
| 40 | $5.9519\cdot10^{-2}$ | 0.38 |
| 60 | $4.7400\cdot10^{-2}$ | 0.37 |

$L_2$-error of the reproduction $f_p^3$ if the data are exact.

Next we investigated our procedure in a more realistic case. We considered a second test object simulating the case of a tumor in the brain.
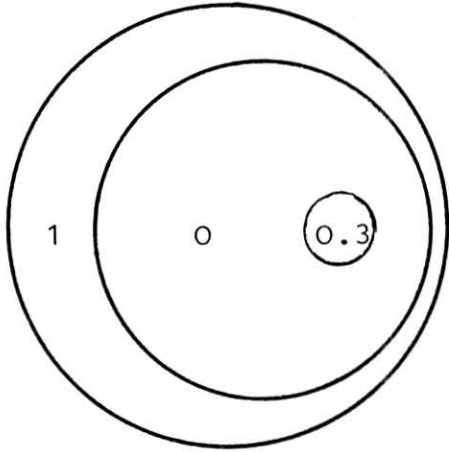


Figure 3.    Second test object

The function f is again in $H_o^s(\Omega)$, s < 1/2, and defined as 1 in $\{ x \in \mathbb{R}^2 : |x| < 0.3 \wedge |x-(0.05,0)| > 0.225 \}$, as 0.3 in $\{ x \in \mathbb{R}^2 : |x-(0.15,0)| < 0.05 \}$ and 0 otherwise.

Here the integrals in (3.3) have to be computed numerically. We used the trapezoidal rule with stepsize $\frac{1}{256}$ . The range where the data are given is again $[0,\frac{2}{3}\pi]$. Here we used p = 30, 40, 60 and equidistributed angles. The picture is reconstructed with the filtered backprojection and $\frac{3}{2}p$ equidistributed angles. These results are compared with that obtained from p+1 projections in $[0,\pi[$. The $L_2$-errors are given in table 4 and 3-D pictures over a 64×64 grid in the figures 4-8. Figure 4 shows the original function f, figures 5, 6 the reconstructed function from 40 projections and figure 7, 8 from 60 projections in $[0,\frac{2}{3}\pi]$ resp. $[0,\pi[$. Finally figure 9 shows a cut of these five functions at y = 0.

Table  4

| Projections | $\|f-f_p^1\|_{L_2}$ | $\|f-f_p^2\|_{L_2}$ |
|---|---|---|
| 30 | $1.2651\cdot10^{-1}$ | $9.3582\cdot10^{-2}$ |
| 40 | $1.0433\cdot10^{-1}$ | $7.8882\cdot10^{-2}$ |
| 60 | $9.3725\cdot10^{-2}$ | $6.9419\cdot10^{-2}$ |

$L_2$-errors of the reproduction of test-object 2 from p projections in $[0,\frac{2}{3}\pi]$ and $[0,\pi[$ resp.

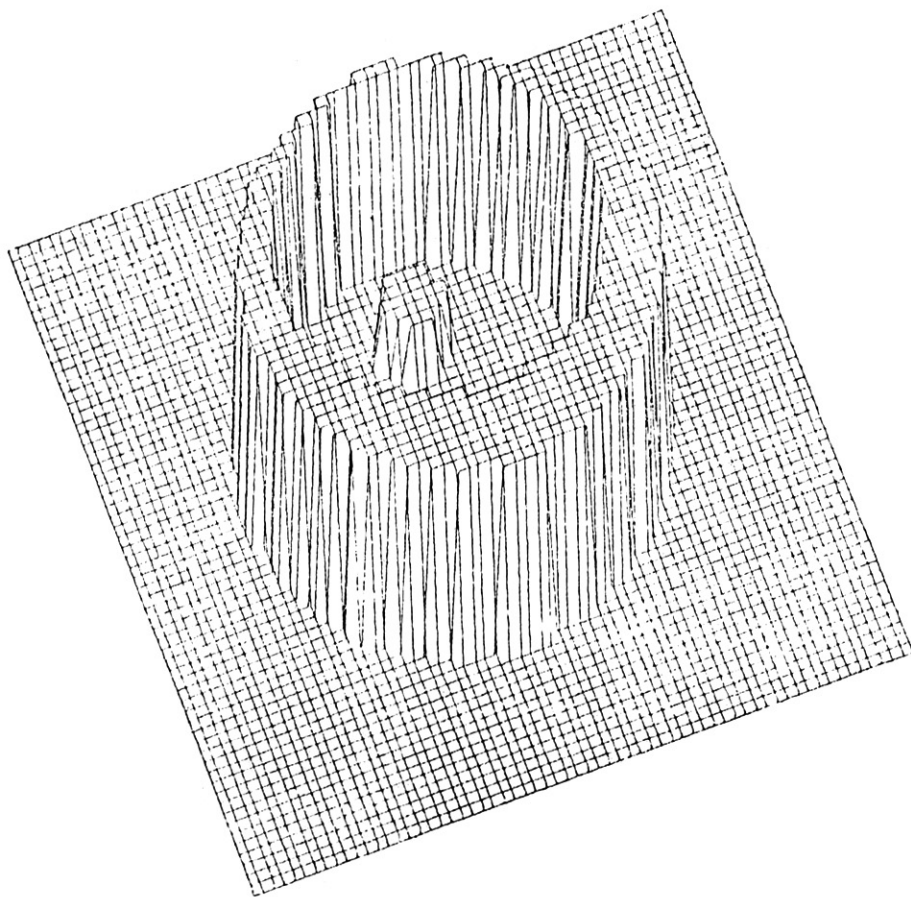Figure  4

3-D picture of the second test object, see figure 3
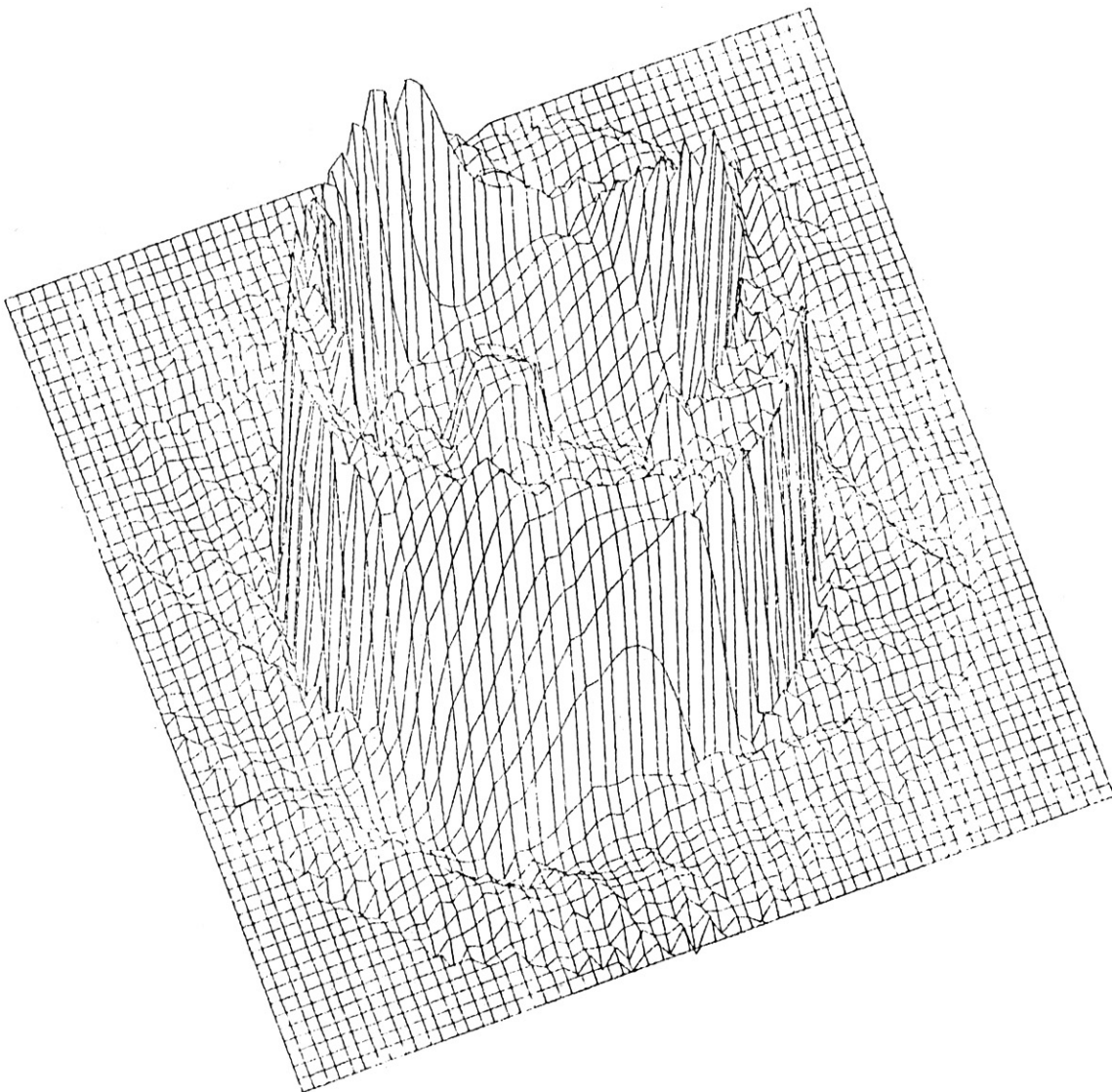
Figure 5

Reconstruction of the second test object from 40 projections in $[0, \frac{2}{3}\pi]$

Figure  6

Reconstruction of the second test object from 40 projections
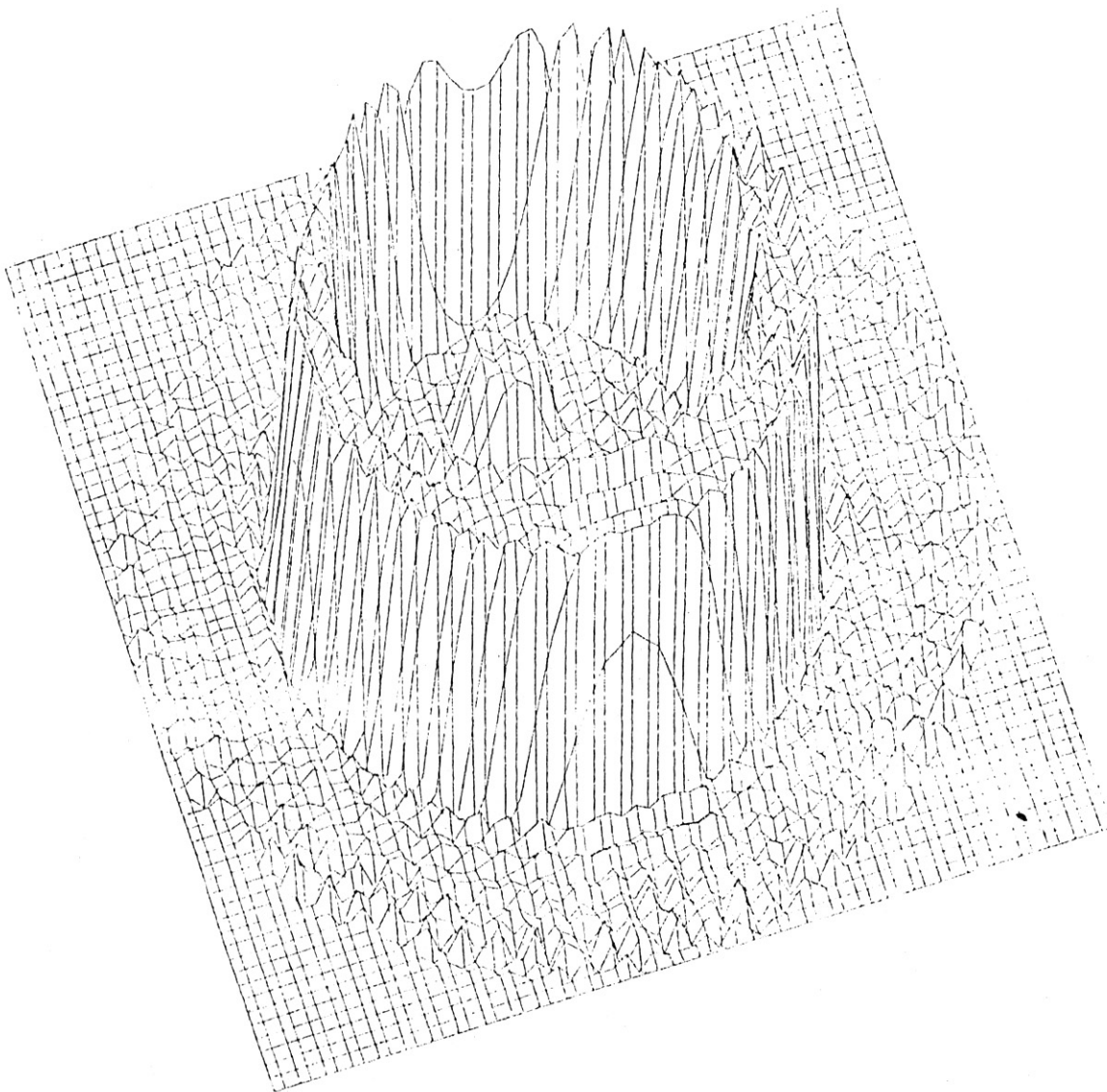in [O,π[

Figure 7

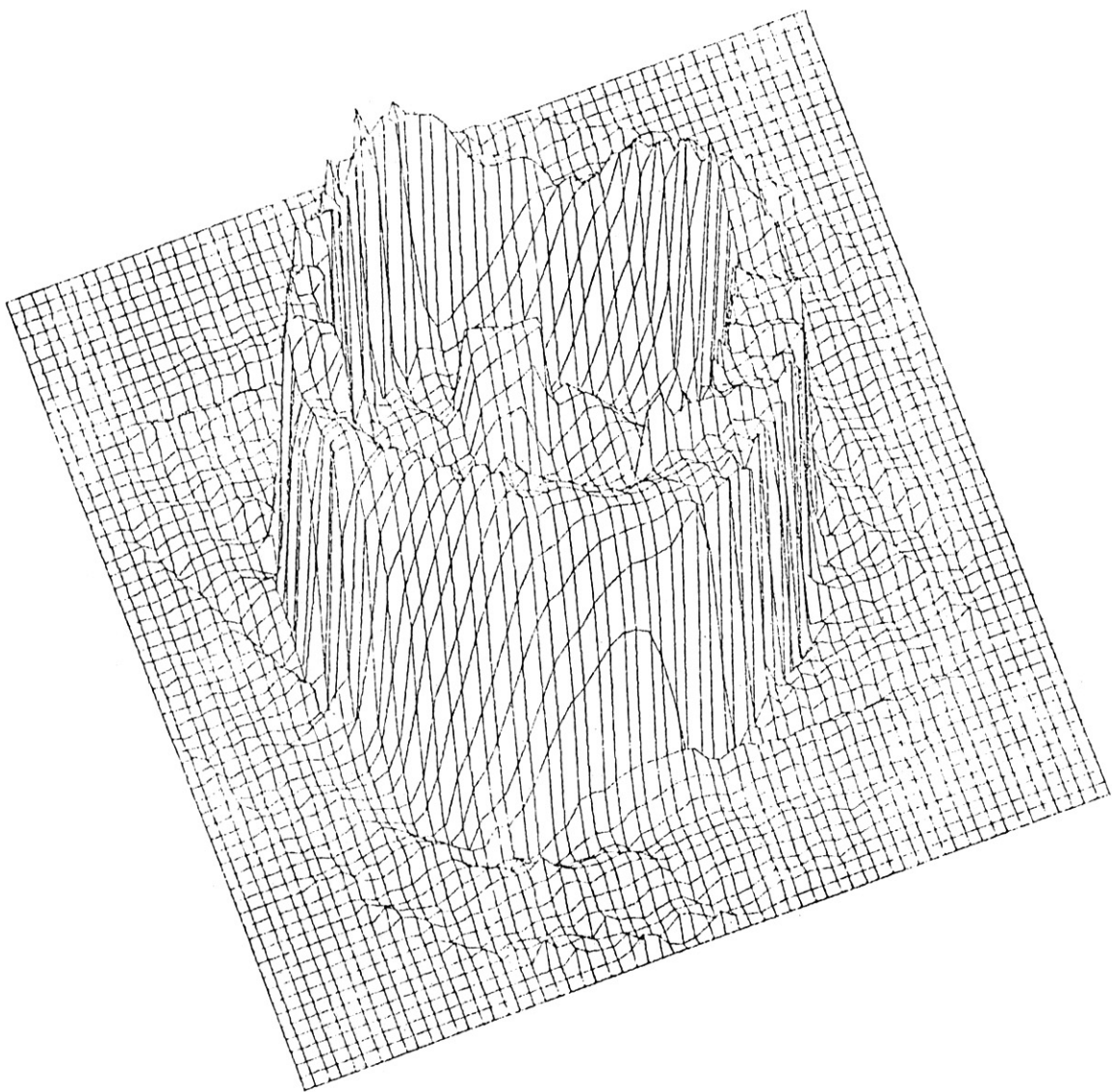Reconstruction of the second test object from 60 projections
in $[0, \frac{2}{3}\pi]$

Figure  8

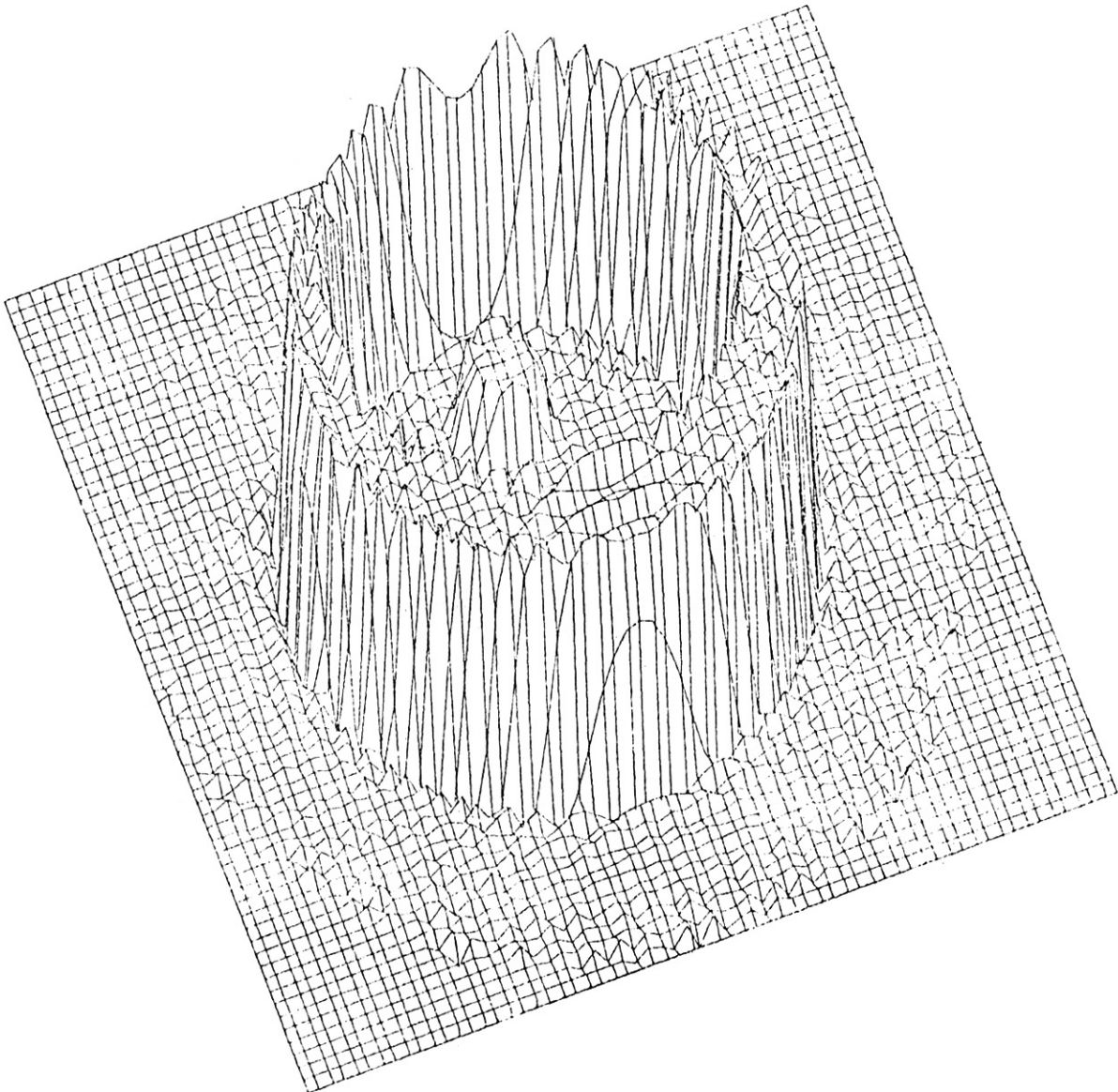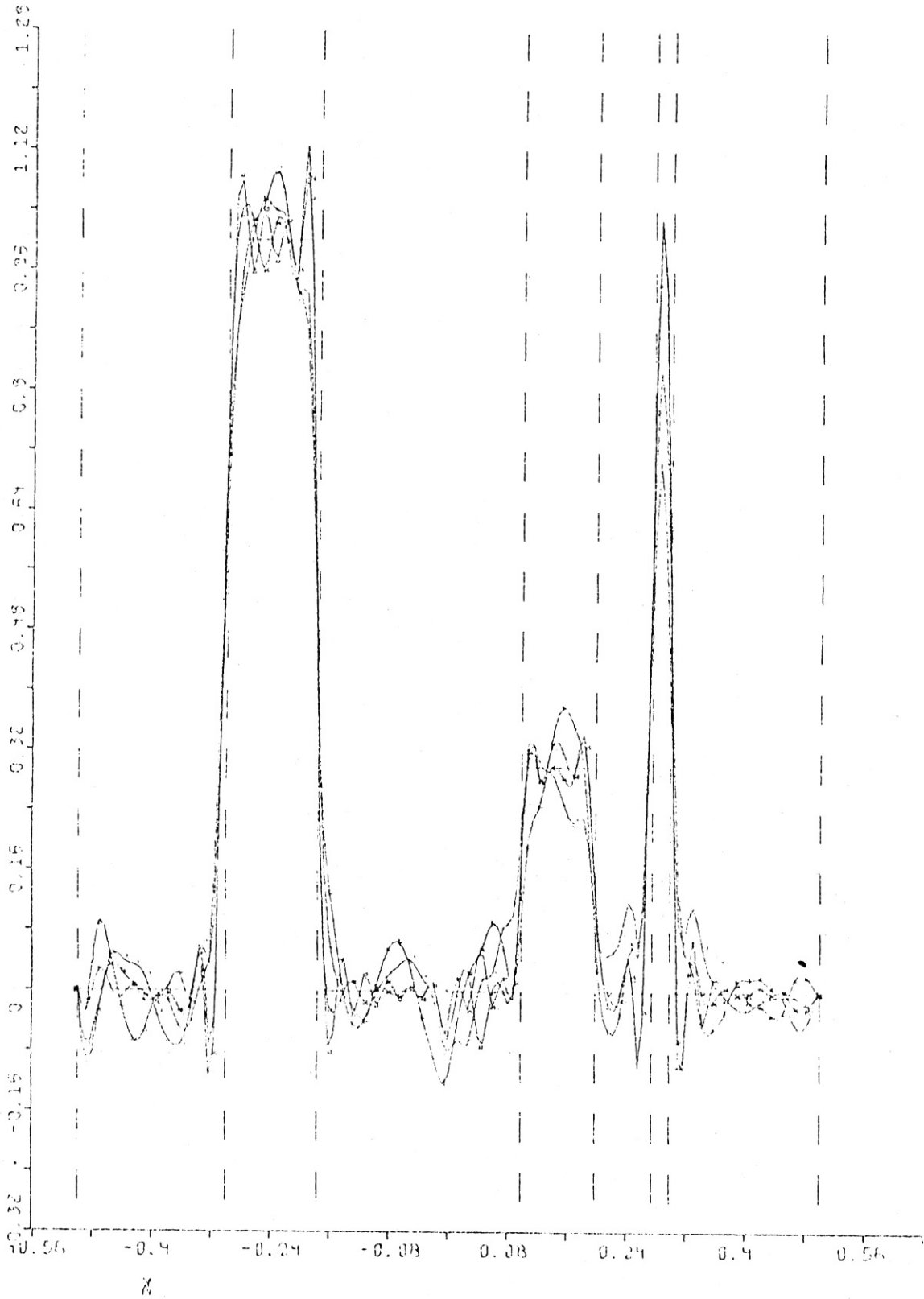Reconstruction of the second test object from 60 projections
in [0,π[

Figure 9         Cut at y = 0

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