Beurling-type representation of invariant subspaces in reproducing kernel Hilbert spaces

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To

my parents

Abstract

By Beurling's theorem, the orthogonal projection onto a multiplier invariant subspace M of the Hardy space $H^2(\mathbb{D})$ over the complex unit disk can be represented as $P_M = M_\phi M_\phi^*$, where ϕ is a suitable inner function. This result essentially remains true for arbitrary Nevanlinna-Pick spaces but fails in more general settings such as the Bergman space. We therefore introduce the notion of Beurling decomposability of subspaces: An invariant subspace M of a reproducing kernel space $\mathcal{H}$ is called Beurling decomposable if there exist (operator-valued) multipliers ϕ_1, ϕ_2 such that $P_M = M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^*$ and $M = \mathrm{ran} M_{\phi_1}$. Our aim is to characterize Beurling-decomposable subspaces by means of the core function and the core operator. More precisely, an invariant subspace M of $\mathcal{H}$ is Beurling decomposable precisely if its core function induces a completely bounded Schur multiplication on $B(\mathcal{H})$, defined in an appropriate way. These Schur multiplications turn out to be $(\mathcal{M}(\mathcal{H}), \overline{\mathcal{M}(\mathcal{H})}^{op})$-module homomorphisms on $B(\mathcal{H})$ (where $\mathcal{M}(\mathcal{H})$ denotes the multiplier algebra of $\mathcal{H}$). This allows us, in formal analogy to the case of classical Schur multipliers and to the study of multipliers of the Fourier algebra $A(G)$, to make use of the representation theory for completely bounded module homomorphisms. As an application, we show that, for the standard reproducing kernel Hilbert spaces over bounded symmetric domains, every finite-codimensional submodule M is Beurling decomposable and, in many concrete situations, can be represented as $M = \sum_{i=1}^{r} p_i \mathcal{H}$ with suitable polynomials p_i. We thus extend well-known results of Ahern and Clark, Axler and Bourdon and Guo. Furthermore, we prove that, in vector-valued Hardy spaces over bounded symmetric domains, defect functions of Beurling decomposable subspaces have boundary values almost everywhere on the Shilov boundary of D and, moreover, that these boundary values are projections of constant rank. This is a complete generalization of results of Guo and of Greene, Richter and Sundberg. Finally, we characterize the Beurling decomposable subspaces of the Bergman space $L_a^2(\mathbb{D})$. As a byproduct of the techiques developed in this paper, we obtain a new proof of the 'Wandering Subspace Theorem' for the Bergman space.

Abstract

Eine mögliche Formulierung des Satzes von Beurling für den Hardyraum $H^2(\mathbb{D})$ besagt, dass die Orthogonalprojektion P_M auf jeden invarianten Teilraum M von $H^2(\mathbb{D})$ vermöge eines geeigneten Multiplikationsoperators M_ϕ als $P_M = M_\phi M_\phi^*$ faktorisiert werden kann. Läßt man an dieser Stelle vektorwertige Multiplier ϕ zu, so charakterisiert diese Faktorisierungseigenschaft genau die Klasse der Nevanlinna-Pick-Räume. In der vorliegenden Arbeit werden allgemeiner invariante Teilräume M funktionaler Hilberträume untersucht, für die mit geeigneten operatorwertigen Multipliern $P_M = M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^*$ und zusätzlich $M = \operatorname{ran} M_{\phi_1}$ gilt (die sogenannten Beurling-zerlegbaren Teilräume). Es stellt sich heraus, dass im allgemeinen zwar nicht alle invarianten Teilräume Beurling-zerlegbar sind, aber in den meisten praktischen Fällen alle endlich kodimensionalen Teilräume Beurling-zerlegbar sind. Dieses Ergebnis führt in sehr allgemeinen Situationen zu einer Lösung des Gleason-Problems und zu Verallgemeinerungen von bekannten Resultaten von Guo, von Ahern und Clark und von Axler und Bourdon. Ein weiteres Hauptresultat dieser Arbeit besagt, dass über die von Guo betrachtete 'core function' ein enger Zusammenhang zwischen Beurling-zerlegbaren Teilräumen und einer verallgemeinerten Form von Schur-Multipliern besteht. Das Konzept der Beurling-Zerlegbarkeit ermöglicht es uns außerdem, Resultate von Guo und von Greene, Richter und Sundberg über das Randwertverhalten von 'defect functions' in Hardyräumen zu verallgemeinern. Abschließend wird das Phänomen der Beurling-Zerlegbarkeit in der Situation des Bergmanraumes $L^2_a(\mathbb{D})$ diskutiert. Die im Verlauf der Arbeit entwickelten Techniken führen zu einem neuen und elementaren Beweis des von Aleman, Richter und Sundberg bewiesenen 'Wandering Subspace'-Theorems für den Bergmanraum.

Zusammenfassung

Der Satz von Beurling in seiner klassischen Form besagt, dass für jeden invarianten Teilraum M des Hardyraumes $H^2(\mathbb{D})$ eine innere Funktion ϕ existiert, so dass die Orthogonalprojektion P_M auf M als $P_M = M_\phi M_\phi^*$ faktorisiert werden kann. Ähnliche Ergebnisse gelten für den Arvesonraum $H(\mathbb{B}_d)$ über der komplexen Kugel bzw. für beliebige Nevanlinna-Pick-Räume. Über allgemeineren Räumen wie dem Bergmanraum zeigen einfachste Beispiele, dass der Satz von Beurling in dieser Form nicht gelten kann. Da also im allgemeinen die Darstellbarkeit von P_M als $P_M = M_\phi M_\phi^*$ (mit einem operatorwertigen Multiplier ϕ) eine zu starke Forderung zu sein scheint, betrachten wir in der vorliegenden Arbeit Teilräume M von funktionalen Hilberträumen $\mathcal{H}$, die einer schwächeren Darstellungsbedingung genügen, die sogenannten Beurling-zerlegbaren Teilräume. Ein Teilraum M eines funktionalen Hilbertraumes $\mathcal{H}$ heißt hierbei Beurling-zerlegbar, wenn operatorwertige Multiplier ϕ_1, ϕ_2 existieren, so dass $P_M = M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^*$ und $M = \operatorname{ran} M_{\phi_1}$ gilt. Solche Teilräume sind offenbar Multiplier-invariant, aber, wie sich herausstellt, sind im allgemeinen nicht alle invarianten Teilräume Beurling-zerlegbar. Nichtsdestotrotz ist die Klasse der Beurling-zerlegbaren Teilräume in den praktischen Beispielen viel größer als die Klasse der Teilräume, deren Orthogonalprojektion eine Darstellung der Form $P_M = M_\phi M_\phi^*$ besitzt.

Das Hauptziel dieser Arbeit ist es nun, Charakterisierungen Beurling-zerlegbarer Teilräume zu entwickeln. Das erste Ergebnis in dieser Richtung besagt, dass unter gewissen Voraussetzungen an den zugrundeliegenden Raum $\mathcal{H}$ ein invarianter Teilraum genau dann Beurling-zerlegbar ist, wenn seine 'core function' G_M eine Darstellung $G_M(z,w) = \phi_1(z)\phi_1(w)^* - \phi_2(z)\phi_2(w)^*$ mit Multipliern ϕ_1, ϕ_2 besitzt, oder, äquivalent dazu, als $G_M(z,w) = \phi(z)\psi(w)^*$ mit Multipliern ϕ, ψ faktorisiert werden kann. Das motiviert die allgemeinere Frage, welche beliebigen Kerne G auf diese Art faktorisiert werden können. Es stellt sich nun heraus, dass diese Kerne G genau als vollständig beschränkte punktweise Multiplier einer kanonisch mit $B(\mathcal{H})$ assoziierten C^*-Algebra von Kernen auftreten. Als Spezialfall beinhaltet dieses Ergebnis große Teile der Theorie der klassischen Schurmultiplikationen über endlichen und unendlichen Matrizen.

Neben dieser allgemeinen Beschreibung Beurling-zerlegbarer Teilräume ist es möglich, für spezielle Klassen von invarianten Teilräumen, wie etwa endlichkodimensionale Teilräume und Teilräume endlichen Ranges, sehr viel konkretere Charakterisierungen zu geben. Ein weiteres Hauptresultat der vorliegenden Arbeit besagt, dass für viele analytische Hilbertmoduln alle endlichkodimensionalen Untermoduln Beurling-zerlegbar sind. Insbesondere gilt dies für die natürlichen funktionalen Hilberträume über beschränkten symmetrischen Gebieten. Dies erlaubt es uns, in vielen Situationen das wesentliche rechte Spektrum des Multiplikationstupels $M_{\mathbf{z}}$ zu berechnen, was dann einerseits sofort die Lösbarkeit des Gleason-Problems impliziert und andererseits zeigt,

dass die endlichkodimensionalen Untermoduln der betreffenden Räume genau die Teilräume von der Form $M = \sum_{i=1}^{r} p_i \mathcal{H}$ (mit geeigneten Polynomen p_i) sind. Resultate dieser Art wurden in Spezialfällen von Ahern und Clark, von Axler und Bourdon sowie von Guo bewiesen.

Als weitere Anwendung zeigen wir, dass in den Hardyräumen über beschränkten symmetrischen Gebieten die Defektfunktion D_M eines Beurling-zerlegbaren Teilraumes M fast überall auf dem Shilovrand Randwerte besitzt und dass diese Randwerte fast überall Orthogonalprojektionen von konstantem Rang sind. Dieses Ergebnis verallgemeinert gleichzeitig Resultate von Guo und von Greene, Richter und Sundberg.

Schlussendlich charakterisieren wir die Beurling-zerlegbaren Teilräume M des Bergmanraumes $L_a^2(\mathbb{D})$ anhand ihrer 'extremal function' g_M. Es stellt sich heraus, dass ein Teilraum M von $L_a^2(\mathbb{D})$ genau dann Beurling-zerlegbar ist, wenn g_M beschränkt ist. Diese Erkenntnis erlaubt es uns insbesondere, Beispiele Beurling-zerlegbarer Teilräume von unendlicher Kodimension anzugeben. Die im Verlauf der Arbeit und insbesondere bei der Behandlung des Bergmanraumes entwickelten Techniken liefern außerdem einen neuen und elementaren Beweis des 'Wandering Subspace'- Theorems, welches ursprünglich von Aleman, Richter und Sundberg bewiesen wurde.

Contents

Introduction

One of the central problems of modern functional analysis is to explore the invariant subspace lattice of single operators or of classes of operators on Hilbert spaces. The discovery of the Jordan canonical form of matrices exhaustively solved this problem for operators on finite-dimensional Hilbert spaces, and the theory of spectral measures, exhibited in the middle of the last century, opened the road to understanding the invariant subspace lattice of normal operators. Until today, the probably best-understood operator which does not belong to the classes mentioned above is the Hardy shift, that is the operator of multiplication with the coordinate function

$$M_z : H^2(\mathbb{D}) \to H^2(\mathbb{D}) \ , \ M_z f = z \cdot f$$

defined on the Hardy space $H^2(\mathbb{D})$ over the complex unit disk. The invariant subspace lattice of the Hardy shift is completely described by Beurling's famous theorem [20], which can be stated in the following three equivalent formulations:

Theorem (Beurling). *Suppose that M is an invariant subspace of $H^2(\mathbb{D})$ (this means by definition that M is closed and invariant for the Hardy shift M_z).*

(a) There exists an inner function η on $\mathbb{D}$ such that $M = \eta \cdot H^2(\mathbb{D})$ holds.

(b) There exists a bounded holomorphic function η such that the orthogonal projection P_M of $H^2(\mathbb{D})$ onto M admits a factorization $P_M = M_\eta M_\eta^*$ (where $M_\eta : H^2(\mathbb{D}) \to H^2(\mathbb{D})$, $f \mapsto \eta \cdot f$, denotes the multiplication operator with symbol η).

(c) $[M \ominus z \cdot M] = M$ (where $[Y]$ denotes the smallest invariant subspace of $H^2(\mathbb{D})$ containing a given subset Y of $H^2(\mathbb{D})$).

After Beurling's result appeared in the late 1940's, the question arose which of the versions presented above remain true if the operator M_z is considered on different spaces of holomorphic functions such as the Bergman space $L^2_a(\mathbb{D})$. It soon turned out that the corresponding statements (a) and (b) must fail in the Bergman space. For if M is an invariant (that is, M_z-invariant) subspace of $L^2_a(\mathbb{D})$, then both (a) and (b) would imply that M contains bounded holomorphic functions and, as a consequence, that the common zero set of M is a Blaschke sequence. However, there are known examples of Bergman space zero sets which are not Blaschke. A detailed discussion of these topics can be found in Chapter 5 of this paper. A remarkable and surprising result was exhibited in 1996 by Aleman, Richter and Sundberg [3], who recognized that version (c) of Beurling's theorem, known as the *wandering subspace theorem*, remains valid in the Bergman space setting. More general results were proved by Shimorin [67] and McCullough and Richter [52]. In Corollary 5.2.5 of this paper we present a new and elementary proof of the Bergman space version of the *wandering subspace theorem*.

The search for multivariable analogues of Beurling's theorem turns out to be even more delicate. The first problem one encounters is that there are different canonical multivariable generalizations of the Hardy space $H^2(\mathbb{D})$ such as the Hardy space $H^2(\mathbb{B}_d)$ over the complex unit ball, the Hardy space $H^2(\mathbb{D}^d)$ over the unit polydisk or the symmetric Fock space $H(\mathbb{B}_d)$, also known as the Arveson space. As observed by Drury [31], Müller and Vasilescu [54] and more recently by Arveson [13], the symmetric Fock space is the 'correct' multivariable replacement of the Hardy space $H^2(\mathbb{D})$ in the dilation theory for spherical contractions. Moreover, a Beurling-type theorem, analogous to parts (a) and (b) of the classical theorem, can be proved for the Arveson space:

Theorem (Greene, Richter, Sundberg). *Suppose that M is an invariant subspace of the Arveson space $H(\mathbb{B}_d)$ (that is, M is closed and invariant under the multiplications $M_{\mathbf{z}_i}$ by the coordinate functions). Then there exists a (possibly finite) sequence $(\phi_n)_n$ of multipliers of $H(\mathbb{B}_d)$ such that $P_M = \sum_n M_{\phi_n} M_{\phi_n}^*$ holds (the series converging in the strong operator topology). Moreover, every sequence $(\phi_n)_n$ with this property is an inner sequence, which means by definition that the function $\sum_n |\phi_n|^2$ has non-tangential limit 1 almost everywhere on the boundary of $\mathbb{B}_d$.*

This result was proved by Arveson [14] in special cases and by Green, Richter and Sundberg [39] in the general case. An example given by Rudin (see [62], p. 71) shows that a result of this type must fail when passing to the Hardy space $H^2(\mathbb{D}^d)$. Therefore, the question arises which differences between the Hardy spaces and the Arveson space are responsible for this unexpected failure of Beurling's theorem in $H^2(\mathbb{B}_d)$ and $H^2(\mathbb{D}^d)$. One answer is possibly given through the fact that the Arveson space belongs to the class of Nevanlinna-Pick spaces. A Nevanlinna-Pick space (NP space) over an arbitrary set X is by definition a reproducing kernel Hilbert space $\mathcal{H}$ of complex-valued functions such that the reproducing kernel K of $\mathcal{H}$ has no zeroes and such that $1 - \frac{1}{K}$ is a positive definite function. It is not hard to see that, for $d \geq 2$, the Hardy spaces discussed above (and also the Bergman space over the disk) are not of this type.

Without much effort, it is now possible to prove similar results for arbitrary NP spaces. The first results of this general Beurling type are due to McCullough and Trent [53]:

Theorem. *Suppose that $\mathcal{H}$ is an NP space and that M is an invariant subspace of $\mathcal{H}$. Then there exist a Hilbert space $\mathcal{D}$ and a multiplier $\phi : X \to B(\mathcal{D}, \mathbb{C})$ of $\mathcal{H}$ such that $P_M = M_\phi M_\phi^*$ and, in particular, $M = \operatorname{ran} M_\phi$.*

Note that in general, a closed subspace M of a reproducing kernel Hilbert space $\mathcal{H}$ is called invariant if $\alpha \cdot M \subset M$ holds for all multipliers α of $\mathcal{H}$. For the Arveson space and also for the Hardy and Bergman spaces, this definition clearly coincides with the previous definition of invariant subspaces.

In the sequel it was observed by Guo et al. (cf. [40] and [41]) that in an arbitrary reproducing kernel Hilbert space $\mathcal{H}$ with zero-free reproducing kernel K the orthogonal projection P_M onto an invariant subspace M can be factorized in the form $P_M = M_\phi M_\phi^*$ (as in the NP situation) precisely if the so-called *core function*

$$G_M : X \times X \to \mathbb{C}\ ,\ G_M(z,w) = \frac{K_M(z,w)}{K(z,w)}$$

is positive definite. Here K_M denotes the reproducing kernel of M. Therefore, the NP space version of Beurling's theorem could be rephrased as follows:

Theorem. *Suppose that $\mathcal{H}$ is an NP space. Then every invariant subspace M of G_M has a positive definite core function.*

It can easily be seen (cf. Proposition 3.3.9) that NP spaces are essentially the only reproducing kernel Hilbert spaces in which all invariant subspaces possess a positive definite core function. In fact, if D is a bounded symmetric domain in $\mathbb{C}^d$ $(d \geq 2)$, then the prototypical invariant subspace $M = \{f \in H^2(D)\ ; f(0) = 0\}$ of the Hardy space $H^2(D)$ has a non-positive definite core function. The situation is even worse in the Bergman space $L_a^2(D)$, where no non-trivial invariant subspace has a positive definite core function (cf. Example 3.3.10).

Now consider a reproducing kernel Hilbert space $\mathcal{H}$ with zero-free reproducing kernel K. The core function G_M of an invariant subspace M of $\mathcal{H}$ is positive definite precisely if it can be written in the form $G_M(z,w) = \phi(z)\phi(w)^*$ with a suitable multiplier $\phi : X \to B(\mathcal{D}, \mathbb{C})$ (cf. Proposition 3.3.7). On the other hand, in non-NP spaces, there are in general only few invariant subspaces of this type. In many specific spaces however, there is a rich supply of invariant subspaces M whose core function admits a decomposition

$$G_M(z,w) = \phi_1(z)\phi_1(w)^* - \phi_2(z)\phi_2(w)^* \quad (z,w \in X) \tag{1}$$

with multipliers $\phi_i : X \to B(\mathcal{D}_i, \mathbb{C})$. The last property is equivalent to the condition that the orthogonal projection P_M onto M can be written as

$$P_M = M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^*,$$

which implies that $M \subset \operatorname{ran} M_{\phi_1}$. If additionally $\operatorname{ran} M_{\phi_1} = M$ holds for some choice of ϕ_1 and ϕ_2, then the space M is said to be *Beurling decomposable*, and the pair (ϕ_1, ϕ_2) is called a *Beurling decomposition* of M (cf. Definition 3.3.1). It is one of our central results (cf. Theorem 3.3.5) that invariant subspaces of Beurling spaces whose core function can be decomposed as in (1) are automatically Beurling decomposable. In this context, a reproducing kernel Hilbert space $\mathcal{H}$ of complex-valued functions on a set X is called a *Beurling space* (Definition 3.1.1) if the following conditions are fulfilled:

(i) The reproducing kernel K of $\mathcal{H}$ has no zeroes.

(ii) The inverse kernel $\frac{1}{K}$ admits a representation of the form

$$\frac{1}{K(z,w)} = \beta(z)\beta(w)^* - \gamma(z)\gamma(w)^* \quad (z, w \in X)$$

with suitable multipliers $\beta : X \to B(\mathcal{B}, \mathbb{C})$ and $\gamma : X \to B(\mathcal{C}, \mathbb{C})$.

(iii) The functions $K(\cdot, w)$ $(w \in X)$ are multipliers of $\mathcal{H}$.

We shall see in Section 3.1 that NP spaces as well as the standard reproducing kernel Hilbert spaces over bounded symmetric domains (which of course include the Arveson space and the Hardy or Bergman spaces discussed above) are Beurling spaces.

Clearly it is now a mandatory task to determine which subspaces of Beurling spaces are Beurling decomposable. First of all, it follows immediately from the definition that every Beurling decomposable subspace contains non-trivial multipliers. Hence the same counter-examples we used earlier to point out that there are invariant subspaces with non-positive definite core function, now reveal that not all invariant subspaces are Beurling decomposable. However, as a positive result, we are able to prove that in the standard reproducing kernel Hilbert spaces over bounded symmetric domains every finite-codimensional invariant subspace is automatically Beurling decomposable (cf. Theorem 4.2.5). As a preparation of this result we exhibit the following characterization of finite-codimensional Beurling decomposable subspaces of arbitrary Beurling spaces (cf. Proposition 3.3.11):

Theorem. *Suppose that $\mathcal{H}$ is a Beurling space and that M is a finite-codimensional invariant subspace of $\mathcal{H}$. Then M is Beurling decomposable if and only if $M^\perp$ consists entirely of multipliers.*

Let $\mathcal{H}$ be a Beurling space with reproducing kernel K. One can show that condition (ii) from the definition of Beurling spaces ensures that, given a closed subspace M of $\mathcal{H}$, there exists a (necessarily unique) operator $\Delta_M \in B(\mathcal{H})$ satisfying

$$G_M(z,w) = \langle \Delta_M K(\cdot, w), K(\cdot, z) \rangle \quad (z, w \in X).$$

Following common terminology, we call this operator the *core operator* associated with M (see [40], [41] and also [76]). The *rank* of an invariant subspace M of $\mathcal{H}$ is then defined as the rank of its core operator. With these definitions it is possible to prove the following result (cf. Proposition 3.3.13):

Theorem. *Suppose that $\mathcal{H}$ is a Beurling space and that M is a finite-rank invariant subspace of $\mathcal{H}$. Then M is Beurling decomposable if and only if $\operatorname{ran}\Delta_M$ consists entirely of multipliers.*

In order to find a characterization of arbitrary Beurling decomposable subspaces, we turn towards the more general question which kernels $G : X \times X \to \mathbb{C}$ can be decomposed in the form

$$G(z,w) = \phi_1(z)\phi_1(w)^* - \phi_2(z)\phi_2(w)^* \quad (z,w \in X) \tag{2}$$

with multipliers $\phi_i : X \to B(\mathcal{D}_i, \mathbb{C})$ of $\mathcal{H}$ or, equivalently, can be factorized as

$$G(z,w) = \phi(z)\psi(w)^* \quad (z,w \in X) \tag{3}$$

with multpliers $\phi, \psi : X \to B(\mathcal{D}, \mathbb{C})$.

To answer this question, let us consider a reproducing kernel Hilbert space $\mathcal{H}$ with reproducing kernel K. A kernel $L : X \times X \to \mathbb{C}$ is called *subordinate* to K if there exists a (necessarily unique) operator $T \in B(\mathcal{H})$ such that $L = \Lambda_T$, where

$$\Lambda_T : X \times X \to \mathbb{C}\ ,\ \Lambda_T(z,w) = \langle TK(\cdot,w), K(\cdot,z)\rangle.$$

The set of all such kernels will be denoted by $B(K)$. Then the correspondence $T \leftrightarrow \Lambda_T$ between $B(\mathcal{H})$ and $B(K)$ can be used to turn $B(K)$ into a C^*-algebra. The concept of subordinate kernels originally appeared in [17] and is systematically developed in Section 1.6 of this paper. A kernel $G : X \times X \to \mathbb{C}$ is called a *Schur kernel* or *Schur multiplier* (with respect to K) if it maps $B(K)$ into itself by pointwise multiplication. It is not difficult to verify that the corresponding multiplication operator

$$S_G : B(K) \to B(K)\ ,\ L \mapsto G \cdot L$$

is automatically continuous. Note that the name 'Schur kernel' is motivated by the following observation: If $X = \{1, \dots, n\}$ is a finite set and if K is the diagonal kernel given by $K : X \times X \to \mathbb{C}\ ,\ K(i,j) = \delta_{ij}$, then $B(K)$ can be identified canonically with the space of $n \times n$-matrices. Under this identification, the pointwise product of kernels becomes the usual Schur (or Hadamard) product of matrices, and every kernel $G : X \times X \to \mathbb{C}$ is a Schur kernel.

Returning to the question raised above, one observes that every kernel G that admits a factorization of the form (3) is a Schur kernel and moreover, that the multiplication operator S_G is completely bounded. It is the main result of the second chapter that the converse of this statement is true for a large class of reproducing kernel Hilbert spaces (cf. Theorem 2.3.9):

Theorem. *Let $\mathcal{H}$ be a reproducing kernel Hilbert space with reproducing kernel K and suppose that the multipliers form a dense subset of $\mathcal{H}$ or that $\mathcal{H}$ is regular. Then a kernel $G : X \times X \to \mathbb{C}$ can be factorized as*

$$G(z,w) = \phi(z)\psi(w)^* \quad (z,w \in X)$$

with suitable multipliers $\phi, \psi : X \to B(\mathcal{D}, \mathbb{C})$ precisely if it is a Schur kernel and the operator S_G is completely bounded. Moreover, the multipliers ϕ, ψ can be chosen in such a way that $\|S_G\|_{cb} = \|M_\phi\|\|M_\psi\|$ holds.

Note that the case of *regular reproducing kernel Hilbert spaces* (see Section 1.3 for an exact definition) is not essential for the results of this paper, since the holomorphic function spaces in which we are mainly interested always contain their multipliers as a dense subset. However, the concept of regular reproducing kernel Hilbert spaces allows us to cover the known representation results for classical Schur multipliers of infinite matrices (cf. Paulsen, Corollary 8.8): An infinite matrix G defines a contractive Schur multiplier on $B(l^2)$, realized as space of infinite matrices with respect to the standard orthonormal basis of l^2, if and only if there exist a Hilbert space $\mathcal{D}$ and sequences $(x_n)_n$ and $(y_n)_n$ in the unit ball of $\mathcal{D}$ such that $G_{ij} = \langle x_j, y_i \rangle$ holds for all $i, j \in \mathbb{N}$. Clearly, this result becomes a special case of the above theorem if one realizes the following facts: The space l^2 is the reproducing kernel Hilbert space associated with the diagonal kernel $K : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$, $K(i,j) = \delta_{ij}$. Furthermore, l^2 is regular in the sense of Definition 1.3.5. At this point, we note that the multiplier algebra of l^2 is l^∞, which is clearly not contained (and therefore not dense) in l^2. Finally, it is well known that every Schur multiplication S_G on $B(l^2)$ is automatically completely bounded with $\|S_G\|_{cb} = \|S_G\|$.

The fact that the norm and the *cb*-norm of classical Schur multiplications coincide, motivates the question whether the same is true for our general Schur multipliers. Although we conjecture that, at least for the holomorphic function spaces we are interested in, the answer is affirmative, we are unfortunately not able to present a proof. The problem is that in the classical case $\mathcal{H} = l^2$ the algebra $M(\mathcal{H})$, consisting of all multiplication operators on $\mathcal{H}$, coincides with the space of diagonal matrices and is hence a C^*-subalgebra of $B(\mathcal{H})$. It is very elementary to see that this cannot hold true for the holomorphic function spaces considered in this paper. However, the known proofs (see for example [29], [68], [57] and [55], Chapter 8) of the equality of norm and *cb*-norm of classical Schur multiplications rely heavily on this natural C^*-structure of the multplier algebra. We note that there is another, formally similar situation, in which the automatic complete boundedness of Schur-type multiplication operators is known. Namely, it is proved in [22] that every Herz-Schur multiplier S over a locally compact group is automatically completely bounded with $\|S\|_{cb} = \|S\|$. However, the proofs given there depend to a large extent on the underlying group structure and can probably not be transferred to our situation.

We conclude this introduction with a brief outline of this paper.

The first chapter is preliminary and provides the necessary basics from the theory of reproducing kernel Hilbert spaces and their multipliers. Of particular interest are Sections 1.4 and 1.5. They establish (partially known) facts about hermitian kernels and, moreover, the connection between hermitian kernels and reproducing kernel Kreĭn spaces. Section 1.6 contains the above mentioned introduction to the theory of subordinate kernels which was developed in [17] in the scalar case. Section 1.7 supplies the necessary material about multipliers of reproducing kernel Hilbert spaces. We mention in particular Proposition 1.7.9, which provides a seemingly new and purely algebraic characterization of multiplication operators.

The second chapter deals with the class of Schur kernels discussed above. Section 2.1 contains the necessary definitions and basic properties of Schur kernels. In Section 2.3 we concentrate on completely bounded Schur multiplications. The key tool in the proof of the main result (Theorem 2.3.9) is a representation theorem for completely bounded normal module maps (cf. Theorem 2.3.5), which appears in [48] in a slightly different form.

Chapter 3 can be regarded as the central part of this paper. It contains the definitions of Beurling spaces and of Beurling decomposable subspaces. The main results of this chapter are Theorem 3.3.5 and Propositions 3.3.11 and 3.3.13. It should be remarked that, unlike the presentation in this introduction, the results of the third chapter are formulated in a fully vector-valued context.

The fourth chapter is dedicated to the study of Beurling decomposability in the setting of analytic Beurling modules. Analytic Beurling modules arise as a special type of analytic Hilbert modules satisfying some very natural additional conditions. For the definition and basic properties of analytic Hilbert modules, the reader is referred to the book of Guo [25]. The main result of this chapter is Theorem 4.2.5, which states that finite-codimensional submodules of analytic Beurling modules are Beurling decomposable. It should be noted that the theory developed in this chapter applies to the standard reproducing kernel Hilbert spaces over bounded symmetric domains. As an application of Theorem 4.2.5, we are able to prove the following result (cf. Proposition 4.2.6):

Theorem. *Suppose that $\mathcal{H}$ is an analytic Beurling module over some bounded open set $D \subset \mathbb{C}^d$ such that the inverse kernel $\frac{1}{K}$ is a polynomial in z and $\overline{w}$. Then the right essential spectrum $\sigma_{re}(M_{\mathbf{z}})$ of the tuple $M_{\mathbf{z}} = (M_{\mathbf{z}_1}, \ldots, M_{\mathbf{z}_d})$ is ∂D.*

This leads immediately to a supplement (Corollary 4.2.7) of the famous result of

Ahern and Clark (Theorem 2.2.3 in [25]):

Theorem. *Suppose that $\mathcal{H}$ is an analytic Beurling module over D such that the inverse kernel $\frac{1}{K}$ is a polynomial in z and $\overline{w}$. Then the finite-codimensional submodules of $\mathcal{H}$ are precisely the closed subspaces M of the form $M = \sum_{i=1}^{r} p_i \cdot \mathcal{H}$, where $r \in \mathbb{N}$ and $p = (p_1, \ldots, p_r)$ is a tuple of polynomials with $Z(p) \subset D$.*

As a consequence, we deduce that Gleason's problem can be solved for these spaces.

The fifth and final chapter explores the phenomenon of Beurling decomposability in Hardy and Bergman spaces. Section 5.1 is devoted to the Hardy spaces $H^2(D)$ over bounded symmetric domains. The main result (Theorem 5.1.1) of this section describes (also in the vector-valued case) the boundary values of the defect function and can hence be regarded as a generalization of results of Guo [40],[41] and of Greene, Richter and Sundberg [39]. As an application we prove that every invariant subspace of $H^2(D)$ with positive definite core function automatically has rank one (cf. Proposition 5.1.3). This result clearly is a complete generalization of Beurling's classical theorem on $H^2(\mathbb{D})$. The main aim of Section 5.2 is to characterize the Beurling decomposable subspaces of the Bergman space $L_a^2(\mathbb{D})$. It turns out (cf. Proposition 5.2.7) that an invariant subspace M of $L_a^2(\mathbb{D})$ is Beurling decomposable precisely if its extremal function g_M is bounded. This allows us to present examples of Beurling decomposable subspaces which are infinite codimensional (cf. Example 5.2.10). As mentioned above, this section contains a new and elementary proof (Corollary 5.2.5) of the *wandering subspace theorem* for the Bergman space.

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1 Preliminaries and basic constructions

The aim of this first chapter is to recall the basic concepts of the theory of reproducing kernel Hilbert spaces, which can be found in many places in the literature. In fact, many of the stated results go back to the groundbreaking papers of Aronszajn [11] and Schwartz [66]. We will therefore sometimes only give sketches of the proofs or even omit the proofs entirely. Some of the presented results, including proofs, can also be found in [16].

Unless otherwise specified, let X denote an arbitrary non-empty set.

1.1 Reproducing kernel Hilbert spaces

Definition 1.1.1. *Let $\mathcal{E}$ be a Hilbert space. A Hilbert space $\mathcal{H} \subset \mathcal{E}^X$ is called a reproducing kernel (Hilbert) space if the point evaluations*

$$\delta_z : \mathcal{H} \to \mathcal{E}\ ,\ f \mapsto f(z) \quad (z \in X)$$

are continuous.

The name 'reproducing kernel Hilbert space' is justified by the following fundamental fact.

Proposition 1.1.2. *Suppose that $\mathcal{E}$ and $\mathcal{H} \subset \mathcal{E}^X$ are Hilbert spaces. Then the following are equivalent:*

(i) $\mathcal{H}$ is a reproducing kernel Hilbert space.

(ii) There exists a function $K : X \times X \to B(\mathcal{E})$ such that

- *the function $X \to \mathcal{E}\ ,\ z \mapsto K(z,w)y$ belongs to $\mathcal{H}$ for all $w \in X$ and $y \in \mathcal{E}$*
- *the equality*
$$\langle f(w), y\rangle = \langle f, K(\cdot, w)y\rangle$$
holds for all $f \in \mathcal{H}$, $w \in X$ and $y \in \mathcal{E}$.

In this case, the identity $K(z,w) = \delta_z\delta_w^*$ *holds for all* $z, w \in X$*. In particular, the function* K *is uniquely determined. Whenever* $(e_i)_i$ *is an orthonormal basis of* $\mathcal{H}$*, then* K *can be written as*

$$K(z,w) = \sum_i e_i(z) \otimes e_i(w) \quad (z, w \in X),$$

where the series converges in the weak operator topology. Furthermore, the set

$$\{K(\cdot, w)y \ ; \ w \in X, y \in \mathcal{E}\}$$

spans $\mathcal{H}$ *topologically.*

The function K associated with a reproducing kernel Hilbert space $\mathcal{H}$ as described in the preceding proposition will be called the reproducing kernel of $\mathcal{H}$.

Although in this preliminary chapter, we formulate all results in a fully vector-valued context, we shall focus in the sequel mainly on scalar reproducing kernel Hilbert spaces, that is, reproducing kernel Hilbert spaces of complex-valued functions. In this case, the reproducing kernel is regarded as a function with values in $\mathbb{C} \simeq B(\mathbb{C})$.

Example 1.1.3.

(a) For a given non-empty set I*, define as usual*

$$l^2(I) = \{f : I \to \mathbb{C} \ ; \ \|f\|^2 = \sum_{i \in I} |f(i)|^2 < \infty\}.$$

The point evaluations on $l^2(I)$ *are obviously continuous, and the reproducing kernel is given by*

$$K : I \times I \to \mathbb{C} \ , \ K(i,j) = \delta_{ij}.$$

This space has the very particular property that the family $(K(\cdot, i))_{i \in I}$ *is an orthonormal set in* $l^2(I)$*, which has far-reaching consequences as we shall see later. In a general reproducing kernel Hilbert space* $\mathcal{H} \subset \mathbb{C}^X$ *with reproducing kernel* K*, the set* $\{K(\cdot, z) \ ; \ z \in X\}$ *is always total in* $\mathcal{H}$*, but almost never orthonormal. In fact, the family* $(K(\cdot, z))_{z \in X}$ *is orthonormal if and only if* $\mathcal{H} = l^2(X)$*.*

(b) Let D *be an open subset of* $\mathbb{C}^d$ *and let* $\mathcal{E}$ *be a Hilbert space. Suppose that* $\mathcal{H} \subset \mathcal{E}^D$ *is a reproducing kernel space with reproducing kernel* K *and point evaluations* δ_z $(z \in D)$*. Then the following assertions are equivalent:*

  (i) $\mathcal{H} \subset \mathcal{O}(D, \mathcal{E})$*.*

  (ii) The mapping $D \to B(\mathcal{H}, \mathcal{E})$, $z \mapsto \delta_z$*, is analytic.*

  (iii) The kernel K *is sesquianalytic which means by definition analytic in the first* d *variables and antianalytic in the last* d *variables (where a function* $F : D \to B(\mathcal{E})$ *is called antianalytic if the conjugate function*

$$\tilde{F} : D \to B(\mathcal{E}) \ , \tilde{F}(z) = F(z)^*$$

is analytic).

In this case, $\mathcal{H}$ is separable whenever $\mathcal{E}$ is.

There is a fundamental relation between reproducing kernel Hilbert spaces and positive definite functions. Before we recapitulate this connection, let us recall the definition of positive definiteness.

Definition 1.1.4. *Suppose that $\mathcal{E}$ is Hilbert space. A function $K : X \times X \to B(\mathcal{E})$ is called positive definite if*

$$\sum_{i,j} \langle K(z_i, z_j)x_j, x_i \rangle \geq 0$$

for all finite sequences $(z_i)_{i=1}^n$ in X and $(x_i)_{i=1}^n$ in $\mathcal{E}$.

A fundamental result due to Moore and Aronszajn ([11], Section I.2) shows that there is a bijective correspondence between reproducing kernel Hilbert spaces and positive definite functions.

Theorem 1.1.5. *Suppose that $\mathcal{E}$ is a Hilbert space.*

(a) If $\mathcal{H} \subset \mathcal{E}^X$ is a reproducing kernel Hilbert space, then the reproducing kernel K of $\mathcal{H}$ is a positive definite function.

(b) If $K : X \times X \to B(\mathcal{E})$ is a positive definite function, then there exists a unique reproducing kernel Hilbert space with reproducing kernel K (the reproducing kernel Hilbert space associated with K).

A proof of this result (for the vector-valued case) can be found in [16].

Remark 1.1.6. *In particular, this result includes Kolmogorov's factorization theorem for positive definite functions:*

> *If $K : X \times X \to B(\mathcal{E})$ is a positive definite function, then there exist a Hilbert space $\mathcal{H}$ and a function $\phi : X \to B(\mathcal{H}, \mathcal{E})$ such that*
>
> $$K(z, w) = \phi(z)\phi(w)^* \tag{1.1.1}$$
>
> *holds for all $z, w \in X$ and such that*
>
> $$\mathcal{H} = \bigvee \{\phi(z)^* x \ ; \ z \in X, x \in \mathcal{E}\}. \tag{1.1.2}$$

In fact, by Proposition 1.1.2, one can simply choose $\mathcal{H}$ as the reproducing kernel Hilbert space associated with K and $\phi(z) = \delta_z$ (the point evaluation at $z \in X$).

A tuple $(\mathcal{H}, \phi)$ consisting of a Hilbert space $\mathcal{H}$ and a function $\phi : X \to B(\mathcal{H}, \mathcal{E})$ satisfying (1.1.1) will be called a (Kolmogorov) factorization of K. If in addition, *(1.1.2) is fulfilled, then we call* $(\mathcal{H}, \phi)$ *a minimal (Kolmogorov) factorization of* K*. It is easy to see that all minimal Kolmogorov factorizations of* K *are unitarily equivalent in the sense that, for any two minimal Kolmogorov factorizations* $(\mathcal{H}_1, \phi_1)$ *and* $(\mathcal{H}_2, \phi_2)$*, there exists a unitary* $U : \mathcal{H}_1 \to \mathcal{H}_2$ *such that*

$$U\phi_1(z)^* = \phi_2(z)^*$$

holds for all $z \in X$*. Hence it makes sense to define the rank of* K *by setting* $\operatorname{rank} K = \dim \mathcal{H}$*, where* $(\mathcal{H}, \phi)$ *is any minimal Kolmogorov factorization of* K*. Clearly* $\operatorname{rank} K$ *equals the dimension of the reproducing kernel Hilbert space associated with* K*.*

As a first consequence of this factorization result, we see that every positive definite function $K : X \times X \to B(\mathcal{E})$ *satisfies the following inequality of Cauchy-Schwarz type:*

$$|\langle K(z,w)y, x\rangle|^2 \leq \langle K(z,z)x, x\rangle\langle K(w,w)y, y\rangle \quad (z, w \in X, x, y \in \mathcal{E}).$$

Besides the Moore-Aronszajn result, probably the most important milestone in the investigation of positive definite functions is a famous result of I. Schur which, in its classical formulation, says that the entrywise product (also called Schur or Hadamard product) of two positive semi-definite matrices is positive semi-definite again. Actually, Schur's result can be deduced from Kolmogorov's factorization theorem. Before we do so, we introduce some notations that will used throughout the paper.

Definition 1.1.7. *Suppose that* $\mathcal{E}_1, \mathcal{E}_2, \mathcal{F}_1, \mathcal{F}_2$ *are Hilbert spaces and that* X, Y_1, Y_2 *are non-empty sets.*

(a) Let $f_i : Y_i \to \mathcal{E}_i$*,* $\phi_i : Y_i \to B(\mathcal{E}_i, \mathcal{F}_i)$ *and* $K_i : Y_i \times Y_i \to B(\mathcal{E}_i, \mathcal{F}_i)$ *be arbitrary functions* $(i = 1, 2)$*. We set* $Y = Y_1 \times Y_2$ *and define the outer products*

$$f_1 \circledast f_2 : Y \to \mathcal{E}_1 \otimes \mathcal{E}_2 \ , \ (z_1, z_2) \mapsto f_1(z_1) \otimes f_2(z_2),$$
$$\phi_1 \circledast \phi_2 : Y \to B(\mathcal{E}_1 \otimes \mathcal{E}_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \ , \ (z_1, z_2) \mapsto \phi_1(z_1) \otimes \phi_2(z_2)$$

and

$$K_1 \circledast K_2 : Y \times Y \to B(\mathcal{E}_1 \otimes \mathcal{E}_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \ ,$$
$$((z_1, z_2), (w_1, w_2)) \mapsto K_1(z_1, w_1) \otimes K_2(z_2, w_2).$$

(b) Let $f_i : X \to \mathcal{E}_i$*,* $\phi_i : X \to B(\mathcal{E}_i, \mathcal{F}_i)$ *and* $K_i : X \times X \to B(\mathcal{E}_i, \mathcal{F}_i)$ *be arbitrary functions* $(i = 1, 2)$*. Then the inner products are defined as*

$$f_1 * f_2 : X \to \mathcal{E}_1 \otimes \mathcal{E}_2 \ , \ z \mapsto f_1(z) \otimes f_2(z)$$
$$\phi_1 * \phi_2 : X \to B(\mathcal{E}_1 \otimes \mathcal{E}_2, \mathcal{F}_1 \otimes \mathcal{F}_2) \ , \ z \mapsto \phi_1(z) \otimes \phi_2(z)$$

and

$$K_1 * K_2 : X \times X \to B(\mathcal{E}_1 \otimes \mathcal{E}_2, \mathcal{F}_1 \otimes \mathcal{F}_2)\ ,\ (z,w) \mapsto K_1(z,w) \otimes K_2(z,w).$$

Remark 1.1.8. *Suppose that $\mathcal{E}_1, \mathcal{E}_2$ are Hilbert spaces and that $f_i : X \to \mathcal{E}_i$ and $K_i : X \times X \to B(\mathcal{E}_i)$ are arbitrary functions, $i = 1, 2$. Let $D = \{(z,z)\ ;\ z \in X\}$ denote the diagonal in $X \times X$. Then, via the bijection*

$$X \to D\ ,\ z \mapsto (z,z),$$

$f_1 * f_2$ is the restriction of $f_1 \circledast f_2$ to D, and $K_1 * K_2$ is the restriction of $K_1 \circledast K_2$ to $D \times D$. It is furthermore clear that the inner products defined above coincide with the usual pointwise product whenever $\mathcal{E}_1 = \mathbb{C}$ or $\mathcal{E}_2 = \mathbb{C}$.

Now Schur's result can be stated as follows.

Proposition 1.1.9. *Let X, Y_1, Y_2 be non-empty sets and let $\mathcal{E}_1, \mathcal{E}_2$ be Hilbert spaces.*

(a) Let $K_1 : Y_1 \times Y_1 \to B(\mathcal{E}_1)$ and $K_2 : Y_2 \times Y_2 \to B(\mathcal{E}_2)$ be positive definite functions. Then the outer product $K_1 \circledast K_2$ is positive definite.

(b) Let $K_1 : X \times X \to B(\mathcal{E}_1)$ and $K_2 : X \times X \to B(\mathcal{E}_2)$ be positive definite functions. Then is inner product $K_1 * K_2$ is positive definite.

Proof. In order to prove (a), let $(\mathcal{H}_1, \phi_1)$ and $(\mathcal{H}_2, \phi_2)$ be Kolmogorov factorizations of K_1, K_2. Then the mapping

$$\phi = \phi_1 \circledast \phi_2 : Y_1 \times Y_2 \to B(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{E}_1 \otimes \mathcal{E}_2)$$

defines a Kolmogorov factorization of $K_1 \circledast K_2$. This implies that $K_1 \circledast K_2$ is positive definite. Part (b) follows, since $K_1 * K_2$ is the restriction of the positive definite function $K_1 \circledast K_2$ to the diagonal of $X \times X$. $\square$

We shall frequently use this result tacitly or with the comment 'Products of positive definite functions are positive definite'.

Of particular interest among the spaces introduced in Example 1.1.3(b) are the standard reproducing kernel spaces over irreducible bounded symmetric domains in $\mathbb{C}^d$.

Example 1.1.10. *A bounded domain $D \subset \mathbb{C}^d$ is called symmetric if every two points in D can be interchanged by a self-inverse biholomorphic automorphism of D. A bounded symmetric domain is by definition irreducible if it is not biholomorphically equivalent to a product of two non-trivial bounded symmetric domains. For* *more details on bounded symmetric domains, the reader is referred to [50], [37] and to the surveys [7] and [9].*

Every irreducible bounded symmetric domain D is biholomorphically equivalent to a so-called Cartan domain (the Harish-Chandra realization of D). Cartan domains are always circular (which means closed under the action of the circle group) and convex and contain the origin. In particular, every Cartan domain is the open unit ball with respect to a suitable norm on $\mathbb{C}^d$. In the sequel, when we consider irreducible bounded symmetric domains, we shall always use their realization as a Cartan domain.

For any Cartan domain D, let $\mathrm{Aut}(D)$ denote the group of biholomorphic self-maps of D and let $\mathfrak{G}$ denote the connected component of $\mathrm{Aut}(D)$ containing the identity. Furthermore, let $\mathfrak{K} \subset \mathfrak{G}$ denote the stabilizer of the origin. By Cartan's linearity theorem, $\mathfrak{K} = \mathfrak{G} \cap \mathrm{GL}(\mathbb{C}^d)$.

Using Jordan theoretic methods, one can assign to every Cartan domain several non-negative integer-valued invariants, namely the rank r, the characteristic multiplicities a and b and the genus, usually denoted by g. For the precise definitions of these quantities, we refer to [50] or [7].

Let μ denote the Lebesgue measure on $\mathbb{C}^d$, normalized such that $\mu(D) = 1$. The Bergman space

$$L^2_a(D) = L^2(D, \mu) \cap \mathcal{O}(D),$$

endowed with the relative inner product of $L^2(D, \mu)$, is a reproducing kernel Hilbert space. This is true for every bounded open set $D \subset \mathbb{C}^d$.

If D is a Cartan domain, then Jordan theory yields the existence of a $\mathfrak{K}$-invariant polynomial h in z and $\overline{w}$ (the so-called Jordan triple determinant) such that h has no zeroes on D and such that the reproducing kernel K of the Bergman space $L^2_a(D)$ is of the form

$$K(z, w) = h(z, w)^{-g} \quad (z, w \in D).$$

In particular, $h(z, z) > 0$ for all $z \in D$ and, by the symmetry of D, $h(0, 0) = 1$. This implies that, for $\nu \in \mathbb{C}$, there exist a neighbourhood U of 0 and a sesquianalytic power $K_\nu = h^{-\nu} : U \times U \to \mathbb{C}$ of h satisfying $K_\nu(0, 0) = 1$. Central for the theory of function spaces on Cartan domains is a famous result of Faraut and Koranyi [37] stating that the functions K_ν possess a representation as a series

$$K_\nu : D \times D \to \mathbb{C}\ ,\ K_\nu(z, w) = \sum_{\mathbf{m}} (\nu)_{\mathbf{m}} K_{\mathbf{m}}(z, w), \tag{1.1.3}$$

that is locally uniformly convergent on $D \times D$ and thus extends the power $h^{-\nu}$ to a sesquianalytic function on $D \times D$. The formula (1.1.3) requires of course some more explanation: The sum ranges over all signatures $\mathbf{m}$ of length r, that is, all finite sequences $\mathbf{m} = (m_1, \ldots, m_r)$ of integers satisfying $m_1 \geq \ldots \geq m_r \geq 0$. The *numbers* $(\nu)_{\mathbf{m}}$ *are the so-called generalized Pochhammer symbols, defined by*

$$(\lambda)_{\mathbf{m}} = \prod_{j=1}^{r} \prod_{l=0}^{m_j-1} \left(\lambda + l - (j-1)\frac{a}{2}\right), \tag{1.1.4}$$

for $\lambda \in \mathbb{C}$ *and all signatures* $\mathbf{m}$*. From the above considerations, it is clear that the functions* K_ν *satisfy the functional equations*

$$K_\nu \cdot K_\mu = K_{\nu+\mu} \quad \text{and} \quad K_\nu^\rho = K_{\rho\nu} \quad (\nu, \mu \in \mathbb{C}, \rho \in \mathbb{Z}).$$

Since $K_\nu \cdot K_{-\nu} = K_0 = \mathbf{1}$*, the functions* K_ν *obviously have no zeroes.*

Now let $\mathcal{P}$ *denote the space of polynomials, restricted to* D*. Then the stabilizer* $\mathfrak{K}$ *of the origin acts in a natural way on* $\mathcal{P}$ *by the assignment* $p \mapsto p \circ k$ $(k \in \mathfrak{K})$*. With respect to this action, the Peter-Weyl theorem yields a decomposition of* $\mathcal{P}$ *into a multiplicity-free direct sum of irreducible linear subspaces* $\mathcal{P} = \oplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}$*, where the sum ranges over all signatures of length* r *(see [65] or [72] for details). The spaces* $\mathcal{P}_{\mathbf{m}}$ *consist of homogeneous polynomials of degree* $|\mathbf{m}| = \sum_{i=1}^{r} m_i$ *and hence are finite dimensional. Endowing* $\mathcal{P}$ *with the so-called Fock inner product*

$$\langle p, q \rangle_F = \frac{1}{\pi^d} \int_{\mathbb{C}^d} p(z)\overline{q(z)} e^{-|z|^2}\, dm(z)$$

(where the norm $|\cdot|$ *on* $\mathbb{C}^d$ *and the measure* m *on* $\mathbb{C}^d$ *are canonically assigned to* D *by its Jordan triple structure, see [7], p. 24), one obtains a natural reproducing kernel Hilbert space structure on each of the spaces* $\mathcal{P}_{\mathbf{m}}$*. The reproducing kernels with respect to this structure are denoted by* $K_{\mathbf{m}}$*. It follows from the details of the construction that* $K_{\mathbf{m}}(kz, kw) = K_{\mathbf{m}}(z, w)$ *holds for all signatures* $\mathbf{m}$*,* $z, w \in D$ *and all* $k \in \mathfrak{K}$*. It is clear by (1.1.3) that then* $K_\nu(kz, kw) = K_\nu(z, w)$ *holds for all* $\nu \in \mathbb{C}$*,* $z, w \in D$ *and* $k \in \mathfrak{K}$*.*

It is proved in [37] that K_ν *is a positive definite function precisely if the coefficients* $(\nu)_{\mathbf{m}}$ *are non-negative for all* $\mathbf{m}$*. By the defining equality (1.1.4) of the Pochhammer symbols, one deduces that this is the case if and only if* ν *is contained in the so-called Wallach set*

$$\mathcal{W} = \{\frac{j-1}{2}a \; ; \; j = 1, \ldots, r\} \cup (\frac{r-1}{2}a, \infty).$$

The first set in this disjoint union is called the discrete Wallach set $\mathcal{W}_d$ *and the second part is called the continuous Wallach set* $\mathcal{W}_c$*. Consequently, for* $\nu \in \mathcal{W}$ *there exists a unique reproducing kernel Hilbert space* $\mathcal{H}_\nu$ *with reproducing kernel* K_ν*. In the sequel, we refer to these spaces* $\mathcal{H}_\nu$ *as the standard reproducing kernel Hilbert spaces on the irreducible bounded symmetric domain* D*. It is clear from the Faraut-Koranyi expansion (1.1.3) that* $\mathcal{H}_\nu$ *contains the constant functions (note that* $K_\nu(\cdot, 0) \equiv \mathbf{1}$*). It follows from the definition of the Pochhammer symbols (1.1.4) that* $(\nu)_{\mathbf{m}} > 0$ *for all* $\mathbf{m}$ *precisely if* $\nu \in \mathcal{W}_c$*. Using this fact, it can be proved that* $\mathcal{H}_\nu$ *contains the polynomials if and only if* $\nu \in \mathcal{W}_c$ *and that, in this case, the polynomials are dense in* $\mathcal{H}_\nu$*. One can furthermore show that the (finite-dimensional) spaces* $\mathcal{P}_{\mathbf{m}}$

then form an orthogonal decomposition of the Hilbert space $\mathcal{H}_\nu$. Therefore, every function $f \in \mathcal{H}_\nu$ has a unique orthogonal representation $f = \sum_{\mathbf{m}} f_{\mathbf{m}}$ with $f_{\mathbf{m}} \in \mathcal{P}_{\mathbf{m}}$ for all $\mathbf{m}$, where the series converges in $\mathcal{H}_\nu$ and uniformly on compact subsets of D. For f and g in $\mathcal{H}_\nu$, one has

$$\langle f, g\rangle_\nu = \sum_{\mathbf{m}} \frac{1}{(\nu)_{\mathbf{m}}} \langle f_{\mathbf{m}}, g_{\mathbf{m}}\rangle_F, \tag{1.1.5}$$

where $\langle \cdot, \cdot\rangle_F$ denotes the Fock inner product as defined above.

Among the spaces $\mathcal{H}_\nu$, there are two classes of particular interest: For $\nu > g-1$, one can show that $\mathcal{H}_\nu$ is the Bergman space with respect to the weighted measure

$$d\mu_\nu(z) = c_\nu h(z,z)^{\nu-g} d\mu(z),$$

where c_ν is a normalizing constant. In fact, for $\nu = g$ we obtain the unweighted Bergman space discussed above, that is, $\mathcal{H}_g = L^2_a(D)$. The second class consists of the Hardy-type spaces $\mathcal{H}_\nu$, where $\frac{d}{r} \leq \nu \leq g-1$ (note that always $\frac{d}{r} > \frac{r-1}{2}a$). These spaces can be realized as the closure of the analytic polynomials in $L^2(S_\nu, \sigma_\nu)$, where S_ν is a subset of the topological boundary ∂D and σ_ν is an appropriate probability measure on S_ν. In particular, $S_{\frac{d}{r}}$ is is the full Shilov boundary of D, defined as the smallest closed subset S of ∂D such that every function $\phi \in C(\overline{D})$ which is holomorphic on D assumes its maximum on S. One can prove that the Shilov boundary S consists precisely of those points in ∂D having maximal Euclidean distance to the origin (cf. [50], Theorem 6.5). Furthermore, $\sigma = \sigma_{\frac{d}{r}}$ is the unique $\mathfrak{K}$-invariant probability measure on the Shilov boundary (see [9], p. 223 for details). In the sequel, we refer to the space $\mathcal{H}_{\frac{d}{r}}$ as the Hardy space $H^2(D)$ over D. It is known (cf. [9], p.223) that the functions $f \in H^2(D)$ possess radial limits f^* σ-almost everywhere on S and that the mapping

$$H^2(D) \mapsto L^2(S,\sigma) \ , \ f \mapsto f^*$$

defines an isometry.

It should be stressed that for $\frac{r-1}{2}a < \nu < \frac{d}{r}$, the spaces $\mathcal{H}_\nu$ cannot be realized as holomorphic subspaces of any L^2, which means that there is no positive measure μ on $\mathbb{C}^d$ such that

$$\|p\|^2 = \int_{\mathbb{C}^d} |p(z)|^2 \, d\mu(z)$$

holds for all polynomials p.

Finally, we focus on the special case of the unit ball $\mathbb{B}_d \subset \mathbb{C}^d$, which is probably the best understood Cartan domain. It is the only Cartan domain with rank $r = 1$ and has characteristic multiplicities $a = 0$, $b = d-1$ and genus $g = d+1$. The Jordan triple determinant of the ball is given by

$$h : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C} \ , \ h(z,w) = 1 - (z,w),$$

where $(\cdot,\cdot)$ denotes the standard Euclidean inner product on $\mathbb{C}^d$. Since the rank of $\mathbb{B}_d$ is one, signatures have length one and the spaces $\mathcal{P}_m$ consist of all homogeneous polynomials of degree m. The kernels K_m are given by

$$K_m : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}\ ,\ K_m(z,w) = \frac{(z,w)}{m!}.$$

The expression for the Pochhammer symbols reduces to

$$(\lambda)_m = \prod_{j=0}^{m-1} (\lambda + j). \tag{1.1.6}$$

Since $\operatorname{Re} h(z,w) > 0$ *for all $z, w \in \mathbb{B}_d$, the powers $h^{-\nu}$ can be defined for all $\nu \in \mathbb{C}$ by*

$$h^{-\nu} : \mathbb{B}_d \times \mathbb{B}_d \to \mathbb{C}\ ,\ h^{-\nu}(z,w) = e^{-\nu \log h(z,w)},$$

where log *denotes the standard branch of the complex logarithm. The Faraut-Koranyi expansion (*1.1.3*) can now be verified by a straightforward computation.*

Although it is not needed in this place, we point out that the function $-\log h$ itself is positive definite. To prove this, recall the integral representation of the standard branch of the complex logarithm

$$\log(z) = \int_1^\infty \frac{z-1}{t(t+z-1)}\, dt \quad (\operatorname{Re} z > 0).$$

Since the functions

$$\begin{aligned} -\frac{h(z,w)-1}{t(t+h(z,w)-1)} &= \frac{(z,w)}{t(t-(z,w))} \\ &= \frac{(z,w)}{t^2(1-\frac{(z,w)}{t})} \\ &= \frac{(z,w)}{t^2} \sum_{j=0}^{\infty} \frac{(z,w)^j}{t^j} \end{aligned}$$

are positive definite for all $t \geq 1$, the same is true for the integral

$$\int_1^\infty -\frac{h(z,w)-1}{t(t+h(z,w)-1)}\, dt = -\log h(z,w).$$

These considerations show directly that $\mathcal{W}_d = \{0\}$ and $\mathcal{W}_c = (0,\infty)$. Of course, the spaces $\mathcal{H}_\nu$ on the unit ball $(0 < \nu < \infty)$ contain the classical ones: For $\nu = \frac{d}{r} = d$, we regain the Hardy space $H^2(\mathbb{B}_d)$ and, for $\nu = g = d+1$, $\mathcal{H}_\nu$ is the Bergman space $L_a^2(\mathbb{B}_d)$. The space $\mathcal{H}_1$ over $\mathbb{B}_d$, also known as the Arveson space $H(\mathbb{B}_d)$, has attracted some attention in recent years, since it triggered a remarkable progress in the model theory for spherical contractions (cf. [13] and [54]).

The Moore-Aronszajn result establishes a link between the purely algebraic property of positive definiteness and the topological property of a linear function space to be a reproducing kernel Hilbert space. With this in mind, it is not surprising that those functions $f : X \to \mathcal{E}$ which belong to a given reproducing kernel space $\mathcal{H} \subset \mathcal{E}^X$ can be characterized by the following positivity condition (cf. [23] or [16]).

Proposition 1.1.11. *Suppose that $\mathcal{E}$ is a Hilbert space, that $\mathcal{H} \subset \mathcal{E}^X$ is a reproducing kernel Hilbert space with reproducing kernel K and that $f : X \to \mathcal{E}$ is an arbitrary function. Then the following are equivalent:*

(i) *f belongs to $\mathcal{H}$.*

(ii) *There exists a constant $c \geq 0$ such that the mapping*

$$X \times X \to B(\mathcal{E}) \ , \ (z, w) \mapsto c^2 K(z, w) - f(z) \otimes f(w)$$

is positive definite.

In this case, $\|f\|$ is the infimum of all such constants $c \geq 0$. Moreover, the infimum is achieved.

This leads to an approximation result which will be frequently used in the sequel.

Corollary 1.1.12. *Suppose that $\mathcal{E}$ is a Hilbert space, that $\mathcal{H} \subset \mathcal{E}^X$ is a reproducing kernel Hilbert space and that $f : X \to \mathcal{E}$ is a function. Then the following are equivalent:*

(i) *f belongs to $\mathcal{H}$.*

(ii) *There exists a bounded sequence $(f_n)_n$ in $\mathcal{H}$ such that $(f_n(z))_n$ converges weakly to $f(z)$ for all $z \in X$.*

(iii) *There exists a bounded net $(f_\alpha)_\alpha$ in $\mathcal{H}$ such that $(f_\alpha(z))_\alpha$ converges weakly to $f(z)$ for all $z \in X$.*

In this case,

$$\|f\| \leq \liminf_n \|f_n\| \qquad \text{and} \qquad \|f\| \leq \liminf_\alpha \|f_\alpha\|$$

holds for all sequences and nets as in (ii) and (iii), respectively.

Proof. Recall that, the limes inferior of a net $(x_i)_i$ of real numbers is defined as

$$\liminf_i x_i = \liminf_j \{x_i \ ; \ i \in I \text{ and } i \geq j\}$$

and exists in $\mathbb{R}$ whenever the net is bounded below.

To prove the non-trivial parts of the corollary, let $(f_\alpha)_{\alpha \in A}$ be a bounded net in $\mathcal{H}$ coverging pointwise weakly to f. One checks that the set $A_d = \{\alpha \in A \ ; \ \|f_\alpha\| < d\}$ is cofinal in A for every $d > \liminf_\alpha \|f_\alpha\|$. Thus the set $(f_\alpha)_{\alpha \in A_d}$ is a subnet of $(f_\alpha)_{\alpha \in A_d}$ satisfying $\|f_\alpha\| < d$ for all $\alpha \in A_d$. It is easily verified that the function

$$X \times X \to B(\mathcal{E}) \ ; \ (z, w) \mapsto d^2 K(z, w) - f(z) \otimes f(w)$$

is positive definite as pointwise WOT limit of the positive definite functions

$$X \times X \to B(\mathcal{E}) \ ; \ (z, w) \mapsto d^2 K(z, w) - f_\alpha(z) \otimes f_\alpha(w),$$

which implies by Proposition 1.1.11 that $f \in \mathcal{H}$ and $\|f\| \leq d$. $\square$

Although Corollary 1.1.12 follows immediately from Proposition 1.1.11, it could also be deduced from the weak compactness of the unit ball in Hilbert spaces by using the following obvious characterization of weak convergence in reproducing kernel spaces.

Proposition 1.1.13. *Suppose that $\mathcal{E}$ is a Hilbert space and that $\mathcal{H} \subset \mathcal{E}^X$ is a reproducing kernel Hilbert space.*

(a) A bounded net $(f_\alpha)_\alpha$ in $\mathcal{H}$ converges weakly to $f \in \mathcal{H}$ if and only if $(f_\alpha(z))_\alpha$ converges weakly to $f(z)$ for all $z \in X$.

(b) A sequence $(f_n)_n$ in $\mathcal{H}$ converges weakly to $f \in \mathcal{H}$ if and only if it is bounded and $(f_n(z))_n$ converges weakly to $f(z)$ for all $z \in X$.

1.2 Standard constructions with reproducing kernel Hilbert spaces

1.2.1 Inflations of reproducing kernel Hilbert spaces

In Example 1.1.3(b), we introduced the class of reproducing kernel Hilbert spaces consisting of analytic functions on some open subset $D \subset \mathbb{C}^d$. Maybe the simplest of these spaces is the Hardy space $H^2(\mathbb{D})$ on the unit disk $\mathbb{D}$ in $\mathbb{C}$. It is commonly known that

$$H^2(\mathbb{D}) = \{f = \sum_n a_n z^n \in \mathcal{O}(\mathbb{D})\ ;\ \|f\|^2 = \sum_n |a_n|^2 < \infty\}$$

and that

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C}\ ,\ K(z,w) = \frac{1}{1 - z\overline{w}}$$

is the reproducing kernel of $H^2(\mathbb{D})$. If $\mathcal{F}$ is some Hilbert space, then we can consider the $\mathcal{F}$-valued Hardy space

$$H^2_{\mathcal{F}}(\mathbb{D}) = \{f = \sum_n a_n z^2 \in \mathcal{O}(\mathbb{D}, \mathcal{F})\ ;\ \|f\|^2 = \sum_n \|a_n\|^2 < \infty\}$$

on $\mathbb{D}$. Then $H^2_{\mathcal{F}}(\mathbb{D})$ is in fact a reproducing kernel Hilbert space contained in $\mathcal{O}(\mathbb{D}, \mathcal{F})$. Its reproducing kernel has the very simple form $K_{\mathcal{F}} = K \cdot 1_{\mathcal{F}}$. This motivates the following general definition.

Definition 1.2.1. *Suppose that $\mathcal{E}$ is a Hilbert space and that $\mathcal{H} \subset \mathcal{E}^X$ is a reproducing kernel Hilbert space with reproducing kernel K. Then for any Hilbert space $\mathcal{F}$, the reproducing kernel Hilbert space $\mathcal{H}_{\mathcal{F}} \subset (\mathcal{E} \otimes \mathcal{F})^X$ which is associated with the positive definite function*

$$K_{\mathcal{F}} : X \times X \to B(\mathcal{E} \otimes \mathcal{F})\ ,\ (z,w) \mapsto K(z,w) \otimes 1_{\mathcal{F}},$$

is called the inflation of $\mathcal{H}$ *along* $\mathcal{F}$*. For simplicity, we shall use the abbreviations* $K^{(n)} : X \times X \to B(\mathcal{E}^n)$ *and* $\mathcal{H}^{(n)} \subset (\mathcal{E}^n)^X$ *for* $K_{\mathbb{C}^n}$ *and* $\mathcal{H}_{\mathbb{C}^n}$*, respectively.*

Note that, using the notations introduced in Definition 1.1.7, the kernel $K_{\mathcal{F}}$ is of course the inner product of K and the constant $B(\mathcal{F})$-valued kernel with value $1_{\mathcal{F}}$. The following result (cf. [39], p. 314) provides an intrinsic description of inflations.

Proposition 1.2.2. *Suppose that* $\mathcal{E}$ *is a Hilbert space, that* $\mathcal{H} \subset \mathcal{E}^X$ *is a reproducing kernel Hilbert space with reproducing kernel* K *and that* $\mathcal{F}$ *is an arbitrary Hilbert space.*

(a) For $x \in \mathcal{F}$*, let* p_x *denote the projection defined by*

$$p_x : \mathcal{E} \otimes \mathcal{F} \to \mathcal{E} \ , \ (e \otimes f) \mapsto e\langle f, x\rangle.$$

For a function $f : X \to \mathcal{E} \otimes \mathcal{F}$ *and* $x \in \mathcal{F}$*, we define the slice function*

$$f_x : X \to \mathcal{E} \ , \ f_x(z) = p_x f(z).$$

Then

$$\begin{aligned} \mathcal{H}_{\mathcal{F}} \quad &= \quad \{f : X \to \mathcal{E} \otimes \mathcal{F} \ ; \ f_x \in \mathcal{H} \text{ for all } x \in \mathcal{F} \text{ and} \\ &\qquad \sum_i \|f_{x_i}\|^2 < \infty \text{ for some orthonormal basis } (x_i)_i \text{ of } \mathcal{F}\}. \end{aligned}$$

Moreover,

$$\|f\|^2 = \sum_i \|f_{x_i}\|^2$$

for all $f \in \mathcal{H}_{\mathcal{F}}$ *and every orthonormal basis* $(x_i)_i$ *of* $\mathcal{F}$*.*

(b) There is a unique isometric isomorphism $j : \mathcal{H} \otimes \mathcal{F} \to \mathcal{H}_{\mathcal{F}}$ *with*

$$j(f \otimes x)(z) = f(z) \otimes x \quad (z \in X).$$

(c) If $\mathcal{F}'$ *is another Hilbert space, then by the associativity and symmetry of the Hilbert space tensor product, there exist natural isometric isomorphisms*

$$(\mathcal{H}_{\mathcal{F}})_{\mathcal{F}'} \simeq \mathcal{H}_{\mathcal{F} \otimes \mathcal{F}'} \simeq \mathcal{H}_{\mathcal{F}' \otimes \mathcal{F}} \simeq (\mathcal{H}_{\mathcal{F}'})_{\mathcal{F}}.$$

The identifications stated in parts (b) and (c) of the preceding proposition will be used throughout this paper without further mentioning.

1.2.2 Restrictions of reproducing kernel Hilbert spaces

Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{H} \subset \mathcal{E}^X$ be a reproducing kernel space with reproducing kernel K. Then, for every subset Y of X, the restriction

$$K_{|Y} : Y \times Y \to B(\mathcal{E}) \ , \ K_{|Y}(z, w) = K(z, w)$$

is positive definite again and hence the reproducing kernel of a reproducing kernel space $\mathcal{H}_{|Y} \subset \mathcal{E}^Y$.

The following proposition shows that the notation $\mathcal{H}_{|Y}$ is in fact justified.

Proposition 1.2.3. *Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{H} \subset \mathcal{E}^X$ be a reproducing kernel Hilbert space with reproducing kernel K. Then, for $Y \subset X$, the reproducing kernel space $\mathcal{H}_{|Y}$ associated with the restriction $K_{|Y}$ is given by*

$$\mathcal{H}_{|Y} = \{f_{|Y} \ ; \ f \in \mathcal{H}\},$$

and the norms on $\mathcal{H}$ and $\mathcal{H}_{|Y}$ are related by

$$\|g\|_{\mathcal{H}_{|Y}} = \inf\{\|f\|_{\mathcal{H}} \ ; \ f_{|Y} = g\} \quad (g \in \mathcal{H}_{|Y}).$$

The restriction mapping

$$\rho_Y : \mathcal{H} \to \mathcal{H}_{|Y} \ , \ f \mapsto f_{|Y}$$

is a coisometry satisfying

$$\rho_Y^* K_{|Y}(\cdot, z)x = K(\cdot, z)x$$

for all $z \in Y$ and $x \in \mathcal{E}$.

This result is proved in [16], Theorem 1.12. It is self-evident that restrictions and inflations of reproducing kernel Hilbert spaces interact in the expected way. That is, if we are given a reproducing kernel Hilbert space $\mathcal{H} \subset \mathcal{E}^X$ and another Hilbert space $\mathcal{F}$, then the identity

$$(\mathcal{H}_{\mathcal{F}})_{|Y} = (\mathcal{H}_{|Y})_{\mathcal{F}}$$

holds.

1.2.3 Sums of reproducing kernel Hilbert spaces

Consider a Hilbert space $\mathcal{E}$ and reproducing kernel Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2 \subset \mathcal{E}^X$ with reproducing kernels K_1, K_2. Then the sum $K_1 + K_2$ is a positive definite function again and therefore the reproducing kernel of a reproducing kernel Hilbert space, which is described by the following proposition.

Proposition 1.2.4. *Let $\mathcal{E}$ be a Hilbert space and let $\mathcal{H}_1, \mathcal{H}_2 \subset \mathcal{E}^X$ be reproducing kernel Hilbert spaces with kernels K_1, K_2. Then*

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 = \{f_1 + f_2 \ ; \ f_1 \in \mathcal{H}_1 \text{ and } f_2 \in \mathcal{H}_2\},$$

endowed with the norm

$$\|f\|^2 = \inf\{\|f_1\|^2_{\mathcal{H}_1} + \|f_2\|^2_{\mathcal{H}_2} \ ; \ f_1 \in \mathcal{H}_1, f_2 \in \mathcal{H}_2 \text{ and } f = f_1 + f_2\},$$

is the reproducing kernel Hilbert space with reproducing kernel $K = K_1 + K_2$. In particular, both $\mathcal{H}_1$ and $\mathcal{H}_2$ are contractively embedded into $\mathcal{H}_1 + \mathcal{H}_2$.

A proof of this result (in the scalar case) can be found in [11], Section I.6. The general case is verified analogously.

1.2.4 Products of reproducing kernel Hilbert spaces

In the following, we shall describe the reproducing kernel Hilbert spaces associated with the inner and outer products of positive definite kernels introduced in Definition 1.1.7. The scalar versions of the following results appear in [11], Section I.8. The proofs given there apply to the vector-valued situation without major changes.

Proposition 1.2.5. *Suppose that Y_1, Y_2 are non-empty sets, that $\mathcal{E}_1, \mathcal{E}_2$ are Hilbert spaces and that $\mathcal{H}_i \subset \mathcal{E}_i^{Y_i}$, $i = 1, 2$, are reproducing kernel Hilbert spaces with reproducing kernels K_1, K_2. Write $Y = Y_1 \times Y_2$. The reproducing kernel Hilbert space associated with the positive definite kernel $K_1 \circledast K_2$ is called the outer product of $\mathcal{H}_1$ and $\mathcal{H}_2$ and is denoted by $\mathcal{H}_1 \circledast \mathcal{H}_2$. There exists a unitary mapping*

$$U : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \circledast \mathcal{H}_2 \quad \text{with} \quad U(f_1 \otimes f_2) = f_1 \circledast f_2.$$

In particular, the functions of the form $f_1 \circledast f_2$, where $f_1 \in \mathcal{H}_1$ and $f_2 \in \mathcal{H}_2$, are total in $\mathcal{H}_1 \circledast \mathcal{H}_2$.

The prototypical example for such a product is the Hardy space $H^2(\mathbb{D}^d)$ over the unit polydisk in $\mathbb{C}^d$. It is the d-fold outer product of the Hardy space $H^2(\mathbb{D})$ over the unit disk in $\mathbb{C}$. We turn now to the second type of product.

Proposition 1.2.6. *Suppose that $\mathcal{E}_1, \mathcal{E}_2$ are Hilbert spaces and that $\mathcal{H}_1 \subset \mathcal{E}_1^X$ and $\mathcal{H}_2 \subset \mathcal{E}_2^X$ are reproducing kernel Hilbert spaces with reproducing kernels K_1, K_2. Then the reproducing kernel Hilbert space associated with the positive definite kernel $K_1 * K_2$ is called the inner product of $\mathcal{H}_1$ and $\mathcal{H}_2$ and is denoted by $\mathcal{H}_1 * \mathcal{H}_2$. There exists a coisometry*

$$V : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 * \mathcal{H}_2 \quad \text{with} \quad V(f_1 \otimes f_2) = f_1 * f_2.$$

In particular, the functions of the form $f_1 * f_2$, where $f_1 \in \mathcal{H}_1$ and $f_2 \in \mathcal{H}_2$, form a total subset of $\mathcal{H}_1 * \mathcal{H}_2$.

In fact, these assertions all become clear, if one realizes that $\mathcal{H}_1 * \mathcal{H}_2$ is the restriction of $\mathcal{H}_1 \circledast \mathcal{H}_2$ to the diagonal $\{(z, z) \; ; \; z \in X\} \subset X \times X$.

Maybe the simplest example of an inner product is the Bergman space $L_a^2(\mathbb{D})$ over the complex unit disk. It is well known that the reproducing kernel of the Bergman space is given by

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C} \; , \; (z, w) \mapsto \left(\frac{1}{1 - z\overline{w}} \right)^2 ,$$

which means that $L_a^2(\mathbb{D}) = H^2(\mathbb{D}) * H^2(\mathbb{D})$ is the inner product of the Hardy space with itself.

1.3 Non-degenerate and regular reproducing kernel Hilbert spaces

In practice, one sometimes has to deal with scalar-valued reproducing kernel Hilbert spaces having a common zero point. This usually leads to (at least technical) difficulties. In order to avoid these problems, results are often formulated supposing the absence of common zero points. It is easily seen that, in the scalar case, a reproducing kernel Hilbert space does not have common zeroes if and only if its reproducing kernel does not vanish anywhere on the diagonal. We now introduce the concept of non-degenerate reproducing kernel Hilbert spaces, which generalizes this condition to the setting of vector-valued reproducing kernel Hilbert spaces.

Definition 1.3.1. *Suppose that $\mathcal{E}$ is a Hilbert space. A reproducing kernel Hilbert space $\mathcal{H} \subset \mathcal{E}^X$ is called non-degenerate if the point evaluations δ_z are onto for all $z \in X$.*

Proposition 1.3.2. *Suppose that $\mathcal{E}$ is a Hilbert space and that $\mathcal{H} \subset \mathcal{E}^X$ is a reproducing kernel Hilbert space with reproducing kernel K and point evaluations δ_z ($z \in X$). Then the following are equivalent:*

(i) *$\mathcal{H}$ is non-degenerate.*

(ii) *For all $z \in X$, the point evaluation δ_z has a right inverse $i_z \in B(\mathcal{E}, \mathcal{H})$.*

(iii) *$K(z,z)$ is invertible for all $z \in X$.*

In particular, if $\mathcal{H}$ contains the constant functions, then $\mathcal{H}$ is non-degenerate. Furthermore, inflations and restrictions of non-degenerate reproducing kernel Hilbert spaces are non-degenerate again.

Proof. The equivalence of (i) and (ii) is clear by the definition. If $K(z,z)$ is invertible, then obviously

$$i_z : \mathcal{E} \to \mathcal{H} \ , \ i_z = \delta_z^* K(z,z)^{-1}$$

is a right inverse for δ_z. Finally, suppose that (i) holds. If δ_z is onto, then δ_z^* is one-to-one and has closed range $\operatorname{ran} \delta_z^* = (\ker \delta_z)^\perp$. Hence $K(z,z) = \delta_z \delta_z^*$ is invertible. The remaining assertions are obvious. □

Remark 1.3.3. *Suppose that $\mathcal{E}$ is a Hilbert space and that $\mathcal{H} \subset \mathcal{E}^X$ is a reproducing kernel Hilbert space with reproducing kernel K.*

(a) *If $\mathcal{H}$ contains the constant functions, then*

$$i : \mathcal{E} \to \mathcal{H} \ , \ x \mapsto \underline{x},$$

mapping $x \in \mathcal{E}$ to the constant function $\underline{x}$, is easily seen to be bounded by the closed graph theorem. It is a natural common right inverse for all point evaluations δ_z ($z \in X$).

(b) *It is not hard to prove (using the Cauchy-Schwarz inequality for K) that $\mathcal{H}$ has no common zeroes if and only if $K(z,z) \neq 0$ for all $z \in X$. Hence, if $\mathcal{H}$ is non-degenerate, then it has no common zeroes. The converse is false in general, although it is true if $\mathcal{H}$ is the inflation of some scalar reproducing kernel space.*

Example 1.3.4. *Recall that a positive definite function $K : X \times X \to \mathbb{C}$ is called a Nevanlinna-Pick kernel (shortly NP kernel) if it has no zeroes and if the function $1 - \frac{1}{K}$ is positive definite as well. The reproducing kernel Hilbert space $\mathcal{H}$ associated with K is then called a Nevanlinna-Pick space (NP space, for short). The denotation 'Nevanlinnna-Pick space' originates in the fact that these spaces allow a solution of a generalized Nevanlinna-Pick interpolation problem (cf. [59], and [16] for a detailed treatment of NP spaces). Prototypical Nevanlinna-Pick spaces are the Hardy space $H^2(\mathbb{D})$ over the unit disk in $\mathbb{C}$ and, more generally, the Arveson space $H(\mathbb{B}_d)$ over the unit ball in $\mathbb{C}^d$. Since for any NP kernel K, the function*

$$X \times X \to \mathbb{C}\ ,\ (z,w) \mapsto K(z,w) - 1 = K(z,w)(1 - \frac{1}{K(z,w)})$$

is positive definite by Schur's lemma (Proposition 1.1.9), the constant function $\mathbf{1}$ belongs to the associated kernel space $\mathcal{H}$ by Proposition 1.1.11. Therefore every NP space is non-degenerate.

The defining property of non-degenerate reproducing kernel Hilbert spaces actually is a kind of interpolation condition: For every pair (z,x) consisting of a point $z \in X$ and a vector $x \in \mathcal{E}$, there exists a function $f \in \mathcal{H}$ with $f(z) = x$. A natural strengthening of this property is the concept of regular reproducing kernel Hilbert spaces.

Definition 1.3.5. *Suppose that $\mathcal{E}$ is a Hilbert space. A reproducing kernel Hilbert space $\mathcal{H} \subset \mathcal{E}^X$ is called regular if, for every finite family of pairwise distinct points $z_1, \dots, z_n \in X$ and any choice of vectors $x_1, \dots, x_n \in \mathcal{E}$, there exists a function $f \in \mathcal{H}$ such that $f(z_i) = x_i$ holds for all $i = 1, \dots, n$.*

Proposition 1.3.6. *Suppose that $\mathcal{E}$ is a Hilbert space and that $\mathcal{H} \subset \mathcal{E}^X$ is a reproducing kernel Hilbert space with reproducing kernel K. Then the following are equivalent:*

(i) *$\mathcal{H}$ is regular.*

(ii) *For all finite subsets Y of X, the equality $\mathcal{H}_{|Y} = \mathcal{E}^Y$ holds.*

(iii) *For every finite family of pairwise distinct points $z_1, \dots, z_n \in X$, the operator matrix $[K(z_i, z_j)] \in B(\mathcal{E}^n)$ is invertible.*

Inflations and restrictions of regular spaces are again regular.

Proof. The equivalence of (i) and (ii) follows by Proposition 1.2.3, since elements of $\mathcal{H}_{|Y}$ are exactly the restrictions of functions in $\mathcal{H}$. To see the equivalence of (i) and (iii), note that for any finite subset $Y = \{z_1, \dots, z_n\}$ of X, the operator matrix $\kappa_Y = [K(z_i, z_j)] \in B(\mathcal{E}^n)$ is invertible if and only if the operator

$$\delta_Y : \mathcal{H} \to \mathcal{E}^n \ , \ f \mapsto (f(z_i))_i$$

is onto since $\kappa_Y = \delta_Y \delta_Y^*$. □

Example 1.3.7. *If D is an open subset of $\mathbb{C}^d$ and $\mathcal{H} \subset \mathcal{O}(D)$ is a reproducing kernel Hilbert space containing the polynomials, then $\mathcal{H}$ is obviously regular.*

The remainder of this section is devoted to the study of regularity of scalar reproducing kernel Hilbert spaces.

Proposition 1.3.8. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a reproducing kernel Hilbert space with reproducing kernel K. Then $\mathcal{H}$ is regular if and only if the family $\{K(\cdot, z) \ ; \ z \in X\}$ is linearly independent in $\mathcal{H}$.*

Proof. Suppose that $\mathcal{H}$ is regular and choose some subset $Y = \{z_1, \dots, z_n\}$ of X. Then $\mathcal{H}_{|Y} = \mathbb{C}^Y$ is an n-dimensional linear space. Since it is spanned by the elements $K_{|Y}(\cdot, z_1), \dots, K_{|Y}(\cdot, z_n)$, they are linearly independent in $\mathcal{H}_{|Y}$. Letting $\rho_Y : \mathcal{H} \to \mathcal{H}_{|Y}$ the restriction mapping, it follows that $K(\cdot, z_1), \dots, K(\cdot, z_n)$ are linearly independent in $\mathcal{H}$, since $\rho_Y K(\cdot, z_i) = K_{|Y}(\cdot, z_i)$ holds for all $i = 1, \dots, n$.

Conversely, suppose that the family $\{K(\cdot, z) \ ; \ z \in X\}$ is linearly independent in $\mathcal{H}$ and fix some subset $Y = \{z_1, \dots, z_n\}$ of X. Since $\rho_Y^* K_{|Y}(\cdot, z_i) = K(\cdot, z_i)$ holds for all $i = 1, \dots, n$, it follows at once that the functions $K_{|Y}(\cdot, z_1), \dots, K_{|Y}(\cdot, z_n)$ are linearly independent in $\mathcal{H}_{|Y}$ and hence that $\mathcal{H}_{|Y} = \mathbb{C}^Y$ by equality of dimensions. □

1.4 The cone of positive definite kernels

Again, let X denote some non-empty set. Adapting common terminology, every function $K : X \times X \to B(\mathcal{E})$ (where $\mathcal{E}$ is a Hilbert space) is called a kernel on X.

Definition 1.4.1. *Suppose that $\mathcal{E}$ is a Hilbert space.*

(a) A kernel $K : X \times X \to B(\mathcal{E})$ is called a positive kernel if it is a positive definite function.

(b) *The adjoint of a kernel* $K : X \times X \to B(\mathcal{E})$ *is defined as*

$$K^* : X \times X \to B(\mathcal{E}) \ , \ K^*(z,w) = K(w,z)^*.$$

We refer to the operation $K \mapsto K^*$ *as the natural involution of kernels.* K *is called hermitian (or self-adjoint) if* $K = K^*$*. The real and imaginary parts of* K *are defined as*

$$\operatorname{Re} K = \frac{1}{2}(K + K^*) \quad \textit{and} \quad \operatorname{Im} K = \frac{1}{2i}(K - K^*).$$

(c) *If* $\mathscr{S}$ *is a subspace of* $B(\mathcal{E})^{X\times X}$*, then* $\mathscr{S}_+$ *denotes the set of all positive kernels in* $\mathscr{S}$*, and* $\mathscr{S}_h$ *denotes the set of all hermitian kernels in* $\mathscr{S}$*.*

Remark 1.4.2. *Suppose that* $\mathcal{E}$ *is a Hilbert space.*

(a) *Obviously, every positive kernel* $K : X \times X \to B(\mathcal{E})$ *is hermitian. To see this, choose a Kolmogorov factorization* $(\mathcal{H}, \phi)$ *of* K*, that is, a pair of a Hilbert space* $\mathcal{H}$ *and a function* $\phi : X \to B(\mathcal{H}, \mathcal{E})$ *with*

$$K(z,w) = \phi(z)\phi(w)^* \quad (z, w \in X).$$

Then

$$K(w,z)^* = (\phi(w)\phi(z)^*)^* = \phi(z)\phi(w)^* = K(z,w) \quad (z, w \in X),$$

which proves the self-adjointness of K*.*

(b) *Obviously, the real and imaginary parts of a kernel* $K : X \times X \to B(\mathcal{E})$ *are hermitian with* $K = \operatorname{Re} K + i \operatorname{Im} K$*. If* $\mathscr{S} \subset B(\mathcal{E})^{X\times X}$ *is a self-adjoint subspace (that is, closed under involution), then* $\mathscr{S}$ *is closed under the forming of real and imaginary parts and* $\mathscr{S} = \mathscr{S}_h + i\mathscr{S}_h$*.*

(c) *The set of all positive kernels clearly is a pointed salient convex cone in the linear space* $B(\mathcal{E})^{X\times X}$*, that is,*

$$(B(\mathcal{E})^{X\times X})_+ \cap -(B(\mathcal{E})^{X\times X})_+ = \{0\}.$$

The set of all hermitian kernels is a real linear subspace of $B(\mathcal{E})^{X\times X}$*.*

(d) *Both the cone of positive kernels and the real linear space of hermitian kernels are closed in* $B(\mathcal{E})^{X\times X}$ *with respect to the topology of pointwise WOT convergence, that is, the locally convex topology induced by the seminorms*

$$B(\mathcal{E})^{X\times X} \to [0, \infty) \ , \ K \mapsto |\langle K(z,w)y, x\rangle| \quad (z, w \in X, x, y \in \mathcal{E}).$$

The fact that the positive kernels form a pointed salient convex cone yields a natural partial ordering on $B(\mathcal{E})^{X\times X}$.

Definition 1.4.3. *Suppose that $\mathcal{E}$ is a Hilbert space. Given $K_1, K_2 : X \times X \to B(\mathcal{E})$, we write $K_1 \leq K_2$ ($K_2 \geq K_1$, resepectively) to indicate that $K_2 - K_1$ is a positive kernel.*

The restriction of this ordering to the set of positive kernels corresponds to the inclusion ordering of the associated reproducing kernel Hilbert spaces.

Lemma 1.4.4. *Suppose that $\mathcal{E}$ is a Hilbert space and that $\mathcal{H}_1, \mathcal{H}_2 \subset \mathcal{E}^X$ are reproducing kernel Hilbert spaces with reproducing kernels K_1, K_2. Then the following are equivalent:*

(i) $K_1 \leq K_2$.

(ii) $\mathcal{H}_1 \subset \mathcal{H}_2$ *and the inclusion mapping* $i : \mathcal{H}_1 \hookrightarrow \mathcal{H}_2$ *is contractive.*

Proof. Suppose that (i) holds and choose $f \in \mathcal{H}_1$ with $\|f\|_{\mathcal{H}_1} = 1$. By Proposition 1.1.11, the kernel

$$F_1 : X \times X \to B(\mathcal{E}) \ , \ (z, w) \mapsto K_1(z, w) - f(z) \otimes f(w)$$

is positive. But then also the kernel

$$F_2 : X \times X \to B(\mathcal{E}) \ , \ (z, w) \mapsto K_2(z, w) - f(z) \otimes f(w)$$

is positive since $F_2 = F_1 + (K_2 - K_1)$. Another application of Proposition 1.1.11 yields that $f \in \mathcal{H}_2$ with $\|f\|_{\mathcal{H}_2} \leq 1$.

Conversely, if $\mathcal{H}_1$ is continuously included in $\mathcal{H}_2$ and if i denotes the inclusion mapping, then one observes that

$$i^* K_2(\cdot, z)x = K_1(\cdot, z)x \quad (z \in X, x \in \mathcal{E}).$$

If in addition, i is contractive, then $1_{\mathcal{H}_2} - ii^* \in B(\mathcal{H}_2)$ is positive and the identity

$$\langle (K_2 - K_1)(z, w)y, x \rangle = \langle (1_{\mathcal{H}_2} - ii^*) K_2(\cdot, w)y, K_2(\cdot, z)x \rangle \quad (z, w \in X, x, y \in \mathcal{E})$$

shows that $K_2 - K_1$ is positive. □

Obviously the difference of two positive kernels is hermitian. However, the converse of this statement is known to be false. That is, there are hermitian kernels which cannot be written as a difference of two positive kernels. We shall see an example at the end of this section. But there are natural examples of subspaces $\mathscr{S}$ of $B(\mathcal{E})^{X \times X}$ satisfying $\mathscr{S}_h = \mathscr{S}_+ - \mathscr{S}_+$, and a frequent problem in applications is to decide whether a given subspace $\mathscr{S}$ of $B(\mathcal{E})^{X \times X}$ has this property. One of the main objectives of this paper is to study certain kernel classes $\mathscr{S}$ arising in the context of invariant subspaces of reproducing kernel Hilbert spaces with respect to this question.

Example 1.4.5. *Let $U \subset \mathbb{C}^d$ be an open set and consider the class $\mathscr{S} \subset \mathbb{C}^{U\times U}$ consisting of all polynomials in z and $\overline{w}$, restricted to $U \times U$. We claim that every hermitian kernel $G \in \mathscr{S}$ can be written as the difference of two positive kernels in $\mathscr{S}$. Indeed, write*

$$G(z,w) = \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha \overline{w}^\beta \quad (z,w \in U),$$

where only finitely many of the coefficients $c_{\alpha,\beta}$ are non-zero. Since G was supposed to be hermitian and since U is open, we infer that $c_{\alpha,\beta} = \overline{c_{\beta,\alpha}}$ holds for all α,β. Then $C = [c_{\alpha,\beta}]$ is a hermitian matrix, and we obtain

$$G(z,w) = (C(\overline{w}^\alpha)_\alpha, (\overline{z}^\alpha)_\alpha) \quad (z,w \in U),$$

where $(\cdot,\cdot)$ denotes the Euclidean inner product. Clearly, there are finite positive matrices C_1, C_2, indexed by α, β such that $C = C_1 - C_2$. Setting

$$G_i(z,w) = (C_i(\overline{w}^\alpha)_\alpha, (\overline{z}^\alpha)_\alpha) \quad (z,w \in U, i = 1,2)$$

obviously defines positive kernels in $\mathscr{S}$ such that $G = G_1 - G_2$. Furthermore, since $\mathscr{S}$ is closed under the natural involution of kernels, the linear span of $\mathscr{S}_+$ is all of $\mathscr{S}$.

If a hermitian kernel K can be written as the difference of two positive kernels K_1, K_2, then K_1, K_2 can be chosen minimal in a certain sense.

Definition 1.4.6.

(a) Two positive kernels $K_1, K_2 : X \times X \to B(\mathcal{E})$ are called disjoint if 0 *is the only positive kernel K with $K \le K_1$ and $K \le K_2$.*

(b) A decomposition $K = K_1 - K_2$ of a hermitian kernel $K : X \times X \to B(\mathcal{E})$ into positive kernels K_1, K_2 is called disjoint if K_1, K_2 are disjoint kernels.

Every decomposition of a hermitian kernel can be replaced by a disjoint decomposition.

Proposition 1.4.7. *Suppose that $\mathcal{E}$ is a Hilbert space and that we are given two positive kernels $K_1, K_2 : X \times X \to B(\mathcal{E})$. Then there exist disjoint positive kernels $\tilde{K}_1, \tilde{K}_2 : X \times X \to B(\mathcal{E})$ such that*

$$K_1 - K_2 = \tilde{K}_1 - \tilde{K}_2$$

and $\tilde{K}_1 \le K_1$, $\tilde{K}_2 \le K_2$.

This result is due to Schwartz [66] and was rediscovered later in the study of reproducing Kreĭn spaces (see Section 1.5 of this paper, [4], and [21] for a general treatment of Kreĭn spaces). Since we were not able to find an appropriate citation for the case of operator-valued kernels, we include a proof.

Proof (of Proposition 1.4.7). It suffices to show that the set

$$\Sigma(K_1, K_2) = \{L \geq 0 \; ; \; L \leq K_1 \text{ and } L \leq K_2\}$$

contains maximal elements. Indeed, if L_0 is such a maximal element, then the kernels $\tilde{K}_1 = K_1 - L_0$ and $\tilde{K}_2 = K_2 - L_0$ are clearly positive with $\tilde{K}_1 \leq K_1$, $\tilde{K}_2 \leq K_2$ and

$$\tilde{K}_1 - \tilde{K}_2 = (K_1 - L_0) - (K_2 - L_0) = K_1 - K_2.$$

Furthermore, if K is a positive kernel with $K \leq \tilde{K}_1$ and $K \leq \tilde{K}_2$, then $K + L_0$ obviously belongs to $\Sigma(K_1, K_2)$. Since $K + L_0 \geq L_0$, the maximality of L_0 implies $K = 0$. This shows the disjointness of $\tilde{K}_1$ and $\tilde{K}_2$.

Now the existence of maximal elements in $\Sigma(K_1, K_2)$ follows by Zorn's lemma if we can prove that every chain in $\Sigma(K_1, K_2)$ possesses an upper bound in $\Sigma(K_1, K_2)$.

So let $\mathcal{C}$ be a chain in $\Sigma(K_1, K_2)$. By Remark 1.4.2(d), it suffices to show that for all $z, w \in X$, the net $(L(z, w))_{L \in \mathcal{C}}$ has a WOT limit $L_1(z, w) \in B(\mathcal{E})$. Fix $z, w \in X$. We claim that it is enough to show that the limit

$$(x, y) = \lim_{L \in \mathcal{C}} \langle L(z, w)x, y\rangle$$

exists for all $x, y \in \mathcal{E}$. In fact, if this is the case, then the estimate

$$\begin{aligned} |\langle L(z, w)x, y\rangle|^2 &\leq \langle L(z, z)y, y\rangle\langle L(w, w)x, x\rangle \qquad \text{(by Remark 1.1.6)} \\ &\leq \langle K_1(z, z)y, y\rangle\langle K_1(w, w)x, x\rangle \\ &\leq \|K_1(z, z)\|\|K_1(w, w)\|\|x\|^2\|y\|^2 \quad (L \in \mathcal{C}, x, y \in \mathcal{E}) \end{aligned}$$

proves that $(\cdot, \cdot)$ is a bounded sesquilinear form on $\mathcal{E}$. The existence of the WOT limit $L_1(z, w)$ then follows by the Lax-Milgram theorem. We fix $x, y \in \mathcal{E}$. Let L, L' be kernels in $\mathcal{C}$. Without restriction, we may assume that $L \geq L'$. Then

$$\begin{aligned} &|\langle L(z, w)x, y\rangle - \langle L'(z, w)x, y\rangle|^2 \\ &= |\langle (L - L')(z, w)x, y\rangle|^2 \\ &\leq \langle (L - L')(z, z)y, y\rangle\langle (L - L')(w, w)x, x\rangle \qquad \text{(by Remark 1.1.6)} \\ &= (\langle L(z, z)y, y\rangle - \langle L'(z, z)y, y\rangle)(\langle L(w, w)x, x\rangle - \langle L'(w, w)x, x\rangle). \end{aligned}$$

Now observe, that for fixed $\zeta \in X$ and $\xi \in \mathcal{E}$, the net $(\langle L(\zeta, \zeta)\xi, \xi\rangle)_{L \in \mathcal{C}}$ is increasing and bounded above and thus convergent. Together with the above estimate, this implies that the net $(\langle L(z, w)x, y\rangle)_{L \in \mathcal{C}}$ is a Cauchy net in $\mathbb{R}$ and hence convergent.

□

Example 1.4.8.

(a) *It should be stressed that a hermitian kernel may have different disjoint decompositions. For example, consider the kernel*

$$K : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \; , \; K(z, w) = 1 - z\overline{w}.$$

Then each of the decompositions $L = L_1 - L_2$ *and* $L = L_1' - L_2'$ *with*

$$L_1(z,w) = 1 \quad \textit{and} \quad L_2(z,w) = z\overline{w}$$

and

$$L_1'(z,w) = (\sqrt{2}-z)\overline{(\sqrt{2}-w)} \quad \textit{and} \quad L_2'(z,w) = (1-\sqrt{2}z)\overline{(1-\sqrt{2}w)}$$

is disjoint.

However, we shall see later (cf. Proposition 1.5.7*) that*

$$\operatorname{rank} L_1 = \operatorname{rank} L_1' \quad \textit{and} \quad \operatorname{rank} L_2 = \operatorname{rank} L_2'$$

holds for all disjoint decompositions $L = L_1 - L_2$ *and* $L = L_1' - L_2'$ *of a given hermitian kernel* L*.*

(b) *As in Example* 1.4.5*, let* U *be some open subset of* $\mathbb{C}^d$ *and let* $\mathscr{S}$ *denote the subclass of* $\mathbb{C}^{U\times U}$ *consisting of restrictions of polynomials in* z *and* $\overline{w}$ *to* $U \times U$*. Let us first observe that a positive kernel* $G : U \times U \to \mathbb{C}$ *belongs to* $\mathscr{S}$ *if and only if the associated reproducing kernel space* $\mathcal{G}$ *is finite-dimensional and consists of restrictions of polynomials to* U*. In fact, similar considerations as in Example* 1.4.5 *show that, for every kernel* $G \in \mathscr{S}_+$*, there exists a finite positive semi-definite matrix* C *such that*

$$G(z,w) = (C(\overline{w})^\alpha, (\overline{z})^\alpha) = (C^{\frac{1}{2}}(\overline{w})^\alpha, C^{\frac{1}{2}}(\overline{z})^\alpha).$$

This clearly induces a finite-rank Kolmogorov factorization

$$G(z,w) = \sum_{i=1}^{r} g_i(z)\overline{g_i(w)} \quad (z,w \in U)$$

with suitable polynomials $g_1, \ldots, g_r$*. Hence the reproducing kernel space* $\mathcal{G}$ *associated with* G *is finite-dimensional and, being the linear span of the functions* $G(\cdot,w)$ $(w \in U)$*, consists of polynomials. Conversely, suppose that* $\mathcal{G}$ *has finite dimension and consists of polynomials. Let* $(e_i)_{i=1}^r$ *be an orthonormal basis of* $\mathcal{G}$*. Since, according to Proposition* 1.1.2*,* G *can be written as*

$$G(z,w) = \sum_{i=1}^{r} e_i(z)\overline{e_i(w)} \quad (z,w \in X),$$

it follows that G *belongs to* $\mathscr{S}$*.*

The class $\mathscr{S}_+$ *has the following 'completeness' property: Whenever* G_0 *belongs to* $\mathscr{S}_+$ *and* $G : U \times U \to \mathbb{C}$ *is an arbitrary positive kernel with* $G \leq G_0$*, then* $G \in \mathscr{S}_+$*. In fact, let* $\mathcal{G}$ *and* $\mathcal{G}_0$ *denote the associated reproducing kernel spaces. By Lemma* 1.4.4*,* $\mathcal{G}$ *is contained in* $\mathcal{G}_0$ *which is, as seen above, finite dimensional and consists of polynomials. The same is then true for* $\mathcal{G}$*, being a subspace of* $\mathcal{G}_0$*. By the preceding discussion,* $G \in \mathscr{S}_+$*.*

> *Therefore, Proposition 1.4.7 and Example 1.4.5 yield that every hermitian kernel $G \in \mathscr{S}$ can be written as $G = G_1 - G_2$, where G_1, G_2 are positive disjoint kernels in $\mathscr{S}$.*

The fact that two positive kernels are disjoint can also be expressed in terms of the associated reproducing kernel spaces.

Proposition 1.4.9. *Suppose that $\mathcal{E}$ is a Hilbert space, that $\mathcal{H}_1, \mathcal{H}_2 \subset \mathcal{E}^X$ are reproducing kernel Hilbert spaces with reproducing kernels K_1, K_2. Then the following are equivalent:*

(i) K_1 and K_2 are disjoint.

(ii) $\mathcal{H}_1$ and $\mathcal{H}_2$ have trivial intersection, that is, $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$.

Proof. Suppose that (i) holds. We endow the intersection $\mathcal{H}_1 \cap \mathcal{H}_2$ with the norm

$$\|f\|^2 = \|f\|^2_{\mathcal{H}_1} + \|f\|^2_{\mathcal{H}_2} \quad (f \in \mathcal{H}_1 \cap \mathcal{H}_2).$$

It is easily verified that this turns $\mathcal{H}_1 \cap \mathcal{H}_2$ into a reproducing kernel Hilbert space, which is contractively contained in both $\mathcal{H}_1$ and $\mathcal{H}_2$. By Lemma 1.4.4, the reproducing kernel K of $\mathcal{H}_1 \cap \mathcal{H}_2$ satisfies $K \leq K_1$ and $K \leq K_2$. Since K_1 and K_2 are disjoint, we must have $K = 0$ and hence $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$.

To prove the opposite direction, suppose that L is a positive kernel with $L \leq K_1$ and $L \leq K_2$. By Lemma 1.4.4, the reproducing kernel Hilbert space $\mathcal{L}$ associated with L is contractively included in $\mathcal{H}_1$ and $\mathcal{H}_2$. Hence by hypothesis, $\mathcal{L} = 0$ and therefore also $L = 0$. □

We conclude this section with the announced example of a hermitian kernel which cannot be decomposed as difference of two positive kernels. The example is essentially due to L. Schwartz [66].

Example 1.4.10. *We first recall some definitions from the theory of indefinite inner products. We refer the reader to [21] for a detailed introduction to this topic. A sesquilinear form $[\cdot,\cdot]$ on a linear space X is called an (indefinite) inner product if $\overline{[x,y]} = [y,x]$ holds for all $x, y \in X$. An inner product is said to be non-degenerate if the only element $y \in X$ with $[x,y] = 0$ for all $x \in X$ is $y = 0$. A locally convex topology τ on X is called a majorant for $[\cdot,\cdot]$ if the sesquilinear form $[\cdot,\cdot]$ is jointly continuous with respect to τ. This implies in particular that the linear forms $[\cdot,y]$ are τ-continuous for all $y \in X$. Furthermore, τ is called an admissible majorant, if all τ-continuous linear functionals are of this type.*

We suppose now that X is a Banach space with a non-degenerate inner product $[\cdot,\cdot]$ such that the norm topology is an admissible majorant for $[\cdot,\cdot]$ and such that the *norm of X is not equivalent to a Hilbert space norm. We show later that such a space does exist.*

In this situation, it is clear that

$$L : X \times X \to \mathbb{C}\ ,\ \ L(x,y) = [x,y]$$

defines a hermitian kernel on X. Assume that L can be written as the difference of two positive kernels K_1, K_2. By Proposition 1.4.7, we may assume that K_1, K_2 are disjoint. The positive kernel $K = K_1 + K_2$ is the reproducing kernel of some reproducing kernel Hilbert space $\mathcal{H} \subset \mathbb{C}^X$.

We claim first that the (antilinear) mapping

$$j : X \to \mathcal{H}\ ,\ \ x \mapsto L(\cdot, x)$$

is continuous. Note that j well defined since $K_1(\cdot,x)$ and $K_2(\cdot,x)$ obviously belong to $\mathcal{H}$ by Lemma 1.4.4. The continuity of j is then an easy consequence of the closed graph theorem: If $(x_n)_n$ converges to x in X and $(L(\cdot,x_n))_n$ converges to a function $f \in \mathcal{H}$, then

$$\begin{aligned}
\langle f, K(\cdot,y)\rangle_{\mathcal{H}} &= \lim_n \langle L(\cdot,x_n), K(\cdot,y)\rangle_{\mathcal{H}} \\
&= \lim_n L(y,x_n) \\
&= \lim_n [y,x_n] \\
&= [y,x] \quad \textit{(by the continuity of } [\cdot,\cdot]\textit{)} \\
&= \langle L(\cdot,x), K(\cdot,y)\rangle_{\mathcal{H}}
\end{aligned}$$

holds for all $y \in X$. Hence $f = L(\cdot,x)$, which shows that the graph of j is closed.

Secondly, we prove that $\mathcal{H} \subset X^*$ and that the inclusion $i : \mathcal{H} \hookrightarrow X^*$ is continuous. If we can show that the linear subspace $\mathcal{H}_0 = \{L(\cdot,y)\ ;\ y \in X\}$ is dense in $\mathcal{H}$, then an application of the uniform boundedness principle shows that $\mathcal{H}$ consists of continuous linear functionals on X, and the asserted coninuity of i follows by the closed graph theorem. So suppose that $\langle f, L(\cdot,y)\rangle_{\mathcal{H}} = 0$ for all $y \in X$ and let $\mathcal{H}_1, \mathcal{H}_2$ denote the reproducing kernel space associated with the kernels K_1, K_2. By Proposition 1.4.9 and Proposition 1.2.4, $\mathcal{H}$ is the orthogonal sum of $\mathcal{H}_1$ and $\mathcal{H}_2$. Write f as $f = f_1 + f_2$ with $f_1 \in \mathcal{H}_1$ and $f_2 \in \mathcal{H}_2$. We obtain that

$$f_1(y) - f_2(y) = \langle f_1 + f_2, K_1(\cdot,y) - K_2(\cdot,y)\rangle_{\mathcal{H}} = \langle f, L(\cdot,y)\rangle_{\mathcal{H}} = 0$$

for all $y \in X$ and hence that $f_1 = f_2$. Since $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$, it follows that $f = 0$. Hence $\mathcal{H}_0$ is in fact dense in $\mathcal{H}$.

Now the hypothesis that the norm topology is an admissible majorant for $[\cdot,\cdot]$ implies that the (antilinear) mapping

$$\alpha : X \to X^*\ ,\ \ x \mapsto [\cdot, x]$$

is bijective and continuous and hence a homeomorphism with respect to the norm topologies. Considering the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "j"] & X^* \\ \mathcal{H} \arrow[ur, "i"'] & \end{tikzcd}$$

reveals that i and j are bijective and hence that X is isomorphic to the Hilbert space $\mathcal{H}$, in contradiction to the assumptions.

It remains to check that there exist Banach spaces X having the desired properties. To this end, we start with a reflexive Banach space E which is not isomorphic to a Hilbert space, such as $l^p(\mathbb{N})$ for $1 < p < \infty$ and $p \neq 2$. Letting $\overline{E}$ denote the conjugate linear version of E, we define $X = E \times \overline{E}^*$, endowed with the norm

$$\|(x, \lambda)\| = \left(\|x\|^2 + \|\lambda\|^2\right)^{\frac{1}{2}} \quad ((x, \lambda) \in X).$$

Standard calculations reveal that

$$[\cdot, \cdot] : X \times X \to \mathbb{C} \ , \ [(x, \lambda), (y, \mu)] = \lambda(y) + \overline{\mu(x)}.$$

defines a non-degenerate inner product on X and that the norm defined above is a majorant for $[\cdot, \cdot]$. Furthermore, this majorant is admissible by the reflexivity of E. Finally, the norm on X cannot be equivalent to a Hilbert space norm, since then the same would be true for the norm of E, which is clearly a closed subspace of X.

1.5 Hermitian kernels and Kreĭn spaces

As remarked earlier, Kreĭn spaces play an important role in the study of hermitian kernels. Instead of giving a detailed introduction to the theory of Kreĭn spaces, we refer the reader to the book of Bognár [21], which contains an excellent and self-contained survey of this topic.

Throughout this section, indefinite inner products are denoted by the symbol $[\cdot, \cdot]$, while positive inner products are represented by $\langle \cdot, \cdot \rangle$. The adjoint of an operator T between Kreĭn spaces will be denoted by $T^{\#}$. The positive and negative index of a Kreĭn space $\mathcal{K}$ will be denoted by $\dim_+ \mathcal{K}$ and $\dim_- \mathcal{K}$, respectively.

Definition 1.5.1. *Let $\mathcal{E}$ be a Hilbert space. A Kreĭn space $\mathcal{K} \subset \mathcal{E}^X$ of $\mathcal{E}$-valued functions is called a reproducing kernel Kreĭn space if the point evalutions*

$$\delta_z : \mathcal{K} \to \mathcal{E} \ , \ f \mapsto f(z) \quad (z \in X)$$

are continuous.

In analogy to the case of reproducing kernel Hilbert spaces, one can show the following result.

Proposition 1.5.2. *Suppose that $\mathcal{E}$ is a Hilbert space and that $\mathcal{K} \subset \mathcal{E}^X$ is a Kreĭn space. Then the following are equivalent:*

(i) $\mathcal{K}$ is a reproducing kernel Kreĭn space.

(ii) There exists a function $L : X \times X \to B(\mathcal{E})$ such that

- *the function $L(\cdot, w)y$ belongs to $\mathcal{K}$ for all $w \in X$ and $y \in \mathcal{E}$*
- *the equality*
$$\langle f(w), y\rangle = [f, L(\cdot, w)y]$$
holds for all $f \in \mathcal{K}$, $w \in X$ and $y \in \mathcal{E}$.

In this case, the identity $L(z, w) = \delta_z \delta_w^\#$ holds for all $z, w \in X$. In particular, the function L is uniquely determined and is called the reproducing kernel of $\mathcal{K}$. Moreover, L is hermitian and can be written as the difference of two disjoint positive kernels L_1, L_2 satisfying $\operatorname{rank} L_1 = \dim_+ \mathcal{K}$ *and* $\operatorname{rank} L_2 = \dim_- \mathcal{K}$.

Proof. The implication (i) to (ii) follows by setting

$$L : X \times X \to B(\mathcal{E}) \ , \ L(z, w) = \delta_z \delta_w^\# .$$

Clearly, L is then a hermitian kernel with the properties required in (ii). Since $[\cdot, \cdot]$ is non-degenerate, there can be no other function with the same properties.

Conversely, suppose that $\mathcal{K} \subset \mathcal{E}^X$ is a Kreĭn space such that there exists a function L as described in (ii). Since the Kreĭn space topology of $\mathcal{K}$ is a Hilbert space topology such that the inner product $[\cdot, \cdot]$ is continuous, an easy application of the closed graph theorem shows that the point evaluations

$$\delta_z : \mathcal{K} \to \mathcal{E} \ , \ f \mapsto f(z) \quad (z \in X)$$

are continuous. Hence $\mathcal{K}$ is a reproducing kernel Kreĭn space. Let $\mathcal{K} = \mathcal{L}_1 \dot{+} \mathcal{L}_2$ be a fundamental decomposition of the Kreĭn space $\mathcal{K}$. Then $\mathcal{L}_1$ and $\mathcal{L}_2$ are reproducing kernel Hilbert spaces with scalar products given by the restrictions of $[\cdot, \cdot]$ onto $\mathcal{L}_1$ and $-[\cdot, \cdot]$ onto $\mathcal{L}_2$, respectively. Let L_1 and L_2 denote the reproducing kernels of $\mathcal{L}_1$ and $\mathcal{L}_2$. Because of the identity

$$\begin{aligned}
[f, L_1(\cdot, w)y - L_2(\cdot, w)y] &= [f_1, L_1(\cdot, w)y] - [f_2, L_2(\cdot, w)y] \\
&= \langle f_1(w), y\rangle + \langle f_2(w), y\rangle \\
&= \langle f(w), y\rangle \\
&= [f, L(\cdot, w)y],
\end{aligned}$$

valid for $f = f_1 + f_2 \in \mathcal{K}$, $w \in X$ and $y \in \mathcal{E}$, we obtain that $L = L_1 - L_2$ is the difference of two positive kernels. By Proposition 1.4.9, the kernels L_1 and L_2 are disjoint. □

Motivated by the Hilbert space setting, the question arises which hermitian kernels are the reproducing kernel of some reproducing kernel Kreĭn space. The answer is given by the following proposition.

Proposition 1.5.3. *Suppose that $\mathcal{E}$ is a Hilbert space and that $L : X \times X \to B(\mathcal{E})$ is a hermitian kernel. Then the following are equivalent:*

(i) L can be written as difference of two positive kernels.

(ii) L is the reproducing kernel of a reproducing kernel Kreĭn space.

In this case, if $L = L_1 - L_2$ is a disjoint decomposition of L in positive kernels L_1, L_2, and if $\mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{E}^X$ denote the associated reproducing kernel Hilbert spaces, then the space $\mathcal{K} = \mathcal{L}_1 \dotplus \mathcal{L}_2$, endowed with the indefinite inner product

$$[f_1 + f_2, g_1 + g_2] = \langle f_1, g_1 \rangle - \langle f_2, g_2 \rangle \quad (f_1, g_1 \in \mathcal{L}_1 \text{ and } f_2, g_2 \in \mathcal{L}_2),$$

is a reproducing kernel Kreĭn space with reproducing kernel L. Furthermore, the space $\mathcal{K}$ satisfies $\dim_+ \mathcal{K} = \operatorname{rank} L_1$ and $\dim_- \mathcal{K} = \operatorname{rank} L_2$.

Proof. The implication (ii) to (i) is already proved. Conversely, Proposition 1.4.7 yields two disjoint positive kernels $L_1, L_2 : X \times X \to B(\mathcal{E})$ such that $L = L_1 - L_2$. Letting $\mathcal{L}_1, \mathcal{L}_2$ denote the associated reproducing kernel Hilbert spaces, one easily constructs a Kreĭn space $\mathcal{K} = \mathcal{L}_1 \dotplus \mathcal{L}_2$ with the inner product

$$[f_1 + f_2, g_1 + g_2] = \langle f_1, g_1 \rangle - \langle f_2, g_2 \rangle \quad (f_1, g_1 \in \mathcal{L}_1 \text{ and } f_2, g_2 \in \mathcal{L}_2).$$

Proposition 1.4.9 then immediately shows that this is well defined and that $\mathcal{K}$ is in fact a Kreĭn space. An obvious calculation reveals that L is the reproducing kernel of $\mathcal{K}$. □

In contrast to the case of reproducing kernel Hilbert spaces, where every positive kernel is the reproducing kernel of a uniquely determined reproducing kernel Hilbert space, not every hermitian kernel is the reproducing kernel of a reproducing kernel Kreĭn space (by Example 1.4.10 and Proposition 1.5.3). Furthermore, there can be different reproducing kernel Kreĭn spaces with the same reproducing kernel as an example given by Alpay [4] shows. But if the hermitian kernel L admits a decomposition into positive kernels at least one of which has finite rank (cf. Remark 1.1.6), then the uniqueness of the associated reproducing kernel Kreĭn space is guaranteed. Before we prove this assertion, we introduce the following definition.

Definition 1.5.4. *Let $\mathcal{E}$ be a Hilbert space and let $L : X \times X \to B(\mathcal{E})$ be a hermitian kernel. Then we define the positive (negative) rank of L, denoted by* $\operatorname{rank}_+ L$ *(*$\operatorname{rank}_- L$*), as the smallest number $r \in \mathbb{N}_0 \cup \{\infty\}$ such that every matrix of the form*

$$[\langle L(z_i, z_j)x_j, x_i\rangle]_{i,j=1}^n$$

with $n \in \mathbb{N}$, $z_1, \dots, z_n \in X$ and $x_1, \dots, x_n \in \mathcal{E}$ has at most r positive (negative) eigenvalues (counting multiplicity).

It is elementary to see that a hermitian kernel L is positive precisely if $\operatorname{rank}_- L = 0$ and that, in this case, $\operatorname{rank} L = \operatorname{rank}_+ L$ holds. The aim of this section is to prove that, more generally,

$$\operatorname{rank} L_1 = \operatorname{rank}_+ L \quad \text{and} \quad \operatorname{rank} L_2 = \operatorname{rank}_- L$$

holds for every disjoint decomposition $L = L_1 - L_2$ of a hermitian kernel L into positive kernels L_1, L_2. A first step in this direction is the following result.

Lemma 1.5.5. *Let $\mathcal{E}$ be a Hilbert space.*

(a) Suppose that $L : X \times X \to B(\mathcal{E})$ is a hermitian kernel and that $L = L_1 - L_2$ is a decomposition of L into positive kernels L_1, L_2. Then the inequalities

$$\operatorname{rank}_+ L \leq \operatorname{rank} L_1 \quad \text{and} \quad \operatorname{rank}_- L \leq \operatorname{rank} L_2$$

hold.

(b) Suppose that $\mathcal{K} \subset \mathcal{E}^X$ is a reproducing kernel Kreĭn space. If $L : X \times X \to B(\mathcal{E})$ denotes the reproducing kernel of $\mathcal{K}$, then the inequalities

$$\operatorname{rank}_+ L \leq \dim_+ \mathcal{K} \quad \text{and} \quad \operatorname{rank}_- L \leq \dim_- \mathcal{K}$$

hold.

Proof. We start by proving the second inequality of (a). The first one then follows by passing from L to $-L$. Clearly, we may assume that $r = \operatorname{rank} L_2 < \infty$ and, according to Theorem 1.4.7 and Lemma 1.4.4, that L_1 and L_2 are disjoint kernels. Let $\mathcal{K} \subset \mathcal{E}^X$ denote the reproducing kernel Kreĭn space formed with respect to the decomposition $L = L_1 - L_2$ according to Proposition 1.5.3. We infer that $\dim_- \mathcal{K} = \operatorname{rank} L_2 = r < \infty$, which means that $\mathcal{K}$ is a reproducing kernel Pontryagin space in the sense of [5]. Now fix $n \in \mathbb{N}$ and choose points $z_1, \dots, z_n \in X$ and vectors $x_1, \dots, x_n \in \mathcal{E}$. By [5], Lemma 1.1.1, the matrix

$$[\langle L(z_i, z_j)x_j, x_i\rangle_{\mathcal{E}}]_{i,j=1}^n = [[L(\cdot, z_j)x_j, L(\cdot, z_i)x_i]_{\mathcal{K}}]_{i,j=1}^n$$

can have no more than r negative eigenvalues. By definition, this means that $\operatorname{rank}_- L \leq r$.

Part (b) is a direct consequence of Proposition 1.5.2 and (a). $\square$

The following uniqueness result appears in [69] in the scalar case and in [5] in the vector-valued setting; see also [66], Proposition 40.

Lemma 1.5.6. *Suppose that $\mathcal{E}$ is a Hilbert space and that $L : X \times X \to B(\mathcal{E})$ is a hermitian kernel. Then the following are equivalent:*

(i) $\operatorname{rank}_+ L < \infty$ *(*$\operatorname{rank}_- L < \infty$*).*

(ii) There exists a decomposition $L = L_1 - L_2$ of L with disjoint positive kernels L_1, L_2 such that $\operatorname{rank} L_1 < \infty$ *(*$\operatorname{rank} L_2 < \infty$*).*

In this case, there exists a uniquely determined reproducing kernel Kreĭn space $\mathcal{K}$ of $\mathcal{E}$-valued functions admitting L as its reproducing kernel. Moreover, in this case, the identities $\dim_+ \mathcal{K} = \operatorname{rank}_+ L$ *(*$\dim_- \mathcal{K} = \operatorname{rank}_- L$*) hold.*

In the literature, one usually treats only the case $\operatorname{rank}_- L < \infty$. The resulting reproducing kernel Kreĭn spaces are also known as reproducing kernel Pontryagin spaces. But as usual, the case $\operatorname{rank}_+ L < \infty$ follows immediately by passing from L to $-L$.

Proof (of Lemma 1.5.6). Suppose that $\operatorname{rank}_- L < \infty$. Then, by Theorem 1.1.3 in [5], there exists a unique reproducing kernel Pontryagin space $\mathcal{K}$ with reproducing kernel L. An inspection of the proof given in [5] reveals that $\dim_- \mathcal{K} = \operatorname{rank}_- L$. Condition (ii) now follows immediately by Proposition 1.5.2.

That (ii) implies (i) is clear by Lemma 1.5.5.

In order to complete the proof, we have to show that every reproducing kernel Kreĭn space $\mathcal{K}'$ with reproducing kernel L automatically is a Pontryagin space. To this end, let $L = L_1 - L_2$ and $L = L_1' - L_2'$ be disjoint decompositions of L such that the corresponding reproducing kernel Hilbert spaces $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_1', \mathcal{L}_2'$ define fundamental decompositions for $\mathcal{K}, \mathcal{K}'$, that is,

$$\mathcal{K} = \mathcal{L}_1 \dotplus \mathcal{L}_2 \quad \text{and} \quad \mathcal{K}' = \mathcal{L}_1' \dotplus \mathcal{L}_2'.$$

Define $r = \operatorname{rank}_- L < \infty$. Then $\dim \mathcal{L}_2 = \dim_- \mathcal{K} = \operatorname{rank}_- L = r < \infty$. Note that $K = L_1 + L_2' = L_1' + L_2$ is a positive kernel. Let $\mathcal{H} \subset \mathcal{E}^X$ denote the associated reproducing kernel Hilbert space. By Proposition 1.2.4, we conclude that $\mathcal{H} = \mathcal{L}_1 + \mathcal{L}_2' = \mathcal{L}_1' + \mathcal{L}_2$. Therefore, the codimension of $\mathcal{L}_1'$ in $\mathcal{H}$ is at most r. Since $\mathcal{L}_1'$ and $\mathcal{L}_2'$ have trivial intersection and are both included in $\mathcal{H}$, we infer that $\dim \mathcal{L}_2' \leq r$. Then $\mathcal{K}'$ is a Pontryagin space and, by the cited uniqueness result from [5], $\mathcal{K} = \mathcal{K}'$. □

We are now able to prove the announced main result of this section.

Proposition 1.5.7. *Suppose that $\mathcal{E}$ is a Hilbert space and that $L : X \times X \to B(\mathcal{E})$ is a hermitian kernel. Suppose that $L = L_1 - L_2$ is a decomposition of L with disjoint positive kernels L_1, L_2 and that $\mathcal{K}$ is a reproducing kernel Kreĭn space with reproducing kernel L. Then the identities*

$$\operatorname{rank}_+ L = \dim_+ \mathcal{K} = \operatorname{rank} L_1 \quad \text{and} \quad \operatorname{rank}_- L = \dim_- \mathcal{K} = \operatorname{rank} L_2$$

hold.

Proof. We show first that $\operatorname{rank}_- L = \operatorname{rank} L_2$. By Lemma 1.5.5, it suffices to prove that $\operatorname{rank} L_2 \leq \operatorname{rank}_- L$. Clearly, we may assume that $\operatorname{rank}_- L < \infty$. By Lemma 1.5.6, there exists a uniquely determined reproducing kernel Kreĭn space $\mathcal{L} \subset \mathcal{E}^X$ with reproducing kernel L, and this space satisfies $\dim_- \mathcal{L} = \operatorname{rank}_- L$. Then, by Proposition 1.5.3, we conclude that $\dim_- \mathcal{L} = \operatorname{rank} L_2$ and consequently, that $\operatorname{rank}_- L = \operatorname{rank} L_2$. The assertion $\operatorname{rank}_+ L = \operatorname{rank} L_1$ is proved analogously or by passing from L to $-L$.

Finally, let $\mathcal{K} \subset \mathcal{E}^X$ be an arbitrary reproducing kernel Kreĭn space with reproducing kernel L. By Proposition 1.5.2, there exists a disjoint decomposition $L = L_1' - L_2'$ satisfying $\operatorname{rank} L_1' = \dim_+ \mathcal{K}$ and $\operatorname{rank} L_2' = \dim_- \mathcal{K}$. By the first part of the proof, we obtain

$$\dim_+ \mathcal{K} = \operatorname{rank} L_1' = \operatorname{rank}_+ L \quad \text{and} \quad \dim_- \mathcal{K} = \operatorname{rank} L_2' = \operatorname{rank}_- L.$$

□

1.6 Subordinate kernels

The intention of this section is to examine the structure of the space $B(\mathcal{H})$, where $\mathcal{H} \subset \mathcal{E}^X$ is a reproducing kernel Hilbert space. It will turn out that $B(\mathcal{H})$ can be canonically identified with a C^*-algebra consisting of $B(\mathcal{E})$-valued kernels. This observation yields an additional structure on $B(\mathcal{H})$, which is very useful in the study of many problems concerning the underlying space $\mathcal{H}$. We note that many results of this section already appeared in [17] in the scalar case.

Definition 1.6.1. *Suppose that $\mathcal{E}$ is Hilbert space and that $K : X \times X \to B(\mathcal{E})$ is a positive kernel.*

(a) A kernel $L : X \times X \to B(\mathcal{E})$ is called subordinate to K if there exists a constant $c \geq 0$ such that the inequality

$$|\sum_{i,j} \langle L(z_i, z_j) y_j, x_i \rangle|^2 \leq c^2 \sum_{i,j} \langle K(z_i, z_j) x_j, x_i \rangle \sum_{i,j} \langle K(z_i, z_j) y_j, y_i \rangle \qquad (1.6.1)$$

holds for all finite sequences $(z_i)_{i=1}^n$ in X and $(x_i)_{i=1}^n, (y_i)_{i=1}^n$ in $\mathcal{E}$. In this case, we write $L \prec K$. The infimum of all such c is denoted by $\|L\|_K$.

(b) We write $B(K)$ for the class of all kernels $L : X \times X \to B(\mathcal{E})$ with $L \prec K$.

We collect some immediate consequences of this definition.

Remark 1.6.2. *Suppose that $\mathcal{E}$ is a Hilbert space and that $K : X \times X \to B(\mathcal{E})$ is a positive kernel.*

(a) Let L be a kernel in $B(K)$. Then $c = \|L\|_K$ satisfies inequality (1.6.1), that is, the infimum in the definition of $\|L\|_K$ actually is a minimum.

(b) The set $B(K)$ obviously is a complex linear space. Moreover, it is not hard to check that $\|\cdot\|_K$ defines a norm on $B(K)$.

(c) Suppose that $L : X \times X \to B(\mathcal{E})$ is a kernel and that $c \geq 0$. Then inequality (1.6.1) is satisfied for all finite sequences $(z_i)_{i=1}^n$ in X and $(x_i)_{i=1}^n, (y_i)_{i=1}^n$ in $\mathcal{E}$ if and only if

$$|\sum_{i,j} \langle L(z_i, w_j)y_j, x_i\rangle|^2 \leq c^2 \sum_{i,j} \langle K(z_i, z_j)x_j, x_i\rangle \sum_{i,j} \langle K(w_i, w_j)y_j, y_i\rangle \quad (1.6.2)$$

is satisfied for all finite sequences $(z_i)_{i=1}^m, (w_i)_{i=1}^n$ in X and $(x_i)_{i=1}^m, (y_i)_{i=1}^n$ in $\mathcal{E}$. In order to prove the non-trivial part of this claim, just consider the finite sequences

$$\begin{aligned}(\tilde{z}_i) &= (z_1, \dots, z_m, w_1, \dots, w_n)\\ (\tilde{x}_i) &= (x_1, \dots, x_m, 0, \dots, 0)\\ (\tilde{y}_i) &= (0, \dots, 0, y_1, \dots, y_n).\end{aligned}$$

This means that we could replace (1.6.1) by (1.6.2) in the above definition. In particular, we obtain that the point evalutions

$$\delta_{z,w} : B(K) \to B(\mathcal{E}) \ , \ L \mapsto L(z, w)$$

are continuous. Indeed, by (1.6.2), we have

$$|\langle L(z, w)y, x\rangle|^2 \leq \|L\|_K^2 \langle K(z, z)x, x\rangle \langle K(w, w)y, y\rangle$$

for all $x, y \in \mathcal{E}$, and hence

$$\|L(z, w)\| \leq \|L\|_K \|K(z, z)\|^{\frac{1}{2}} \|K(w, w\|^{\frac{1}{2}}.$$

(d) Suppose that $L : X \times X \to B(\mathcal{E})$ is an arbitrary kernel. Then we have $L \in B(K)$ if and only if $L^* \in B(K)$. In this case, $\|L\|_K = \|L^*\|_K$.

(e) Suppose that $L : X \times X \to B(\mathcal{E})$ is a hermitian kernel and that $c \geq 0$. Then inequality (1.6.1) is fulfilled for all finite sequences $(z_i)_{i=1}^n$ in X and $(x_i)_{i=1}^n, (y_i)_{i=1}^n$ in $\mathcal{E}$ if and only if

$$|\sum_{i,j} \langle L(z_i, z_j)x_i, x_j\rangle| \leq c \sum_{i,j} \langle K(z_i, z_j)x_j, x_i\rangle \quad (1.6.3)$$

holds for all finite sequences $(z_i)_{i=1}^n$ *in* X *and* $(x_i)_{i=1}^n$ *in* $\mathcal{E}$*. Hence, in the case of a hermitian kernel* L*, we could replace (*1.6.1*) by (*1.6.3*) in the definition of subordinate kernels.*

To prove the assertion, we fix $n \in \mathbb{N}$ *and a sequence* $(z_i)_{i=1}^n$ *in* X *and write*

$$A = (K(z_i, z_j))_{i,j} \quad \text{and} \quad B = (L(z_i, z_j))_{i,j}.$$

Then $A \in B(\mathcal{E}^n)$ *is a positive and* $B \in B(\mathcal{E}^n)$ *is a self-adjoint operator. By hypothesis, we know that*

$$|\langle Bx, x\rangle| \leq c\langle Ax, x\rangle \tag{1.6.4}$$

holds for all $x \in \mathcal{E}^n$*. It suffices to show that*

$$\operatorname{Re}\langle Bx, y\rangle \leq c\ \langle Ax, x\rangle^{\frac{1}{2}}\langle Ay, y\rangle^{\frac{1}{2}} \tag{1.6.5}$$

is fulfilled for all $x, y \in \mathcal{E}^n$*. Without restriction, we may assume that* $A \neq 0$ *(otherwise, we have* $B = 0$ *by (*1.6.4*) and the assertion is trivial). Furthermore, it suffices to show (*1.6.5*) for* $x, y \in \mathcal{E}^n$ *with*

$$\langle Ax, x\rangle \neq 0 \neq \langle Ay, y\rangle.$$

Indeed, since $A \neq 0$*, it is always possible to find sequences* $(x^{(k)})_k$ *and* $(y^{(k)})_k$ *in* $\mathcal{E}^n$ *approximating* x *and* y *such that*

$$\langle Ax^{(k)}, x^{(k)}\rangle \neq 0 \quad \text{and} \quad \langle Ay^{(k)}, y^{(k)}\rangle \neq 0$$

for all $k \in \mathbb{N}$*. By an obvious scaling of* x, y*, we may assume*

$$\langle Ax, x\rangle = 1 = \langle Ay, y\rangle.$$

Then the parallelogram equality and the self-adjointness of B *yield*

$$\begin{aligned}
4\operatorname{Re}\langle Bx, y\rangle &= 2\langle Bx, y\rangle + 2\langle By, x\rangle \\
&= \langle B(x+y), (x+y)\rangle - \langle B(x-y), (x-y)\rangle \\
&\leq c\langle A(x+y), (x+y)\rangle + c\langle A(x-y), (x-y)\rangle \\
&= 2c(\langle Ax, x\rangle + \langle Ay, y\rangle) \\
&= 4c \\
&= 4c\langle Ax, x\rangle^{\frac{1}{2}}\langle Ay, y\rangle^{\frac{1}{2}},
\end{aligned}$$

as desired.

(f) Suppose that $L : X \times X \to B(\mathcal{E})$ *is a positive kernel and that* $c \geq 0$*. Then* $L \in B(K)$ *and* $\|L\|_K \leq c$ *if and only if* $L \leq cK$*. This is an immediate consequence of part (*e*).*

We are now in a position to state the main result of this section.

Theorem 1.6.3. *Suppose that $\mathcal{E}$ is a Hilbert space, that $K : X \times X \to B(\mathcal{E})$ is a positive kernel and that $\mathcal{H} \subset \mathcal{E}^X$ denotes the reproducing kernel Hilbert space associated with K. For an operator $T \in B(\mathcal{H})$, we define*

$$\Lambda_T : X \times X \to B(\mathcal{E})\ ,\ \Lambda_T(z,w)x = \left(T(K(\cdot,w)x)\right)(z),$$

or equivalently,

$$\Lambda_T : X \times X \to B(\mathcal{E})\ ,\ \Lambda_T(z,w) = \delta_z T \delta_w^*.$$

Then the linear mapping

$$B(\mathcal{H}) \to B(K)\ ,\ T \mapsto \Lambda_T$$

is an isometric isomorphism preserving involution and positivity.

If $L \in B(K)$ is a kernel, then the unique operator $T \in B(\mathcal{H})$ with $\Lambda_T = L$ is called the representing operator of L.

Proof. It follows by the definition that, given an operator $T \in B(\mathcal{H})$, the kernel Λ_T belongs to $B(K)$ with $\|\Lambda_T\|_K \leq \|T\|$. This shows that the mapping

$$j : B(\mathcal{H}) \to B(K)\ ,\ T \mapsto \Lambda_T$$

is well defined and contractive. Since the elements of the form $K(\cdot,w)x$ span the space $\mathcal{H}$ topologically, the map j is injective. It is furthermore clear that

$$j(T)^* = (\Lambda_T)^* = \Lambda_{T^*}$$

holds for all $T \in B(\mathcal{H})$ and that Λ_T is a positive kernel if and only if T is a positive operator.

So it remains to check that j is isometric and onto. To this end, fix some kernel $L \in B(K)$. We claim that there exists a sesquilinear form $(\cdot,\cdot)$ on $\mathcal{H}$, bounded by $\|L\|_K$, such that

$$(K(\cdot,w)y, K(\cdot,z)x) = \langle L(z,w)y, x\rangle \quad (z,w \in X, x,y \in \mathcal{E}).$$

In fact, by Remark 1.6.2(c), it follows that the sesquilinear form

$$(\sum_i K(\cdot,w_i)y_i, \sum_i K(\cdot,z_i)x_i) = \sum_{i,j} \langle L(z_i,w_j)y_j, x_i\rangle,$$

defined on the dense subspace $\mathcal{H}_0 = \mathrm{span}\{K(\cdot,z)x\ ;\ z \in X, x \in \mathcal{E}\}$, is well defined and bounded by $\|L\|_K$ (and hence extends to the whole of $\mathcal{H} \times \mathcal{H}$). By the Lax-Milgram theorem, there exists an operator $T \in B(\mathcal{H})$ with $\|T\| \leq \|L\|_K$ such that

$$\langle Tf, g\rangle = (f,g)$$

for all $f,g \in \mathcal{H}$. In particular, $\Lambda_T = L$ which completes the proof. $\square$

Corollary 1.6.4. *Suppose that $\mathcal{E}$ is a Hilbert space, that $K : X \times X \to B(\mathcal{E})$ is a positive kernel and that $\mathcal{H} \subset \mathcal{E}^X$ is the reproducing kernel Hilbert space associated with K.*

(a) For given $L \in B(K)$, $z \in X$ and $x \in \mathcal{E}$, the slice function $L(\cdot, z)x$ belongs to $\mathcal{H}$ with $\|L(\cdot, z)x\| \leq \|L\|_K \|K(\cdot, z)x\|$.

(b) For $L, L' \in B(K)$, there exists a unique kernel $L \circ L' \in B(K)$ with

$$\langle (L \circ L')(z, w)y, x\rangle = \langle L'(\cdot, w)y, L^*(\cdot, z)x\rangle \quad (z, w \in X, x, y \in \mathcal{E}).$$

Endowed with the product $\circ$, the space $B(K)$ (with the natural involution of kernels) becomes a C^*-algebra and the isomorphism

$$B(\mathcal{H}) \to B(K)\ ,\ T \mapsto \Lambda_T$$

is a C^*-isomorphism.

Proof. Choose $T \in B(\mathcal{H})$ with $L = \Lambda_T$. Then we have $L(\cdot, z)x = T(K(\cdot, z)x) \in \mathcal{H}$. This proves (a). To prove (b), just note that the defined product $\circ$ is nothing but the product of $B(\mathcal{H})$ via the identification $B(\mathcal{H}) \to B(K)\ ,\ T \mapsto \Lambda_T$. □

Remark 1.6.5. *Suppose that $\mathcal{E}$ is a Hilbert space, that $K : X \times X \to B(\mathcal{E})$ is a positive kernel and that $\mathcal{H} \subset \mathcal{E}^X$ is the reproducing kernel Hilbert space associated with K. For functions $f, g : X \to \mathcal{E}$, we write*

$$f \odot g : X \times X \to B(\mathcal{E})\ ,\ (z, w) \mapsto f(z) \otimes g(w).$$

(a) If f, g are elements of $\mathcal{H}$, then $\Lambda_{f \otimes g} = f \odot g$. In particular, $\|f \odot g\|_K = \|f\|\|g\|$.

(b) In view of Proposition 1.1.11 *and Remark* 1.6.2*(*f*) , it becomes clear that a function $f : X \to \mathcal{E}$ belongs to $\mathcal{H}$ if and only if the positive kernel $f \odot f$ belongs to $B(K)$. In this case, $\|f\|^2 = \|f \odot f\|_K$, and $f \odot f$ is represented by the rank-one operator $f \otimes f$.*

Example 1.6.6. *As in Example* 1.1.3*(*a*), we consider an index set I and the positive kernel*

$$K : I \times I \to \mathbb{C}\ ,\ K(i, j) = \delta_{ij}.$$

Obviously, $B(K)$ is then the set of all kernels $L : I \times I \to \mathbb{C}$ such that the matrix $[L(i, j)]$ represents a bounded operator on $l^2(I)$ with respect to the standard orthonormal basis. In particular, if $I = \{1, \ldots, n\}$ is a finite set, then $B(K)$ can be canonically identified with $M_n(\mathbb{C})$, endowed with the spectral norm.

Before we draw further consequences of Theorem 1.6.3, we discuss the relation between the cone of positive kernels in $B(K)$ and the class of all reproducing kernel Hilbert spaces included in $\mathcal{H}$. The following proposition can be regarded as a supplement to Lemma 1.4.4.

Proposition 1.6.7. *Let $\mathcal{E}$ be a Hilbert space and let $K, L : X \times X \to B(\mathcal{E})$ be positive kernels with associated reproducing kernel Hilbert spaces $\mathcal{H}$ and $\mathcal{L}$, respectively. Then the following are equivalent:*

(i) $\mathcal{L} \subset \mathcal{H}$.

(ii) $L \in B(K)$.

In this case, the inclusion mapping $i : \mathcal{L} \hookrightarrow \mathcal{H}$ is continuous and $\|i\|^2 = \|L\|_K$. Moreover, if $T \in B(\mathcal{H})$ is the unique (positive) operator with $L = \Lambda_T$, then we have $T = ii^*$ and $\mathcal{L} = \operatorname{ran} T^{\frac{1}{2}}$. Furthermore,

$$\langle T^{\frac{1}{2}} f, T^{\frac{1}{2}} g \rangle_{\mathcal{L}} = \langle P_{\overline{\operatorname{ran} T}} f, g \rangle_{\mathcal{H}}$$

for all $f, g \in \mathcal{H}$, that is, the operator

$$\overline{\operatorname{ran} T} \to \mathcal{L}\ ,\ \ f \mapsto T^{\frac{1}{2}} f$$

is unitary.

Proof. Suppose that $\mathcal{L} \subset \mathcal{H}$. By the closed graph theorem, the inclusion $i : \mathcal{L} \to \mathcal{H}$ is bounded. A short calculation shows that

$$i^* K(\cdot, z)x = L(\cdot, z)x \quad (z \in X, x \in \mathcal{E}).$$

Now we define $T = ii^* \in B(\mathcal{H})$ and obtain

$$\langle \Lambda_T(z, w)y, x \rangle = \langle TK(\cdot, w)y, K(\cdot, z)x \rangle_{\mathcal{H}} = \langle L(\cdot, w)y, L(\cdot, z)x \rangle_{\mathcal{L}} = \langle L(z, w)y, x \rangle$$

for all $z, w \in X$ and $x, y \in \mathcal{E}$. Thus $L = \Lambda_T \in B(K)$ and $\|L\|_K = \|T\| = \|i\|^2$. If conversely $L \in B(K)$, then we have that $\frac{1}{c} L \leq K$ for some $c > 0$. By Lemma 1.4.4, the reproducing kernel Hilbert space associated with $\frac{1}{c} L$ (which coincides with $\mathcal{L}$ as a set by Proposition 1.1.11) is included in $\mathcal{H}$.

To prove the remaining assertions, one checks that

$$(T^{\frac{1}{2}} f, T^{\frac{1}{2}} g) = \langle P_{\overline{\operatorname{ran} T}} f, g \rangle \quad (f, g \in \mathcal{H})$$

is a well-defined scalar product on $\mathcal{L}' = \operatorname{ran} T^{\frac{1}{2}}$ and that $\mathcal{L}'$ actually is complete with the induced norm and hence a Hilbert space. Fix $f \in \mathcal{H}$, $z \in X$ and $x \in \mathcal{E}$. The fact that $L(\cdot, z)x = TK(\cdot, z)x$ belongs to $\mathcal{L}'$ and the calculation

$$\begin{aligned} (T^{\frac{1}{2}} f, L(\cdot, z)x) &= (T^{\frac{1}{2}} f, TK(\cdot, z)x) \\ &= \langle f, T^{\frac{1}{2}} K(\cdot, z)x \rangle \\ &= \langle T^{\frac{1}{2}} f, K(\cdot, z)x \rangle \\ &= \langle (T^{\frac{1}{2}} f)(z), x \rangle \end{aligned}$$

show that $\mathcal{L}'$ is the reproducing kernel Hilbert space with kernel L. By uniqueness, $\mathcal{L} = \mathcal{L}'$. □

In Example 1.4.10, we discussed the problem that, in general, not all hermitian kernels can be written as the difference of two positive kernels. Theorem 1.6.3 now allows us to apply spectral theory in order to prove that the class $B(K)$ satisfies $B(K)_h = B(K)_+ - B(K)_+$ and that there exists a decomposition which is canonical in a certain sense.

Proposition 1.6.8. *Suppose that $\mathcal{E}$ is a Hilbert space, that $K : X \times X \to B(\mathcal{E})$ is a positive kernel and that $\mathcal{H} \subset \mathcal{E}^X$ is the reproducing kernel Hilbert space associated with K.*

Let L be a hermitian kernel $L \in B(K)$ with representing operator $T \in B(\mathcal{H})$. If $T = T_+ - T_-$ denotes the spectral decomposition of T in positive operators, then the positive kernels $L_+ = \Lambda_{T_+}$ and $L_- = \Lambda_{T_-}$ are unique in $B(K)_+$ with the property that $L = L_+ - L_-$ and $L_+ \circ L_- = 0 = L_- \circ L_+$.

The associated reproducing kernel Hilbert spaces $\mathcal{L}_+, \mathcal{L}_-$ are orthogonal in $\mathcal{H}$ and, in particular, L_+ and L_- are disjoint. Moreover, the equalities

$$\operatorname{rank}_\pm L = \operatorname{rank} L_\pm = \dim \mathcal{L}_\pm = \operatorname{rank} T_\pm \qquad \text{and} \qquad \operatorname{rank}_+ L + \operatorname{rank}_- L = \operatorname{rank} T$$

hold.

Proof. The existence and uniqueness of the decomposition $L = L_+ - L_-$ follows by Theorem 1.6.3 and the existence and uniqueness of the spectral decomposition $T = T_+ - T_-$ into two positive operators with $T_+T_- = 0 = T_-T_+$.

The spectral decomposition of T satisfies $\operatorname{ran} T_+ \oplus \operatorname{ran} T_- = \operatorname{ran} T$, as an orthogonal sum. Recall that $\mathcal{L}_\pm = \operatorname{ran} T_\pm^{\frac{1}{2}}$ by Proposition 1.6.7. Thus the spaces $\mathcal{L}_+, \mathcal{L}_-$ must be orthogonal in $\mathcal{H}$. In particular, they have trivial intersection which, by an application of Proposition 1.4.9, shows that L_+ and L_- are disjoint. An application of Proposition 1.5.7 proves the equality $\operatorname{rank}_\pm L = \operatorname{rank} L_\pm$. All remaining equalities are trivial. □

In the sequel, the decomposition $L = L_+ - L_-$ of a hermitian kernel $L \in B(K)$ will be referred to as the spectral decomposition of L (with respect to K). The preceding result yields, besides Proposition 1.5.3, another characterization of those hermitian kernels that can be written as as difference of two positive kernels. It turns out that every disjoint decomposition of a hermitian kernel is the spectral decomposition with respect to a suitable positive kernel.

Corollary 1.6.9. *Suppose that $\mathcal{E}$ is a Hilbert space and that $L : X \times X \to B(\mathcal{E})$ is a hermitian kernel. Then the following are equivalent:*

(i) L can be represented as a difference of two positive kernels.

(ii) There exists a positive kernel $K : X \times X \to B(\mathcal{E})$ such that $L \in B(K)$.

In particular, if $L = L_1 - L_2$ is a disjoint decomposition of L and if $K = L_1 + L_2$, then $L \in B(K)$ and $L = L_1 - L_2$ is the spectral decomposition of L with respect to K.

Proof. That (ii) implies (i) follows by Proposition 1.6.8. Conversely, let $L = L_1 - L_2$ be such a decomposition of L in positive kernels L_1, L_2. Then $K = L_1 + L_2$ is positive and obviously satisfies $L_1, L_2 \in B(K)$ and hence $L = L_1 - L_2 \in B(K)$. If in addition, L_1 and L_2 are disjoint, then the reproducing kernel Hilbert space $\mathcal{H}$ associated with K is the orthogonal sum of the reproducing kernel Hilbert spaces $\mathcal{L}_1$ and $\mathcal{L}_2$ associated with L_1 and L_2, respectively (cf. Proposition 1.2.4). This clearly implies that the decomposition $L = L_1 - L_2$ coincides with the spectral decomposition of L with respect to K. $\square$

Remark 1.6.10. *In [17] (proof of Theorem 3.3), it is shown that all hermitian sesquianalytic kernels $L : D \times D \to \mathbb{C}$ on domains $D \subset \mathbb{C}^d$ can be written as a difference of disjoint positive sesquianalytic kernels. The main idea in the proof of this surprising result is to construct a weighted Bergman space over D such that L is subordinate to the reproducing kernel K of that Bergman space.*

If $f : D \to \mathbb{C}$ is a non-zero holomorphic function, then the hermitian kernel

$$L' : D \times D \to \mathbb{C}\ ,\ \ L'(z,w) = \alpha(z)L(z,w)\overline{\alpha(w)}$$

satisfies $\operatorname{rank}_\pm L' = \operatorname{rank}_\pm L$. *In particular, if L is positive definite, then so is L' and* $\operatorname{rank} L' = \operatorname{rank} L$. *In fact, by the result cited above, we can choose disjoint positive sesquianalytic kernels L_1, L_2 such that $L = L_1 - L_2$. Now define*

$$L_i' : D \times D \to \mathbb{C}\ ,\ \ L_i'(z,w) = \alpha(z)L_i(z,w)\overline{\alpha(w)} \quad (i = 1,2),$$

and let $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_1', \mathcal{L}_2'$ denote the reproducing kernel Hilbert spaces associated with the positive kernels L_1, L_2 and L_1', L_2', respectively. We claim that the mappings

$$U_i : \mathcal{L}_i \to \mathcal{L}_i'\ ,\ \ f \mapsto \alpha \cdot f$$

are unitary. This follows immediately by a double application of Proposition 1.1.11 *if α has no zeroes. For the general case, let D_0 denote the complement of the zero set of α in D and note that restriction mappings*

$$\mathcal{L}_i \to (\mathcal{L}_i)_{|D_0} \quad \text{and} \quad \mathcal{L}_i' \to (\mathcal{L}_i')_{|D_0}$$

are unitary by the identity theorem. This shows that L_1, L_2 form a disjoint decomposition of L' and that $\operatorname{rank} L_i = \operatorname{rank} L_i'$ *for $i = 1, 2$. Now the assertion follows by Proposition* 1.5.7.

Suppose that $K : X \times X \to B(\mathcal{E})$ is a positive kernel and that $\mathcal{H} \subset \mathcal{E}^X$ is the associated reproducing kernel Hilbert space. Via the identification from Theorem

1.6.3, we can equip $B(K)$ with the weak operator and weak-$*$ topologies of $B(\mathcal{H})$. Since we shall frequently make use of these topologies on $B(K)$, we discuss them now in some detail.

Proposition 1.6.11. *Suppose that $\mathcal{E}$ is a Hilbert space and that $K : X \times X \to B(\mathcal{E})$ is a positive kernel.*

(a) Let $(L_\alpha)_\alpha$ be a bounded net in $B(K)$ and let $L \in B(K)$ be a kernel. Then the following are equivalent:

(i) $(L_\alpha)_\alpha$ converges WOT to L.

(ii) $(L_\alpha)_\alpha$ converges weak-$*$ to L.

(iii) $(L_\alpha(z, w))_\alpha$ converges WOT to $L(z, w)$ for all $z, w \in X$.

(iv) $(L_\alpha(z, w))_\alpha$ converges weak-$*$ to $L(z, w)$ for all $z, w \in X$.

(b) Let $(L_n)_n$ be a sequence in $B(K)$ and let $L \in B(K)$ be a kernel. Then the following are equivalent:

(i) $(L_n)_n$ converges WOT to L.

(ii) $(L_n)_n$ converges weak-$*$ to L.

(iii) $(L_n)_n$ is bounded and the sequence $(L_n(z, w))_n$ converges WOT to $L(z, w)$ for all $z, w \in X$.

(iv) $(L_n)_n$ is bounded and the sequence $(L_n(z, w))_n$ converges weak-$*$ to $L(z, w)$ for all $z, w \in X$.

Proof. Let $\mathcal{H} \subset \mathcal{E}^X$ the reproducing kernel Hilbert space associated with K. To prove (a), one uses that the weak operator and the weak-$*$ topologies coincide on bounded sets. This shows the equivalence of (i) and (ii) and of (iii) and (iv). Let T_α, T denote the representing operators of L_α, L. Now by definition

$$\langle L_\alpha(z, w)y, x\rangle = \langle T_\alpha K(\cdot, w)y, K(\cdot, z)x\rangle \xrightarrow{\alpha} \langle TK(\cdot, w)y, K(\cdot, z)x\rangle = \langle L(z, w)y, x\rangle$$

holds for all $z, w \in X$ and $x, y \in \mathcal{E}$. Hence we obtain the implication (i) to (iii). Since the set $\{K(\cdot, z)x\ ;\ z \in X, x \in \mathcal{E}\}$ is total in $\mathcal{H}$ and the net $(L_\alpha)_\alpha$ was supposed to be bounded, the remaining implication also follows. The missing statements of part (b) are clear by the uniform boundedness principle. $\square$

The following approximation result is similar to Corollary 1.1.12.

Lemma 1.6.12. *Suppose that $\mathcal{E}$ is a Hilbert space and that $K : X \times X \to B(\mathcal{E})$ is a positive kernel. Let $L : X \times X \to B(\mathcal{E})$ be an arbitrary kernel. Then the following are equivalent:*

(i) L belongs to $B(K)$.

(ii) There exists a bounded sequence $(L_n)_n$ in $B(K)$ such that $(L_n(z,w))_n$ converges WOT (equivalently, weak-$*$) to $L(z,w)$ for all $z,w \in X$.

(iii) There exists a bounded net $(L_\alpha)_\alpha$ in $B(K)$ such that $(L_\alpha(z,w))_\alpha$ converges WOT (equivalently, weak-$*$) to $L(z,w)$ for all $z,w \in X$.

In this case,

$$\|L\|_K \leq \liminf_n \|L_n\|_K \quad \text{and} \quad \|L\|_K \leq \liminf_\alpha \|L_\alpha\|_K$$

holds for all sequences and nets as in (ii) and (iii), respectively.

Proof. The only non-trivial implication is (iii) to (i). To prove it, let $(L_\alpha)_{\alpha\in A}$ be a net satisfying (iii). One verifies that, for every $d > \liminf_{\alpha\in A} \|L_\alpha\|_K$, there exists a subnet $(L_\alpha)_{\alpha\in A_d}$ satisfying $\|L_\alpha\|_K < d$ for all $\alpha \in A_d$. Now the assertion follows directly from the definition of subordinate kernels. □

Alternatively, one could also use the WOT compactness of the unit ball of $B(K)$ and Proposition 1.6.11 in order to prove the lemma.

We conclude this section by a short discussion of the operator space structure of $B(K)$. As before, $\mathcal{E}$ is an arbitrary Hilbert space and $K : X \times X \to B(\mathcal{E})$ is a positive kernel with associated reproducing kernel Hilbert space $\mathcal{H} \subset \mathcal{E}^X$.

Clearly $B(K)$ carries the natural operator space structure inherited from $B(\mathcal{H})$ via the canonical isomorphism of Theorem (1.6.3). That is, the n-th matrix norm of a matrix $[L_{ij}] \in M_n(B(K))$ is given by

$$\|[L_{ij}]\| = \|[T_{ij}]\|,$$

where $T_{ij} \in B(\mathcal{H})$ are the representing operators of the kernels L_{ij} and the matrix $[T_{ij}]$ is understood as an operator on $\mathcal{H}^n$.

On the other hand, we could identify the matrix $[L_{ij}]$ with a $B(\mathcal{E}^n)$-valued kernel

$$L : X \times X \to B(\mathcal{E}^n)\ ,\ L(z,w) = [L_{ij}(z,w)] \in B(\mathcal{E}^n),$$

which is subordinate to the inflation $K^{(n)}$ defined in Section 1.2. In fact, the reproducing kernel Hilbert space $\mathcal{H}^{(n)} \subset (\mathcal{E}^n)^X$ associated with $K^{(n)}$ is unitarily equivalent to $\mathcal{H}^n$ by Proposition 1.2.2, and L is, under this identification, represented by the operator matrix $[T_{ij}] \in B(\mathcal{H}^n)$. In particular, the matrix norms

$$\|[L_{ij}]\| = \|L\|_{K^{(n)}}$$

are the same as the induced matrix norms considered above.

1.7 Multipliers of reproducing kernel Hilbert spaces

Definition 1.7.1. *Suppose that $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2$ are Hilbert spaces and that $\mathcal{H} \subset \mathcal{E}^X$, $\mathcal{H}_1 \subset \mathcal{E}_1^X$ and $\mathcal{H}_2 \subset \mathcal{E}_2^X$ are reproducing kernel Hilbert spaces.*

(a) A function $\phi : X \to B(\mathcal{E}_1, \mathcal{E}_2)$ is called a multiplier between $\mathcal{H}_1$ and $\mathcal{H}_2$ if the pointwise product $\phi \cdot f$ belongs to $\mathcal{H}_2$ for all $f \in \mathcal{H}_1$.

(b) The collection of all multipliers between $\mathcal{H}_1, \mathcal{H}_2$ is denoted by $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$. We further use the abbreviation $\mathcal{M}(\mathcal{H}) = \mathcal{M}(\mathcal{H}, \mathcal{H})$.

(c) For $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$, the linear mapping

$$M_\phi : \mathcal{H}_1 \to \mathcal{H}_2 \ , \ f \mapsto \phi \cdot f$$

is called the multiplication operator with symbol ϕ.

(d) We write

$$M(\mathcal{H}_1, \mathcal{H}_2) = \{M_\phi \ ; \ \phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)\}$$

for the set of all multiplication operators (and $M(\mathcal{H}) = M(\mathcal{H}, \mathcal{H})$).

The class $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ actually is a linear space and enjoys the following obvious multiplication property: if $\mathcal{H}_3 \subset \mathcal{E}_3^X$ is another reproducing kernel Hilbert space, then $\psi \cdot \phi$ belongs to $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_3)$ for all $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ and $\psi \in \mathcal{M}(\mathcal{H}_2, \mathcal{H}_3)$. In particular, the space $\mathcal{M}(\mathcal{H})$ is an algebra.

It is folklore that multiplication operators are automatically continuous (by the closed graph theorem). Hence $M(\mathcal{H}_1, \mathcal{H}_2)$ is a linear subspace of $B(\mathcal{H}_1, \mathcal{H}_2)$.

We shall use this fact to equip $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ via the mapping

$$\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \to M(\mathcal{H}_1, \mathcal{H}_2) \ , \ \phi \mapsto M_\phi,$$

with the topological and operator space structures of $B(\mathcal{H}_1, \mathcal{H}_2)$. At this point, we should stress the fact that the above assignment may fail to be one-to-one: For example, consider some reproducing kernel Hilbert space $\mathcal{H} \subset \mathbb{C}^X$ having a common zero at some point $z_0 \in X$. Then the function being 1 at z_0 and 0 elsewhere obviously is a multiplier of $\mathcal{H}$ with $M_\phi = 0$ (but $\phi \neq 0$).

To ensure the injectivity of the mapping

$$\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \to M(\mathcal{H}_1, \mathcal{H}_2) \ , \ \phi \mapsto M_\phi,$$

it obviously suffices to require that the space $\mathcal{H}_1$ is non-degenerate.

Definition 1.7.2. *Suppose that $\mathcal{E}_1, \mathcal{E}_2$ are Hilbert spaces and $\mathcal{H}_1 \subset \mathcal{E}_1^X$, $\mathcal{H}_2 \subset \mathcal{E}_2^X$ are reproducing kernel Hilbert spaces.*

(a) *We define a semi-norm* $\|\cdot\|_{\mathcal{M}}$ *on the multiplier space* $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ *by setting*

$$\|\phi\|_{\mathcal{M}} = \|M_\phi\|$$

for $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$.

(b) *The weak operator and weak-$*$ topologies on* $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ *are the initial topologies of the corresponding topologies on* $B(\mathcal{H}_1, \mathcal{H}_2)$ *induced by the mapping*

$$\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \to B(\mathcal{H}_1, \mathcal{H}_2)\ ,\ \phi \mapsto M_\phi.$$

Note that $\|\cdot\|_{\mathcal{M}}$ is a norm and the weak operator and weak-$*$ topologies on $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ are Hausdorff whenever $\mathcal{H}_1$ is non-degenerate.

Example 1.7.3.

(a) *Let* $\mathcal{H} = l^2(I)$ *be the reproducing kernel space considered in Example* 1.1.3 (a) *and let* $\mathcal{E}_1, \mathcal{E}_2$ *be Hilbert spaces. Then* $\mathcal{M}(\mathcal{H}_{\mathcal{E}_1}, \mathcal{H}_{\mathcal{E}_2}) = l^\infty(I, B(\mathcal{E}_1, \mathcal{E}_2))$ *isometrically, where*

$$l^\infty(I, B(\mathcal{E}_1, \mathcal{E}_2)) = \{\phi : I \to B(\mathcal{E}_1, \mathcal{E}_2)\ ;\ \|\phi\|_{\infty,I} = \sup_{i\in I} \|\phi(i)\| < \infty\}.$$

Furthermore, it is easy to see that for $\phi \in \mathcal{M}(\mathcal{H}_{\mathcal{E}_1}, \mathcal{H}_{\mathcal{E}_2})$, *the adjoint of* M_ϕ *is the multiplication operator with symbol*

$$\overline{\phi} : I \to B(\mathcal{E}_2, \mathcal{E}_1)\ ,\ i \mapsto \phi(i)^*.$$

In particular, $M(\mathcal{H})$ *is a* C^**-subalgebra of* $B(\mathcal{H})$, *which is a very special situation.*

Indeed, let $\mathcal{H} \subset \mathbb{C}^X$ *be a reproducing kernel Hilbert space with kernel* K *such that* K *has no zeroes. Note that the typical analytic function spaces presented in Example* 1.1.10 *are of this type. Assume that* $T = M_\phi$ *is an operator in* $M(\mathcal{H}) \cap M(\mathcal{H})^*$ *(where* $M(\mathcal{H})^*$ *is the space of adjoints of multiplication operators). Then there exists some* $\psi \in \mathcal{M}(\mathcal{H})$ *such that* $M_\phi^* = M_\psi$. *This implies*

$$\overline{\phi(w)} K(z,w) = \langle M_\phi^* K(\cdot,w), K(\cdot,z)\rangle = \langle M_\psi K(\cdot,w), K(\cdot,z)\rangle = \psi(z)K(z,w)$$

for all $z, w \in X$. *Hence* ϕ *is constant. Therefore* $M(\mathcal{H}) \cap M(\mathcal{H})^* = \mathbb{C} \cdot 1_{\mathcal{H}}$, *and* $M(\mathcal{H})$ *is is a* C^**-subalgebra of* $B(\mathcal{H})$ *precisely if* $\mathcal{M}(\mathcal{H})$ *consists solely of constant functions. Clearly, this happens only in pathological examples.*

(b) *Suppose that* D *is an open subset of* $\mathbb{C}^d$ *and that* $\mathcal{H} \subset \mathcal{O}(D)$ *is a reproducing kernel space consisting of analytic functions such that the coordinate functions* $\mathbf{z}_i$ $(1 \le i \le d)$ *are multipliers of* $\mathcal{H}$.

If the Taylor spectrum $\sigma(M_{\mathbf{z}})$ *of the commuting tuple* $M_{\mathbf{z}} = (M_{\mathbf{z}_1}, \dots, M_{\mathbf{z}_d})$ *is contained in* $\overline{D}$, *then every function* $\phi \in \mathcal{O}(\overline{D}, B(\mathcal{E}_1, \mathcal{E}_2))$ $(\mathcal{E}_1, \mathcal{E}_2$ *Hilbert spaces)* *defines a multiplier between* $\mathcal{H}_{\mathcal{E}_1}$ *and* $\mathcal{H}_{\mathcal{E}_2}$ *and moreover,* $M_\phi = \phi(M_{\mathbf{z}})$*, where the operator on the right is formed by means of a suitable operator-valued analytic functional calculus. Details can be found in [36].*

Note that a partial converse of this statement holds. Namely, suppose that $\overline{D}$ *is a Stein compact set. Then* $\mathcal{O}(\overline{D}) \subset \mathcal{M}(\mathcal{H})$ *implies that* $\sigma(M_{\mathbf{z}}) \subset \overline{D}$*. In fact, fix some* $\lambda \notin \overline{D}$*. By the definition of Stein compactness, there exists a domain of holomorphy* $U \supset \overline{D}$ *such that* $\lambda \notin U$*. But then there exist functions* $\phi_1, \ldots, \phi_d \in \mathcal{O}(U)$ *such that*

$$\mathbf{1} = \sum_{i=1}^{d} (\lambda_i - \mathbf{z}_i) \cdot \phi_i$$

holds. By hypothesis, the functions ϕ_i *are multipliers and hence*

$$1_{\mathcal{H}} = \sum_{i=1}^{d} (\lambda_i - M_{\mathbf{z}_i}) M_{\phi_i}.$$

Then by [35], Lemma 2.2.4, λ *does not belong to the Taylor spectrum of the commuting tuple* $M_{\mathbf{z}}$*.*

Finally, we point out that the inclusion $\overline{D} \subset \sigma(M_{\mathbf{z}})$ *always holds if one requires in addition that there is no component* C *of* D *such that all functions in* $\mathcal{H}$ *vanish on* C*. Indeed, if* K *denotes the reproducing kernel of* $\mathcal{H}$*, then*

$$M_{\mathbf{z}_i}^* K(\cdot, w) = \overline{w_i} K(\cdot, w)$$

holds for $w \in D$ *and* $1 \leq i \leq d$*. Letting* $Z(\mathcal{H})$ *denote the common zero set of* $\mathcal{H}$*, this shows that the set*

$$D \cap Z(\mathcal{H})^c = \{w \in D \, ; K(\cdot, w) \neq 0\}$$

is included in the Taylor spectrum of the tuple $M_{\mathbf{z}}$*. Our assumption and the identity theorem imply*

$$\overline{D} = \overline{D \cap Z(\mathcal{H})^c} \subset \sigma(M_{\mathbf{z}}).$$

(c) *Let* D *be a Cartan domain in* $\mathbb{C}^d$ *and fix* ν *in the continuous Wallach set of* D*. Let* $\mathcal{H} = \mathcal{H}_\nu$ *be the corresponding space of analytic functions as defined in Example* 1.1.10 *and write* $K = K_\nu$*. Then by [10], the coordinate functions belong to* $\mathcal{M}(\mathcal{H})$ *and hence, every polynomial is a multiplier. Moreover, in [10] it is shown that the Taylor spectrum of the multiplication tuple* $\sigma(M_{\mathbf{z}})$ *equals* $\overline{D}$*. Hence, by part (*b*), every function holomorphic on a neighbourhood of* $\overline{D}$ *defines a multiplier of* $\mathcal{H}$*. In particular, the functions* $K(\cdot, w)$ *(*$w \in D$*) belong to* $\mathcal{M}(\mathcal{H})$ *since they can be analytically extended on a neighbourhood of* $\overline{D}$*. Indeed, the Faraut-Koranyi expansion (*1.1.3*), in connection with the homogeneity of the kernels* $K_{\mathbf{m}}$*, shows that*

$$K(rz, \frac{1}{r} w) = \sum_{\mathbf{m}} (\nu)_{\mathbf{m}} K_{\mathbf{m}}(rz, \frac{1}{r} w) = \sum_{\mathbf{m}} (\nu)_{\mathbf{m}} K_{\mathbf{m}}(z, w) = K(z, w)$$

holds for all $z \in D$ *and all* $0 < r < 1$ *for which* $\frac{1}{r}w$ *belongs to* D*. Since* D *is the open unit ball with respect to an appropriate norm on* $\mathbb{C}^d$*, the set* $\frac{1}{r}D$ *is an open neighbourhood of* $\overline{D}$*, and the analytic function*

$$\frac{1}{r}D \to \mathbb{C}\ ,\ \ z \mapsto K(rz, \frac{1}{r}w)$$

is an extension of $K(\cdot, w)$*.*

(d) *Let* D *be a Cartan domain in* $\mathbb{C}^d$ *of rank* r *and fix* $\nu \geq \frac{d}{r}$*. By the discussion in Example* 1.1.10*, this means that* $\mathcal{H} = \mathcal{H}_\nu$ *is either of Hardy or Bergman type. It is elementary to check that, in these cases,*

$$\mathcal{M}(\mathcal{H}_{\mathcal{E}_1}, \mathcal{H}_{\mathcal{E}_2}) = H^\infty(D, B(\mathcal{E}_1, \mathcal{E}_2))$$

(with equality of norms) holds for all Hilbert spaces $\mathcal{E}_1, \mathcal{E}_2$*.*

(e) *For the Arveson space* $\mathcal{H} = H(\mathbb{B}_d)$*, it is shown in [36] that*

$$\mathcal{M}(\mathcal{H}_{\mathcal{E}_1}, \mathcal{H}_{\mathcal{E}_2}) = \{\phi \in \mathcal{O}(\mathbb{B}_d, B(\mathcal{E}_1, \mathcal{E}_2))\ ;\ \sup_{r,T,H} \|\phi(rT)\| < \infty\},$$

where the supremum ranges over all $0 < r < 1$ *and all d-contractions* T *on Hilbert spaces* H*. Moreover, for such* ϕ*, this supremum actually equals* $\|\phi\|_{\mathcal{M}}$*. Recall that a commuting tuple* $T = (T_1, \ldots, T_d) \in L(H)^d$ *on some Hilbert space* H *is called a d-contraction if* $\sum_i T_i T_i^* \leq 1_H$ *holds. Note further that the operators* $\phi(rT)$ *in the above expression are formed by means of an operator-valued extension of Taylor's analytic functional calculus, also described in [36].*

In practice, one often faces the following two problems: First, to decide whether or not a function $\phi : X \to B(\mathcal{E}_1, \mathcal{E}_2)$ defines a multiplier between reproducing kernel Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, and secondly, to characterize $M(\mathcal{H}_1, \mathcal{H}_2)$ as a subspace of $B(\mathcal{H}_1, \mathcal{H}_2)$. The following results (Lemma 1.7.4, Proposition 1.7.6 and Proposition 1.7.9) provide tools to deal with these questions.

Lemma 1.7.4. *Let* $\mathcal{E}_1, \mathcal{E}_2$ *be Hilbert spaces and let* $\mathcal{H}_1 \subset \mathcal{E}_1^X$ *and* $\mathcal{H}_2 \subset \mathcal{E}_2^X$ *be reproducing kernel Hilbert spaces. Suppose that* $\phi : X \to B(\mathcal{E}_1, \mathcal{E}_2)$ *is an operator-valued function. If there exist an operator* $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ *and a dense subset* M *of* $\mathcal{H}_1$ *such that* $\phi \cdot f = Tf$ *holds for all* $f \in M$*, then* ϕ *belongs to* $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ *and* $T = M_\phi$*.*

Proof. It suffices to show that $\phi \cdot f \in \mathcal{H}_2$ for all $f \in \mathcal{H}_1$. For $f \in \mathcal{H}_1$, we can choose a sequence $(f_n)_n$ in M such that $\lim_n f_n = f$. Since $Tf = \lim_n \phi \cdot f_n$ in $\mathcal{H}_2$, and since the sequence $(\phi \cdot f_n)_n$ converges pointwise to $\phi \cdot f$, it follows that $\phi \cdot f = Tf \in \mathcal{H}_2$. □

Corollary 1.7.5. *Suppose that* $\mathcal{H} \subset \mathbb{C}^X$ *is a reproducing kernel Hilbert space with the property that* $\mathcal{M}(\mathcal{H})$ *is a dense subset of* $\mathcal{H}$*. Let* $\mathcal{E}_1, \mathcal{E}_2$ *be Hilbert spaces. If*

$T \in B(\mathcal{H}_{\mathcal{E}_1}, \mathcal{H}_{\mathcal{E}_2})$ is an operator which satisfies

$$T(M_\alpha \otimes 1_{\mathcal{E}_1}) = (M_\alpha \otimes 1_{\mathcal{E}_2})T$$

for all $\alpha \in \mathcal{M}(\mathcal{H})$, then $T \in M(\mathcal{H}_{\mathcal{E}_1}, \mathcal{H}_{\mathcal{E}_2})$.

Proof. By hypothesis, $\mathcal{H}$ contains the constant functions. We define

$$\phi : X \to B(\mathcal{E}_1, \mathcal{E}_2)\ ,\ \phi(z)x = T(\mathbf{1} \otimes x)(z).$$

Now consider functions $f \in \mathcal{H}_{\mathcal{E}_1}$ of the form $f = \alpha \otimes x$, where $\alpha \in \mathcal{M}(\mathcal{H})$ and $x \in \mathcal{E}_1$. Such functions satisfy

$$\phi(z)f(z) = \alpha(z)\phi(z)x = (M_\alpha \otimes 1_{\mathcal{E}_2})T(\mathbf{1} \otimes x)(z) = T(\alpha \otimes x)(z) = Tf(z)$$

for all $z \in X$. Clearly, the identity $\phi \cdot f = Tf$ holds then for linear combinations of such functions, which form a dense subset of $\mathcal{H}_{\mathcal{E}_1}$. By Lemma 1.7.4, we have $\phi \in \mathcal{M}(\mathcal{H}_{\mathcal{E}_1}, \mathcal{H}_{\mathcal{E}_2})$ and $T = M_\phi$. □

In particular, we observe that for reproducing kernel Hilbert spaces $\mathcal{H} \subset \mathbb{C}^X$ as in Corollary 1.7.5, the space $M(\mathcal{H})$ coincides with its commutant in $B(\mathcal{H})$ and is therefore WOT closed. Later in this section, we shall prove that the second fact remains true in more general settings.

The question if an operator-valued function is a multiplier between given reproducing kernel spaces can be answered by checking the positivity of an assigned kernel. The next result can therefore be regarded as a structural analogue of Proposition 1.1.11.

Proposition 1.7.6. *Suppose that $\mathcal{E}_1, \mathcal{E}_2$ are Hilbert spaces and that $\mathcal{H}_1 \subset \mathcal{E}_1^X$ and $\mathcal{H}_2 \subset \mathcal{E}_2^X$ are reproducing kernel Hilbert spaces. Let $K_1 : X \times X \to B(\mathcal{E}_1)$ and $K_2 : X \times X \to B(\mathcal{E}_2)$ denote the corresponding reproducing kernels. Then, for a function $\phi : X \to B(\mathcal{E}_1, \mathcal{E}_2)$, the following are equivalent:*

(i) ϕ belongs to $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$.

(ii) There exists a constant $c \geq 0$ such that the mapping

$$X \times X \to B(\mathcal{E}_2)\ ,\ (z, w) \mapsto c^2 K_2(z, w) - \phi(z)K_1(z, w)\phi(w)^*$$

is positive definite. In this case, $\|\phi\|_{\mathcal{M}}$ is the infimum of all such constants $c \geq 0$. Moreover, the infimum is achieved.

For the proof of this well-known result, the reader is referred to [16]. As a consequence, we see that, in the case of scalar inflations, the multiplier norm dominates the sup norm.

Corollary 1.7.7. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a reproducing kernel Hilbert space with reproducing kernel $K : X \times X \to \mathbb{C}$. Suppose further that $\mathcal{E}_1, \mathcal{E}_2$ are Hilbert spaces. Then every multiplier $\phi \in \mathcal{M}(\mathcal{H}_{\mathcal{E}_1}, \mathcal{H}_{\mathcal{E}_2})$ is pointwise bounded by $\|\phi\|_{\mathcal{M}}$ outside the set $\{z \in X \ ; \ K(z,z) = 0\}$. In particular, if $\mathcal{H}$ is non-degenerate, then*

$$\|\phi\|_\infty \leq \|\phi\|_{\mathcal{M}}$$

holds for all $\phi \in \mathcal{M}(\mathcal{H}_{\mathcal{E}_1}, \mathcal{H}_{\mathcal{E}_2})$.

Remark 1.7.8. *The statement of Proposition* 1.7.6 *can also be rephrased in terms of subordinate kernels. That is, the following are equivalent in the situation of Proposition* 1.7.6*:*

(i) *ϕ belongs to $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$.*

(ii) *The kernel*

$$G_\phi : X \times X \to B(\mathcal{E}_2) \ , \ G_\phi(z,w) = \phi(z) K_1(z,w) \phi(w)^*$$

belongs to $B(K_2)$.

In this case, we have $\|\phi\|^2_{\mathcal{M}} = \|G_\phi\|_{K_2}$.

Next, we want to study the topological properties of $M(\mathcal{H}_1, \mathcal{H}_2)$ as a subspace of $B(\mathcal{H}_1, \mathcal{H}_2)$. It is easily seen (and well known) that $M(\mathcal{H}_1, \mathcal{H}_2) \subset B(\mathcal{H}_1, \mathcal{H}_2)$ is norm-closed if $\mathcal{H}_1$ is non-degenerate. We shall prove the stronger result that $M(\mathcal{H}_1, \mathcal{H}_2)$ is WOT closed. To do so, we establish the following purely algebraic characterization of multiplication operators, which seems to be new so far.

Proposition 1.7.9. *Suppose that $\mathcal{E}_1, \mathcal{E}_2$ are Hilbert spaces and that $\mathcal{H}_1 \subset \mathcal{E}_1^X$ and $\mathcal{H}_2 \subset \mathcal{E}_2^X$ are reprocing kernel Hilbert spaces, $\mathcal{H}_1$ non-degenerate. Let $\delta_{1,z} : \mathcal{H}_1 \to \mathcal{E}_1$ and $\delta_{2,z} : \mathcal{H}_2 \to \mathcal{E}_2$ denote the point evaluations at $z \in X$. Then for an operator $T \in B(\mathcal{H}_1, \mathcal{H}_2)$, the following are equivalent:*

(i) *Whenever $f \in \mathcal{H}_1$, $z \in X$ and $f(z) = 0$, then $Tf(z) = 0$.*

(ii) *$T^* \operatorname{ran} \delta_{2,z}^* \subset \operatorname{ran} \delta_{1,z}^*$ for all $z \in X$.*

(iii) *$T \in M(\mathcal{H}_1, \mathcal{H}_2)$.*

In particular, $M(\mathcal{H}_1, \mathcal{H}_2)$ is WOT closed in $B(\mathcal{H}_1, \mathcal{H}_2)$, and $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ is a dual space.

Proof. Condition (i) clearly says $T \ker \delta_{1,z} \subset \ker \delta_{2,z}$ for all $z \in X$. Equivalently, $T^* \operatorname{ran} \delta_{2,z}^* \subset \overline{\operatorname{ran} \delta_{1,z}^*}$ for all $z \in X$. Note that $\delta_{1,z}$ is onto for all $z \in X$ since $\mathcal{H}_1$ was supposed to be non-degenerate. In particular, the operators $\delta_{1,z}$ have closed range and, by standard duality arguments, the same is true for their adjoints. This proves the implication (i) to (ii). Now suppose (ii). Since $\mathcal{H}_1$ was supposed to be non-degenerate, there exist right inverses $i_{1,z} \in B(\mathcal{E}_1, \mathcal{H}_1)$ for the point evaluations $\delta_{1,z}$. We define

$$\phi : X \to B(\mathcal{E}_1, \mathcal{E}_2)\ ,\ \phi(z) = \delta_{2,z} T i_{1,z}.$$

Now fix $f \in \mathcal{H}_1$, $z \in X$ and $y \in \mathcal{E}_2$. By (ii), we can choose some $x \in \mathcal{E}_1$ with $T^*\delta_{2,z}^* y = \delta_{1,z}^* x$. We obtain that

$$\begin{aligned} \langle \phi(z) f(z), y\rangle &= \langle i_{1,z} f(z), T^*\delta_{2,z}^* y\rangle \\ &= \langle i_{1,z} f(z), \delta_{1,z}^* x\rangle \\ &= \langle \delta_{1,z} i_{1,z} f(z), x\rangle \\ &= \langle f(z), x\rangle \\ &= \langle f, \delta_{1,z}^* x\rangle \\ &= \langle f, T^*\delta_{2,z}^* y\rangle \\ &= \langle Tf(z), y\rangle. \end{aligned}$$

Thus $\phi \cdot f = Tf$ for all $f \in \mathcal{H}_1$ which shows that ϕ belongs to $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ and that $T = M_\phi$. The implication (iii) to (i) is obvious.

□

Although we will not need it, we provide the following result which seems to be unknown in this general form.

Corollary 1.7.10. *Suppose that $\mathcal{E}$ is a Hilbert space and that $\mathcal{H} \subset \mathcal{E}^X$ is a non-degenerate reproducing kernel Hilbert space. Then $M(\mathcal{H})$ is a reflexive subalgebra of $B(\mathcal{H})$.*

Proof. Suppose that $T \in B(\mathcal{H})$ leaves invariant all closed linear subspaces of $\mathcal{H}$ that are invariant under the algebra $M(\mathcal{H})$. Fix $f \in \mathcal{H}$ and $z \in X$ with $f(z) = 0$. Since the subspace

$$M_z = \{g \in \mathcal{H}\ ;\ g(z) = 0\}$$

obviously is invariant for $M(\mathcal{H})$, we have $TM_z \subset M_z$. In particular, $Tf(z) = 0$. □

The following proposition provides a characterization of convergence with respect to the weak operator and weak-$*$ topologies on $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$.

Proposition 1.7.11. *Suppose that $\mathcal{E}_1, \mathcal{E}_2$ are Hilbert spaces and that $\mathcal{H}_1 \subset \mathcal{E}_1^X$ and $\mathcal{H}_2 \subset \mathcal{E}_2^X$ are reproducing kernel spaces such that $\mathcal{H}_1$ is non-degenerate.*

(a) Let $(\phi_\alpha)_\alpha$ be a bounded net in $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$. Then, for $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$, the following are equivalent:

(i) $(\phi_\alpha)_\alpha$ *converges WOT to* ϕ.

(ii) $(\phi_\alpha)_\alpha$ *converges weak-$*$ to* ϕ.

(iii) $(\phi_\alpha(z))_\alpha$ *converges WOT to* $\phi(z)$ *for all* $z \in X$.

(iv) $(\phi_\alpha(z))_\alpha$ *converges weak-$*$ to* $\phi(z)$ *for all* $z \in X$.

(b) *Let* $(\phi_n)_n$ *be a sequence in* $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$. *Then, for* $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$, *the following are equivalent:*

(i) $(\phi_n)_n$ *converges WOT to* ϕ.

(ii) $(\phi_n)_n$ *converges weak-$*$ to* ϕ.

(iii) $(\phi_n)_n$ *is bounded and* $(\phi_n(z))_n$ *converges WOT to* $\phi(z)$ *for all* $z \in X$.

(iv) $(\phi_n)_n$ *is bounded and* $(\phi_n(z))_n$ *converges weak-$*$ to* $\phi(z)$ *for all* $z \in X$.

The proof is similar to the one of Proposition 1.6.11 and therefore omitted.

Similarly to Corollary 1.1.12 in the context of reproducing kernel Hilbert spaces and to Lemma 1.6.12 in the setting of subordinate kernels, the following approximation result holds for multipliers.

Lemma 1.7.12. *Suppose that* $\mathcal{E}_1, \mathcal{E}_2$ *are Hilbert spaces and that* $\mathcal{H}_1 \subset \mathcal{E}_1^X$ *and* $\mathcal{H}_2 \subset \mathcal{E}_2^X$ *are reproducing kernel Hilbert spaces. Let* $\phi : X \to B(\mathcal{E}_1, \mathcal{E}_2)$ *be an arbitrary function. Then the following are equivalent:*

(i) $\phi \in \mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$.

(ii) *There exists a bounded sequence* $(\phi_n)_n$ *in* $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ *such that* $(\phi_n(z))_n$ *converges WOT (equivalently, weak-$*$) to* $\phi(z)$ *for all* $z \in X$.

(iii) *There exists a bounded net* $(\phi_\alpha)_\alpha$ *in* $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ *such that* $(\phi_\alpha(z))_\alpha$ *converges WOT (equivalently, weak-$*$) to* $\phi(z)$ *for all* $z \in X$.

In this case,

$$\|\phi\|_{\mathcal{M}} \leq \liminf_n \|\phi_n\|_{\mathcal{M}} \quad \text{and} \quad \|\phi\|_{\mathcal{M}} \leq \liminf_\alpha \|\phi_\alpha\|_{\mathcal{M}}$$

holds for all sequences and nets as in (ii) and (iii), respectively.

Proof. The only non-trivial implication is (iii) to (i). Let K_1, K_2 denote the reproducing kernels of $\mathcal{H}_1, \mathcal{H}_2$ and let $d > \liminf_{\alpha \in A} \|\phi_\alpha\|_{\mathcal{M}}$ be arbitrary. After passing to a suitable subnet, we may assume that $\|\phi_\alpha\|_{\mathcal{M}} < d$ holds for all α. Clearly, the net $(M_{\phi_\alpha})_\alpha$ has a WOT cluster point $T \in B(\mathcal{H}_1, \mathcal{H}_2)$ with $\|T\| \leq d$. Then, for an appropriate subnet, we obtain

$$\begin{aligned} \langle Tf(z), x \rangle &= \lim_i \langle M_{\phi_{\alpha_i}} f, K_2(\cdot, z)x \rangle \\ &= \lim_i \langle \phi_{\alpha_i}(z) f(z), x \rangle \\ &= \langle \phi(z) f(z), x \rangle \end{aligned}$$

for all $f \in \mathcal{H}$, $z \in X$ and $x \in \mathcal{E}$. Hence $\phi \in \mathcal{M}(\mathcal{H})$ and $T = M_\phi$ and moreover, $\|\phi\|_{\mathcal{M}} = \|T\| \leq d$. □

It is a well-known fact that the space $H^\infty(\mathbb{B}_d, B(\mathcal{F}_1, \mathcal{F}_2))$ can be expressed as the normal spatial tensor product $H^\infty(\mathbb{B}_d)\overline{\otimes}B(\mathcal{F}_1, \mathcal{F}_2)$, where $H^\infty(\mathbb{B}_d)$ is realized as a subspace of $B(H^2(\mathbb{B}_d))$. We now want to generalize this result to arbitrary multiplier algebras. To do so, we need the following lemma.

Lemma 1.7.13. *Suppose that $\mathcal{E}$ is a Hilbert space and that $\mathcal{H} \subset \mathcal{E}^X$ is a reproducing kernel Hilbert space. Suppose further that $\mathcal{F}_1, \mathcal{F}_2$ are Hilbert spaces. Then for every operator $A \in B(\mathcal{F}_1, \mathcal{F}_2)$, the constant function*

$$\alpha : X \to B(\mathcal{E} \otimes \mathcal{F}_1, \mathcal{E} \otimes \mathcal{F}_2) \ , \ z \mapsto (1_{\mathcal{E}} \otimes A)$$

belongs to $\mathcal{M}(\mathcal{H} \otimes \mathcal{F}_1, \mathcal{H} \otimes \mathcal{F}_2)$ with $\|\alpha\|_{\mathcal{M}} \leq \|A\|$ and moreover, $M_\alpha = 1_{\mathcal{H}} \otimes A$.

Proof. We clearly may assume that $\|A\| = 1$. Let $K : X \times X \to B(\mathcal{E})$ denote the reproducing kernel of $\mathcal{H}$. Then the kernel

$$\begin{aligned} &K_{\mathcal{F}_2}(z,w) - \alpha(z)K_{\mathcal{F}_1}(z,w)\alpha(w)^* \\ &= \quad (K(z,w) \otimes 1_{\mathcal{F}_2}) - (1_{\mathcal{E}} \otimes A)(K(z,w) \otimes 1_{\mathcal{F}_1})(1_{\mathcal{E}} \otimes A^*) \\ &= \quad K(z,w) \otimes (1_{\mathcal{F}_2} - AA^*) \end{aligned}$$

obviously is positive definite. Hence, by Proposition 1.7.6, α is a contractive multiplier between $\mathcal{H}_{\mathcal{F}_1}$ and $\mathcal{H}_{\mathcal{F}_2}$. □

Proposition 1.7.14. *Let $\mathcal{E}_1, \mathcal{E}_2$ be Hilbert spaces and let $\mathcal{H}_1 \subset \mathcal{E}_1^X$, $\mathcal{H}_2 \subset \mathcal{E}_2^X$ be reproducing kernel Hilbert spaces such that $\mathcal{H}_1$ is non-degenerate. For Hilbert spaces $\mathcal{F}_1, \mathcal{F}_2$, the mapping*

$$\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \otimes B(\mathcal{F}_1, \mathcal{F}_2) \to \mathcal{M}(\mathcal{H}_1 \otimes \mathcal{F}_1, \mathcal{H}_2 \otimes \mathcal{F}_2) \ , \ \phi \otimes T \mapsto (z \mapsto \phi(z) \otimes T)$$

extends to a completely isometric isomorphism between the normal spatial tensor product $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)\overline{\otimes}B(\mathcal{F}_1, \mathcal{F}_2)$ and $\mathcal{M}(\mathcal{H}_1 \otimes \mathcal{F}_1, \mathcal{H}_2 \otimes \mathcal{F}_2)$.

Proof. By definition, the normal spatial tensor product of $M(\mathcal{H}_1, \mathcal{H}_2) \subset B(\mathcal{H}_1, \mathcal{H}_2)$ and $B(\mathcal{F}_1, \mathcal{F}_2)$ is realized as the weak-$*$ closure of the algebraic tensor product $M(\mathcal{H}_1, \mathcal{H}_2) \otimes B(\mathcal{F}_1, \mathcal{F}_2)$ as a subspace of $B(\mathcal{H}_1 \otimes \mathcal{F}_1, \mathcal{H}_2 \otimes \mathcal{F}_2)$.

Fix some elementary tensor $\phi \otimes T$ in $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2) \otimes B(\mathcal{F}_1, \mathcal{F}_2)$. Using Lemma 1.7.13, the assigned function

$$X \to B(\mathcal{E}_1 \otimes \mathcal{F}_1, \mathcal{E}_2 \otimes \mathcal{F}_2) \ , \ z \mapsto \phi(z) \otimes T$$

belongs to $\mathcal{M}(\mathcal{H}_1 \otimes \mathcal{F}_1, \mathcal{H}_2 \otimes \mathcal{F}_2)$ as the composition of multipliers

$$X \to B(\mathcal{E}_1 \otimes \mathcal{F}_2, \mathcal{E}_2 \otimes \mathcal{F}_2) \ , \ z \mapsto \phi(z) \otimes 1_{\mathcal{F}_2}$$

and

$$X \to B(\mathcal{E}_1 \otimes \mathcal{F}_1, \mathcal{E}_2 \otimes \mathcal{F}_2)\ ,\ z \mapsto 1_{\mathcal{E}_1} \otimes T.$$

Since the subspace $M(\mathcal{H}_1 \otimes \mathcal{F}_1, \mathcal{H}_2 \otimes \mathcal{F}_2)$ is weak-$*$ closed in $B(\mathcal{H}_1 \otimes \mathcal{E}_1, \mathcal{H}_2 \otimes \mathcal{E}_2)$ by Propositions 1.3.2 and 1.7.9, we have the inclusion

$$M(\mathcal{H}_1, \mathcal{H}_2) \overline{\otimes} B(\mathcal{F}_1, \mathcal{F}_2) \subset M(\mathcal{H}_1 \otimes \mathcal{F}_1, \mathcal{H}_2 \otimes \mathcal{F}_2).$$

Conversely fix some $\psi \in \mathcal{M}(\mathcal{H}_1 \otimes \mathcal{F}_1, \mathcal{H}_2 \otimes \mathcal{F}_2)$. For arbitrary finite rank projections $P_1 \in B(\mathcal{F}_1)$ and $P_2 \in B(\mathcal{F}_2)$, we consider the function

$$\psi_{P_1,P_2} : X \to B(\mathcal{E}_1 \otimes \mathcal{F}_1, \mathcal{E}_2 \otimes \mathcal{F}_2)\ ,\ z \mapsto (1_{\mathcal{E}_2} \otimes P_2)\psi(z)(1_{\mathcal{E}_1} \otimes P_1).$$

Again by Lemma 1.7.13, it follows that ψ_{P_1,P_2} belongs to $\mathcal{M}(\mathcal{H}_1 \otimes \mathcal{F}_1, \mathcal{H}_2 \otimes \mathcal{F}_2)$ and that $M_{\psi_{P_1,P_2}} = (1_{\mathcal{H}_2} \otimes P_2)M_\psi(1_{\mathcal{H}_1} \otimes P_1)$. Clearly, the net $(M_{\psi_{P_1,P_2}})_{P_1,P_2}$ approximates M_ψ in the weak-$*$ sense since it is bounded and converges WOT to M_ψ.

In order to show that the operators $M_{\psi_{P_1,P_2}}$ belong to the algebraic tensor product $M(\mathcal{H}_1, \mathcal{H}_2) \otimes B(\mathcal{F}_1, \mathcal{F}_2)$, we may assume that $P_1 = v \otimes v$ and $P_2 = u \otimes u$ are rank-one projections. Define $\psi_0 : X \to B(\mathcal{E}_1, \mathcal{E}_2)$ by

$$\langle \psi_0(z)y, x\rangle = \langle \psi(z)y \otimes v, x \otimes u\rangle.$$

Equivalently, writing

$$A : \mathbb{C} \to \mathcal{F}_1\ ,\ \zeta \mapsto \zeta v \quad \text{and} \quad B : \mathcal{F}_2 \to \mathbb{C}\ ,\ u' \mapsto \langle u', u\rangle,$$

we could realize ψ_0 as

$$\psi_0(z) = (1_{\mathcal{E}_2} \otimes B)\psi(z)(1_{\mathcal{E}_1} \otimes A).$$

By Lemma 1.7.13, we see that ψ_0 belongs to $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ with $\|\psi_0\|_{\mathcal{M}} \leq \|\psi\|_{\mathcal{M}}$. Letting $T = u \otimes v \in B(\mathcal{F}_1, \mathcal{F}_2)$, we obtain that

$$\begin{aligned}
\langle (\psi_0(z) \otimes T)(f(z) \otimes v'), x \otimes u'\rangle &= \langle \psi_0(z)f(z), x\rangle \langle Tv', u'\rangle \\
&= \langle \psi(z)(f(z) \otimes v), x \otimes u\rangle \langle v', v\rangle \langle u, u'\rangle \\
&= \langle \psi(z)(f(z) \otimes P_1 v'), x \otimes P_2 u'\rangle \\
&= \langle (1_{\mathcal{E}_2} \otimes P_2)\psi(z)(1_{\mathcal{E}_1} \otimes P_1)(f(z) \otimes v'), x \otimes u'\rangle
\end{aligned}$$

for all $f \in \mathcal{H}_1, z \in X$ and all $x \in \mathcal{E}_2, v' \in \mathcal{F}_1, u' \in \mathcal{F}_2$. Since $\mathcal{H}_1$ was supposed to be non-degenerate, it follows that

$$\psi_0(z) \otimes T = \psi_{P_1,P_2}(z)$$

holds for all $z \in X$, as desired. □

Finally, we take a closer look at the natural operator space structure of $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$. Let $\mathcal{E}_1, \mathcal{E}_2$ be Hilbert spaces and let $\mathcal{H}_1 \subset \mathcal{E}_1^X$, $\mathcal{H}_2 \subset \mathcal{E}_2^X$ be reproducing kernel Hilbert spaces. The natural n-th matrix norm of $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ is given by identifying an $n \times n$-matrix $[M_{\phi_{ij}}]$ with an operator in $B(\mathcal{H}_1^n, \mathcal{H}_2^n)$. It is not hard to see that $[M_{\phi_{ij}}]$ is unitarily equivalent to the multiplication operator $M_\phi \in B(\mathcal{H}_1^{(n)}, \mathcal{H}_2^{(n)})$, where $\mathcal{H}_1^{(n)}$ and $\mathcal{H}_2^{(n)}$ are the inflations of $\mathcal{H}_1, \mathcal{H}_2$ as defined in Section 1.2 and

$$\phi : X \to B(\mathcal{E}_1^n, \mathcal{E}_2^n) \ , \ \phi(z) = [\phi_{ij}(z)].$$

Using the canonical identification between $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ and $M(\mathcal{H}_1, \mathcal{H}_2)$, we obtain matrix (semi-)norms

$$\|[\phi_{ij}]\|_{\mathcal{M}} = \|\phi\|_{\mathcal{M}}$$

on $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ which are, of course, matrix norms whenever $\mathcal{H}_1$ is non-degenerate.

Of particular interest is the case that this natural operator space structure of the multiplier space coincides with the minimal operator space structure.

Example 1.7.15. *Consider a Cartan domain $D \subset \mathbb{C}^d$ of rank r. Then, for $\nu \geq \frac{d}{r}$ and $\mathcal{H} = \mathcal{H}_\nu$, we saw in Example* 1.7.3 *(*d*) that $\mathcal{M}(\mathcal{H}) = H^\infty(D)$ completely isometrically (where the operator space structure on $H^\infty(D)$ comes from the commutative C^*-algebra $L^\infty(D)$). By [32], Theorem 3.3.1, this means that $\mathcal{M}(\mathcal{H})$ is a minimal operator space.*

2 Schur kernels

2.1 Definition and basic properties

We turn towards one of the central notions of this paper, namely, the concept of Schur kernels. Roughly speaking, Schur kernels are the pointwise multipliers of the kernel class $B(K)$ (where K denotes a positive scalar kernel).

Definition 2.1.1. *Suppose that $K : X \times X \to \mathbb{C}$ is a positive kernel and that $\mathcal{E}$ is a Hilbert space.*

(a) We say that a kernel $G : X \times X \to B(\mathcal{E})$ is a Schur kernel (with respect to K) if the pointwise product

$$G \cdot L : X \times X \to B(\mathcal{E}) \ , \ (z, w) \mapsto G(z, w)L(z, w)$$

belongs to $B(K_{\mathcal{E}})$ for all $L \in B(K)$. In this case, S_G denotes the linear mapping

$$S_G : B(K) \to B(K_{\mathcal{E}}) \ , \ L \mapsto G \cdot L.$$

(b) The class of all $B(\mathcal{E})$-valued Schur kernels (with respect to K) will be denoted by $\mathscr{S}_{\mathcal{E}}(K)$. Instead of $\mathscr{S}_{\mathbb{C}}(K)$, we shall use the abbreviation $\mathscr{S}(K)$.

(c) We write

$$S_{\mathcal{E}}(K) = \{S_G \ ; \ G \in \mathscr{S}_{\mathcal{E}}(K)\}$$

for the set of all multiplication operators induced by Schur kernels. For simplicity, we write $S(K)$ instead of $S_{\mathbb{C}}(K)$.

Remark 2.1.2. *Suppose that $K : X \times X \to \mathbb{C}$ is a positive kernel with associated reproducing kernel Hilbert space $\mathcal{H} \subset \mathbb{C}^X$ and that $\mathcal{E}$ is Hilbert space.*

(a) For $G \in \mathscr{S}_{\mathcal{E}}(K)$, the linear mapping

$$S_G : B(K) \to B(K_{\mathcal{E}}) \ , \ L \mapsto G \cdot L$$

is continuous. This follows by the closed graph theorem and the fact that the point evaluations are continuous on $B(K)$ and $B(K_{\mathcal{E}})$ (cf. Remark 1.6.2 (c)).

(b) *The mapping*

$$\mathscr{S}_{\mathcal{E}}(K) \to S_{\mathcal{E}}(K) \subset B(B(K), B(K_{\mathcal{E}}))\ ,\ G \mapsto S_G$$

may certainly fail to be injective. When we suppose in addition that $\mathcal{H}$ is non-degenerate, then the above mapping is one-to-one. Indeed, if $\mathcal{H}$ is non-degenerate, then for all $z \in X$, we can find functions $f_z \in \mathcal{H}$ such that $f_z(z) = 1$. Evaluating S_G on the kernels $f_z \odot f_w \in B(K)$ shows that $G = 0$ whenever $S_G = 0$.

(c) *It is clear that $\mathscr{S}_{\mathcal{E}}(K)$ is a complex linear space. Moreover, the obvious identity $G^* \cdot L = (G \cdot L^*)^*$ shows that $\mathscr{S}_{\mathcal{E}}(K)$ is closed under the natural involution of kernels. Hence, every kernel $G \in \mathscr{S}_{\mathcal{E}}(K)$ can be written as $G = \operatorname{Re} G + i \operatorname{Im} G$, where $\operatorname{Re} G$ and $\operatorname{Im} G$ belong to $\mathscr{S}_{\mathcal{E}}(K)$.*

It was a successful procedure in the previous sections to equip function or kernel spaces, such as the multiplier algebra $\mathcal{M}(\mathcal{H})$ or the space of subordinate kernels $B(K)$, with the weak-$*$ or weak operator topologies of $B(\mathcal{H})$. We want to repeat this step for the set of Schur kernels $\mathscr{S}_{\mathcal{E}}(K)$. We shall see soon that $\mathscr{S}_{\mathcal{E}}(K)$ is canonically contained in $B(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}}))$. This allows us to transport the natural weak-$*$ topology of $B(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}}))$, the so-called BW topology, to $\mathscr{S}_{\mathcal{E}}(K)$.

For the reader's convenience, we now recapitulate the definition of the BW topology (see also [55], Chapter 7). Let X, Y be Banach spaces. Clearly, there exists a linear mapping $j : X \otimes Y \to B(X, Y^*)^*$, defined on the algebraic tensor product of X and Y, such that

$$j(x \otimes y)(T) = (Tx)(y) \quad (x \in X, y \in Y, T \in B(X, Y^*)).$$

It turns out that j is one-to-one and satisfies $\|j(x \otimes y)\| = \|x\|\|y\|$ for all $x \in X$ and $y \in Y$. One defines

$$X \tilde{\otimes} Y = \overline{j(X \otimes Y)} \subset B(X, Y^*)^*$$

and proves that the mapping

$$\rho : B(X, Y^*) \to (X \tilde{\otimes} Y)^*\ ,\ \rho(T)u = u(T)$$

is an isometric isomorphism. The weak-$*$ topology on $B(X, Y^*)$ with respect to this duality is called the BW topology. We are mainly interested in the case that $X = B(H_1)$ and $Y = T(H_2)$ (the space of trace class operators), where H_1, H_2 are Hilbert spaces. Then the BW topology on $B(B(H_1), B(H_2))$ has the property that a bounded net $(\Phi_i)_i$ in $B(B(H_1), B(H_2))$ converges to some Φ in $B(B(H_1), B(H_2))$ if and only if for every $T \in B(H_1)$, the net $(\Phi_i(T))_i$ converges to $\Phi(T)$ in the weak operator topology of $B(H_2)$ (cf. [55], Proposition 7.3). This behaviour explains the name 'BW topology', standing for 'bounded weak'.

Definition 2.1.3. *Let $K : X \times X \to \mathbb{C}$ be a positive kernel with associated reproducing kernel Hilbert space $\mathcal{H} \subset \mathbb{C}^X$ and let $\mathcal{E}$ be a Hilbert space.*

(a) For $G \in \mathscr{S}_{\mathcal{E}}(K)$, we define $\|G\|_{\mathscr{S}} = \|S_G\|$.

(b) Let G be a kernel in $\mathscr{S}_{\mathcal{E}}(K)$. Then $\Sigma_G \in B(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}}))$ denotes the unique mapping that makes the diagram

$$\begin{tikzcd} B(K) \arrow[r, "\simeq"] \arrow[d, "S_G"'] & B(\mathcal{H}) \arrow[d, "\Sigma_G"] \\ B(K_{\mathcal{E}}) \arrow[r, "\simeq"] & B(\mathcal{H}_{\mathcal{E}}) \end{tikzcd}$$

commutative (where $\simeq$ is the canonical identification explained in Theorem 1.6.3*). Furthermore, we define*

$$\Sigma_{\mathcal{E}}(K) = \{\Sigma_G \ ; \ G \in \mathscr{S}_{\mathcal{E}}(K)\} \subset B(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}}))$$

and, as usual, we use the abbreviation $\Sigma(K) = \Sigma_{\mathbb{C}}(K)$.

(c) The BW topology on $\mathscr{S}_{\mathcal{E}}(K)$ is defined as the initial topology of the BW topology on $B(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}}))$ with respect to the mapping

$$\mathscr{S}_{\mathcal{E}}(K) \to B(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}})) \ , \ G \mapsto \Sigma_G.$$

Clearly we encounter the usual difficulties if the underlying space $\mathcal{H}$ is degenerate. In this case, $\|\cdot\|_{\mathscr{S}}$ might only be a semi-norm, and the BW topology on $\mathscr{S}_{\mathcal{E}}(K)$ need not be Hausdorff. Since the spaces we are interested in are non-degenerate, and in order to keep the proofs simple, we require non-degenerateness whenever it is convenient.

Proposition 2.1.4. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a non-degenerate reproducing kernel space with reproducing kernel K and that $\mathcal{E}$ is a Hilbert space. Then $\Sigma_{\mathcal{E}}(K)$ is a BW closed subspace of $B(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}}))$. In particular, $\mathscr{S}_{\mathcal{E}}(K)$ is a dual space.*

Proof. Recall that the BW topology is a weak-$*$ topology. By the Kreĭn -Smulian theorem, it is sufficient to show that $\operatorname{ball}\Sigma_{\mathcal{E}}(K)$ is BW closed in $B(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}}))$. Let $(G_\alpha)_\alpha$ be a net in $\operatorname{ball}\mathscr{S}_{\mathcal{E}}(K)$ such that the net $(\Sigma_{G_\alpha})_\alpha$ converges BW to some $Y \in \operatorname{ball} B(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}}))$. By the preceding discussion of the BW topology, this means that

$$\lim_\alpha \Sigma_{G_\alpha}(T) = Y(T)$$

in the WOT sense for all $T \in B(\mathcal{H})$. For all $z \in X$, we fix functions $f_z \in \mathcal{H}$ such that $f_z(z) = 1$. For $z, w \in X$, we then define $G(z,w)$ as the unique operator in $B(\mathcal{E})$ satisfying

$$\langle G(z,w)y, x\rangle = \langle Y(f_z \otimes f_w)K(\cdot, w)y, K(\cdot, z)x\rangle$$

for all $x, y \in \mathcal{E}$. We obtain that

$$\langle G(z,w)y,x\rangle = \lim_\alpha \langle \Sigma_{G_\alpha}(f_z \otimes f_w)K(\cdot,w)y, K(\cdot,z)x\rangle = \lim_\alpha \langle G_\alpha(z,w)y,x\rangle$$

for all $z, w \in X$ and $x, y \in \mathcal{E}$. For $L = \Lambda_T \in B(K)$, we have

$$\begin{aligned} \langle G(z,w)L(z,w)y,x\rangle &= \lim_\alpha \langle G_\alpha(z,w)L(z,w)y,x\rangle \\ &= \lim_\alpha \langle \Sigma_{G_\alpha}(T)K(\cdot,w)y, K(\cdot,z)x\rangle \\ &= \langle Y(T)K(\cdot,w)y, K(\cdot,z)x\rangle \\ &= \langle \Lambda_{Y(T)}(z,w)y,x\rangle \quad (z,w \in X, x,y \in \mathcal{E}). \end{aligned}$$

Hence $G \cdot L = \Lambda_{Y(T)} \in B(K_\mathcal{E})$. This means that $G \in \mathscr{S}_\mathcal{E}(K)$ and hence that $Y = \Sigma_G$. □

For bounded nets, convergence with respect to the BW topology can be described as follows.

Proposition 2.1.5. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a non-degenerate reproducing kernel space with reproducing kernel K and that $\mathcal{E}$ is a Hilbert space. Then for a bounded net $(G_\alpha)_\alpha$ in $\mathscr{S}_\mathcal{E}(K)$ and $G \in \mathscr{S}_\mathcal{E}(K)$, the following are equivalent:*

(i) $(G_\alpha)_\alpha$ converges BW to G.

(ii) $(G_\alpha(z,w))_\alpha$ converges WOT (equivalently, weak-$*$) to $G(z,w)$ for all $z, w \in X$.

Proof. According to our initial discussion of the BW topology, (i) is fulfilled if and only if for all $L \in B(K)$, the bounded net $(G_\alpha \cdot L)_\alpha$ converges WOT in $B(K_\mathcal{E})$ to $G \cdot L$. By Proposition 1.6.11, this is the case if and only if for all $L \in B(K)$, the net $(G_\alpha(z,w)L(z,w))_\alpha$ converges WOT (equvalently, weak-$*$) in $B(\mathcal{E})$ to $G(z,w)L(z,w)$ for all $z, w \in X$. Since $\mathcal{H}$ is non-degenerate, this is clearly equivalent to (ii). □

Related to the preceding characterization of BW convergence is the following approximation result.

Proposition 2.1.6. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a reproducing kernel space with kernel K and that $\mathcal{E}$ is a Hilbert space. Then, for a kernel $G : X \times X \to B(\mathcal{E})$, the following are equivalent:*

(i) G belongs to $\mathscr{S}_\mathcal{E}(K)$.

(ii) There exists a bounded sequence $(G_n)_n$ in $\mathscr{S}_\mathcal{E}(K)$ such that $(G_n(z,w))_n$ converges WOT to $G(z,w)$ for all $z, w \in X$.

(iii) There exists a bounded net $(G_\alpha)_\alpha$ in $\mathscr{S}_\mathcal{E}(K)$ such that $(G_\alpha(z,w))_\alpha$ converges WOT to $G(z,w)$ for all $z, w \in X$.

In this case,

$$\|G\|_{\mathscr{S}} \leq \liminf_n \|G_n\|_{\mathscr{S}} \quad \text{and} \quad \|G\|_{\mathscr{S}} \leq \liminf_\alpha \|G_\alpha\|_{\mathscr{S}}$$

holds for all sequences and nets as in (ii) and (iii), respectively.

Proof. We have to prove (iii) to (i). Let $d > \liminf_\alpha \|G\|_{\mathscr{S}}$ be arbitrary. By passing to a suitable subnet, we may assume that $\|G_\alpha\|_{\mathscr{S}} < d$ holds for all α. Let $\Phi \in B(B(\mathcal{H}), B(\mathcal{H}_\mathcal{E}))$ be a BW cluster point of the net $(\Sigma_{G_\alpha})_\alpha$. Clearly, $\|\Phi\| \leq d$. By choosing an appropriate subnet, we deduce that

$$\begin{aligned}
\langle \Phi(T)K(\cdot,w)y, K(\cdot,z)x\rangle &= \lim_i \langle \Sigma_{G_{\alpha_i}}(T)K(\cdot,w)y, K(\cdot,z)x\rangle \\
&= \lim_i \langle G_{\alpha_i}(z,w)y,x\rangle \Lambda_T(z,w) \\
&= \langle G(z,w)y,x\rangle \Lambda_T(z,w) \\
&= \langle G(z,w)\Lambda_T(z,w)y,x\rangle
\end{aligned}$$

holds for all $T \in B(\mathcal{H})$, $z,w \in X$ and $x,y \in \mathcal{E}$. This shows that $G \in \mathscr{S}_\mathcal{E}(K)$ and $\Sigma_G = \Phi$, and hence that $\|G\|_{\mathscr{S}} = \|\Phi\| \leq d$. □

We now introduce a class of prototypical Schur kernels which will turn out to be central in the following examinations of the class $\mathscr{S}_\mathcal{E}(K)$.

Lemma 2.1.7. *Let $K : X \times X \to \mathbb{C}$ be a positive kernel with associated reproducing kernel space $\mathcal{H} \subset \mathbb{C}^X$ and let $\mathcal{E}, \mathcal{G}$ be Hilbert spaces. Then, for any pair of multipliers $\phi, \psi \in \mathcal{M}(\mathcal{H}_\mathcal{G}, \mathcal{H}_\mathcal{E})$, the kernel*

$$G : X \times X \to B(\mathcal{E}) \ , \ G(z,w) = \phi(z)\psi(w)^*$$

belongs to $\mathscr{S}_\mathcal{E}(K)$. Moreover,

$$\Sigma_G(T) = M_\phi(T \otimes 1_\mathcal{G})M_\psi^*$$

holds for all $T \in B(\mathcal{H})$. In particular, Σ_G is completely bounded and

$$\|G\|_{\mathscr{S}} = \|\Sigma_G\| \leq \|\Sigma_G\|_{cb} \leq \|\phi\|_\mathcal{M} \|\psi\|_\mathcal{M}$$

holds.

Proof. Let $L = \Lambda_T$ be a kernel in $B(K)$. The calculation

$$\begin{aligned}
\langle \Lambda_{M_\phi(T\otimes 1_\mathcal{G})M_\psi^*}(z,w)y,x\rangle &= \langle M_\phi(T\otimes 1_\mathcal{G})M_\psi^* K(\cdot,w)y, K(\cdot,z)x\rangle \\
&= \langle (T\otimes 1_\mathcal{G})K(\cdot,w)\otimes\psi(w)^*y, K(\cdot,z)\otimes\phi(z)^*x\rangle \\
&= \langle TK(\cdot,w), K(\cdot,z)\rangle\langle\phi(z)\psi(w)^*y,x\rangle \\
&= \langle G(z,w)L(z,w)y,x\rangle \quad (z,w \in X, x,y \in \mathcal{E})
\end{aligned}$$

shows that the kernel $G \cdot L$ belongs to $B(K_\mathcal{E})$ and, in fact, is represented by the operator $M_\phi(T \otimes 1_\mathcal{G})M_\psi^*$. □

Example 2.1.8. *We consider the set $I = \{1, \dots, n\}$ and the positive kernel*

$$K : I \times I \to \mathbb{C} \ , \ K(i,j) = \delta_{i,j}.$$

Then clearly $l^2(I) = l^2_n$ is the reproducing kernel Hilbert space associated with K (cf. Example 1.1.3*). For the moment, let us identify kernels on $I \times I$ with $n \times n$-matrices, that is, we identify an kernel $L : I \times I \to \mathbb{C}$ with the matrix $[L(i,j)]$. Under this completely isometric identification, the pointwise product $G \cdot L$ of two kernels G and L is nothing but the entrywise (Schur, Hadamard) matrix product $G \bullet L$. Obviously, every kernel G belongs to $\mathscr{S}(K)$, and the mapping $\Sigma_G : B(l^2) \to B(l^2)$ coincides completely isometrically with the Schur product mapping*

$$\mathfrak{S}_G : M_n \to M_n \ , \ L \mapsto G \bullet L.$$

The study of Schur products goes back to Schur and Hadamard, and the use of operator space methods in the past twenty years led to a remarkable progress in this area (cf. [55], [68], [57]).

The key step in the understanding of Schur product mappings is the following observation:

> *For a linear mapping $\Phi : M_n \to M_n$, the following are equivalent:*
>
> *(i) Φ is a Schur product mapping, that is, there exists some matrix $G \in M_n$ such that $\Phi = \mathfrak{S}_G$.*
>
> *(ii) Φ is a D_n-bimodule homomorphism, that is, for all diagonal matrices D, E and all matrices L, the identity*
>
> $$\Phi(DLE) = D\Phi(L)E$$
>
> *holds. Since $D_n = M(l^2_n)$ via obvious identification, we could also say that Φ is an $M(l^2_n)$-bimodule homomorphism.*

This has at least the following two consequences:

(1) Every Schur product mapping $\mathfrak{S}_G$ satisfies $\|\mathfrak{S}_G\| = \|\mathfrak{S}_G\|_{cb}$. In fact, this is shown in greater generality in [55], Chapter 8, and in [68] or [58]. The original proof of this non-trivial fact is probably due to Haagerup and can be found in [57].

(2) Every Schur product mapping is an elementary operator of the form

$$\mathfrak{S}_G(L) = \sum_{i=1}^{k} D_i L E_i^*$$

with suitable diagonal matrices D_i, E_i $(1 \leq i \leq k)$. The use of E_i^* instead of E_i has structural reasons that will become clear in the sequel. Moreover, the matrices D_i, E_i can be chosen in such a way that

$$\|\mathfrak{S}_G\|_{cb}^2 = \|\sum_{i=1}^{k} D_i D_i^*\| \|\sum_{i=1}^{k} E_i E_i^*\|$$

holds (note that every choice of D_i, E_i satisfies of course $\leq$). This follows by suitable versions of the Stinespring representation theorem or Wittstock's decomposition theorem (see for example [55], Exercise 8.6).

This example suggests the following program to study the class $\Sigma(K)$.

(1) Characterize operators of the form $\Sigma_G \in B(B(\mathcal{H}))$ by suitable module properties. We shall see that, under some mild assumptions, $\Sigma(K)$ coincides with the class of all normal (that is, weak-$*$ continuous) $(M(\mathcal{H}), M(\mathcal{H})^*)$-module homomorphisms in $B(B(\mathcal{H}))$. Thus we generalize classical results on Schur multipliers to the setting of arbitrary Schur kernels.

(2) Try to decide whether Schur kernels satisfy the equality $\|\Sigma_G\| = \|\Sigma_G\|_{cb}$. Unfortunately, this problem seems to be rather deep. The techniques which are used in the case of classical Schur multipliers heavily rely on the fact that $M(\mathcal{H})$ is a C^*-subalgebra of $B(\mathcal{H})$. In Example 1.7.3 (a), we indicated that, in our more general setting, this will almost never happen.

(3) Find representations of completely bounded Schur multiplications.

Before we approach the first of the above points, we need a precise definition of module homomorphisms.

Definition 2.1.9. *Suppose that $H, \mathcal{E}_1, \mathcal{E}_2$ are Hilbert spaces and that $\mathcal{A}, \mathcal{B} \subset B(H)$ are operator algebras. An operator $\Phi \in B(B(H \otimes \mathcal{E}_1), B(H \otimes \mathcal{E}_2))$ is called an $(\mathcal{A}, \mathcal{B})$-module homomorphism if*

$$\Phi((A \otimes 1_{\mathcal{E}_1})T(B \otimes 1_{\mathcal{E}_1})) = (A \otimes 1_{\mathcal{E}_2})\Phi(T)(B \otimes 1_{\mathcal{E}_2})$$

holds for all $T \in B(H \otimes \mathcal{E}_1)$ and $A \in \mathcal{A}, B \in \mathcal{B}$.

Proposition 2.1.10. *Let $K : X \times X \to \mathbb{C}$ be a positive kernel with associated reproducing kernel space $\mathcal{H} \subset \mathbb{C}^X$ and let $\mathcal{E}$ be a Hilbert space. For all $G \in \Sigma_{\mathcal{E}}(K)$, the mapping Σ_G is is a normal $(M(\mathcal{H}), M(\mathcal{H})^*)$-module homomorphism. If, in addition, $\mathcal{M}(\mathcal{H})$ is dense in $\mathcal{H}$, then every normal $(M(\mathcal{H}), M(\mathcal{H})^*)$-module homomorphism belongs to $\Sigma_{\mathcal{E}}(K)$.*

In this context, $M(\mathcal{H})^*$ denotes the subspace of $B(\mathcal{H})$ consisting of all adjoints of multiplication operators, that is,

$$M(\mathcal{H})^* = \{M_\phi^* \; ; \; \phi \in \mathcal{M}(\mathcal{H})\}.$$

Proof. For $G \in \mathscr{S}_{\mathcal{E}}(K)$, the calculation

$$\begin{aligned}
&\langle \Sigma_G(M_\alpha T M_\beta^*)K(\cdot,w)y, K(\cdot,z)x\rangle \\
&\quad = \langle M_\alpha T M_\beta^* K(\cdot,w), K(\cdot,z)\rangle \langle G(z,w)y, x\rangle \\
&\quad = \alpha(z)\overline{\beta(w)}\langle TK(\cdot,w), K(\cdot,z)\rangle\langle G(z,w)y,x\rangle \\
&\quad = \alpha(z)\overline{\beta(w)}\langle \Sigma_G(T)K(\cdot,w)y, K(\cdot,z)x\rangle \\
&\quad = \langle (M_\alpha \otimes 1_{\mathcal{E}})\Sigma_G(T)(M_\beta^* \otimes 1_{\mathcal{E}})K(\cdot,w)y, K(\cdot,z)x\rangle,
\end{aligned}$$

valid for all $T \in B(\mathcal{H})$, $\alpha,\beta \in \mathcal{M}(\mathcal{H})$, $z,w \in X$ and $x,y \in \mathcal{E}$, shows that Σ_G an $(M(\mathcal{H}), M(\mathcal{H})^*)$-module homomorphism. Next we prove that Σ_G is normal. To this end, let $(T_\alpha)_\alpha$ be a net in the unit ball of $B(\mathcal{H})$ which is weak-$*$ convergent to some T in the unit ball of $B(\mathcal{H})$. An application of Proposition 1.6.11 shows that the net $\left(S_G(\Lambda_{T_\alpha})\right)_\alpha$ is weak-$*$ convergent towards $S_G(\Lambda_T)$, or equivalently, that the net $\left(\Sigma_G(T_\alpha)\right)_\alpha$ is weak-$*$-convergent towards $\Sigma_G(T)$. Thus, the restriction of Σ_G to the unit ball of $B(\mathcal{H})$ is weak-$*$ continuous. Now a well-known corollary of the Kreĭn -Smulian theorem implies that Σ_G is weak-$*$ continuous.

For second assertion, let $Q_0 = \mathbf{1}\otimes\mathbf{1} \in B(\mathcal{H})$ denote the unique operator satisfying $Q_0K(\cdot,w) = \mathbf{1}$ for all $w \in X$ (note that $\mathcal{H}$ contains the constant functions by the hypothesis that $\mathcal{M}(\mathcal{H}) \subset \mathcal{H}$).

We define $G = \Lambda_{\Phi(Q_0)} \in B(K_{\mathcal{E}})$ and claim that

$$G \cdot \Lambda_T = \Lambda_{\Phi(T)}$$

holds for all finite-rank operators $T \in B(\mathcal{H})$. Clearly, since $\mathcal{M}(\mathcal{H})$ was assumed to be dense in $\mathcal{H}$, it suffices to show this for rank-one operators of the form $T = \alpha\otimes\beta$ with $\alpha,\beta \in \mathcal{M}(\mathcal{H})$. The trivial identity

$$M_\alpha Q_0 M_\beta^* = \alpha\otimes\beta$$

leads to

$$\begin{aligned}
\langle \Lambda_{\Phi(T)}(z,w)y,x\rangle &= \langle \Phi(M_\alpha Q_0 M_\beta^*)K(\cdot,w)y, K(\cdot,z)x\rangle \\
&= \langle (M_\alpha\otimes 1_{\mathcal{E}})\Phi(Q_0)(M_\beta^*\otimes 1_{\mathcal{E}})K(\cdot,w)y, K(\cdot,z)x\rangle \\
&= \alpha(z)\overline{\beta(w)}\langle \Phi(Q_0)K(\cdot,w)y, K(\cdot,z)x\rangle \\
&= \Lambda_T(z,w)\langle G(z,w)y,x\rangle \quad (z,w\in X, x,y\in\mathcal{E}),
\end{aligned}$$

which proves this first assertion. Next, we want to show that G belongs to $\mathscr{S}_{\mathcal{E}}(K)$. To this end, fix some kernel $L = \Lambda_T \in B(K)$. Clearly, we can approximate T in the WOT sense by a net of finite-rank operators $(T_i)_i$ such that $\|T_i\| \le \|T\|$ holds for all i. Writing $L_i = \Lambda_{T_i}$ for all i, we observe that $G\cdot L_i = \Lambda_{\Phi(T_i)} \in B(K_{\mathcal{E}})$ and that $\|G\cdot L_i\|_{K_{\mathcal{E}}} \le \|\Phi\|\|T\|$. Hence $(G\cdot L_i)_i$ is a bounded net in $\mathscr{S}_{\mathcal{E}}(K)$ which converges pointwise WOT to $G\cdot L$ by Proposition 1.6.11. By Lemma 1.6.12, we have $G\cdot L \in B(K_{\mathcal{E}})$. This means that $G \in \mathscr{S}_{\mathcal{E}}(K)$. Finally, since the normal mappings Φ

and Σ_G coincide on the weak-$*$ dense subspace of finite-rank operators, they must be equal. □

The following example shows we cannot drop the hypothesis that $\mathcal{M}(\mathcal{H})$ is dense in $\mathcal{H}$.

Example 2.1.11. *Consider the Segal-Bargmann-Fock space $\mathcal{H} = L^2_a(\mathbb{C}, \mu)$, which is by definition the Bergman space on $\mathbb{C}$ with respect to the normalized Gaussian measure $d\mu = \frac{1}{\pi} e^{-|z|^2} d\lambda$, λ denoting the ordinary Lebesgue measure on $\mathbb{C}$ (see also Example* 1.1.10*). This is a non-degenerate reproducing kernel space with kernel*

$$K : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \ , \ K(z, w) = e^{z\overline{w}}.$$

By Liouville's theorem, it has no non-constant multipliers (since multipliers are bounded entire functions). Thus every $\Phi \in B(B(\mathcal{H}))$ has the $(M(\mathcal{H}), M(\mathcal{H})^*)$-module property. The following proposition however, implies that for the Segal-Bargmann kernel, the space $\Sigma(K)$ consists only of multiples of the identity operator.

The space $\mathscr{S}(K)$ enjoys a kind of slice property, which establishes a first connection between Schur kernels and multipliers.

Proposition 2.1.12. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a non-degenerate reproducing kernel space with reproducing kernel K. Then for all kernels $G \in \mathscr{S}(K)$, the slice functions*

$$G_w : X \to \mathbb{C} \ , \ G_w(z) = G(z, w) \quad (w \in X)$$

belong to $\mathcal{M}(\mathcal{H})$ with $\|G_w\|_{\mathcal{M}} \le \|G\|_{\mathscr{S}}$.

Proof. Fix some $w \in X$ and consider the mappings

$$i : \mathcal{H} \to B(K) \ , \ f \mapsto f \odot K(\cdot, w)$$

and

$$\pi : B(K) \to \mathcal{H} \ , \ L \mapsto L(\cdot, w).$$

Then both i and π have norm at most $K(w, w)^{\frac{1}{2}}$ by Remark 1.6.5 and Corollary 1.6.4, respectively. Under the composition

$$\mathcal{H} \xrightarrow{i} B(K) \xrightarrow{S_G} B(K) \xrightarrow{\pi} \mathcal{H},$$

an element f of $\mathcal{H}$ is mapped to $K(w, w)\ G_w \cdot f$. Since $\mathcal{H}$ is supposed to be non-degenerate, we have $K(w, w) > 0$ and hence

$$G_w \cdot f = \frac{1}{K(w, w)} \pi S_G i f \in \mathcal{H}$$

for all $f \in \mathcal{H}$. Therefore $G_w \in \mathcal{M}(\mathcal{H})$ and $\|G_w\|_{\mathcal{M}} \le \|G\|_{\mathscr{S}}$. □

Corollary 2.1.13. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a non-degenerate reproducing kernel space with reproducing kernel K. Then every kernel $G \in \mathscr{S}(K)$ is bounded with $\|G\|_\infty \leq \|G\|_{\mathscr{S}}$.*

Proof. By Corollary 1.7.7, we have

$$\|G_w\|_\infty \leq \|G_w\|_{\mathcal{M}} \quad (w \in X).$$

Consequently,

$$\|G\|_\infty = \sup_{w \in X} \|G_w\|_\infty \leq \sup_{w \in X} \|G_w\|_{\mathcal{M}} \leq \|G\|_{\mathscr{S}}.$$

$\square$

Example 2.1.14. *We continue the discussion of the Segal-Bargman-Fock space started in Example* 2.1.11. *Suppose that G is a hermitian kernel in $\mathscr{S}(K)$. By Proposition* 2.1.12*, the functions $G(\cdot, w)$ are multipliers of $\mathcal{H}$ and therefore constant. For $w \in \mathbb{C}$, let $g(w)$ denote the constant value of the function $G(\cdot, w)$. We obtain*

$$g(w) = G(z, w) = G^*(z, w) = \overline{G(w, z)} = \overline{g(z)}$$

for all $z, w \in \mathbb{C}$. This shows that g and therefore G is constant. Since $\mathscr{S}(K)$ is the span of its hermitian kernels, every kernel in $\mathscr{S}(K)$ must be constant.

It may appear annoying that we did not formulate Proposition 2.1.12 in the expected full generality for $B(\mathcal{E})$-valued Schur kernels. We will partially fill this gap later by restricting our attention to completely bounded Schur multiplications.

2.2 Positive Schur kernels

This section is devoted to the study of the positive Schur kernels.

Proposition 2.2.1. *Suppose that $K : X \times X \to \mathbb{C}$ is a positive kernel with associated reproducing kernel space $\mathcal{H} \subset \mathbb{C}^X$ and that $\mathcal{E}$ is a Hilbert space. For a positive kernel $G : X \times X \to B(\mathcal{E})$, the following are equivalent:*

(i) $G \in \mathscr{S}_{\mathcal{E}}(K)$.

(ii) $G \cdot K \in B(K_{\mathcal{E}})$.

(iii) There exist a Hilbert space $\mathcal{G}$ and a multiplier $\phi \in \mathcal{M}(\mathcal{H}_{\mathcal{G}}, \mathcal{H}_{\mathcal{E}})$ such that $G(z, w) = \phi(z)\phi(w)^*$ holds for all $z, w \in X$.

In this case, every Kolmogorov factorization $(\mathcal{G}, \phi)$ of G defines a multiplier ϕ in $\mathcal{M}(\mathcal{H}_{\mathcal{G}}, \mathcal{H}_{\mathcal{E}})$ such that

$$\Sigma_G(T) = M_\phi(T \otimes 1_{\mathcal{G}})M_\phi^*$$

for all $T \in B(\mathcal{H})$. In particular, Σ_G is completely positive and

$$\|G \cdot K\|_{K_{\mathcal{E}}} = \|G\|_{\mathscr{S}} = \|\Sigma_G\| = \|\Sigma_G\|_{cb} = \|\phi\|^2_{\mathcal{M}}$$

holds for all such ϕ.

Proof. The implication (i) to (ii) is clear by the definition of $\mathscr{S}_{\mathcal{E}}(K)$. Suppose that (ii) holds and choose a Kolmogorov factorization (not necessarily minimal) of $(\mathcal{G}, \phi)$ of G. Then, by Proposition 1.7.6 and Remark 1.6.2(f), it follows that $\phi \in \mathcal{M}(\mathcal{H}_{\mathcal{G}}, \mathcal{H}_{\mathcal{E}})$ and that

$$\|\phi\|^2_{\mathcal{M}} \leq \|G \cdot K\|_{K_{\mathcal{E}}} \leq \|G\|_{\mathscr{S}} \tag{2.2.1}$$

holds. Finally, Lemma 2.1.7 shows that (iii) implies (i) and that

$$\|G\|_{\mathscr{S}} = \|\Sigma_G\| \leq \|\Sigma_G\|_{cb} \leq \|\phi\|^2_{\mathcal{M}}. \tag{2.2.2}$$

Combining (2.2.1) and (2.2.2) finishes the proof. □

Note that the equality $\|\Sigma_G\| = \|\Sigma_G\|_{cb}$ reflects the fact that

$$\|\Phi\|_{cb} = \|\Phi\| = \|\Phi(1_{H_1})\|$$

holds for every completely positive map $\Phi : B(H_1) \to B(H_2)$ (where H_1 and H_2 are Hilbert spaces).

2.3 Completely bounded Schur multiplications

The aim of this section is to find representation theorems for Schur kernels. We shall use operator space theoretic methods to describe explicitly the structure of completely bounded normal module homomorphisms (see also [55], [48] for earlier results of this type). Proposition 2.1.10 allows us to use these results to study completely bounded Schur multiplications. Since we are not able to show that the operators in $\Sigma_{\mathcal{E}}(K)$ are automatically completely bounded (as in the case of classical Schur multipliers), we shall restrict ourselves to the class of completely bounded Schur multiplications, in the following denoted by the symbol $\mathscr{S}^{(0)}_{\mathcal{E}}(K)$.

Definition 2.3.1. *Suppose that $K : X \times X \to \mathbb{C}$ is a positive kernel with associated reproducing kernel space $\mathcal{H} \subset \mathbb{C}^X$ and that $\mathcal{E}$ is a Hilbert space. We define*

$$\mathscr{S}^{(0)}_{\mathcal{E}}(K) = \{G \in \mathscr{S}_{\mathcal{E}}(K) \ ; \ \Sigma_G \text{ is completely bounded } \}$$

and set $\|G\|_{\mathscr{S}^{(0)}} = \|\Sigma_G\|_{cb}$ for $G \in \mathscr{S}^{(0)}_{\mathcal{E}}(K)$. Furthermore, we write

$$\Sigma^{(0)}_{\mathcal{E}}(K) = \{\Sigma_G \ ; \ G \in \mathscr{S}^{(0)}_{\mathcal{E}}(K)\},$$

which is clearly a subspace of $CB(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}}))$. As usual, the abbreviations $\mathscr{S}^{(0)}(K) = \mathscr{S}^{(0)}_{\mathbb{C}}(K)$ and $\Sigma^{(0)}(K) = \Sigma^{(0)}_{\mathbb{C}}(K)$ will be used.

Unless otherwise stated, the space $\mathscr{S}^{(0)}_{\mathcal{E}}(K)$ will be equipped with the relative operator space structure of $CB(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}}))$. We shall see later that under additional hypotheses on the underlying space $\mathcal{H}$, $\mathscr{S}^{(0)}_{\mathcal{E}}(K)$ is complete and even a dual space. However, it is not clear if the space $\mathscr{S}^{(0)}_{\mathcal{E}}(K)$ is closed or even BW closed in $\mathscr{S}_{\mathcal{E}}(K)$. In fact, if it were, then in many concrete cases we could prove that $\mathscr{S}^{(0)}_{\mathcal{E}}(K) = \mathscr{S}_{\mathcal{E}}(K)$. In the positive direction, we can at least show the following approximation result (cf. Proposition 2.1.6).

Proposition 2.3.2. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a reproducing kernel space with reproducing kernel $K : X \times X \to \mathbb{C}$ and that $\mathcal{E}$ is a Hilbert space. Then for a given kernel $G : X \times X \to B(\mathcal{E})$, the following are equivalent:*

(i) $G \in \mathscr{S}^{(0)}_{\mathcal{E}}(K)$.

(ii) There exists a bounded sequence $(G_n)_n$ in $\mathscr{S}^{(0)}_{\mathcal{E}}(K)$ such that $(G_n(z,w))_n$ converges WOT to $G(z,w)$ for all $z, x \in X$.

(iii) There exists a bounded net $(G_\alpha)_\alpha$ in $\mathscr{S}^{(0)}_{\mathcal{E}}(K)$ such that $(G_\alpha(z,w))_\alpha$ converges WOT to $G(z,w)$ for all $z, w \in X$.

In this case,

$$\|G\|_{\mathscr{S}^{(0)}} \leq \liminf_n \|G_n\|_{\mathscr{S}^{(0)}} \quad \textit{and} \quad \|G\|_{\mathscr{S}^{(0)}} \leq \liminf_\alpha \|G_\alpha\|_{\mathscr{S}^{(0)}}$$

holds for all sequences and nets as in (ii) and (iii), respectively.

Proof. We have to prove the implication (iii) to (i). Let $d > \liminf_\alpha \|G_\alpha\|_{\mathscr{S}^{(0)}}$ be arbitrary. By passing to a suitable subnet, we may assume that $\|\Sigma_{G_\alpha}\|_{cb} < d$ holds for all α. By [55], Theorem 7.4, there exists a completely bounded operator $\Phi : B(\mathcal{H}) \to B(\mathcal{H}_{\mathcal{E}})$ with $\|\Phi\|_{cb} \leq d$ such that an appropriate subnet $(\Sigma_{G_{\alpha_i}})_i$ converges to Φ in the BW topology of $B(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}}))$. Routine arguments now show that $G \cdot \Lambda_T = \Lambda_{\Phi(T)}$ holds for all $T \in B(\mathcal{H})$ and hence, that G belongs to $\mathscr{S}^{(0)}_{\mathcal{E}}(K)$ with $\Phi = \Sigma_G$. In particular, $\|G\|_{\mathscr{S}^{(0)}} = \|\Sigma_G\|_{cb} = \|\Phi\|_{cb} \leq d$. □

The following result provides a characterization of the class $\mathscr{S}^{(0)}_{\mathcal{E}}(K)$ avoiding the notion of complete boundedness.

Proposition 2.3.3. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a reproducing kernel space with reproducing kernel $K : X \times X \to \mathbb{C}$ and that $\mathcal{E}$ is a Hilbert space. For a given kernel $G : X \times X \to \mathcal{E}$, the following are equivalent:*

(i) $G \in \mathscr{S}^{(0)}_{\mathcal{E}}(K)$.

(ii) There exists an infinite dimensional Hilbert space $\mathcal{F}$ such that

$$G * L \in B(K_{\mathcal{E}\otimes\mathcal{F}}) \quad \textit{for all} \quad L \in B(K_{\mathcal{F}}).$$

(iii) For every Hilbert space $\mathcal{F}$, we have

$$G * L \in B(K_{\mathcal{E}\otimes\mathcal{F}}) \quad \textit{for all} \quad L \in B(K_{\mathcal{F}}).$$

In this case,

$$\|G\|_{\mathscr{S}^{(0)}} = \sup \|G * L\|_{K_{\mathcal{E}\otimes\mathcal{F}}},$$

where the supremum ranges over all Hilbert spaces $\mathcal{F}$ and all kernels $L \in B(K_{\mathcal{F}})$ with $\|L\|_{K_{\mathcal{F}}} \leq 1$.

Proof. Suppose that (i) holds and let $\mathcal{F}$ be an arbitrary Hilbert space. For simplicity, let us assume that $\|G\|_{\mathscr{S}^{(0)}} = 1$. Since $\Sigma_G : B(\mathcal{H}) \to B(\mathcal{H}_{\mathcal{E}})$ is normal (cf. Proposition 2.1.10) and completely contractive, the mapping

$$\Sigma_G \otimes 1_{B(\mathcal{F})} : B(\mathcal{H}) \otimes B(\mathcal{F}) \to B(\mathcal{H}_{\mathcal{E}}) \otimes B(\mathcal{F})$$

extends to a normal complete contraction

$$\Phi : B(\mathcal{H})\overline{\otimes}B(\mathcal{F}) \to B(\mathcal{H}_{\mathcal{E}})\overline{\otimes}B(\mathcal{F})$$

(by [30], Lemma 1.5). Clearly there are natural completely isometric identifications

$$B(\mathcal{H})\overline{\otimes}B(\mathcal{F}) \simeq B(\mathcal{H}_{\mathcal{F}}) \quad \text{and} \quad B(\mathcal{H}_{\mathcal{E}})\overline{\otimes}B(\mathcal{F}) \simeq B(\mathcal{H}_{\mathcal{E}\otimes\mathcal{F}})$$

by the definition of the normal spatial tensor product and Proposition 1.2.2. Using these identifications, one checks that $G * \Lambda_{T\otimes X} = \Lambda_{\Phi(T\otimes X)}$ holds for all $T \in B(\mathcal{H})$ and $X \in B(\mathcal{F})$. A standard continuity argument now proves that $G*L \in B(K_{\mathcal{E}\otimes\mathcal{F}})$ and that $\|G * L\|_{K_{\mathcal{E}\otimes\mathcal{F}}} \leq \|L\|_{K_{\mathcal{F}}}$ holds for all $L \in B(K_{\mathcal{F}})$. This proves the implication (i) to (iii). That (iii) implies (ii) is trivial. Finally, suppose that (ii) holds for some infinite dimensional Hilbert space $\mathcal{F}_0$. Then the mapping

$$T_G : B(K_{\mathcal{F}_0}) \to B(K_{\mathcal{E}\otimes\mathcal{F}_0})\ ,\ L \mapsto G * L$$

is continuous by Remark 1.6.2 (c) and the closed graph theorem. Using the canonical embeddings of $B(K^{(n)})$ into $B(K_{\mathcal{F}_0})$ and of $B(K_{\mathcal{E}}^{(n)})$ into $B(K_{\mathcal{E}\otimes\mathcal{F}_0})$, one easily deduces that $G \in \mathscr{S}_{\mathcal{E}}^{(0)}(K)$ and $\|G\|_{\mathscr{S}^{(0)}} \leq \|T_G\|$. The asserted equality

$$\|G\|_{\mathscr{S}^{(0)}} = \sup_{\mathcal{F},L} \|G * L\|_{K_{\mathcal{E}\otimes\mathcal{F}}}$$

is now obvious. $\square$

Before we continue, we stress the fact that the positive cones of $\mathscr{S}_{\mathcal{E}}(K)$ and $\mathscr{S}_{\mathcal{E}}^{(0)}(K)$ are the same, isometrically. In fact, we saw in Proposition 2.2.1 that for every $G \in \mathscr{S}_{\mathcal{E}}(K)_+$, the corresponding operator Σ_G is automatically completely positive with $\|\Sigma_G\|_{cb} = \|\Sigma_G\|$. In particular, linear combinations of kernels in $\mathscr{S}_{\mathcal{E}}(K)_+$ belong to $\mathscr{S}_{\mathcal{E}}^{(0)}(K)$. This observation leads to the following example.

Example 2.3.4. *Suppose that D is an open set in $\mathbb{C}^d$ and that K is a sesquianalytic kernel over D. Let $\mathcal{H} \subset \mathbb{C}^D$ denote the associated reproducing kernel Hilbert space consisting of analytic functions and assume that the coordinate functions $\mathbf{z}_i$ with $1 \leq i \leq d$ are multipliers of $\mathcal{H}$.*

(a) By Lemma 2.1.7, the kernels

$$D \times D \to \mathbb{C}\ ,\ (z,w) \mapsto z_i \quad \text{and} \quad D \times D \to \mathbb{C}\ ,\ (z,w) \mapsto \overline{w_i}$$

belong to $\mathscr{S}^{(0)}(K)$ for all $1 \leq i \leq d$. Since $\mathscr{S}^{(0)}(K)$ is obviously stable under the forming of pointwise products, every polynomial in z and $\overline{w}$ belongs to $\mathscr{S}^{(0)}(K)$. We already know by Example 1.4.5 that every polynomial kernel G can be written as a linear combination of positive definite polynomial kernels. In particular, G is a linear combination of elements of $\mathscr{S}(K)_+$. Later we shall see that, in general, the linear hull of $\mathscr{S}(K)_+$ is all of $\mathscr{S}^{(0)}(K)$.

(b) If we assume in addition that D is a domain and that the Taylor spectrum of the tuple $M_{\mathbf{z}}$ is contained in $\overline{D}$, then every kernel G that extends sesquianalytically to a neighbourhood of $\overline{D} \times \overline{D}$ belongs to $\mathscr{S}^{(0)}(K)$. To prove this, we choose an open set $U \supset \overline{D}$ such that G is sesquianalytic on $U \times U$. Let us first consider the case that G is positive. Fix a minimal Kolmorogov factorization $(\mathcal{G}, \phi)$ of G. Then the function $\phi : U \to B(\mathcal{G}, \mathbb{C})$ obviously is an analytic function on an open neighbourhood of $\overline{D}$ and therefore belongs to $\mathcal{M}(\mathcal{H} \otimes \mathcal{G}, \mathcal{H})$ by Example 1.7.3 (b). Proposition 2.2.1 now shows that $G \in \mathscr{S}(K)_+$. If G is not necessarily positive, then by Remark 1.6.10, it is possible to represent G as a linear combination of positive kernels which are sesquianalytic on $U \times U$. Consequently, G itself belongs to $\mathscr{S}^{(0)}(K)$.

We already pointed out that the space $\mathscr{S}^{(0)}_{\mathcal{E}}$ carries a natural operator space structure, which is induced via the assignment

$$\mathscr{S}^{(0)}_{\mathcal{E}}(K) \to CB(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}}))\ ,\ G \mapsto \Sigma_G.$$

Clearly, to be precise, the induced matrix norms may happen to be only seminorms. When the underlying Hilbert space $\mathcal{H}$ is supposed to be non-degenerate, then they are norms. The matrix norms on $\mathscr{S}^{(0)}_{\mathcal{E}}(K)$ can be expressed as follows: Let $[G_{ij}] \in M_n(\mathscr{S}^{(0)}_{\mathcal{E}}(K))$ be an $n \times n$-matrix of kernels in $\mathscr{S}^{(0)}_{\mathcal{E}}(K)$. By the definition of the operator space structure of $CB(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}}))$, the n-th norm of $[G_{ij}]$ is the cb-norm of the operator

$$B(\mathcal{H}) \to M_n(B(\mathcal{H}_{\mathcal{E}}))\ , T \mapsto [\Sigma_{G_{ij}}(T)],$$

which is, under the obvious identification $M_n(B(\mathcal{H}_{\mathcal{E}})) = B(\mathcal{H}_{\mathcal{E}^n})$ the same as $\|G\|_{\mathscr{S}^{(0)}}$, where G denotes the Schur kernel

$$G : X \times X \to B(\mathcal{E}^n)\ ,\ G(z,w) = [G_{ij}(z,w)].$$

In Proposition 2.1.10, we found a characterization of Schur multiplications as normal $(M(\mathcal{H}), M(\mathcal{H})^*)$-module homomorphisms in $B(B(\mathcal{H}), B(\mathcal{H}_\mathcal{E}))$. The following representation theorem for completely bounded module homomorphisms will be our main tool in the description of $\mathscr{S}_\mathcal{E}^{(0)}(K)$.

Theorem 2.3.5. *Suppose that $H, \mathcal{E}_1, \mathcal{E}_2$ are Hilbert spaces and that $\mathcal{A}, \mathcal{B}$ are unital subalgebras of $B(H)$. Let $\Phi \in CB(B(H \otimes \mathcal{E}_1), B(H \otimes \mathcal{E}_2))$ be a normal $(\mathcal{A}, \mathcal{B})$-module homomorphism. Then there exist a Hilbert space $\mathcal{G}$, operators*

$$V \in B(H \otimes \mathcal{E}_1 \otimes \mathcal{G}, H \otimes \mathcal{E}_2) \quad \text{and} \quad W \in B(H \otimes \mathcal{E}_2, H \otimes \mathcal{E}_1 \otimes \mathcal{G})$$

such that

$$\Phi(T) = V(T \otimes 1_\mathcal{G})W \quad (T \in B(H \otimes \mathcal{E}_1))$$

with $\|\Phi\|_{cb} = \|V\|\|W\|$ and such that

$$(A \otimes 1_{\mathcal{E}_2})V = V(A \otimes 1_{\mathcal{E}_1} \otimes 1_\mathcal{G}) \quad \text{and} \quad (B \otimes 1_{\mathcal{E}_1} \otimes 1_\mathcal{G})W = W(B \otimes 1_{\mathcal{E}_2})$$

holds for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

The particular case $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{C}$ of this result can already be found in [48]. Since this reference is not widely available and since we need a slightly more general version, we provide a detailed proof. For better readability, we will split the proof into several parts.

Definition 2.3.6. *Let H_1, H_2 be Hilbert spaces and let $\Phi : B(H_1) \to B(H_2)$ be a linear mapping. A representation of Φ is a tuple (K, π, V, W) consisting of a Hilbert space K, a $*$-homomorphism $\pi : B(H_1) \to B(K)$ and operators $V \in B(K, H_2)$, $W \in B(H_2, K)$ such that*

$$\phi(T) = V\pi(T)W$$

holds for all $T \in B(H_1)$. A representation will be called minimal if

$$[\pi(B(H_1))V^*H_2] = K = [\pi(B(H_1))WH_2]$$

holds.

A key step in the proof of Theorem 2.3.5 is the observationt that representations can always be minimized.

Lemma 2.3.7. *Let H_1, H_2 be Hilbert spaces and suppose that the linear mapping $\Phi : B(H_1) \to B(H_2)$ admits a representation (K, π, V, W). Then there exists a minimal representation (K_0, π_0, V_0, W_0) of Φ such that $K_0 \subset K$, $\|V_0\| \leq \|V\|$ and $\|W_0\| \leq \|W\|$.*

Proof. Set $L = [\pi(B(H_1))WH_2]$. Then L is an $\pi(B(H_1))$-invariant subspace of K. Hence the orthogonal projection P from K onto L commutes with $\pi(B(H_1))$.

Now define $K_0 = [P\pi(B(H_1))V^*H_2]$. Then also $K_0 \subset L$ is a $\pi(B(H_1))$-invariant subspace of K, and the orthogonal projection Q from K onto K_0 commutes with $\pi(B(H_1))$. We define

$$\pi_0 : B(H_1) \to B(K_0)\ ,\ \pi_0(T) = Q\pi(T)_{|K_0}$$

and

$$V_0 = V_{|K_0} \quad \text{and} \quad W_0 = QW.$$

First we observe that π_0 is a $*$-homomorphism. Since $K_0^{\perp} = \bigcap_{T\in B(H_1)} \ker V\pi(T)P$, we obtain that

$$\begin{aligned}
\langle V_0\pi_0(T)W_0x, y\rangle &= \langle V\pi(T)QWx, y\rangle \\
&= \langle V\pi(T)PWx, y\rangle \\
&= \langle V\pi(T)Wx, y\rangle \\
&= \langle \Phi(T)x, y\rangle \quad (T \in B(H_1), x, y \in H_2).
\end{aligned}$$

Hence (K_0, π_0, V_0, W_0) is still a representation of Φ. It remains to show that the constructed representation is minimal. To this end, note that

$$[\pi_0(B(H_1))V_0^*H_2] = [\pi(B(H_1))QV^*H_2] = [QP\pi(B(H_1))V^*H_2] = K_0.$$

Similarly, one obtains that

$$[\pi_0(B(H_1))W_0H_2] = [\pi(B(H_1))QWH_2] = [Q\pi(B(H_1))WH_2] = QK = K_0.$$

$\square$

Before we prove Theorem 2.3.5, we recapitulate some facts from the theory of von Neumann algebras. Let $\mathcal{M}$ be a von Neumann algebra. Then the norm closed subspace

$$\mathcal{M}_* = \{\lambda : \mathcal{M} \to \mathbb{C}\ ;\ \lambda \text{ weak-}* \text{ continuous }\}$$

of $\mathcal{M}^*$ is the unique predual of $\mathcal{M}$. Elements of $\mathcal{M}_*$ are called normal functionals. A positive functional $\lambda \in \mathcal{M}^*$ is called singular if the only normal functional λ' with $0 \leq \lambda' \leq \lambda$ is $\lambda' = 0$. The linear span of the positive singular functionals is denoted by $\mathcal{M}_*^{\perp}$. Elements of $\mathcal{M}_*^{\perp}$ are called singular functionals. One can show that $\mathcal{M}_*^{\perp}$ is a norm closed subspace of $\mathcal{M}^*$ and that $\mathcal{M}^* = \mathcal{M}_* \oplus_1 \mathcal{M}_*^{\perp}$ ([71], Theorem 2.14).

Now suppose that $\Phi : \mathcal{M} \to \mathcal{N}$ is a bounded operator between two von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$. Then Φ is called normal if $\Phi^*\mathcal{N}_* \subset \mathcal{M}_*$, and it is called singular if $\Phi^*\mathcal{N}_* \subset \mathcal{M}_*^{\perp}$. Clearly, Φ is normal precisely if it is weak-$*$ continuous. By a famous result of Tomiyama-Takesaki ([71], Definition 2.15), every operator $\Phi \in B(\mathcal{M}, \mathcal{N})$ can be uniquely decomposed as $\Phi = \Phi^{\sigma} + \Phi^s$, where Φ^{σ} is normal and Φ^s is singular.

In the case that $\mathcal{M} = B(H)$ for some Hilbert space H, normal and singular functionals can be described in a more concrete way: a functional $\lambda \in B(H)^*$ is normal if and only if there exist sequences $(x_n)_n$ and $(y_n)_n$ in H such that

$$\sum_n \|x_n\|^2 < \infty \quad , \quad \sum_n \|y_n\|^2 < \infty$$

and such that

$$\lambda(T) = \sum_n \langle Tx_n, y_n \rangle$$

holds for all $T \in B(H)$ (cf. [64], Corollary 1.15.5). Furthermore it is well known that a functional $\lambda \in B(H)^*$ is singular if and only if it vanishes on the ideal of compact operators $K(H)$. In order to prove this, one could use [71], Theorem 3.8. In particular, it follows that an operator $\Phi \in B(B(H_1), B(H_2))$ (H_1, H_2 Hilbert spaces) is singular if and only if it vanishes on $K(H_1)$.

Lemma 2.3.8. *Suppose that $H, \mathcal{E}_1, \mathcal{E}_2$ are Hilbert spaces and that $\mathcal{A}, \mathcal{B}$ are unital subalgebras of $B(H)$. Let $\Phi \in CB(B(H \otimes \mathcal{E}_1), B(H \otimes \mathcal{E}_2))$ be an $(\mathcal{A}, \mathcal{B})$-module homomorphism. Then there exists a minimal representation (K, π, V, W) of Φ satisfying $\|\Phi\|_{cb} = \|V\|\|W\|$ such that*

$$(A \otimes 1_{\mathcal{E}_2})V = V\pi(A \otimes 1_{\mathcal{E}_1}) \quad \text{and} \quad \pi(B \otimes 1_{\mathcal{E}_1})W = W(B \otimes 1_{\mathcal{E}_2})$$

holds for all $A \in \mathcal{A}, B \in \mathcal{B}$. If Φ is supposed to be normal, then π as well can be chosen as a normal $*$-homomorphism.

Proof. We start by choosing a Stinespring representation (K_1, π_1, V_1, W_1) of the completely bounded map Φ, that is, a representation satisfying $\|\Phi\|_{cb} = \|V_1\|\|W_1\|$ (see [55], Theorem 8.4 for the existence of such a representation). By Lemma 2.3.7, we find a minimal representation (K, π, V, W) of Φ such that $\|V\| \leq \|V_1\|$ and $\|W\| \leq \|W_1\|$. Hence $\|V\|\|W\| \leq \|V_1\|\|W_1\| = \|\Phi\|_{cb}$. On the other hand, we always have

$$\|\Phi\|_{cb} \leq \|V\|\|\pi\|_{cb}\|W\| \leq \|V\|\|W\|,$$

and thus equality, $\|\Phi\|_{cb} = \|V\|\|W\|$.

Now for $A \in \mathcal{A}$, we observe

$$\begin{aligned} V\pi(A \otimes 1_{\mathcal{E}_1})\pi(T)Wx &= \Phi((A \otimes 1_{\mathcal{E}_1})T)x \\ &= (A \otimes 1_{\mathcal{E}_2})\Phi(T)x \\ &= (A \otimes 1_{\mathcal{E}_2})V\pi(T)Wx \quad (T \in B(H \otimes \mathcal{E}_1), x \in H_2). \end{aligned}$$

By the minimality of the representation, we obtain that

$$V\pi(A \otimes 1_{\mathcal{E}_1}) = (A \otimes 1_{\mathcal{E}_2})V.$$

In the same way, it follows that, for $B \in \mathcal{B}$,

$$\begin{aligned} W^*\pi(B\otimes 1_{\mathcal{E}_1})^*\pi(T)V^*x &= \Phi(T^*(B\otimes 1_{\mathcal{E}_1}))^*x \\ &= (B\otimes 1_{\mathcal{E}_2})^*\Phi(T^*)^*x \\ &= (B\otimes 1_{\mathcal{E}_2})^*W^*\pi(T)V^*x \quad (T\in B(H\otimes\mathcal{E}_1), x\in H_2), \end{aligned}$$

and hence, by the minimality of the representation, we find that

$$W^*\pi(B\otimes 1_{\mathcal{E}_1})^* = (B\otimes 1_{\mathcal{E}_2})^*W^*.$$

To complete the proof, we suppose in addition that Φ is normal. Let $\pi = \pi^\sigma + \pi^s$ denote the Tomiyama decomposition of the representation π. For a moment, we fix operators $R, S \in B(H\otimes\mathcal{E}_1)$ and define

$$\psi^\sigma : B(H\otimes\mathcal{E}_1) \to B(H\otimes\mathcal{E}_2)\ ,\ \psi^\sigma(T) = V\pi(R^*)\pi^\sigma(T)\pi(S)W$$

and

$$\psi^s : B(H\otimes\mathcal{E}_1) \to B(H\otimes\mathcal{E}_2)\ ,\ \psi^s(T) = V\pi(R^*)\pi^s(T)\pi(S)W.$$

By the remarks preceding this lemma, it is not hard to see that ψ^σ is normal. The operator ψ^s is singular, since it vanishes on the compact operators. As a composition of normal operators, the mapping

$$\psi : B(H\otimes\mathcal{E}_1) \to B(H\otimes\mathcal{E}_2)\ ,\ \psi(T) = \Phi(R^*TS)$$

certainly is normal as well, and $\psi = \psi^\sigma + \psi^s$ is easily recognized as the unique Tomiyama decomposition of ψ. We see that $\psi^s = 0$ and hence that

$$\langle \pi^s(T)\pi(S)Wy, \pi(R)V^*x\rangle = \langle \psi^s(T)y, x\rangle = 0$$

holds for all $T \in B(H\otimes\mathcal{E}_1)$ and $x, y \in H\otimes\mathcal{E}_2$. Since R, S were arbitrary and since

$$[\pi(B(H\otimes\mathcal{E}_1))WH\otimes\mathcal{E}_2] = K = [\pi(B(H\otimes\mathcal{E}_1))V^*H\otimes\mathcal{E}_2],$$

we conclude that $\pi^s = 0$ or, equivalently, that $\pi = \pi^\sigma$ is normal. $\square$

We are now in the position to accomplish the proof of the representation theorem.

Proof (of Theorem 2.3.5). By Lemma 2.3.8, there exists a minimal normal representation (K_0, π_0, V_0, W_0) of Φ. We claim that π_0 is unitarily equivalent to an amplification of the identical representation, that is, there exist a Hilbert space $\mathcal{G}$ and a unitary operator $U \in B(K_0, H\otimes\mathcal{E}_1\otimes\mathcal{G})$ such that

$$U^*(T\otimes 1_{\mathcal{G}})U = \pi_0(T)$$

holds for all $T \in B(H\otimes\mathcal{E}_1)$. In fact, by [12], Corollary 1 of Theorem 1.4.4, there exists a Hilbert space $\mathcal{G}$ and a unitary $U \in B(K_0, \mathcal{H}\otimes\mathcal{E}_1\otimes\mathcal{G})$ such that

$$\pi_0(T) = U^*(T\otimes 1_{\mathcal{G}})U \tag{2.3.1}$$

holds for all $T \in K(H \otimes \mathcal{E}_1)$ (the algebra of compact operators on $H \otimes \mathcal{E}_1$). Since $K(H \otimes \mathcal{E}_1)$ is weak-$*$ dense in $B(H \otimes \mathcal{E}_1)$ and since both sides of (2.3.1) are weak-$*$ continuous, the claim is proved.

Next we want to show that the operators

$$V = V_0 U^* : H \otimes \mathcal{E}_1 \otimes \mathcal{G} \to H \otimes \mathcal{E}_2$$

and

$$W = U W_0 : H \otimes \mathcal{E}_2 \to H \otimes \mathcal{E}_1 \otimes \mathcal{G}$$

have the desired properties. First of all,

$$\Phi(T) = V_0 \pi_0(T) W_0 = V_0 U^*(T \otimes 1_{\mathcal{G}}) U W_0 = V(T \otimes 1_{\mathcal{G}}) W$$

for all $T \in B(H \otimes \mathcal{E}_1)$, and also

$$\|\Phi\|_{cb} \leq \|V\| \|W\| = \|V_0\| \|W_0\| = \|\Phi\|_{cb},$$

and thus $\|\Phi\|_{cb} = \|V\| \|W\|$. Finally,

$$\begin{aligned} (A \otimes 1_{\mathcal{E}_2}) V &= (A \otimes 1_{\mathcal{E}_2}) V_0 U^* \\ &= V_0 \pi_0(A \otimes 1_{\mathcal{E}_1}) U^* \\ &= V_0 U^*(A \otimes 1_{\mathcal{E}_1} \otimes 1_{\mathcal{G}}) \\ &= V(A \otimes 1_{\mathcal{E}_1} \otimes 1_{\mathcal{G}}) \end{aligned}$$

and

$$\begin{aligned} W(B \otimes 1_{\mathcal{E}_2}) &= U W_0 (B \otimes 1_{\mathcal{E}_2}) \\ &= U \pi_0(B \otimes 1_{\mathcal{E}_1}) W_0 \\ &= (B \otimes 1_{\mathcal{E}_1} \otimes 1_{\mathcal{G}}) U W_0 \\ &= (B \otimes 1_{\mathcal{E}_1} \otimes 1_{\mathcal{G}}) W \end{aligned}$$

holds for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, which finishes the proof. □

We are now able to prove the main result of this chapter, namely, a factorization theorem for $\mathscr{S}_{\mathcal{E}}^{(0)}(K)$.

Theorem 2.3.9. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a reproducing kernel Hilbert space with reproducing kernel $K : X \times X \to \mathbb{C}$ such that $\mathcal{M}(\mathcal{H})$ is dense in $\mathcal{H}$ or such that $\mathcal{H}$ is regular. Let $\mathcal{E}$ be a Hilbert space. Then, for a given kernel $G : X \times X \to B(\mathcal{E})$, the following are equivalent:*

(i) $G \in \mathscr{S}_{\mathcal{E}}^{(0)}(K)$ *and* $\|G\|_{\mathscr{S}^{(0)}} \leq 1$.

(ii) There exist kernels $H_1, H_2 \in \mathscr{S}_{\mathcal{E}}(K)_+$ with $\|H_1\|_{\mathscr{S}} \leq 1$, $\|H_2\|_{\mathscr{S}} \leq 1$ such that the kernel

$$H : X \times X \to B(\mathcal{E}^2)\ ,\ H(z,w) \mapsto \begin{pmatrix} H_1(z,w) & G(z,w) \\ G^*(z,w) & H_2(z,w) \end{pmatrix}$$

belongs to $\mathscr{S}_{\mathcal{E}^2}(K)_+$ and satisfies $\|H\|_{\mathscr{S}} \leq 2$.

(iii) There exist a Hilbert space $\mathcal{G}$ and contractive multipliers $\phi, \psi \in \mathcal{M}(\mathcal{H}_{\mathcal{G}}, \mathcal{H}_{\mathcal{E}})$ such that

$$G(z,w) = \phi(z)\psi(w)^*$$

holds for all $z, w \in X$.

Proof. We first prove the equivalence of (ii) and (iii). Suppose that (ii) holds. By Proposition 2.2.1, there exist a Hilbert space $\mathcal{G}$ and a multiplier $\gamma \in \mathcal{M}(\mathcal{H}_{\mathcal{G}}, \mathcal{H}_{\mathcal{E}^2})$ such that

$$H(z,w) = \gamma(z)\gamma(w)^*$$

holds for all $z, w \in X$. Define functions $\phi, \psi : X \to B(\mathcal{G}, \mathcal{E})$ by

$$\gamma(z)x = \begin{pmatrix} \phi(z)x \\ \psi(z)x \end{pmatrix}.$$

This yields

$$\begin{pmatrix} H_1(z,w) & G(z,w) \\ G^*(z,w) & H_2(z,w) \end{pmatrix} = \begin{pmatrix} \phi(z)\phi(w)^* & \phi(z)\psi(w)^* \\ \psi(z)\phi(w)^* & \psi(z)\psi(w)^* \end{pmatrix} \tag{2.3.2}$$

for all $z, w \in X$. Hence, on the one hand

$$H_1(z,w) = \phi(z)\phi(w)^* \quad \text{and} \quad H_2(z,w) = \psi(z)\psi(w)^*$$

for all $z, w \in X$ which shows, again by Proposition 2.2.1, that ϕ and ψ define contractive multipliers in $\mathcal{M}(\mathcal{H}_{\mathcal{G}}, \mathcal{H}_{\mathcal{E}})$. On the other hand, (2.3.2) shows of course that $G(z,w) = \phi(z)\psi(w)^*$ holds for $z, w \in X$.

Conversely suppose that G can be factorized as in (iii) with contractive multipliers ϕ and ψ. Then Proposition 2.2.1 implies that the functions $H_1, H_2 : X \times X \to B(\mathcal{E})$, defined by

$$H_1(z,w) = \phi(z)\phi(w)^* \quad \text{and} \quad H_2(z,w) = \psi(z)\psi(w)^* \quad (z, w \in X),$$

are positive Schur kernels with $\|H_1\|_{\mathscr{S}} \leq 1$ and $\|H_2\|_{\mathscr{S}} \leq 1$. Then

$$\gamma : X \times X \to B(\mathcal{G}, \mathcal{E}^2)\ ,\ \gamma(z)x = \begin{pmatrix} \phi(z)x \\ \psi(z)x \end{pmatrix}$$

defines a multiplier $\gamma \in \mathcal{M}(\mathcal{H}_{\mathcal{G}}, \mathcal{H}_{\mathcal{E}^2})$. Indeed, consider the operators

$$j_1 : \mathcal{E} \to \mathcal{E}^2\ ,\ x \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix} \quad \text{and} \quad j_2 : \mathcal{E} \to \mathcal{E}^2\ ,\ x \mapsto \begin{pmatrix} 0 \\ x \end{pmatrix}.$$

According to Lemma 1.7.13, these operators define constant multipliers between $\mathcal{H}_{\mathcal{E}}$ and $\mathcal{H}_{\mathcal{E}^2}$. Writing γ as

$$\gamma(z) = j_1\phi(z) + j_2\psi(z) \quad (z \in X),$$

we see that γ is a multiplier. Moreover, since M_{j_1} and M_{j_2} are isometries with orthogonal ranges, we infer that

$$M_\gamma^* M_\gamma = M_\phi^* M_\phi + M_\psi^* M_\psi.$$

This shows of course that $\|\gamma\|_{\mathcal{M}} \leq \sqrt{2}$. Then Proposition 2.2.1 implies that the kernel $H : X \times X \to B(\mathcal{E}^2)$,

$$H(z,w) = \begin{pmatrix} H_1(z,w) & G(z,w) \\ G^*(z,w) & H_2(z,w) \end{pmatrix} = \gamma(z)\gamma(w)^*$$

belongs to $\mathscr{S}_{\mathcal{E}^2}(K)_+$ with $\|H\|_{\mathscr{S}} \leq 2$.

The implication (iii) to (i) is precisely the statement of Lemma 2.1.7.

For the remaining implication (i) to (iii), let us first consider the case that $\mathcal{M}(\mathcal{H})$ is a dense subset of $\mathcal{H}$. By Proposition 2.1.10, the mapping Σ_G is a completely bounded normal $(M(\mathcal{H}), M(\mathcal{H})^*)$-module homomorphism. Theorem 2.3.5 now furnishes a Hilbert space $\mathcal{G}$ and operators

$$V \in B(\mathcal{H} \otimes \mathcal{G}, \mathcal{H} \otimes \mathcal{E}) \quad \text{and} \quad W \in B(\mathcal{H} \otimes \mathcal{E}, \mathcal{H} \otimes \mathcal{G})$$

with

$$\Sigma_G(T) = V(T \otimes 1_{\mathcal{G}})W \quad (T \in B(\mathcal{H})$$

such that $\|\Sigma_G\|_{cb} = \|V\|\|W\|$ and such that

$$(M_\alpha \otimes 1_{\mathcal{E}})V = V(M_\alpha \otimes 1_{\mathcal{G}}) \quad \text{and} \quad W^*(M_\beta \otimes 1_{\mathcal{G}}) = (M_\beta \otimes 1_{\mathcal{E}})W^*$$

holds for all $\alpha, \beta \in \mathcal{M}(\mathcal{H})$. By Corollary 1.7.5, there exist $\phi, \psi \in \mathcal{M}(\mathcal{H} \otimes \mathcal{G}, \mathcal{H} \otimes \mathcal{E})$ such that $V = M_\phi$ and $W = M_\psi^*$. An evaluation of Σ_G on the rank-one operator $Q_0 = \mathbf{1} \otimes \mathbf{1}$ yields that

$$\begin{aligned} \langle G(z,w)y, x\rangle &= \langle \Sigma_G(Q_0)K(\cdot,w)y, K(\cdot,z)x\rangle \\ &= \langle (Q_0 \otimes 1_{\mathcal{G}})M_\psi^* K(\cdot,w)y, M_\phi^* K(\cdot,z)x\rangle \\ &= \langle \psi(w)^* y, \phi(z)^* x\rangle \end{aligned}$$

for all $z, w \in X$ and $x, y \in \mathcal{E}$. Thus $G(z,w) = \phi(z)\psi(w)^*$ for all $z, w \in X$ and $\|G\|_{\mathscr{S}^{(0)}} = \|\Sigma_G\|_{cb} = \|V\|\|W\| = \|\phi\|_{\mathcal{M}}\|\psi\|_{\mathcal{M}}$, which proves all assertions.

We now consider the case that $\mathcal{H}$ is regular. Then the implication (i) to (ii) can be proved by an approximation argument: Let $\mathfrak{Y}$ denote the collection of all finite subsets of X. If $Y \in \mathfrak{Y}$, then $\mathcal{H}_{|Y}$ denotes as usual the space associated with the restriction $K_{|Y}$ of K onto $Y \times Y$. By Proposition 1.2.3, the space $\mathcal{H}_{|Y}$ consists precisely of all restrictions of functions in $\mathcal{H}$. Similarly, $(\mathcal{H}_{\mathcal{E}})_{|Y}$ is the space associated with the restriction $(K_{\mathcal{E}})_{|Y}$ of $K_{\mathcal{E}}$ onto $Y \times Y$. Clearly, $(\mathcal{H}_{\mathcal{E}})_{|Y} = (\mathcal{H}_{|Y})_{\mathcal{E}}$. Again by Proposition 1.2.3, the restriction maps

$$\rho_Y : \mathcal{H} \to \mathcal{H}_{|Y} \quad \text{and} \quad \rho_{Y,\mathcal{E}} : \mathcal{H}_{\mathcal{E}} \to (\mathcal{H}_{|Y})_{\mathcal{E}}$$

are coisometries. It is then very easy to see that the maps

$$i_Y : B(\mathcal{H}_{|Y}) \to B(\mathcal{H}) \ , \ T \mapsto \rho_Y^* T \rho_Y$$

and

$$i_{Y,\mathcal{E}} : B((\mathcal{H}_{|Y})_{\mathcal{E}}) \to B(\mathcal{H}_{\mathcal{E}}) \ , \ T \mapsto \rho_{Y,\mathcal{E}}^* T \rho_{Y,\mathcal{E}}$$

are faithful $*$-homomorphisms. Analogously, the maps

$$\pi_Y : B(\mathcal{H}) \to B(\mathcal{H}_{|Y}) \ , \ T \mapsto \rho_Y T \rho_Y^*$$

and

$$\pi_{Y,\mathcal{E}} : B(\mathcal{H}_{\mathcal{E}}) \to B((\mathcal{H}_{|Y})_{\mathcal{E}}) \ , \ T \mapsto \rho_{Y,\mathcal{E}} T \rho_{Y,\mathcal{E}}^*$$

define completely positive unital mappings such that

$$\pi_Y \ i_Y = 1_{B(\mathcal{H}_{|Y})} \quad \text{and} \quad \pi_{Y,\mathcal{E}} \ i_{Y,\mathcal{E}} = 1_{B((\mathcal{H}_{|Y})_{\mathcal{E}})}.$$

One verifies that, for $Y \in \mathfrak{Y}$, the kernel $G_{|Y} = G_{|Y \times Y}$ belongs to $\mathscr{S}_{\mathcal{E}}(K_Y)$ and, moreover, satisfies the relation $\Sigma_{G_{|Y}} = \pi_{Y,\mathcal{E}} \Sigma_G i_Y$. Hence it follows that actually $G_{|Y} \in \mathscr{S}_{\mathcal{E}}^{(0)}(K_Y)$ and $\|G_{|Y}\|_{\mathscr{S}^{(0)}} \leq \|G\|_{\mathscr{S}^{(0)}}$. Since $\mathcal{H}_{|Y} = \mathbb{C}^Y$ by Proposition 1.3.6, it follows that $\mathcal{M}(\mathcal{H}_{|Y}) = \mathbb{C}^Y = \mathcal{H}_{|Y}$.

By the previous part of the proof, for every set $Y \in \mathfrak{Y}$, there exist Schur kernels $H_1^Y, H_2^Y \in \mathscr{S}_{\mathcal{E}}(K_{|Y})_+$ with $\|H_1^Y\|_{\mathscr{S}} \leq 1$ and $\|H_2^Y\|_{\mathscr{S}} \leq 1$ such that

$$H^Y : Y \times Y \to B(\mathcal{E}^2) \ , \ H^Y(z,w) = \begin{pmatrix} H_1^Y(z,w) & G_{|Y}(z,w) \\ G_{|Y}^*(z,w) & H_2^Y(z,w) \end{pmatrix}$$

belongs to $\mathscr{S}_{\mathcal{E}^2}(K_{|Y})_+$ and $\|H^Y\|_{\mathscr{S}} \leq 2$. Defining

$$\Phi_i^Y : B(\mathcal{H}) \to B(\mathcal{H}_{\mathcal{E}}) \ , \ \Phi_i^Y = i_{Y,\mathcal{E}} \Sigma_{H_i^Y} \pi_Y \quad (i = 1,2)$$

yields nets $(\Phi_1^Y)_{Y \in \mathfrak{Y}}$ and $(\Phi_2^Y)_{Y \in \mathfrak{Y}}$ of completely positive operators in the unit ball of $CB(B(\mathcal{H}), B(\mathcal{H}_{\mathcal{E}}))$. Since the latter unit ball is BW compact (by [55], Theorem 7.4), there are subnets $(\Phi_i^{Y_j})_j$ of $(\Phi_i^Y)_Y$ such that the BW limits

$$\Phi_i = \lim_j \Phi_i^{Y_j} \quad (i = 1,2) \tag{2.3.3}$$

exist. As expected, the operators Φ_1, Φ_2 allow us to define kernels H_1, H_2 satisfying condition (ii). Since $\mathcal{H}$ was supposed to be non-degenerate, for every $z \in X$, we can choose a function $f_z \in \mathcal{H}$ with $f_z(z) = 1$. Let us define

$$H_i : X \times X \to B(\mathcal{E}) \ , \ H_i(z,w) = \Lambda_{\Phi_i(f_z \otimes f_w)}(z,w) \quad (i = 1,2).$$

For every $F \in \mathfrak{Y}$, let us furthermore fix some $j_F \in J$ with the property that $Y_{j_F} \supset F$. The set $\{j \in J \ ; \ j \geq j_F\}$ is then cofinal in J for all $F \in \mathfrak{Y}$. Now, by Proposition 7.3 in [55], equation (2.3.3) means that

$$\lim_j \Phi_i^{Y_j}(T) = \Phi_i(T)$$

holds in the WOT sense for $i = 1, 2$ and for all $T \in B(\mathcal{H})$. Hence

$$\begin{aligned} \lim_{j \geq j_F} \langle H_i^{Y_j}(z,w)\Lambda_T(z,w)y, x\rangle &= \lim_{j \geq j_F} \langle \Phi_i^{Y_j}(T)K(\cdot,w)y, K(\cdot,z)x\rangle \\ &= \langle \Phi_i(T)K(\cdot,w)y, K(\cdot,z)x\rangle \\ &= \langle \Lambda_{\Phi_i(T)}(z,w)y, x\rangle \end{aligned}$$

holds for $i = 1, 2$ and all $T \in B(\mathcal{H})$, $z, w \in F$ and $x, y \in \mathcal{E}$. In particular, if $z, w \in X$, then setting $F = \{z, w\}$ yields

$$H_i(z,w) = \lim_{j \geq j_F} H_i^{Y_j}(z,w) \quad (i = 1, 2)$$

in the WOT sense (since $\Lambda_{f_z \otimes f_w}(z,w) = 1$) which shows that the kernels H_i are independent of the special choice of the functions f_z. It follows that

$$H_i(z,w)\Lambda_T(z,w) = \lim_{j \geq j_F} H_i^{Y_j}(z,w)\Lambda_T(z,w) = \Lambda_{\Phi_i(T)}(z,w)$$

(all limits in the weak operator sense) holds for all $T \in B(\mathcal{H})$. Since z, w are arbitrary, this shows that $H_i \in \mathscr{S}_{\mathcal{E}}(K)$ with $\Sigma_{H_i} = \Phi_i$ and $\|H_i\|_{\mathscr{S}} \leq 1$ for $i = 1, 2$.

It remains to show that the kernels H_i and the associated kernel $H : X \times X \to B(\mathcal{E}^2)$ are positive and that $\|H\|_{\mathscr{S}} \leq 2$. First let us prove the positivity of the kernels H_1, H_2. Clearly, it suffices to show the positivity on each finite subset F of X. But for every finite set $F \subset X$, we have

$$\lim_{j \geq j_F} H_i^{Y_j}(z,w) = H_i(z,w) \quad (z, w \in F),$$

where the limit is meant in the weak operator sense. Since the cone of $B(\mathcal{E})$-valued positive functions on $F \times F$ is closed in the topology of pointwise WOT convergence as mentioned in Remark 1.4.2(d), and since the restrictions of the kernels $H_i^{Y_j}$ on $F \times F$ are clearly positive, the assertion follows.

The proof of the positivity of H is completely analogous, after one has checked that

$$\lim_{j \geq j_F} H^{Y_j}(z,w) = H(z,w)$$

holds in the WOT sense for all finite sets $F \subset X$ and all $z, w \in F$.

Finally, by Proposition 2.2.1, the kernels $2\ (K_{|Y})_{\mathcal{E}^2} - H^Y \cdot K_{|Y}$ are positive for all $Y \in \mathfrak{Y}$. The same argument as above yields that $2\ K_{\mathcal{E}^2} - H \cdot K$ is positive. Another application of Proposition 2.2.1 shows that $H \in \mathscr{S}_{\mathcal{E}}(K)$ with $\|H\|_{\mathscr{S}} \leq 2$. □

The equivalence of (i) and (iii) in the preceding theorem can be regarded as a Schur kernel version of Wittstock's decomposition theorem. The statement in (ii) is an adapted realization of Paulsen's off-diagonal technique (cf. [55], Chapter 8).

Corollary 2.3.10. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a reproducing kernel space with reproducing kernel $K : X \times X \to \mathbb{C}$ such that $\mathcal{M}(\mathcal{H})$ is dense in $\mathcal{H}$ or such that $\mathcal{H}$ is regular. Let $\mathcal{E}$ be a Hilbert space. Then, for a given kernel $G : X \times X \to B(\mathcal{E})$, the following are equivalent:*

(i) $G \in \mathscr{S}_{\mathcal{E}}^{(0)}(K)$.

(ii) There exist a Hilbert space $\mathcal{G}$ and multipliers $\phi, \psi \in \mathcal{M}(\mathcal{H}_{\mathcal{G}}, \mathcal{H}_{\mathcal{E}})$ such that

$$G(z, w) = \phi(z)\psi(w)^*$$

holds for all $z, w \in X$.

In this case,

$$\|G\|_{\mathscr{S}^{(0)}} = \inf\{\|\phi\|_{\mathcal{M}}\|\psi\|_{\mathcal{M}} \ ; \ \phi, \psi \text{ as in (ii) }\}.$$

Moreover, the infimum is attained.

Example 2.3.11. *We return to the case of classical Schur multiplications as discussed in Example 2.1.8. That is, we consider the reproducing kernel Hilbert space $\mathcal{H} = l^2(I)$ on the set $I = \{1, \dots, n\}$ and its reproducing kernel*

$$K : I \times I \to \mathbb{C} \ , \ K(i, j) = \delta_{i,j}.$$

As before, we identify kernels on $I \times I$ with $n \times n$-matrices.

So suppose that $G \in M_n$ is a matrix with $\|S_G\| = \|S_G\|_{cb} \leq 1$. Then Theorem 2.3.9 yields contractive multipliers $\phi, \psi \in \mathcal{M}(\mathcal{H}_{\mathcal{G}}, \mathcal{H})$ ($\mathcal{G}$ a suitable Hilbert space) such that

$$G(i, j) = \phi(i)\psi(j)^*(1)$$

holds for all $i, j \in I$. In Example 1.7.3 (a) we pointed out that $\mathcal{M}(\mathcal{H}_{\mathcal{G}}, \mathcal{H})$ can be canonically identified with $l^\infty(I, B(\mathcal{G}, \mathbb{C}))$ and therefore with $l^\infty(I, \mathcal{G})$. Using these identifications, it follows immediately that there exist finite sequences $(x_i)_{i=1}^n$ and $(y_j)_{j=1}^n$ in the unit ball of $\mathcal{G}$ such that

$$G_{i,j} = \langle y_j, x_i \rangle$$

holds for all $i, j = 1, \dots, n$. So we regain the well-knonw representation result for classical Schur multipliers as proved by Haagerup [43], Pisier [57] and Paulsen (cf. [55], Corollary 8.8).

We also deduce that in the situation of Theorem 2.3.9, the space $\mathscr{S}_{\mathcal{E}}^{(0)}(K)$ is the linear span of its positive kernels.

Proposition 2.3.12. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a reproducing kernel space with reproducing kernel $K : X \times X \to \mathbb{C}$ and that $\mathcal{E}$ is a Hilbert space. Then, for an arbitrary kernel $G : X \times X \to B(\mathcal{E})$, the following are equivalent:*

(i) $G \in \mathscr{S}_{\mathcal{E}}(K)$ *and there exists a kernel* $\hat{G} \in \mathscr{S}_{\mathcal{E}}(K)_+$ *such that the kernels*

$$\hat{G} \pm \operatorname{Re} G \quad \text{and} \quad \hat{G} \pm \operatorname{Im} G$$

are positive.

(ii) There exists a kernel $\hat{G} \in \mathscr{S}_{\mathcal{E}}(K)_+$ *such that the kernels*

$$\hat{G} \pm \operatorname{Re} G \quad \text{and} \quad \hat{G} \pm \operatorname{Im} G$$

belong to $\mathscr{S}_{\mathcal{E}}(K)_+$.

(iii) There exist kernels $G_1, \dots, G_4 \in \mathscr{S}_{\mathcal{E}}(K)_+$ *such that*

$$G = G_1 - G_2 + iG_3 - iG_4.$$

In this case, $G \in \mathscr{S}^{(0)}_{\mathcal{E}}(K)$. *If in addition,* $\mathcal{M}(\mathcal{H})$ *is dense in* $\mathcal{H}$ *or if* $\mathcal{H}$ *is regular, then each of the above conditions is equivalent to:*

(iv) $G \in \mathscr{S}^{(0)}_{\mathcal{E}}(K)$.

Proof. Obviously, condition (i) implies condition (ii). If (ii) holds, then

$$G_1 = \frac{1}{2}(\hat{G} + \operatorname{Re} G),\ G_2 = \frac{1}{2}(\hat{G} - \operatorname{Re} G),\ G_3 = \frac{1}{2}(\hat{G} + \operatorname{Im} G),\ G_4 = \frac{1}{2}(\hat{G} - \operatorname{Im} G)$$

are positive Schur kernels as required in (iii). If $G_1, \dots, G_4$ satisfy condition (iii), then

$$\hat{G} = G_1 + G_2 + G_3 + G_4$$

has the properties demanded in condition (i). Since $\mathscr{S}^{(0)}_{\mathcal{E}}(K)$ is a linear space and since by Proposition 2.2.1, $\mathscr{S}_{\mathcal{E}}(K)_+ \subset \mathscr{S}^{(0)}_{\mathcal{E}}(K)$, condition (iii) obviously implies (iv). Finally, if $\mathcal{M}(\mathcal{H})$ is dense in $\mathcal{H}$ or if $\mathcal{H}$ is regular, then by Theorem 2.3.9, every kernel $G \in \mathscr{S}^{(0)}_{\mathcal{E}}(K)$ can be factorized by multipliers $\phi, \psi \in \mathcal{M}(\mathcal{H}_{\mathcal{G}}, \mathcal{H}_{\mathcal{E}})$, where $\mathcal{G}$ is an appropriate Hilbert space. Then the kernel

$$\hat{G} : X \times X \to B(\mathcal{E})\ ,\ \hat{G}(z,w) = \frac{1}{2}\left(\phi(z)\phi(w)^* + \psi(z)\psi(w)^*\right)$$

satisfies (i). □

The question whether $\mathscr{S}_{\mathcal{E}}(K)$ is the linear span of its positive kernels (equivalently, whether every hermitian kernel in $\mathscr{S}_{\mathcal{E}}(K)$ can be written as the difference of two positive kernels in $\mathscr{S}_{\mathcal{E}}(K)$) is therefore equivalent to the question whether the mappings Σ_G are automatically completely bounded for all $G \in \mathscr{S}_{\mathcal{E}}(K)$.

2.4 Positive Schur kernels - revisited

We make the following definitions: If H is a Hilbert space and h is an element of H, then we denote by h_c and h_r the operators

$$h_c : \mathbb{C} \to H \ , \ \zeta \mapsto \zeta h \quad \text{ and } \quad h_r : \overline{H} \to \mathbb{C} \ , \ h' \mapsto \langle h, h' \rangle_H .$$

It is then clear from the definitions that

$$j_c : H \to B(\mathbb{C}, H) \ , \ h \mapsto h_c \quad \text{ and } \quad j_r : H \to B(\overline{H}, \mathbb{C}) \ , \ h \mapsto h_r$$

are linear isometric isomorphisms. Here, as usual in the literature, $\overline{H}$ denotes the conjugate linear version of H.

With these notations, the column Hilbert space structure on H is by definition the operator space structure of $B(\mathbb{C}, H)$ induced by j_c. Analogously, the row Hilbert space structure on H is the operator space structure of $B(\overline{H}, \mathbb{C})$ under the identification j_r. It is then clear that the identity mapping $H_r \to (H_c)^{op}$ is a complete isometry, where V^{op} denotes the transposed operator space structure of a given operator space V.

Proposition 2.4.1. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a non-degenerate reproducing kernel space with reproducing kernel $K : X \times X \to \mathbb{C}$ and that $\mathcal{E}$ is a Hilbert space. Let $G : X \times X \to B(\mathcal{E})$ be a positive kernel with associated reproducing kernel space $\mathcal{G} \subset \mathcal{E}^X$. Then the following assertions are equivalent:*

(i) $G \in \mathscr{S}_{\mathcal{E}}(K)_+$ *and* $\|G\|_{\mathscr{S}} \leq 1$.

(ii) The embedding

$$i : \mathcal{G}_c \to \mathcal{M}(\mathcal{H}, \mathcal{H}_{\mathcal{E}}) \ , \ (ig)(z) = g(z)_c$$

between the column Hilbert space $\mathcal{G}_c$ and $\mathcal{M}(\mathcal{H}, \mathcal{H}_{\mathcal{E}})$ is well defined and completely contractive.

(iii) The inclusion mapping $j : B(G) \hookrightarrow \mathscr{S}^{(0)}_{\mathcal{E}}(K)$ is completely contractive.

Proof. We prove (i) to (ii). Fix $n \in \mathbb{N}$ and suppose that (i) holds. Let $[g_{ij}] \in M_n(\mathcal{G}_c)$ be an $n \times n$-matrix of functions in $\mathcal{G}$ such that $\|[g_{ij}]\| = 1$ and write

$$\gamma : X \to B(\mathbb{C}^n, \mathcal{E}^n) \ , \ \gamma(z) = [g_{ij}(z)_c].$$

Recalling Proposition 1.7.6 and the discussion of the operator space structure of multiplier spaces at the end of Section 1.7, it suffices to show that the kernel

$$\Gamma : X \times X \to B(\mathcal{E}^n) \ , \ \Gamma(z, w) = K^{(n)}_{\mathcal{E}}(z, w) - K(z, w)\gamma(z)\gamma(w)^*$$

is positive. By Proposition 2.2.1, the kernel

$$K^{(n)}_{\mathcal{E}} - G^{(n)} \cdot K = (K_{\mathcal{E}} - G \cdot K)^{(n)}$$

is positive. Rewriting Γ as

$$\Gamma(z,w) = \left(K_{\mathcal{E}}^{(n)}(z,w) - G^{(n)}(z,w)K(z,w)\right) + K(z,w)\left(G^{(n)}(z,w) - \gamma(z)\gamma(w)^*\right)$$

reveals that it suffices to show the positivity of the kernel

$$\Gamma' : X \times X \to B(\mathcal{E}^n) \ , \ (z,w) \mapsto G^{(n)}(z,w) - \gamma(z)\gamma(w)^*.$$

Under canonical identifications, the kernel Γ' is represented by the operator

$$1_{\mathcal{G}^n} - [\sum_k g_{ik} \otimes g_{jk}] \in B(\mathcal{G}^n).$$

Now the assumption $\|[g_{ij}]\| = 1$ means, by the definition of the column Hilbert space, that the operator

$$g : \mathbb{C}^n \to \mathcal{G}^n \ , \ (\zeta_i) \mapsto (\sum_j g_{ij}\zeta_j)$$

has norm 1. An easy calculation then shows that $[\sum_k g_{ik} \otimes g_{jk}] = gg^*$ is a positive contraction. Consequently, the operator $1_{\mathcal{G}^n} - [\sum_k g_{ik} \otimes g_{jk}]$ is positive, which clearly implies that Γ' is a positive kernel.

Now suppose that (ii) holds. We introduce some new notations: For $g \in \mathcal{G}$, write

$$\tilde{g} : X \to B(\mathbb{C}, \mathcal{E}) \ , \ \tilde{g}(z) = g(z)_c.$$

Then (ii) precisely means that the mapping

$$\mathcal{G}_c \to \mathcal{M}(\mathcal{H}, \mathcal{H}_{\mathcal{E}}) \ , \ g \mapsto \tilde{g}$$

is well defined and completely contractive. We claim now that then also the mapping

$$\overline{\mathcal{G}}_r \to \overline{M(\mathcal{H}, \mathcal{H}_{\mathcal{E}})}^{op} \ , \ g \mapsto \tilde{g}$$

is well defined and completely contractive. In fact, by our preliminary remarks, the identity from $\overline{\mathcal{G}}_r$ to $(\overline{\mathcal{G}}_c)^{op}$ is a complete isometry. By [32], Proposition 3.4.3, the idenity from $(\overline{\mathcal{G}}_c)^{op}$ to $\overline{\mathcal{G}_c}^{op}$ is completely isometric too. Putting this together, we have indeed a complete contraction

$$\overline{G}_r \overset{g \mapsto g}{\longrightarrow} (\overline{\mathcal{G}}_c)^{op} \overset{g \mapsto g}{\longrightarrow} \overline{\mathcal{G}_c}^{op} \overset{g \mapsto \tilde{g}}{\longrightarrow} \overline{\mathcal{M}(\mathcal{H}, \mathcal{H}_{\mathcal{E}})}^{op}.$$

Now for $\phi, \psi \in \mathcal{M}(\mathcal{H}, \mathcal{H}_{\mathcal{E}})$, let us write

$$G_{\phi,\psi} : X \times X \to B(\mathcal{E}) \ , \ (z,w) \mapsto \phi(z)\psi(w)^*.$$

Since the Haagerup tensor product is functorial ([32], Proposition 9.2.5), the mapping

$$\mathcal{G}_c \otimes \overline{\mathcal{G}}_r \to \mathcal{M}(\mathcal{H}, \mathcal{H}_{\mathcal{E}}) \otimes^h \overline{\mathcal{M}(\mathcal{H}, \mathcal{H}_{\mathcal{E}})}^{op} \ , \ g \otimes h \mapsto \tilde{g} \otimes \tilde{h}$$

extends completely contractively to $\mathcal{G}_c \otimes^h \overline{\mathcal{G}}_r$. As an immediate consequence of Lemma 2.1.7 and the definition of the Haagerup tensor product, the mapping

$$\mathcal{M}(\mathcal{H},\mathcal{H}_{\mathcal{E}}) \otimes \overline{\mathcal{M}(\mathcal{H},\mathcal{H}_{\mathcal{E}})}^{op} \to \mathscr{S}^{(0)}_{\mathcal{E}}(K)\ ,\ \phi\otimes\psi \mapsto G_{\phi,\psi}$$

extends completely contractively to $\mathcal{M}(\mathcal{H},\mathcal{H}_{\mathcal{E}}) \otimes^h \overline{\mathcal{M}(\mathcal{H},\mathcal{H}_{\mathcal{E}})}^{op}$. On the other hand, it is well known ([32], Proposition 9.3.4) that there exists a complete isometric isomorphism

$$\mathcal{G}_c \otimes^h \overline{\mathcal{G}}_r \to K(\mathcal{G}) \ \text{ with } \ g\otimes h \mapsto g\otimes h.$$

Letting $K(G)$ denote the closed subspace of $B(G)$ consisting of compactly represented kernels, the diagram

$$\begin{tikzcd}
K(G) \arrow[d, "g\odot h\mapsto g\otimes h"'] & & \mathscr{S}^{(0)}_{\mathcal{E}}(K) \\
K(\mathcal{G}) \arrow[r, "g\otimes h\mapsto g\otimes h"] & \mathcal{G}_c \otimes^h \overline{\mathcal{G}}_r \arrow[r, "g\otimes h\mapsto \tilde{g}\otimes \tilde{h}"] & \mathcal{M}(\mathcal{H},\mathcal{H}_{\mathcal{E}}) \otimes^h \overline{\mathcal{M}(\mathcal{H},\mathcal{H}_{\mathcal{E}})}^{op} \arrow[u, "\phi\otimes\psi\mapsto G_{\phi,\psi}"']
\end{tikzcd}$$

together with a short calculation shows that there is a (unique) complete contraction

$$j_0 : K(G) \to \mathscr{S}^{(0)}_{\mathcal{E}}(K)\ ,\ G \mapsto G.$$

Now fix a kernel $L \in B(G)$. Clearly, there exists a net of kernels $(L_\alpha)_\alpha$ in $K(G)$ satisfying $\|L_\alpha\|_G \le \|L\|_G$ and converging pointwise WOT to L. By the contractivity of j_0 and Proposition 2.3.2, we deduce that $L \in \mathscr{S}^{(0)}_{\mathcal{E}}(K)$ with $\|L\|_{\mathscr{S}^{(0)}_{\mathcal{E}}} \le \|L\|_G$. This shows that the inclusion mapping $j : B(G) \hookrightarrow \mathscr{S}^{(0)}_{\mathcal{E}}(K)$ is well defined and contractive. That j is actually completely contractive follows analogously: By the discussion of the operator space structures of $B(\mathcal{G})$ (p. 49) and $\mathscr{S}^{(0)}_{\mathcal{E}}(K)$ (p. 74), we have to show that for every $n \in \mathbb{N}$, the inclusion $B(G^{(n)}) \hookrightarrow \mathscr{S}^{(0)}_{\mathcal{E}^n}(K)$ is well defined and contractive. In order to repeat the arguments given above, it suffices to show that the mapping

$$i_n : (\mathcal{G}^{(n)})_c \to \mathcal{M}(\mathcal{H},\mathcal{H}_{\mathcal{E}^n})\ ,\ g \mapsto (z \mapsto g(z)_c)$$

is completely contractive. And in fact, this is clear since i_n is under the completely isometric identifications $(\mathcal{G}^{(n)})_c \simeq M_{n\times 1}(\mathcal{G}_c)$ and $\mathcal{M}(\mathcal{H},\mathcal{H}_{\mathcal{E}^n}) \simeq M_{n\times 1}(\mathcal{M}(\mathcal{H},\mathcal{H}_{\mathcal{E}}))$ the same as the inflation

$$i^{(n,1)} : M_{n,1}(\mathcal{G}_c) \to M_{n,1}(\mathcal{M}(\mathcal{H},\mathcal{H}_{\mathcal{E}}))\ ,\ [g_k] \mapsto [i(g_k)]$$

of i, which is clearly completely contractive. □

Example 2.4.2. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a non-degenerate reproducing kernel space with reproducing kernel K such that the minimal and the natural operator space structure on $\mathcal{M}(\mathcal{H})$ coincide. In this situation, the preceding result can be remarkably strengthened: Suppose that $G : X \times X \to \mathbb{C}$ is a positive kernel such that the associated reproducing kernel space $\mathcal{H}$ is included in $\mathcal{M}(\mathcal{H})$. Then, at first, the* *inclusion mapping* $i : \mathcal{G} \hookrightarrow \mathcal{M}(\mathcal{H})$ *is automatically continuous by the closed graph theorem. To prove this, note that the point evaluations on* $\mathcal{M}(\mathcal{H})$ *are bounded since* $\mathcal{H}$ *was supposed to be non-degenerate. Second, since* $\mathcal{M}(\mathcal{H})$ *is a minimal operator space, it follows by [32], (3.3.7) that* i *is completely bounded with* $\|i\|_{cb} = \|i\|$. *Hence the kernel* G *belongs to* $S(K)_+$.

It might be interesting that this could also be proved in some cases without the use of operator space theory. In fact, suppose that $\mathcal{H}$ *is a Bergman or Hardy space over some Cartan domain* $D \subset \mathbb{C}^d$. *(In theses cases, we know of course that* $\mathcal{M}(\mathcal{H}) = H^\infty(D)$ *holds completely isometrically, which means that* $\mathcal{M}(\mathcal{H})$ *is a minimal operator space). Let* $\gamma_z : \mathcal{G} \to \mathbb{C}$ $(z \in X)$ *denote the point evaluations of* $\mathcal{G}$. *We want to prove that* $G \in \mathscr{S}(K)_+$. *By Proposition* 2.2.1, *it suffices to show that the mapping*

$$\gamma : X \to B(\mathcal{G}, \mathbb{C}) \ , \ z \mapsto \gamma_z$$

belongs to $\mathcal{M}(\mathcal{H}_\mathcal{G}, \mathcal{H})$ *or, equivalently, that*

$$\|\gamma\|_\infty = \sup_{z \in D} \|\gamma(z)\| = \sup_{z \in D} \|\gamma_z\| < \infty.$$

According to the uniform boundedness principle, it suffices to show that

$$\|g\|_\infty = \sup_{z \in D} |g(z)| = \sup_{z \in D} \|\gamma_z(g)\| < \infty$$

holds for all $g \in \mathcal{G}$. *But this is obviously true since* $\mathcal{G} \subset \mathcal{M}(\mathcal{H})$ *and since multipliers are of course bounded functions.*

3 Beurling decomposable subspaces

3.1 Beurling spaces

For the rest of this paper, we will mainly be concerned with reproducing kernel spaces having certain additional properties, formulated in the following definition. We shall apply the theory developed in Chapter 2 to study invariant subspaces of these spaces. As before, X denotes an arbitrary non-empty set.

Definition 3.1.1.

(a) Let $\mathcal{H} \subset \mathbb{C}^X$ be a reproducing kernel Hilbert space with reproducing kernel K. We call $\mathcal{H}$ a Beurling space if the following conditions are satisfied:

(i) The reproducing kernel K has no zeroes.

(ii) The inverse kernel $\frac{1}{K}$ belongs to $\mathscr{S}^{(0)}(K)$.

(iii) For all $w \in X$, the function $K(\cdot, w)$ belongs to $\mathcal{M}(\mathcal{H})$.

(b) A positive definite kernel $K : X \times X \to \mathbb{C}$ is called a Beurling kernel if its associated reproducing kernel Hilbert space is a Beurling space.

We begin by listing some first consequences of this definition.

Remark 3.1.2.

(a) Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a Beurling space with reproducing kernel K. Since the constant kernel $\mathbf{1} = \frac{1}{K} \cdot K$ belongs to $B(K)$, Corollary 1.6.4 *implies that $\mathcal{H}$ contains the constant functions. In particular, $\mathcal{H}$ is non-degenerate. Furthermore, $\mathcal{M}(\mathcal{H})$ is included in $\mathcal{H}$ and moreover, a dense subset of $\mathcal{H}$ by condition (iii) of the definition. Observe that $\frac{1}{K}$ is a hermitian kernel. By Propositions* 2.3.12 *and* 2.2.1, *condition (ii) is equivalent to the existence of Hilbert spaces $\mathcal{B}$, $\mathcal{C}$ and multipliers $\beta \in \mathcal{M}(\mathcal{H}_{\mathcal{B}}, \mathcal{H})$, $\gamma \in \mathcal{M}(\mathcal{H}_{\mathcal{C}}, \mathcal{H})$ such that*

$$\frac{1}{K(z,w)} = \beta(z)\beta(w)^*(1) - \gamma(z)\gamma(w)^*(1) \tag{3.1.1}$$

holds for all $z, w \in X$. Finally, by Proposition 2.1.12, *also the functions $\frac{1}{K(\cdot,w)}$ belong to $\mathcal{M}(\mathcal{H})$ for all $w \in X$.*

(b) The class of Beurling spaces is closed under restrictions. That is, if $\mathcal{H} \subset \mathbb{C}^X$ is a Beurling space and Y is a non-empty subset of X, then the reproducing kernel Hilbert space $\mathcal{H}_{|Y}$ is a Beurling space as well. This follows from the fact that, given Hilbert spaces $\mathcal{E}_1, \mathcal{E}_2$, restrictions $\phi_{|Y}$ of multipliers $\phi \in \mathcal{M}(\mathcal{H}_{\mathcal{E}_1}, \mathcal{H}_{\mathcal{E}_2})$ belong to $\mathcal{M}((\mathcal{H}_{|Y})_{\mathcal{E}_1}, (\mathcal{H}_{|Y})_{\mathcal{E}_2})$ by Proposition 1.7.6.

(c) The class of Beurling kernels is stable with respect to the forming of outer products (cf. Section 1.2*). In order to prove this, suppose that Y_1, Y_2 are non-empty sets and that $K_1 : Y_1 \times Y_1 \to \mathbb{C}$ and $K_2 : Y_2 \times Y_2 \to \mathbb{C}$ are Beurling kernels with associated reproducing kernel spaces $\mathcal{H}_1, \mathcal{H}_2$. We write $Y = Y_1 \times Y_2$. Then the kernel $K_1 \circledast K_2$, as defined in Definition* 1.1.7*, has no zeroes. Next we claim that, for $G_i \in \mathscr{S}^{(0)}(K_i)$ $(i = 1, 2)$, the kernel $G_1 \circledast G_2$ belongs to $\mathscr{S}^{(0)}(K_1 \circledast K_2)$. By Proposition* 2.3.12*, we may assume that G_1, G_2 are positive kernels and that $\|G_i\|_{\mathscr{S}} = 1$ for $i = 1, 2$. Then the kernels $G_1 \circledast G_2$ and*

$$\begin{aligned}
&(K_1 \circledast K_2) - (G_1 \circledast G_2) \cdot (K_1 \circledast K_2) \\
&= \quad (K_1 \circledast K_2) - (G_1 \cdot K_1) \circledast (G_2 \cdot K_2) \\
&= \quad K_1 \circledast (K_2 - G_2 \cdot K_2) + (K_1 - G_1 \cdot K_1) \circledast (G_2 \cdot K_2)
\end{aligned}$$

are positive by Proposition 1.1.9 *and Remark* 1.6.2*(*f*). An application of Proposition* 2.2.1 *reveals that $G_1 \circledast G_2 \in \mathscr{S}^{(0)}(K)$. Therefore $\frac{1}{K_1 \circledast K_2} = \frac{1}{K_1} \circledast \frac{1}{K_2}$ belongs to $\mathscr{S}^{(0)}(K_1 \circledast K_2)$. In order to verify condition (iii), it suffices to show that, given multipliers $\phi_1 \in \mathcal{M}(\mathcal{H}_1)$ and $\phi_2 \in \mathcal{M}(\mathcal{H}_2)$, the function $\phi_1 \circledast \phi_2$ belongs to $\mathcal{M}(\mathcal{H}_1 \circledast \mathcal{H}_2)$. And in fact, this is easily checked by Proposition* 1.7.6.

(d) The class of Beurling kernels is also closed under pointwise (inner) products. This follows immediately by Remark 1.1.8 *and (*b*) and (*c*).*

We now provide some examples to illustrate the general character of the notion of Beurling spaces.

Example 3.1.3.

(a) Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is an NP space. By definition, the kernel $1 - \frac{1}{K}$ is positive and hence admits a Kolmogorov factorization $(\mathcal{C}, \gamma)$, that is,

$$1 - \frac{1}{K(z,w)} = \gamma(z)\gamma(w)^*(1)$$

or, equivalently,

$$\frac{1}{K(z,w)} = 1 - \gamma(z)\gamma(w)^*(1) \quad (z, w \in X). \tag{3.1.2}$$

Multiplication with K shows that

$$X \times X \to \mathbb{C}\ ,\ (z,w) \mapsto K(z,w)(1 - \gamma(z)\gamma(w)^*(1)) = 1$$

is a positive kernel. By Proposition 1.7.6*, γ belongs to $\mathcal{M}(\mathcal{H}_{\mathcal{C}}, \mathcal{H})$ and defines a contractive multiplier. Thus equation (*3.1.2*) proves that $\frac{1}{K} \in \mathscr{S}^{(0)}(K)$.*

It remains to show that the functions $K(\cdot, w)$ $(w \in X)$ are multpliers on $\mathcal{H}$. To this end, we first note that

$$\|\gamma(z)\|^2 = \|\gamma(z)\gamma(z)^*\| = 1 - \frac{1}{K(z,z)} < 1$$

for all $z \in X$. For fixed $w \in X$, the function

$$\gamma_w : X \to \mathbb{C} \ , \ \ z \mapsto \gamma(z)\gamma(w)^*(1)$$

is then a multplier with $\|\gamma_w\|_{\mathcal{M}} < 1$ by Lemma 1.7.13*. In particular, the series $\sum_{k=0}^{\infty} \gamma_w^k$ converges absolutely in $\mathcal{M}(\mathcal{H})$ and, since $\mathcal{M}(\mathcal{H})$ is a Banach space, defines a multiplier, which obviously coincides pointwise with $K(\cdot, w)$. Summing up, we have proved that $\mathcal{H}$ is a Beurling space.*

(b) Suppose that $D \subset \mathbb{C}^d$ is a domain and that $K : D \times D \to \mathbb{C}$ is a non-vanishing sesquianalytic positive kernel over D with the property that $\frac{1}{K}$ can be sesquianalytically extended to an open neighbourhood of $\overline{D} \times \overline{D}$. This is trivially fulfilled whenever $\frac{1}{K}$ is a polynomial in z and $\overline{w}$. Let $\mathcal{H} \subset \mathbb{C}^D$ denote the reproducing kernel space associated with K and suppose that the coordinate functions $\mathbf{z}_i$ $(1 \leq i \leq d)$ define multipliers of $\mathcal{H}$ and that $\sigma(M_{\mathbf{z}}) \subset \overline{D}$. Then by Example 2.3.4 *(*b*), $\frac{1}{K}$ belongs to $\mathscr{S}^{(0)}(K)$. Note that in this situation, condition (iii) of Definition* 3.1.1 *is not automatically fulfilled. Example* 1.7.3 *(*c*) suggests that this condition depends in an intimate way on symmetry properties of D and K.*

(c) Let D be a Cartan domain in $\mathbb{C}^d$ and let r, a, b, g denote the rank, the characteristic multiplicities and the genus of D. Fix ν in the continuous Wallach set of D and write $K = K_\nu$ and $\mathcal{H} = \mathcal{H}_\nu$. By Example 1.1.10*, we know that K has no zeroes on D. In Example* 1.7.3 *(*c*), we observed that the functions $K(\cdot, w)$ belong to $\mathcal{M}(\mathcal{H})$ for all $w \in D$. Therefore conditions (i) and (iii) of Definition* 3.1.1 *are fulfilled. It remains to verify that $\frac{1}{K}$ belongs to $\mathscr{S}^{(0)}(K)$. If $\frac{1}{K}$ is a polynomial in z and $\overline{w}$, then this is trivially true by Example* 2.3.4*. But $\frac{1}{K}$ is a polynomial in z and $\overline{w}$ exactly if ν is an integer. In fact, considering the Faraut-Koranyi expansion (*1.1.3*) reveals that*

$$\frac{1}{K(z,w)} = \sum_{\mathbf{m}} (-\nu)_{\mathbf{m}} K_{\mathbf{m}}(z,w)$$

*holds for all $z, w \in D$. Since the kernels $K_{\mathbf{m}}$ are homogeneous polynomials of degree $2|m|$ in z and $\overline{w}$, we conclude that $\frac{1}{K}$ is a polynomial in z and $\overline{w}$ precisely if the coefficients $(-\nu)_{\mathbf{m}}$ vanish for almost all $\mathbf{m}$. A closer look at the definition of the Pochhammer symbols (*1.1.4*) shows that, given $\lambda \in \mathbb{C}$, the following are equivalent:*

(i) $(\lambda)_{\mathbf{m}} = 0$ for almost all $\mathbf{m}$.

(ii) $\lambda \in -\mathbb{N}_0$.

Hence $(-\nu)_{\mathbf{m}} = 0$ *for almost all signatures* $\mathbf{m}$ *if and only if* ν *is an integer.*

If ν *is not an integer, then the situation is more complicated. It is shown in [33], Theorem 6 that for* $\nu \geq \frac{r-1}{2}a + 1$, *the inverse kernel* $\frac{1}{K}$ *possesses a representation as in (*3.1.1*) and therefore belongs to* $\mathscr{S}^{(0)}(K)$. *Unfortunately it is far from clear what happens for* $\nu \in (\frac{r-1}{2}a, \frac{r-1}{2}a+1)$. *Nevertheless, for* $\nu \geq \frac{r-1}{2}a + 1$, *we have shown that* $\mathcal{H} = \mathcal{H}_\nu$ *is a Beurling space.*

(d) It should be noted that the Segal-Bargmann space $\mathcal{H}$, *considered in Example* 2.1.11, *is not a Beurling space. In fact, its reproducing kernel is given by*

$$K : \mathbb{C} \times \mathbb{C} \to \mathbb{C}\ ,\ K(z,w) = e^{z\overline{w}},$$

and we saw that in this case, $\mathcal{M}(\mathcal{H})$ *as well as* $\mathscr{S}(K)$ *are trivial (that is, consist only of constant functions). Hence neither condition (ii) nor condition (iii) of Definition* 3.1.1 *can be fulfilled. However, the Segal-Bargmann space is not an interesting example for the study of invariant subspaces since every closed subspace is invariant under every multiplier.*

3.2 Invariant subspaces

Before we proceed in our studies, we want to provide a precise definition of the expression 'invariant subspace'.

Definition 3.2.1. *Let* $\mathcal{H} \subset \mathbb{C}^X$ *be a reproducing kernel space, let* $\mathcal{E}$ *be a Hilbert space and let* Y *be a subset of* $\mathcal{M}(\mathcal{H})$.

(a) We say that a closed subspace M *of* $\mathcal{H}_\mathcal{E}$ *is* Y*-invariant if*

$$(M_\alpha \otimes 1_\mathcal{E})M \subset M$$

holds for every multiplier $\alpha \in Y$. *We simply call* M *invariant if it is* $\mathcal{M}(\mathcal{H})$*-invariant in the above sense.*

(b) Let N *be a subset of* $\mathcal{H}_\mathcal{E}$. *We define* $[N]_Y$ *as the smallest* Y*-invariant subspace containing* N. *We write* $[N]$ *for the smallest invariant subspace which contains* N.

Remark 3.2.2. *Let* $\mathcal{H} \subset \mathbb{C}^X$ *be a reproducing kernel space, let* $\mathcal{E}$ *be a Hilbert space and let* Y *be a subset of* $\mathcal{M}(\mathcal{H})$. *If* $\mathcal{A} \subset \mathcal{M}(\mathcal{H})$ *denotes the unital subalgebra generated by* Y *and if* $N \subset \mathcal{H}_\mathcal{E}$ *is an arbitrary subset, then of course*

$$[N]_Y = \bigvee\{\alpha \cdot f\ ;\ \alpha \in \mathcal{A}\ ,\ f \in N\}$$

and, in particular,

$$[N] = \bigvee\{\alpha \cdot f\ ;\ \alpha \in \mathcal{M}(\mathcal{H}),\ f \in N\}.$$

Suppose that $D \subset \mathbb{C}^d$ is an open set and that $\mathcal{H} \subset \mathcal{O}(D)$ is a non-degenerate reproducing kernel Hilbert space with the property that the coordinate functions $\mathbf{z}_i$ $(1 \le i \le d)$ (and hence all polynomials) are multipliers of $\mathcal{H}$. Suppose furthermore that $\mathcal{E}$ is a Hilbert space. Then the inflation $\mathcal{H}_\mathcal{E}$ is a $\mathbb{C}[z]$-module via the polynomial functional calculus of the multiplication tuple $(M_{\mathbf{z}_1} \otimes 1_\mathcal{E}, \ldots, M_{\mathbf{z}_d} \otimes 1_\mathcal{E})$. In this case, the closed $\mathbb{C}[z]$-submodules of $\mathcal{H}_\mathcal{E}$ are precisely the closed linear subspaces of $\mathcal{H}_\mathcal{E}$ that are $\mathbb{C}[z]$-invariant in the sense of Definition 3.2.1. In general, such a $\mathbb{C}[z]$-submodule is not necessarily invariant (that is, invariant under all multipliers). However, if the polynomials are weak-$*$ dense in $\mathcal{M}(\mathcal{H})$, then every $\mathbb{C}[z]$-invariant closed linear subspcae of $\mathcal{H}_\mathcal{E}$ is even invariant. This is obvious in the scalar-valued case $\mathcal{E} = \mathbb{C}$. In the vector-valued case, it suffices to observe that the linear map $B(\mathcal{H}) \to B(\mathcal{H}_\mathcal{E})$, $A \mapsto A \otimes 1_\mathcal{E}$ is weak-$*$ continuous, since its restriction to the closed unit ball of $B(\mathcal{H})$ is obviously continuous with respect to the relative weak-$*$ topology.

We now exhibit some cases in which the polynomials are in fact weak-$*$ dense in the multiplier algebra.

Example 3.2.3. *Let $D \subset \mathbb{C}^d$ a circular and convex domain containing the origin. Suppose further that $\mathcal{H} \subset \mathcal{O}(D)$ is a reproducing kernel Hilbert space with the following properties:*

- *The coordinate functions (and hence all polynomials) are multipliers of $\mathcal{H}$.*
- *$\mathcal{H}$ contains the constant functions.*
- *The reproducing kernel K of $\mathcal{H}$ respects the circular symmetry of D, that is,*

$$K(e^{it}z, e^{it}w) = K(z,w)$$

holds for all $t \in \mathbb{R}$ and $z, w \in D$.

For a given function $u : D \to \mathbb{C}$ and $t \in \mathbb{R}$, let us define

$$u_t : D \to \mathbb{C} \ , \ u_t(z) = u(e^{it}z).$$

Then Propositions 1.1.11 *and* 1.7.6 *imply that for $f \in \mathcal{H}$ and $\phi \in \mathcal{M}(\mathcal{H})$, the functions f_t and ϕ_t belong to $\mathcal{H}$ and $\mathcal{M}(\mathcal{H})$, respectively, and moreover, that*

$$\|f_t\| = \|f\| \quad \text{and} \quad \|\phi_t\|_\mathcal{M} = \|\phi\|_\mathcal{M}.$$

Consequently, the mappings

$$U_t : \mathcal{H} \to \mathcal{H} \ , \ f \mapsto f_t \quad \text{and} \quad V_t : \mathcal{M}(\mathcal{H}) \to \mathcal{M}(\mathcal{H}) \ , \ \phi \mapsto \phi_t$$

are isometric isomorphisms. Furthermore, the unitary one-parameter group

$$U : \mathbb{R} \to B(\mathcal{H}) \ , \ U(t) = U_t$$

is obviously weakly continuous. By [56], Chapter 2, Theorem 1.4, we conclude that U *is strongly continuous, that is,* $\lim_{t\to 0} U_t f = f$ *holds for all* $f \in \mathcal{H}$. *Now let us denote by*

$$F_N : [-\pi,\pi] \to [0,\infty)\ ,\ F_N(t) = \frac{1}{N}\sum_{n=0}^{N-1} D_n(t) = \frac{1}{N}\frac{\left(\sin\frac{Nt}{2}\right)^2}{\left(\sin\frac{t}{2}\right)^2} \quad (N \geq 1)$$

the Fejér kernel, where

$$D_n : [-\pi,\pi] \to \mathbb{C}\ ,\ D_n(t) = \sum_{\nu=-n}^{n} e^{i\nu t} \quad (n \geq 0)$$

is the Dirichlet kernel. We claim that for every function $u \in \mathcal{O}(D)$ *and* $N \geq 1$, *the function*

$$u_N : D \to \mathbb{C}\ ,\ u_N(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it}z)F_N(t)\ dt$$

is a polynomial of degree at most $N-1$ *and that the sequence* $(u_N)_N$ *converges pointwise on* D *to* u. *In fact, every* u_N *is a holomorphic function on* D *by standard arguments, and*

$$\begin{aligned} u_N^{(\alpha)}(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} u^{(\alpha)}(e^{it}z)e^{i|\alpha|t}F_N(t)\ dt \\ &= \frac{1}{2\pi N}\sum_{n=0}^{N-1}\sum_{\nu=-n}^{n}\int_{-\pi}^{\pi} u^{(\alpha)}(e^{it}z)e^{i(|\alpha|+\nu)t}\ dt \end{aligned}$$

holds for all $z \in D$ *and all multiindices* $\alpha \in \mathbb{N}_0^d$. *By Cauchy's theorem, the integrals are all zero if* $|\alpha| \geq N$, *which means that* $u_N^{(\alpha)} \equiv 0$ *if* $|\alpha| \geq N$. *By the identity theorem,* u_N *is a polynomial. To prove that* $(u_N)_N$ *converges pointwise to* u, *we fix* $z \in D$ *and* $\epsilon > 0$. *Then, by the continuity of the mapping* $[-\pi,\pi] \to \mathbb{C}\ ,\ t \mapsto u(e^{it}z)$, *there exists* $\delta > 0$ *such that* $|u(e^{it}z) - u(z)| < \frac{\epsilon}{2}$ *holds for all* $|t| < \delta$. *Furthermore, we define* $M = \sup\{|u(e^{it}z) - u(z)|\ ;\ t \in [-\pi,\pi]\}$. *Then, using the well-known properties*

$$\int_{-\pi}^{\pi} F_N(t)\ dt = 2\pi \quad (N \geq 1)$$

and

$$\int_{t_0 \leq |t| \leq \pi} F_N(t)\ dt \leq \frac{2\pi}{N\left(\sin\frac{t_0}{2}\right)^2} \quad (N \geq 1, t_0 > 0)$$

of the Fejér kernel (cf. [47], pp. 16-17), we obtain that

$$\begin{aligned} |u_N(z) - u(z)| &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(e^{it}z) - u(z)|F_N(t)\ dt \\ &= \frac{1}{2\pi}\int_{|t|\leq\delta} |u(e^{it}z) - u(z)|F_N(t)\ dt \\ &\qquad + \frac{1}{2\pi}\int_{\delta\leq|t|\leq\pi} |u(e^{it}z) - u(z)|F_N(t)\ dt \\ &\leq \frac{\epsilon}{2} + \frac{M}{2\pi}\int_{\delta\leq|t|\leq\pi} F_N(t)\ dt \\ &\leq \frac{\epsilon}{2} + \frac{M}{N\left(\sin\frac{\delta}{2}\right)^2} \end{aligned}$$

for all $N \geq 1$. Certainly, the second term is less than $\frac{\epsilon}{2}$ for large N, which proves the claim.

Now let us fix $\phi \in \mathcal{M}(\mathcal{H})$. Then the functions ϕ_N are polynomials (in particular multipliers). We claim that $\|\phi_N\|_{\mathcal{M}} \leq \|\phi\|_{\mathcal{M}}$ holds for all $N \geq 1$. In fact, for $f \in \mathcal{H}$, we may define

$$g = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\phi_t \cdot f) F_N(t)\ dt \in \mathcal{H}.$$

Note that this makes sense because the function

$$\mathbb{R} \to \mathcal{H}\ ,\ t \mapsto \phi_t \cdot f$$

is continuous. This is the case since

$$\phi_t \cdot f = U_t M_\phi U_{-t} f$$

holds for all $t \in \mathbb{R}$ and since multiplication on $B(\mathcal{H})$ is SOT continuous on bounded sets. The continuity of the point evaluations implies

$$g(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_t(z) f(z) F_N(t)\ dt = \phi_N(z) f(z)$$

for all $z \in D$, and therefore

$$\begin{aligned} \|\phi_N \cdot f\| = \|g\| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|\phi_t \cdot f\| F_N(t)\ dt \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \|\phi_t\|_{\mathcal{M}} \|f\|\ F_N(t)\ dt \\ &= \|\phi\|_{\mathcal{M}} \|f\|, \end{aligned}$$

proving that $\|\phi_N\|_{\mathcal{M}} \leq \|\phi\|_{\mathcal{M}}$. By Proposition 1.7.11, the sequence $(\phi_N)_N$ converges to ϕ in the weak-$*$ topology of $\mathcal{M}(\mathcal{H})$, as desired.

This example shows in particular that for the standard reproducing kernel spaces over Cartan domains, there is no need to distinguish between $\mathbb{C}[z]$-submodules and invariant subspaces.

We conclude this section with a technical lemma which demonstrates how invariant subspaces interact with inflations.

Lemma 3.2.4. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a reproducing kernel space and that $\mathcal{E}, \mathcal{F}_1, \mathcal{F}_2$ are Hilbert spaces. Suppose further that $M \subset \mathcal{H}_{\mathcal{E}}$ is an invariant subspace and that $\phi \in \mathcal{M}(\mathcal{H}_{\mathcal{F}_1}, \mathcal{H}_{\mathcal{F}_2})$ is a multiplier. Then the multiplier*

$$\tilde{\phi} : X \to B(\mathcal{E} \otimes \mathcal{F}_1, \mathcal{E} \otimes \mathcal{F}_2)\ ,\ \tilde{\phi}(z) = 1_{\mathcal{E}} \otimes \phi(z)$$

satisfies $M_{\tilde{\phi}} M_{\mathcal{F}_1} \subset M_{\mathcal{F}_2}$.

Proof. First, it follows by Proposition 1.7.6 that $\tilde{\phi}$ belongs to $\mathcal{M}(\mathcal{H}_{\mathcal{E}\otimes\mathcal{F}_1}, \mathcal{H}_{\mathcal{E}\otimes\mathcal{F}_2})$. Next, we fix $f \in M$ and $x \in \mathcal{F}_1$. We claim that

$$\phi_{x,y} \cdot f = \left(\tilde{\phi} \cdot (f \otimes x)\right)_y$$

holds for all $y \in \mathcal{F}_2$, where for a function $g : X \to \mathcal{E} \otimes \mathcal{F}$, g_y is the slice function as defined in Proposition 1.2.2, and $\phi_{x,y}$ denotes the scalar multiplier

$$\phi_{x,y} : X \to \mathbb{C} \ , \ \phi_{x,y}(z) = \langle \phi(z)x, y\rangle.$$

In fact, for $u \in \mathcal{E}$ and $z \in X$, we have

$$\begin{aligned} \langle \left(\tilde{\phi} \cdot (f \otimes x)\right)_y (z), u\rangle &= \langle \tilde{\phi}(z)(f(z) \otimes x), u \otimes y\rangle \\ &= \langle f(z) \otimes \phi(z)x, u \otimes y\rangle \\ &= \langle f(z), u\rangle\langle \phi(z)x, y\rangle \\ &= \langle\langle \phi(z)x, y\rangle f(z), u\rangle \\ &= \langle \phi_{x,y}(z)f(z), u\rangle, \end{aligned}$$

which proves the claim. Since M is invariant, we obtain that

$$\left(\tilde{\phi} \cdot (f \otimes x)\right)_y = \phi_{x,y} \cdot f$$

belongs to M for all $y \in \mathcal{F}_2$. Proposition 1.2.2 now shows that $\tilde{\phi} \cdot (f \otimes x)$ belongs to $M_{\mathcal{F}_2}$. Since the functions $f \otimes x$ $(f \in M, x \in \mathcal{F}_1)$ form a total subset of $M_{\mathcal{F}_1}$, the proof is complete. □

Remark 3.2.5. *We have actually proved a stronger result. Namely, it suffices to require that M is invariant under the multiplication with the scalar multipliers*

$$\phi_{x,y} : X \to \mathbb{C} \ , \ \phi_{x,y}(z) = \langle \phi(z)x, y\rangle,$$

where $y \in \mathcal{F}_2$ is arbitrary and x varies over a total subset of $\mathcal{F}_1$.

3.3 Beurling decomposable subspaces

The main aim of this section will be to obtain natural generalizations of Beurling's invariant subspace theorem.

Definition 3.3.1. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a reproducing kernel Hilbert space and that $\mathcal{E}$ is a Hilbert space. Then a closed subspace M of $\mathcal{H}_\mathcal{E}$ is called Beurling decomposable if there exist Hilbert spaces $\mathcal{D}_1, \mathcal{D}_2$ and multipliers $\phi_1 \in \mathcal{M}(\mathcal{H}_{\mathcal{D}_1}, \mathcal{H}_\mathcal{E})$, $\phi_2 \in \mathcal{M}(\mathcal{H}_{\mathcal{D}_2}, \mathcal{H}_\mathcal{E})$ such that $M = \operatorname{ran} M_{\phi_1}$ and*

$$P_M = M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^*. \tag{3.3.1}$$

In this case, the tuple (ϕ_1, ϕ_2) is called a Beurling decomposition of M.

Every Beurling decomposable subspace M of $\mathcal{H}_\mathcal{E}$ is obviously invariant. In fact, if (ϕ_1, ϕ_2) is a Beurling decomposition of M, then we have

$$\begin{aligned}(M_\alpha \otimes 1_\mathcal{E})M &= (M_\alpha \otimes 1_\mathcal{E})M_{\phi_1}(\mathcal{H} \otimes \mathcal{D}_1) \\ &= M_{\phi_1}(M_\alpha \otimes 1_{\mathcal{D}_1})(\mathcal{H} \otimes \mathcal{D}_1) \subset M_{\phi_1}(\mathcal{H} \otimes \mathcal{D}_1) = M\end{aligned}$$

for all $\alpha \in \mathcal{M}(\mathcal{H})$.

The aim of this section is to relate the Beurling decomposability of a subspace $M \subset \mathcal{H}_\mathcal{E}$ to the so-called core function and the core operator of M. For related results on these objects, see [40],[41],[76].

Definition 3.3.2. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a Beurling space with reproducing kernel K. Let $\mathcal{E}$ be a Hilbert space.*

(a) For every closed subspace M of $\mathcal{H}_\mathcal{E}$, we define

$$K_M = \Lambda_{P_M} \in B(K_\mathcal{E}).$$

(b) Let M be a closed subspace of $\mathcal{H}_\mathcal{E}$. Then the hermitian kernel

$$G_M = \frac{K_M}{K} : X \times X \to B(\mathcal{E})$$

is called the core function of M. The unique self-adjoint operator $\Delta_M \in B(\mathcal{H}_\mathcal{E})$ with $\Lambda_{\Delta_M} = G_M$ is called the core operator of M. The rank of M is defined as the rank of Δ_M. The positive (negative) rank of M is defined as

$$\operatorname{rank}_\pm M = \operatorname{rank}(\Delta_M)_\pm.$$

The function

$$D_M : X \to B(\mathcal{E}) \ , \ D_M(z) = G_M(z, z)$$

is called the defect function of M.

The definition requires perhaps some explanation. First, the kernel K_M is simply the reproducing kernel of M, as the calculation

$$\begin{aligned}\langle f, K_M(\cdot, z)x\rangle_M &= \langle f, P_M K(\cdot, z)x\rangle_{\mathcal{H}_\mathcal{E}} \\ &= \langle f, K(\cdot, z)x\rangle_{\mathcal{H}_\mathcal{E}} = \langle f(z), x\rangle_\mathcal{E} \quad (f \in M, z \in X, x \in \mathcal{E})\end{aligned}$$

shows.

Secondly, the core function is well defined since K was in particular supposed to have no zeroes. By Proposition 2.3.3, it follows that G_M in fact belongs to $B(K_\mathcal{E})$, which ensures the existence of the core operator. Furthermore, we mention that G_M (or D_M) is also known as the Berezin transform of P_M in the literature.

Recalling Proposition 1.6.8, we observe that the definition of the (positive, negative) rank of M relates well to the definition of the (positive, negative) rank of G_M we gave in Definition 1.5.4, that is,

$$\operatorname{rank}_\pm M = \operatorname{rank}(\Delta_M)_\pm = \operatorname{rank}(G_M)_\pm = \operatorname{rank}_\pm G_M$$

and

$$\operatorname{rank} M = \operatorname{rank}_+ M + \operatorname{rank}_- M.$$

In some situations, the core operator admits a very concrete realization in terms of the inverse kernel.

Example 3.3.3.

(a) *Suppose that $D \subset \mathbb{C}^d$ is an open set and that $\mathcal{H} \subset \mathcal{O}(D)$ is a Beurling space with reproducing kernel K. In addition, we suppose that $\frac{1}{K}$ is a polynomial in z and $\overline{w}$, say*

$$\frac{1}{K(z,w)} = \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha \overline{w}^\beta,$$

and that the coordinate functions $\mathbf{z}_i$ $(1 \leq i \leq d)$ are multipliers of $\mathcal{H}$. Let $\mathcal{E}$ be a Hilbert space and let M be a closed subspace of $\mathcal{H}_\mathcal{E}$. Then it is easy to check that the core operator of M is given by the formula

$$\Delta_M = \sum_{\alpha,\beta} c_{\alpha,\beta} (M_\mathbf{z} \otimes 1_\mathcal{E})^\alpha P_M (M_\mathbf{z}^* \otimes 1_\mathcal{E})^\beta. \tag{3.3.2}$$

By the definition of the core function, it is furthermore obvious that the identity $G_M + G_{M^\perp} = \mathbf{1}$ holds. Equivalently, if Q_0 denotes the orthogonal projection of $\mathcal{H}$ onto the one-dimensional subspace consisting of all constant functions in $\mathcal{H}$, then

$$\Delta_M + \Delta_{M^\perp} = \|\mathbf{1}\|^2_\mathcal{H} (Q_0 \otimes 1_\mathcal{E}).$$

This observation and (3.3.2) show that, if $\mathcal{E}$ and at least one of the spaces $M, M^\perp$ are finite-dimensional, then both Δ_M and $\Delta_{M^\perp}$ have finite rank. In particular, when $\dim \mathcal{E} < \infty$, then every finite-codimensional subspace of $\mathcal{H}_\mathcal{E}$ has automatically finite rank.

Formula (3.3.2) reveals two further properties of the core operator in the case that $M \subset \mathcal{H}_\mathcal{E}$ is invariant under multiplication with the coordinate functions. First, Δ_M vanishes on $M^\perp$ or, equivalently, $\operatorname{ran} \Delta_M \subset M$. Secondly, if we suppose in addition that D contains the origin and that $K(\cdot, 0) \equiv \mathbf{1}$, then the space $M \ominus \sum_{i=1}^d \mathbf{z}_i \cdot M$ consists of eigenvectors of Δ_M with respect to the eigenvalue 1. In fact, choose $f \in M \ominus \sum_{i=1}^d \mathbf{z}_i \cdot M$. Since

$$M \ominus \sum_{i=1}^d \mathbf{z}_i \cdot M = M \cap \bigcap_{i=1}^d (\mathbf{z}_i \cdot M)^\perp,$$

it follows that $(M_{\mathbf{z}}^* \otimes 1_{\mathcal{E}})^\beta f \in M^\perp$ *for all* $\beta \neq 0$*. Now the assumption that* $K(\cdot, 0) \equiv \mathbf{1}$ *implies that*

$$\frac{1}{K(z,w)} = 1 + \sum_{\alpha,\beta \neq 0} c_{\alpha,\beta} z^\alpha \overline{w}^\beta$$

holds for all $z, w \in D$*. Hence*

$$\Delta_M = P_M + \sum_{\alpha,\beta \neq 0} c_{\alpha,\beta} (M_{\mathbf{z}} \otimes 1_{\mathcal{E}})^\alpha P_M (M_{\mathbf{z}}^* \otimes 1_{\mathcal{E}})^\beta$$

which implies $\Delta_M f = f$*. Furthermore, the space* $M \ominus \sum_{i=1}^d \mathbf{z}_i \cdot M$ *is never trivial (if* $M \neq \{0\}$*, of course). In fact, if we define for* $N \subset \mathcal{H}_{\mathcal{E}}$*,*

$$\operatorname{ord}(N) = \min\{\operatorname{ord}_0(f) \ ; \ f \in N\}$$

(where $\operatorname{ord}_0(f) = \inf\{|\alpha| \ ; \ D^\alpha f(0) \neq 0\}$*), then*

$$\operatorname{ord}\left(\overline{\sum_{i=1}^d \mathbf{z}_i \cdot M}\right) = 1 + \operatorname{ord}(M)$$

because convergence in $\mathcal{H}_{\mathcal{E}}$ *implies uniform convergence on compact subsets of* D*. This implies that* $\overline{\sum_{i=1}^d \mathbf{z}_i \cdot M} \neq M$*. Summing up, we have shown that in this situation,* 1 *is an eigenvalue of* Δ_M *and that, in particular,* $\|\Delta_M\| \geq 1$*.*

(b) *Suppose now that* D *is a Cartan domain with rank* r*, characteristic multiplicities* a, b *and genus* g*. Fix some* $\nu \geq \frac{r-1}{2}a+1$ *and let* $\mathcal{E}$ *be a Hilbert space. To simplify the notation, we write* $\mathcal{H} = \mathcal{H}_\nu$ *and* $K = K_\nu$*. We claim that, for every closed subspace* M *of* $\mathcal{H}_{\mathcal{E}}$*, the identity*

$$\Delta_M = \sum_{\mathbf{m}} (-\nu)_{\mathbf{m}} C_{\mathbf{m}}(L_{M_{\mathbf{z}} \otimes 1_{\mathcal{E}}}, R_{M_{\mathbf{z}}^* \otimes 1_{\mathcal{E}}})(P_M), \qquad (3.3.3)$$

holds, where (cf. Example 1.1.10*)*

$$C_{\mathbf{m}} : D \times D \to \mathbb{C} \ , \ C_{\mathbf{m}}(z,w) = K_{\mathbf{m}}(z, \overline{w})$$

and $L_{M_{\mathbf{z}} \otimes 1_{\mathcal{E}}}$ *and* $R_{M_{\mathbf{z}}^* \otimes 1_{\mathcal{E}}}$ *denote the tuples of left and right multiplications with the operators* $M_{\mathbf{z}_i} \otimes 1_{\mathcal{E}}$ *and* $M_{\mathbf{z}_i}^* \otimes 1_{\mathcal{E}}$*. To see that the series in (*3.3.3*) is well defined, note first that the functions* $C_{\mathbf{m}}$ *are polynomials. Since the kernels* $K_{\mathbf{m}}$ *are positive definite, we infer that (cf. Example* 1.4.8*)*

$$0 \leq C_{\mathbf{m}}(L_{M_{\mathbf{z}} \otimes 1_{\mathcal{E}}}, R_{M_{\mathbf{z}}^* \otimes 1_{\mathcal{E}}})(P_M) \leq C_{\mathbf{m}}(L_{M_{\mathbf{z}} \otimes 1_{\mathcal{E}}}, R_{M_{\mathbf{z}}^* \otimes 1_{\mathcal{E}}})(1_{\mathcal{H}_{\mathcal{E}}}).$$

It follows from [33], Theorem 1, that the series

$$\sum_{\mathbf{m}} |(-\nu)_{\mathbf{m}}| \|C_{\mathbf{m}}(L_{M_{\mathbf{z}} \otimes 1_{\mathcal{E}}}, R_{M_{\mathbf{z}}^* \otimes 1_{\mathcal{E}}})(1_{\mathcal{H}_{\mathcal{E}}})\|$$

*converges, which implies the (absolute) convergence of the series in (*3.3.3*). This also shows that* Δ_M *is compact whenever* M *is finite-dimensional. Similarly to (a), the identity*

$$\Delta_M + \Delta_{M^\perp} = \|\mathbf{1}\|_{\mathcal{H}}^2 (Q_0 \otimes 1_{\mathcal{E}})$$

proves that both Δ_M and $\Delta_{M^\perp}$ are compact if $\mathcal{E}$ has finite dimension and M is finite codimensional.

Now suppose that $\{0\} \neq M \subset \mathcal{H}_{\mathcal{E}}$ is a closed subspace which is invariant under multiplication by the coordinate functions. Then, analogously to (a), one shows that the space $M \ominus \sum_{i=1}^d \mathbf{z}_i \cdot M$ is non-zero and consists of eigenvectors of Δ_M with respect to the eigenvalue 1.

(c) *Consider again a Cartan domain $D \subset \mathbb{C}^d$ of rank r and characteristic multiplicities a, b. Fix some $\nu \geq \frac{r-1}{2}a+1$ and write $\mathcal{H} = \mathcal{H}_\nu$ and $K = K_\nu$. Consider the invariant subspace*

$$M = \{f \in \mathcal{H} \ ; \ f(0) = 0\} = \{\mathbf{1}\}^\perp.$$

Then it is easily verified that

$$G_M = 1 - \frac{1}{K} = \sum_{\mathbf{m} \neq 0} -(-\nu)_{\mathbf{m}} K_{\mathbf{m}},$$

using the Faraut-Koranyi expansion (1.1.3). Consequently, if $\mathfrak{K}$ denotes the stablizer of the origin in the identity component of $\mathrm{Aut}(G)$, then it follows that G_M is $\mathfrak{K}$-invariant (this means that $G_M(kz, kw) = G_M(z, w)$ holds for all $z, w \in D$ and $k \in \mathfrak{K}$). Equivalently, the core operator Δ_M is invariant under the action of $\mathfrak{K}$ on $\mathcal{H}$, that is,

$$\Delta_M(f \circ k) = (\Delta_M f) \circ k$$

holds for all $f \in \mathcal{H}$ and $k \in \mathfrak{K}$. As an immediate consequence of Schur's Lemma, Δ_M is diagonal with respect to the Peter-Weyl decomposition $\mathcal{H} = \bigoplus_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}$ of $\mathcal{H}$, which means that there exist real numbers $\lambda_{\mathbf{m}}$ such that $\Delta_{M|\mathcal{P}_{\mathbf{m}}} = \lambda_{\mathbf{m}} \cdot 1_{\mathcal{P}_{\mathbf{m}}}$. Using formula 1.1.5, one obtains that

$$\begin{aligned} \frac{-(-\nu)_{\mathbf{m}}}{(\nu)_{\mathbf{m}}} K_{\mathbf{m}}(z, w) &= \langle -(-\nu)_{\mathbf{m}} K_{\mathbf{m}}(\cdot, w), K_{\mathbf{m}}(\cdot, z)\rangle \\ &= \langle G_M(\cdot, w), K_{\mathbf{m}}(\cdot, z)\rangle \\ &= \langle \Delta_M K(\cdot, w), K_{\mathbf{m}}(\cdot, z)\rangle \\ &= \lambda_{\mathbf{m}} K_{\mathbf{m}}(z, w) \end{aligned}$$

for all $z, w \in D$ and $\mathbf{m} \neq 0$. Moreover a short calculation reveals that $\lambda_0 = 0$, and therefore

$$\lambda_{\mathbf{m}} = \begin{cases} -\frac{(-\nu)_{\mathbf{m}}}{(\nu)_{\mathbf{m}}} & ; \ \mathbf{m} \neq 0 \\ 0 & ; \ \mathbf{m} = 0 \end{cases}.$$

Clearly, Δ_M has finite rank if and only if almost all $\lambda_{\mathbf{m}}$ are 0. As seen in Example 3.1.3 (c), this is the case if and only if ν is an integer. If ν is not an integer, then Δ_M is still compact by (b), since M is finite codimensional. In this situation, the compactness of Δ_M could also be proved directly. In fact, for

$x, y \in \mathbb{R}$, it is known that

$$\frac{(x)_{\mathbf{m}}}{(y)_{\mathbf{m}}} \approx \prod_{j=1}^{r} (m_j + 1)^{x-y}$$

asymptotically as $|\mathbf{m}| \to \infty$. This can be proved by use of the Gindikin Gamma function and Stirling's formula (see [9], p. 229 for details). Hence

$$\lim_{|\mathbf{m}| \to \infty} \lambda_{\mathbf{m}} = 0,$$

proving that Δ_M is compact. Furthermore, the above calculations reveal that

$$\sigma(\Delta_M) = \{\lambda_{\mathbf{m}} \ ; \ \mathbf{m} \text{ is a signature of length } r\}.$$

This shows that for non-integer ν, there are of course finite-codimensional invariant subspaces of $\mathcal{H}$ having infinite rank.

Before we deduce a first characterization of Beurling decomposable subspaces, we need one further definition.

Definition 3.3.4. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a Beurling space with reproducing kernel K and that $\mathcal{E}$ is a Hilbert space. A closed subspace M of $\mathcal{H}_{\mathcal{E}}$ is called K-invariant if it is $\{K(\cdot, w) \ ; \ w \in X\}$-invariant and $\{\frac{1}{K(\cdot,w)} \ ; \ w \in X\}$-invariant.*

Note that this definition makes sense, since in Beurling spaces the functions $K(\cdot, w)$ as well as the functions $\frac{1}{K(\cdot,w)}$ $(w \in X)$ are multipliers.

The above notion of K-invariant subspaces originates in the problem discussed after Remark 3.2.2. Namely, if D is open in $\mathbb{C}^d$ and if $\mathcal{H} \subset \mathcal{O}(D)$ is a reproducing kernel Hilbert space which is a $\mathbb{C}[z]$-module at the same time, then it is not clear that every $\mathbb{C}[z]$-submodule is automatically invariant (although the considerations in Example 3.2.3 ensure that the spaces standing in the centre of our attention behave well in this sense). However, in many situations it is true that $\mathcal{O}(\overline{D}) \subset \mathcal{M}(\mathcal{H})$ and that every $\mathbb{C}[z]$-submodule automatically is $\mathcal{O}(\overline{D})$-invariant (using results like the Oka-Weil theorem) or that at least (as we shall see in Chapter 4) every finite-codimensional submodule is $\mathcal{O}(\overline{D})$-invariant. Since in the main cases of interest, the functions $K(\cdot, w)$ and $\frac{1}{K(\cdot,w)}$ belong to $\mathcal{O}(\overline{D})$, the considered $\mathbb{C}[z]$-submodules are K-invariant in the above sense. Therefore, formulating the following result for K-invariant (and not only for invariant) subspaces will turn out to be essential for some applications in Chapter 4.

Theorem 3.3.5. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a Beurling space with reproducing kernel K and that $\mathcal{E}$ is a Hilbert space. Then, for a given K-invariant subspace M of $\mathcal{H}_{\mathcal{E}}$, the following assertions are equivalent:*

(i) M is Beurling decomposable.

(ii) $G_M \in \mathscr{S}_{\mathcal{E}}^{(0)}(K)$.

Moreover, if G_1 and G_2 are disjoint kernels in $\mathscr{S}_{\mathcal{E}}(K)_+$ with $G_M = G_1 - G_2$, and if $(\mathcal{D}_1, \phi_1)$ and $(\mathcal{D}_2, \phi_2)$ are Kolmogorov factorizations of G_1 and G_2, then (ϕ_1, ϕ_2) is a Beurling decomposition of M.

Proof. Suppose that M is Beurling decomposable and that (ϕ_1, ϕ_2) is a Beurling decomposition of M. Then (3.3.1) implies that

$$K_M(z,w) = (\phi_1(z)\phi_1(w)^* - \phi_2(z)\phi_2(w)^*)\, K(z,w)$$

holds for all $z, w \in X$. Hence

$$G_M(z,w) = \phi_1(z)\phi_1(w)^* - \phi_2(z)\phi_2(w)^* \quad (z, w \in X),$$

and therefore $G_M \in \mathscr{S}_{\mathcal{E}}^{(0)}(K)$ by Proposition 2.2.1.

Suppose conversely that $G_M \in \mathscr{S}_{\mathcal{E}}^{(0)}(K)$. We first claim that there exist disjoint positive kernels $G_1, G_2 \in \mathscr{S}_{\mathcal{E}}(K)_+$ such that $G_M = G_1 - G_2$. By Proposition 2.3.12, there exist positive kernels $G_1', G_2' \in \mathscr{S}_{\mathcal{E}}(K)_+$ such that $G_M = G_1' - G_2'$. Then Proposition 1.4.7 yields the existence of disjoint positive kernels $G_1 \leq G_1'$ and $G_2 \leq G_2'$ satisfying $G_M = G_1 - G_2$. Clearly $G_i \in B(G_i')$ for $i = 1, 2$ and, by Proposition 2.4.1, we have $B(G_i') \subset \mathscr{S}_{\mathcal{E}}^{(0)}(K)$. Hence the kernels G_1, G_2 belong to $\mathscr{S}_{\mathcal{E}}^{(0)}(K)$. A slightly different argument which does not use Proposition 1.4.7 is as follows: Obviously, the kernel $G' = G_1' + G_2'$ belongs to $\mathscr{S}_{\mathcal{E}}(K)_+$ and G_M belongs to $B(G')$ which is contained in $\mathscr{S}_{\mathcal{E}}^{(0)}(K)$ by Proposition 2.4.1. Then, using Proposition 1.6.8, we could also choose (G_1, G_2) as the spectral decomposition $((G_M)_+, (G_M)_-)$ of the self-adjoint kernel G_M, formed in $B(G')$.

So let us choose disjoint kernels G_1, G_2 in $\mathscr{S}_{\mathcal{E}}(K)_+$ satisfying $G_M = G_1 - G_2$, and fix Kolmogorov factorizations $(\mathcal{D}_i, \phi_i)$ of G_i, $i = 1, 2$. By Proposition 2.2.1, the functions ϕ_1, ϕ_2 belong to $\mathcal{M}(\mathcal{H}_{\mathcal{D}_i}, \mathcal{H}_{\mathcal{E}})$, $i = 1, 2$. An obvious calculation reveals that

$$P_M = M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^*.$$

So it remains to show that $\operatorname{ran} M_{\phi_1} = M$. It is rather easy to see that M is contained in $\operatorname{ran} M_{\phi_1}$. Since

$$M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^* = P_M$$

is a positive operator, there exists a contraction $C \in B(\mathcal{H}_{\mathcal{D}_1}, \mathcal{H}_{\mathcal{D}_2})$ such that

$$M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^* = M_{\phi_1}(1_{\mathcal{H}_{\mathcal{D}_1}} - C^* C) M_{\phi_1}^*.$$

Hence we find that

$$\begin{aligned} M = \operatorname{ran} P_M &= \operatorname{ran}(M_{\phi_1} M_{\phi_1}^* - M_{\phi_2} M_{\phi_2}^*) \\ &= \operatorname{ran} M_{\phi_1}(1_{\mathcal{H}_{\mathcal{D}_1}} - C^* C) M_{\phi_1}^* \subset \operatorname{ran} M_{\phi_1}. \end{aligned}$$

Conversely, we note first that the kernels G_1, G_2 belong to $B(K_{\mathcal{E}})$, since the constant kernel $\mathbf{1}$ belongs to $B(K)$. Let $\Delta_1, \Delta_2 \in B(\mathcal{H}_{\mathcal{E}})$ denote the representing operators of G_1, G_2 and let $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{E}^X$ be the reproducing kernel Hilbert spaces associated with G_1, G_2. By Proposition 1.6.7, we obtain that $\mathcal{G}_i = \operatorname{ran} \Delta_i^{\frac{1}{2}}$ $(i = 1, 2)$. According to Proposition 1.4.9, we have

$$\operatorname{ran} \Delta_1 \cap \operatorname{ran} \Delta_2 \subset \operatorname{ran} \Delta_1^{\frac{1}{2}} \cap \operatorname{ran} \Delta_2^{\frac{1}{2}} = \mathcal{G}_1 \cap \mathcal{G}_2 = \{0\}.$$

Now it is an elementary exercise to verify that the ranges of Δ_1, Δ_2 must be contained in the closure of the range of $\Delta_M = \Delta_1 - \Delta_2$. Because of

$$\Delta_M(K(\cdot, w)y) = G_M(\cdot, w)y = \frac{1}{K(\cdot, w)} \cdot K_M(\cdot, w)y \in M \quad (w \in X, y \in \mathcal{E}),$$

the range of Δ_M is contained in M. Hence also $\operatorname{ran} \Delta_1 \cup \operatorname{ran} \Delta_2 \subset M$. Therefore, for every $w \in X$ and $y \in \mathcal{E}$,

$$G_1(\cdot, w)y = \Delta_1(K(\cdot, w)y) \in M,$$

which leads to

$$\begin{aligned} M_{\phi_1} M_{\phi_1}^*(K(\cdot, w)y) &= K(\cdot, w) \cdot \phi_1(\cdot)\phi_1(w)^* y \\ &= K(\cdot, w) \cdot G_1(\cdot, w)y \in M \quad (w \in X, y \in \mathcal{E}), \end{aligned}$$

since M was supposed to be K-invariant. The observation

$$\overline{\operatorname{ran} M_{\phi_1}} = \overline{\operatorname{ran} M_{\phi_1} M_{\phi_1}^*} = \bigvee \{M_{\phi_1} M_{\phi_1}^*(K(\cdot, w)y) \ ; \ w \in X, y \in \mathcal{E}\} \subset M$$

completes the proof. □

Corollary 3.3.6. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a Beurling space with reproducing kernel K and let $\mathcal{E}$ be a Hilbert space.*

(a) If $M \subset \mathcal{H}_{\mathcal{E}}$ is an invariant subspace, then $M = [\operatorname{ran} \Delta_M]$.

(b) Let $M \subset \mathcal{H}_{\mathcal{E}}$ be a Beurling decomposable subspace and let $G_M = G_1 - G_2$ be a disjoint decomposition of G_M with $G_1, G_2 \in \mathscr{S}_{\mathcal{E}}(K)_+$. Furthermore, let $\mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{H}_{\mathcal{E}}$ denote the associated reproducing kernel Hilbert spaces and let $\Delta_1, \Delta_2 \in B(\mathcal{H}_{\mathcal{E}})$ be the representing operators of G_1, G_2. Then

$$M = [\operatorname{ran} \Delta_1] = [\mathcal{G}_1].$$

Moreover, via the canonical identification of $\mathcal{E}$ with $B(\mathbb{C}, \mathcal{E})$, the spaces $\mathcal{G}_1, \mathcal{G}_2$ are contained in $\mathcal{M}(\mathcal{H}, \mathcal{H}_{\mathcal{E}})$. Hence $\operatorname{ran} \Delta_i \subset \mathcal{M}(\mathcal{H}, \mathcal{H}_{\mathcal{E}})$ for $i = 1, 2$ and $\operatorname{ran} \Delta_M \subset \mathcal{M}(\mathcal{H}, \mathcal{H}_{\mathcal{E}})$. In particular, $M \cap \mathcal{M}(\mathcal{H}, \mathcal{H}_{\mathcal{E}})$ is dense in M.

Proof. Obviously,

$$K_M(\cdot, w)y = K(\cdot, w) \cdot \Delta_M(K(\cdot, w)y)$$

holds for all $w \in X$ and $y \in \mathcal{E}$. Since the functions $K(\cdot, w)$ and $\frac{1}{K(\cdot,w)}$ belong to $\mathcal{M}(\mathcal{H})$, this shows that $[\operatorname{ran} \Delta_M] = M$.

We are going to prove part (b). Let $(\mathcal{D}_i, \phi_i)$ be Kolmogorov factorizations of G_i $(i = 1, 2)$. As seen in the proof of the previous theorem, the equality $M = \operatorname{ran} M_{\phi_1}$ implies

$$\begin{aligned} M &= \bigvee\{M_{\phi_1} M_{\phi_1}^*(K(\cdot, w)y) \ ; \ w \in X, y \in \mathcal{E}\} \\ &= \bigvee\{K(\cdot, w) \cdot G_1(\cdot, w)y \ ; \ w \in X, y \in \mathcal{E}\} \subset [\mathcal{G}_1]. \end{aligned}$$

On the other hand, we observed that

$$\operatorname{ran} \Delta_1 \subset \overline{\operatorname{ran} \Delta_M} \subset M.$$

Using Proposition 1.6.7, we find that

$$\mathcal{G}_1 = \operatorname{ran} \Delta_1^{\frac{1}{2}} \subset M.$$

This proves $[\operatorname{ran} \Delta_1^{\frac{1}{2}}] = [\mathcal{G}_1] = M$. Clearly, since $\overline{\operatorname{ran} \Delta_1} = \overline{\operatorname{ran} \Delta_1^{\frac{1}{2}}}$, it follows that also $[\operatorname{ran} \Delta_1] = M$. By Proposition 2.4.1, the spaces $\mathcal{G}_i$ are, up to canonical identifaction, contained in $\mathcal{M}(\mathcal{H}, \mathcal{H}_{\mathcal{E}})$ for $i = 1, 2$. To see that $M \cap \mathcal{M}(\mathcal{H}, \mathcal{H}_{\mathcal{E}})$ is dense in M, it suffices to observe that the functions

$$K(\cdot, w) \cdot G_M(\cdot, w)y \quad (w \in X, y \in \mathcal{E})$$

form a total subset of M and obviously belong to $\mathcal{M}(\mathcal{H}, \mathcal{H}_{\mathcal{E}})$. $\square$

In particular, Corollary 3.3.6 implies that, given a Beurling space $\mathcal{H}$ and a Beurling decomposable subspace M of $\mathcal{H}$, the intersection $M \cap \mathcal{M}(\mathcal{H})$ is dense in $\mathcal{H}$. An example given by Rudin ([62], Theorem 4.1.1) shows that there exists an invariant subspace of the Hardy space $H^2(\mathbb{D}^2)$ over the bidisk which does not contain any non-zero multiplier $\phi \in \mathcal{M}(H^2(\mathbb{D}^2)) = H^\infty(\mathbb{D}^2)$. As we shall see in Chapter 5, there exist similar examples in the Bergman space $L_a^2(\mathbb{D})$. Therefore, we can in general not expect all invariant subspaces to be Beurling decomposable.

Before we proceed, we explain how Theorem 3.3.5 is related to known Beurling-type results. The starting point is the following observation.

Proposition 3.3.7. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a Beurling space with reproducing kernel K and that $\mathcal{E}$ is a Hilbert space. Suppose further that $M \subset \mathcal{H}_{\mathcal{E}}$ is a closed subspace with positive definite core function G_M. Then every Kolomogorov factorization $(\mathcal{D}, \phi)$ of G_M defines a multiplier $\phi \in \mathcal{M}(\mathcal{H}_{\mathcal{D}}, \mathcal{H}_{\mathcal{E}})$ such that $P_M = M_\phi M_\phi^*$. In particular, M_ϕ is a partial isometry and $\operatorname{ran} M_\phi = M$. This means that M is Beurling decomposable and $(\phi, 0)$ is a Beurling decomposition of M. In particular, M is automatically invariant.*

Proof. Clearly, the identity $G_M(z,w) = \phi(z)\phi(w)^*$, valid for all $z, w \in X$, shows that the kernel

$$X \times X \to \mathcal{E}\ ,\ (z,w) \mapsto K_M(z,w) - K(z,w)\phi(z)\phi(w)^*$$

vanishes and is therefore positive. According to Proposition 1.7.6, ϕ belongs to $\mathcal{M}(\mathcal{H}_\mathcal{D}, M) \subset \mathcal{M}(\mathcal{H}_\mathcal{D}, \mathcal{H}_\mathcal{E})$. The identity $P_M = M_\phi M_\phi^*$ is checked by a straightforward computation. Consequently, M_ϕ is partially isometric which shows that $M = \operatorname{ran} M_\phi$. □

This leads to the generic Beurling-type result for NP spaces as proved in [39] and [53].

Theorem 3.3.8. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is an NP space with reproducing kernel K. Let $\mathcal{E}$ be a Hilbert space and let $M \subset \mathcal{H}_\mathcal{E}$ be a K-invariant subspace. Then G_M is positive. Moreover, there exist a Hilbert space $\mathcal{D}$ and a multiplier $\phi \in \mathcal{M}(\mathcal{H}_\mathcal{D}, \mathcal{H}_\mathcal{E})$ such that $P_M = M_\phi M_\phi^*$.*

Proof. It clearly suffices to show that G_M is positive. To this end, choose a minimal Kolomogorov factorization $(\mathcal{C}, \gamma)$ of the positive kernel $1 - \frac{1}{K}$. Reordering shows that the kernel

$$X \times X \to \mathbb{C}\ ,\ (z,w) \mapsto K(z,w)(1 - \gamma(z)\gamma(w)^*(1)) = 1$$

is positive. By Proposition 1.7.6, $\gamma \in \mathcal{M}(\mathcal{H}_\mathcal{C}, \mathcal{H})$ with $\|\gamma\|_\mathcal{M} \leq 1$. Now let us consider the function

$$\tilde{\gamma} : X \to B(\mathcal{E} \otimes \mathcal{C}, \mathcal{E})\ ,\ \tilde{\gamma}(z) = 1_\mathcal{E} \otimes \gamma(z).$$

Then, again by Proposition 1.7.6, $\tilde{\gamma}$ belongs to $\mathcal{M}(\mathcal{H}_{\mathcal{E}\otimes\mathcal{C}}, \mathcal{H}_\mathcal{E})$ and $\|\tilde{\gamma}\|_\mathcal{M} \leq 1$. The K-invariance of M implies that M is invariant under multiplication with the functions

$$\gamma(\cdot)\gamma(w)^*(1) = 1 - \frac{1}{K(\cdot, w)} \quad (w \in X).$$

Furthermore, the family $\{\gamma(w)^*(1)\ ;\ w \in X\}$ is total in $\mathcal{C}$ by the minimality of the Kolmogorov factorization $(\mathcal{C}, \gamma)$. Lemma 3.2.4 and the subsequent Remark 3.2.5 show that

$$P_M - M_{\tilde{\gamma}}(P_M \otimes 1_\mathcal{C})M_{\tilde{\gamma}}^* = P_M(1_{\mathcal{H}_\mathcal{E}} - M_{\tilde{\gamma}}(P_M \otimes 1_\mathcal{C})M_{\tilde{\gamma}}^*)P_M$$

is positive or, equivalently, that

$$G_M(z,w) = K_M(z,w)(1 - \gamma(z)\gamma(w)^*(1))$$

is positive. □

The simplest NP space probably is the Hardy space $H^2(\mathbb{D})$ over the complex unit disk. The classical theorem of Beurling states that, for every invariant subspace M of $H^2(\mathbb{D})$, the orthogonal projection P_M can be factorized as $P_M = M_\eta M_\eta^*$, where $\eta : \mathbb{D} \to \mathbb{C}$ is an inner function, that is, a bounded holomorphic function with boundary values of modulus 1 almost everywhere. A short reflection shows that this is equivalent to the statement of Theorem 3.3.8 plus the assertion that one can choose $\mathcal{D} = \mathbb{C}$. Hence Beurling's theorem seems to be stronger than our result. However, we shall demonstrate later how the classical result and also similar results for the Arveson space can be recovered from Theorem 3.3.8.

We have just seen that in NP spaces, all invariant subspaces have a positive core function. This property essentially characterizes the class of NP spaces.

Proposition 3.3.9. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a Beurling space with reproducing kernel K, which is normalized at a point $z_0 \in X$, that is, $K(\cdot, z_0) \equiv \mathbf{1}$. Suppose that every invariant subspace has a positive core function. Then $\mathcal{H}$ is an NP space.*

Proof. This follows by considering the invariant subspace

$$M = \{f \in \mathcal{H} \ ; \ f(z_0) = 0\} = \{\mathbf{1}\}^\perp .$$

One easily checks that $G_M = 1 - \frac{1}{K}$. Hence, if G_M is positive, then K must be an NP kernel. □

We mention that the requirement that K is normalized at some point is not really necessary. It can be eliminated by a slightly more general definition of NP kernels. Namely, we could say that a kernel $K : X \times X \to \mathbb{C}$ is an NP kernel if it has no zeroes and if there exists a function $a : X \to \mathbb{C}$ such that the kernel

$$X \times X \to \mathbb{C} \ , \ (z, w) \mapsto a(z)\overline{a(w)} - \frac{1}{K(z, w)}$$

is positive definite. With this definition, the theory of NP kernels can be developed without major changes, but we decided to use the more special definition in order to keep the notations and proofs simple.

Proposition 3.3.9 shows that in non-Nevanlinna Pick spaces, one cannot expect the classical Beurling theorem to hold. All known results of Beurling type rely on the positivity of the core function. While this approach is perfect in the setting of NP spaces, there are simple examples of reproducing kernel spaces with few or without any invariant subspaces admitting a positive core function.

Example 3.3.10.

(a) *Let us consider the Hardy space $H^2(\mathbb{B}_d)$ over the unit ball in $\mathbb{C}^d$, $d \geq 2$. Then $H^2(\mathbb{B}_d)$ is not an NP space. If $\eta : \mathbb{B}_d \to \mathbb{C}$ is an inner function, then M_η obviously is an isometry and $M = \operatorname{ran} M_\eta$ defines an invariant subspace such* *that* $P_M = M_\eta M_\eta^*$. *Hence,* $G_M(z,w) = \eta(z)\overline{\eta(w)}$ *is a positive definite function. Aleksandrov's [1] famous solution of the inner function problem on the unit ball guarantees the existence of inner functions on* $\mathbb{B}_d$. *Hence there is a rich supply of invariant subspaces of* $H^2(\mathbb{B}_d)$ *having a positive core function. In the opposite direction, we shall show later that all invariant subspaces with a positive core function are of the form* $M = \operatorname{ran} M_\eta$ *for some inner function* η. *However, note that by Proposition* 3.3.9 *and its proof, the zero-based invariant subspace*

$$M = \{f \in H^2(\mathbb{B}_d)\ ;\ f(0) = 0\}$$

cannot be represented this way.

(b) The situation is even worse when passing from the Hardy to the Bergman space $L_a^2(D)$, *over some bounded symmetric domain* $D \subset \mathbb{C}^d$. *It turns out that no non-trivial invariant subspace of the Bergman space has a positive definite core function. In fact, suppose that* $\{0\} \neq M$ *is an invariant subspace of* $L_a^2(D)$. *Then, by Example* 3.3.3 *(*b*), the space* $M \ominus \sum_{i=1}^d \mathbf{z}_i \cdot M$ *is not zero and consists of eigenvectors of* Δ_M *for the eigenvalue* 1. *Therefore, we can choose a unit vector* $f \in M \ominus \sum_{i=1}^d \mathbf{z}_i \cdot M$. *Since by assumption* $\Delta_M \geq 0$, *it follows that also* $\Delta_M - f \otimes f \geq 0$. *Since the defect function* D_M *is obviously pointwise bounded by* 1, *we obtain that*

$$\begin{aligned} 1 = \|f\|^2 &= \int_D |f(z)|^2 \, d\mu(z) \\ &= \int_D \langle (f \otimes f) K(\cdot, z), K(\cdot, z)\rangle \, d\mu(z) \\ &\leq \int_D \langle \Delta_M K(\cdot, z), K(\cdot, z)\rangle \, d\mu(z) \\ &= \int_D D_M(z) \, d\mu(z) \leq 1. \end{aligned}$$

Hence $D_M(z) = 1$ *for all* $z \in D$. *The fact that*

$$D_M(z) = \frac{\|P_M K(\cdot, z)\|^2}{\|K(\cdot, z)\|^2}$$

holds for all $z \in D$ *shows that necessarily* $M = L_a^2(D)$.

We conclude this section by discussing the phenomenon of Beurling decomposability in two special classes of K-invariant subspaces. The first consists of all finite-codimensional and the second of all finite-rank K-invariant subspaces.

Proposition 3.3.11. *Suppose that* $\mathcal{H} \subset \mathbb{C}^X$ *is a Beurling space with reproducing kernel* K, *that* $\mathcal{E}$ *is a Hilbert space and that* $M \subset \mathcal{H}_\mathcal{E}$ *is a finite-codimensional* K*-invariant subspace. Then the following are equivalent:*

(i) M *is Beurling decomposable.*

(ii) $M^\perp \subset \mathcal{M}(\mathcal{H}, \mathcal{H}_\mathcal{E})$ *via the canonical identification* $\mathcal{E} \simeq B(\mathbb{C}, \mathcal{E})$.

Proof. Suppose that (i) holds. Then for $w \in X$ and $y \in \mathcal{E}$,

$$K_{M^\perp}(\cdot, w)y = K(\cdot, w)y - K_M(\cdot, w)y = K(\cdot, w) \cdot (y - G_M(\cdot, w)y)$$

belongs to $\mathcal{M}(\mathcal{H}, \mathcal{H}_\mathcal{E})$ by Corollary 3.3.6 and the fact that $K(\cdot, w)$ belongs to $\mathcal{M}(\mathcal{H})$. Since $M^\perp$ is finite dimensional, it is the linear span of the functions $K_{M^\perp}(\cdot, w)y$, and the assertion follows. Conversely, suppose that $M^\perp \subset \mathcal{M}(\mathcal{H}, \mathcal{H}_\mathcal{E})$. Then, for $g \in M^\perp$, the function

$$\tilde{g} : X \to B(\mathbb{C}, \mathcal{E}) \ , \ \tilde{g}(z) = g(z)_c$$

(with the notation introduced in the beginning of Section 2.4) satisfies

$$g(z) \otimes g(w) = \tilde{g}(z)\tilde{g}(w)^* \quad (z, w \in X).$$

By Proposition 2.2.1, this means that the positive kernel

$$X \times X \to B(\mathcal{E}) \ , \ (z, w) \mapsto g(z) \otimes g(w)$$

belongs to $\mathscr{S}_\mathcal{E}(K)_+$. Now, choosing an orthonormal basis $(e_i)_{i=1}^r$ of the finite-dimensional space $M^\perp$, it follows that (cf. Proposition 1.1.2)

$$K_{M^\perp}(z, w) = \sum_{i=1}^{r} e_i(z) \otimes e_i(w)$$

belongs to $\mathscr{S}_\mathcal{E}(K)_+$. By the hypothesis that $\frac{1}{K} \in \mathscr{S}^{(0)}(K)$ and by Proposition 2.3.3, we conclude that

$$G_M = 1_\mathcal{E} - \frac{1}{K} \cdot K_{M^\perp} \in \mathscr{S}_\mathcal{E}^{(0)}(K).$$

An application of Theorem 3.3.5 completes the proof. □

As an application, we show that finite zero-based subspaces of Beurling spaces are Beurling decomposable.

Corollary 3.3.12. *Suppose that* $\mathcal{H} \subset \mathbb{C}^X$ *is a Beurling space with reproducing kernel* K *and let* $z_1, \ldots, z_r$ *be distinct points in* X*. Then the invariant subspace*

$$M = \{f \in \mathcal{H} \ ; \ f(z_i) = 0 \text{ for } i = 1, \ldots, r\}$$

is Beurling decomposable.

Proof. Clearly,

$$M^\perp = \operatorname{span}\{K(\cdot, z_i) \ ; \ i = 1, \ldots, r\}$$

is contained in $\mathcal{M}(\mathcal{H})$, since the functions $K(\cdot, w)$ $(w \in X)$ are supposed to be multipliers. Now the assertion follows by Proposition 3.3.11. □

We shall prove later that, in the setting of analytic Hilbert modules, even all finite-codimensional submodules are Beurling decomposable. But first, we turn our attention to the class of finite-rank Beurling decomposable subspaces.

Proposition 3.3.13. *Suppose that $\mathcal{H} \subset \mathbb{C}^X$ is a Beurling space with reproducing kernel K and that $\mathcal{E}$ is a Hilbert space. Suppose further that $M \subset \mathcal{H}_\mathcal{E}$ is a K-invariant subspace with finite positive (negative) rank r. Then the following assertions are equivalent:*

(i) *M is Beurling decomposable.*

(ii) $\operatorname{ran}(\Delta_M)_+ \subset \mathcal{M}(\mathcal{H}, \mathcal{H}_\mathcal{E})$ $(\operatorname{ran}(\Delta_M)_- \subset \mathcal{M}(\mathcal{H}, \mathcal{H}_\mathcal{E}))$ *via canonical identification.*

(iii) $\operatorname{ran}\Delta_M \subset \mathcal{M}(\mathcal{H}, \mathcal{H}_\mathcal{E})$ *via canonical identification.*

In this case, the positive and negative parts $(G_M)_+$ and $(G_M)_-$ of the spectral decomposition of G_M, formed in $B(K_\mathcal{E})$, belong to $\mathscr{S}_\mathcal{E}(K)_+$. In particular, if $\operatorname{rank}_+ M = r < \infty$, *then there exist multipliers $\phi_1, \ldots, \phi_r \in$* $\operatorname{ran}(\Delta_M)_+$ *such that*

$$M = \sum_{i=1}^{r} \phi_i \cdot \mathcal{H}.$$

Proof. The implication (i) to (iii) is Corollary 3.3.6 and that (iii) implies (ii) is trivial. So suppose that (ii) holds. For simplicity, we assume $\operatorname{ran}(\Delta_M)_+ \subset \mathcal{M}(\mathcal{H}, \mathcal{H}_\mathcal{E})$ (the other case is analogous). Then we can find functions $g_1, \ldots, g_r \subset \operatorname{ran}(\Delta_M)_+$ such that

$$(\Delta_M)_+ = \sum_{i=1}^{r} g_i \otimes g_i.$$

Then the functions

$$\phi_i : X \to B(\mathbb{C}, \mathcal{E}) \ , \ \phi_i(z) = g_i(z)_c \quad (1 \leq i \leq r)$$

(with the notations introduced at the beginning of Section 2.4) are multipliers satisfying

$$(G_M)_+(z, w) = \sum_{i=1}^{r} \phi_i(z)\phi_i(w)^* \quad (z, w \in X).$$

By Proposition 2.2.1, $(G_M)_+ \in \mathscr{S}_\mathcal{E}(K)_+$. Since

$$(G_M)_- \cdot K = (G_M)_+ \cdot K - G_M \cdot K = (G_M)_+ \cdot K - K_M$$

obviously belongs to $B(K_\mathcal{E})$ and since $(G_M)_-$ is positive, another application of Proposition 2.2.1 shows that also $(G_M)_-$ belongs to $\mathscr{S}_\mathcal{E}(K)_+$. Consequently, G_M belongs to $\mathscr{S}_\mathcal{E}^{(0)}(K)$. By Theorem 3.3.5, M is indeed Beurling decomposable and $M = \sum_{i=1}^{r} \phi_i \cdot \mathcal{H}$. □

4 Applications to analytic Hilbert modules

4.1 Analytic Hilbert modules

To begin with, we recall the definition of analytic Hilbert modules, following the monograph of Chen and Guo [25].

Throughout this section, unless otherwise stated, we shall denote by D a fixed non-empty bounded open subset of $\mathbb{C}^d$.

Definition 4.1.1. *A reproducing kernel Hilbert space $\mathcal{H} \subset \mathcal{O}(D)$ is called an analytic Hilbert module if the following conditions are satisfied:*

(A) $\mathcal{H}$ contains the constant functions.

(B) The coordinate functions $\mathbf{z}_i$ $(1 \leq i \leq d)$ are multipliers of $\mathcal{H}$. Equivalently, $\mathcal{H}$ is a $\mathbb{C}[z]$-module with respect to pointwise multiplication.

(C) The polynomials are dense in $\mathcal{H}$.

(D) There are no points $z \in \mathbb{C}^d \backslash D$ for which the mapping

$$\mathbb{C}[z] \to \mathbb{C}\ ,\ \ p \mapsto p(z) \tag{4.1.1}$$

extends to a continuous linear form on all of $\mathcal{H}$.

In the language of [25], condition (D) means that the set of virtual points of $\mathcal{H}$ coincides with D. We are now going to show that the class of analytic Hilbert modules is closed under the usual product operations (see Section 1.2 for the definitions and compare Remark 3.1.2 for the corresponding statements in the setting of Beurling spaces).

Remark 4.1.2.

(a) The class of analytic Hilbert modules is closed with respect to the forming of outer products. In fact, suppose that $D_1 \subset \mathbb{C}^{d_1}$ and $D_2 \subset \mathbb{C}^{d_2}$ are bounded open

sets and that $\mathcal{H}_1, \mathcal{H}_2$ *are analytic Hilbert modules over* D_1 *and* D_2*, respectively. Define* $D = D_1 \times D_2$. *By Proposition* 1.2.5*, there exists a unitary operator*

$$U : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \circledast \mathcal{H}_2 \quad \text{with} \quad U(f_1 \otimes f_2) = f_1 \circledast f_2.$$

It is clear that U *maps the algebraic tensor product* $\mathbb{C}[z_1] \otimes \mathbb{C}[z_2]$ *onto* $\mathbb{C}[z_1, z_2]$. *This proves (A) and (C). Condition (B) is also fulfilled since more generally, for multipliers* $\phi_1 \in \mathcal{M}(\mathcal{H}_1)$ *and* $\phi_2 \in \mathcal{M}(\mathcal{H}_2)$*, the outer product* $\phi_1 \circledast \phi_2 : D \to \mathbb{C}$ *belongs to* $\mathcal{M}(\mathcal{H}_1 \circledast \mathcal{H}_2)$. *Assume finally that* $z = (z_1, z_2) \notin D$ *is a virtual point of* $\mathcal{H}_1 \circledast \mathcal{H}_2$. *This means that there exists a constant* $c > 0$ *such that* $|p(z)| \leq c\|p\|_{\mathcal{H}_1 \circledast \mathcal{H}_2}$ *holds for all* $p \in \mathbb{C}[z_1, z_2]$. *Hence, for* $q \in \mathbb{C}[z_1]$*, we see*

$$|q(z_1)| = |(q \circledast \mathbf{1})(z)| \leq c\|(q \circledast \mathbf{1})\|_{\mathcal{H}_1 \circledast \mathcal{H}_2} = c\|\mathbf{1}\|_{\mathcal{H}_2}\|q\|_{\mathcal{H}_1}.$$

This shows that z_1 *is a virtual point of* $\mathcal{H}_1$ *and hence that* $z_1 \in D_1$. *Analogously, we obtain* $z_2 \in D_2$*, and thus* $z \in D$.

(b) The class of analytic Hilbert modules is also stable with respect to the forming of inner products. Indeed, we shall prove a stronger result. Namely, suppose that $\mathcal{H}_1$ *is an analytic Hilbert module over* D *and that* $\mathcal{H}_2 \subset \mathcal{O}(D)$ *is a reproducing kernel Hilbert space satisfying condition (C). Then the inner product* $\mathcal{H}_1 * \mathcal{H}_2$ *is an analytic Hilbert module. By Proposition* 1.2.6*, there exists a coisometry*

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 * \mathcal{H}_2 \quad \text{with} \quad f_1 \otimes f_2 \mapsto f_1 * f_2 = f_1 \cdot f_2,$$

mapping the algebraic tensor product $\mathbb{C}[z] \otimes \mathbb{C}[z]$ *onto* $\mathbb{C}[z]$. *This shows that* $\mathcal{H}_1 * \mathcal{H}_2$ *contains the polynomials and also that they are dense in it. Condition (B) is satisfied since* $\mathcal{M}(\mathcal{H}_1)$ *is contained in* $\mathcal{M}(\mathcal{H}_1 * \mathcal{H}_2)$ *by standard arguments. To prove that (D) is fulfilled, choose some virtual point* z *of* $\mathcal{H}_1 * \mathcal{H}_2$. *Hence there exists some* $c > 0$ *such that* $|p(z)| \leq c\|p\|_{\mathcal{H}_1 * \mathcal{H}_2}$ *holds for all polynomials* p. *We obtain*

$$|p(z)| \leq c\|p\|_{\mathcal{H}_1 * \mathcal{H}_2} \leq c\|\mathbf{1}\|_{\mathcal{H}_2}\|p\|_{\mathcal{H}_1}$$

for all polynomials p*, proving that* z *is a virtual point of* $\mathcal{H}_1$ *and hence, that* $z \in D$.

Of special interest for us are of course the standard reproducing kernel spaces over bounded symmetric domains. The following example demonstrates that these spaces are analytic Hilbert modules.

Example 4.1.3. *Suppose that* $D \subset \mathbb{C}^d$ *is a Cartain domain of rank* r *and that* ν *belongs to the continuous Wallach set* $\mathcal{W}_c$ *of* D. *As before, we write* $\mathcal{H} = \mathcal{H}_\nu$ *and* $K = K_\nu$. *Clearly, conditions (A),(B) and (C) are fulfilled as previously observed in Examples* 1.1.10 *and* 1.7.3 *(*c*). Condition (D) was verified in [42] for the Bergman space over* D. *However, the proof given there contains some inconsistencies. Therefore, we prefer to give an independent proof.*

As before, let us write $\mathfrak{K}$ *to denote the stabilizer of the origin in the identitiy component of* $\mathrm{Aut}(D)$*. Furthermore, let* $\mathrm{vp}(\mathcal{H})$ *denote the set of all virtual points of* $\mathcal{H}$*, that is, the set of all points* $z \in \mathbb{C}^d$ *such that the linear form (*4.1.1*) has a continuous extension on* $\mathcal{H}$*. We first claim that* $r \cdot \mathrm{vp}(\mathcal{H}) \subset \mathrm{vp}(\mathcal{H})$ *holds for all* $0 < r < 1$*. To prove this, one checks that the mapping*

$$\mathcal{H} \to \mathcal{H} \ , \ f \mapsto f_r$$

(where, as usual, $f_r(z) = f(rz)$*) is well-defined and contractive. In fact, this follows by Proposition* 1.1.11*, since the kernel*

$$D \times D \to \mathbb{C} \ , \ (z, w) \mapsto K(z, w) - K(rz, rw)$$

*is positive definite, which in turn can be easily verified by the Faraut-Koranyi expansion (*1.1.3*). So if* $z \in \mathrm{vp}(\mathcal{H})$*, then there exists some* $c > 0$ *such that* $|p(z)| \leq c\|p\|$ *holds for all polynomials* p*. Hence*

$$|p(rz)| = |p_r(z)| \leq c\|p_r\| \leq c\|p\|$$

for all polynomials p *or, equivalently,* $rz \in \mathrm{vp}(\mathcal{H})$*, which proves the claim.*

So, in order to prove $\mathrm{vp}(\mathcal{H}) \subset D$ *(the other inclusion is trivial), it suffices to show that* $\mathrm{vp}(\mathcal{H}) \cap \partial D = \emptyset$ *(recall that* D *is the unit ball with respect to a suitable norm on* $\mathbb{C}^d$*). Let us pick some virtual point* $z_0 \in \partial D$*. Then there exists a constant* $c > 0$ *such that* $|p(z_0)| \leq c\|p\|$ *holds for all polynomials* p*. Since* D *is polynomially convex, the Oka-Weil theorem shows that then* $|f(z_0)| \leq c\|f\|$ *for all* $f \in \mathcal{O}(\overline{D})$*. Now let us fix some Jordan frame* $e_1, \ldots, e_r$ *of* D *(see [7], pp. 14-20 for details). Then there exist a unitary* $k \in \mathfrak{K}$ *and unique numbers* $1 = t_1 \geq \ldots \geq t_r \geq 0$ *such that* $z_0 = k(t_1 e1 + \ldots + t_r e_r)$*. Let us define* $f^{(r)} = K(\cdot, rz_0)$ *for* $0 < r < 1$*. Then* $f^{(r)} \in \mathcal{O}(\overline{D})$ *for all* $0 < r < 1$*. By the* $\mathfrak{K}$*-invariance of the Jordan triple determinant* h*, we obtain*

$$\begin{aligned} f^{(r)}(z_0) &= K(z_0, rz_0) = K(\sqrt{r}z_0, \sqrt{r}z_0) \\ &= h(\sqrt{r}z_0, \sqrt{r}z_0)^{-\nu} = \left(\prod_{j=1}^{r} (1 - rt_j^2) \right)^{-\nu} , \end{aligned}$$

using formula 3.12 of [7]. Analogously,

$$\|f^{(r)}\|^2 = K(rz_0, rz_0) = h(rz_0, rz_0) = \left(\prod_{j=1}^{r} (1 - r^2 t_j^2) \right)^{-\nu} .$$

Hence

$$|\frac{|f^{(r)}(z_0)|^2}{\|f^{(r)}\|^2} = \left(\prod_{j=1}^{r} \frac{1 - r^2 t_j^2}{(1 - rt_j^2)^2} \right)^{\nu} \geq \left(\frac{1 - r^2 t_1^2}{(1 - rt_1^2)^2} \right)^{\nu} = \left(\frac{1+r}{1-r} \right)^{\nu} \overset{r \to 1}{\longrightarrow} \infty,$$

since, for all $0 \leq t \leq 1$ *and all* $0 < r < 1$*, the inequality*

$$(1 - rt^2)^2 \leq (1 - r^2t^2)$$

is satisfied. On the other hand, we must have $|f^{(r)}(z_0)| \leq c\|f^{(r)}\|$ *for all* $0 < r < 1$*, a contradiction.*

The aim of this chapter is to prove that, under some mild additional hyptotheses, every finite-codimensional submodule of an analytic Hilbert module is Beurling decomposable. Recall that, given an analytic Hilbert module $\mathcal{H}$ over D, a submodule of $\mathcal{H}$ is a closed invariant subspace of the tuple $M_{\mathbf{z}} = (M_{\mathbf{z}_1}, \ldots, M_{\mathbf{z}_d})$ (equivalently, is a norm-closed submodule with respect to the $\mathbb{C}[z]$-module structure of $\mathcal{H}$). We again stress the fact that, in general, submodules do not need to be invariant in our sense (see the discussion in Section 3.2).

We recapitulate some definitions taken from [25].

Suppose that $\mathcal{H}$ is an analytic Hilbert module with reproducing kernel K. Then, for every multiindex $\alpha \in \mathbb{N}_0^d$ and every $w \in D$, the functional

$$\mathcal{H} \to \mathbb{C}\ ,\ f \mapsto D^\alpha f(w)$$

is continuous (since the embedding $\mathcal{H} \hookrightarrow \mathcal{O}(D)$ is continuous). Hence there exists a unique element $K_w^{(\alpha)} \in \mathcal{H}$ such that

$$D^\alpha f(w) = \langle f, K_w^{(\alpha)} \rangle$$

holds for all $f \in \mathcal{H}$. Furthermore, if $(w_1, \alpha_1), \ldots, (w_m, \alpha_m)$ are pairwise different, then the functions $K_{w_1}^{(\alpha_1)}, \ldots, K_{w_m}^{(\alpha_m)}$ are linearly independent in $\mathcal{H}$. To see this, choose polynomials $p_1, \ldots, p_m$ such that

$$D^{\alpha_i} p_j(w_i) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{else} \end{cases} \qquad (1 \leq i, j \leq m).$$

The observation that

$$\overline{c_j} = \sum_{i=1}^m \overline{c_i} D^{\alpha_i} p_j(w_i) = \langle p_j, \sum_{i=1}^m c_i K_{w_i}^{(\alpha_i)} \rangle \quad (1 \leq j \leq m)$$

holds for any choice of complex numbers $c_1, \ldots, c_m$ proves the claimed linear independence.

Let $w \in D$ be arbitrary. For a polynomial $p = \sum_\alpha c_\alpha z^\alpha \in \mathbb{C}[z]$, we define

$$K_w^{(p)} = \sum_\alpha \overline{c_\alpha} K_w^{(\alpha)}.$$

Then

$$\langle f, K_w^{(p)} \rangle = \sum_\alpha c_\alpha D^\alpha f(w)$$

holds for all $f \in \mathcal{H}$, and the mapping

$$\gamma_w : \mathbb{C}[z] \to \mathcal{H}\ ,\ p \mapsto K_w^{(p)}$$

is antilinear and one-to-one because the family $\{K_w^{(\alpha)}\ ;\ \alpha \in \mathbb{N}_0^d\}$ is linearly independent. Now, for a given submodule M of $\mathcal{H}$ and $w \in D$, the set

$$M_w = \gamma_w^{-1}(M^\perp)$$

is a linear subspace of $\mathbb{C}[z]$, and the enveloping space of M, defined by

$$M_w^e = (\gamma_w(M_w))^\perp \subset \mathcal{H},$$

is a submodule containing M (cf. [25], p. 25). For an arbitrary subset N of $\mathcal{H}$, we denote by $Z(N)$ the zero variety of N, that is,

$$Z(N) = \{z \in D\ ;\ f(z) = 0 \text{ for all } f \in N\}.$$

Now if M is a finite-codimensional submodule of $\mathcal{H}$, then the enveloping spaces M_w^e have a very simple structure. More precisely, we claim that

$$Z(M_w^e) = \begin{cases} \{w\} & ; \quad \text{if } w \in Z(M) \\ \emptyset & ; \quad \text{otherwise} \end{cases}$$

holds for all $w \in D$. In fact, if we suppose that $z \in Z(M_w^e)$, then the function $K(\cdot, z)$ is contained in $\overline{\gamma_w(M_w)} = \gamma_w(M_w)$ since M_w has finite dimension by hypothesis. Therefore $K(\cdot, z)$ is a linear combination of the elements $K_w^{(\alpha)}$ $(\alpha \in \mathbb{N}_0^d)$, and hence $z = w$. This proves the inclusion $Z(M_w^e) \subset \{w\}$. For obvious reasons, we have $Z(M_w^e) \subset Z(M)$. So it remains to show that $w \in Z(M_w^e)$ whenever $w \in Z(M)$. But $w \in Z(M)$ is equivalent to $\mathbf{1} \in M_w$, which implies $K(\cdot, w) \in \gamma_w(M_w)$, and thus $w \in Z(M_w^e)$.

The following result appears in [25] as Corollary 2.2.6 and completely describes the structure of finite-codimensional submodules of analytic Hilbert modules in terms of the enveloping spaces.

Lemma 4.1.4. *Suppose that $\mathcal{H}$ is an analytic Hilbert module over D and that M is a finite-codimensional submodule of $\mathcal{H}$. Then*

(a) $Z(M)$ is a finite subset of D.

(b) $M = \bigcap_{w \in Z(M)} M_w^e$.

(c) $\dim M^\perp = \sum_{w \in Z(M)} \dim M_w$.

Actually, we shall only use the following elementary consequence of Lemma 4.1.4.

Corollary 4.1.5. *Suppose that $\mathcal{H}$ is an analytic Hilbert module over D and that M is a finite-codimensional submodule of $\mathcal{H}$. Then*

$$M^\perp \subset \operatorname{span}\{K_w^{(\alpha)}\ ;\ w \in D\ ,\ \alpha \in \mathbb{N}_0^d\}.$$

In particular, M is invariant.

Proof. Suppose first that $Z(M) = \{w\}$ holds for some $w \in D$. From Lemma 4.1.4, we know that $M = M_w^e = (\gamma_w(M_w))^\perp$ and hence that $M^\perp = \gamma_w(M_w)$ (since M_w is finite dimensional). But this implies immediately that

$$M^\perp \subset \operatorname{ran} \gamma_w = \{K_w^{(p)} \; ; \; p \in \mathbb{C}[z]\} = \operatorname{span}\{K_w^{(\alpha)} \; ; \; \alpha \in \mathbb{N}_0^d\}$$

by the definition of γ_w. Let us choose a basis $g_1, \ldots, g_n$ of $M^\perp$, say $g_i = K_w^{(p_i)}$ for suitable polynomials p_i, $1 \leq i \leq n$. Fix $\phi \in \mathcal{M}(\mathcal{H})$ and $f \in M$. Then ϕ can be approximated uniformly by polynomials on a neighbourhood U of w, that is, there exists a sequence $(q_k)_k$ of polynomials converging uniformly on U to ϕ. Since $q_k \cdot f \in M$ for all k, we obtain

$$\langle \phi \cdot f, K_w^{(p_i)} \rangle = \lim_k \langle q_k \cdot f, K_w^{(p_i)} \rangle = 0$$

for all $1 \leq i \leq n$, and hence $\phi \cdot f \in M$. This shows that M is invariant.

If $Z(M)$ is arbitrary, then for every $w \in Z(M)$, the subspace M_w^e is a finite-codimensional submodule with $Z(M_w^e) = \{w\}$. By what we just proved, M_w^e is an invariant subspace and

$$(M_w^e)^\perp \subset \operatorname{span}\{K_w^{(\alpha)} \; ; \; \alpha \in \mathbb{N}_0^d\}.$$

Lemma 4.1.4 yields that $Z(M)$ is a finite set and that

$$M = \bigcap_{w \in Z(M)} M_w^e.$$

Hence M is invariant as an intersection of invariant subspaces and

$$M^\perp = \sum_{w \in Z(M)} (M_w^e)^\perp \subset \operatorname{span}\{K_w^{(\alpha)} \; ; \; w \in D, \alpha \in \mathbb{N}_0^d\}.$$

This observation completes the proof. □

4.2 Analytic Beurling modules

In this section, we shall extend the definition of analytic Hilbert modules by some very natural conditions in order to obtain what we call analytic Beurling modules. This will allow us to prove that every finite-codimensional submodule of an analytic Beurling module is Beurling decomposable.

As before, D denotes some bounded open subset of $\mathbb{C}^d$.

Definition 4.2.1. *An analytic Hilbert module $\mathcal{H}$ over D is called an analytic Beurling module if it satisfies the following conditions:*

(E) The reproducing kernel K of $\mathcal{H}$ has no zeroes and the inverse kernel $\frac{1}{K}$ belongs to $\mathscr{S}^{(0)}(K)$.

(F) *The Taylor spectrum* $\sigma(M_{\mathbf{z}})$ *of the commuting tuple* $M_{\mathbf{z}} = (M_{\mathbf{z}_1}, \dots, M_{\mathbf{z}_d})$ *is contained in* $\overline{D}$.

(G) *For all* $z \in D$, *there exist open neighbourhoods* $U \subset D$ *of* z *and* V *of* $\overline{D}$ *such that* $K_{|U \times D}$ *admits a zero-free sesquianalytic extension to* $U \times V$.

As observed in Example 1.7.3 (b), condition (F) of the preceding definition implies that every function $\phi \in \mathcal{O}(\overline{D}, B(\mathcal{E}_1, \mathcal{E}_2))$ ($\mathcal{E}_1, \mathcal{E}_2$ Hilbert spaces) belongs to $\mathcal{M}(\mathcal{H}_{\mathcal{E}_1}, \mathcal{H}_{\mathcal{E}_2})$. By condition (G), the functions $K(\cdot, w)$ as well as the functions $\frac{1}{K(\cdot,w)}$ belong to $\mathcal{O}(\overline{D})$ and hence are multipliers of $\mathcal{H}$. In particular, every analytic Beurling module is a Beurling space.

As earlier in the setting of Beurling spaces and analytic Hilbert modules, we observe that the class of analytic Beurling modules is (essentially) closed under inner and outer products.

Remark 4.2.2.

(a) *Let* $D_1 \subset \mathbb{C}^{d_1}$ *and* $D_2 \subset \mathbb{C}^{d_2}$ *be non-empty bounded open sets and suppose that* $\mathcal{H}_1 \subset \mathcal{O}(D_1)$ *and* $\mathcal{H}_2 \subset \mathcal{O}(D_2)$ *are analytic Beurling modules with reproducing kernels* K_1, K_2. *We define* $D = D_1 \times D_2$ *and claim that also* $\mathcal{H}_1 \circledast \mathcal{H}_2 \subset \mathcal{O}(D)$ *is an analytic Beurling module. We have already observed in Remark* 4.1.2 *that* $\mathcal{H}_1 \circledast \mathcal{H}_2$ *is an analytic Hilbert module. As in Remark* 3.1.2*(*c*), one checks that* $\frac{1}{K_1 \circledast K_2} = \frac{1}{K_1} \circledast \frac{1}{K_2}$ *belongs to* $\mathscr{S}^{(0)}(K_1 \circledast K_2)$. *This shows that* $\mathcal{H}_1 \circledast \mathcal{H}_2$ *satisfies condition (E), and condition (G) is elementary to verify. Turning towards condition (F), we have to prove that the Taylor spectrum of the commuting* $d_1 + d_2$*-tuple* $(M_{\mathbf{z}_{11}}, \dots, M_{\mathbf{z}_{1d_1}}, M_{\mathbf{z}_{21}}, \dots, M_{\mathbf{z}_{2d_2}})$ *is contained in* $\overline{D} = \overline{D_1} \times \overline{D_2}$. *Under the unitary identification* $\mathcal{H}_1 \otimes \mathcal{H}_2 \simeq \mathcal{H}_1 \circledast \mathcal{H}_2$, *this tuple is equivalent to the commuting tuple*

$$(M_{z_{11}} \otimes 1_{\mathcal{H}_2}, \dots, M_{z_{1d_1}} \otimes 1_{\mathcal{H}_2}, 1_{\mathcal{H}_1} \otimes M_{z_{21}}, \dots, 1_{\mathcal{H}_1} \otimes M_{z_{2d_2}}),$$

and the assertion follows by [34], Theorem 3.2.

(b) *The case of inner products is more difficult to treat. In fact, suppose that* $\mathcal{H}_1, \mathcal{H}_2$ *are analytic Beurling modules over* D. *Then, by the usual arguments, conditions (E) and (G) are satisfied. In order to show that the Taylor spectrum of* $M_{\mathbf{z}}$, *acting on* $\mathcal{H}_1 * \mathcal{H}_2$ *is included in* $\overline{D}$, *we have to require that* $\overline{D}$ *is Stein compact. By Example* 1.7.3 *(*b*), it suffices to show in this case, that* $\mathcal{O}(\overline{D})$ *is contained in* $\mathcal{M}(\mathcal{H}_1 * \mathcal{H}_2)$. *But this is trivially satisfied since on the one hand,* $\mathcal{O}(\overline{D}) \subset \mathcal{M}(\mathcal{H}_1)$ *and, on the other hand,* $\mathcal{M}(\mathcal{H}_1) \subset \mathcal{M}(\mathcal{H}_1 * \mathcal{H}_2)$.

Next we check that the standard reproducing kernel Hilbert spaces over bounded symmetric domains are in fact analytic Beurling modules whenever they are Beurling spaces.

Example 4.2.3. *Suppose that $D \subset \mathbb{C}^d$ is a Cartan domain of rank r and characteristic multiplicities a, b. Suppose further that $\nu \geq \frac{r-1}{2}a + 1$. For simplicity, we write $\mathcal{H} = \mathcal{H}_\nu$ and $K = K_\nu$. We saw in Example* 4.1.3 *that $\mathcal{H}$ is an analytic Hilbert module (this is even true for all ν in the continuous Wallach set). We already mentioned several times that $\sigma(M_{\mathbf{z}}) = \overline{D}$, as shown in [10]. Furthermore, we saw in Example* 3.1.3*(*c*) that $\mathcal{H}$ is a Beurling space. In particular, condition (E) is fulfilled. Turning towards condition (G), we fix some $z \in D$ and a positive number $0 < \rho < 1$ such that $\frac{z}{\rho} \in D$. Similarly to Example* 1.7.3 *(*c*), we see that the function*

$$\rho D \times \frac{1}{\rho} D \to \mathbb{C}\ ,\ (\zeta, \omega) \mapsto K(\frac{\zeta}{\rho}, \rho w)$$

*is a sesquianalytic non-vanishing extension of $K_{|\rho D \times D}$. In fact, this follows immediately by the Faraut-Koranyi expansion (*1.1.3*) and the homogeneity of the kernels $K_{\mathbf{m}}$.*

The additional conditions that characterize analytic Beurling modules now allow us to prove the following auxiliary result.

Lemma 4.2.4. *Suppose that $\mathcal{H}$ is an analytic Beurling module over D. Then the higher order kernels $K_w^{(\alpha)}$ defined in Section* 4.1 *belong to $\mathcal{O}(\overline{D})$ for all $w \in D$ and $\alpha \in \mathbb{N}_0^d$.*

Proof. Fix $w \in D$ and $\alpha \in \mathbb{N}_0^d$. We observe that

$$K_w^{(\alpha)}(z) = \langle K_w^{(\alpha)}, K(\cdot, z)\rangle = \overline{\langle K(\cdot, z), K_w^{(\alpha)}\rangle} = \overline{(D^\alpha K(\cdot, z))\,(w)}$$

holds for all $z \in D$. By condition (G) of Definition 4.2.1, there exist open neighbourhoods V of $\overline{D}$ and $U \subset D$ of w such that $K_{|U \times D}$ extends to a sesquianalytic function $H : U \times V \to \mathbb{C}$. But then

$$h : \tilde{V} \to \mathcal{O}(U)\ ,\ z \mapsto H(\cdot, \overline{z}),$$

defined on the open set $\tilde{V} = \{\overline{z}\ ;\ z \in V\}$, is analytic as a function with values in the Fréchet space $\mathcal{O}(U)$. Since continuous linear maps preserve analycity, it follows that the function

$$V \to \mathbb{C}\ ,\ z \mapsto \overline{(D^\alpha H(\cdot, z))\,(w)}$$

is analytic again and, as seen above, extends the function $K_w^{(\alpha)}$. □

The main result of this section can now be stated and proved.

Theorem 4.2.5. *Suppose that $\mathcal{H}$ is an analytic Beurling module over D with reproducing kernel K and that $M \subset \mathcal{H}$ is a finite-codimensional submodule.*

(a) M is Beurling decomposable. Moreover, $M^\perp \subset \mathcal{O}(\overline{D})$. The core function G_M can be sesquianalytically extended to a neighbourhood of $\overline{D} \times \overline{D}$ whenever $\frac{1}{K}$ can. In all cases, $G_M(\cdot, w) \in \mathcal{O}(\overline{D})$ for all $w \in D$.

(b) *Let us suppose in addition that M has finite rank and let us write $s = \operatorname{rank}_+ M$ and $t = \operatorname{rank}_- M$. Then G_M can be sesquianalytically extended on a neighbourhood of $\overline{D} \times \overline{D}$. Furthermore, there exist multipliers $\phi_1, \dots, \phi_s$ and $\psi_1, \dots \psi_t$ in $\mathcal{O}(\overline{D})$ such that*

$$P_M = \sum_{i=1}^{s} M_{\phi_i} M_{\phi_i}^* - \sum_{i=1}^{t} M_{\psi_i} M_{\psi_i}^*$$

and

$$M = \sum_{i=1}^{s} \phi_i \cdot \mathcal{H}$$

holds.

Proof. For the proof of (a), first note that $M^\perp \subset \mathcal{O}(\overline{D})$. Indeed, this follows directly from Corollary 4.1.5 and Lemma 4.2.4. Since M is invariant (and in particular K-invariant) by Corollary 4.1.5, the Beurling decomposability of M is guaranteed by Proposition 3.3.11. Furthermore, by Proposition 1.1.2,

$$K_{M^\perp}(z, w) = \sum_{i=1}^{n} g_i(z)\overline{g_i(w)} \quad (z, w \in D)$$

holds, whenever $g_1, \dots, g_n$ is an orthonormal basis of $M^\perp$. Therefore, $K_{M^\perp}$ can be sesquianalytically extended to a neighbourhood of $\overline{D} \times \overline{D}$. Clearly, if also $\frac{1}{K}$ admits such an extension, then so does $G_M = 1 - \frac{K_{M^\perp}}{K}$. Since in any case the functions $\frac{1}{K(\cdot, w)}$ $(w \in D)$ can be extended analytically to a neighbourhood of $\overline{D}$ by condition (G) of Definition 4.2.1, also the functions

$$G_M(\cdot, w) = 1 - \frac{1}{K(\cdot, w)} \cdot K_{M^\perp}(\cdot, w) \quad (w \in D)$$

belong to $\mathcal{O}(\overline{D})$. If in addition, M has finite rank, then the (finite-dimensional) range of Δ_M is given by

$$\operatorname{ran} \Delta_M = \operatorname{span}\{G_M(\cdot, w) \ ; \ w \in D\},$$

and hence included in $\mathcal{O}(\overline{D})$. Choosing functions $\phi_1, \dots, \phi_s$ and $\psi_1, \dots, \psi_t$ in $\operatorname{ran} \Delta_M$ such that

$$(\Delta_M)_+ = \sum_{i=1}^{s} \phi_i \otimes \phi_i \quad \text{and} \quad (\Delta_M)_- = \sum_{i=1}^{t} \psi_i \otimes \psi_i$$

shows that G_M admits a sesquianalytic extension on a neighbourhood of $\overline{D} \times \overline{D}$. The remaining assertions follow immediately by Theorem 3.3.5. □

Of particular interest are of course those analytic Hilbert modules $\mathcal{H}$ over D having the property that every finite-codimensional submodule automatically has finite rank. In Example 3.3.3 (a) we saw that this is the case if the inverse kernel $\frac{1}{K}$ is a polynomial in z and $\overline{w}$. And in fact, most of the occurring examples such as

- the (unweighted) Bergman spaces over arbitrary Cartan domains
- the Hardy spaces over Cartan domains of type $\mathrm{I}_{n,m}$, II_n with n even, III_n with n odd, IV_n with n even and V and VI (cf. [7], p. 17)
- the Arveson space over the unit ball

enjoy this property.

For such spaces, Theorem 4.2.5 allows us to compute the right essential spectrum of the tuple $M_{\mathbf{z}}$. Recall that the right essential spectrum $\sigma_{re}(T)$ of a commuting tuple $T \in B(H)^d$ is the set of all $\lambda \in \mathbb{C}^d$ for which the last cohomology group in the Koszul complex of the tuple $\lambda - T$ has infinite dimension. Equivalently, $\lambda \in \mathbb{C}^d$ does not belong to the right essential spectrum of T exactly if the row operator $(\lambda_1 - T_1, \dots, \lambda_d - T_d) \in B(H^d, H)$ has finite-codimensional range.

Proposition 4.2.6. *Suppose that $\mathcal{H}$ is an analytic Beurling module over D such that the inverse kernel $\frac{1}{K}$ is a polynomial in z and $\overline{w}$. Then $\sigma_{re}(M_{\mathbf{z}}) = \partial D$.*

Proof. First of all, observe that $\sigma_{re}(M_{\mathbf{z}}) \subset \sigma(M_{\mathbf{z}}) \subset \overline{D}$. We are now going to prove that $\sigma_{re}(M_{\mathbf{z}}) \cap D = \emptyset$. To this end, fix $\lambda \in D$ and let M_λ denote the finite-codimensional submodule

$$M_\lambda = \{f \in \mathcal{H} \ ; \ f(\lambda) = 0\} = \{K(\cdot, \lambda)\}^\perp.$$

Theorem 4.2.5 shows that there exist multipliers $\phi_1, \dots, \phi_s \in \mathcal{O}(\overline{D})$, such that

$$M_\lambda = \sum_{i=1}^{s} \phi_i \cdot \mathcal{H}.$$

The row operator $(M_{\phi_1}, \dots, M_{\phi_s}) \in B(\mathcal{H}^s, \mathcal{H})$ consequently has finite-codimensional range. This means that 0 is not in the right essential spectrum of the commuting tuple

$$M_\phi = (M_{\phi_1}, \dots, M_{\phi_s}) \in B(\mathcal{H})^s.$$

By the spectral mapping theorem for the right essential spectrum (Corollary 2.6.9 in [35]), we have

$$\sigma_{re}(M_\phi) = \phi(\sigma_{re}(M_{\mathbf{z}})).$$

Since $\phi(\lambda) = 0$, it follows that $\lambda \notin \sigma_{re}(M_{\mathbf{z}})$. This proves that $\sigma_{re}(M_{\mathbf{z}}) \subset \partial D$.

Suppose conversely that λ is in the boundary of D. Then λ is not a virtual point of $\mathcal{H}$. As observed in [25], Remark 2.2.2, this is equivalent to the fact that the maximal ideal of $\mathbb{C}[z]$ at λ is dense in $\mathcal{H}$ or, in other words, that

$$\overline{\sum_{i=1}^{d} (\lambda_i - M_{\mathbf{z}_i})\mathcal{H}} = \overline{\sum_{i=1}^{d} (\lambda_i - M_{\mathbf{z}_i})\mathbb{C}[z]} = \mathcal{H}.$$

Assume now that $\lambda \notin \sigma_{re}(M_{\mathbf{z}})$. Then the space

$$\sum_{i=1}^{d}(\lambda_i - M_{\mathbf{z}_i})\mathcal{H} \subset \mathcal{H}$$

is closed and therefore equals $\mathcal{H}$. Since the surjectivity spectrum is closed, there exists some $r > 0$ such that

$$\sum_{i=1}^{d}(\mu_i - M_{\mathbf{z}_i})\mathcal{H} = \mathcal{H}$$

holds for all $\mu \in \mathbb{C}^d$ with $|\mu - \lambda| < r$. Hence there would have to be a point $\mu \in D$ with $\mathbf{1} \in \sum_{i=1}^{d}(\mu_i - M_{\mathbf{z}_i})\mathcal{H}$. This contradiction completes the proof. $\square$

We are now able to give the following supplement to the Ahern-Clark type result stated in [25] as Theorem 2.2.3.

Corollary 4.2.7. *Suppose $\mathcal{H}$ is an analytic Beurling module over D such that the inverse kernel $\frac{1}{K}$ is a polynomial in z and $\overline{w}$. Then the finite-codimensional submodules of $\mathcal{H}$ are exactly the closed subspaces M of the form $M = \sum_{i=1}^{r} p_i \cdot \mathcal{H}$, where $r \in \mathbb{N}$ and $p = (p_1, \ldots, p_r)$ is a tuple of polynomials with $Z(p) \subset D$.*

Proof. Suppose that M is a finite-codimensional submodule of $\mathcal{H}$. By Theorem 2.2.3 in [25], the intersection $I = M \cap \mathbb{C}[z]$ is a finite-codimensional ideal in $\mathbb{C}[z]$ with $Z(I) \subset D$ and $M = \overline{I}$. Now we choose a generating set $p = (p_1, \ldots, p_r)$ of I and claim that $M = \sum_{i=1}^{r} p_i \cdot \mathcal{H}$. Since

$$M = \overline{I} = \overline{\sum_{i=1}^{r} p_i \cdot \mathbb{C}[z]} = \overline{\sum_{i=1}^{r} p_i \cdot \mathcal{H}},$$

it suffices to show that the row operator $(M_{p_1}, \ldots, M_{p_r}) \in B(\mathcal{H}^r, \mathcal{H})$ has closed range. But this is obvious, because $Z(p) = Z(I) \subset D$ and $\sigma_{re}(M_{\mathbf{z}}) = \partial D$, and hence

$$0 \notin p(\sigma_{re}(M_{\mathbf{z}})) = \sigma_{re}(M_{p_1}, \ldots, M_{p_r}).$$

$\square$

Remark 4.2.8. *The proof shows that the polynomials $p_1, \ldots, p_r$ can be chosen as any generating set of the ideal $M \cap \mathbb{C}[z]$. If in particular $d = 1$, then we can achieve that $r = 1$.*

We also point out the connection of the preceding corollary with results proved in [15], in the setting of Bergman spaces.

Note also that, in this situation, Gleason's problem can be solved in $\mathcal{H}$. Recall that Gleason's problem is the question whether, for a every $f \in \mathcal{H}$ and $\lambda \in D$, there exist functions $g_1, \ldots, g_d \in \mathcal{H}$ satisfying

$$f(z) - f(\lambda) = \sum_{i=1}^{d}(z_i - \lambda_i)g_i(z) \quad (z \in D).$$

Corollary 4.2.9. *Suppose that $\mathcal{H}$ is an analytic Beurling module over D such that the inverse kernel $\frac{1}{K}$ is a polynomial in z and $\overline{w}$. Then Gleason's problem is solvable in $\mathcal{H}$.*

Proof. Apply Corollary 4.2.7 to the submodule $M_\lambda = \{h \in \mathcal{H}\ ;\ h(\lambda) = 0\}$. $\square$

An examination of the proofs of Proposition 4.2.6 and Corollary 4.2.7 reveals the following result.

Corollary 4.2.10. *For an analytic Beurling module $\mathcal{H}$ over D, the following are equivalent:*

(i) $\sigma_{re}(M_{\mathbf{z}}) = \partial D$.

(ii) Gleason's problem can be solved in $\mathcal{H}$.

The motivates the conjecture that Proposition 4.2.6 and Corollaries 4.2.7 and 4.2.9 remain valid without the assumption that the inverse kernel is a polynomial in z and $\overline{w}$.

5 Beurling decomposability in Hardy and Bergman spaces

5.1 Hardy spaces

5.1.1 Preliminaries

The aim of this section is to discuss the phenomenon of Beurling decomposability in the setting of the Hardy space over some bounded symmetric domain.

So throughout this section, let $D \subset \mathbb{C}^d$ be a Cartan domain of rank r, and let $H^2(D) = \mathcal{H}_{\frac{d}{r}}$ denote the Hardy space over D as explained in Example 1.1.10. Furthermore, $K = K_{\frac{d}{r}}$ denotes the reproducing kernel of $H^2(D)$. As usual, S is the Shilov boundary of D and can be defined as the smallest closed subset of the topological boundary ∂D such that every function $\phi \in C(\overline{D})$ that is holomorphic on D assumes its maximum on S. As in Example 1.1.10, we denote by σ the canonical probability measure on S.

A function $f : D \to V$ having values in some topological vector space V is said to have radial limit $v \in V$ at $\zeta \in S$ if $\lim_{r \nearrow 1} f(r\zeta) = v$ holds in the topology of V. We define

$$f^* : S \to V \ , \ f^*(\zeta) = \begin{cases} \lim_{r \nearrow 1} f(r\zeta) & , \quad \text{if the limit exists} \\ 0 & , \quad \text{else} \end{cases} .$$

The Poisson kernel P on D is defined by

$$P : S \times D \to \mathbb{C} \ , \ P(\zeta, z) = \frac{|K(\zeta, z)|^2}{K(z, z)}.$$

Note that the right-hand side is well defined, since the functions $K(\cdot, z)$ $(z \in D)$ can be analytically extended onto an open neighbourhood of $\overline{D}$ as seen in Example 1.7.3 (c). We shall sometimes write

$$P_z : S \to \mathbb{C} \ , \ P_z(\zeta) = P(\zeta, z) \quad (z \in D).$$

Since P_z is continuous on S, it makes sense to define

$$P[h] : D \to \mathbb{C} \ , \ P[h](z) = \int_S h(\zeta) P_z(\zeta) \, d\sigma(\zeta)$$

for every function $h \in L^1(S, \sigma)$. The function $P[h]$ is called the Poisson transform of h. One then proves that $P[h]$ has radial limits almost everywhere on S and that $P[h]^* = h$ almost everywhere. Analogous to the classical theory on the unit disk, one defines $H^2(S)$ as the closure of the analytic polynomials in $L^2(S, \sigma)$ and proves that the restricted Poisson transform

$$H^2(S) \to H^2(D) \ , \ h \mapsto P[h]$$

is well defined and unitary. Equivalently, every function $f \in H^2(D)$ has radial limits almost everywhere on S, and the mapping

$$H^2(D) \to H^2(S) \ , \ f \mapsto f^* \tag{5.1.1}$$

is unitary.

In analogy to [61], Chapter 4, the following Fatou-type theorem can be proved: Suppose that $\mathcal{D}, \mathcal{E}$ are separable Hilbert spaces and that either $f \in H^2(D)_{\mathcal{E}}$ or $f \in H^\infty(D, B(\mathcal{D}, \mathcal{E}))$. Then f has radial limits in almost all points $\zeta \in S$ (with respect to the norm topology of $\mathcal{E}$ in the first case and with respect to the strong operator topology of $B(\mathcal{D}, \mathcal{E})$ in the second case). Furthermore, if f^* vanishes on a subset of positive measure, then $f = 0$. Note that the required separability of the underlying Hilbert spaces $\mathcal{D}$ and $\mathcal{E}$ is essential for this theorem. In analogy to the scalar case, the mapping

$$H^2(D)_{\mathcal{E}} \to L^2(S, \sigma, \mathcal{E}) \ , \ f \mapsto f^*$$

is an isometry.

We finish this preliminary paragraph with an observation which will turn out to be useful for the rest of this section:

For $w \in D$, let us write

$$k_w : D \to \mathbb{C} \ , \ k_w(z) = \frac{K(z,w)}{K(w,w)^{\frac{1}{2}}}.$$

Then k_w is a multiplier of $H^2(D)$, has unit norm in $H^2(D)$ and can be analytically extended onto a neighbourhood of $\overline{D}$. Moreover, $P_z(\zeta) = |k_z(\zeta)|^2$ holds for all $z \in D$ and $\zeta \in S$. Suppose that f is a function in $H^2(D)_{\mathcal{E}}$. Then

$$\lim_{r \nearrow 1} \|k_{r\zeta} \cdot f\| = \|f^*(\zeta)\| \tag{5.1.2}$$

holds for almost all $\zeta \in S$. In fact,

$$\begin{aligned} \|k_z \cdot f\|^2 &= \int_S \|k_z(\zeta) f^*(\zeta)\|^2 \, d\sigma(\zeta) \\ &= \int_S \|f^*(\zeta)\|^2 P_z(\zeta) \, d\sigma(\zeta) \\ &= P[\|f^*\|^2](z) \end{aligned}$$

holds for all $z \in D$, and the last expression has radial limit $\|f^*(\zeta)\|^2$ at almost every $\zeta \in S$.

5.1.2 Boundary values of the defect function

In recent time, it was observed independenty by several authors that, given some invariant subspace M of the Hardy space over the unit ball or the polydisk in $\mathbb{C}^d$, the core function G_M has much better boundary behaviour than the reproducing kernel K_M of M. We mention in particular the work of Guo et al. ([40] and [41]), which can be regarded as a starting point for this paper. An analogous observation for the Arveson space can be found in [39].

We next show how to extend the known results using the framework of Beurling decomposability.

Theorem 5.1.1. *Suppose that $\mathcal{E}$ is a separable Hilbert space and that M is a non-zero invariant subspace of $H^2(D)_{\mathcal{E}}$. Define $M_z = \{f(z)\ ;\ f \in M\}$ for $z \in D$ and $m = \sup_{z\in D} \dim M_z$.*

(a) We have $\lim_{r\nearrow 1} \|D_M(r\zeta)\| = 1$ for almost all $\zeta \in S$.

(b) If in addition, M is Beurling decomposable, then D_M has radial limits at almost every point $\zeta \in S$ (with respect to the strong operator topology). Moreover, $D_M^*(\zeta)$ is an orthogonal projection of rank m for almost all $\zeta \in S$.

Proof. As before, we write

$$k_w : D \to \mathbb{C}\ ,\ k_w(z) = \frac{K(z,w)}{K(w,w)^{\frac{1}{2}}} \quad (w \in D)$$

for the normalized kernel functions. Then clearly, $k_w \in \mathcal{M}(H^2(D))$ for all $w \in D$. Our first observation is the following: For all $0 \neq f \in M$, we have

$$D_M(z) \geq \frac{1}{\|k_z \cdot f\|^2} f(z) \otimes f(z) \tag{5.1.3}$$

for all $z \in D$. In fact, since the invariant subspace

$$[f] = \overline{\{\phi \cdot f\ ;\ \phi \in \mathcal{M}(H^2(D))\}}$$

is contained in M, it follows from the definition of D_M that

$$\begin{aligned}
\langle D_M(z)x, x\rangle &= \frac{\|P_M K(\cdot,z)x\|^2}{K(z,z)} \\
&\geq \frac{\|P_{[f]} K(\cdot,z)x\|^2}{K(z,z)} \\
&= \sup\{\frac{|\langle K(\cdot,z)x, \phi\cdot f\rangle|^2}{K(z,z)\|\phi\cdot f\|^2}\ ;\ \phi \in \mathcal{M}(H^2(D)), \phi\cdot f \neq 0\} \\
&\geq \frac{|\langle K(\cdot,z)x, k_z\cdot f\rangle|^2}{K(z,z)\|k_z\cdot f\|^2} \\
&= \frac{k_z(z)^2|\langle x, f(z)\rangle|^2}{K(z,z)\|k_z\cdot f\|^2} \\
&= \frac{\langle f(z)\otimes f(z)x, x\rangle}{\|k_z\cdot f\|^2}
\end{aligned}$$

holds for all $z \in D$ and $x \in \mathcal{E}$ which proves the claim.

From the definition of D_M (cf. Section 3.3) and from equation (5.1.3) it is clear that, for every function $0 \neq f \in M$, the estimates

$$1 \geq \|D_M(z)\| \geq \frac{\|f(z)\|^2}{\|k_z \cdot f\|^2} \tag{5.1.4}$$

hold for all $z \in D$. By (5.1.2), $\lim_{r \nearrow 1} \|k_{r\zeta} \cdot f\| = \|f^*(\zeta)\|$ holds for almost all $\zeta \in S$. Furthermore, $f^*(\zeta) \neq 0$ for almost all $\zeta \in S$ by our preliminary remarks. This implies that the right-hand side of (5.1.4) has radial limit 1 at almost all points $\zeta \in S$. Hence $\lim_{r \nearrow 1} \|D_M(r\zeta)\| = 1$ exists almost everywhere, and part (a) is proved.

Turning towards (b), we fix a Beurling decomposition (ϕ_1, ϕ_2) of M with multipliers $\phi_i \in \mathcal{M}(H^2(D)_{\mathcal{D}_i}, H^2(D)_{\mathcal{E}}) = H^\infty(D, B(\mathcal{D}_i, \mathcal{E}))$, $i = 1, 2$. Since $\mathcal{E}$ was supposed to be separable, we can achieve that $\mathcal{D}_1, \mathcal{D}_2$ are separable as well. In fact, by Theorem 3.3.5, one could choose $\mathcal{D}_1, \mathcal{D}_2$ as the reproducing kernel Hilbert spaces assoicated with suitable positive kernels G_1, G_2, where G_1, G_2 are a disjoint decomposition of G_M in $\mathscr{S}_{\mathcal{E}}^{(0)}(K)$. Since the kernels G_1, G_2 are necessarily sesquianalytic, the chosen spaces $\mathcal{D}_1, \mathcal{D}_2$ are indeed separable by Example 1.1.3 (b).

As noted above, it is proved in [61], Chapter 4, that the bounded holomorphic functions ϕ_1, ϕ_2 have radial limits (SOT) almost everywhere on S. By similar techniques, it is possible to show that also the functions

$$D \to B(\mathcal{D}_i, \mathcal{E})\ ,\ z \mapsto \phi_i(z)^* \quad (i = 1, 2),$$

have radial limits (SOT) almost everywhere on S (note that this does not follow trivially, since the involution is not SOT continuous). We fix some measurable subset S_0 of S such that $S \backslash S_0$ is a σ-zero set and such that the radial limits

$$\lim_{r \nearrow 1} \phi_i(r\zeta) \quad \text{and} \quad \lim_{r \nearrow 1} \phi_i(r\zeta)^* \quad (i = 1, 2)$$

exist in the SOT sense for all $\zeta \in S_0$. Since $\|\phi_i(z)\| \leq \|\phi_i\|_{\infty, D}$ for all $z \in D$ and $i = 1, 2$ and since multiplication is SOT continuous on bounded sets, it follows that

$$D_M^*(\zeta) = \lim_{r \nearrow 1} D_M(r\zeta) = \phi_1^*(\zeta)\phi_1^*(\zeta)^* - \phi_2^*(\zeta)\phi_2^*(\zeta)^* \tag{5.1.5}$$

exists in the SOT sense for all $\zeta \in S_0$. We claim now that $D_M^*(\zeta) \leq P_\zeta$ for all $\zeta \in S_0$, where $P_\zeta \in B(\mathcal{E})$ denotes the orthogonal projection onto $\overline{\operatorname{ran} \phi_1^*(\zeta)}$. In fact, the positivity of $D_M^*(\zeta)$ and (5.1.5) imply the existence of some positive contraction $X \in B(\mathcal{D}_1)$ such that

$$D_M^*(\zeta) = \phi_1^*(\zeta)(1_{\mathcal{D}_1} - X)\phi_1^*(\zeta)^*,$$

proving that $\operatorname{ran} D_M^*(\zeta) \subset \operatorname{ran} \phi_1^*(\zeta)$. Furthermore, since $D_M^*(\zeta)$ is contractive as SOT limit of contractions for all $\zeta \in S_0$, it follows that $D_M^*(\zeta) \leq P_\zeta$.

In order to prove equality, we fix a dense subset $\{x_n \ ; \ n \in \mathbb{N}\}$ of $\mathcal{D}_1$. Note that this is possible, since $\mathcal{D}_1, \mathcal{D}_2$ were chosen as separable Hilbert spaces. Using (5.1.2), we can find some measurable set $S_1 \subset S_0$ such that $S \backslash S_1$ is a σ-zero set and such that

$$\lim_{r \nearrow 1} \|k_{r\zeta} \cdot \phi_1(\cdot)x_n\| = \|\phi_1^*(\zeta)x_n\|$$

holds for all $\zeta \in S_1$ and all $n \in \mathbb{N}$. For arbitrary $\zeta \in S_1$ and for $n \in \mathbb{N}$ with $\phi_1^*(\zeta)x_n \neq 0$, an application of inequality (5.1.3) with $f = \phi_1(\cdot)x_n \in M \backslash \{0\}$ yields that

$$\begin{aligned} \langle D_M^*(\zeta)\phi_1^*(\zeta)x_n, \phi_1^*(\zeta)x_n \rangle &= \lim_{r \nearrow 1} \langle D_M(r\zeta)\phi_1^*(\zeta)x_n, \phi_1^*(\zeta)x_n \rangle \\ &\geq \lim_{r \nearrow 1} \frac{|\langle \phi_1^*(\zeta)x_n, \phi_1(r\zeta)x_n \rangle|^2}{\|k_{r\zeta} \cdot \phi_1(\cdot)x_n\|^2} \\ &= \|\phi_1^*(\zeta)x_n\|^2. \end{aligned}$$

By continuity, we see that

$$\langle D_M^*(\zeta)y, y \rangle \geq \|y\|^2$$

holds for all $\zeta \in S_1$ and all $y \in \overline{\operatorname{ran} \phi_1^*(\zeta)}$. Since we already know that

$$D_M^*(\zeta) = P_\zeta D_M^*(\zeta) P_\zeta,$$

holds for all $\zeta \in S_0$, it follows that $D_M^*(\zeta) \geq P_\zeta$ for all $\zeta \in S_1$.

Thus we have proved that $D_M^*(\zeta) = P_\zeta$ holds almost everywhere. To prove that the rank of P_ζ equals m almost everywhere, it clearly suffices to show that, for every multiplier $\phi \in \mathcal{M}(H^2(D)_{\mathcal{D}}, H^2(D)_{\mathcal{E}})$,

$$\operatorname{rank} \phi^*(\zeta) = \sup_{z \in D} \operatorname{rank} \phi(z)$$

for almost all $\zeta \in S$. In the case of the unit ball, this is exactly Lemma 3.1 in [39], and the general case follows analogously. □

Remark 5.1.2.

(a) *If $\mathcal{E} = \mathbb{C}$, then part (a) of Theorem* 5.1.1 *means that, for every non-zero invariant subspace M of $H^2(D)$, $\lim_{r \nearrow 1} D_M(r\zeta) = 1$ holds at almost every $\zeta \in S$. Therefore (b) is fulfilled for all invariant subspaces and not only for Beurling decomposable ones. Therefore we strongly conjecture that the required Beurling decomposability in part (b) of the theorem can be dropped in the general case.*

(b) *A careful examination of the proof of Theorem* 5.1.1 *shows that the assertion remains valid if $H^2(D)$ is replaced by some Beurling space $\mathcal{H} \subset \mathcal{O}(D)$ which is contained in $H^2(D)$ and satisfies condition (*5.1.2*), that is, for every separable Hilbert space $\mathcal{E}$ and every $f \in \mathcal{H}_{\mathcal{E}}$,*

$$\lim_{r \nearrow 1} \|\frac{K(\cdot, r\zeta)}{K(r\zeta, r\zeta)^{\frac{1}{2}}} \cdot f\| = \|f^*(\zeta)\|$$

holds for almost all $\zeta \in S$ (where K denotes the reproducing kernel of $\mathcal{H}$).

It is shown in [39], Proposition 2.4, that the Arveson space $H(\mathbb{B}_d)$ satisfies these conditions, and it is not hard to check that Theorem 5.1.1 *then reduces to the main result of [39], since all invariant subspaces of $H(\mathbb{B}_d)_{\mathcal{E}}$ are Beurling decomposable by Theorem* 3.3.8.

(c) *By the same reasonings it is clear that Theorem* 5.1.1 *is also true if $H^2(D)$ is replaced by the Hardy space over some reducible bounded symmetric domain such as the polydisk in $\mathbb{C}^d$.*

As mentioned already in Example 3.3.10 (a), positive definite core functions of invariant subspaces of $H^2(D)$ can be factorized by a single inner function. Recall that a function $\eta \in H^\infty(D)$ is called inner if $|f^*| = 1$ almost everywhere on S. The following result contains, in view of Theorem 3.3.8, the classical Beurling theorem on the unit disk and also results of Guo [40],[41].

Proposition 5.1.3. *Suppose that $M \subset H^2(D)$ is a non-zero closed subspace of $H^2(D)$. Then the following are equivalent:*

(i) G_M is positive definite.

(ii) There exists an inner function $\eta \in H^\infty(D)$ such that $G_M(z,w) = \eta(z)\overline{\eta(w)}$ holds for all $z, w \in D$.

In this case, $M = \eta \cdot H^2(D)$ and M is invariant.

Proof. It is clear that G_M is a positive kernel if condition (ii) is fulfilled. By Proposition 3.3.7, it is also clear that, in this case, $P_M = M_\eta M_\eta^*$ holds and hence that $M = \operatorname{ran} M_\eta$. Suppose conversely that G_M is positive definite. Then it clearly suffices to show that $\operatorname{rank} G_M = 1$. In fact, if G_M has rank one, then M is invariant by Proposition 3.3.7, and there exists a multiplier $\eta \in \mathcal{M}(H^2(D)) = H^\infty(D)$ such that $G_M(z,w) = \eta(z)\overline{\eta(w)}$ holds for all $z, w \in D$. By Theorem 5.1.1, we obtain

$$\lim_{r \nearrow 1} |\eta(r\zeta)|^2 = \lim_{r \nearrow 1} D_M(r\zeta) = 1$$

for almost all $\zeta \in S$, proving that η is inner.

To see that $\operatorname{rank} G_M = 1$ (or, equivalently, that the positive operator Δ_M has rank one), it suffices to prove that $\operatorname{trace} \Delta_M = \|\Delta_M\|$. Since

$$K_M - G_M = G_M(K - 1)$$

is positive definite (since $H^2(D)$ contains the constant functions with $\|\mathbf{1}\| = 1$), we see that $\|G_M\|_K \leq 1$ and hence that $\|\Delta_M\| \leq 1$. Example 3.3.3 (b) shows that 1 is an eigenvalue of Δ_M, which implies that $\|\Delta_M\| = 1$. So we want to prove that trace $\Delta_M = 1$. To this end, choose a Kolmogorov factorization $(\mathcal{D}, \phi)$ of G_M such that $\mathcal{D}$ has a countable orthonormal basis $(e_n)_n$ (cf. Example 1.1.3(b)). Then ϕ and also the functions ϕ_n defined by

$$\phi_n : D \to \mathbb{C} \ , \ \phi_n(z) = \phi(z) e_n \quad (n \in \mathbb{N}),$$

are multipliers by Lemma 1.7.13. It is easily checked that

$$\Delta_M = \sum_n \phi_n \otimes \phi_n \quad (SOT).$$

Furthermore, the proof of Theorem 5.1.1 shows that

$$D_M^*(\zeta) = \phi^*(\zeta)\phi^*(\zeta)^*(1) = \sum_n |\phi_n^*(\zeta)|^2$$

holds for almost all $\zeta \in S$. We obtain

$$\begin{aligned}
\operatorname{trace} \Delta_M &= \sum_{n=0}^{\infty} \|\phi_n\|^2 \\
&= \sum_{n=0}^{\infty} \int_S |\phi_n^*(\zeta)|^2 \, d\sigma(\zeta) \\
&= \int_S \sum_{n=0}^{\infty} |\phi_n^*(\zeta)|^2 \, d\sigma(\zeta) \qquad \text{(by Beppo-Levi)} \\
&= \int_S D_M^*(\zeta) \, d\sigma(\zeta) \\
&= \int_S 1 \, d\sigma(\zeta) \qquad \text{(by Theorem 5.1.1)} \\
&= 1,
\end{aligned}$$

which finishes the proof. $\square$

Remark 5.1.4. *In view of the preceding proposition, it would be natural to conjecture that, given a Beurling decomposable subspace M of $H^2(D)$, the positive rank of M is* 1*. However, it follows by Example* 3.3.3 *(*c*) that the core operator Δ_M of the Beurling decomposable subspace*

$$M = \{f \in H^2(\mathbb{B}_2) \ ; \ f(0) = 0\}$$

of $H^2(\mathbb{B}_2)$ has spectrum $\sigma\{0, 1, \frac{-1}{3}\}$, and the eigenspace for the eigenvalue 1 *consists of all homogeneous polynomials of degree* 1 *and has is therefore two-dimensional.*

Furthermore, the core operator of the Beurling decomposable subspace

$$M = \{f \in H^2(\mathbb{B}_3) \ ; \ f(0) = 0\}$$

of $H^2(\mathbb{B}_3)$ has spectrum $\sigma(\Delta_M) = \{0, 1, -\frac{1}{2}, \frac{1}{10}\}$, which shows that there can even be different positive eigenvalues.

5.2 The Bergman space

In this section we shall discuss the phenomenon of Beurling decomposable subspaces in the context of the Bergman space $L^2_a(\mathbb{D})$. In strong contrast to the Hardy space $H^2(\mathbb{D})$, the lattice of invariant subspaces of the Bergman space is very complicated.

By Theorem 3.3.8 and Proposition 5.1.3, Beurling's Theorem is equivalent to the statement that the core function of every invariant subspace of $H^2(\mathbb{D})$ is positive. As seen in Example 3.3.10 (b), a similar statement must fail in the Bergman space. Motivated by this defect, it seems to be a good strategy to identify the Beurling decomposable subspaces of the Bergman space.

A second aim of this section is to establish a connection between our results and the 'Wandering Subspace Theorem' [3] which is often regarded as a weak replacement for Beurling's Theorem in the setting of the Bergman space. In particular, we present a simplified proof which involves reproducing kernel techniques in a more effective way than the known proofs do.

5.2.1 Preliminaries

The aim of this section is to provide some introductory results on the Bergman space $L^2_a(\mathbb{D})$ over the unit disk. The norm of $L^2_a(\mathbb{D})$ is defined as

$$\|f\| = \left(\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z)\right)^{\frac{1}{2}} \quad (f \in L^2_a(\mathbb{D})),$$

where $\mu = \frac{\lambda}{\pi}$ is the normalized planar Lebesgue measure on $\mathbb{D}$. The reproducing kernel K of $L^2_a(\mathbb{D})$ is given by

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C} \ , \ K(z,w) = \left(\frac{1}{1 - z\overline{w}}\right)^2 .$$

The multiplier algebra of $L^2_a(\mathbb{D})$ is $H^\infty(\mathbb{D})$ completely isometrically, that is,

$$H^\infty(\mathbb{D}, B(\mathcal{E}_1, \mathcal{E}_2)) = \mathcal{M}(L^2_a(\mathbb{D})_{\mathcal{E}_1}, L^2_a(\mathbb{D})_{\mathcal{E}_2})$$

holds with equality of norms for all Hilbert spaces $\mathcal{E}_1, \mathcal{E}_2$. It follows by Example 3.2.3 that we do not have to distinguish between invariant subspaces and subspaces being invariant under the multiplication with z.

Recall that Beurling's Theorem for the Hardy space $H^2(\mathbb{D})$ can equivalently be formulated as follows:

> Whenever M is an invariant subspace of $H^2(\mathbb{D})$, then the space $M \ominus z \cdot M$ has dimension 1 and is spanned by an inner function. In particular, $M = \operatorname{ran} M_\phi$ and $M = [M \ominus z \cdot M]$.

In particular, M is completely determined by the space $M \ominus z \cdot M$. It is therefore natural to ask how much information the space $M \ominus z \cdot M$ contains in the case of an invariant subspace M of the Bergman space $L^2_a(\mathbb{D})$. A first and highly non-trivial observation is that, contrary to the Hardy space situation, the index of M, defined as

$$\operatorname{ind} M = \dim M \ominus z \cdot M,$$

can attain every value in $\mathbb{N}_0 \cup \{\infty\}$. We refer the reader to [46], Chapter 6 for details. However, there are important classes of invariant subspaces having index one. First, subspaces $M_f = [f]$ generated by a single function $f \in L^2_a(\mathbb{D})$ and secondly, the so-called zero-based subspaces

$$M_A = \{f \in L^2_a(\mathbb{D})\ ;\ f \text{ has a zero of order at least } n_k \text{ at } z_k \text{ for all } k\},$$

where $A = (z_1, \ldots, z_1, z_2, \ldots, z_2, \ldots)$ is a (possibly finite) sequence of points in $\mathbb{D}$ such that the multiplicity of z_k in A is n_k. Those sequences A for which M_A is non-trivial are called $L^2_a(\mathbb{D})$-zero sequences (or zero sets).

Let us for the moment consider an invariant subspace M of the Hardy space $H^2(\mathbb{D})$ and write $n = \operatorname{ord}(M)$ for the minimal order of 0 as a zero of functions in M. It is not hard to see that the inner function $\phi \in M \ominus z \cdot M$ satisfying $\phi^{(n)}(0) > 0$ is the unique solution of the extremal problem

$$\sup\{\operatorname{Re} f^{(n)}(0)\ ;\ f \in M, \|f\| \leq 1\}. \tag{5.2.1}$$

An important breakthrough in the study of the Bergman space was the observation that it makes sense to pose the extremal problem (5.2.1) for invariant subspaces M of $L^2_a(\mathbb{D})$. More precisely, it can be shown (see [46], Theorem 3.5) that the above extremal problem always has a unique solution, which is called the extremal function g_M of M. Moreover, the extremal function g_M belongs to $M \ominus z \cdot M$. If M has index one, then $M \ominus z \cdot M$ is clearly spanned by g_M. If $M = M_A$ is a zero-based invariant subspace of $L^2_a(\mathbb{D})$, then the extremal function of M is often denoted by g_A instead of g_{M_A}.

5.2.2 Decomposition of the core function

Let M be an invariant subspace of $L^2_a(\mathbb{D})$. We observed in Example 3.3.3 (a) that the non-zero space $M \ominus z \cdot M$ consists of eigenvectors of Δ_M for the eigenvalue 1. Buth then, an elementary calculation shows that $K_{M \ominus z \cdot M} \leq (G_M)_+$. The following result, which strenghtens Theorem 0.6 in [52], shows that much more is true in the Bergman space.

Proposition 5.2.1. *Suppose that M is an invariant subspace of $L^2_a(\mathbb{D})$. Then there exists a positive definite kernel $v \in B(K_M)$ such that*

$$(G_M)_+(z,w) = K_{M \ominus z \cdot M}(z,w) \quad \textit{and} \quad (G_M)_-(z,w) = 2z\overline{w}\ v(z,w)$$

holds for all $z, w \in \mathbb{D}$. In particular, the identities

$$\operatorname{rank}_+ M = \operatorname{ind} M \quad \textit{and} \quad \operatorname{rank}_- M = \operatorname{rank} v$$

hold.

Proof. By Theorem 0.6 in [52], there exists a positive kernel v such that

$$G_M(z,w) = K_{M\ominus z\cdot M}(z,w) - 2z\overline{w}\; v(z,w)$$

holds for all $z, w \in \mathbb{D}$. Furthermore, a closer look at the proof of Lemma 1.4 in [52] reveals that in fact $v \in B(K_M)$.

We already observed that $K_{M\ominus z\cdot M} \leq (G_M)_+$. Writing

$$L(z,w) = 2z\overline{w}\; v(z,w) \quad (z, w \in \mathbb{D}),$$

we obtain that

$$(G_M)_+ - (G_M)_- = G_M = K_{M\ominus z\cdot M} - L \leq (G_M)_+ - L,$$

and hence that $L \leq (G_M)_-$. The disjointness of the kernels $(G_M)_+$ and $(G_M)_-$ shows that $(G_M)_+ = K_{M\ominus z\cdot M}$ and $(G_M)_- = L$. ☐

Remark 5.2.2. *Proposition* 5.2.1 *characterizes the Bergman space $L^2_a(\mathbb{D})$ in the following sense:*

Let $\mathcal{H}_\nu \subset \mathcal{O}(\mathbb{D})$ denote the reproducing kernel Hilbert spaces with reproducing kernels

$$K_\nu : \mathbb{D}\times\mathbb{D} \to \mathbb{C}\ ,\ K_\nu(z,w) = \left(\frac{1}{1-z\overline{w}}\right)^{\nu} \quad (\nu \geq 1).$$

Then the following are equivalent:

(i) $\nu \leq 2$.

(ii) The statement of Proposition 5.2.1 *holds for $\mathcal{H}_\nu$.*

(iii) For every invariant subspace M of $\mathcal{H}_\nu$, we have $\sigma(\Delta_M) \cap (0,\infty) = \{1\}$.

If $1 \leq \nu \leq 2$, then K_ν is a B-kernel in the sense of [52], and we can repeat the proof of Proposition 5.2.1*. This shows the implication (i) to (ii). The implication (ii) to (iii) is trivial. To prove the remaining implication (iii) to (i), consider the invariant subspace $M = \{f \in \mathcal{H}_\nu\ ;\ f(0) = 0\}$. In Example* 3.3.3 *(*c*), we determined the spectrum of Δ_M and obtained*

$$\sigma(\Delta_M) = \left\{-\frac{(-\nu)_m}{(\nu)_m}\ ;\ m \geq 1\right\} \cup \{0\}.$$

*A closer look at the definition of the Pochhammer symbols (*1.1.4*) reveals that the eigenvalue corresponding to $m = 3$ is*

$$\lambda_3 = \frac{(1-\nu)(2-\nu)}{(1+\nu)(2+\nu)},$$

which is clearly positive and different from 1 *if* $\nu > 2$.

Therefore, the Bergman space is maximal among all spaces $\mathcal{H}_\nu$, $\nu \geq 1$, *admitting the special decomposition of the core function as described in Proposition* 5.2.1.

Without doubt, one of the great achievements in the study of the Hardy space $H^2(\mathbb{D})$ is, besides Beurling's Theorem, the factorization theorem for $H^2(\mathbb{D})$, which asserts that the zero sets of $H^2(\mathbb{D})$ are exactly the Blaschke sequences and that Blaschke products are contractive zero divisors of $H^2(\mathbb{D})$. More precisely, for a function $f \in H^2(\mathbb{D})$, the Blaschke product B_A corresponding to the zero sequence A of f (counting multiplicities) divides f in $H^2(\mathbb{D})$ without increasing the norm, that is, the function $\frac{f}{B_A}$, extended analytically across the singularities, belongs to $H^2(\mathbb{D})$ with $\|\frac{f}{B_A}\| \leq \|f\|$. It was recognized first by Hedenmalm (see [44] and [46]) that, in the Bergman space, the role of Blaschke products is played by the extremal functions of zero-based invariant subspaces: If A is an $L^2_a(\mathbb{D})$-zero set, then the extremal function g_A of the zero-based subspace M_A has the 'contractive divisor property' for M_A, which means that the function $\frac{f}{g_A}$, extended analytically across the singularitites, belongs to $L^2_a(\mathbb{D})$ with $\|\frac{f}{g_A}\| \leq \|f\|$ for all $f \in M_A$.

We now want to demonstrate how the 'contractive divisor property' of the extremal function (even for arbitrary index-one invariant subspaces) can be deduced directly from Proposition 5.2.1.

Corollary 5.2.3. *Suppose that* M *is an invariant subspace of* $L^2_a(\mathbb{D})$ *of index one.*

(a) We have

$$K_{M\ominus z\cdot M}(z,w) = g_M(z)\overline{g_M(w)} \quad (z,w \in \mathbb{D}).$$

Moreover, the decomposition

$$G_M(z,w) = g_M(z)\overline{g_M(w)} - 2z\overline{w}\ v(z,w) \quad (z,w \in \mathbb{D})$$

(with v *as in Proposition* 5.2.1*) is the spectral decomposition of the core function.*

(b) For every $f \in M$, *the function* $\frac{f}{g_M}$ *extends analytically across the singularities and belongs to* $L^2_a(\mathbb{D})$ *with* $\|\frac{f}{g_M}\| \leq \|f\|$.

(c) For every $\phi \in M \cap H^\infty(\mathbb{D})$, *the function* $\frac{\phi}{g_M}$ *extends analytically across the singularities and belongs to* $H^\infty(\mathbb{D})$ *with* $\|\frac{\phi}{g_M}\|_{\infty,\mathbb{D}} \leq \|\phi\|_{\infty,\mathbb{D}}$.

Proof. The first equality of (a) is clear since g_M is a unit vector in the one-dimensional space $M \ominus z \cdot M$. The second assertion follows by Proposition 5.2.1.

In order to prove (b), we set $D_0 = \{z \in \mathbb{D}\ ;\ g_M(z) \neq 0\}$ and define $\mathcal{H}_0 = L^2_a(\mathbb{D})_{|D_0}$ and $M_0 = M_{|D_0}$. Since the restriction map $L^2_a(\mathbb{D}) \to \mathcal{H}_0\ ,\ f \mapsto f_{|D_0}$ is an isometric isomorphism, M_0 is a closed subspace of $\mathcal{H}_0$ and $K_{M_0} = (K_M)_{|D_0}$. By (a), the kernel

$$D_0 \times D_0 \to \mathbb{C}\ ,\ (z,w) \mapsto K(z,w) - K_M(z,w)\frac{1}{g_M(z)}\overline{\frac{1}{g_M(w)}} = \frac{2z\overline{w}\ v(z,w)}{g_M(z)\overline{g_M(w)}}$$

is positive, which means by Proposition 1.7.6 that $\frac{1}{g_M}$ defines a contractive multiplier in $\mathcal{M}(M_0, \mathcal{H}_0)$. Hence for $f \in M$, the function $\frac{f}{g_M}$ (defined on D_0) belongs to $\mathcal{H}_0$ with $\|\frac{f}{g_M}\| \leq \|f\|$. But then there exists a unique function in $L^2_a(\mathbb{D})$ extending $\frac{f}{g_M}$ without increasing the norm. Turning to part (c), we let ψ denote the analytic extension of $\frac{\phi}{g_M}$. Then for every polynomial p, the function $\phi \cdot p$ belongs to M since M is invariant. By (b), we observe

$$\|\psi \cdot p\| = \|\frac{\phi \cdot p}{g_M}\| \leq \|\phi \cdot p\| \leq \|\phi\|_{\infty,\mathbb{D}}\|p\|.$$

This clearly shows that ψ is a multiplier with

$$\|\psi\|_{\infty,\mathbb{D}} = \|\psi\|_{\mathcal{M}} \leq \|\phi\|_{\infty,\mathbb{D}}.$$

□

As a consequence, we obtain an expression of the extremal function of index-one subspaces in terms of the core function.

Corollary 5.2.4. *Suppose that M is an invariant subspace of the Bergman space $L^2_a(\mathbb{D})$ of index one. Let $n = \operatorname{ord}(M)$ denote the minimal order of 0 as a zero of functions in M. Then*

$$g_M(z) = \frac{\frac{\partial^n}{\partial \overline{w}^n} G_M(z,0)}{\left(\frac{\partial^n}{\partial z^n}\frac{\partial^n}{\partial \overline{w}^n} G_M(0,0)\right)^{\frac{1}{2}}}$$

holds for all $z \in \mathbb{D}$.

Proof. By the Leibniz generalized product rule, we obtain that

$$\frac{\partial^n}{\partial \overline{w}^n} G_M(z,w) = g_M(z)\overline{g_M^{(n)}(w)} - 2nz\frac{\partial^{n-1}}{\partial \overline{w}^{n-1}} v(z,w) - 2z\overline{w}\frac{\partial^n}{\partial \overline{w}^n} v(z,w) \qquad (5.2.2)$$

for all $z, w \in \mathbb{D}$. Since, by the positive definiteness of v,

$$\frac{\partial^k}{\partial \overline{w}^k} v(z,w) = \overline{\left(v(\cdot,z)^{(k)}\right)(w)}$$

holds for all $k \geq 0$ and $z, w \in \mathbb{D}$ and since the function $v(\cdot, z)$ belongs to M for fixed $z \in \mathbb{D}$ (by Proposition 5.2.1 and Corollary 1.6.4), we see that

$$\frac{\partial^{n-1}}{\partial \overline{w}^{n-1}} v(z,0) = 0$$

holds for all $z \in \mathbb{D}$. Now (5.2.2) shows that

$$\frac{\partial^n}{\partial \overline{w}^n} G_M(z,0) = g_M(z)\overline{g_M^{(n)}(0)} \quad (z \in \mathbb{D}). \qquad (5.2.3)$$

Similarly, we see

$$\begin{aligned}\frac{\partial^n}{\partial z^n}\frac{\partial^n}{\partial \overline{w}^n}G_M(z,w) &= g_M^{(n)}(z)\overline{g_M^{(n)}(w)}\\ &\quad -2nz\frac{\partial^n}{\partial z^n}\frac{\partial^{n-1}}{\partial \overline{w}^{n-1}}v(z,w) - 2n^2\frac{\partial^{n-1}}{\partial z^{n-1}}\frac{\partial^{n-1}}{\partial \overline{w}^{n-1}}v(z,w)\\ &\quad -2z\overline{w}\frac{\partial^n}{\partial z^n}\frac{\partial^n}{\partial \overline{w}^n}v(z,w) - 2n\overline{w}\frac{\partial^{n-1}}{\partial z^{n-1}}\frac{\partial^n}{\partial \overline{w}^n}v(z,w).\end{aligned}$$

for all $z, w \in \mathbb{D}$. Since for all $w \in \mathbb{D}$ and $k \geq 0$, the function

$$\mathbb{D} \to \mathbb{C}\ ,\ z \mapsto \frac{\partial^k}{\partial \overline{w}^k}v(z,w)$$

belongs to M (in fact, this follows since the mapping $\mathbb{D} \to M\ ,\ w \mapsto v(\cdot, w)$, is antiholomorphic and since M is closed), we have

$$\frac{\partial^{n-1}}{\partial z^{n-1}}\frac{\partial^{n-1}}{\partial \overline{w}^{n-1}}v(0,0) = 0$$

and hence

$$\frac{\partial^n}{\partial z^n}\frac{\partial^n}{\partial \overline{w}^n}G_M(0,0) = |g_M^{(n)}(0)|^2 = \left(g_M^{(n)}(0)\right)^2.$$

Now recall that g_M is normalized in the sense that $g^{(n)}(0) > 0$. By combining the last observation with (5.2.3), we obtain the desired result.

□

Also the 'Wandering Subspace Theorem' is an immediate consequence of Proposition 5.2.1

Corollary 5.2.5. *Suppose that M is an invariant subspace of $L^2_a(\mathbb{D})$. Then the space $M \ominus z \cdot M$ generates M as an invariant subspace, that is, $M = [M \ominus z \cdot M]$.*

Proof. In the following, we write

$$S : \mathbb{D} \times \mathbb{D} \to \mathbb{C}\ ,\ S(z,w) = \frac{1}{1 - z\overline{w}}$$

for the reproducing kernel of the Hardy space $H^2(\mathbb{D})$ (the Szegö kernel). Then clearly $K = S^2$. Furthermore, we define

$$F : \mathbb{D} \times \mathbb{D} \to \mathbb{C}\ ,\ F(z,w) = \frac{K_M(z,w)}{S(z,w)} = K_M(z,w)(1 - z\overline{w}).$$

The right-hand side of this equation reveals that F is a positive kernel (by Proposition 1.7.6), since M is invariant. Obviously, F is represented by the operator $T = P_M - M_z P_M M_z^* \in B(L^2_a(\mathbb{D}))$, that is, $\Lambda_T = F$. A routine calculation shows that $M \ominus z \cdot M$ consists of eigenvectors of T to the eigenvalue 1. Hence $P_{M\ominus z\cdot M} \leq T$ or, equivalently, $K_{M\ominus z\cdot M} \leq F$.

In particular, since $K_{M\ominus z\cdot M} \cdot S \leq F \cdot S = K_M$, the kernel $K_{M\ominus z\cdot M} \cdot S$ belongs to $B(K)$. By Proposition 5.2.1, we can choose a positive kernel $L : \mathbb{D}\times\mathbb{D} \to \mathbb{C}$ such that

$G_M = K_{M\ominus z\cdot M} - L$. Since $G_M \cdot S = F \in B(K)$, we conclude that also $L\cdot S$ belongs to $B(K)$. Choosing Kolmogorov factorizations $(\mathcal{D}, \phi)$ of $K_{M\ominus z\cdot M}$ and $(\mathcal{E}, \psi)$ of L, Proposition 1.7.6 shows that $\phi \in \mathcal{M}(H^2(\mathbb{D})_{\mathcal{D}}, L^2_a(\mathbb{D}))$ and $\psi \in \mathcal{M}(H^2(\mathbb{D})_{\mathcal{E}}, L^2_a(\mathbb{D}))$.

A simple calculation reveals that $T = M_\phi M_\phi^* - M_\psi M_\psi^*$ and, since T is positive, we obtain that $\operatorname{ran} T \subset \operatorname{ran} M_\phi$. Now we see on the one hand that

$$K_M(\cdot, w) = S(\cdot, w)\cdot F(\cdot, w) = S(\cdot, w)\cdot (TK(\cdot, w)) \in [\operatorname{ran} T] \quad (w \in \mathbb{D}),$$

which implies $M \subset [\operatorname{ran} T]$. On the other hand, it follows that

$$M_\phi M_\phi^* K(\cdot, w) = S(\cdot, w)\cdot K_{M\ominus z\cdot M}(\cdot, w) \in [M \ominus z\cdot M] \quad (w \in \mathbb{D}),$$

and therefore $\operatorname{ran} M_\phi \subset [M\ominus z\cdot M]$. All in all, we have proved that $M \subset [M\ominus z\cdot M]$. Since the reverse inclusion is obvious, the proof is complete. □

We mention that our proof of the 'Wandering Subspace Theorem' also works in the more general setting of Bergman-type kernels, as defined in [52]. Furthermore, we raise the question whether, in spite of the failure of the statement of Proposition 5.2.1, the 'Wandering Subspace Theorem' remains valid in the spaces $\mathcal{H}_\nu$ over $\mathbb{D}$ for $\nu > 2$. We conjecture that the answer is yes.

5.2.3 Classification of invariant subspaces

Our first aim in this paragraph is to characterize the Beurling decomposable subspaces of $L^2_a(\mathbb{D})$ by means of their extremal function. In order to do so, we need the following lemma which is actually a special case of a more general result in [38].

Lemma 5.2.6. *Suppose that M is an invariant subspace of the Bergman space $L^2_a(\mathbb{D})$ and that $M \cap H^\infty(\mathbb{D}) \neq \{0\}$. Then M has index one.*

Proof. Choose some function $\phi \in M \cap H^\infty(\mathbb{D})$ such that $\phi \neq 0$. By Theorem 1.1 in [38], it follows that $\dim M \ominus (z-\lambda)\cdot M = 1$ for every $\lambda \in D$ with $\phi(\lambda) \neq 0$.

It is well known (and follows, for example, from Proposition 4.2.6) that the approximate point spectrum of $M_z \in B(L^2_a(\mathbb{D}))$ is precisely the unit circle $\partial\mathbb{D}$. Hence $\mathbb{D}$ is contained in the semi-Fredholm domain of $(M_z)_{|M}$. Since the Fredholm Index, denoted by 'Ind', is constant on every connected component of the semi-Fredholm domain, we find that

$$\dim M \ominus (z-\lambda)\cdot M = -\operatorname{Ind}(M_z - \lambda)_{|M} = 1$$

for all $\lambda \in \mathbb{D}$. In particular, we deduce that $\operatorname{ind} M = \dim M \ominus z\cdot M = 1$. □

Proposition 5.2.7. *Suppose that M is an invariant subspace of $L^2_a(\mathbb{D})$. Then the following are equivalent:*

(i) M is Beurling decomposable.

(ii) g_M is bounded, that is, $g_M \in H^\infty(\mathbb{D})$.

In this case, $M = \operatorname{ran} M_{g_M}$.

Proof. Without loss of generality, we may assume that $M \neq \{0\}$. If M is Beurling decomposable, then $M \cap H^\infty(\mathbb{D})$ is dense in M by Corollary 3.3.6 (b). In particular, M contains non-zero bounded holomorphic functions. By Lemma 5.2.6, M has index one. Now Proposition 5.2.1 yields $(\Delta_M)_+ = g_M \otimes g_M$, and Proposition 3.3.13 shows that $g_M \in H^\infty(\mathbb{D})$.

Conversely, if g_M is bounded, then the index of M is one by Lemma 5.2.6. By Corollary 5.2.3, $(\Delta_M)_+ = g_M \otimes g_M$, and therefore $\operatorname{ran}(\Delta_M)_+ \subset H^\infty(\mathbb{D})$. By Proposition 3.3.13, M is Beurling decomposable. The assertion that $M = \operatorname{ran} M_{g_M}$ follows by Proposition 3.3.13 (and the fact that $(\Delta_M)_+$ has rank one). □

Next we want to describe the finite-codimensional invariant subspaces of $L^2_a(\mathbb{D})$. Before we do, we formulate the following lemma which is of its own interest.

Lemma 5.2.8. *Let $g \in \mathcal{O}(\overline{\mathbb{D}})$ be a function holomorphic on a neighbourhood of $\overline{\mathbb{D}}$ and assume $g \neq 0$. Let $A = Z(g)$ be the (finite) sequence of zeroes of g belonging to $\mathbb{D}$ (counting multiplicities). Then $\overline{\operatorname{ran} M_g} = M_A$. Moreover, when g has no zeroes on $\partial\mathbb{D}$, then $\operatorname{ran} M_g = M_A$.*

Proof. Let $U \supset \overline{\mathbb{D}}$ be an open set such that $g \in \mathcal{O}(U)$. Without restriction we may assume that $Z(g) \subset \overline{\mathbb{D}}$ (otherwise, we replace U by a smaller open set $\tilde{U} \supset \overline{\mathbb{D}}$ with $Z(g) \cap \tilde{U} = Z(g) \cap \overline{\mathbb{D}}$).

First we consider the case $Z(g) \subset \mathbb{D}$. The inclusion $\operatorname{ran} M_g \subset M_A$ is trivial. Conversely, let f be a function in M_A and let $\lambda_1, \ldots, \lambda_r$ denote the distinct zeroes of g and $n_{\lambda_1}, \ldots, n_{\lambda_r}$ the corresponding multiplicities. Define

$$b(z) = (z - \lambda_1)^{n_{\lambda_1}} \cdots (z - \lambda_r)^{n_{\lambda_r}} \quad (z \in \mathbb{C}).$$

It is a well-known fact that the quotient $f_0 = \frac{f}{b}$, extended analytically across the singularities, belongs to $L^2_a(\mathbb{D})$ (in fact, this follows since the operator $M_z - \lambda$ is bounded below for every $\lambda \in \mathbb{D}$). On the other hand, the function $g_0 = \frac{g}{b}$ belongs to $\mathcal{O}(U)$ and has no zeroes. Therefore, the inverse function $\frac{1}{g_0}$ defines a multiplier of $L^2_a(\mathbb{D})$, which implies that the function $\tilde{f}_0 = \frac{f}{g_0 \cdot b} = \frac{1}{g_0} \cdot f_0$ belongs to $L^2_a(\mathbb{D})$. Since $g \cdot \tilde{f}_0 = f$, we conclude $f \in \operatorname{ran} M_g$.

For the general case, let $\mu_1, \ldots, \mu_s$ denote the (finitely many) distinct zeroes of g on $\partial\mathbb{D}$, and let $n_{\mu_1}, \ldots, n_{\mu_s}$ be the corresponding multiplicities. We define

$$c(z) = (z - \mu_1)^{n_{\mu_1}} \cdots (z - \mu_s)^{n_{\mu_s}} \quad (z \in \mathbb{C}).$$

We recall that, since the Bergman space has no virtual points outside $\mathbb{D}$, the maximal ideal

$$I_\mu = \{(z-\mu)\cdot q \; ; \; q \in \mathbb{C}[z]\}$$

of $\mathbb{C}[z]$ at μ is dense in $L^2_a(\mathbb{D})$ for all $\mu \in \mathbb{C}\backslash\mathbb{D}$ by Remark 2.2.2 in [25]. We observe that the function $h = \frac{g}{z-\mu_1}$ belongs to $\mathcal{O}(U)$ and satisfies

$$\overline{\operatorname{ran} M_h} = \overline{M_h I_{\mu_1}} = \overline{M_g \mathbb{C}[z]} = \overline{\operatorname{ran} M_g}.$$

Iterating this argument yields the identity

$$\overline{\operatorname{ran} M_{g_1}} = \overline{\operatorname{ran} M_g},$$

where $g_1 = \frac{g}{c} \in \mathcal{O}(U)$. We have $Z(g_1) = Z(g) \cap \mathbb{D} = A$ and, in particular, $Z(g_1) \subset \mathbb{D}$. We have already proved that $\operatorname{ran} M_{g_1} = M_A$, and therefore $M_A = \overline{\operatorname{ran} M_g}$. $\square$

Proposition 5.2.9. *Suppose that M is a non-zero invariant subspace of $L^2_a(\mathbb{D})$. Then the following are equivalent:*

(i) $\operatorname{codim} M < \infty$.

(ii) *$M^\perp$ consists of rational functions with poles off $\overline{\mathbb{D}}$.*

(iii) *G_M is a rational function in z and $\overline{w}$ with poles off $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$.*

(iv) *G_M admits a sesquianalytic extension on a neighbourhood of $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$.*

(v) *g_M is a rational function with poles off $\overline{\mathbb{D}}$.*

(vi) $g_M \in \mathcal{O}(\overline{\mathbb{D}})$.

In this case, there is a finite sequence A in $\mathbb{D}$ such that $M = M_A$.

Proof. The equivalence of (i) and (ii) is Corollary 2.5.4 in [25]. If conditions (i) and (ii) are fulfilled, then by Proposition 1.1.2, the kernel $K_{M^\perp}$ is certainly a rational function in z and $\overline{w}$ with poles off $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$. Since $\frac{1}{K}$ is a polynomial in z and $\overline{w}$, it follows that $G_M = 1 - \frac{K_{M^\perp}}{K}$ is rational with poles off $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$. Furthermore, the implications (iii) to (v) to (vi) and (iii) to (iv) to (vi) are clear by Corollary 5.2.4. So suppose that $g_M \in \mathcal{O}(\overline{\mathbb{D}})$. By Proposition 5.2.7, M is Beurling decomposable and $M = \operatorname{ran} M_{g_M}$. Lemma 5.2.8 shows that $M = M_A$, where A is the finite set $Z(g_M) \cap \mathbb{D}$ (counting multiplicities). But obviously, zero-based subspaces which are determined by a finite sequence are finite codimensional. $\square$

Propositions 5.2.7 and 5.2.9 provide characterizations of Beurling decomposable and finite-codimensional submodules in terms of their extremal functions. These characterizations allow us to present an example of a Beurling decomposable submodule of $L^2_a(\mathbb{D})$ that has infinite codimension.

Example 5.2.10. *It is proved in [2], Corollary 2.4 that, if $A = (a_n)_n$ is a Blaschke sequence in $\mathbb{D}$, then the extremal function g_A of the zero-based invariant subspace M_A satisfies*

$$|g_A(z)| \leq \exp\left(\sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-\overline{a_n}z|}\right)$$

for all $z \in \mathbb{D}$. So whenever $A = (a_n)_n$ is an infinite Blaschke sequence such that

$$\sup_{z\in\mathbb{D}}\left(\sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-\overline{a_n}z|}\right) < \infty, \tag{5.2.4}$$

then the zero-based subspace M_A is Beurling decomposable by Proposition 5.2.7 *and has infinite codimension by Proposition* 5.2.9*. In order to find such sequences, one should first note that*

$$\sup_{z\in\mathbb{D}}\left(\sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-\overline{a_n}z|}\right) = \sup_{\zeta\in\partial\mathbb{D}}\left(\sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-\overline{a_n}\zeta|}\right) \tag{5.2.5}$$

holds for every sequence $(a_n)_n$ in $\mathbb{D}$. Indeed, this follows easily by the subharmonicity of the functions

$$f_n : \mathbb{D} \to \mathbb{R}\ ,\ f_n(z) = \frac{1-|a_n|^2}{|1-\overline{a_n}z|} \quad (n \in \mathbb{N}).$$

The condition

$$\sup_{\zeta\in\partial\mathbb{D}}\left(\sum_{n=1}^{\infty} \frac{1-|a_n|^2}{|1-\overline{a_n}\zeta|}\right) < \infty$$

characterizes exactly the so-called Frostman sequences. These sequences were first studied in [49] and determine the inner multipliers of the space of Cauchy type integrals on the torus. Examples of infinite Frostman sequences can be constructed as follows: suppose that, for all $n \in \mathbb{N}$, we are given numbers $t_n \in [-\pi, \pi]$ and $\epsilon_n > 0$ such that the intervals $I_n = [t_n - \epsilon_n, t_n + \epsilon_n]$ are pairwise disjoint. Suppose further that $(r_n)_n$ is a sequence in $(0,1)$ such that the series $\sum_n \frac{1-r_n}{\epsilon_n}$ is convergent. Then, by Lemma 5.1 in [49], the sequence $(a_n)_n$, defined by $a_n = r_n e^{it_n}$, is a Frostman sequence. To be even more concrete, one could choose

$$1 - r_n = \frac{1}{n^3(n+1)} \quad , \quad t_n = \frac{1}{n} \quad \textit{and} \quad \epsilon_n = \frac{1}{3n(n+1)}.$$

Now that we have successfully characterized the finite-codimensional and the Beurling decomposable submodules of the Bergman space, we turn our attention to another, larger class of submodules. We mention that parts of the following result are well known and appeared in [45].

Proposition 5.2.11. *Suppose that M is a non-zero invariant subspace of $L^2_a(\mathbb{D})$. Then the following are equivalent:*

(i) $[M \cap H^\infty(\mathbb{D})] = M$.

(ii) $[M \cap H^2(\mathbb{D})] = M$.

(iii) $[M \cap N(\mathbb{D})] = M$ *(where* $N(\mathbb{D})$ *denotes the Nevanlinna class).*

(iv) M *is generated by an inner function.*

(v) M *is generated by a bounded holomorphic function.*

(vi) M *is generated by a Hardy space function.*

(vii) M *is generated by a Nevanlinna class function.*

(viii) M *contains an inner function.*

(ix) M *contains a non-zero bounded holomorphic function.*

(x) M *contains a non-zero Hardy space function.*

(xi) M *contains a non-zero Nevanlinna class function.*

(xii) g_M *belongs to the Nevanlinna class.*

Proof. The proof will be organized as follows:

$$
\begin{tikzcd}
\text{(i)} \arrow[r] & \text{(ii)} \arrow[r] \arrow[dl] \arrow[d] & \text{(iv)} \arrow[r] \arrow[d] & \text{(v)} \arrow[r] & \text{(vi)} \arrow[r] & \text{(vii)} \arrow[lllll, bend right] \\
\text{(x)} \arrow[dr] & \text{(iii)} \arrow[d] & \text{(viii)} \arrow[d] & & & \\
& \text{(xi)} \arrow[r] & \text{(ix)} \arrow[r] & \text{(xii)} \arrow[uurr] & &
\end{tikzcd}
$$

Most of the implications are obvious. The first non-trivial implication is (ii) to (iv). In order to prove it, let us suppose that the space $M' = M \cap H^2(\mathbb{D})$ is dense in M. Clearly M' is a closed invariant subspace of $H^2(\mathbb{D})$, since the inclusion mapping $H^2(\mathbb{D}) \hookrightarrow L_a^2(\mathbb{D})$ is continuous. By Beurling's Theorem, there exists an inner function ϕ such that $M' = \phi \cdot H^2(\mathbb{D})$. Then

$$M = [M'] = [\phi \cdot H^2(\mathbb{D})] = [\phi],$$

which shows that (iv) is fulfilled. The implication (vii) to (i) is Theorem 5.1 in [45]. Next, we verify the implication (xi) to (ix). Whenever u is a Nevanlinna class function in M, then u can be written as quotient of two bounded holomorphic functions ϕ and ψ. Since M is invariant, we infer that $\phi = \psi \cdot u \in M$. The remaining implications (ix) to (vii) and (xii) to (vii) are proved as follows: If ϕ is a non-zero bounded holomorphic function in M, then, by Lemma 5.2.6, the index of M is one. By Corollary 5.2.3, the function $\psi = \frac{\phi}{g_M}$ is bounded. Hence $g_M = \frac{\phi}{\psi}$ is the quotient of two bounded holomorphic functions and therefore belongs to $N(\mathbb{D})$. Finally, if

$g_M \in N(\mathbb{D})$, then by the already proven implication (xi) to (ix) and Lemma 5.2.6, it follows that M has index one. Then the 'Wandering Subspace Theorem' (Corollary 5.2.5) implies that M is generated by the Nevanlinna class function g_M. □

Clearly, every Beurling decomposable subspace satisfies the equivalent conditions of the preceding proposition. It is therefore a natural to ask whether the submodules described in Proposition 5.2.11 are necessarily Beurling decomposable. The following example reveals that this is not the case.

Example 5.2.12. *Consider the submodule M of $L^2_a(\mathbb{D})$ generated by the singular inner function*

$$S : \mathbb{D} \to \mathbb{C}\ ,\ S(z) = \exp\left(-\frac{1+z}{1-z}\right).$$

Then M belongs to the submodules characterized in Proposition 5.2.11*. By Lemma* 5.2.6*, the index of M is one. It is shown in [77] that the reproducing kernel of M is given by*

$$K_M : \mathbb{D} \times \mathbb{D} \to \mathbb{C}\ ,\ K_M(z,w) = K(z,w)S(z)\overline{S(w)}\left(1 + \frac{1+z}{1-z} + \frac{1+\overline{w}}{1-\overline{w}}\right).$$

Since M has no common zero at 0*, Corollary* 5.2.4 *yields*

$$g_M(z) = \frac{G_M(z,0)}{\sqrt{G_M(0,0)}} = \frac{1}{\sqrt{3}}\frac{S(z)}{1-z}(3-z).$$

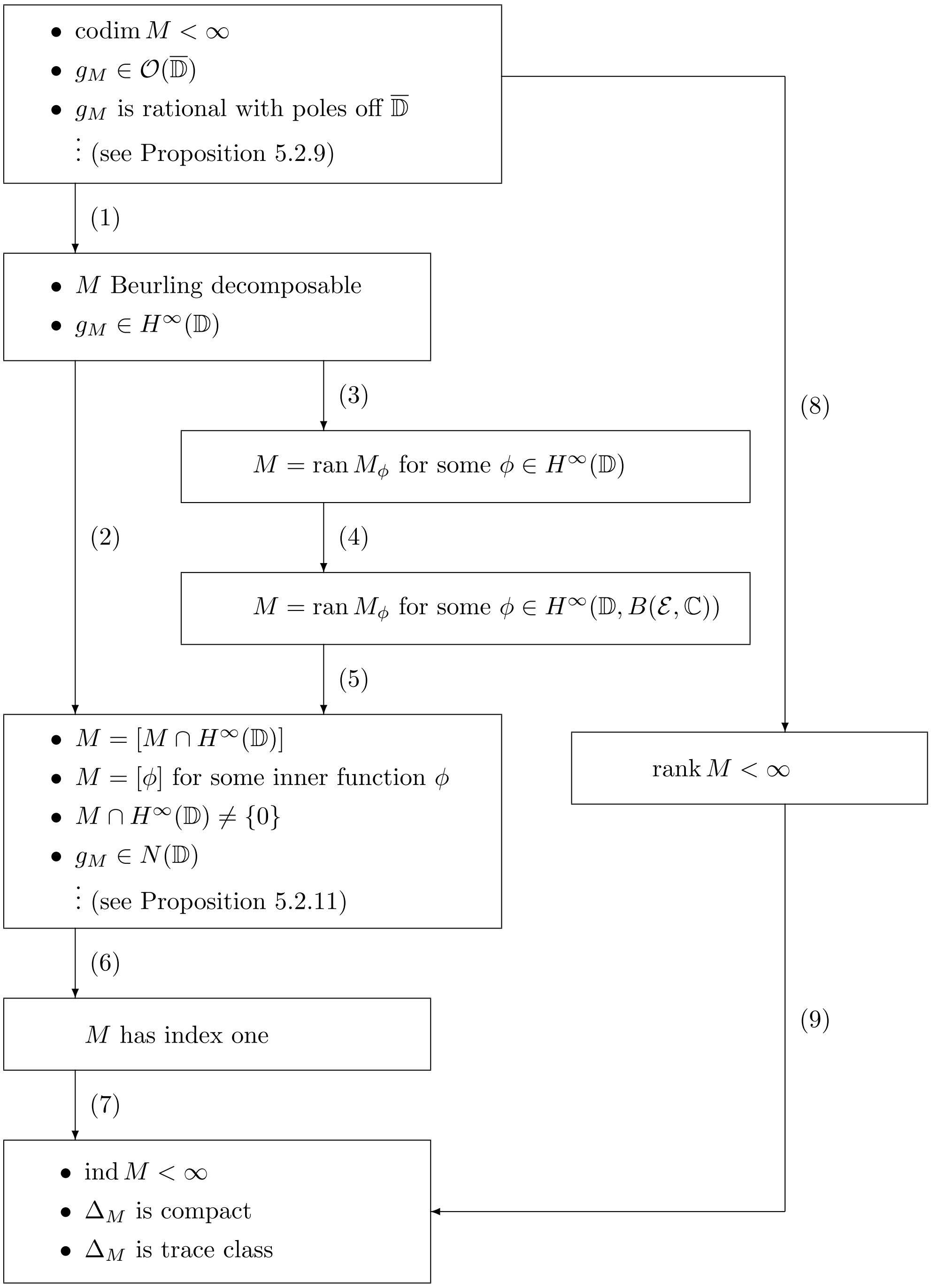
In order to prove that M is not Beurling decomposable, it therefore suffices to show that the function $h(z) = \frac{S(z)}{1-z}$ is not bounded on $\mathbb{D}$. And in fact, this follows by evaluating h at the points $z_n = \frac{in}{1+in}$ ($n \in \mathbb{N}$).

As observed in [76], the submodule M considered in the previous example is interesting for another reason. Namely, it is an example of a finite-rank invariant subspace that is not zero-based. In fact, it is clear that M has no common zeroes at all (since the generating function S does not). This implies that M is not zero based because in that case, we would have $M = L^2_a(\mathbb{D})$. Since the core function of M can be written as

$$G_M(z,w) = S(z)\overline{S(w)}\left(\left(1 + \frac{1+z}{1-z}\right)\left(1 + \frac{1+\overline{w}}{1-\overline{w}}\right) - \frac{1+z}{1-z}\,\frac{1+\overline{w}}{1-\overline{w}}\right) \quad (z,w \in \mathbb{D}),$$

it is clear that M has rank 2 (cf. Remark 1.6.10).

Now consider a non-zero invariant subspace M of $L^2_a(\mathbb{D})$. The following diagram summarizes the results we found in this section. The conditions in each of the boxes are equivalent, and the arrows are implications.



It should be mentioned that the equivalence of the conditions in the bottom box is the main result of the recent paper [76] (Theorem 9).

Finally, we want to examine which of the implications in the above diagram are proper. Clearly, the implications (1) and (2) are by Examples 5.2.10 and 5.2.12, respectively. Therefore, at least one of the implications (3) to (5) is proper. Although we do not know which, we conjecture that they are all. To see that the implication

(6) is proper, consider an $L_a^2(\mathbb{D})$-zero set A that is not a Blaschke sequence (see Chapter 4 in [46] for the existence of such a zero set). Then $g_A \notin N(\mathbb{D})$, but M_A clearly has index one. Furthermore, implication (7) is proper since there exist invariant subspaces of every arbitrary index (see [46], Chapter 6). That implication (8) is proper follows again by Example 5.2.12. In order to show that (9) is proper, it suffices to find a zero-based invariant subspace of infinite rank. And in fact, a recent result ([24], Theorem 3.4) shows that, for every $L_a^2(\mathbb{D})$-zero set A, the rank of the kernel

$$l : \mathbb{D} \times \mathbb{D} \ , \ l(z,w) = \frac{(G_{M_A})_-(z,w)}{zg(z)\overline{wg(w)}},$$

extended sesquianalytically across the singularities, and the cardinality of A (not counting multiplicities) coincide. We infer that $\operatorname{rank} l = \operatorname{rank}(G_M)_-$ by Remark 1.6.10. Hence, for every infinite $L_a^2(\mathbb{D})$-zero set A, we deduce that

$$\operatorname{rank} M_A \geq \operatorname{rank}_- M_A = \operatorname{rank}(G_{M_A})_- = \infty.$$

List of symbols

General notations

span Y .... linear hull of a subset Y of a linear space
$\overline{X}$ .... conjugate linear version of a linear space X
$X \otimes Y$ .... algebraic tensor product of linear spaces X and Y
$\ker T$ .... kernel of a linear mapping
$\operatorname{ran} T$ .... range of a linear mapping
$\operatorname{ball} X$ .... the closed unit ball of a normed space X
$\bigvee Y$ .... closed linear span of a subset Y in a normed space
$B(X_1, X_2)$ .... bounded operators between normed spaces X_1, X_2
X^* .... topological dual of a topological vector space X
$CB(V_1, V_2)$ .... completely bounded operators between operator spaces V_1, V_2
$V_1 \otimes^h V_2$ .... Haagerup tensor product of operator spaces V_1, V_2
V^{op} .... transposed operator space
$[a_{ij}]$, $[a_{ij}]_{i,j}$ .... matrix with entries a_{ij}
$M_{n,m}(X)$, $M_n(X)$ .... matrices with entries in the set X
$M_{n,m}$, M_n .... matrices with entries in $\mathbb{C}$
D_n .... diagonal matrices
$x_1 \otimes x_2$ .... rank one operator between Hilbert spaces
$T(H_1, H_2)$ .... trace class operators between Hilbert spaces H_1, H_2
$H_1 \otimes H_2$ .... Hilbert space tensor product of Hilbert spaces H_1, H_2
H_c, H_r .... column and row Hilbert space
$M \overline{\otimes} N$ .... normal spatial tensor product
$\mathcal{O}(D, X)$ .... X-valued holomorphic functions on the open set D
$\mathcal{O}(F, X)$ .... X-valued functions holomorphic on neighbourhoods of the closed set F
$\mathbb{D}$ .... open unit disk in $\mathbb{C}$
$\mathbb{B}_d$ .... open unit ball in $\mathbb{C}^d$
$\mathbb{C}[z]$, $\mathbb{C}[z_1, \ldots, z_d]$ .... polynomials over $\mathbb{C}$ in d variables
$\sigma(T)$ .... Taylor spectrum of a commuting tuple T
$\sigma_{re}(T)$ .... right essential spectrum of a commuting tuple T
$Z(N)$ .... common zero set of a class of functions

Spaces

Operations

Special operators, kernels and functions

Relations

Norms

Miscellaneous

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