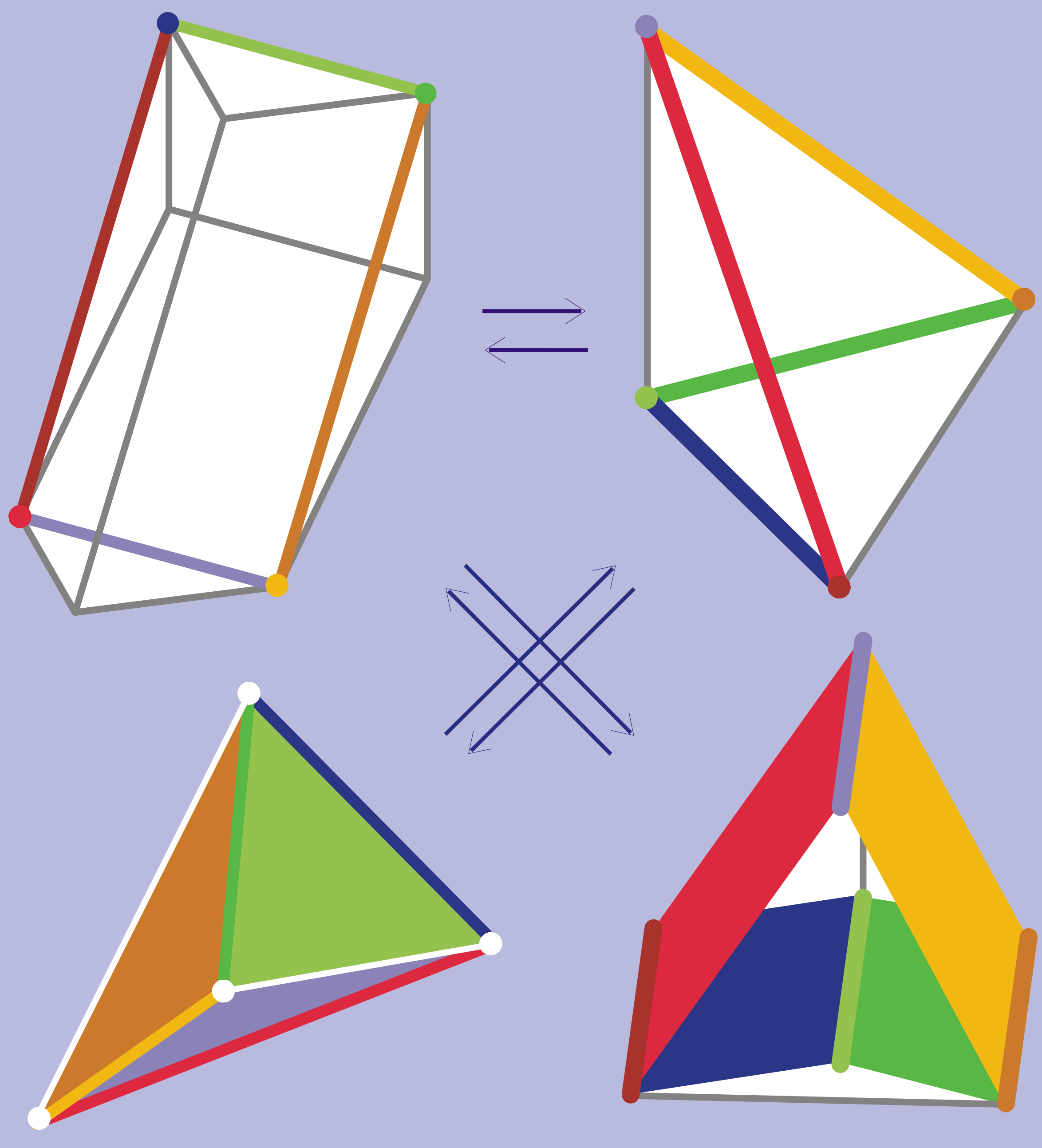
Janko Böhm

Mirror Symmetry and Tropical Geometry



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Mirror Symmetry and Tropical Geometry

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Mirror Symmetry and Tropical Geometry

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To my parents

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0 Introduction

Mirror symmetry is a phenomenon postulated by theoretical physics in the context of string theory. The goal of string theory is the unification of general relativity, describing gravity, with the standard model, which describes the electroweak and strong coupling. These theories model nature in the large respectively in small scales in such astonishing precision that it is very hard to obtain experimental data on this unification. String theory follows the idea to replace point particles by extended objects like a 1-sphere and to replace the 4-dimensional spacetime by a Riemannian manifold of dimension 10, which is locally the product of a 4-dimensional Minkowski space and a 6-dimensional compact Riemannian manifold X, too small to appear in measurements. For two out of five possible string theories the manifold X turns out to be a 3-dimensional complex manifold with trivial canonical sheaf. These kind of manifolds are called Calabi-Yau manifolds and were studied in mathematics for a long time before their appearance in theoretical physics. Hodge theory associates to X the Hodge numbers

$$h^{p,q}(X) = \dim H_{\bar{\partial}}^{p,q}(X) = \dim H^q(X, \Omega_X^p), \; p, q = 0, ..., 3$$

The general framework of string theory predicts that one type of string theory obtained from a Calabi-Yau manifold X is equivalent to the second type of string theory on another Calabi-Yau manifold X° and the Hodge numbers of these are related by

$$h^{p,q}(X) = h^{3-p,q}(X^\circ) \;\; \forall \; 0 \leq p, q \leq 3$$

Such a pair X and X° is called a mirror pair, and the question arises how to get X° from X and vice versa.

The first mirror construction was given by Greene and Plesser in [Greene, Plesser, 1990] for the general quintic threefold $X \subset \mathbb{P}^4$. As for Calabi-Yau manifolds $T_{X^\circ} \cong \Omega^2_{X^\circ}$, the mirror X° should satisfy

$$\dim H^1(X^\circ, T_{X^\circ}) = h^{2,1}(X^\circ) = h^{1,1}(X) = 1$$

Greene and Plesser construct the mirror as a general element in the 1-parameter family of quintics

$$X_\lambda = \left\{ x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + \lambda x_0 x_1 x_2 x_3 x_4 = 0 \right\}$$

with fibers in a $\mathbb{Z}_5^3$-quotient of $\mathbb{P}^4$. This 1-dimensional parameter space contains the degeneration point $\lambda = \infty$ corresponding to the union of 5 hyperplanes $\{x_0 x_1 x_2 x_3 x_4 = 0\}$. Indeed, degenerations of Calabi-Yau manifolds to varieties given by monomial ideals appear naturally in the context of various mirror constructions.

Generalizing the construction by Greene and Plesser, Batyrev considers in [Batyrev, 1994] anticanonical hypersurfaces in Gorenstein toric Fano varieties. There is a one-to-one correspondence of the Gorenstein toric Fano varieties $\mathbb{P}(\Delta)$ of dimension n, polarized by $-K_{\mathbb{P}(\Delta)}$ with the reflexive polytopes $\Delta \subset M \otimes \mathbb{R}$, where $M = \mathbb{Z}^n$. Recall that a polytope Δ is called reflexive if Δ and its dual polytope Δ^* are integral and contain 0 in their interior. So duality of reflexive polytopes is an involution of the set of Gorenstein toric Fano varieties. Batyrev proves that general elements in $\left|-K_{\mathbb{P}(\Delta)}\right|$ and $\left|-K_{\mathbb{P}(\Delta^*)}\right|$ form a mirror pair in the sense of mirrored Hodge numbers generalized to singular varieties. In the following we associate to Batyrev's data a monomial degeneration. Denote by $\Sigma \subset N \otimes \mathbb{R}$, where $N = \operatorname{Hom}(M, \mathbb{Z})$, the fan representing $Y = \mathbb{P}(\Delta)$, i.e., the set of cones over the faces of the dual polytope Δ^*. Generalizing the homogeneous coordinate ring of projective space, the Cox ring of Y is the polynomial ring

$$S = \mathbb{C}[y_r \mid r \in \Sigma(1)]$$

with variables corresponding to the 1-dimensional cones $\Sigma(1)$ in Σ and graded by the Chow group of divisors $A_{n-1}(Y)$ of Y via the exact sequence

$$0 \to M \stackrel{A}{\to} \mathbb{Z}^{\Sigma(1)} \stackrel{\deg}{\to} A_{n-1}(Y) \to 0$$

A reflexive polytope has 0 as its unique interior lattice point, so a generic anticanonical hypersurface in Y comes with a natural monomial degeneration

$$\left\{ t \cdot \left(\sum_{m \in \partial\Delta} c_m \cdot A(m) \cdot m_0 \right) + m_0 = 0 \right\}$$

where $m_0 = \prod_{r \in \Sigma(1)} y_r$ and c_m are generic coefficients.

Note that toric varieties also appear in the context of monomial degenerations in the sense that the special fiber is a union of toric varieties. Indeed, toric geometry plays an important role in the context of mirror symmetry as toric varieties have non-trivial geometry and still can contain a reducible special fiber of a degeneration in a natural description as a union of toric strata.

As general setup, we consider a one parameter degeneration $\mathfrak{X}$ of Calabi-Yau varieties with fibers in a toric Fano variety Y with Cox ring S and with reduced monomial special fiber X_0. The toric Fano variety Y is given by the fan over the faces of a Fano polytope P, which is an integral polytope in $N \otimes \mathbb{R}$ containing 0 as the unique interior lattice point. So we generalize

Gorenstein toric Fano varieties to the Mori category of $\mathbb{Q}$-Gorenstein toric Fano varieties. Let $I \subset \mathbb{C}[t] \otimes S$ be the ideal of the total space of the degeneration and $I_0 \subset S$ the ideal of the monomial special fiber.

Given a polytope Δ we denote by $\operatorname{Poset}(\Delta)$ the complex of faces of Δ, which is a partially ordered set with respect to inclusion. For the polytope $\Delta = P^*$ the complex $\operatorname{Poset}(\Delta)$ is isomorphic to the complex of the closed strata of Y, which we denote by $\operatorname{Strata}(Y)$. So the complex of closed strata $\operatorname{Strata}(X_0)$ of the monomial special fiber of $\mathfrak{X}$ can be considered as a subcomplex of $\operatorname{Poset}(\Delta)$.

Using Gröbner basis techniques, we construct from the degeneration $\mathfrak{X}$ a new polytope ∇ with a new subcomplex $B(I) \subset \operatorname{Poset}(\nabla)$. We begin by associating to $\mathfrak{X}$ the Gröbner cone $C_{I_0}(I)$ of weights on $\mathbb{C}[t] \otimes S$ (looking for the minimal weight terms) selecting the monomial special fiber ideal I_0 as lead ideal of I. For every face F of $C_{I_0}(I)$ we have an initial ideal $\operatorname{in}_F(I)$ of I, which is no longer monomial for the proper faces of $C_{I_0}(I)$. Denote by $\operatorname{Poset}(C_{I_0}(I))$ the fan of the faces of $C_{I_0}(I)$. We consider the subfan $BF_{I_0}(I) \subset \operatorname{Poset}(C_{I_0}(I))$ of those faces of $C_{I_0}(I)$, which have no monomial in their initial ideal. This fan is the intersection of the fan $\operatorname{Poset}(C_{I_0}(I))$ with the Bergman fan $BF(I)$, introduced in [Bergman, 1971]. Essentially equivalent to Bergman's original definition, we define the Bergman fan $BF(I)$ as the closure of the image of the vanishing locus of I over the field $L = \mathbb{C}\{\{s\}\}$ of Puiseux series under the valuation map

$$\begin{aligned} L^* \times (L^*)^n &\rightarrow \mathbb{R}^{n+1} \\ (t, y_1, ..., y_n) &\mapsto (val(t), val(y_1), ..., val(y_n)) \end{aligned}$$

Here we consider the torus $(L^*)^n \cong (L^*)^{\Sigma(1)} / \operatorname{Hom}_{\mathbb{Z}}(A_{n-1}(Y), L^*)$ and val denotes the valuation associating to a power series its vanishing order, i.e., the exponent of its lowest order term.

The cone $C_{I_0}(I)$ is contained in the half-space of t-local orderings. Hence, intersecting it transversally with the hyperplane of t-weight $w_t = 1$, i.e., identifying the parameters t and s, we obtain a convex polytope ∇. The polytope ∇ is naturally contained in $N \otimes \mathbb{R}$ and it turns out that ∇^* is again a Fano polytope. Corresponding to $BF_{I_0}(I) = BF(I) \cap \operatorname{Poset}(C_{I_0}(I))$ we also obtain a subcomplex $B(I)$ of the complex $\operatorname{Poset}(\nabla)$ of faces of ∇. Let $K = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series in the parameter t.

If w is a weight vector in a face of the Bergman complex $B(I)$, we can consider the power series solutions of I lying over w via the valuation map

$$\begin{aligned} (K^*)^n &\rightarrow \mathbb{R}^n \\ (y_1, ..., y_n) &\mapsto (val(y_1), ..., val(y_n)) \end{aligned}$$

Taking the limit $t \to 0$ of these solutions induces an inclusion reversing map

$$\lim : \quad B(I) \quad \to \quad \mathrm{Strata}(X_0)$$

from the complex $B(I)$ to the complex of strata of the special fiber X_0 of $\mathfrak{X}$, which is a subcomplex of faces of $\Delta = P^*$. It turns out that the complex $B(I)$ essentially is dual to the complex of strata of X_0.

The complex $\mathrm{Poset}(\nabla^*)$ describes the initial ideals of I at the faces of ∇. Consider the reduced standard basis of I in $S \otimes \mathbb{C}[t] / \langle t^2 \rangle$ with respect to a monomial ordering in the interior of ∇. If F is a face of ∇, then all initial forms with respect to F of the standard basis elements involve a minimal generator of I_0. Hence, dividing for all initial forms the non-special fiber monomials by the special fiber monomial, we obtain a set of degree 0 Cox Laurent monomials. These monomials correspond via the Chow presentation sequence

$$0 \to M \overset{A}{\to} \mathbb{Z}^{\Sigma(1)} \overset{\deg}{\to} A_{n-1}(Y) \to 0$$

to the lattice points of F^*.

In the following we denote the first order deformations of X_0, which are characters of the torus $(\mathbb{C}^*)^{\Sigma(1)}$ as $(\mathbb{C}^*)^{\Sigma(1)}$-deformations. Note that the vector space $\mathrm{Hom}(I_0, S/I_0)_0$ of degree 0 deformations has a basis of $(\mathbb{C}^*)^{\Sigma(1)}$-deformations.

The lattice points of the faces of ∇^* have a two-fold interpretation:

- Let $F \in \nabla$ be a face, $\delta \in F^*$ a lattice point and let $A(\delta) = \frac{m_1}{m_0} \in \mathbb{Z}^{\Sigma(1)}$ with relative prime m_0 and m_1 be the corresponding degree 0 Cox Laurent monomial. Then δ can be considered as a $(\mathbb{C}^*)^{\Sigma(1)}$-deformation of X_0 of degree 0 by associating to it the element of $\mathrm{Hom}(I_0, S/I_0)_0$ defined for minimal generators $m \in I_0$ by $\delta(m) = \frac{m_1}{m_0} \cdot m$ if $m_0 \mid m$ and 0 otherwise. Here we use $\delta \in F^*$.

  It turns out that $B(I)^*$ represents the tangent space of the component of the Hilbert scheme of I_0 containing $\mathfrak{X}$, assuming that we took the tangent vector of $\mathfrak{X}$ general enough.

- The fan over the faces of ∇^* defines a toric Fano variety Y°, so the vertices of ∇^* are the variables of the Cox ring of Y°, i.e., the torus invariant prime Weil divisors on Y°. Hence in particular the vertices of the faces of $B(I)^*$ have an interpretation as torus invariant divisors on Y°. Passing from vertices to lattice points amounts to a toric blowup.

This is compatible with the identification of $H^{\dim(X)-1,1}(X)$ and $H^{1,1}(X^\circ)$ for a mirror pair X and X° and more generally of the complex moduli space of X with the Kähler moduli space of X°. From this point of view Y° is the toric Fano variety with sufficient divisors to represent locally the component of the Hilbert scheme at X_0 containing $\mathfrak{X}$. In the same way as $B(I)^*$ describes the tangent space at X_0 of the component of the Hilbert scheme along $\mathfrak{X}$, we expect that the Kähler classes given by the lattice points of $B(I)^*$ suffice to represent the Kähler moduli of the mirror.

The subcomplex $B(I) \subset \operatorname{Poset}(\nabla)$ defines a monomial ideal I_0° in the Cox ring S° of Y°. The ideal I_0° is the intersection over all facets (i.e., faces of maximal dimension) F of $B(I)$ of the ideals generated by the set of all facets of ∇ containing F. This generalizes the idea of Stanley-Reisner rings.

So we have constructed a toric Fano variety Y° and a monomial ideal I_0°, whose zero locus X_0° essentially is combinatorially dual to the complex of strata of the special fiber X_0 of $\mathfrak{X}$.

We know that the lattice points of $B(I)^* \subset \operatorname{Poset}(\nabla^*)$ have an interpretation as first order deformations of X_0 contributing to tangent vector of the family $\mathfrak{X}$. Hence the first order deformations of the mirror special fiber X_0° contributing to the tangent vector the mirror degeneration $\mathfrak{X}^\circ$ should be given by the lattice points of the dual $(\lim(B(I)))^* \subset \operatorname{Poset}(\Delta^*)$ of the image of the limit map. Again the lattice points of $(\lim(B(I)))^*$ have the two fold interpretation as deformations of X_0° and torus divisors on a blowup of Y. Applying these deformations to I_0° we obtain the conjectural mirror degeneration up to first order.

If the ideal I_0° obeys a structure theorem, e.g., the Koszul resolution for complete intersections or the structure theorem of Buchsbaum and Eisenbud for codimension 3 subcanonical varieties, we can (in the case of complete intersection trivially) extend the first order mirror family to a degeneration over $\operatorname{Spec}\mathbb{C}[t]$.

The tropical mirror construction formalizes as follows:

- Let $N = \mathbb{Z}^n$, let $P \subset N \otimes \mathbb{R}$ be a Fano polytope and Y the corresponding toric Fano variety with Cox ring S. Let $\mathfrak{X}$ be a one parameter monomial degeneration of Calabi-Yau varieties with fibers of codimension c in Y and let $\mathfrak{X}$ be given by the ideal $I \subset \mathbb{C}[t] \otimes S$. Suppose that the ideal I_0 of the special fiber is a reduced monomial ideal.

- Fix a monomial ordering $>$ on $\mathbb{C}[t] \otimes S$, which is respecting the Chow grading on S and which is local in t, and denote by $>_w$ the weight ordering by w refined by $>$. Then define

$$C_{I_0}(I) = \left\{ -(w_t, w_y) \in \mathbb{R} \oplus N_{\mathbb{R}} \mid L_{>_{(w_t, \varphi(w_y))}}(I) = I_0 \right\}$$

Note that we add the minus sign as L is defined as selecting the monomial of maximal weight.

- Intersecting $C_{I_0}(I)$ with the hyperplane of t-weight one, we obtain a polytope
$$\nabla = C_{I_0}(I) \cap \{w_t = 1\} \subset N_{\mathbb{R}}$$
and ∇^* is a Fano polytope, so gives a toric Fano variety Y°.

- The complex of the faces of the polytope ∇ has the subcomplex
$$\begin{aligned} B(I) &= (BF(I) \cap \operatorname{Poset}(C_{I_0}(I))) \cap \{w_t = 1\} \\ &= \{F \text{ face of } \nabla \mid \operatorname{in}_F(I) \text{ does not contain a monomial}\} \end{aligned}$$
the Bergman subcomplex or tropical subcomplex of ∇. The intersection of the fan $BF(I) \cap \operatorname{Poset}(C_{I_0}(I))$ with $\{w_t = 1\}$ is defined as the complex, whose faces are the intersections of the cones of $BF(I) \cap \operatorname{Poset}(C_{I_0}(I))$ with the hyperplane $\{w_t = 1\}$.

- The complex $B(I)$ is a subdivision of the dual of the complex of strata $\operatorname{Strata}(X_0)$ of the special fiber X_0 of $\mathfrak{X}$ via the map
$$\begin{array}{rccl} \lim: & B(I) & \to & \operatorname{Strata}(X_0) \subset \operatorname{Strata}(Y) \\ & F & \mapsto & \left\{\lim_{t\to 0} a(t) \mid a \in \operatorname{val}^{-1}(\operatorname{int}(F))\right\} \end{array}$$
taking the limit of arc solutions of I. Here $\operatorname{int}(F)$ denotes the relative interior of F.

- Denote by Σ° the fan over the faces of ∇^* defining Y° and by
$$S^\circ = \mathbb{C}[z_r \mid r \in \Sigma^\circ(1)]$$
the Cox ring of Y° graded via
$$0 \to \quad N \quad \overset{A^\circ}{\to} \quad \mathbb{Z}^{\Sigma^\circ(1)} \quad \overset{\deg}{\to} \quad A_{n-1}(Y^\circ) \quad \to 0$$
Then the monomial ideal defining the special fiber $X_0^\circ \subset Y^\circ$ of the mirror degeneration $\mathfrak{X}^\circ$ of $\mathfrak{X}$ is
$$\begin{aligned} I_0^\circ &= \left\langle \prod_{v\in J} z_v \mid J \subset \Sigma^\circ(1) \text{ with } \operatorname{supp}(B(I)) \subset \bigcup_{r\in J} F_r \right\rangle \\ &= \textstyle\bigcap_{F\in B(I)} \langle z_{G^*} \mid G \text{ a facet of } \nabla \text{ with } F \subset G\rangle \subset S^\circ \end{aligned}$$
where F_r denotes the facet of ∇ corresponding to the 1-dimensional cone r of $\Sigma^\circ = \operatorname{NF}(\nabla)$. Note that in the second description of I_0 it is sufficient to take the intersection over the maximal faces of $B(I)$.

- Let $M = \operatorname{Hom}(N,\mathbb{Z})$ and $\Delta = P^* \subset M \otimes \mathbb{R}$. The image of lim naturally is a subcomplex of the complex of faces of Δ. Hence we obtain a subcomplex $(\lim(B(I)))^*$ of the complex of faces of $\Delta^* = P$, which describes the first order deformations of the mirror degeneration at X_0° as degree zero Cox Laurent monomials. So the conjectural mirror degeneration up to first order is given by

  $$\left\langle m + t \cdot \sum_{\alpha \in \operatorname{supp}((\lim(B(I)))^*)\cap N} c_\alpha \cdot \alpha(m) \mid m \text{ min. gen. of } I_0^\circ \right\rangle \subset \mathbb{C}[t] / \langle t^2 \rangle \otimes S^\circ$$

  with generic coefficients c_α.

  Note that the description of the first order deformations as lattice points of Δ^* is independent of the toric variety Y°. This easily allows to replace Y° by different birational models in the Mori category.

  To give a model of the mirror family in a simplicial toric variety represented by a flat affine cone, consider a projective simplicial subdivision $\hat{\Sigma}^\circ \subset M_\mathbb{R}$ of Σ° and $\hat{Y}^\circ = X(\hat{\Sigma}^\circ) \to X(\Sigma^\circ) = Y^\circ$. Let

  $$\operatorname{Poset}(\nabla^*)^\wedge = \left\{ \sigma \cap F \mid \sigma \in \hat{\Sigma}^\circ,\ F \in \operatorname{Poset}(\nabla^*) \right\}$$
  $$\operatorname{dual}(B(I))^\wedge = \left\{ G \in \operatorname{Poset}(\nabla^*)^\wedge \mid G \subset F,\ G \notin \partial F \text{ for some } F \in \operatorname{dual}(B(I)) \right\}$$

  be the induced subdivisions of the complex of faces of ∇^* and of the complex $\operatorname{dual}(B(I))$ and denote by

  $$\hat{I}_0 = \bigcap\nolimits_{F \in \operatorname{dual}(B(I))_c^\wedge} \langle y_r \mid \hat{r} \in F \rangle$$

  the corresponding special fiber ideal.

  Zariski closed subsets of $\hat{Y}^\circ$ are in one-to-one correspondence with the graded radical ideals in S° contained in the irrelevant ideal

  $$B(\hat{\Sigma}^\circ) = \left\langle \prod\nolimits_{r \in \Sigma^\circ(1),\ r \not\subset \sigma} y_r \mid \sigma \in \hat{\Sigma}^\circ \right\rangle \subset S^\circ$$

  The first order mirror degeneration of $\mathfrak{X}$ with fibers in $\hat{Y}^\circ$ is the Zariski closed subset $\hat{\mathfrak{X}}^{1\circ} \subset \hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t] / \langle t^2 \rangle$ defined by the ideal

  $$\left\langle m + t \cdot \sum_{\alpha \in \operatorname{supp}((\lim(B(I)))^*)\cap N} c_\alpha \cdot \alpha(m) \mid m \in \hat{I}_0 \cap B(\hat{\Sigma}^\circ) \right\rangle \cap B(\hat{\Sigma}^\circ)$$

  in $\mathbb{C}[t] / \langle t^2 \rangle \otimes S^\circ$.

Note that one can also represent the mirror family via the one-to-one correspondence of subschemes of $\hat{Y}^{\circ}$ and saturated ideals in $R^{\circ} = \bigoplus_{\alpha \in \mathrm{Pic}(\hat{Y}^{\circ})} S^{\circ}_{\alpha}$, as explained in Section 9.12.

The choice of a projective simplicial subdivision of Σ° is known in the context of hypersurfaces and complete intersections as the multiple mirror phenomenon which is essential to the global understanding of the complex and Kähler moduli. Indeed, the mirror should be seen as the totality of all models of the tropical mirror family in some simplicial or non-simplicial projective toric variety $\hat{Y}^{\circ} = X(\hat{\Sigma}^{\circ})$ with $\hat{\Sigma}^{\circ}(1) \subset \Sigma^{\circ}(1)$.

The tropical mirror construction reproduces known mirror constructions. Batyrev and Borisov extend in [Borisov, 1993] and [Batyrev, Borisov, 1996-II] the mirror construction for hypersurfaces in toric Fano varieties to complete intersections given by nef partitions. In an analogous way as we obtained the degeneration associated to an anticanonical hypersurface in a Gorenstein toric Fano variety, there is also a monomial degeneration for a complete intersection. We show that, when applied to this complete intersection degeneration, the tropical mirror construction gives the degeneration associated to the Batyrev-Borisov mirror. In particular, this also holds true in the case of Batyrev's mirror construction for hypersurfaces.

We introduce the notion of Fermat deformations in order to relate the mirror degenerations to birational models with fibers in toric Fano varieties with Chow group of rank 1. Applying this, we connect the mirror degeneration associated to the complete intersection of two general cubics in $\mathbb{P}^5$ to a Greene Plesser type orbifolding mirror family given in [Libgober, Teitelbaum, 1993].

In the same way, applying the tropical mirror construction to a monomial degeneration of non-complete intersection Calabi-Yau threefolds of degree 14 in $\mathbb{P}^6$ defined by the Pfaffians of a general linear skew symmetric map $7\mathcal{O}(-1) \to 7\mathcal{O}$, we reproduce the orbifolding mirror given by Rødland in [Rødland, 1998].

We also apply the tropical mirror construction to a monomial degeneration of non-complete intersection Calabi-Yau threefolds of degree 13 in $\mathbb{P}^6$ defined by the Pfaffians of a general skew symmetric map $\mathcal{O}(-2) \oplus 4\mathcal{O}(-1) \to \mathcal{O}(1) \oplus 4\mathcal{O}$. From the mirror degeneration given by the tropical mirror construction we obtain, via the concept of Fermat deformations, a flat monomial degeneration with fibers in an orbifold of $\mathbb{P}^6$, which again obeys the structure theorem of Buchsbaum-Eisenbud.

In the following, we give a short overview of the individual sections.

Section 1. This section provides an introduction to various concepts used in the tropical mirror construction.

Section 1.1 recalls some facts on Calabi-Yau manifolds and their relation to string theory and mirror symmetry. A manifold X of dimension d is called a Calabi-Yau manifold if $K_X = \mathcal{O}_X$ and $h^i(X, \mathcal{O}_X) = 0$ for $0 < i < d$.

In Section 1.2 we give a short introduction to the concept of stringy Hodge numbers introduced by Batyrev to generalize Hodge numbers to singular varieties. Given a normal projective variety X with log-terminal singularities one associates to X, via a resolution $f : Y \to X$ of singularities, a function $E_{st}(X; u, v)$, which Batyrev proves to be independent of the choice of the resolution. If E_{st} is a polynomial, then stringy Hodge numbers can be defined via the coefficients of E_{st}. In any case, topological mirror symmetry of a pair of Calabi-Yau varieties X and X° of dimension d can be defined via the stringy E-functions as the relation $E_{st}(X; u, v) = (-u)^d E_{st}(X^\circ; u^{-1}, v)$. If X admits a crepant resolution $f : Y \to X$, then

$$E_{st}(X; u, v) = \sum_{0 \le p,q \le d} (-1)^{p+q} h^{p,q}(Y) u^p v^q$$

In Section 1.3 we continue with an overview of toric geometry. The Sections 1.3.1–1.3.3 give the standard description of toric varieties and morphisms. If $N \cong \mathbb{Z}^n$, $N_{\mathbb{R}} = N \otimes \mathbb{R}$, $\sigma \subset N_{\mathbb{R}}$ is a rational convex polyhedral cone, $M = \mathrm{Hom}(N, \mathbb{Z})$ and

$$\check{\sigma} = \{m \in M_{\mathbb{R}} \mid \langle m, w \rangle \ge 0 \ \forall w \in \sigma\}$$

is the dual cone, then $\check{\sigma} \cap M$ is a finitely generated semigroup and defines an affine toric variety $U(\sigma) = \mathrm{Spec}(\mathbb{C}[\check{\sigma} \cap M])$. Given a fan Σ in $N_{\mathbb{R}}$, i.e., a finite set of strongly convex rational polyhedral cones such that every face of a cone in Σ is again a cone in Σ and the intersection of any two cones is a face of each, the $U(\sigma)$, $\sigma \in \Sigma$ glue to a toric variety $Y = X(\Sigma)$. The torus $\mathrm{Spec}(\mathbb{C}[\mathbb{Z}^n]) \hookrightarrow Y$ acts on Y. There is an inclusion reversing bijection between the cones of Σ and the torus orbit closures. Let $\Sigma(1)$ be the set of rays in Σ, i.e., the set of 1-dimensional cones. We denote by D_r the torus invariant divisor on Y corresponding to the ray $r \in \Sigma(1)$.

As explained in Section 1.3.5, one can describe the dualizing sheaf of a toric variety $X(\Sigma)$ as

$$\hat{\Omega}^n_{X(\Sigma)} \cong \mathcal{O}_{X(\Sigma)}\left(-\sum_{v \in \Sigma(1)} D_v\right)$$

Section 1.3.4 shows how to represent Weil and Cartier divisors, the Chow group $A_{n-1}(Y)$ of Weil divisors modulo linear equivalence on a toric variety $Y = X(\Sigma)$ and the Picard group $\mathrm{Pic}(Y)$. Classes in $A_{n-1}(Y)$ can be represented by torus invariant Weil divisors via the exact sequence

$$0 \rightarrow \quad M \quad \overset{A}{\rightarrow} \quad \mathbb{Z}^{\Sigma(1)} \quad \rightarrow \quad A_{n-1}(Y) \quad \rightarrow 0$$

where the rows of A are formed by the minimal lattice generators of the rays.

In Section 1.3.6 we describe the correspondence of integral polytopes in $M_{\mathbb{R}}$ and projective toric varieties. To an integral polytope $\Delta \subset M_{\mathbb{R}}$ one can associate the graded ring

$$S(\Delta) = \mathbb{C}\left[t^k x^m \mid m \in k\Delta\right] \quad \deg t^k x^m = k$$

with $k\Delta = \{km \mid m \in \Delta\}$ and $t^k x^m \cdot t^l x^{m'} = t^{k+l} x^{m+m'}$, and hence the projective toric variety $\mathbb{P}(\Delta) = \mathrm{Proj}(S(\Delta))$. Consider for any face F of Δ the cone of linear forms $w \in N_{\mathbb{R}}$, which take their minimum on Δ at the points of F. These cones form a fan, the normal fan $\Sigma = \mathrm{NF}(\Delta)$ of Δ. If 0 is in the interior of Δ, then $\mathrm{NF}(\Delta)$ is the fan formed by the cones over the faces of the dual polytope

$$\Delta^* = \{n \in N_{\mathbb{R}} \mid \langle m, n \rangle \geq -1 \ \forall m \in \Delta\}$$

of Δ. Furthermore, Δ defines a divisor on $X(\Sigma)$

$$D_\Delta = \sum_{r \in \Sigma(1)} - \min_{m \in \Delta} \langle m, \hat{r} \rangle D_r$$

Then as a toric variety $\mathbb{P}(\Delta) \cong X(\Sigma)$ with choice of an ample line bundle $\mathcal{O}_{\mathbb{P}(\Delta)}(1) \cong \mathcal{O}_{X(\Sigma)}(D_\Delta)$.

The Cox ring of a toric variety is explained in Section 1.3.7 and homogeneous coordinate presentations of toric varieties in Section 1.3.9. The Cox ring of a toric variety $Y = X(\Sigma)$ is the polynomial ring $S = \mathbb{C}[y_r \mid r \in \Sigma(1)]$ graded via the above presentation of the Chow group considering monomials in S as elements of $\mathbb{Z}^{\Sigma(1)}$. In an analogous way to the representation of projective space as

$$\mathbb{P}^n = \left(\mathbb{C}^{n+1} - V(\langle y_0, ..., y_n \rangle)\right) / \mathbb{C}^*$$

there is a similar description of toric varieties as a categorial quotient

$$X(\Sigma) = \left(\mathbb{C}^{\Sigma(1)} - V(B(\Sigma))\right) // G(\Sigma)$$

with some irrelevant ideal $B(\Sigma) \subset S$ and the action of

$$G(\Sigma) = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(Y), \mathbb{C}^*)$$

induced by the above sequence.

The application of the Cox ring to represent subvarieties and sheaves is treated in Sections 1.3.8 and 1.3.10. For example the vector space of global sections of the reflexive sheaf of sections $\mathcal{O}_{X(\Sigma)}(D)$ of a Weil divisor D on Y is isomorphic to the degree $[D]$-part of the Cox ring.

Section 1.3.11 gives an algorithm to compute the Mori cone $\overline{NE}(Y)_{\mathbb{R}} \subset A_1(Y) \otimes \mathbb{R}$ of effective 1-cycles for a simplicial toric variety Y.

The one-to-one correspondence of Gorenstein toric Fano varieties $\mathbb{P}(\Delta)$ of dimension n, polarized by $-K_{\mathbb{P}(\Delta)}$ and reflexive polytopes $\Delta \subset \mathbb{Z}^n \otimes \mathbb{R}$ is treated in Section 1.3.12. The involution of Gorenstein toric Fano varieties induced by duality of reflexive polytopes is the foundation of Batyrev's mirror construction for anticanonical hypersurfaces.

In order to understand, which torus invariant deformations represented by Cox Laurent monomials are trivial, we have to describe the automorphism group. If $Y = X(\Sigma)$ is simplicial, then the connected component of the identity of $\mathrm{Aut}(Y)$ is generated by automorphisms induced by the torus in Y and by the so called root automorphisms. Represented as Cox Laurent monomials a root automorphism is a degree 0 Cox Laurent monomial in $\mathbb{Z}^{\Sigma(1)}$ of the form

$$\frac{\prod_{r\in\Sigma(1)} y_r^{a_r}}{y_v}$$

with relative prime numerator and denominator, and the corresponding 1-parameter family of automorphisms is

$$\begin{array}{rll} y_v \mapsto & y_v + \lambda \prod_{s\in\Sigma(1)-\{v\}} y_s^{a_r} & \\ y_r \mapsto & y_r & \text{for} \quad r \in \Sigma(1) - \{v\} \end{array}$$

Toric Mori theory will be used to relate Calabi-Yau degenerations to orbifolding mirror families by relating the polarizing toric Fano variety of the degeneration to a different birational model. Section 1.3.14 gives an overview of Reid's toric interpretation of Mori theory, i.e., cone theorem, contraction theorem, existence and termination of flips and the minimal model program. Given a finite set $\mathcal{R}$ of 1-dimensional rational cones of a projective fan, the set of all closures of Kähler cones $\mathrm{cpl}(\Sigma)$ of projective simplicial fans Σ with $\Sigma(1) \subset \mathcal{R}$ fit together as $(|\mathcal{R}| - n)$-dimensional cones of a fan in $A_{n-1}(\mathcal{R})_{\mathbb{R}} \cong \mathbb{R}^{\mathcal{R}}/M_{\mathbb{R}}$. To justify the notation $A_{n-1}(\mathcal{R})_{\mathbb{R}}$, observe that the presentation of the Chow group of a toric variety $X(\Sigma)$ only depends on the 1-dimensional cones of the fan Σ. The fan generated by the maximal cones $\mathrm{cpl}(\Sigma)$ is called the Gelfand-Kapranov-Zelevinsky decomposition associated to $\mathcal{R}$ and can be extended to a complete fan, called the secondary fan $\Sigma(\mathcal{R})$. We explain an algorithm to compute the secondary fan via triangulations of marked polytopes.

The next Section 1.4 gives a short account of Gröbner bases, weight orderings and the Mora algorithm computing standard bases in the non-global setting. The concept of Gröbner bases plays an important role both for the theory of flat degenerations and for computing tropical varieties, so also for the tropical mirror construction. With regard to flat degenerations see also the remarks about Section 5 below. Gröbner basis theory is the algorithmic object connecting tropical geometry to degenerations and mirror symmetry.

Section 2. In this section we summarize the mirror constructions, which will be generalized in a common framework by the tropical mirror construction.

We begin in Section 2.1 with a short overview of the mirror construction given by Batyrev for anticanonical hypersurfaces in toric Fano varieties. Reflexive polytopes $\Delta \subset M_{\mathbb{R}}$ correspond to Gorenstein toric Fano varieties $Y = \mathbb{P}(\Delta)$ polarized by $-K_{\mathbb{P}(\Delta)}$. A general element of $\left|-K_{\mathbb{P}(\Delta)}\right|$ is a Calabi-Yau hypersurface in Y. Duality is an involution of the set of reflexive polytopes. Batyrev proves that general elements of $\left|-K_{\mathbb{P}(\Delta)}\right|$ and $\left|-K_{\mathbb{P}(\Delta^*)}\right|$ form a mirror pair in the sense of stringy Hodge numbers. In the original approach, Batyrev constructs, via maximal projective subdivisions of the fan of Y, a partial crepant resolution of the hypersurface. A maximal projective subdivision of the normal fan Σ of Δ is a simplicial refinement $\bar{\Sigma}$ of Σ defining a projective toric variety $X\left(\bar{\Sigma}\right)$ with the property that the non-zero lattice points of Δ^* span the 1-dimensional cones of $\bar{\Sigma}$.

Batyrev's construction for hypersurfaces has a generalization to the case of complete intersections given by nef partitions of reflexive polytopes. This mirror construction was given by Borisov and is explained in Section 2.2. Let $\Delta \subset M_{\mathbb{R}}$ be a reflexive polytope and $\Sigma = \mathrm{NF}(\Delta)$ its normal fan. Let

$$\Sigma(1) = I_1 \cup ... \cup I_c$$

be a disjoint union and suppose that the corresponding divisors $E_j = \sum_{v \in I_j} D_v$ are Cartier, spanned by global sections, and let $\Delta_j \subset M_{\mathbb{R}}$ be the polytope of sections of E_j. Note that $\sum_{j=1}^{c} E_j = -K_Y$. With $\nabla_j = \mathrm{convexhull}\{\{0\} \cup I_j\}$ the Minkowski sum

$$\nabla_{BB} = \nabla_1 + ... + \nabla_c$$

is again a reflexive polytope with $\nabla_{BB}^* = \mathrm{convexhull}(\Delta_1 \cup ... \cup \Delta_c)$. Let $\Sigma^\circ = \mathrm{NF}(\nabla_{BB})$, let

$$\Sigma^\circ(1) = J_1 \cup ... \cup J_c$$

be the disjoint union corresponding to the partition $\Delta_j \cap \text{vertices}(\nabla^*_{BB})$ of the vertices of ∇^*_{BB} and $E^\circ_j = \sum_{v \in J_j} D^\circ_v$. Then X in $Y = \mathbb{P}(\Delta)$ given by general sections of $\mathcal{O}(E_1), ..., \mathcal{O}(E_c)$ and X° in $Y^\circ = \mathbb{P}(\nabla_{BB})$ defined by general sections of $\mathcal{O}(E^\circ_1), ..., \mathcal{O}(E^\circ_c)$ form a mirror pair with respect to stringy Hodge numbers.

Section 2.3 introduces the Greene-Plesser orbifolding mirror family given by Rødland for the general Calabi-Yau threefold X of degree 14 in $\mathbb{P}^6$ defined by the Pfaffians of a general linear skew symmetric map

$$7\mathcal{O}_{\mathbb{P}^6}(-1) \to 7\mathcal{O}_{\mathbb{P}^6}$$

The mirror is given as a general element of the $1 = h^{1,1}(X)$-parameter family with fibers in a $\mathbb{Z}_7$-quotient of $\mathbb{P}^6$ defined by the Pfaffians

$$\begin{pmatrix} 0 & tx_1 & x_2 & 0 & 0 & -x_5 & -tx_6 \\ -tx_1 & 0 & tx_3 & x_4 & 0 & 0 & -x_0 \\ -x_2 & -tx_3 & 0 & tx_4 & x_6 & 0 & 0 \\ 0 & -x_4 & -tx_4 & 0 & tx_0 & x_1 & 0 \\ 0 & 0 & -x_6 & -tx_0 & 0 & tx_2 & x_3 \\ x_5 & 0 & 0 & -x_1 & -tx_2 & 0 & tx_4 \\ tx_6 & x_0 & 0 & 0 & -x_3 & -tx_4 & 0 \end{pmatrix}$$

in $\mathbb{C}[t] \otimes \mathbb{C}[x_0, ..., x_6]$, i.e., by the square roots of the 6×6 diagonal minors.

Section 3. The next main section introduces examples of monomial degenerations of Calabi-Yau varieties, which will serve as an input for the tropical mirror construction. Section 3.1 defines the natural monomial degenerations associated to hypersurfaces given by reflexive polytopes and to complete intersections given by nef partitions. Let $\Delta \subset M_{\mathbb{R}}$ be a reflexive polytope, $Y = \mathbb{P}(\Delta)$ a toric Fano variety with Cox ring S, $\Sigma = \text{NF}(\Delta)$ and $\Sigma(1) = I_1 \cup ... \cup I_c$ a nef partition. Then we obtain a degeneration given by

$$I = \langle t \cdot g_j + m_j \mid j = 1, ..., c \rangle \subset \mathbb{C}[t] \otimes S$$
$$\text{with } m_j = \prod_{v \in I_j} y_v$$

and monomial special fiber

$$I_0 = \langle m_j \mid j = 1, ..., c \rangle$$

Section 3.2 gives monomial degenerations of some non-complete intersection Pfaffian Calabi-Yau varieties. A monomial degeneration of a general Pfaffian elliptic curve in $\mathbb{P}^4$ defined by the Pfaffians of a general linear skew symmetric map $A : 5\mathcal{O}_{\mathbb{P}^4}(-1) \to 5\mathcal{O}_{\mathbb{P}^4}$ is given by the Pfaffians of

$$t \cdot A + \begin{pmatrix} 0 & 0 & x_1 & -x_4 & 0 \\ 0 & 0 & 0 & x_2 & -x_0 \\ -x_1 & 0 & 0 & 0 & x_3 \\ x_4 & -x_2 & 0 & 0 & 0 \\ 0 & x_0 & -x_3 & 0 & 0 \end{pmatrix}$$

If $A : 7\mathcal{O}_{\mathbb{P}^6}(-1) \to 7\mathcal{O}_{\mathbb{P}^6}$ is a general skew symmetric map, then the Pfaffians of

$$t \cdot A + \begin{pmatrix} 0 & 0 & x_2 & 0 & 0 & -x_5 & 0 \\ 0 & 0 & 0 & x_4 & 0 & 0 & -x_0 \\ -x_2 & 0 & 0 & 0 & x_6 & 0 & 0 \\ 0 & -x_4 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & -x_6 & 0 & 0 & 0 & x_3 \\ x_5 & 0 & 0 & -x_1 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 & -x_3 & 0 & 0 \end{pmatrix}$$

define a monomial degeneration of a general degree 14 Pfaffian Calabi-Yau threefold. In the same way there is a monomial degeneration of a general Calabi-Yau threefold of degree 13 in $\mathbb{P}^6$ defined by the Pfaffians of a general skew symmetric map $\mathcal{O}_{\mathbb{P}^6}(-2) \oplus 4\mathcal{O}_{\mathbb{P}^6}(-1) \to \mathcal{O}_{\mathbb{P}^6}(1) \oplus 4\mathcal{O}_{\mathbb{P}^6}$.

Section 4. The next main section introduces fundamental facts from tropical geometry used to formulate the mirror construction. Section 4.1 defines the amoeba of a subvariety of a torus as its image under the map

$$\begin{gathered} \log_t : (\mathbb{C}^*)^n \to \mathbb{R}^n \\ (z_1, ..., z_n) \mapsto (\log_t |z_1|, ..., \log_t |z_n|) \end{gathered}$$

Let K be the metric completion of the field of Puiseux series $\mathbb{C}\{\{t\}\}$ with respect to the norm $\|f\| = e^{-val(f)}$, where $val(f)$ denotes the exponent of the lowest weight term of f, and let $I \subset K[x_1, ..., x_n]$ be an ideal. Section 4.2 relates the limit for $t \to \infty$ of the amoeba given by I to the non-Archimedian amoeba. This is the image under

$$\begin{gathered} \text{val}_- : (K^*)^{n+1} \to \mathbb{R}^{n+1} \\ (f_1, ..., f_n) \mapsto (-val(f_1), ..., -val(f_n)) \end{gathered}$$

of the vanishing locus $V_K(I)$ of I over K. The non-Archimedian amoeba of I is also called the tropical variety $\text{tropvar}(I)$ of I. Note that here we take the negative of the vanishing order in the definition of the valuation map, as in the context of tropical geometry one usually considers the point of view of the $(\max, +)$ algebra.

Section 4.3 lists the basic properties of tropical varieties, in particular their characterization as the set of weight vectors $w \in \mathbb{R}^n$ such that $\mathrm{in}_w(I)$ contains no monomial. In Section 4.4 we recall the algebraic description of tropical varieties. Given a polynomial $f \in K[x_1, ..., x_n]$ we replace $+$ by max, multiplication by $+$ and the coefficients c by $-val(c)$ hence associating to f a piecewise linear function $\mathrm{trop}(f)$. Then $\mathrm{tropvar}(\langle f \rangle)$ is the non-differentiability locus of $\mathrm{trop}(f)$. In the same way $\mathrm{tropvar}(I)$ is the intersection $T(\mathrm{trop}(I))$ of the non-differentiability loci of all $\mathrm{trop}(f)$ for $f \in I$. In Section 4.5 we relate the tropical variety of $I \subset \mathbb{C}[t, x_1, ..., x_n]$ to a complex $BC_-(I)$, defined via its underlying set, which is the set of those points on the unit sphere that are the limit of projections of points of $\log(V(I))$ on an expanding sphere jS^n for $j \to \infty$. The fan $BF_-(I)$ is defined as the fan over $BC_-(I)$. Note that the fan $BF_-(I)$ is known in the literature as the Bergman fan, which differs by reflection at the origin from the Bergman fan $BF(I)$ as we defined above. The relation between $\mathrm{tropvar}(I)$ and $BC_-(I)$ is given by stereographic projection π_- of the lower half unit sphere from 0 to the plane $\{w_t = -1\} = \mathbb{R}^n$ of t-weight -1.

In the definition of the amoeba, of the non-Archimedian amoeba, of the tropical variety, of the non-differentiability locus of $\mathrm{trop}(f)$ and in the definition of $BC_-(I)$ and $BF_-(I)$ we adopt the Gröbner basis point of view, looking at the maximal weight term and take $weight(c) = -val(c)$ for constants $c \in K$. From the point of view of degenerations and local arc solutions of the total space of a degeneration at the special fiber, it is more natural to consider the minimal weight term combined with the definition $weight(c) = val(c)$ for constants $c \in K$. Summarizing, in our notation we have

$$\begin{aligned}
\mathrm{val}(V_K(I)) &= \pi(BF(I) \cap S^n \cap \{w_t > 0\}) \\
&= -\lim_{t \to \infty} (\log_t V(I_t)) = -\mathrm{val}_-(V_K(I)) \\
&= -\mathrm{tropvar}(I) = -T(\mathrm{trop}(I)) \\
&= -\pi_-(BF_-(I) \cap S^n \cap \{w_t < 0\})
\end{aligned}$$

where π is the stereographic projection of the upper half unit sphere from 0 to the plane $\{w_t = 1\} = \mathbb{R}^n$ of t-weight 1 and in the same way π_- from the lower half unit sphere.

Section 5. This section gives the standard characterization of flatness via Gröbner bases, e.g., a first order degeneration $\mathfrak{X}$ defined by

$$\langle f_1 + t \cdot g_1, ..., f_r + t \cdot g_r \rangle \subset R \otimes \mathbb{C}[t] / \langle t^2 \rangle$$

with special fiber given by $\langle f_1, ..., f_r \rangle$ is flat if and only if any syzygy $\sum_i a_i f_i = 0 \in R$ lifts to a syzygy between $f_1 + tg_1, ..., f_r + tg_r$, i.e., there are $b_i \in R$ such that

$$\sum_i (a_i + tb_i)(f_i + tg_i) = 0 \in R \otimes k[t] / \langle t^2 \rangle$$

Section 6. We recall the definition of the Gröbner fan of an ideal introduced by Mora, its dual description via state polytopes and the construction of multigraded Hilbert schemes. Furthermore, we connect stability of the Hilbert point with state polytopes. The existence of the multigraded Hilbert scheme shows that ideals in the Cox ring provide the right framework to describe subvarieties in toric varieties.

Our main interest in the Gröbner fan is the computation of tropical varieties, so in Section 6.1 we begin with a concept for computing the Bergman fan. Consider an ideal $J \subset \mathbb{C}[x_1, ..., x_n]$ such that every weight vector is equivalent to a non-negative weight vector, e.g., a homogeneous ideal. Section 6.2 introduces the Gröbner cone of weight vectors equivalent to a given global ordering $>$ and the Gröbner fan $GF(J)$. The maximal cones of the fan $GF(J)$ correspond to the monomial initial ideals of J. Section 6.3 gives a simple algorithm terminating with the fan $GF(J)$. We take a cone C in a non-complete subfan of $GF(J)$ and move into the complement of the fan along an outer normal vector of a face, which appears in the fan only once. Then we compute the corresponding Gröbner cone. Note that this is well suited for using Gröbner walk algorithms.

The second part of Section 6 deals with the Hilbert scheme and stability. Generalizing step by step, we begin in Section 6.4 with the setup of homogeneous ideals J with fixed Hilbert polynomial $P_{S/J}$ in $S = \mathbb{C}[x_0, ..., x_n]$ with respect to the grading

$$0 \to \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^{n+1} \xrightarrow{\deg} \mathbb{Z} \to 0$$

with

$$A = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ -1 & \cdots & -1 \end{pmatrix}$$

i.e., ideals in the Cox ring of $\mathbb{P}^n$. We recall the construction of the Hilbert scheme, the state polytope and the characterization of stability via the state polytope as given by Bayer and Morrison in [Bayer, Morrison, 1988].

After summarizing in Section 6.5.1 some facts on G-linearizations of line bundles for an affine algebraic group G acting rationally on an algebraic variety, we generalize to the toric setting. In the same way as for the case of subvarieties of $\mathbb{P}^n$, the key ingredients are the Grassmann functor and the Hilbert functor explained in Sections 6.6.1 and 6.6.2 as given by Haiman and Sturmfels in [Haiman, Sturmfels, 2004]. Let k be a commutative ring, A a set and S a polynomial ring graded by a set A. If $h : A \to \mathbb{N}$ is a function and R is a k-algebra, then the Hilbert functor $\mathbb{H}^h_{(S,F)}$ is defined via

$$\mathbb{H}^h_{(S,F)}(R) = \left\{ L \mid \begin{array}{l} L \subset R \otimes S \text{ is an } F\text{-submodule with} \\ (R \otimes S_a)/L_a \text{ locally free of rank } h(a)\ \forall a \in A \end{array} \right\}$$

Here the notion of an F-submodule is defined via sets of operators $F_{a,b} \subset \operatorname{Hom}_k(S_a, S_b)$. Under appropriate conditions $\mathbb{H}^h_{(S,F)}$ is represented by a closed subscheme of a projective Grassmann scheme. The key point is the restriction to a finite set of degrees. In the case of the homogeneous coordinate ring of $\mathbb{P}^n$ one can restrict to one degree.

Considering an example by Haiman and Sturmfels, Section 6.6.3 explains the application of this construction to the Hilbert scheme of admissible ideals, i.e., ideals with the property that $(S/I)_a = S_a/I_a$ is a locally free k-module of finite rank for all $a \in A$. Note that this setup is not directly applicable even to the homogeneous coordinate ring of projective space. As a second example, Section 6.6.4 applies the above construction of $\mathbb{H}^h_{(S,F)}$ to obtain the classical Hilbert scheme via truncation $I_{\geq a}$ of ideals at an appropriate degree a.

The tangent space at $I \in \mathbb{H}^h_{(S,F)}(k)$ of the scheme representing $\mathbb{H}^h_{(S,F)}$ is described as $\operatorname{Hom}_S(I, S/I)_0$ in Section 6.6.5.

Sections 6.6.6 and 6.6.7 give an overview of Stanley filtrations and multigraded regularity as introduced by Maclagan and Smith in [Maclagan, Smith, 2004] and [Maclagan, Smith, 2005]. If Y is a smooth toric variety and M is a finitely generated S-module then for $m \in A_{n-1}(Y)$ the notion of m-regularity of M is defined via local cohomology. The regularity of M is the set of all degrees $m \in A_{n-1}(Y)$ such that M is m-regular. In Section 6.6.8 the multigraded Hilbert functor $R \mapsto \mathbb{H}^P_Y(R)$ associating to a $\mathbb{C}$-algebra R the set of ideal sheaves $\mathcal{J}$ of families of subschemes $X \subset Y \times_{\mathbb{C}} \operatorname{Spec} R \to \operatorname{Spec} R$ with fixed multivariate Hilbert polynomial P is given. Using the above construction of $\mathbb{H}^h_{(S,F)}$, the functor $\mathbb{H}^P_Y$ is represented by a projective scheme over $\mathbb{C}$. The finite set of degrees to represent $\mathbb{H}^P_Y$ as a subscheme of a Grassmann scheme can be computed algorithmically.

In Section 6.6.9 we introduce the state polytope in the multigraded setting and characterize the sets of stable and semistable points via the state polytope. If I is an ideal in the Cox ring of a smooth toric variety with Hilbert function h and $\mathbb{H}$ the corresponding Hilbert scheme, then the Hilbert point $H(I) \in \mathbb{H}$ is in the stable locus $\mathbb{H}^s$ if and only if 0 is in the interior of the state polytope $\mathrm{State}(I)$ of I.

Finally, given a toric variety Y defined by a fan $\Sigma \subset N_{\mathbb{R}}$, we identify in Section 6.7 the weight vectors on the Cox ring S of Y with the vectors in $N_{\mathbb{R}}$ by dualizing the presentation of the Chow group. Hence, the Gröbner fan of an ideal in the Cox ring of Y can be considered as a fan in $N_{\mathbb{R}}$. Note that this generalizes tropical projective space $\mathbb{TP}^n = \mathbb{R}^{n+1}/\mathbb{R}(1, ..., 1)$.

Section 7. This section considers toric Fano varieties Y in the sense that some multiple of $-K_Y$ is an ample Cartier divisor. It explains how Fano polytopes $P \subset N_{\mathbb{R}}$ represent $\mathbb{Q}$-Gorenstein toric Fano varieties defined by the fan over the faces of P. This is the right category of toric Fano varieties with respect to toric Mori theory.

Section 8. Here we formulate the tropical mirror construction for complete intersections. The construction takes as an input the degenerations associated in Section 3.1 to complete intersections defined by nef partitions. So let $\Delta \subset M_{\mathbb{R}}$ be a reflexive polytope, $Y = \mathbb{P}(\Delta)$ the corresponding toric Fano variety with Cox ring S, presentation matrix A of $A_{n-1}(Y)$ and $\Sigma = \mathrm{NF}(\Delta) \subset N_{\mathbb{R}}$. Let $\Sigma(1) = I_1 \cup ... \cup I_c$ be a nef partition and $\Delta_j \subset M_{\mathbb{R}}$ the polytopes of sections of the corresponding divisors and $\nabla_j = \mathrm{convexhull}\{\{0\} \cup I_j\}$. Denote by $\mathfrak{X} \subset Y \times \mathrm{Spec}(\mathbb{C}[t])$ the associated degeneration as defined above by the ideal

$$I = \langle f_j = t \cdot g_j + m_j \mid j = 1, ..., c\rangle \subset \mathbb{C}[t] \otimes S$$

and by $I_0 \subset S$ the ideal with minimal generators m_j. We begin in Section 8.1 by exploring the properties of these degenerations and describe in Section 8.2 the special fiber Gröbner cone $C_{I_0}(I) \subset \mathbb{R} \oplus N_{\mathbb{R}}$ and the special fiber polytope

$$\nabla = C_{I_0}(I) \cap \{w_t = 1\} \subset N_{\mathbb{R}}$$

The special fiber cone $C_{I_0}(I)$ is cut out by the half-space equations corresponding to first order deformations contributing to the tangent vector of $\mathfrak{X}$ at the special fiber. We show that the reflexive polytope ∇ coincides with the Batyrev-Borisov mirror polytope. Section 8.3 gives an explicit description of the initial ideals of the faces of ∇ and Section 8.4 introduces the map

$$\mathrm{dual} : \mathrm{Poset}(\nabla) \to \mathrm{Poset}(\nabla^*)$$

between the complexes of faces of ∇ and ∇^* associating to a face F of ∇ the convex hull of all first order deformations appearing the initial ideal of I with respect to F. So, if

$$\mathrm{in}_F(f_j) = t \sum_{m \in G_j(F)} c_m m + m_j$$

then

$$\operatorname{dual}(F) = \operatorname{convexhull}\left(A^{-1}\left(\frac{m}{m_j}\right) \mid m \in G_j(F),\ j = 1, ..., c\right) \subset M_{\mathbb{R}}$$

Note that first order deformations correspond to Cox Laurent monomials of degree 0 hence via the presentation matrix A of the Chow group to elements of M. Indeed, we show that $\operatorname{dual}(F) = F^*$ is the face of ∇^* dual to F.

By considering the faces of ∇, which correspond to cones in the Bergman fan, we obtain in Section 8.5 the Bergman subcomplex $B(I) \subset \operatorname{Poset}(\nabla)$ of the complex of faces of ∇. In Section 8.6 we give an inclusion reversing map from $B(I)$ to the complex of faces of Δ

$$\mu : B(I) \to \operatorname{Poset}(\Delta)$$

by taking the Minkowski sum over the faces of ∇^* corresponding to deformations of the individual equations, i.e.,

$$\mu(F) = \sum_{j=1}^{c} \operatorname{convexhull}\left(A^{-1}\left(\frac{m}{m_j}\right) \mid m \in G_j(F)\right)$$

We relate $\mu(F)$ and $\operatorname{dual}(F)$ in Section 8.7 via

$$\mu(F) = \sum_{j=1}^{c} \operatorname{dual}(F) \cap \Delta_j$$

In Section 8.8 we relate the maps lim and μ. If F is a face of $B(I)$, then

$$\lim(F) = V\left((\mu(F))^*\right)$$

is the torus orbit closure of Y corresponding to the face $\mu(F)$ of Δ. Figure 0.1 shows the complexes $B(I)$ and $\lim(B(I))$ and the polyhedra ∇ and Δ for the monomial degeneration of the complete intersection of two general quadrics in $\mathbb{P}^3$ as given above.

Section 8.10 gives a $c : 1$ covering of the complex $B(I)^{\vee}$ by faces of $\operatorname{dual}(B(I))$. The covering is induced by associating to $F \in B(I)$ the faces

$$\{F^* \cap \Delta_j \mid j = 1, ..., c\}$$

Note that this covering can have degenerate faces in the sense that $\dim(F^* \cap \Delta_j)$ can be less than $\dim(F^{\vee}) = d - \dim(F)$, but the faces are always non-empty. We give an algorithm computing this covering from the complex $\operatorname{dual}(B(I))$.

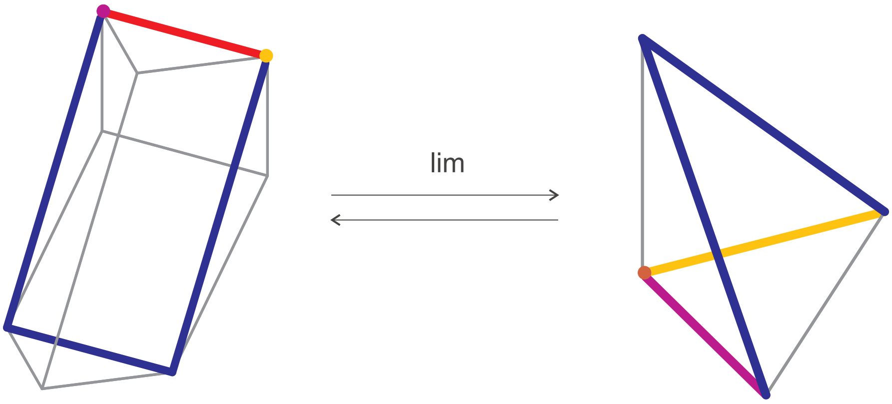

Figure 0.1: Limit correspondence for the monomial degeneration of the complete intersection of two general quadrics in $\mathbb{P}^3$ and its mirror

Let $Y^\circ = \mathbb{P}(\nabla)$ and A° be the presentation matrix of $A_{n-1}(Y^\circ)$. Applying in Section 8.11 the covering algorithm on the mirror side to the complex $(\mu(F))^*$, via the sheets we obtain the ideal

$$\begin{aligned} I^\circ &= \left\langle t \cdot \sum_{\delta \in (\mu(B(I)))^* \cap M} c_\delta \cdot \delta\left(m_j^\circ\right) + m_j^\circ \mid j = 1, ..., c \right\rangle \\ &= \left\langle t \cdot \sum_{\delta \in (\mu(B(I)))^* \cap \nabla_j \cap M} c_\delta \cdot \delta\left(m_j^\circ\right) + m_j^\circ \mid j = 1, ..., c \right\rangle \subset S^\circ \otimes \mathbb{C}[t] \end{aligned}$$

defining the mirror degeneration $\mathfrak{X}^\circ \subset \mathbb{P}(\nabla) \times \operatorname{Spec} \mathbb{C}[t]$. Here the monomials m_j° are the least common multiples of denominators of the Cox Laurent monomials $A^\circ(\delta)$ for $\delta \in (\mu(B(I)))^* \cap \nabla_j$. Passing from I° to the ideal of the tropical mirror as defined above by applying the deformations in $(\mu(B(I)))^* \cap N$ to the special fiber ideal

$$\bigcap_{F \in B(I)} \langle z_{G^*} \mid G \text{ a facet of } \nabla \text{ with } F \subset G \rangle \subset S^\circ$$

is the toric analogue of saturation, also valid for non-simplicial toric varieties. It does not change the geometry of the degeneration $\mathfrak{X}^\circ$ or the objects involved in the tropical mirror construction.

Along Section 8 we visualize the objects introduced in the tropical mirror construction for the example of the general complete intersection of two quadrics in $\mathbb{P}^3$.

In Section 8.12 we apply the tropical mirror construction to some complete intersection examples. In particular, by considering a set of Fermat deformations in order to relate Y° to a different birational model, we obtain the Greene-Plesser type orbifolding mirror family of the complete intersection of two cubics in $\mathbb{P}^5$ as given in [Libgober, Teitelbaum, 1993]. Note that the text is computer generated by the implementation of the tropical mirror construction in the Maple package tropicalmirror.

Section 9. This section gives the tropical mirror construction in its general form, as outlined above.

We begin in Section 9.1 with a summary and continue in the following sections by introducing fundamental concepts used in the tropical mirror construction. Section 9.2 represents torus invariant first order deformations of monomial ideals by lattice monomials. Let $Y = X(\Sigma)$ be a toric variety given by the fan Σ in $N_{\mathbb{R}}$, let $S = \mathbb{C}[y_r \mid r \in \Sigma(1)]$ be the Cox ring of Y and $I_0 \subset S$ a monomial ideal. The space of degree 0 first order deformations $\mathrm{Hom}(I_0, S/I_0)_0$ has a basis of $(\mathbb{C}^*)^{\Sigma(1)}$-deformations. Any such homomorphism $\delta : I_0 \to S/I_0$ is representable by a degree 0 Cox Laurent monomial $\frac{q_1}{q_0}$ with relatively prime monomials $q_0, q_1 \in S$ via

$$\delta(m) = \left\{ \begin{array}{ll} \frac{q_1}{q_0} \cdot m & \text{if } q_0 \mid m \\ 0 & \text{otherwise} \end{array} \right\}$$

for minimal generators $m \in I_0$. Via the sequence

$$0 \to M \stackrel{A}{\to} \mathbb{Z}^{\Sigma(1)} \stackrel{\deg}{\to} A_{n-1}(X(\Sigma)) \to 0$$

$\frac{q_1}{q_0}$ corresponds to a lattice monomial in $M = \mathrm{Hom}(N, \mathbb{Z})$.

In Section 9.3 we give a combinatorial description of the vanishing loci in Y of reduced monomial ideals $I_0 \subset S$. Given a monomial $m \in I_0$, denote by

$$\mathrm{rays}_m(\Sigma) = \{r \in \Sigma(1) \mid y_r \text{ divides } m\}$$

the set of rays of Σ such that y_r divides m. We define the stratified toric primary decomposition $SP(I_0)$ as the complex, which has as faces of dimension s the ideals $\langle y_r \mid r \subset \sigma\rangle$ for all cones $\sigma \in \Sigma$ of dimension $n - s$ which contain a ray in $\mathrm{rays}_m(\Sigma)$ for all monomials $m \in I_0$.

Suppose Δ is a polytope with $\Sigma = \mathrm{NF}(\Delta)$. Then $SP(I_0)$ is naturally isomorphic to the complex $\mathrm{Strata}_\Delta(I_0)$ of strata of I_0. $\mathrm{Strata}_\Delta(I_0)$ is defined as the complex which has as faces of dimension s those faces F of Δ such that for all monomials $m \in I_0$ the set

$$\{G \mid G \text{ facet of } \Delta \text{ with } y_{G^*} \mid m\}$$

contains a facet G with $F \subset G$. Suppose that the vanishing locus X_0 of I_0 in Y is equidimensional of dimension d. The complexes $SP(I_0) \cong \mathrm{Strata}_\Delta(I_0)$

describe the vanishing locus of I_0 and they define the ideal

$$\begin{aligned} I_0^{\Sigma} &= \bigcap_{J \in SP(I_0)_d} J \\ &= \bigcap_{F \in \mathrm{Strata}_{\Delta}(I_0)_d} \langle y_{G^*} \mid G \text{ a facet of } \Delta \text{ with } F \subset G \rangle \end{aligned}$$

naturally associated to X_0. Passing from I_0 to I_0^{Σ} is the toric analogue of saturation. Note that we do not assume Y to be simplicial.

If Δ is a simplex and I_0 is a Stanley-Reisner ideal given by a simplicial subcomplex Z of the complex of cones of $\Sigma = \mathrm{NF}(\Delta)$, then we relate $\mathrm{Strata}_{\Delta}(I_0)$ to Z via the map associating to a face $F \in \mathrm{Strata}_{\Delta}(I_0)$ the hull of the rays of Σ not contained in $\mathrm{hull}(F^*)$.

In Section 9.4 we introduce the notion of locally irrelevant deformations. Let $I_0 \subset S$ be a reduced monomial ideal defining $X_0 \subset Y$, X_i a stratum of X_0 and $\mathfrak{X}$ a first order deformation of X_0. Then $\mathfrak{X}$ is called locally irrelevant at X_i, if there is a formal analytic open neighborhood $\tilde{U} \subset Y$ of X_i and an isomorphism

$$\left(\tilde{U} \cap X_0\right) \times \mathrm{Spec}\left(\mathbb{C}[t] / \langle t^2 \rangle\right) \cong \mathfrak{X} \cap \left(\tilde{U} \times \mathrm{Spec}\left(\mathbb{C}[t] / \langle t^2 \rangle\right)\right)$$

extending $X_i \times \mathrm{Spec}(\mathbb{C}[t] / \langle t^2 \rangle) \subset \mathfrak{X}$.

In Section 9.5 we give the setup for the tropical mirror construction. Consider $N \cong \mathbb{Z}^n$, $M = \mathrm{Hom}(N, \mathbb{Z})$, P a Fano polytope, $\Sigma = \Sigma(P)$ the fan over the faces of P and $Y = X(\Sigma)$ with Cox ring S. Let $I_0 \subset S$ be a reduced monomial ideal with $I_0 = I_0^{\Sigma}$ and equidimensional vanishing locus. Let $\mathfrak{X} \subset Y \times \mathrm{Spec}\,\mathbb{C}[[t]]$ be a degeneration of Calabi-Yau varieties of codimension c, which is given by $I \subset S \otimes \mathbb{C}[t]$ and with special fiber X_0 defined by I_0.

We give the conditions assumed to be satisfied for the input degeneration $\mathfrak{X}$. Formulated in an explicit and testable form, these conditions are:

1. $C_{I_0}(I) \cap \{w_t = 0\} = \{0\}$
2. $C_{I_0}(I)$ is the cone defined by the half-space equations corresponding to the torus invariant first order deformations appearing in the reduced standard basis of I in $S \times \mathbb{C}[t] / \langle t^2 \rangle$ with respect to a monomial ordering in the interior of $C_{I_0}(I)$.

   All lattice points of F^* appear as deformations in I.
3. $\nabla^* \subset \Delta$, which is equivalent to the condition that any first order deformation appearing in I is also a deformation of the anticanonical Calabi-Yau hypersurface in Y.

4. Any facet of $\text{Strata}_\Delta(I_0)$ is contained in precisely c facets of Δ.

5. Any facet of $B(I)$ is contained in precisely c facets of ∇.

An interpretation of these conditions with respect to the geometry of $\mathfrak{X}$ is given. We can satisfy requirement

1. via a condition on the position of the Hilbert point of I_0 with respect to the state polytope of the general fiber,

2. via a genericity condition on the tangent vector with respect to the tangent space of the component of the Hilbert scheme containing $\mathfrak{X}$,

3. via the condition that $\mathcal{O}_{X_0}$ has a resolution

$$0 \to \mathcal{O}_Y(-K_Y) \to \ldots \to \mathcal{F}_1 \to \mathcal{O}_Y \to \mathcal{O}_{X_0} \to 0$$

   with direct sums $\mathcal{F}_j = \bigoplus_i \mathcal{O}_Y(D_{ji})$,

4. via the components of X_0 being given by c linear equations,

5. via a condition on the locally relevant deformations of $\mathfrak{X}$ at the zero dimensional strata of X_0.

In Sections 9.6-9.12 we formulate the tropical mirror construction in the general setting as already outlined above. Section 9.6 describes the special fiber Gröbner cone $C_{I_0}(I) \subset \mathbb{R} \oplus N_{\mathbb{R}}$ and the special fiber polytope

$$\nabla = C_{I_0}(I) \cap \{w_t = 1\} \subset N_{\mathbb{R}}$$

The polytope $\nabla^* \subset M_{\mathbb{R}}$ is a Fano polytope, so the Fan $\Sigma^\circ = \Sigma(\nabla^*)$ over the faces of ∇^* defines a toric Fano variety $Y^\circ = X(\Sigma^\circ)$ with Cox ring S°.

Let

$$0 \to M \overset{A}{\to} \mathbb{Z}^{\Sigma(1)} \overset{\deg}{\to} A_{n-1}(X(\Sigma)) \to 0$$

be the presentation of the Chow group of divisors of $X(\Sigma)$. Consider a face F of ∇^*. The initial forms of the reduced standard basis of I in $S \otimes \mathbb{C}[t] / \langle t^2 \rangle$ with respect to a monomial ordering in the interior of ∇ involve a minimal generator of I_0. Dividing the non-special fiber monomials by the special fiber monomial of the initial forms and applying A^{-1} we obtain a set lattice monomials. In analogy to complete intersections associating to F of ∇ the convex hull of these lattice monomials, we define in Section 9.7 the map

$$\text{dual} : \text{Poset}(\nabla) \to \text{Poset}(\nabla^*)$$

We observe that for $F \in \text{Poset}(\nabla)$

$$\text{dual}(F) = F^*$$

and all lattice points appear in the initial ideal. The non-special fiber monomials of the initial forms decompose into characters of the torus $(\mathbb{C}^*)^{\Sigma(1)}$, which are just the Cox Laurent monomials associated to the lattice points of F^*. The characters correspond to deformations of X_0 in $\text{Hom}(I_0, S/I_0)_0$.

The complex $\text{dual}(B(I))$ can be seen as a polyhedral representation of the structure of the ideal I, as described by structure theorems like the Koszul resolution for complete intersections or the Buchsbaum-Eisenbud theorem for codimension 3 subcanonical varieties.

We define in Section 9.8 the special fiber Bergman complex

$$B(I) = (BF(I) \cap \text{Poset}(C_{I_0}(I))) \cap \{w_t = 1\} \subset \text{Poset}(\nabla)$$

of those faces F of ∇ such that $\text{in}_F(I)$ does not contain a monomial. Note that the Bergman fan $BF(I)$ contains more information than the combinatorial objects derived from $B(I)$.

Section 9.9 explores the covering structure in $\text{dual}(B(I))$ over $B(I)^\vee$, generalizing the $c : 1$ trivial covering in the case of complete intersection.

In Section 9.10 we describe the limit map $\lim : B(I) \to \text{Strata}(X_0)$. Let K be the metric completion of the field of Puiseux series as defined above. We introduce the notion of Cox arcs as elements of

$$(K^*)^{\Sigma(1)} / \text{Hom}_{\mathbb{Z}}(A_{n-1}(Y), K^*) \cong \text{Hom}_{\mathbb{Z}}(M, K^*) = (K^*)^n$$

representing via the presentation of $A_{n-1}(Y)$ elements of the torus $(K^*)^n$ of dimension $n = \dim(Y)$. Then $V_K(I) \subset (K^*)^n$ is the image of the vanishing locus of $I \subset \mathbb{C}[t] \otimes S$ in $(K^*)^{\Sigma(1)} / \text{Hom}_{\mathbb{Z}}(A_{n-1}(X(\Sigma)), K^*)$. The limit map

$$\begin{array}{rccl} \lim : & B(I) & \to & \text{Strata}(X_0) \subset \text{Strata}(Y) \\ & F & \mapsto & \left\{\lim_{t\to 0} a(t) \mid a \in \text{val}^{-1}(\text{int}(F))\right\} \end{array}$$

associating to a face F of $B(I)$ the stratum of X_0 of limit points of arc solutions of I over the interior of F. If F is a face of the special fiber Bergman complex $B(I)$, then there is a unique cone τ of Σ such that $\text{int}(F) \subset \text{int}(\tau)$ and

$$\lim(F) = V(\tau)$$

is the torus stratum of Y corresponding to τ.

In Sections 9.11 and 9.12 we define the mirror special fiber and the first order conjectural mirror degeneration. This generalizes the case of complete intersections. Denote by $d = \dim(B(I))$ the fiber dimension of $\mathfrak{X}$. As noted above, the mirror special fiber is the vanishing locus of the ideal

$$\begin{aligned} I_0^\circ &= \left\langle \prod_{v\in J} z_v \mid J \subset \Sigma^\circ(1) \text{ with } \operatorname{supp}(B(I)) \subset \bigcup_{v\in J} F_v \right\rangle \\ &= \bigcap\nolimits_{F\in B(I)_d} \langle z_{G^*} \mid G \text{ a facet of } \nabla \text{ with } F \subset G\rangle \subset S^\circ \end{aligned}$$

in Y° and the first order mirror family is defined by

$$\left\langle m + t \cdot \sum_{\alpha\in\operatorname{supp}((\lim(B(I)))^*)\cap N} a_\alpha \cdot \alpha(m) \mid m \text{ min. gen. of } I_0^\circ \right\rangle \subset \mathbb{C}[t]/\langle t^2\rangle \otimes S^\circ$$

We give a geometric realization of the mirror family with fibers in a simplicial toric variety given by a projective simplicial subdivision of Σ°. Here we can represent the mirror family in three equivalent ways: By a graded radical ideal contained in the irrelevant ideal, by a $\operatorname{Pic}(Y)$-generated and saturated ideal and by a saturated ideal in the Picard-Cox ring $R^\circ = \bigoplus_{\alpha\in\operatorname{Pic}(Y^\circ)} S_\alpha^\circ$.

In Section 9.13 we propose the notion of a set of Fermat deformations associated to a monomial degeneration $\mathfrak{X}$. The goal is to relate, if possible, the tropical mirror degeneration $\mathfrak{X}^\circ$ to an orbifolding mirror family. For simplicity we assume that the fibers of $\mathfrak{X} \subset \mathbb{P}^n \times \operatorname{Spec}\mathbb{C}[t]$ are embedded in projective space. A set of Fermat deformations of $\mathfrak{X}$ is a set $\mathfrak{F}$ of non-trivial first order deformations of X_0 in $\mathfrak{X}$ corresponding to vertices of faces of $\operatorname{dual}(B(I))$. So the elements of $\mathfrak{F}$ have an interpretation as Cox variables of Y°. We require $\mathfrak{F}$ to satisfy the following properties:

- For all zero dimensional strata p of $\mathbb{P}^n$ precisely one of the vertices of the faces $F \in \operatorname{dual}(B(I))$ with $\lim(F) = p$ is an element of $\mathfrak{F}$.
- The convex hull of $\mathfrak{F}$ in $M_\mathbb{R}$ is a full dimensional polytope containing 0 in its interior, so $\mathfrak{F}$ spans a fan $\hat{\Sigma}^\circ$ over a lattice simplex.
- The elements of $\mathfrak{F}$ are incomparable with respect to the preordering of Cox Laurent monomials given by divisibility of the denominators (each assumed to be relatively prime to the numerator).

The fan $\hat{\Sigma}^\circ$ defines a toric Fano variety $\hat{Y}^\circ = X(\hat{\Sigma}^\circ)$, which is an orbifold of a weighted projective space. We describe the special fiber of the monomial degeneration $\hat{\mathfrak{X}}^\circ$ induced by $\mathfrak{X}^\circ$ via a birational map $Y^\circ \to \hat{Y}^\circ$. The degeneration $\hat{\mathfrak{X}}^\circ$ involves the first order deformations represented by the degree 0 Cox Laurent monomials

$$\left\{\prod\nolimits_{r\in\hat{\Sigma}^\circ(1)} z_r^{\langle\hat{r},w\rangle} \mid w \in (\lim(B(I)))^* \cap N\right\}$$

in the Cox ring of $\hat{Y}^\circ$.

Section 10. Here, we apply the tropical mirror construction to non-complete intersection Pfaffian examples.

We begin in Section 10.1 by recalling the structure theorem of Buchsbaum and Eisenbud for Pfaffian subschemes of $\mathbb{P}^n$. Excluding special cases, locally Gorenstein subcanonical schemes of codimension 3 of $\mathbb{P}^n$ are locally given by the Pfaffians of order $2k$ of a skew symmetric map $\varphi : \mathcal{E}(-t) \to \mathcal{E}^*$ for some vector bundle $\mathcal{E} \to \mathbb{P}^n$ of rank $2k+1$. By the theorem of Buchsbaum-Eisenbud, they have a locally free resolution of the form

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-t-2s) \to \mathcal{E}(-t-s) \xrightarrow{\varphi} \mathcal{E}^*(-s) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

with $s = c_1(\mathcal{E}) + kt$. We recall the list of Pfaffian Calabi-Yau threefolds given in [Tonoli, 2000].

In Section 10.2 we make some remarks on the deformation theory of Pfaffian ideals. The deformations of arithmetically Gorenstein Pfaffian varieties are unobstructed and the base space is smooth given by the independent entries of the skew symmetric syzygy matrix. These observations allow to apply the tropical mirror construction to monomial degenerations of Pfaffian ideals and to extend first order mirror families $\mathfrak{X}^{\circ 1} \subset Y^\circ \times \operatorname{Spec}\mathbb{C}[t] / \langle t^2 \rangle$, which obey the structure theorem of Buchsbaum and Eisenbud, to flat families $\mathfrak{X}^\circ \subset Y \times \operatorname{Spec}\mathbb{C}[t]$.

Section 10.3 applies the tropical mirror construction to the monomial degeneration $\mathfrak{X}$ of the general Pfaffian elliptic curve in $\mathbb{P}^4$ as defined in Section 3.2. Note that the total space of $\mathfrak{X}$ is a local complete intersection and the tropical mirror construction treats this example much like a complete intersection. The covering of $B(I)^\vee$ in $\operatorname{dual}(B(I))$ is $c:1$ unbranched, but not a trivial $c:1$ covering like it would be for a codimension $c=3$ complete intersection. Figure 0.2 shows a projection into 3-space of the complex $\lim(B(I)) \subset \operatorname{Poset}(\Delta)$. Figure 0.3 visualizes a projection of the facets of the polytope ∇^* and the subcomplex $\operatorname{dual}(B(I)) \subset \operatorname{Poset}(\nabla^*)$ of the boundary of ∇^*. Figure 0.4 shows the topology of $\operatorname{dual}(B(I))$. The faces of the complex are labeled by their image under lim and the lattice points of the faces are labeled by the corresponding deformations of I_0. The complex $\operatorname{dual}(B(I))$ has 5 prisms as facets of dimension 3 and 5 triangles as faces of dimension 2. Every prism intersects two of the other prisms along triangles. The vertices of the triangles and the edges of the prisms connecting these vertices form the $3:1$ unbranched covering of $B(I)^\vee$.

In Section 10.4 we apply the tropical mirror construction to the monomial degeneration given in Section 3.2 for a general Calabi-Yau threefold of degree 14 in $\mathbb{P}^6$ defined by the Pfaffians a general skew symmetric map $7\mathcal{O}(-1) \to$

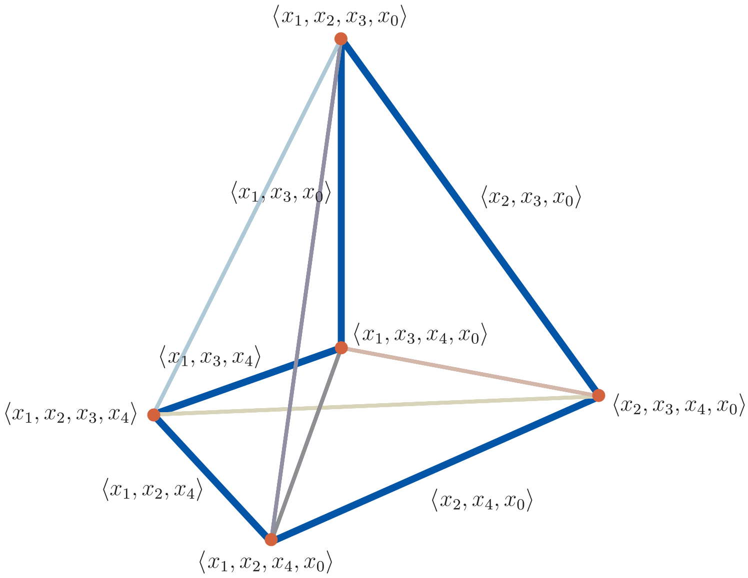

Figure 0.2: Projection of the complex $\lim(B(I)) \subset \mathrm{Poset}(\Delta)$ for the monomial degeneration of the general Pfaffian elliptic curve in $\mathbb{P}^4$

$7\mathcal{O}$. Using the concept of Fermat deformations from Section 9.13, we relate the mirror degeneration to the orbifolding mirror given by Rødland.

Section 10.5 applies the tropical mirror construction to a monomial degeneration of the general Pfaffian Calabi-Yau threefold of degree 13 in $\mathbb{P}^6$ defined via a general skew symmetric map $\mathcal{O}(-2) \oplus 4\mathcal{O}(-1) \to \mathcal{O}(1) \oplus 4\mathcal{O}$, as given in Section 3.2. Applying the concept of Fermat deformations to switch to another birational model of Y° with Chow group of rank 1, we relate the mirror degeneration to a Greene-Plesser orbifolding mirror family. This degeneration satisfies the structure theorem of Buchsbaum-Eisenbud, in particular, allows extension of the first order mirror degeneration.

Note again that the text of these examples is computer generated from the output of the Maple package tropicalmirror, so all examples use the same text fragments.

Section 11. The next main section contains some remarks on the tropical computation of stringy E-functions.

As this gives the general direction, we recall in Section 11.1 the relation of $h^{d-1,1}(X)$, $h^0\left(X, N_{X/\mathbb{P}^n}\right)$ and $\mathrm{Aut}(\mathbb{P}^n)$ for Calabi-Yau manifolds of dimension d in projective space $\mathbb{P}^n$.

Section 11.2 explains Batyrev's original formulas for $h^{1,1}\left(\bar{X}\right)$ and $h^{d-1,1}\left(\bar{X}\right)$ via MPCP (maximal projective crepant partial) desingularizations $\bar{X} \to X$.

In Section 11.3 we explain a tropical method to compute $h^{1,\dim(X)-1}(X)$

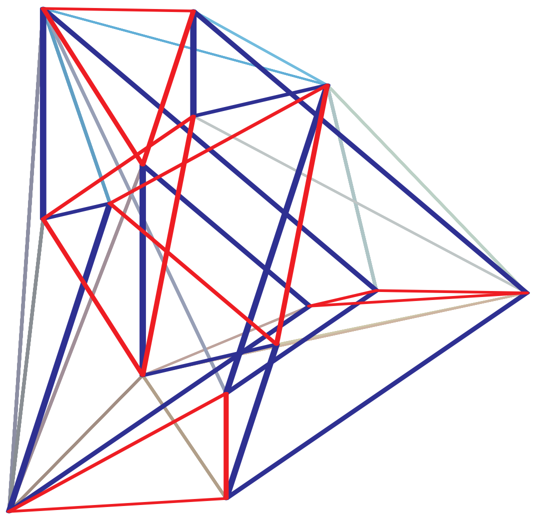

Figure 0.3: Projection of the complex dual $(B(I))$ for a monomial degeneration of the generic Pfaffian elliptic curve in $\mathbb{P}^4$

for the general fiber X of a Calabi-Yau monomial degeneration with fibers in $Y = \mathbb{P}^n$, which is given by the ideal I. We consider the lattice points of the dual complex dual $(B(I))$, which do not correspond to roots of the toric variety Y (which are trivial deformations), and then divide out the torus.

In Section 11.4 we recall some known formulas for stringy E-functions and give some ideas on a formula for the stringy E-function of a Calabi-Yau variety computed from the tropical data we associated to a monomial degeneration. We begin in Section 11.4.1 with the example of the stringy E-function of a toric variety. Section 11.4.2 gives Batyrev's general concepts for the computation of stringy E-functions and we consider the example of Calabi-Yau hypersurfaces in toric varieties in Section 11.4.3. Section 11.4.4 recalls Batyrev's and Borisov's computation of the stringy E-function of a complete intersection, which works by relating the complete intersection to a hypersurface. Finally, Section 11.4.5 makes some remarks on a tropical formula for the stringy E-function.

Section 12. This section gives some remarks on computer algebra libraries, which have been written by the author in the context of the tropical mirror construction. See also Remark 1.65 for an algorithm checking whether a divisor is Cartier, $\mathbb{Q}$-Cartier or not $\mathbb{Q}$-Cartier and which is implemented in the Maple package tropicalmirror. This package also contains an implementation of Algorithm 1.89 computing the global sections of a divisor. See Section 1.3.14 for some remarks on the implementation within the package

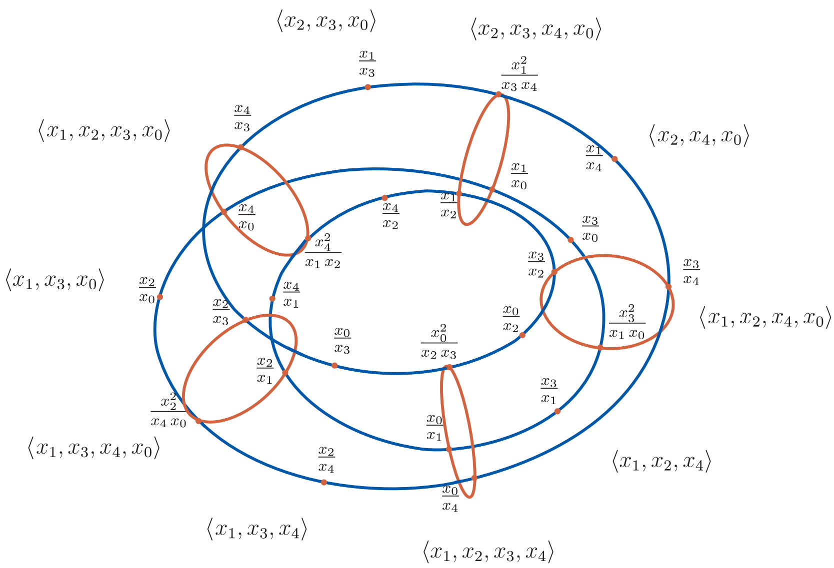

Figure 0.4: Topology of the complex dual $(B(I))$ for a monomial degeneration of the generic Pfaffian elliptic curve in $\mathbb{P}^4$

tropicalmirror of Algorithm 1.148 computing the secondary fan and the GKZ decomposition.

Section 12.1 explains the syntax of the Macaulay2 library mora.m2, which implements the standard basis algorithm. See also the remarks in Section 1.4. The goal was to provide a simple, transparent and flexible implementation capable of intermediate output useful for testing and didactical purposes.

The Macaulay2 library homology.m2, described in Section 12.2, computes the homology groups of a simplicial cell complex and is useful in the context of Stanley filtrations.

Section 12.3 gives an outline of the Macaulay2 library stanleyfiltration.m2 providing functions computing a Stanley decomposition and Stanley filtration of an ideal and the set of monomial ideals in a multigraded polynomial ring with given multigraded Hilbert polynomial.

Finally, Section 12.4 explains the syntax of the key functions of the Maple package tropicalmirror, which gives a full implementation of the tropical mirror construction. It takes as an input a Fano polytope P and a Calabi-Yau monomial degeneration $\mathfrak{X}$ with fibers in the toric Fano variety $Y = X(\Sigma)$, where $\Sigma = \Sigma(P)$ is the fan over the faces of P. The degeneration is defined by equations $m_i + tg_i \in S \otimes \mathbb{C}[t]$, where the m_i are the minimal generators of a monomial ideal $I_0 \subset S$ in the Cox ring S of Y defining the special fiber X_0 of $\mathfrak{X}$. The library outputs the mirror Fano polytope P° and the first order mirror degeneration $\mathfrak{X}^\circ$ with fibers in $X(\Sigma^\circ)$, $\Sigma^\circ = \Sigma(P^\circ)$ specified in an analogous way to $\mathfrak{X}$. The package tropicalmirror provides functions to compute the various intermediate objects introduced in the tropical mirror construction. The library also contains a function to find sets of Fermat deformations and the corresponding contracted degeneration $\hat{\mathfrak{X}}^\circ$. We provide a function, which tests whether a first order degeneration satisfies the Koszul or Buchsbaum-Eisenbud structure theorem and extends degenerations to a flat family over $\operatorname{Spec}\mathbb{C}[t]$, provided the arithmetically Gorenstein Buchsbaum-Eisenbud structure theorem applies.

Section 13. The last main section explores the perspectives of the tropical mirror construction, of the underlying concepts and technical formalisms.

In Section 13.1 we note that the natural next step is to compute from the tropical objects the stringy E-functions and more generally string cohomology in order to deal with the singular general fibers appearing in the tropical mirror construction. The tropical formula should generalize Batyrev's formula for the stringy E-function of anticanonical hypersurfaces in Gorenstein toric Fano varieties.

Section 13.2 raises the question of the computation of the local Hilbert scheme and moduli stack. The concepts of Section 6.6 introduced by Haiman and Sturmfels allow algorithmic computation of the local equations of the Hilbert scheme for ideals in the Cox ring of a smooth toric variety. Using the ideas noted in Sections 6.6.8 and 9.3, one should be able to generalize the multigraded regularity and Hilbert scheme to the setting of simplicial and further to non-simplicial toric varieties.

Section 13.3 raises the question of relating the tropical mirror construction to the mirror construction by Gross and Siebert via integrally affine structures.

As noted in Section 13.4, the non-Archimedian amoeba map gives a degenerate torus fibration of the special fiber, hence we ask the question to obtain from this, via the amoeba map, a torus fibration of the general fiber.

Section 13.5 suggests to apply the tropical mirror construction to further Calabi-Yau degenerations with fibers in projective space. Altmann and Christophersen compute the first order deformations and obstructions of Stanley-Reisner rings. Applying these algorithms in the case of triangulations of spheres one can obtain the necessary data to apply the tropical mirror construction for smoothable examples. We also ask in Section 13.6 how to generalize the work of Altmann and Christophersen to the non-simplicial toric setting.

Section 13.7 suggests to extend the structure theorem of Buchsbaum-Eisenbud to describe codimension 3 Calabi-Yau ideals in the Cox ring of a toric variety.

Tropical geometry is known to reflect the p-adic geometry, so Section 13.8 raises the relation of the tropical mirror construction to mirror symmetry over finite fields and ζ-functions as explored by Candelas et al.

A central question is the extension of topological mirror symmetry to the stronger condition of mathematical mirror symmetry via Frobenius manifolds. So Section 13.9 makes some remarks on the question of computation of instanton numbers and the A-model correlation functions. Section 13.10 raises the question of describing quantum cohomology rings from the tropical data via GKZ hypergeometric differential equations associated to toric data.

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1 Prerequisites

1.1 Calabi-Yau varieties and mirror symmetry

Definition 1.1 *A normal projective d-dimensional algebraic variety X is called a* ***Calabi-Yau variety*** *if it has at worst Gorenstein canonical singularities, $K_X = \mathcal{O}_X$ and $h^i(X, \mathcal{O}_X) = 0$ for $0 < i < d$.*

Remark 1.2 *The* ***Hodge diamond*** *of a Calabi-Yau d-fold X, formed by the Hodge numbers $h^{p,q}(X) = \dim H^{p,q}_{\bar{\partial}}(X) = \dim H^q(X, \Omega^p_X)$, has horizontal and vertical symmetry by Serre duality and Hodge duality and $\Omega^d_X = K_X = \mathcal{O}_X$ so*

$$H^{0,i}(X) \cong H^i(X, \mathcal{O}_X) \cong H^i\left(X, \Omega^d_X\right) \cong H^{d,i}(X)$$

e.g., for $d = 3$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2}(X) & & 0 & \\ 1 & & h^{2,1}(X) & & h^{1,2}(X) & & 1 \\ & 0 & & h^{1,1}(X) & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Definition 1.3 *A pair of smooth Calabi-Yau d-folds X and X° is called a* ***topological mirror pair*** *if their Hodge numbers satisfy*

$$h^{p,q}(X) = h^{d-p,q}(X^\circ) \ \ \forall 0 \le p, q \le d \tag{1.1}$$

i.e., the Hodge diamond is mirrored at the diagonal.

This definition can be extended via the stringy E-functions for varieties with log-terminal singularities. For the precise definition see Section 1.2.4.

Definition 1.4 *Calabi-Yau varieties X and X° of dimension d are called a* ***stringy topological mirror pair*** *if the stringy E-functions satisfy*

$$E_{st}(X; u, v) = u^d E_{st}\left(X^\circ; u^{-1}, v\right)$$

For Gorenstein varieties E_{st} is the generating function for the stringy Hodge numbers, which coincide with the Hodge numbers of a crepant resolution if such one exists.

Remark 1.5 *To a Calabi-Yau manifold we can associate two Frobenius manifolds called A- and B-model (see, e.g., [Manin, 1999] and [Cox, Katz, 1999]).* X *and* X° *are called a* ***mathematical mirror pair*** *if the A-model of* X *is isomorphic to the B-model of* X° *and vice versa.*

Remark 1.6 *String theory replaces particles by extended objects, e.g., by an* S^1 *or an interval. Whereas a point sweeps out a real* 1*-dimensional object in spacetime, a propagating string gives a surface, called its worldsheet.*

There are 5 *possible superstring theories, which are defined on a real* 10*-dimensional Riemannian manifold. One assumes that this manifold is locally the product of a real* 4*-dimensional Riemannian manifold* M_4 *and a* 6*-dimensional compact Riemannian manifold* X *too small to appear in measurements.*

In the case of type IIA *and* IIB *superstring theory one concludes that* M_4 *is a Minkowski space, that* X *has a complex structure* J*, that there is a Kähler metric* g *on* (X, J) *and that* g *has holonomy group* $\mathrm{Hol}(g) \subset SU(d)$*. As explained in Remark 1.8, these conditions are satisfied by a Calabi-Yau manifold.*

The worldsheets project to algebraic curves on this threefold. The Hodge numbers of X *are important characteristica of the physical theory, e.g.,*

$$\frac{1}{2}\left|\chi(X)\right| = \left|h^{1,1}(X) - h^{2,1}(X)\right|$$

is the number of fermion generations. So this number is identical for two manifolds, which form a mirror pair. Experiments indicate that real world has 3 *fermion generations.*

From the point of view of physics, mirror symmetry of two Calabi-Yau threefolds X *and* X° *is the duality of two of these types of compactified string theories, which is again a stronger condition than* X *and* X° *forming a mathematical mirror pair.*

Remark 1.7 *The most simple case of duality in physics is found in Maxwell's equations, describing the electromagnetic interaction:*

$$\partial_\nu F^{\mu\nu} = -j^\mu \qquad \partial_\nu \tilde{F}^{\mu\nu} = -k^\mu$$

with

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

E is the electric and B the magnetic field, $j = (\rho, j_x, j_y, j_z)$ the electric four-current with charge density ρ and electric three-current (j_x, j_y, j_z) and $k = (\sigma, k_x, k_y, k_z)$ is the magnetic four-current introduced by Dirac, and index manipulations are done with respect to the flat Minkowski metric with signature $(+,-,-,-)$. These equations are invariant under an $\mathrm{SO}(2)$ *rotating E and B, in particular under*

$$E \mapsto B \qquad B \mapsto -E \qquad j \mapsto k \qquad k \mapsto -j$$

One can deduce that electrostatic theory for high interaction energies, which is difficult to solve, is equivalent to magnetic theory for low interaction energies, which is easy to solve. In the case of mirror symmetry, duality allows for example the treatment of enumerative problems in algebraic geometry.

Remark 1.8 *(see also [Gross, Huybrechts, Joyce, 2003]) Let X be a manifold of dimension d, E a vector bundle on X and*

$$\nabla : \mathcal{A}(E) \to \mathcal{A}(E \otimes T^*(X))$$

a connection on E, where $\mathcal{A}(E)$ is the sheaf of smooth sections of E. Then for any smooth curve $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$ with $x, y \in X$ and any $v \in E_x$ there is a unique smooth section $\sigma \in \gamma^*(E)$ with $\nabla_{\gamma(t)}\sigma(t) = 0$ for all $t \in [0,1]$ and $\sigma(0) = v$. So one can associate the ***parallel transport map***

$$\begin{array}{rccc} P_\gamma : & E_x & \to & E_y \\ & v & \mapsto & \sigma(1) \end{array}$$

The ***holonomy group*** $\mathrm{Hol}_x(\nabla)$ *of ∇ based at $x \in X$ is*

$$\mathrm{Hol}_x(\nabla) = \{P_\gamma \mid \gamma \text{ a loop based at } x\} \subset \mathrm{GL}(E_x)$$

If g is a Riemannian metric on X, there is a unique torsion free connection ∇ on X with $\nabla g = 0$, which is called the ***Levi-Civita connection****.*

If $x \in X$, then $\mathrm{Hol}_x(g)$ *is the holonomy group of the Levi-Civita connection on the Riemannian manifold (X, g). As $\nabla g = 0$, it follows that g is invariant under the natural action of the holonomy group, so* $\mathrm{Hol}_x(g)$ *is up to conjugation a subgroup of* $\mathrm{O}(d)$ *and is denoted by* $\mathrm{Hol}(g)$*.*

If (X, g) is a Riemannian manifold of dimension r such that X is simply-connected, g is irreducible (i.e., (X, g) is not locally isometric to a Riemannian product) and g is non-symmetric, then Berger's classification (see [Berger, 1955]) shows that

1. $\mathrm{Hol}(g) = \mathrm{SO}(r)$. *This is the case of the generic Riemannian metric.*

2. $\mathrm{Hol}(g) = \mathrm{U}(d)$ *with* $r = 2d$, $d \geq 2$. *This is the case of a generic Kähler manifold, in particular* X *is a complex manifold.*

3. $\mathrm{Hol}(g) = \mathrm{SU}(d) \subset \mathrm{SO}(r)$ *with* $r = 2d$, $d \geq 2$. *Then* X *is a Ricci-flat Kähler manifold.* X *is a Calabi-Yau manifold (omitting the condition algebraic) if* X *is compact .*

4. $\mathrm{Hol}(g) = \mathrm{Sp}(a) \subset \mathrm{SO}(r)$ *with* $r = 4a$, $a \geq 2$. *Then* X *is a Ricci-flat Kähler manifold of complex dimension* $2a$. *If* X *is compact, then* X *is called compact* ***hyperkähler manifold*** *(it admits many Kähler metrics).*

5. $\mathrm{Hol}(g) = \mathrm{Sp}(a)\,\mathrm{Sp}(1) \subset \mathrm{SO}(r)$ *with* $d = 4a$, $a \geq 2$. *In this case* X *is called* ***quaternionic Kähler*** *(note that* $\mathrm{Sp}(a)$ *and* $\mathrm{Sp}(a)\,\mathrm{Sp}(1)$ *are groups of automorphism of* $\mathbb{H}^d$, *denoting by* $\mathbb{H}$ *the quaternions), it is not Kähler.*

6. $\mathrm{Hol}(g) = G_2 \subset \mathrm{SO}(7)$ *and* $r = 7$, *a so called exceptional case.*

7. $\mathrm{Hol}(g) = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$ *and* $r = 8$, *the other exceptional case.*

A Riemannian manifold (X, g) *of dimension* $r = 2d$ *is Kähler if and only if* $\mathrm{Hol}(g) \subset \mathrm{U}(d) \subset \mathrm{O}(r)$. *Then* X *has a complex structure* J.

If (X, J, g) *is a Kähler manifold and* ρ *its Ricci form, then*

$$[\rho] = 2\pi c_1(X)$$

in $H^2(X, \mathbb{R})$.

Let (X, J, g) *be a Kähler manifold of dimension* d. *If* $\mathrm{Hol}(g) \subset \mathrm{SU}(d)$, *then* g *is Ricci-flat. If* g *is Ricci-flat and* $K_X = \mathcal{O}_X$, *then* $\mathrm{Hol}(g) \subset \mathrm{SU}(d)$. *If* X *is Ricci-flat and simply connected, then* $K_X = \mathcal{O}_X$.

Yau's proof of the Calabi conjecture implies: If (X, J) *is a compact complex manifold, admitting Kähler metrics, and* $c_1(X) = 0$, *then in each Kähler class, there is a unique Ricci-flat Kähler metric. The Ricci-flat Kähler metrics on* X *form a smooth family of dimension* $h^{1,1}(X)$, *which is isomorphic to the Kähler cone of* X.

Any compact Ricci-flat Kähler manifold (X, J, g) *is up to a finite cover isometric to the product of*

- *a flat Kähler torus*

- *a compact simply-connected Riemannian manifold* N.

N is a Riemannian product of non-symmetric Ricci-flat irreducible Riemannian Kähler manifolds X_j of real dimension r_j with $\mathrm{Hol}(g_j) \subset \mathrm{SU}(d_j)$ *and $r_j = 2d_j$, $d_j \geq 2$, which are*

- *a Calabi-Yau manifold (omitting the condition algebraic), i.e.,* $\mathrm{Hol}(g) = \mathrm{SU}(d_j)$*, or*
- *a Hyperkähler manifold, i.e.,* $\mathrm{Hol}(g) = \mathrm{Sp}(a_j)$ *with $r_j = 2d_j = 4a_j$ and $a_j \geq 2$.*

Let (X, J, g) be a compact Kähler manifold of dimension $r = 2d$ with $d \geq 2$ and $\mathrm{Hol}(g) = \mathrm{SU}(d)$*, then X has finite fundamental group, $h^{0,0}(X) = h^{d,0}(X) = 1$ and $h^{i,0}(X) = 0$ for $0 < i < d$. If d is even, then X is simply connected.*

If $d \geq 3$, then (X, J) is isomorphic to a complex submanifold of some $\mathbb{P}^N_{\mathbb{C}}$ and is algebraic.

If $d = 2$, then $\mathrm{SU}(2) = \mathrm{Sp}(1)$*. The moduli space of Calabi-Yau twofolds (omitting the condition algebraic), i.e.,* ***K3-surfaces****, is a connected complex space of dimension* 20*. All Calabi-Yau twofolds are diffeomorphic. The algebraic $K3$ surfaces form a countable dense union of subvarieties of dimension* 19 *inside the moduli space.*

The following example was the first known mirror construction for Calabi-Yau varieties and was given by Greene and Plesser, see [Greene, Plesser, 1990] and [Candelas, de la Ossa, Green, Parkes, 1991].

Example 1.9 *For a general quintic threefold $X \subset \mathbb{P}^4$, by $T_{X^\circ} \cong \Omega^2_{X^\circ}$ the mirror X° should satisfy*

$$\dim H^1(X^\circ, T_{X^\circ}) = h^{2,1}(X^\circ) = h^{1,1}(X) = 1$$

hence in order to construct the mirror one looks for a 1*-parameter family. It turns out that the right choice is*

$$X_\lambda = \left\{x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 + \lambda x_0 x_1 x_2 x_3 x_4 = 0\right\} \tag{1.2}$$

divided out by the action of

$$\frac{\left\{(a_0, ..., a_4) \in \mathbb{Z}_5^5 \mid \sum_{i=0}^4 a_i \equiv 0 \bmod 5\right\}}{\mathbb{Z}_5(1, ..., 1)}$$

via

$$(a_0, ..., a_4)(x_0 : ... : x_4) = (\mu^{a_0} x_0 : ... : \mu^{a_4} x_4)$$

where μ is a 5*th root of unity. Resolving the singularities of this singular quotient without destroying the Calabi-Yau property gives the mirror of X.*

Remark 1.10 *These kind of orbifolding constructions were generalized for some hypersurfaces in weighted projective space, for complete intersections in $\mathbb{P}^n$ and complete intersections in products of weighted projective spaces. See, e.g., [Candelas, Lynker, Schimmrick, 1990], [Berglund, Hübsch, 1993], [Candelas, Dale, Lütken, Schimmrick, 1988], [Libgober, Teitelbaum, 1993], [Klemm, Schimmrigk, 1994].*

As some of the examples did not have a mirror in their classes, these approaches were unified by Batyrev for hypersurfaces in toric varieties and by Batyrev and Borisov for complete intersections in toric varieties, see [Batyrev, 1994], [Borisov, 1993], [Batyrev, Borisov, 1996-II].

1.2 Mirror symmetry for singular Calabi-Yau varieties and stringy Hodge numbers

The following considerations from [Batyrev, Dais, 1996] and [Batyrev, 1998] allow to introduce a well-defined notion of mirror symmetry for a certain class of singular varieties. This justifies to give mirror constructions for and leading to singular Calabi-Yau varieties.

1.2.1 Setup

In constructing mirror pairs we encounter several problems: Even if we start with a manifold, we encounter singular varieties. See, e.g., the quintic in $\mathbb{P}^4$ in Example 1.9. First of all we know that we can resolve the singularities by a sequence of blowups:

Theorem 1.11 (Hironaka) *[Hironaka, 1964] Let X be a normal projective variety over an algebraically closed field of characteristic 0. For any proper subvariety $D \subset X$ there exists a smooth projective variety Y and a birational morphism $f : Y \to X$ such that $f^{-1}(D)$ is a divisor with only simple normal crossings (and f is a composition of blowups in smooth closed centers).*

For a proof and an algorithmic implementation of Hironaka's theorem, see for example [Villamayor, 1989], [Encinas, Hauser, 2002], [Hauser, 2003] and [Frühbis-Krüger, Pfister, 2006]. Of course we want the resolved variety to be still a Calabi-Yau:

Definition 1.12 *A birational projective morphism $f : Y \to X$ with Y smooth and X at worst Gorenstein canonical singularities is called* ***crepant*** *(or non-discrepant)* ***desingularization*** *of X if $f^*K_X = K_Y$ (i.e., if the* ***discrepancy*** *$K_Y - f^*K_X$ is zero).*

If the crepant desingularizations of $Y \to X$ resp. $Y^\circ \to X^\circ$ exist, we can define a topological mirror pair by

$$h^{p,q}(Y) = h^{d-p,q}(Y^\circ) \ \forall 0 \le p,q \le d$$

However it is not obvious that this is well defined: If a crepant desingularization exists, it is not necessarily unique. In particular, given two crepant resolutions $Y_1 \to X$ and $Y_2 \to X$ it is not clear a priori that the Hodge numbers of Y_1 and Y_2 are equal. We will see in Theorem 1.32 that they indeed are.

Example 1.13 *Let X_0 be a smooth projective Fano variety embedded by a very ample line bundle L with $L^k = K_{X_0}^{-l}$ ($k,l \in \mathbb{N}$), let $E = \mathcal{O}_{X_0} \oplus L$ and consider the map*

$$\begin{array}{llll} \pi: & Y = \mathbb{P}(E) & \to & X \subset \mathbb{P}\big(\overbrace{H^0(X_0, \mathcal{O}_{X_0} \oplus L)}^{\cong H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))}\big) \\ & \downarrow\uparrow \sigma & & \\ & X_0 & & \end{array}$$

which is the contraction of $\sigma(X_0) \cong X_0$ to $p \in X$, where $\sigma : X_0 \to \mathbb{P}(E)$ is the section of the $\mathbb{P}^1$-bundle $\mathbb{P}(E)$ corresponding to the natural embedding $\mathcal{O}_{X_0} \hookrightarrow \mathcal{O}_{X_0} \oplus L$. Hence $X = C(X_0)$ is a cone over X_0.

We now calculate the discrepancy: π is the blowup of X in the singular point of X and with exceptional locus $D = \sigma(X_0) \cong X_0$. So

$$\mathcal{O}_Y(D)\mid_D = \mathcal{N}_{D/Y} = L^{-1}$$

Write

$$K_Y = \pi^* K_X \otimes \mathcal{O}_Y(D)^a$$

and restrict to D

$$K_Y \mid_D = \mathcal{O}_Y(D)^a \mid_D = L^{-a}$$

The adjunction formula yields

$$L^{-\frac{k}{l}} = K_D = (K_Y \otimes \mathcal{O}_Y(D))\mid_D = L^{-a} \otimes L^{-1} = L^{-(a+1)}$$

so $a = \frac{k}{l} - 1$.

Now consider the case of a smooth quadric $X_0 \cong \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. Then we can write $X = S(1,1,0)$ as

$$X = \left\{ \det \begin{pmatrix} y_0 & y_2 \\ y_1 & y_3 \end{pmatrix} = 0 \right\} \subset \mathbb{P}^4$$

so $P = (0:0:0:1)$ is the singular point of X. The discrepancy is

$$K_Y - \pi^* K_X = D$$

We now calculate a small and hence crepant resolution of $X = C(X_0)$. Let

$$E_1 := \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$$
$$E_2 := \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$$

The maps from $\mathbb{P}(E_1) = \mathbb{P}(E_2)$ to $\mathbb{P}\left(H^0\left(\mathbb{P}(E_i), \mathcal{O}_{\mathbb{P}(E_i)}(1)\right)\right) = \mathbb{P}\left(H^0\left(\mathbb{P}^1, E_i\right)\right)$ give rise to a diagram

$$\begin{array}{ccc} \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right) & \overset{\sim}{\to} & S(2,2,1) =: Y_{small} \\ \shortparallel & & \\ \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}\right) & \to & S(1,1,0) = X \end{array}$$

and hence to a morphism $Y_{small} \to X$. With

$$S(2,2,1) = \left\{ \text{minors}\left(2, \begin{pmatrix} x_0 & x_1 & x_3 & x_4 & x_6 \\ x_1 & x_2 & x_4 & x_5 & x_7 \end{pmatrix}\right) = 0 \right\} \subset \mathbb{P}^7$$

a morphism $g : Y_{small} \to X$ is given by

$$g(x_0 : ... : x_7) = (x_0 : x_1 : x_3 : x_4 : x_6)$$

and the exceptional locus is $\mathbb{P}^1$.

Finally there are also Calabi-Yau varieties, which do not have crepant desingularizations. Nevertheless we want to have a notion of mirror symmetry also for them.

To resolve these issues, the idea is to define so called stringy Hodge numbers $h_{st}^{p,q}(X)$ for singular varieties. They should coincide with the usual Hodge numbers for smooth varieties, if there is a crepant desingularization $Y \to X$ they should coincide with the Hodge numbers of Y, and even if there is no crepant desingularization there should still be a notion of mirror symmetry.

One can even consider an enlarged class of varieties for which there is in general no notion of stringy Hodge numbers, but as there is a (not necessarily polynomial) generating function encoding equivalent information, there is still a notion of mirror symmetry.

In the following let X be an irreducible normal algebraic variety of dimension d over $\mathbb{C}$.

1.2.2 The Hodge weight filtration and the E-polynomial

The cohomology groups $H^k(X, \mathbb{Q})$ of a complex algebraic variety X carry a natural mixed Hodge structure [Deligne, 1971], [Deligne, 1974], which is given by the following data:

An increasing filtration

$$0 = W_{-1} \subset W_0 \subset ... \subset W_{2k} = H^k(X, \mathbb{Q})$$

on $H^k(X, \mathbb{Q})$ called weight filtration, and a decreasing filtration

$$H^k(X, \mathbb{C}) = F^0 \supset F^1 \supset ... \supset F^k \supset F^{k+1} = 0$$

on $H^k(X, \mathbb{C}) = H^k(X, \mathbb{Q}) \otimes \mathbb{C}$ called Hodge filtration. We then have

$$H^{p,q}\left(H^k(X, \mathbb{C})\right) = F^p Gr_{p+q} H^k(X, \mathbb{C}) \cap \overline{F^q Gr_{p+q} H^k(X, \mathbb{C})}$$

where

$$\begin{aligned} Gr_l H^k(X, \mathbb{Q}) &:= (W_l / W_{l-1}) \\ F^p Gr_l H^k(X, \mathbb{C}) &:= \operatorname{Im}\left(F^p \cap (W_l \otimes \mathbb{C}) \to Gr_l H^k(X, \mathbb{Q}) \otimes \mathbb{C}\right) \end{aligned}$$

and the filtrations have the property that $F^p Gr_l H^k(X, \mathbb{C})$ gives a (pure) Hodge structure of weight l on $Gr_l H^k(X, \mathbb{Q})$.

We therefore have a decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p,q} H^{p,q}\left(H^k(X, \mathbb{C})\right)$$

In [Danilov, Khovanskii, 1987] one can find a proof that also the cohomology with compact support $H_c^i(X, \mathbb{Q})$ admits a mixed Hodge structure.

Definition 1.14 *The* E***-polynomial*** $E(X; u, v) \in \mathbb{Q}[u, v]$ *(coefficients in* $\mathbb{Z}$*) of a complex normal algebraic variety* X *of dimension* d *is then defined as*

$$E(X; u, v) := \sum_{0 \le p,q \le d} \sum_{0 \le i \le 2d} (-1)^i h^{p,q}\left(H_c^i(X)\right) u^p v^q$$

So we have a map from the category of normal algebraic varieties $\mathcal{V}_{\mathbb{C}}$ *to* $\mathbb{Q}[u, v]$ *by*

$$E : ob\mathcal{V}_{\mathbb{C}} \to \mathbb{Q}[u, v], \ X \mapsto E(X; u, v)$$

associating to each X *its* E *polynomial.*

Important properties of the E-polynomial:

Proposition 1.15 *Let* X *and* Y_i *be complex normal algebraic varieties.*

1. *If* $X = \bigcup_i X_i$ *is stratified by a disjoint union of locally closed subvarieties, then*

$$E(X) = \sum_i E(X_i)$$

2. *For products*
$$E(Y_1 \times Y_2) = E(Y_1) \cdot E(Y_2)$$

3. *If $X \to B$ is a locally trivial fibration and F the fiber over the closed point, then*
$$E(X) = E(F) \cdot E(B)$$

A proof can be found in the previously mentioned paper [Danilov, Khovanskii, 1987]. Note that the number of $\mathbb{F}_q$-points of a variety has similar properties as E.

Remark 1.16 *For smooth compact X of dimension d*

$$E(X; u, v) = \sum_{0 \le p,q \le d} (-1)^{p+q} h^{p,q}(X) u^p v^q$$

with $h^{p,q}(X) = \dim H^{p,q}_{\bar{\partial}}(X) = \dim H^q(X, \Omega^p_X)$

- *Hodge duality for X is equivalent to*
$$E(X; u, v) = E(X; v, u)$$

- *Poincaré duality for X is equivalent to*
$$E(X; u, v) = (uv)^d E\left(X; u^{-1}, v^{-1}\right)$$

- *Topological mirror symmetry for a pair of varieties X and X° is equivalent to*
$$E(X; u, v) = (-u)^d E\left(X^\circ; u^{-1}, v\right)$$

Remark 1.17 *Consider a stratification $X = U \cup C$ with X and C compact. The long exact sequence for cohomology with compact support reads as*

$$... \to H^k_c(U) \overset{\varphi_k}{\to} H^k(X) \overset{\psi_k}{\to} H^k(C) \overset{\delta_k}{\to} H^{k+1}_c(U) \to ...$$

where φ_k is given by continuation by 0, ψ_k is given by restriction and the boundary map δ_k is given by $\varpi \mapsto d(\beta \cdot r^*\varpi)$, where r is the retract of a tubular neighborhood of C and β is a bump function on this neighborhood.

We illustrate this with the following two examples:

Example 1.18 *For $X = \mathbb{P}^1$, $U = \mathbb{C}$ and $C = \{pt\}$*

k	$H_c^k(\mathbb{C})$	$\to$	$H^k(\mathbb{P}^1)$	$\to$	$H^k(pt)$
2	uv		uv		0
1	0		0		0
0	0		1		1

where we denote the Hodge filtration by the corresponding E monomials. The E-polynomials are

$$E(\mathbb{C}) = uv \quad E(\mathbb{P}^1) = 1 + uv \quad E(pt) = 1$$

Remark: The long exact sequence decomposes in short ones, if all varieties have only even cohomology.

Example 1.19 *For $X = \mathbb{P}^3$, C an elliptic curve and $U = \mathbb{P}^3 - C$ we have*

k	$H_c^k(U)$	$\to$	$H^k(\mathbb{P}^3)$	$\to$	$H^k(C)$
6	$(uv)^3$		$(uv)^3$		0
5	0		0		0
4	$(uv)^2$		$(uv)^2$		0
3	0		0		0
2	$u+v$		uv		uv
1	0		0		$-u-v$
0	0		1		1

The E-polynomials are

$$E(U) = u + v + (uv)^2 + (uv)^3$$
$$E(\mathbb{P}^3) = 1 + uv + (uv)^2 + (uv)^3$$
$$E(C) = 1 - u - v + uv$$

So a shift in the cohomological weight occurs (notice also the sign of the $u+v$ term).

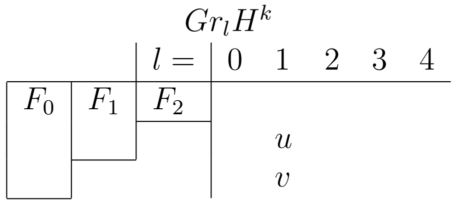
Example 1.20 *Continuing Example 1.19 the corresponding Hodge filtration for the cohomology of U:*

$$k = 0 \qquad Gr_l H^k$$

$l =$	0
F_0	

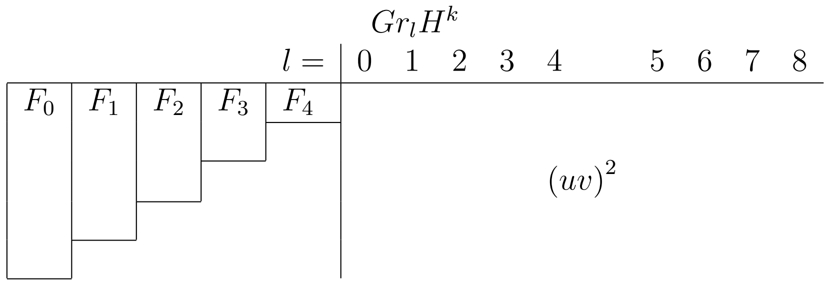
$$0 = W_{-1} \subset W_0 = H^0(U, \mathbb{Q})$$

$k = 2$



$$0 = W_{-1} = W_0 \subset W_1 = ... = W_4 = H^2(U, \mathbb{Q})$$

$k = 4$



$$0 = W_{-1} = ... = W_3 \subset W_4 = ... = W_8 = H^4(U, \mathbb{Q})$$

and similar for $k = 6$.

Example 1.21 *We continue Example 1.13 of the cone over the quadric calculating the E-polynomials:*

The cohomology ring $H^*(X_0)$ of X_0 is generated by $h_i = pr_i^* c_1(\mathbb{P}^1)$, $i = 1, 2$ and hence $1, h_1, h_2, h_1 h_2$ form a basis as a vector space, so

$$E(X_0) = 1 + 2uv + (uv)^2$$

which agrees with the product formula $E(X_0) = E(\mathbb{P}^1)^2 = (1 + uv)^2$.

$H^*(Y)$ is a free module over $H^*(X_0)$ with basis $1, c = c_1(\mathcal{O}_Y(1))$ and hence $1, c, h_1, h_2, ch_1, ch_2, h_1 h_2, ch_1 h_2$ is a vector space basis (where h_i is short for $\pi^* h_i$), so

$$E(Y) = 1 + 3uv + 3(uv)^2 + (uv)^3$$

$H^*(Y_{small})$ is a free module over $H^*(\mathbb{P}^1)$ with basis $1, c, c^2$ with $c = c_1(\mathcal{O}_{Y_{small}}(1))$ and hence $1, h, c, c^2, ch, hc^2$ is a vector space basis ($h = \pi^* c_1(\mathbb{P}^1)$), so

$$E(Y_{small}) = 1 + 2uv + 2(uv)^2 + (uv)^3$$

So the E polynomials

$$\begin{aligned} E(Y \backslash X_0) &= E(Y) - E(X_0) = \left(1 + 3uv + 3(uv)^2 + (uv)^3\right) - \left(1 + 2uv + (uv)^2\right) \\ &= uv + 2(uv)^2 + (uv)^3 \\ E\left(Y_{small} \backslash \mathbb{P}^1\right) &= E(Y_{small}) - E\left(\mathbb{P}^1\right) = \left(1 + 2uv + 2(uv)^2 + (uv)^3\right) - (1 + uv) \\ &= uv + 2(uv)^2 + (uv)^3 \end{aligned}$$

agree as expected because of $Y\backslash X_0 \cong X\backslash P \cong Y_{small}\backslash\mathbb{P}^1$. *Using this we can also calculate*

$$E(X) = 1 + uv + 2(uv)^2 + (uv)^3$$

1.2.3 Varieties with canonical singularities

Definition 1.22 *Let X be a normal projective variety, which is $\mathbb{Q}$-Gorenstein, i.e., $K_X \in Div(X) \otimes \mathbb{Q}$, and let $f : Y \to X$ be a resolution of singularities such that the exceptional locus of f is a divisor E, whose irreducible components $D_1, ..., D_r$ are smooth divisors with only simple normal crossings, and let $K_Y = f^*K_X + \sum_{i=1}^r a_i D_i$. Then X is said to have*

- ***terminal*** *singularities if $a_i > 0$ for all i*
- ***canonical*** *singularities if $a_i \geq 0$ for all i*
- ***log-terminal*** *singularities if $a_i > -1$ for all i*
- ***log-canonical*** *singularities if $a_i \geq -1$ for all i.*

1.2.4 The stringy E-function

In the following, we consider a normal projective d-dimensional variety X with log-terminal singularities, let $f : Y \to X$ be a resolution of singularities and $D_1, ..., D_r$ the smooth components of the exceptional locus with only simple normal crossings, and $K_Y = f^*K_X + \sum_{i=1}^r a_i D_i$.

Let $I = \{1, ..., r\}$ and set for any $J \subset I$

$$D_J = Y \cap \bigcap_{j \in J} D_j$$
$$D_J^\circ = D_J \backslash \bigcup_{i \in I\backslash J} D_i$$

This gives a stratification $D_J = \bigcup_{J', J \subset J'} D_{J'}^\circ$.

Definition 1.23 *Define the **stringy E-function** E_{st} of X as*

$$E_{st}(X; u, v) := \sum_{J \subset I} E(D_J^\circ; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1}$$

Remark 1.24 *If X is Gorenstein, then the $a_j \in \mathbb{Z}_{\geq 0}$ and hence $E_{st}(X; u, v) \in \mathbb{Z}[[u, v]] \cap \mathbb{Q}(u, v)$. $E_{st}(X; u, v)$ is not a rational function in general.*

The following key theorem by Batyrev [Batyrev, 1998], using ideas by Kontsevich and Denef and Loeser, assures that $E_{st}(X; u, v)$ is well defined. See also [Denef, Loeser, 1999].

Theorem 1.25 *$E_{st}(X; u, v)$ does not depend on the resolution $f : Y \to X$, in particular, $E_{st}(X; u, v)$ is well defined.*

This is also true in the case of log-terminal singularities. As direct corollary, we have:

Corollary 1.26 *If X is smooth, then $E_{st}(X; u, v) = E(X; u, v)$.*

Remark 1.27 *First make an easy but important observation: E_{st} is not affected by the blowup $f : Y \to X$ of a point P in smooth X: The exceptional locus of f is $D = \mathbb{P}^{d-1}$ and the discrepancy is*

$$K_Y - f^* K_X = (d-1) D$$

$$\begin{aligned} E_{st}(X) &= E(Y \backslash D) + E(D) \frac{uv-1}{(uv)^{a_1+1}-1} = E(Y \backslash D) + E\left(\mathbb{P}^{d-1}\right) \frac{uv-1}{(uv)^d - 1} \\ &= E(Y \backslash D) + \left(1 + uv + \ldots + (uv)^{d-1}\right) \frac{uv-1}{(uv)^d-1} \\ &= E(Y \backslash D) + 1 = E(X) \end{aligned}$$

Remark 1.28 *The idea of the proof of Theorem 1.25, considering for simplicity Gorenstein canonical singularities, is the following (see [Batyrev, 1998], [Denef, Loeser, 1999] and reviews in [Blicke, 2003] and [Craw, 2004]):*

*Consider the **Grothendieck ring of complex algebraic varieties** $\mathcal{M}$, which is the free abelian group of isomorphism classes of complex algebraic varieties modulo the subgroup generated by $[X] - [V] - [X - V]$ for closed subsets $V \subset X$, with a ring structure given by*

$$[X] \cdot [X'] = [X \times X']$$

Call the neutral element $[point] =: 1$ and $[\mathbb{C}] =: \mathbb{L}$.

The map $[-] : ob\mathcal{V}_{\mathbb{C}} \to \mathcal{M}$ is the universal map being additive on disjoint unions of constructible sets (i.e., sets, which are a finite disjoint union of locally closed subvarieties with respect to the Zariski topology) and multiplicative on products, so any other map $E : ob\mathcal{V}_{\mathbb{C}} \to \mathbb{Q}[u, v]$ with the same *properties factors through $[-]$. So the universality of $[-]$ gives a factorization of $E : \mathcal{V}_{\mathbb{C}} \to \mathbb{Q}[u,v]$ through the Grothendieck ring*

$$\begin{array}{ccc} ob\mathcal{V}_{\mathbb{C}} & \xrightarrow{E} & \mathbb{Q}[u,v] \\ [-] \searrow & & \nearrow E \\ & \mathcal{M} & \end{array}$$

The goal is to write

$$E_{st}(X;u,v) = \sum_{J \subset I} E\left(D_J^{\circ};u,v\right) \prod_{j \in J} \frac{uv-1}{(uv)^{a_j+1}-1} = E\left(\int_{J_\infty(Y)} F_D d\mu \mathbb{L}^d\right)$$

for a suitable function F_D associated to the discrepancy divisor ($J_\infty(Y)$ is the bundle of formal arcs on Y), after extending E to $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$ and then to an appropriate completion. The transformation rule for motivic integration implies that the motivic integral does not depend on the resolution.

1.2.5 Stringy Hodge numbers

Theorem 1.29 (Poincaré Duality) *[Batyrev, 1998] $E_{st}(X;u,v)$ has the following properties:*

$$E_{st}(X;u,v) = (uv)^d E_{st}\left(X;u^{-1},v^{-1}\right)$$
$$E_{st}(X;0,0) = 1$$

This is also true in the case of log-terminal singularities.

Corollary 1.30 *If E_{st} is a polynomial, then $\deg(E) = 2d$.*

Definition 1.31 *If E_{st} is a polynomial, the **stringy Hodge numbers** of X are defined as*

$$h_{st}^{p,q}(X) = (-1)^{p+q} \operatorname{coeff}(E_{st}, u^p v^q)$$

So $h_{st}^{p,q}(X) = 0$ outside the Hodge diamond and $h_{st}^{0,0}(X) = h_{st}^{d,d}(X) = 1$.

1.2.6 Crepant resolutions and mirror symmetry

Theorem 1.32 *[Batyrev, 1998] If X is $\mathbb{Q}$-Gorenstein with at worst log-terminal singularities and $f : Y \to X$ is a projective birational morphism with $K_Y = f^*K_X$, then $E_{st}(X;u,v) = E_{st}(Y;u,v)$. So if X admits a crepant resolution $f : Y \to X$, then $E_{st}(X;u,v) = E(Y;u,v)$.*

Remark 1.33 *In particular if X admits a crepant resolution, then $E_{st}(X;u,v)$ is polynomial and hence the stringy Hodge numbers of X exist. If $E_{st}(X;u,v)$ is not polynomial, then X admits no crepant resolution.*

Definition 1.34 *Two Calabi-Yau varieties X and X° are called **stringy topological mirror pair** if their stringy E-functions satisfy*

$$E_{st}(X;u,v) = (-u)^d E_{st}\left(X^\circ; u^{-1}, v\right)$$

This is well defined even in the case when E_{st} is not polynomial.

Example 1.35 *Now we return to the Example 1.13 and 1.21: Let $X_0 \subset \mathbb{P}^d$ be a smooth quadric ($k = d-1$, $l = 1$) and*

$$\mathbb{P}\left(\mathcal{O}_{X_0}(1) \oplus \mathcal{O}_{X_0}\right) = Y \to X = C(X_0)$$

with discrepancy divisor $(d-2)D$ with $D \cong X_0$.

For $d = 3$ we had $X = S(1,1,0)$, we computed a small resolution

$$S(2,2,1) = Y_{small} \to X$$

and calculated

$$\begin{aligned} E(Y_{small}) &= 1 + 2uv + 2(uv)^2 + (uv)^3 \\ E(Y) &= 1 + 3uv + 3(uv)^2 + (uv)^3 \\ E(D) &= 1 + 2uv + (uv)^2 \\ E(Y \backslash D) &= uv + 2(uv)^2 + (uv)^3 \end{aligned}$$

So the stringy E-function E_{st} is

$$\begin{aligned} E_{st}(X) &= E\left(D^\circ_\emptyset\right) + E\left(D^\circ_{\{1\}}\right) \frac{uv-1}{(uv)^2 - 1} = E(Y\backslash D) + E(D)\frac{1}{uv+1} \\ &= \left(uv + 2(uv)^2 + (uv)^3\right) + (1+uv)^2 \frac{1}{uv+1} \\ &= 1 + 2uv + 2(uv)^2 + (uv)^3 = E(Y_{small}) \end{aligned}$$

and, as predicted by Theorem 1.32, the stringy Hodge numbers of X indeed coincide with the Hodge numbers of the small resolution.

$E_{st}(X)$ is not a polynomial for $d > 3$, in particular X does not admit a crepant resolution: As $X\backslash p \cong Y\backslash D$ is isomorphic to the totalspace of $L \to X_0$, we have

$$E(X\backslash p) = E(Y\backslash D) = (uv)\, E(X_0)$$

and

$$E_{st}\left(X\right)=\left(uv\right)E\left(D\right)+E\left(D\right)\frac{uv-1}{\left(uv\right)^{d-1}-1}=E\left(D\right)\frac{\left(uv\right)^{d}-1}{\left(uv\right)^{d-1}-1}$$

As

$$E\left(D\right)=\left\{\begin{array}{ll}\frac{\left((uv)^{\frac{d-1}{2}}+1\right)\left((uv)^{\frac{d+1}{2}}-1\right)}{uv-1} & \text{for } d \text{ odd}\\ \frac{(uv)^{d}-1}{uv-1} & \text{for } d \text{ even}\end{array}\right\}$$

(see [Batyrev, 1998]) the stringy E-function $E_{st}\left(X\right)$ is not a polynomial.

1.2.7 Birational Calabi-Yau manifolds

Theorem 1.36 *[Batyrev, 1999], [Batyrev, 1998] Birational Calabi-Yau manifolds have equal Hodge numbers.*

Actually one proves that $[X_1]=[X_2]$, i.e., X_1 and X_2 represent the same class in the Grothendieck ring $\mathcal{M}$.

1.3 Some facts and notations from toric geometry

The key example of a toric variety is the projective space $\mathbb{P}^n$. Let $(y_0:...:y_n)$ be the homogeneous coordinates. On the open set $U_i=\{y\in\mathbb{P}^n\mid y_i\neq 0\}$ the functions

$$x_k^i=\frac{y_k}{y_i}$$

give an isomorphism

$$\begin{aligned}U_i&\rightarrow\mathbb{A}^n\\ (y_0:...:y_n)&\mapsto\left(x_0^i,...,\widehat{x_i^i},...,x_n^i\right)\end{aligned}$$

Considering another chart $U_j\rightarrow\mathbb{A}^n$, on $U_i\cap U_j$

$$x_k^j=\frac{y_k}{y_j}=\frac{y_k}{y_i}\frac{y_i}{y_j}=x_k^i\left(x_j^i\right)^{-1}$$

i.e., the coordinate functions in one chart are given as Laurent monomials (i.e., monomials which also can have negative exponents) in the coordinates of the other chart, a key property of toric varieties.

1.3.1 Affine toric varieties

If $S \subset M = \mathbb{Z}^n$ is a finitely generated commutative semigroup with 0, we can associate to S its **semigroup algebra** $\mathbb{C}[S]$, consisting of all finite formal sums $\sum_{m \in S} a_m x^m$, $a_m \in \mathbb{C}$ with multiplication $x^m \cdot x^{m'} = x^{m+m'}$.

Example 1.37 *The semigroup algebra of* $S = \langle (1,0), (1,1), (1,2) \rangle \subset \mathbb{Z}^2$ *is* $\mathbb{C}[S] = \mathbb{C}[x, xy, xy^2]$.

To the semigroup algebra we can associate an **affine toric variety** $Y = \operatorname{Spec} \mathbb{C}[S]$. Considering $\mathbb{C} = \mathbb{C}^* \cup \{0\}$ as a semigroup with respect to multiplication, the maximal points of Y are the semigroup homomorphisms $\operatorname{Hom}_{sg}(S, \mathbb{C})$. If $y \in \operatorname{Hom}_{sg}(S, \mathbb{C})$ and $x^m \in S$, then $x^m(y) = y(m)$.

For generators $m_1, ..., m_r$ of S, the toric ideal of Y is the kernel I_S of

$$\begin{array}{rl} \mathbb{C}[y_1, ..., y_r] \to \mathbb{C}[S] \\ y_i \mapsto x^{m_i} \end{array}$$

It is given by the binomial ideal

$$I_S = \left\langle y^{u^+} - y^{u^-} \mid u \in \ker(m_1, ..., m_r) \right\rangle$$

where $u = u^+ - u^-$ with u^+, u^- with non-negative entries and disjoint support (see [Sturmfels, 1997] and [Gelfand, Kapranov, Zelevinsky, 1994, Sec. 5.1, 5.2]).

Example 1.38 *For* $S = \langle (1,0), (1,1), (1,2) \rangle$ *as in Example 1.37 we have*

$$\ker \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = (1, -2, 1)^t$$

hence

$$\mathbb{C}[S] \cong \mathbb{C}[y_1, y_2, y_3] / \langle y_2^2 - y_1 y_3 \rangle$$

so $Y = \{y_2^2 - y_1 y_3 = 0\}$ *is a quadric cone.*

We may assume that S generates M as an abelian group. The inclusion $S \subset M$ gives an embedding of the torus

$$T = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = (\mathbb{C}^*)^n = \operatorname{Spec}(\mathbb{C}[\mathbb{Z}^n]) \hookrightarrow \operatorname{Spec}(\mathbb{C}[S]) = Y$$

If $t \in T = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$ considered as a group homomorphism $t : M \to \mathbb{C}^*$, and $y \in Y$ considered as a semigroup homomorphism $y : S \to \mathbb{C}$, then T acts on Y by

$$\begin{array}{ccc} T \times Y & \to & Y \\ (t, y) & \longmapsto & \begin{array}{rccc} ty : & S & \to & \mathbb{C} \\ & u & \mapsto & t(u)\, y(u) \end{array} \end{array}$$

i.e.,

$$\begin{aligned} T \times \mathbb{C}[S] &\to \mathbb{C}[S] \\ (t, x^m) &\mapsto t(m)\, x^m \end{aligned}$$

for $m \in S$.

Example 1.39 *In Example 1.38, the torus is* $\{y_1 \neq 0, y_2 \neq 0, y_3 \neq 0\} \subset Y$*, i.e., the complement of the two lines as shown in Figure 1.1.*

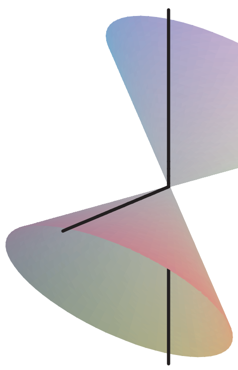


Figure 1.1: Torus orbits of the quadric cone in $\mathbb{A}^3$

We recall some standard facts and notations from polyhedral geometry:

Definition 1.40 *A finite intersection of closed half-spaces in* $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ *is called a* ***polyhedron****.*

A subset $\sigma \subset M_\mathbb{R}$ *is called a* ***polyhedral cone*** *if there are* $u_1, ..., u_s \in M_\mathbb{R}$ *such that*

$$\sigma = \{\lambda_1 u_1 + ... + \lambda_s u_s \mid \lambda_1, ..., \lambda_s \in \mathbb{R}_{\geq 0}\}$$

A polyhedral cone $\sigma \subset M_\mathbb{R}$ *is called* ***rational polyhedral cone*** *if there are* $u_1, ..., u_s \in M$ *with* $\sigma = \{\lambda_1 u_1 + ... + \lambda_s u_s \mid \lambda_1, ..., \lambda_s \in \mathbb{R}_{\geq 0}\}$.

It is called ***strongly convex*** *if* $-\sigma \cap \sigma = \{0\}$ *and the dimension of* σ *is the dimension of the subspace of* $M_\mathbb{R}$*, spanned by the elements of* σ*.*

The ***convex hull*** *of a subset* $V \subset M_\mathbb{R} \cong \mathbb{R}^n$ *is the intersection of all convex sets containing* V*. It is denoted by* $\operatorname{convexhull}(V)$ *and*

$$\operatorname{convexhull}(V) = \left\{ \sum_{i=1}^{s} \lambda_i v_i \mid \lambda_1, ..., \lambda_s \geq 0 \text{ with } \sum_{i=1}^{s} \lambda_i = 1 \text{ and } v_1, ..., v_s \in V \right\}$$

Theorem 1.41 (Carathéodory) *If* $V \subset \mathbb{R}^n$ *then any point of* $\operatorname{convexhull}(V)$ *is a convex combination of at most* $n+1$ *points of* V*.*

Definition 1.42 *A* ***polytope*** $\Delta \subset M_{\mathbb{R}}$ *is the convex hull of a finite set of points. The dimension of* Δ *is the dimension of the subspace spanned by the points* $m - m'$ *with* $m, m' \in \Delta$*.*

A polytope Δ *is called* ***integral*** *or* ***lattice polytope*** *if it is the convex hull of a finite set of points in* M*.*

Theorem 1.43 *Any bounded polyhedron is a polytope and vice versa.*

Any polyhedral cone is a polyhedron.

Definition 1.44 *A* ***face*** *of a polyhedron* Δ *is either* Δ *or a subset* $\Delta \cap h$ *of* Δ*, where* h *is a hyperplane such that* Δ *is contained in one of the closed halfspaces given by* h*. A* ***facet*** *of* Δ *is a codimension one face. Any face of a polyhedron* Δ *is a polyhedron.*

Proposition 1.45 *[Gelfand, Kapranov, Zelevinsky, 1994, Sec. 5.3] Let* $\tau = \operatorname{hull}(S)$ *be the rational polyhedral cone in* $M_{\mathbb{R}}$ *defined as the hull of a semigroup* S *with the properties as above and* $Y = \operatorname{Spec}\mathbb{C}[S]$*. There is a bijective inclusion respecting map*

$$\{\text{faces of } \tau\} \xrightarrow{1:1} \{\text{torus orbit closures in } Y\}$$

If σ *is a face of* τ*, then the corresponding torus orbit is given by* $x^m = 0$ $\forall m \notin S \cap \sigma$ *and* $x^m \neq 0$ $\forall m \in S \cap \sigma$*. The closure of the torus orbit is isomorphic to* $\operatorname{Spec}(\mathbb{C}[S \cap \sigma])$*.*

Example 1.46 *For Example 1.38, the torus orbits in* Y *are given by*

$$\begin{array}{l}\{y_1 \neq 0, y_2 \neq 0, y_3 \neq 0\} \\ \{y_1 \neq 0, y_2 = 0, y_3 = 0\} \\ \{y_1 = 0, y_2 = 0, y_3 \neq 0\} \\ \{(0,0,0)\}\end{array}$$

1.3.2 Toric varieties from fans

Let $N \cong \mathbb{Z}^n$, let $M = \operatorname{Hom}(N, \mathbb{Z})$ be the dual lattice of N, and denote by

$$\langle -, - \rangle : M \times N \to \mathbb{Z}$$

the canonical bilinear pairing. Given a rational convex polyhedral cone σ in $N_{\mathbb{R}}$ consider the dual cone

$$\check{\sigma} = \{m \in M_{\mathbb{R}} \mid \langle m, w \rangle \geq 0 \ \forall w \in \sigma\}$$

of non-negative linear forms on σ.

Proposition 1.47 (Gordan's Lemma) *[Oda, 1988, Sec. 1.1] If $\sigma \subset N_{\mathbb{R}}$ is a rational convex polyhedral cone, then $\check{\sigma} \cap M$ is a finitely generated semigroup.*

Example 1.48 *For $\sigma = \text{hull}\{(0,1),(2,-1)\}$ we get the semigroup $S = \check{\sigma} \cap M = \langle (1,0),(1,1),(1,2) \rangle \subset \mathbb{Z}^2$ given in Example 1.37.*

Given a strongly convex rational polyhedral cone σ, i.e., $\sigma \cap (-\sigma) = \{0\}$, we get a finitely generated semigroup $\check{\sigma} \cap M$ generating M as a group, i.e., $\check{\sigma} \cap M + (-\check{\sigma} \cap M) = M$. It defines an affine toric variety $U(\sigma) = \text{Spec}(\mathbb{C}[\check{\sigma} \cap M])$ containing the torus $T = \text{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) = (\mathbb{C}^*)^n$.

Proposition 1.49 *[Fulton, 1993, Sec. 2.1] $\mathbb{C}[\check{\sigma} \cap M]$ is integrally closed, i.e., $U(\sigma)$ is normal.*

The semigroup $\check{\sigma} \cap M$ is **saturated**, which means that if $a \cdot m \in \check{\sigma} \cap M$ for $a \in \mathbb{Z}_{>0}$ and $m \in M$, then $m \in \check{\sigma} \cap M$. Indeed, a semigroup $S \subset M$ is saturated if and only if $\mathbb{C}[S]$ is integrally closed. The saturation

$$\{m \in M \mid a \cdot m \in S \text{ for some } a \in \mathbb{Z}_{>0}\}$$

of S gives the normalization of $\text{Spec}\,\mathbb{C}[S]$.

The following proposition gives a characterization of the semigroups given by the duals of strongly convex rational polyhedral cones.

Proposition 1.50 *[Oda, 1988, Sec. 1.1] Let S be an additive subsemigroup of M. Then there is a unique strongly convex rational polyhedral cone σ in $N_{\mathbb{R}}$ with $S = \check{\sigma} \cap M$ if and only if the following conditions are satisfied:*

1. *S contains $0 \in M$.*
2. *S is finitely generated as an additive semigroup, i.e., there are $m_1, ..., m_r \in S$ with*
$$S = \{a_1 m_1 + ... + a_r m_r \mid m_1, ..., m_r \in \mathbb{Z}_{\geq 0}\}$$
3. *S generates M as a group, i.e., $S + (-S) = M$.*
4. *S is saturated.*

Lemma 1.51 *If σ is a rational strongly convex polyhedral cone of dimension n in $N_{\mathbb{R}}$, then the dual cone $\check{\sigma}$ is also a rational strongly convex polyhedral cone of dimension n, and there is a canonical bijective inclusion reversing correspondence between the faces of σ and $\check{\sigma}$, given by*

$$F \mapsto F^{\vee} = \{m \in \check{\sigma} \mid \langle m, w \rangle = 0 \ \forall w \in F\}$$

if F is a face of σ.

Proposition 1.52 *[Danilov, 1978] Any toric $U(\sigma)$ is Cohen-Macaulay.*

Proposition 1.53 *[Fulton, 1993, 2.1] $U(\sigma)$ is nonsingular if and only if σ is generated by a subset of a basis of N. Then*

$$U(\sigma) = \mathbb{C}^{\dim\sigma} \times (\mathbb{C}^*)^{n-\dim\sigma}$$

Given two such cones σ_1 and σ_2 intersecting along a face τ of both cones, the inclusions of $\tau \subset \sigma_1, \sigma_2$ give inclusions of $U(\tau) \subset U(\sigma_1), U(\sigma_2)$, hence we can glue the corresponding affine toric varieties $U(\sigma_1)$ and $U(\sigma_2)$ along $U(\tau)$.

A finite set Σ of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ with the property that every face of a cone in Σ is again a cone in Σ and the intersection of any two cones is a face of each is called a **fan**. Given a fan Σ we can glue all $U(\sigma)$, $\sigma \in \Sigma$, and get a **toric variety** $X(\Sigma)$.

Denote by

$$\operatorname{supp}(\Sigma) = \bigcup_{\sigma\in\Sigma} \sigma$$

the **support** of a fan Σ and by $\Sigma(m)$ the set of m-dimensional cones of Σ. The elements of $\Sigma(1)$ are called **rays**, and for each $r \in \Sigma(1)$ let $\hat{r}$ be the minimal lattice generator of r, i.e., the unique generator of the semigroup $r \cap N$.

A cone is called **simplicial** if it is generated by linearly independent generators. A fan Σ and the toric variety $X(\Sigma)$ are called simplicial if all cones of Σ are simplicial. If a cone $\sigma \subset N_{\mathbb{R}}$ is generated by a subset of a basis of N, i.e., $U(\sigma)$ is nonsingular, then we say that σ is **strictly simplicial**.

Example 1.54 *The fan Σ, as depicted in Figure 1.2, formed by the cones $\{0\}, \tau_0, \tau_1, \tau_2, \sigma_0, \sigma_1, \sigma_2$ where*

$$\begin{array}{ll} \tau_0 = \operatorname{hull}\{(-1,-1)\} & \sigma_0 = \operatorname{hull}\{(1,0),(0,1)\} \\ \tau_1 = \operatorname{hull}\{(1,0)\} & \sigma_1 = \operatorname{hull}\{(0,1),(-1,-1)\} \\ \tau_2 = \operatorname{hull}\{(0,1)\} & \sigma_2 = \operatorname{hull}\{(1,0),(-1,-1)\} \end{array}$$

gives $X(\Sigma) = \mathbb{P}^2$. The dual cones of σ_i and τ_i are shown in Figure 1.3 and

$$U(\sigma_i) \cong \mathbb{A}^2 \quad U(\tau_i) \cong \mathbb{C} \times \mathbb{C}^* \quad U(0) \cong (\mathbb{C}^*)^2$$

The torus actions on the affine toric varieties $U(\sigma)$ give an action of the torus on $X(\Sigma)$ extending the product in the torus (see [Fulton, 1993, Sec. 1.4]).

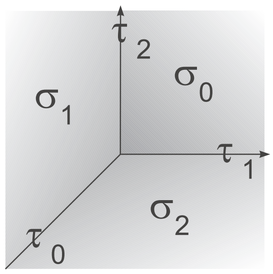


Figure 1.2: Fan representing $\mathbb{P}^2$ as toric variety

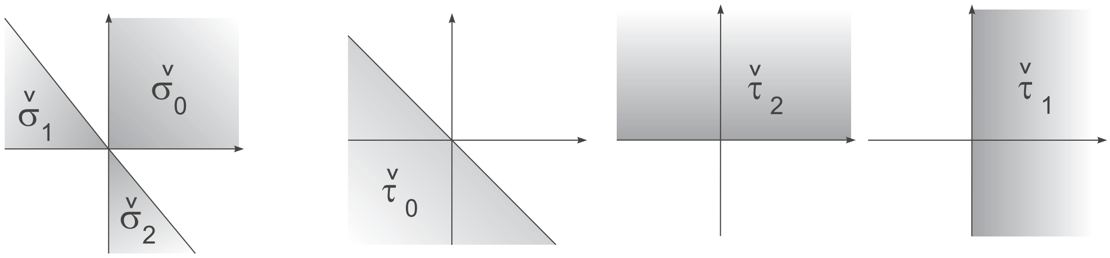


Figure 1.3: Duals of the cones of fan representing $\mathbb{P}^2$

Proposition 1.55 *[Fulton, 1993, Sec. 3.1] The torus acts on $X(\Sigma)$ and we get an inclusion reversing bijection between the cones τ of Σ and the closures $V(\tau)$ of the torus orbits $O(\tau)$*

$$\begin{array}{ccl} \Sigma & \stackrel{1:1}{\longrightarrow} & \{\textit{torus orbit closures in } X(\Sigma)\} \\ \tau & \mapsto & V(\tau) = \overline{O(\tau)} \end{array}$$

For any rational convex polyhedral cone σ in $N_{\mathbb{R}}$, $U(\sigma)$ contains a **distinguished point** x_σ (see [Fulton, 1993, Sec. 2.1]) given by

$$\begin{array}{rccl} x_\sigma : & \check{\sigma} \cap M & \rightarrow & \mathbb{C} \\ & m & \mapsto & \left\{ \begin{array}{ll} 1 & \text{if } m \in \sigma^\perp \\ 0 & \text{otherwise} \end{array} \right\} \end{array}$$

with $\sigma^\perp = \{m \in M_{\mathbb{R}} \mid \langle m, w \rangle = 0 \; \forall w \in \sigma\}$, which is well defined as $\sigma^\perp \cap \check{\sigma}$ is a face of $\check{\sigma}$. If σ spans $N_{\mathbb{R}}$, then x_σ is the unique fixed point of the torus action on $U(\sigma)$.

For the multiplicative group $\mathbb{C}^*$, we have $\mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z}$, hence for $T = \mathrm{Hom}(M, \mathbb{C}^*)$, there is a one-to-one correspondence between lattice points

$w \in N$ and 1-parameter subgroups λ_w of T:

$$\begin{array}{lcl} N \cong \operatorname{Hom}(\mathbb{Z}, N) & \cong & \operatorname{Hom}(\mathbb{C}^*, T) \\ w & \mapsto & \left(\begin{array}{cccl} \lambda_w : & \mathbb{C}^* & \to & \operatorname{Hom}(M, \mathbb{C}^*) \\ & t & \mapsto & \left(\begin{array}{cccc} \lambda_w(t) : & M & \to & \mathbb{C}^* \\ & m & \mapsto & t^{\langle m,w \rangle} \end{array} \right) \end{array} \right) \end{array}$$

Note that by $M = \operatorname{Hom}(N, \mathbb{Z}) = \operatorname{Hom}(T, \mathbb{C}^*)$ there is also a one-to-one correspondence between the elements of M and the elements of the character group $\widehat{T}$ of the torus.

Proposition 1.56 *[Fulton, 1993, Sec. 2.3] If τ is a cone of Σ and $w \in$ $\operatorname{int}(\tau)$ in the relative interior, then*

$$\lim_{t \to 0} \lambda_w(t) = x_\tau$$

If v is not in any cone of Σ, then the limit does not exist in $X(\Sigma)$.

Note that this characterizes $\sigma \cap N$ as the set

$$\sigma \cap N = \left\{ w \in N \mid \lim_{t \to 0} \lambda_w(t) \text{ exists in } U(\sigma) \right\}$$

hence allows to recover the fan from the torus action.

In the above one-to-one correspondence between cones σ of Σ and torus orbits, $O(\sigma)$ is the unique torus orbit containing x_σ. As

$$V(\sigma) = \bigcup_{\substack{\tau \in \Sigma \\ \tau \subset \sigma}} O(\tau)$$

$V(\sigma)$ contains precisely the distinguished points x_τ for $\tau \subset \sigma$.

Example 1.57 *In Example 1.54 the torus orbits and their closures are*

σ	$O(\sigma)$	$V(\sigma)$	x_σ
σ_k	$\{X_i = 0, X_j = 0, X_k \neq 0\}$	$\{X_i = 0, X_j = 0\}$	$x_{\sigma_0} = (0:0:1)$ $x_{\sigma_1} = (1:0:0)$ $x_{\sigma_2} = (0:1:0)$
τ_k	$\{X_i = 0, X_j \neq 0, X_k \neq 0\}$	$\{X_i = 0\}$	$x_{\tau_0} = (0:1:1)$ $x_{\tau_1} = (1:0:1)$ $x_{\tau_2} = (1:1:0)$
0	$\{X_i \neq 0, X_j \neq 0, X_k \neq 0\}$	$\mathbb{P}^2 = X(\Sigma)$	$x_0 = (1:1:1)$

Figure 1.4 shows the real picture of the torus orbits, identifying opposite points of the outer circle.

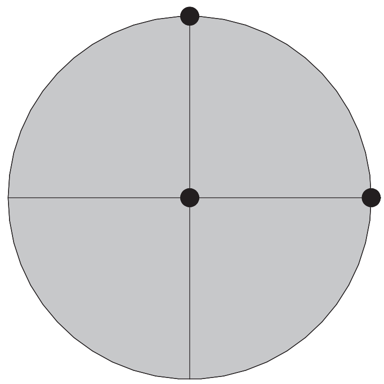


Figure 1.4: Torus orbits of $\mathbb{P}^2$

1.3.3 Morphisms of toric varieties

Suppose $\sigma' \subset N'$ is a strongly convex rational polyhedral cone and

$$\varphi : N' \to N$$

is a homomorphism of lattices such that $\varphi_{\mathbb{R}} : N'_{\mathbb{R}} \to N_{\mathbb{R}}$ is mapping σ' into a strongly convex rational polyhedral cone $\sigma \subset N_{\mathbb{R}}$. Hence the dual $\varphi^* : M \to M'$ maps $\check{\sigma} \cap M$ to $\check{\sigma}' \cap M'$ and gives a homomorphism

$$\mathbb{C}[\check{\sigma} \cap M] \to \mathbb{C}[\check{\sigma}' \cap M']$$

hence a morphism

$$U(\sigma') \to U(\sigma)$$

Proposition 1.58 *[Fulton, 1993, Sec. 1.4] Suppose Σ is a fan in N and Σ' is a fan in N' and $\varphi : N' \to N$ is a homomorphism of lattices. If for each cone σ' in Σ' there is some cone σ in Σ such that $\varphi(\sigma') \subset \sigma$, then there is a morphism $U(\sigma') \to U(\sigma) \subset X(\Sigma)$, and the morphism $U(\sigma') \to X(\Sigma)$ is independent of the choice of σ. These morphisms patch together to a morphism*

$$\varphi_* : X(\Sigma') \to X(\Sigma)$$

If $X(\Sigma)$ is compact, then $\operatorname{supp}(\Sigma) = N_{\mathbb{R}}$. Otherwise, there would be a $w \in (N_{\mathbb{R}} - \operatorname{supp}(\Sigma)) \cap N$ and $\lim_{t\to 0} \lambda_w(t)$ would not exist in $X(\Sigma)$. The converse is given by:

Proposition 1.59 *[Fulton, 1993, Sec. 2.4] Let Σ be a fan in N and Σ' a fan in N' and $\varphi : N' \to N$ a homomorphism of lattices inducing a morphism $\varphi_* : X(\Sigma') \to X(\Sigma)$. The morphism φ_* is proper if and only if $\varphi^{-1}(\operatorname{supp}(\Sigma)) = \operatorname{supp}(\Sigma')$.*

Corollary 1.60 *The toric variety* $X(\Sigma)$ *is complete if and only if* Σ *is complete, i.e.,* $\operatorname{supp}(\Sigma) = N_{\mathbb{R}}$.

Example 1.61 *Suppose* $v_1, ..., v_n$ *are a basis of* N *generating the cone*

$$\sigma = \operatorname{hull}\{v_1, ..., v_n\}$$

and Σ *is the fan generated by the cone* σ *(i.e., the fan consisting of all faces of* σ*), so* $x_\sigma = (0, ..., 0) \in \mathbb{C}^n = U(\sigma) = X(\Sigma)$. *Write* $x_i = x^{e_i^*}$, $i = 1, ..., n$. *Set* $v_0 = v_1 + ... + v_n$ *and consider the subdivision of* σ *with respect to* v_0*, i.e., the fan* Σ' *generated by the cones*

$$\sigma_i = \operatorname{hull}\{v_0, v_1, ..., v_{i-1}, v_{i+1}, ..., v_n\}$$

for $i = 1, ..., n$. *Then* $X(\Sigma')$ *is the blowup of* $X(\Sigma)$ *at* x_σ*:*

To describe $X(\Sigma')$ *note that*

$$\check{\sigma}_i = \operatorname{hull}\left\{v_i^*, v_1^* - v_i^*, ..., v_{i-1}^* - v_i^*, v_{i+1}^* - v_i^*, ..., v_n^* - v_i^*\right\}$$

hence

$$\mathbb{C}[\check{\sigma}_i \cap M] = \mathbb{C}\left[x_i, x_1 x_i^{-1}, ..., x_{i-1} x_i^{-1}, x_{i+1} x_i^{-1}, ..., x_n x_i^{-1}\right]$$

so $U(\sigma_i) = \mathbb{C}^n$.

The blowup of $U(\sigma)$ *at* x_σ *is* $\{x_i y_j - x_j y_i \mid i, j = 1, ..., n\} \subset \mathbb{C}^n \times \mathbb{P}^{n-1}$, *where* $y_1, ..., y_n$ *are homogeneous coordinates on* $\mathbb{P}^{n-1}$, *and it is covered by the open sets* $U_i = \{y_i \neq 0\} = \mathbb{C}^n$ *for* $i = 1, ..., n$, *which by* $x_j = x_i \frac{y_j}{y_i}$ *and* $\frac{y_j}{y_i} = \frac{x_j}{x_i}$ *have coordinates* $x_i, x_1 x_i^{-1}, ..., x_{i-1} x_i^{-1}, x_{i+1} x_i^{-1}, ..., x_n x_i^{-1}$.

Any cone of a given fan Σ can be subdivided such that it becomes simplicial. Given a simplicial cone σ of dimension d with minimal lattice generators $v_1, ..., v_d$ of the rays of σ and the lattice $N_\sigma = \langle \sigma \cap N \rangle$ generated by σ, the **multiplicity** of σ is defined as the index of $\mathbb{Z}v_1 + ... + \mathbb{Z}v_d$ in N_σ

$$\operatorname{mult}(\sigma) = [N_\sigma : \mathbb{Z}v_1 + ... + \mathbb{Z}v_d]$$

Then $U(\sigma)$ is nonsingular if and only if $\operatorname{mult}(\sigma) = 1$.

Example 1.62 *If* $N = \mathbb{Z}e_1 + \mathbb{Z}e_2$ *and* $\sigma = \langle v_1, v_2 \rangle$ *with* $v_2 = e_2$ *and* $v_1 = 2e_1 + e_2$, *then* $N_\sigma = N$ *and* $\mathbb{Z}v_1 + \mathbb{Z}v_2 = \mathbb{Z}(2e_1) + \mathbb{Z}e_2$. *Figure 1.5 shows the cone* σ *and the groups* N_σ *and* $\mathbb{Z}v_1 + \mathbb{Z}v_2$.

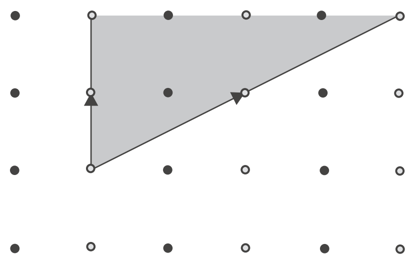


Figure 1.5: Cone of multiplicity 2

Any simplicial cone σ can be subdivided until it has multiplicity 1: If $\operatorname{mult}(\sigma) > 1$ there is some $v \in \sigma \cap N$ such that $v = \sum_{j=1}^{d} a_j v_j$ with $0 \leq a_j < 1$. Subdividing σ with respect to $\operatorname{hull}\{v\}$, we obtain the cones $\sigma_i = \operatorname{hull}\{v, v_1, ..., v_{i-1}, v_{i+1}, ..., v_d\}$ for all i with $a_i \neq 0$, and if v is the minimal lattice generator of $\operatorname{hull}\{v\}$, then

$$\operatorname{mult}(\sigma_i) = a_i \operatorname{mult}(\sigma)$$

Hence:

Proposition 1.63 *[Fulton, 1993, Sec. 2.6] Given a fan Σ in N and a fan Σ' in N refining Σ, the identity $id : N \to N$ on the lattice induces a proper birational morphism $id_* : X(\Sigma') \to X(\Sigma)$. There is a refinement Σ' of Σ inducing a resolution of singularities $X(\Sigma') \to X(\Sigma)$.*

1.3.4 Divisors on toric varieties

Let $X(\Sigma)$ be a toric variety of dimension n. The T-invariant prime Weil divisors on $X(\Sigma)$ are the components of dimension $n-1$ of the complement $X(\Sigma) - O(\{0\})$ of the torus orbit $O(\{0\})$, i.e., they are the closures of the torus orbits of dimension $n-1$. We denote by D_r the T-invariant prime Weil divisor corresponding to the ray $r \in \Sigma(1)$. Denote by $\operatorname{WDiv}_T(X(\Sigma))$ the group of T-invariant Weil divisors on $X(\Sigma)$, which is isomorphic to $\mathbb{Z}^{\Sigma(1)}$ by

$$\begin{array}{ccc} \mathbb{Z}^{\Sigma(1)} & \to & \operatorname{WDiv}_T(X(\Sigma)) \\ (a_r)_r & \mapsto & \sum_r a_r D_r \end{array}$$

T-invariant divisors are also called **T-divisors**.

By a T-invariant Cartier divisor D on $X(\Sigma)$ a collection of rational functions $\varphi_{D,\sigma}$, $\sigma \in \Sigma$ is given such that $\varphi_{D,\sigma}$ defines D on $U(\sigma)$, $\varphi_{D,\sigma}$ is invariant under the torus action up to multiplication by a non-zero constant, $\varphi_{D,\sigma}$ is unique up to multiplication with an invertible function on $U(\sigma)$, and $\frac{\varphi_{D,\sigma_1}}{\varphi_{D,\sigma 2}}$

is invertible on $U(\sigma_1) \cap U(\sigma_2)$. As $\varphi_{D,\sigma}$ is an eigenvector of the action of the torus, we can write

$$\varphi_{D,\sigma} = x^{-m(D,\sigma)}$$

with $m(D,\sigma) \in M$.

As $\varphi_{D,\sigma}$ is unique up to multiplication with an invertible function on $U(\sigma)$, the lattice point $m(D,\sigma) \in M$ is unique modulo the sublattice

$$M_\sigma = \langle \sigma \cap N \rangle^\perp = \{m \in M \mid \langle m, w \rangle = 0 \ \forall w \in \sigma\}$$

of M orthogonal to the sublattice $\langle \sigma \cap N \rangle$ of N generated by σ. Hence giving $D \mid_{U(\sigma)}$ is equivalent to specifying the function $\langle m(D,\sigma), - \rangle$.

Invertibility of

$$\frac{\varphi_{D,\sigma_1}}{\varphi_{D,\sigma 2}} = x^{-m(D,\sigma_1)+m(D,\sigma_2)}$$

on $U(\sigma_1) \cap U(\sigma_2) = U(\tau)$ with $\tau = \sigma_1 \cap \sigma_2$ is equivalent to the condition that $m(D,\sigma_1) - m(D,\sigma_2) \in M_\tau$ or, equivalently, that the functions $\langle m(D,\sigma_1), - \rangle$ and $\langle m(D,\sigma_2), - \rangle$ agree on τ, i.e.,

$$\langle m(D,\sigma_1), - \rangle \mid_\tau = \langle m(D,\sigma_2), - \rangle \mid_\tau$$

Hence associated to D there is a well defined piecewise linear continuous function

$$\Phi_D : \operatorname{supp}(\Sigma) \to \mathbb{R}$$
$$\Phi_D(w) = \langle m(D,\sigma), w \rangle \ \text{ for } w \in \sigma$$

on the support of the fan Σ. The function Φ_D is called the **support function** of D. A piecewise linear continuous function on $\operatorname{supp}(\Sigma)$, which is given on σ by $\langle m(\sigma), - \rangle$ with $m(\sigma) \in M$ is called **integral**.

On the other hand, any piecewise linear continuous integral function $\Phi : \operatorname{supp}(\Sigma) \to \mathbb{R}$ is the support function of a unique Cartier divisor, which, written as a Weil divisor, is given by

$$D = \sum_{r \in \Sigma(1)} -\Phi(\hat{r}) D_r$$

A T-Weil divisor $D = \sum_{r \in \Sigma(1)} a_r D_r$ is Cartier if and only if for all $\sigma \in \Sigma$ there is an $m(D,\sigma) \in M$ such that

$$-a_r = \langle m(D,\sigma), \hat{r} \rangle \ \text{ for all } r \in \Sigma(1) \ \text{ with } \hat{r} \in \sigma$$

Proposition 1.64 *[Voisin, 1996, Sec. 4.2] By associating to a T-Cartier divisor D the function Φ_D, we get a one-to-one correspondence between T-Cartier divisors on $X(\Sigma)$ and piecewise linear continuous integral functions on* $\operatorname{supp}(\Sigma)$.

Remark 1.65 *The Maple package* tropicalmirror *(see also Section 12.4) provides a function* TestCartier *which, given a T-Weil divisor $D = \sum_{r\in\Sigma(1)} a_r D_r$, sets up the linear system of equations*

$$-a_r = \langle m(D,\sigma), \hat{r}\rangle \text{ for all } r \in \Sigma(1) \text{ with } \hat{r} \in \sigma$$

for the $m(D,\sigma)$ and computes the solution. It returns whether D is Cartier, $\mathbb{Q}$-Cartier or not $\mathbb{Q}$-Cartier. In the first two cases there is an integral respectively rational piecewise linear continuous function on the fan, which gives on the minimal lattice generators $\hat{r}$ of the rays of Σ the negative of coefficients a_r of D and this function is also returned by TestCartier.

For any $m \in M$ the Laurent monomial x^m is a holomorphic function on the torus T, hence a rational function on $X(\Sigma)$ defining a T-invariant principal Cartier divisor

$$\operatorname{div}(x^m) = \sum_{r\in\Sigma(1)} \langle m, \hat{r}\rangle D_r$$

The principal Cartier divisor $\operatorname{div}(x^m)$ corresponds to the support function $\Phi_{\operatorname{div}(x^m)} = -\langle m, -\rangle$ defined globally by an element of M.

Proposition 1.66 *[Cox, Katz, 1999, Sec. 3.1], [Fulton, 1993, Sec. 3.4] Classes in the Picard group $\operatorname{Pic}(X(\Sigma))$ of line bundles on $X(\Sigma)$ modulo isomorphism and the Chow group $A_{n-1}(X(\Sigma))$ of Weil divisors on $X(\Sigma)$ modulo linear equivalence can be represented by T-invariant Cartier respectively Weil divisors via the exact sequences*

$$\begin{array}{cccccccc}
 & m & \mapsto & \operatorname{div} x^m & & & \\
0\to & M & \to & \operatorname{Div}_T(X(\Sigma)) & \to & \operatorname{Pic}(X(\Sigma)) & \to 0\\
 & \| & & \cap & & \cap & \\
0\to & M & \stackrel{A}{\to} & \mathbb{Z}^{\Sigma(1)} & \to & A_{n-1}(X(\Sigma)) & \to 0\\
 & m & \mapsto & (\langle m,\hat{r}\rangle)_{r\in\Sigma(1)} & & & \\
 & & & (a_r)_r & \mapsto & \sum_r a_r D_r &
\end{array}$$

$\operatorname{Pic}(X(\Sigma))$ *is torsion free.*

Example 1.67 *For the fan Σ of $\mathbb{P}^2$ given in Example 1.54 we get*

$$A_1(X(\Sigma)) = \operatorname{coker}\begin{pmatrix} 1 & 0\\ 0 & 1\\ -1 & -1\end{pmatrix} \cong \mathbb{Z}$$

Example 1.68 *Considering the fan* Σ *over the faces of the degree* 5 *Veronese polytope of* $\mathbb{P}^4$

$$\text{convexhull}\left((4,-1,-1,-1), ..., (-1,-1,-1,4), (-1,-1,-1,-1)\right)$$

we get

$$A_3(X(\Sigma)) = \text{coker}\begin{pmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \\ -1 & -1 & -1 & -1 \end{pmatrix} \cong \mathbb{Z}\times H$$

with

$$H = \frac{\left\{(a_0, ..., a_4) \in \mathbb{Z}_5^5 \mid \sum_{i=0}^4 a_i = 0 \bmod 5\right\}}{\mathbb{Z}_5(1,1,1,1,1)} \cong \mathbb{Z}_5^3$$

In the following suppose that all maximal dimensional cones of the fan Σ have dimension n, and denote by $\sigma_1, ..., \sigma_s$ the maximal dimensional cones of Σ. As shown above a T-Cartier divisor is given by a collection $m_i \in M/M_{\sigma_i} = M$ for all i such that $m_i = m(D, \sigma_i)$ maps to $m(D, \sigma_i \cap \sigma_j)$ under the canonical map $M/M_{\sigma_i} \to M/M_{\sigma_i\cap\sigma_j}$, hence:

Lemma 1.69 *The group of* T*-Cartier divisors* $\text{Div}_T(X(\Sigma))$ *on* $X(\Sigma)$ *is given by the kernel of the map*

$$\begin{array}{ccc} \bigoplus_{i=1}^s M/M_{\sigma_i} & \to & \bigoplus_{i<j} M/M_{\sigma_i\cap\sigma_j} \\ (m_i)_i & \mapsto & (m_i - m_j)_{i<j} \end{array}$$

Lemma 1.70 *[Fulton, 1993, Sec. 3.2]* $H^2(X(\Sigma), \mathbb{Z})$ *is given by the kernel of the map*

$$\bigoplus_{i<j} M_{\sigma_i\cap\sigma_j} \to \bigoplus_{i<j<l} M_{\sigma_i\cap\sigma_j\cap\sigma_l}$$

Corollary 1.71 *The map*

$$\begin{array}{ccc} \ker\left(\bigoplus_{i=1}^s M/M_{\sigma_i} \to \bigoplus_{i<j} M/M_{\sigma_i\cap\sigma_j}\right) & \to & \ker\left(\bigoplus_{i<j} M_{\sigma_i\cap\sigma_j} \to \bigoplus_{i<j<l} M_{\sigma_i\cap\sigma_j\cap\sigma_l}\right) \\ (m_i)_i & \mapsto & (m_i - m_j)_{i<j} \end{array}$$

induces an isomorphism

$$\text{Pic}(X(\Sigma)) \cong H^2(X(\Sigma), \mathbb{Z})$$

Proposition 1.72 *[Fulton, 1993, Sec. 3.4] The following conditions are equivalent:*

1. *$X(\Sigma)$ is simplicial.*
2. *All Weil divisors on $X(\Sigma)$ are $\mathbb{Q}$-Cartier.*
3. $\mathrm{Pic}(X(\Sigma)) \otimes \mathbb{Q} \to A_{n-1}(X(\Sigma)) \otimes \mathbb{Q}$ *is an isomorphism.*
4. $\mathrm{rank}(\mathrm{Pic}(X(\Sigma))) = |\Sigma(1)| - n$.

Proposition 1.73 *[Cox, Katz, 1999, Sec. 3.2], [Fulton, 1993, Sec. 3.4] Suppose $X(\Sigma)$ is complete. For any divisor $D = \sum_{r\in\Sigma(1)} a_r D_r$ the global sections of the reflexive sheaf $\mathcal{O}_{X(\Sigma)}(D)$ correspond to the lattice points of the polytope*

$$\Delta_D = \{m \in M_{\mathbb{R}} \mid \langle m, \hat{r}\rangle \geq -a_r \forall r \in \Sigma(1)\}$$

i.e.,

$$H^0\left(X(\Sigma), \mathcal{O}_{X(\Sigma)}(D)\right) \cong \bigoplus_{m\in\Delta_D\cap M} \mathbb{C}x^m$$

Remark 1.74 *If D is Cartier, then*

$$\Delta_D = \{m \in M_{\mathbb{R}} \mid \langle m, -\rangle \geq \Phi_D \text{ on } N_{\mathbb{R}}\}$$

Lemma 1.75 *[Cox, Katz, 1999, Sec. 3.2], [Fulton, 1993, Sec. 3.4] A Cartier divisor $D = \sum_{r\in\Sigma(1)} a_r D_r$ on a complete toric variety $X(\Sigma)$ is generated by global sections if and only if for all $\sigma \in \Sigma$*

$$\langle m(D,\sigma), \hat{r}\rangle \geq -a_r \ \forall r \in \Sigma(1) \text{ with } \hat{r} \notin \sigma$$

Note that by definition

$$\langle m(D,\sigma), \hat{r}\rangle = -a_r \ \forall r \in \Sigma(1) \text{ with } \hat{r} \in \sigma$$

Hence by $\Phi_D(\hat{r}) = -a_r$, it follows that D is generated by its global sections if and only if the graph of Φ_D lies below the graphs of the functions $\langle m(D,\sigma), -\rangle$ for all $\sigma \in \Sigma$, i.e., Φ_D is ***upper convex****.*

Reformulating via the polytope of sections: D is generated by global sections if and only if

$$\Delta_D = \mathrm{convexhull}\{m(D,\sigma) \mid \sigma \in \Sigma(n)\}$$

So in particular Δ_D is a lattice polytope and

$$\Phi_D(w) = \min_{\sigma\in\Sigma(n)} \langle m(D,\sigma), w\rangle = \min_{m\in\Delta_D\cap M} \langle m, w\rangle$$

hence Φ_D or equivalently the T-Cartier divisor D can be reconstructed from Δ_D.

Lemma 1.76 *[Cox, Katz, 1999, Sec. 3.2], [Fulton, 1993, Sec. 3.4] A Cartier divisor* $D = \sum_{r \in \Sigma(1)} a_r D_r$ *on a complete toric variety* $X(\Sigma)$ *is ample if and only if for all* $\sigma \in \Sigma$

$$\langle m(D,\sigma), \hat{r} \rangle > -a_r \; \forall r \in \Sigma(1) \;\; \text{with } \hat{r} \notin \sigma$$

i.e., for all $\sigma \in \Sigma$ *the graph of* Φ_D *on the complement of* σ *lies strictly below the graph of* $\langle m(D,\sigma), - \rangle$*, i.e.,* Φ_D *is* ***strictly upper convex****.*

Reformulating via the polytope of sections: D is ample if and only if Δ_D *is a polytope of dimension n with*

$$\text{vertices}(\Delta_D) = \{ m(D,\sigma) \mid \sigma \in \Sigma(n) \}$$

and all $m(D,\sigma)$*,* $\sigma \in \Sigma(n)$ *are pairwise different.*

Lemma 1.77 *[Fulton, 1993, Sec. 3.4] Any ample Cartier divisor on a complete toric variety* $X(\Sigma)$ *is generated by its global sections.*

Lemma 1.78 *[Cox, Katz, 1999, Sec. 3.2], [Fulton, 1993, Sec. 3.4] A Cartier divisor* $D = \sum_{r \in \Sigma(1)} a_r D_r$ *on a complete toric variety* $X(\Sigma)$ *is very ample if and only if* Φ_D *is strictly upper convex and for all* $\sigma \in \Sigma(n)$

$$\check{\sigma} \cap M = \langle m - m(D,\sigma) \mid m \in \Delta_D \cap M \rangle$$

Lemma 1.79 *[Fulton, 1993, Sec. 3.4] If* $X(\Sigma)$ *is complete and non-singular, then a T-divisor is ample if and only if it is very ample.*

1.3.5 Dualizing sheaf of a toric variety

Suppose $X(\Sigma)$ is a nonsingular toric variety, $e_1, ..., e_n$ form a basis of N and $x_i = x^{e_i^*}$, $i = 1, ..., n$ are the corresponding coordinates, then the divisor of the rational section

$$\omega = \frac{dx_1}{x_1} \wedge ... \wedge \frac{dx_n}{x_n}$$

of $\Omega^n_{X(\Sigma)}$ is $-\sum_{v \in \Sigma(1)} D_v$, hence:

Proposition 1.80 *[Fulton, 1993, Sec. 4.3] If* $X(\Sigma)$ *is a non-singular toric variety, then*

$$\Omega^n_{X(\Sigma)} \cong \mathcal{O}_{X(\Sigma)} \left(- \sum_{v \in \Sigma(1)} D_v \right)$$

Suppose $X(\Sigma)$ is any toric variety of dimension n, then $\sum_{v\in\Sigma(1)} D_v$ is not Cartier or $\mathbb{Q}$-Cartier in general, but still the coherent sheaf

$$\hat{\Omega}^n_{X(\Sigma)} = \mathcal{O}_{X(\Sigma)}\left(-\sum_{v\in\Sigma(1)} D_v\right)$$

gives the dualizing sheaf.

Proposition 1.81 *[Fulton, 1993, Sec. 4.4] Suppose $X(\Sigma)$ is a toric variety given by the fan Σ.*

If Σ' is a refinement of Σ inducing a resolution of singularities

$$f : X(\Sigma') \to X(\Sigma)$$

then

$$f_*\left(\Omega^n_{X(\Sigma')}\right) = \hat{\Omega}^n_{X(\Sigma)}$$

and $R^i f_*\left(\Omega^n_{X(\Sigma')}\right) = 0\ \forall i > 0$.

If $X(\Sigma)$ is complete, then for any line bundle L on $X(\Sigma)$

$$H^{n-i}\left(X(\Sigma), L^* \otimes \hat{\Omega}^n_{X(\Sigma)}\right) \cong H^i(X(\Sigma), L)^*$$

1.3.6 Projective toric varieties

The normal fan If $P \subset M_{\mathbb{R}}$ is a polyhedron and $w \in N_{\mathbb{R}}$, then define

$$\text{face}_w(P) = \{m' \in P \mid \langle m', w\rangle \le \langle m, w\rangle \text{ for all } m \in P\}$$

With respect to Minkowski sums, face_w has the property that

$$\text{face}_w(P + P') = \text{face}_w(P) + \text{face}_w(P')$$

If F is a face of P define the **normal cone** of F as

$$\sigma_P(F) = \{w \in N_{\mathbb{R}} \mid \text{face}_w(P) = F\}$$

The normal cone has dimension $\dim(\sigma_P(F)) = n - \dim(F)$. F' is a face of F if and only if $\sigma_P(F)$ is a face of $\sigma_P(F')$. Hence the set of normal cones $\sigma_P(F)$ for all faces F of P forms a fan, the **normal fan** $\text{NF}(P)$ of P.

Any polyhedron P may be written as

$$P = \Delta + C = \{m + m' \mid m \in \Delta \text{ and } m' \in C\}$$

with a polytope Δ and a cone C. The cone C is unique and C^* is the support $\operatorname{supp}(\mathrm{NF}(P))$ of the normal fan of P. It is the set of linear forms on P, which have a bounded minimum on P. If P is a polytope, then $\mathrm{NF}(P)$ is complete.

The normal fan $\Sigma = \mathrm{NF}(\Delta)$ of the polytope Δ consists of all duals

$$\sigma_P(F) = \{w \in N_{\mathbb{R}} \mid \langle m', w\rangle \leq \langle m, w\rangle \text{ for all } m \in \Delta \text{ and } m' \in F\}$$

of the cones

$$\{\lambda(m - m') \in M_{\mathbb{R}} \mid m \in \Delta,\ m' \in F,\ \lambda \geq 0\}$$

for all non-empty faces F of Δ.

If $0 \in \operatorname{int}(\Delta)$, its normal fan $\mathrm{NF}(\Delta)$ is the fan over the **dual polytope**

$$\Delta^* = \{n \in N_{\mathbb{R}} \mid \langle m, n\rangle \geq -1\ \forall m \in \Delta\}$$

The projective toric variety associated to an integral polytope Given an integral polytope $\Delta \subset M$, we can associate to it the **polytope ring**

$$S(\Delta) = \mathbb{C}\left[t^k x^m \mid m \in k\Delta\right] \quad \deg t^k x^m = k$$

with

$$k\Delta = \{km \mid m \in \Delta\} = \overbrace{\Delta + ... + \Delta}^{k}$$

and multiplication $t^k x^m \cdot t^l x^{m'} = t^{k+l} x^{m+m'}$, and hence define a **projective toric variety** $\mathbb{P}(\Delta) = \operatorname{Proj} S(\Delta)$.

On the other hand we can associate to Δ its normal fan $\Sigma = \mathrm{NF}(\Delta)$ and a piecewise linear continuous integral convex function

$$\begin{array}{l}\Phi : N_{\mathbb{R}} \to \mathbb{R} \\ \Phi(w) = \min\limits_{m \in \Delta} \langle m, w\rangle = \min\limits_{m \in \Delta \cap M} \langle m, w\rangle = \min\limits_{m \in \mathrm{vertices}(\Delta)} \langle m, w\rangle\end{array}$$

giving a Cartier divisor

$$D_\Delta = \sum_{r \in \Sigma(1)} -\min_{m \in \Delta} \langle m, \hat{r}\rangle D_r$$

which satisfies $\Delta_{D_\Delta} = \Delta$ and is ample.

Theorem 1.82 *[Batyrev, 1994], [Cox, Katz, 1999, Sec. 3.2], [Fulton, 1993, Sec. 3.4] With this notation*

$$\mathbb{P}(\Delta) \cong X(\mathrm{NF}(\Delta))$$
$$\mathcal{O}_{\mathbb{P}(\Delta)}(1) \cong \mathcal{O}_{\mathbb{P}(\Delta)}(D_\Delta)$$

If $X(\Sigma)$ is complete and D is an ample T-Cartier divisor on $X(\Sigma)$, then $\mathrm{NF}(\Delta_D) = \Sigma$.

Remark 1.83 *Choosing a basis of M gives coordinates $t_1, ..., t_n$ on the torus $T = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$. Writing $m = (a_1, ..., a_n)$ we have $x^m = \prod_{i=1}^n t_i^{a_i} =: t^m$. Given Δ choose k such that kD_Δ is very ample on $\mathbb{P}(\Delta)$. The lattice points $k\Delta \cap M = \{m_0, ..., m_r\}$ of $k\Delta$ correspond to monomials $t^{m_0}, ..., t^{m_r}$. $\mathbb{P}(\Delta)$ is the closure of the image of the map*

$$\begin{aligned} T &\to \mathbb{P}^r \\ t &\mapsto (t^{m_0}, ..., t^{m_r}) \end{aligned}$$

Example 1.84 *For*

$$\Delta = \operatorname{convexhull}((-1,-1), (2,-1), (-1,2))$$

the normal fan is the fan of $\mathbb{P}^2$ given in Example 1.54 and $\mathbb{P}(\Delta)$ is the image of $\mathbb{P}^2$ under the degree 3 monomials in 3 variables, i.e., the degree 3 Veronese embedding of $\mathbb{P}^2$.

1.3.7 The Cox ring of a toric variety

In [Cox, 1995] the representation of $\mathbb{P}^n$ as

$$\mathbb{P}^n = \left(\mathbb{C}^{n+1} - V(\langle y_0, ..., y_n \rangle)\right) / \mathbb{C}^*$$

was generalized to arbitrary toric varieties $X(\Sigma)$. In order to do so, introduce the homogeneous coordinate ring of a toric variety $X(\Sigma)$:

Definition 1.85 *The* ***homogeneous coordinate ring*** *or* ***Cox ring*** *of $X(\Sigma)$ is*

$$S = S(X(\Sigma)) = \mathbb{C}[y_r \mid r \in \Sigma(1)]$$

with the grading

$$\deg\left(\prod\nolimits_r y_r^{a_r}\right) = \left[\sum\nolimits_r a_r D_r\right] \in A_{n-1}(X(\Sigma))$$

If $D = \sum_r a_r D_r$ write $y^D = \prod_r y_r^{a_r}$, so $\deg\left(y^D\right) = [D]$. The homogeneous coordinate ring is the direct sum

$$S = \bigoplus\nolimits_{\alpha \in A_{n-1}(X(\Sigma))} S_\alpha$$

with

$$S_\alpha = \bigoplus_{[D]=\alpha} \mathbb{C} \cdot y^D$$

and it holds $S_\alpha \cdot S_\beta \subset S_{\alpha+\beta}$.

1.3.8 Global sections as Cox monomials

Proposition 1.86 *[Cox, Katz, 1999, Sec. 3.2] The global sections of the reflexive sheaf of sections $\mathcal{O}_{X(\Sigma)}(D)$ of a Weil divisor D is isomorphic to the degree $[D]$-part of the Cox ring*

$$\begin{aligned} H^0\left(X(\Sigma), \mathcal{O}_{X(\Sigma)}(D)\right) &\longrightarrow S_{[D]} \\ x^m &\mapsto \prod_r y_r^{\langle m,\hat{r}\rangle + b_r} \end{aligned}$$

where $D = \sum b_r D_r$.

In particular $S_{[D]}$ is finite dimensional of dimension $\dim_{\mathbb{C}}\left(S_{[D]}\right) = |\Delta_D \cap M|$.

Remark 1.87 *The homogeneous coordinate ring contains all possible polytope rings associated to ample divisors of a projective toric variety. Indeed, if D is ample on Y, then*

$$S(\Delta_D) \cong \bigoplus_{d=0}^{\infty} S_{k[D]}$$

Given $D = \sum b_r D_r$, we can describe $S_{[D]}$ explicitly: In the presentation of the Chow group of divisors of $X(\Sigma)$

$$\begin{array}{cccccccc} 0 \to & M & \xrightarrow{A} & \mathbb{Z}^{\Sigma(1)} & \xrightarrow{\deg} & A_{n-1}(X(\Sigma)) & \to 0 \\ & m & \mapsto & (\langle m,\hat{r}\rangle)_{r\in\Sigma(1)} & & & \\ & & & (a_r)_{r\in\Sigma(1)} & \mapsto & \sum_{r\in\Sigma(1)} a_r D_r & \end{array}$$

with the rows of A being the minimal lattice generators of the elements of $\Sigma(1)$, we have $\operatorname{image}(A) = \ker(\deg)$. Hence the Cox monomials of the same Cox degree as D, i.e., giving divisors linearly equivalent to D (corresponding to lattice monomials forming a vector space basis of the space of global sections of D), are

$$\left\{ y^a \mid a \in (b_r) + \operatorname{image}(A),\ a \in \mathbb{Z}_{\geq 0}^{\Sigma(1)} \right\}$$

Example 1.88 *For $X(\Sigma)$ given by the fan over the faces of*

$$\operatorname{convexhull}((4,-1,-1,-1), ..., (-1,-1,-1,4), (-1,-1,-1,-1))$$

as in Example 1.68 the Cox monomials in $S_{\left[-K_{X(\Sigma)}\right]}$, i.e., the monomials of the same degree as the anticanonical class, are

$$\left(\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} + \operatorname{image}\begin{pmatrix}4 & -1 & -1 & -1\\ -1 & 4 & -1 & -1\\ -1 & -1 & 4 & -1\\ -1 & -1 & -1 & 4\\ -1 & -1 & -1 & -1\end{pmatrix}\right) \cap \mathbb{Z}_{\geq 0}^5 = \left\{\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}5\\0\\0\\0\\0\end{pmatrix}, ..., \begin{pmatrix}0\\0\\0\\0\\5\end{pmatrix}\right\}$$

i.e., the monomials $y_1^5, ..., y_5^5, y_1 \cdot ... \cdot y_5$.

For $\mathbb{P}^4$*, which is given by the fan over the faces of the polytope*

$$\text{convexhull}\left((1,0,0,0), ..., (0,0,0,1), (-1,-1,-1,-1)\right)$$

the Cox monomials in $S_{\left[-K_{\mathbb{P}^4}\right]}$ *are*

$$\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \text{image}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}\right) \cap \mathbb{Z}^5_{\geq 0}$$

yielding all monomials of degree 5 *in* 5 *variables.*

Algorithm 1.89 *In order to compute* $S_{[D]}$ *for given* $D \in \mathbb{Z}^{\Sigma(1)}$ *consider a basis* $v_1, ..., v_s$ *of* $\ker(A^t)$ *and consider the cone* $C \subset \left(\mathbb{R}^{\Sigma(1)}\right)^* \oplus \mathbb{R}$ *with the rays*

$$\text{hull}\left\{(e_r, 0)\right\} \text{ for } r \in \Sigma(1)$$

and lineality space (i.e., largest linear space contained in C*) spanned by*

$$\left\{(v_i, -v_i \cdot D) \mid i = 1, ...s\right\}$$

Then the intersection of C^* *with the hyperplane defined by setting the last coordinate equal to* 1 *is the polytope* $P = (D + \text{image}(A)) \cap \mathbb{R}^{\Sigma(1)}_{\geq 0}$*. Consider the preimage in* $M_{\mathbb{R}}$ *of this polytope under the map* $m \mapsto A \cdot m + D$*. The lattice points of this polytope map via* $m \mapsto A \cdot m + D$ *to a basis of* $S_{[D]}$*.*

Remark 1.90 *This algorithm is implemented in the function* GlobalSections *of the Maple package* ***tropicalmirror*** *(see also Section 12.4). It takes as arguments the Chow presentation matrix* $A : M \to \mathbb{Z}^{\Sigma(1)}$ *and a vector* $D \in \mathbb{Z}^{\Sigma(1)}$*.*

For $y^D, y^E \in S$ define

$$y^D < y^E \Leftrightarrow \exists y^F \in S \text{ such that } [F] = [E], \; y^D \mid y^F \text{ and } y^D \neq y^F$$

Under this condition $[E] - [D] = [F] - [D] = [F - D]$ is the class of an effective divisor.

Lemma 1.91 *[Cox, 1995] If* $X(\Sigma)$ *is complete, then* $>$ *is a transitive, antisymmetric, multiplicative ordering on the monomials of* S*.*

1.3.9 Homogeneous coordinate presentation of toric varieties

Quotient presentations Let $q : Y' \to Y$ be a surjective morphism of toric varieties with tori $T \subset Y$ and $T' \subset Y'$, and denote by $q^* : \mathrm{Div}_T(Y) \to \mathrm{Div}_T(Y')$ the pullback of Cartier divisors. With $U = T \cup \bigcup_{r\in\Sigma(1)} O(r)$ there is the strict transform $q^{\#}$

$$\begin{array}{ccc} \mathrm{Div}_T(U) & \overset{q^*}{\to} & \mathrm{Div}_{T'}(U') \\ & & \cap \\ \| & & \mathrm{WDiv}_{T'}(U') \\ & & \cap \\ \mathrm{WDiv}_T(Y) & \overset{q^{\#}}{\hookrightarrow} & \mathrm{WDiv}_{T'}(Y') \end{array}$$

If Y is a toric variety, then a **quotient presentation** of Y is a quasiaffine toric variety Y' and a surjective, affine toric morphism $q : Y' \to Y$ such that $q^{\#}$ is bijective. This can be tested locally for all invariant affine open $U \subset Y$.

Theorem 1.92 *[A'Campo-Neuen, Hausen, Schröer, 2001] Suppose* $Y = X(\Sigma)$ *and* $Y' = X(\Sigma')$ *are toric varieties given by fans* $\Sigma \subset N_{\mathbb{R}}$ *and* $\Sigma' \subset N'_{\mathbb{R}}$ *and* $q : Y' \to Y$ *is a toric morphism given by a homomorphism of lattices* $\varphi : N' \to N$ *as described in Section 1.3.3. Then* q *is a quotient presentation if and only if the following conditions are satisfied:*

- $\mathrm{coker}(\varphi)$ *is finite.*
- *there is a strongly convex rational polyhedral cone* $\overline{\sigma} \subset N'_{\mathbb{R}}$ *such that* Σ' *is a subfan of a fan* Σ'' *spanned by* $\overline{\sigma}$ *(so* $Y' \subset U(\overline{\sigma})$*).*
- *the map* $\sigma \mapsto \varphi_{\mathbb{R}}(\sigma)$ *is a bijection* $\Sigma'^{\max} \to \Sigma^{\max}$ *and* $\Sigma'(1) \to \Sigma(1)$*.*
- *for all rays* $r' \in \Sigma'(1)$ *the image* $\varphi(\hat{r}')$ *of a minimal lattice generator* $\hat{r}'$ *is a primitive lattice element of* N*.*

If $q : Y' \to Y$ is a quotient presentation of Y, then via the isomorphism $q^{\#}$ we get a commutative diagram

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & M & \overset{\mathrm{div}}{\longrightarrow} & \mathrm{WDiv}_T(Y) \\ & & \downarrow & \nearrow & \\ & & M' & & \end{array}$$

Definition 1.93 *A triangle is a lattice M' and a commutative diagram*

$$\begin{array}{ccc} M & \stackrel{\text{div}}{\longrightarrow} & \text{WDiv}_T(Y) \\ \downarrow & \nearrow & \\ M' & & \end{array}$$

such that $M \longrightarrow M'$ is injective and for all T-invariant open $U \subset Y$ there is an $m' \in M'$ such that the associated divisor on Y is effective with support $Y \backslash U$.

Theorem 1.94 *[A'Campo-Neuen, Hausen, Schröer, 2001] Let Y be a toric variety. Above commutative diagram associated to a quotient presentation is a triangle. Up to isomorphism, this assignment is a bijection between quotient presentations and triangles.*

Example 1.95 *The triangle given by $M' = \text{WDiv}_T(Y) \stackrel{id}{\to} \text{WDiv}_T(Y)$ defines the Cox quotient presentation explored in detail in the following section.*

Example 1.96 *If D is an ample Cartier divisor on Y, then*

$$M \to M \oplus \mathbb{Z}D \to \text{WDiv}_T(Y)$$

is a triangle and the corresponding quotient presentation is the associated $\mathbb{C}^*$-bundle of $\mathcal{O}_Y(D)$.

Suppose $q : Y' \to Y$ is a quotient presentation given by the triangle $M \to M' \to \text{WDiv}_T(Y)$. Denote by T and T' the tori of Y and Y' and let

$$G = \ker(T' \to T)$$

With $A = M'/M$ we have $G = \text{Spec}(\mathbb{C}[A])$ and $\widehat{G} = A$. Then $q_*\mathcal{O}_{Y'}$ is graded by A with $\mathcal{O}_Y$-modules R_a

$$q_*\mathcal{O}_{Y'} = \bigoplus_{a \in A} R_a$$

The group G acts on Y' and the morphism q is a good quotient if

$$(q_*\mathcal{O}_{Y'})^G = \mathcal{O}_Y$$

One can test this condition locally, so assume that q is given by an inclusion $\mathbb{C}[\sigma^\vee \cap M] \subset \mathbb{C}[\sigma'^\vee \cap M']$. One can show that

$$\mathbb{C}[\sigma'^\vee \cap M']^G = \mathbb{C}[\sigma^\vee \cap M]$$

hence:

Proposition 1.97 *[A'Campo-Neuen, Hausen, Schröer, 2001] Any quotient presentation of a toric variety is a good quotient.*

The morphism q is a categorial quotient and for all closed invariant W_i it holds $q\left(\bigcap_i W_i\right) = \bigcap_i q\left(W_i\right)$.

Proposition 1.98 *[A'Campo-Neuen, Hausen, Schröer, 2001] The quotient presentation q gives a geometric quotient if and only if $M' \to \mathrm{WDiv}_T(Y)$ factors through the group of torus invariant $\mathbb{Q}$-Cartier divisors of Y.*

So if Y is simplicial, then q is a geometric quotient. Any quotient presentation is geometric in codimension 2.

Cox quotient presentation of toric varieties Suppose $\Sigma(1)$ spans $N_{\mathbb{R}}$. Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{C}^*)$ to the presentation

$$0 \to M \overset{A}{\to} \underset{\mathrm{WDiv}_T(X(\Sigma))}{\overset{\mathbb{Z}^{\Sigma(1)}}{\cong}} \overset{\deg}{\to} A_{n-1}(X(\Sigma)) \to 0$$

of $A_{n-1}(X(\Sigma))$, we get an exact sequence

$$\begin{array}{ccccccc} 1 \to & G(\Sigma) & \to & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{WDiv}_T(X(\Sigma)), \mathbb{C}^*) & \to & \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*) & \to 1 \\ & & & \parallel & & \parallel & \\ & & & (\mathbb{C}^*)^{\Sigma(1)} & & T & \end{array}$$

with the kernel

$$G(\Sigma) = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X(\Sigma)), \mathbb{C}^*)$$

of the map of tori, hence the inclusion of $G(\Sigma)$ in $\mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*\right)$ gives an action

$$\begin{array}{c} G(\Sigma) \times \mathrm{Hom}_{\mathbb{Z}}(\mathrm{WDiv}_T(X(\Sigma)), \mathbb{C}^*) \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{WDiv}_T(X(\Sigma)), \mathbb{C}^*) \\ (g,a) \mapsto \begin{array}{cccc} ga: & \mathrm{WDiv}_T(X(\Sigma)) & \to & \mathbb{C}^* \\ & D_r & \mapsto & g([D_r])\, a(D_r) \end{array} \end{array}$$

which induces an action of $G(\Sigma)$ on

$$\mathrm{Hom}_{sg}(\mathrm{WDiv}_T(X(\Sigma)), \mathbb{C}) = \mathrm{Specm}(S) = \mathbb{C}^{\Sigma(1)}$$

considering $\mathbb{C} = \mathbb{C}^* \cup \{0\}$ as a semigroup with respect to multiplication. This action is given by

$$\begin{array}{c} G(\Sigma) \times \mathrm{Hom}_{sg}(\mathrm{WDiv}_T(X(\Sigma)), \mathbb{C}) \to \mathrm{Hom}_{sg}(\mathrm{WDiv}_T(X(\Sigma)), \mathbb{C}) \\ (g,a) \mapsto \begin{array}{cccc} ga: & \mathrm{WDiv}_T(X(\Sigma)) & \to & \mathbb{C} \\ & D_r & \mapsto & g([D_r])\, a(D_r) \end{array} \end{array}$$

The group $G(\Sigma)$ is isomorphic to the product of a torus $(\mathbb{C}^*)^{\mathrm{rank}(A_{n-1}(X(\Sigma)))}$ and the finite group $\mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X(\Sigma))_{tor}, \mathbb{Q}/\mathbb{Z})$.

Definition 1.99 *If $\sigma \in \Sigma$ is a cone define the divisor*

$$D_{\widehat{\sigma}} = \sum_{r \in \Sigma(1),\ r \not\subset \sigma} D_r$$

and the ***irrelevant ideal*** *of $X(\Sigma)$ by*

$$B(\Sigma) = \left\langle y^{D_{\widehat{\sigma}}} \mid \sigma \in \Sigma \right\rangle = \left\langle \prod_{r \in \Sigma(1),\ r \not\subset \sigma} y_r \mid \sigma \in \Sigma \right\rangle \subset S$$

If $\sigma \in \Sigma$ is a cone, then

$$U_\sigma = \mathbb{C}^{\Sigma(1)} - V\left(y^{D_{\widehat{\sigma}}}\right)$$

is invariant under the action of $G(\Sigma)$. So

$$\mathbb{C}^{\Sigma(1)} - V(B(\Sigma)) = \bigcup_{\sigma \in \Sigma} U_\sigma$$

is invariant under $G(\Sigma)$. The localization $S_\sigma = S_{y^{\widehat{\sigma}}}$ is the coordinate ring of the affine variety U_σ and the invariants under the action of $G(\Sigma)$ are

$$(S_\sigma)^{G(\Sigma)} = (S_\sigma)_0 \cong \mathbb{C}[\check{\sigma} \cap M]$$

so

$$U_\sigma / G(\Sigma) = \operatorname{Spec}\left((S_\sigma)^{G(\Sigma)}\right) = \operatorname{Spec} \mathbb{C}[\check{\sigma} \cap M] = U(\sigma)$$

is the affine toric variety $U(\sigma) \subset X(\Sigma)$.

Theorem 1.100 *[Cox, Katz, 1999, Sec. 3.2] Suppose $\Sigma(1)$ spans $N_{\mathbb{R}}$. Then with the above action of*

$$G(\Sigma) = \operatorname{Hom}_{\mathbb{Z}}(A_{n-1}(X(\Sigma)), \mathbb{C}^*)$$

on $\mathbb{C}^{\Sigma(1)}$ and above irrelevant ideal $B(\Sigma)$ it holds

$$X(\Sigma) = \left(\mathbb{C}^{\Sigma(1)} - V(B(\Sigma))\right) // G(\Sigma) \tag{1.3}$$

The quotient is geometric if and only if Σ is simplicial.

Example 1.101 *If Σ is the fan over the degree 5 Veronese polytope of $\mathbb{P}^4$ as considered Example 1.68*

$$A_3(X(\Sigma)) \cong \mathbb{Z} \times H$$
$$\text{with } H = \frac{\left\{(a_0, ..., a_4) \in \mathbb{Z}_5^5 \mid \sum_{i=0}^4 a_i = 0 \bmod 5\right\}}{\mathbb{Z}_5(1,1,1,1,1)} \cong \mathbb{Z}_5^3$$

and $\mathrm{Hom}_{\mathbb{Z}}(A_3(X(\Sigma)), \mathbb{C}^*) = \mathbb{C}^* \times \mathbb{Z}_5^3$ *acts on* $\mathbb{C}^5$ *by*

$$(\lambda, (\mu^{a_0}, ..., \mu^{a_4})) \cdot (y_0, ..., y_4) = (\lambda\mu^{a_0} y_0, ..., \lambda\mu^{a_4} y_4)$$

where μ *is a* 5*th root of unity. Furthermore,*

$$B(\Sigma) = \langle y_0, y_1, y_2, y_3, y_4 \rangle \subset \mathbb{C}[y_0, y_1, y_2, y_3, y_4]$$

hence $X(\Sigma) = \mathbb{P}^4/\mathbb{Z}_5^3$*, which is precisely the quotient of* $\mathbb{P}^4$*, the Greene-Plesser mirror of the generic quintic sits inside (see Example 1.9).*

Remark 1.102 *For the practical representation of the action of* $G(\Sigma)$ *on* $\mathbb{C}^{\Sigma(1)}$ *we proceed as follows: Choose a numbering of rays of* Σ*, let* $r = |\Sigma(1)|$ *be the number of rays and denote by* A *the presentation matrix of* $A_{n-1}(Y)$*. By Smith normal form we obtain* $W \in \mathrm{GL}(n, \mathbb{Z})$ *and* $U \in \mathrm{GL}(r, \mathbb{Z})$ *and a commutative diagram*

$$\begin{array}{ccccccc} 0 \to & \mathbb{Z}^n & \xrightarrow{A} & \mathbb{Z}^r & \to & A_{n-1}(Y) & \to 0 \\ & \downarrow W & & \downarrow U & & \downarrow\cong & \\ 0 \to & \mathbb{Z}^n & \xrightarrow{A'} & \mathbb{Z}^r & \to & H & \to 0 \end{array}$$

such that A' *is a matrix with non-zero entries only on the diagonal. Then*

$$G(\Sigma)' = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{C}^*)$$

acts by

$$\begin{array}{l} G(\Sigma)' \times \mathbb{C}^r \to \mathbb{C}^r \\ \quad ((t_j), (a_j)) \mapsto \left(\prod_{i=1}^r t_i^{u_{ij}} a_j\right)_{j=1,\ldots,r} \end{array}$$

where $U = (u_{ij})$*, and it holds*

$$X(\Sigma) = (\mathbb{C}^r - V(B(\Sigma))) // G(\Sigma)'$$

1.3.10 Homogeneous coordinate representations of subvarieties and sheaves

Let $Y = X(\Sigma)$ be simplicial and $I \subset S$ a graded ideal. Then $V(I) - V(B(\Sigma)) \subset \mathbb{C}^{\Sigma(1)} - V(B(\Sigma))$ is $G(\Sigma)$-invariant. As

$$X(\Sigma) = \left(\mathbb{C}^{\Sigma(1)} - V(B(\Sigma))\right) / G(\Sigma)$$

is a geometric quotient, the $G(\Sigma)$-invariant Zariski closed subsets of $\mathbb{C}^{\Sigma(1)} - V(B(\Sigma))$ are in one-to-one correspondence to the Zariski closed subsets of $X(\Sigma)$. Denote by $V_Y(I)$ the Zariski closed subset of Y corresponding to $V(I) - V(B(\Sigma))$.

So $V_Y(I) = \emptyset$ if and only if $V(I) \subset V(B(\Sigma))$, which is equivalent to the existence of an m with $B(\Sigma)^m \subset I$ by the Nullstellensatz.

Proposition 1.103 *[Cox, 1995] Let $Y = X(\Sigma)$ be a simplicial toric variety. Then*

1. *For any graded ideal $I \subset S$*

$$V_Y(I) = \emptyset \Leftrightarrow \exists m : B(\Sigma)^m \subset I$$

2. *There is a one-to-one correspondence*

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{graded radical ideals } I \subset S \\ \text{with } I \subset B(\Sigma) \end{array} \right\} & \to & \{\text{Zariski closed subsets of } Y\} \\ I & \mapsto & V_Y(I) \end{array}$$

A graded S-module F has a decomposition into a direct sum

$$F = \bigoplus\nolimits_{\alpha \in A_{n-1}(X(\Sigma))} F_\alpha$$

with $S_\alpha F_\beta \subset F_{\alpha+\beta}$.

Let $\sigma \in \Sigma$ be a cone. The degree 0 part $(F_\sigma)_0$ of the graded S_σ-module $F_\sigma = F \otimes_S S_\sigma$ is an $(S_\sigma)_0$-module, which defines a quasi-coherent sheaf $\widetilde{(F_\sigma)}_0$ on the affine toric variety $U(\sigma) = \operatorname{Spec}((S_\sigma)_0) \subset X(\Sigma)$. According to the fan the sheaves $\widetilde{(F_\sigma)}_0$ patch to a quasi-coherent sheaf $\widetilde{F}$.

Theorem 1.104 *[Cox, 1995] Let $Y = X(\Sigma)$ with Cox ring S. The map $F \mapsto \widetilde{F}$ is an exact functor from the graded S-modules to quasi-coherent $\mathcal{O}_Y$-modules. It has the following properties:*

- *If Y is simplicial, then every quasi-coherent sheaf $\mathcal{F}$ arises in this way as $\mathcal{F} \cong \widetilde{F}$ with*

$$F = \bigoplus_{\alpha \in A_{n-1}(X(\Sigma))} H^0\left(Y, \mathcal{F} \otimes_{\mathcal{O}_Y} \widetilde{S(\alpha)}\right)$$

  where $S(\alpha)_\beta = S_{\alpha+\beta}$.

- *If F is finitely generated, then $\widetilde{F}$ is coherent.*

- *If Y is simplicial, then every coherent sheaf on Y is of the form $\widetilde{F}$ with F finitely generated.*

  $\widetilde{F} = 0$ if and only if there is some $k > 0$ such that $B(\Sigma)^k F_\alpha = \{0\}$ for all $\alpha \in \operatorname{Pic}(Y)$.

- *If Y is smooth, then $\widetilde{F} = 0$ if and only if there is some $k > 0$ such that $B(\Sigma)^k F = \{0\}$.*

Theorem 1.105 *[Cox, 1995] Let $Y = X(\Sigma)$ with Cox ring S.*

1. *If Y is simplicial, then any closed subscheme of Y is given by a graded ideal $I \subset S$, and graded ideals $I, J \subset S$ correspond to the same closed subscheme of Y if and only if $(I : B(\Sigma)^\infty)_\alpha = (J : B(\Sigma)^\infty)_\alpha$ for all $\alpha \in \operatorname{Pic}(Y)$.*

2. *If Y is smooth, then graded ideals $I, J \subset S$ correspond to the same closed subscheme of Y if and only if $(I : B(\Sigma)^\infty) = (J : B(\Sigma)^\infty)$, so there is a one-to-one correspondence between the graded ideals $I \subset S$ which are saturated in $B(\Sigma)$ and the closed subschemes of Y.*

Definition 1.106 *If $I \subset S$ is generated by homogeneous elements $f \in S$ with $\deg(f) \in \operatorname{Pic}(Y)$, then I is called $\operatorname{Pic}(Y)$**-generated**.*

*I is called $\operatorname{Pic}(Y)$**-saturated** if $I_\alpha = (I : B(\Sigma)^\infty)_\alpha$ for all $\alpha \in \operatorname{Pic}(Y)$.*

Theorem 1.107 *[Cox, 1995] If Y is simplicial, then there is a one-to-one correspondence between the $\operatorname{Pic}(Y)$-generated and $\operatorname{Pic}(Y)$-saturated ideals $I \subset S$ and the closed subschemes of Y.*

Definition 1.108 *The **Picard-Cox ring** of Y is*

$$R = \bigoplus_{\alpha \in \operatorname{Pic}(Y)} S_\alpha$$

It is the invariant ring $R = S^W$ of the finite subgroup

$$W = \operatorname{Hom}_{\mathbb{Z}}(A_{n-1}(Y) / \operatorname{Pic}(Y), \mathbb{C}^*) \subset G(\Sigma)$$

Note that in general R is not a polynomial ring.

Theorem 1.109 *[Cox, 1995] If Y is simplicial, then Y is the geometric quotient*

$$Y = (\operatorname{Spec}(R) - V(B(\Sigma) \cap R)) / H(\Sigma)$$

with

$$H(\Sigma) = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Pic}(Y), \mathbb{C}^*)$$

Theorem 1.110 *[Cox, 1995] If Y is simplicial, then there is a one-to-one correspondence between the graded ideals $I \subset R$ which are saturated in $B(\Sigma) \cap R$ and the closed subschemes of Y.*

1.3.11 Kähler cone and Mori cone

Suppose $Y = X(\Sigma)$ is a simplicial projective toric variety of dimension n given by the fan $\Sigma \subset N_{\mathbb{R}}$. Then

$$A_{n-1}(Y) \otimes \mathbb{R} \cong H^2(Y, \mathbb{R})$$

The **Kähler cone** $K(Y)$ of Y is the cone of all Kähler classes on Y considering $K(Y)$ as a subset in $A_{n-1}(Y) \otimes \mathbb{R}$ or $H^2(Y, \mathbb{R})$.

The cone $A^+_{n-1}(Y) \otimes \mathbb{R}$ is defined as the cone generated by the divisor classes $[D_r] \in A_{n-1}(Y)$ for $r \in \Sigma(1)$.

Proposition 1.111 *If $a = \sum_{r \in \Sigma(1)} a_r [D_r] \in A^+_{n-1}(Y) \otimes \mathbb{R}$, for any $\sigma \in \Sigma$ there is an $m_\sigma \in M_{\mathbb{R}}$ such that $\langle m_\sigma, \hat{r} \rangle = -a_r$ for all rays $r \subset \sigma$. If $\langle m_\sigma, \hat{r} \rangle \geq -a_r$ for all $r \not\subset \sigma$, then a is called* ***convex****. The set* $\mathrm{cpl}(\Sigma)$ *of all convex $a \in A^+_{n-1}(Y) \otimes \mathbb{R}$ is a $|\Sigma(1)| - n$ dimensional convex cone.*

a is in the interior of $\mathrm{cpl}(\Sigma)$ if and only if $\langle m_\sigma, \hat{r} \rangle > -a_r$ for all maximal dimensional cones $\sigma \in \Sigma$ and all $r \not\subset \sigma$.

Proposition 1.112 *[Cox, Katz, 1999, Sec. 3.3.] The Kähler cone of Y is the interior of* $\mathrm{cpl}(\Sigma)$.

Corollary 1.113 *The* ***Mori cone*** $\overline{NE}(Y)_{\mathbb{R}}$ *of effective 1-cycles in $A_1(Y) \otimes \mathbb{R} \cong H_2(Y, \mathbb{R})$ is dual to* $\mathrm{cpl}(\Sigma)$.

Proposition 1.114 *[Reid, 1983] $\overline{NE}(Y)_{\mathbb{R}}$ is generated by the torus orbit closures $V(\sigma)$ where $\sigma \in \Sigma$ is a cone of dimension $n-1$.*

Suppose $\sigma \in \Sigma$ is a cone of dimension $n-1$ generated by $v_1, ..., v_{n-1} \in N$. The cone σ is contained in exactly two n dimensional cones C_1 and C_2. There are $v_n, v_{n+1} \in N$ such that

$$C_1 = \mathrm{hull}(v_1, ..., v_{n-1}, v_n)$$
$$C_2 = \mathrm{hull}(v_1, ..., v_{n-1}, v_{n+1})$$

There are relatively prime integers $\lambda_1, ..., \lambda_{n+1} \in \mathbb{Z}$ with $\lambda_n, \lambda_{n+1} > 0$ such that $\sum_{i=1}^{n+1} \lambda_i v_i = 0$. Denote the relation (λ_i) by λ_σ.

Consider

$$\Lambda_{\mathbb{Q}} = \left\{ (\lambda_v) \in \mathbb{Q}^{\Sigma(1)} \mid \textstyle\sum_{i=1}^{n+1} \lambda_i v_i = 0 \right\}$$

Applying $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to the sequence

$$0 \to M \to \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(X(\Sigma)) \to 0$$

and tensoring with $\mathbb{Q}$ we get a natural isomorphism

$$A_1(Y) \otimes \mathbb{Q} \cong \Lambda_{\mathbb{Q}}$$

and $V(\sigma)$ is mapped to a multiple $c_\sigma \lambda_\sigma \in \Lambda_{\mathbb{Q}}$ of the relation $\lambda_\sigma \in \Lambda_{\mathbb{Q}}$ given by $\sum_{i=1}^{n+1} \lambda_i v_i = 0$.

Proposition 1.115 *[Cox, Katz, 1999, Sec. 3.3] If $\sigma \in \Sigma(n-1)$ and $v_1, ..., v_{n+1} \in N$ with*

$$\sigma = \mathrm{hull}(v_1, ..., v_{n-1}) = \mathrm{hull}(v_1, ..., v_{n-1}, v_n) \cap \mathrm{hull}(v_1, ..., v_{n-1}, v_{n+1})$$

and $\sum_{i=1}^{n+1} \lambda_i v_i = 0$ with $\lambda_1, ..., \lambda_{n+1} \in \mathbb{Z}$ and $\lambda_n, \lambda_{n+1} > 0$, then there is a $c_\sigma > 0$

$$\begin{array}{ccc} A_1(Y) \otimes \mathbb{Q} & \stackrel{\cong}{\to} & \Lambda_{\mathbb{Q}} \subset \mathbb{Q}^{\Sigma(1)} \\ V(\sigma) & \mapsto & \lambda_\sigma = c_\sigma \cdot (\lambda_i) \end{array}$$

A Cartier divisor D is ample if and only if $D.V(\sigma) > 0$ for all $\sigma \in \Sigma(n-1)$, so:

A divisor $D = \sum_{r \in \Sigma(1)} a_r D_r$ is ample if and only if it is Cartier and $(a_r) \cdot \lambda_\sigma > 0$ for all $\sigma \in \Sigma(n-1)$.

Note that by the sequence

$$0 \to \quad \mathrm{Hom}_{\mathbb{R}}(A_{n-1}(Y) \otimes \mathbb{R}, \mathbb{R}) \quad \to \quad \mathrm{Hom}_{\mathbb{R}}\left(\mathbb{R}^{\Sigma(1)}, \mathbb{R}\right) \quad \stackrel{-\circ A}{\to} \quad N_{\mathbb{R}} \quad \to 0$$

the Mori cone of Y is

$$\begin{aligned} \overline{NE}(Y)_{\mathbb{R}} &= \mathrm{hull}\{\lambda_\sigma \mid \sigma \in \Sigma,\ \dim(\sigma) = n-1\} \\ &\subset \ker\left(A^t\right)^* \cong \mathrm{Hom}_{\mathbb{R}}(A_{n-1}(Y) \otimes \mathbb{R}, \mathbb{R}) \cong A_1(Y) \otimes \mathbb{R} \end{aligned}$$

1.3.12 Toric Fano varieties

Recall that any Cohen-Macaulay variety Y of dimension n has a dualizing sheaf $\hat{\Omega}_Y^n$ and that this is a line bundle if and only if Y is Gorenstein.

Definition 1.116 *A complete Gorenstein variety Y is called **Fano** if the dual of $\hat{\Omega}_Y^n$ is ample.*

For any toric variety Y

$$\hat{\Omega}_Y^n = \mathcal{O}_Y\left(-\sum_{v \in \Sigma(1)} D_v\right)$$

so a toric variety Y is Gorenstein if and only if $-K_Y = \sum_{v \in \Sigma(1)} D_v$ is Cartier, hence:

Lemma 1.117 *If Y is a complete toric variety Y, then it is Fano if and only if $\sum_{v\in\Sigma(1)} D_v$ is Cartier and ample.*

Definition 1.118 *A polytope $\Delta \subset M_{\mathbb{R}} \cong \mathbb{R}^n$ of dimension n is called* ***reflexive*** *if Δ and its dual Δ^* are integral and contain 0 in their interior.*

If $\Delta \subset M_{\mathbb{R}}$ is reflexive, then the vertices of Δ^* are in the lattice M, hence, for each facet F of Δ there is an $m_F \in M$ with

$$F = \Delta \cap \{w \in N_{\mathbb{R}} \mid \langle m_F, w\rangle = -1\}$$

and Δ is cut out by the inequalities $\langle m_F, w\rangle \geq -1$ for all F, so for any lattice point in the interior of Δ we have $\langle m_F, w\rangle > -1$ and $\langle m_F, w\rangle \in \mathbb{Z}$ for all facets F of Δ, hence, 0 is the unique interior lattice point of Δ.

Lemma 1.119 *If $\Delta \subset M_{\mathbb{R}}$ is reflexive, then 0 is the unique interior lattice point of Δ.*

Theorem 1.120 *[Cox, Katz, 1999, Sec. 3.5], [Voisin, 1996, Sec. 4.4] The Gorenstein toric Fano varieties $\mathbb{P}(\Delta)$ of dimension n, polarized by $-K_{\mathbb{P}(\Delta)}$ are in one-to-one correspondence with the reflexive polytopes $\Delta \subset M_{\mathbb{R}}$, where* $\operatorname{rank} M = n$. *Hence duality of reflexive polytopes is an involution of the set of Gorenstein toric Fano varieties.*

This involution is used by Batyrev in his mirror construction for hypersurfaces in toric varieties.

Example 1.121 *The polytope*

$$\Delta = \operatorname{convexhull}((2,-1), (-1,2), (-1,-1))$$

which is shown in Figure 1.6, giving the degree 3 Veronese of $\mathbb{P}^2$ is reflexive with dual

$$\Delta^* = \operatorname{convexhull}((1,0), (0,1), (-1,-1))$$

shown in Figure 1.7.

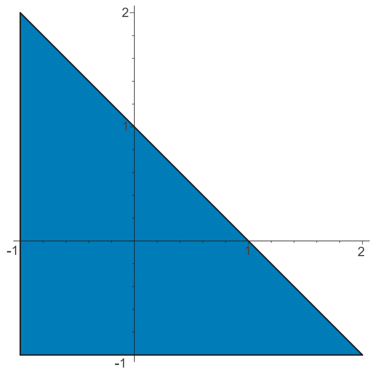


Figure 1.6: Polytope representing the degree 3 Veronese of $\mathbb{P}^2$

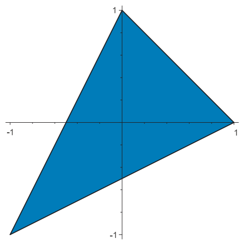


Figure 1.7: Dual polytope of the degree 3 Veronese polytope of $\mathbb{P}^2$

1.3.13 The automorphism group of a toric variety

Suppose $X(\Sigma)$ is a complete toric variety given by a simplicial fan Σ and

$$X(\Sigma) = \left(\mathbb{C}^{\Sigma(1)} - V(B(\Sigma))\right)/G(\Sigma)$$
$$G(\Sigma) = \mathrm{Hom}_{\mathbb{Z}}\left(A_{n-1}(X(\Sigma)), \mathbb{C}^*\right)$$

the homogeneous coordinate representation. The following three possible types of automorphisms of $X(\Sigma)$ are given as automorphisms of $\mathbb{C}^{\Sigma(1)} - V(B(\Sigma))$ commuting with the action of $G(\Sigma)$ on $\mathbb{C}^{\Sigma(1)} - V(B(\Sigma))$:

1. By exactness of

$$1 \to G(\Sigma) \to (\mathbb{C}^*)^{\Sigma(1)} \to T \to 1$$

   the elements of $(\mathbb{C}^*)^{\Sigma(1)}$ induce the automorphisms of $X(\Sigma)$, which are in the torus $T \subset \mathrm{Aut}(X(\Sigma))$.

2. A **root** of $X(\Sigma)$ is a pair $\left(y_v, \prod_{r\in\Sigma(1)} y_r^{a_r}\right)$ of a Cox variable y_v and a Cox monomial $\prod_{r\in\Sigma(1)} y_r^{a_r}$, which are not equal, but have the same

Cox degree, i.e.,

$$\left[\sum_{r\in\Sigma(1)} a_r D_r\right] = [D_v] \in A_{n-1}(X(\Sigma))$$

The Cox monomial $\prod_r y_r^{a_r}$ is not divisible by y_v, as otherwise the quotient would be a nontrivial degree 0 Cox monomial.

Any root $(y_v, \prod_r y_r^{a_r})$ induces a 1-parameter family of automorphisms of $\mathbb{C}^{\Sigma(1)} - V(B(\Sigma))$ commuting with $G(\Sigma)$

$$\begin{array}{lll} y_v \mapsto & y_v + \lambda \prod_{s\in\Sigma(1)-\{v\}} y_s^{a_r} & \\ y_r \mapsto & y_r & \text{for} \quad r \in \Sigma(1) - \{v\} \end{array}$$

where y_v denote coordinates on $\mathbb{C}^{\Sigma(1)}$.

Denote by $\operatorname{Roots}(X(\Sigma))$ the set of roots of $X(\Sigma)$.

3. Any automorphism of N, which permutes the cones of the fan, gives a permutation of the rays of Σ, i.e., of the Cox variables.

Theorem 1.122 *[Cox, 1995], [Cox, Katz, 1999, Sec. 3.6] If $X(\Sigma)$ is simplicial, then torus, root and fan automorphisms generate* $\operatorname{Aut}(X(\Sigma))$. *Torus and root automorphisms generate the connected component of the identity of* $\operatorname{Aut}(X(\Sigma))$ *and*

$$\dim \operatorname{Aut}(X(\Sigma)) = \dim(T) + |\operatorname{Roots}(X(\Sigma))| \tag{1.4}$$

Note that

$$|\operatorname{Roots}(X(\Sigma))| = \sum_{r\in\Sigma(1)} \left(\dim\left(S_{[D_r]}\right) - 1\right)$$

(see also Section 1.3.8).

If $X(\Sigma)$ is simplicial and Gorenstein, then there is a one-to-one correspondence between the lattice points in the relative interior of the facets (i.e., codimension one faces) of the polytope

$$\Delta_{-K_{X(\Sigma)}} = \{m \in M_{\mathbb{R}} \mid \langle m, \hat{r}\rangle \geq -1 \forall r \in \Sigma(1)\}$$

and the roots of $X(\Sigma)$:

- If $\rho = \left(y_v, \prod_{r\in\Sigma(1)-\{v\}} y_r^{a_r}\right)$ is a root of $X(\Sigma)$, then $\deg\left(\frac{\prod_{r\in\Sigma(1)-\{v\}} y_r^{a_r}}{y_v}\right) = 0$, so $(b_r) \in \mathbb{Z}^{\Sigma(1)}$ with

$$b_r = \left\{ \begin{array}{ll} a_r & \text{if } r \in \Sigma(1) - \{v\} \\ -1 & \text{if } r = v \end{array} \right\}$$

is in the image of A in

$$\begin{array}{ccccccc} 0 \to & M & \overset{A}{\to} & \mathbb{Z}^{\Sigma(1)} & \to & A_{n-1}(X(\Sigma)) & \to 0 \\ & m & \mapsto & (\langle m, \hat{r}\rangle)_{r\in\Sigma(1)} & & & \end{array}$$

i.e., there is a unique $m_\rho \in M$ such that

$$\langle m_\rho, \hat{v}\rangle = -1$$

and

$$\langle m_\rho, \hat{r}\rangle = a_r \text{ for all } r \neq v$$

By

$$\Delta_{-K_{X(\Sigma)}} = \{m \in M_{\mathbb{R}} \mid \langle m_\rho, \hat{r}\rangle \geq -1 \forall r \in \Sigma(1)\}$$

and $\langle m_\rho, \hat{r}\rangle = a_r \geq 0 > -1$ for $r \neq v$ and $\langle m_\rho, \hat{v}\rangle = -1$ we conclude that m_ρ is in the interior of the facet of $\Delta_{-K_{X(\Sigma)}}$ given by $\langle m, \hat{v}\rangle = -1$, i.e.,

$$m_\rho \in \text{int}\left(\Delta_{-K_{X(\Sigma)}} \cap \{\langle m, \hat{v}\rangle = -1\}\right) \cap M$$

- If $m_\rho \in \text{int}\left(\Delta_{-K_{X(\Sigma)}} \cap \{\langle m, \hat{v}\rangle = -1\}\right) \cap M$ is a lattice point in the relative interior of a facet of $\Delta_{-K_{X(\Sigma)}}$, then

$$\begin{array}{l} \langle m_\rho, \hat{v}\rangle = -1 \\ \langle m_\rho, \hat{r}\rangle > -1 \ \forall r \in \Sigma(1) \text{ with } r \neq v \end{array}$$

so with

$$a_r = \langle m_\rho, \hat{r}\rangle \in \mathbb{Z}_{\geq -1}$$

for $r \neq v$, we have

$$\frac{\prod_{r\in\Sigma(1)-\{v\}} y_r^{a_r}}{y_v} = A(m_\rho)$$

i.e.,

$$\deg\left(\frac{\prod_{r\in\Sigma(1)-\{v\}} y_r^{a_r}}{y_v}\right) = 0$$

so $\rho = \left(y_v, \prod_{r\in\Sigma(1)-\{v\}} y_r^{a_r}\right)$ is a root of $X(\Sigma)$. Summarizing:

Proposition 1.123 *[Aspinwall, Greene, Morrison, 1993] If $X(\Sigma)$ is simplicial and Gorenstein, then the roots of $X(\Sigma)$ are in one-to-one correspondence with the lattice points in the relative interior of the facets of $\Delta_{-K_{X(\Sigma)}} \subset M_{\mathbb{R}}$.*

The polytope $\Delta_{-K_{X(\Sigma)}}$ is not a lattice polytope in general.

Corollary 1.124 *If $X(\Sigma)$ is simplicial and Gorenstein, then*

$$\dim(\mathrm{Aut}(X(\Sigma))) = \dim(T) + \sum_{Q \text{ facet of } \Delta_{-K_{X(\Sigma)}}} |\mathrm{int}_M(Q)|$$

with int_M denoting the set of lattice points in the relative interior of Q.

Example 1.125 *For $X(\Sigma) = \mathbb{P}^n$ the roots are the pairs (x_i, x_j) for $i \neq j$, hence*

$$\dim(\mathrm{Aut}(\mathbb{P}^n)) = n + (n+1)^2 - (n+1) = (n+1)^2 - 1$$

i.e., the dimension of $\mathrm{PGL}(n+1, \mathbb{C})$.

1.3.14 Toric Mori theory

Recall that for normal varieties X and Y a proper birational morphism $f : X \to Y$ is called **small** if it is an isomorphism in codimension one. A normal variety X is called $\mathbb{Q}$-factorial if all prime divisors on X are $\mathbb{Q}$-Cartier.

Lemma 1.126 *[Reid, 1983] A toric variety Y is $\mathbb{Q}$-factorial if and only if Y is simplicial.*

Recall that for any toric variety X there is a small projective toric morphism $X' \to X$ such that X' is $\mathbb{Q}$-factorial.

Let X and Y be normal varieties and $f : X \to Y$ be a proper morphism. A 1**-cycle** of X/Y is a formal sum $\sum a_i C_i$ with $a_i \in \mathbb{Z}$ of complete curves C_i in the fibers of f. Denote by

$$\begin{aligned} Z_1(X/Y) &= \{\text{1-cycles of } X/Y\} \\ Z_1(X/Y)_{\mathbb{Q}} &= Z_1(X/Y) \otimes \mathbb{Q} \end{aligned}$$

There is a bilinear pairing

$$\begin{array}{ccc} \mathrm{Pic}(X) \times Z_1(X/Y)_{\mathbb{Q}} & \to & \mathbb{Q} \\ (\mathcal{L}, C) & \mapsto & \deg_C(\mathcal{L}) \end{array}$$

Consider two line bundles respectively 1-cycles numerically equivalent $\equiv$ if they induce the same linear form on $Z_1(X/Y)_{\mathbb{Q}}$ respectively $\mathrm{Pic}(X)$. So we get with

$$N^1(X/Y) = (\mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}) / \equiv$$
$$N_1(X/Y) = \left(Z_1(X/Y)_{\mathbb{Q}}\right) / \equiv$$

the induced perfect pairing

$$N^1(X/Y) \times N_1(X/Y) \to \mathbb{Q}$$

Consider the cone of effective 1-cycles

$$NE(X/Y) = \left\{ C \in N_1(X/Y) \mid C = \sum a_i C_i \text{ with } a_i \geq 0 \right\}$$

Definition 1.127 *A subcone W of a cone V is called* ***extremal****, if for all $u, v \in V$ with $u+v \in W$ it holds $u, v \in W$. An* ***extremal ray*** *is an extremal cone of dimension* 1.

So for a strongly convex cone V a subcone W is extremal, if there is a linear form l such that

$$W = \{v \in \partial V \mid l(v) = 0\}$$

The **relative Picard number** of X/Y is

$$\rho(X/Y) = \dim_{\mathbb{Q}}\left(N^1(X/Y)\right)$$

$D \in N^1(X/Y)$ is f**-nef** if $D \geq 0$ on $NE(X/Y)$.

Theorem 1.128 (Cone Theorem) *[Fujino, Sato, 2004] If $f : X \to Y$ is a proper toric morphism, then $NE(X/Y)$ is a convex polyhedral cone. If f is projective, then $NE(X/Y)$ is strongly convex.*

Theorem 1.129 (Contraction Theorem) *[Fujino, Sato, 2004] Let $f : X \to Y$ be a projective toric morphism and let R be an extremal face of $NE(X/Y)$. There is a projective surjective toric morphism $g : X \to W$ over Y such that*

- *W is a toric variety, which is projective over Y,*
- *g has connected fibers,*
- *if C is a curve in a fiber of f, then $[C] \in R$ if and only if $g(C)$ is a point.*

If R is an extremal ray and X is $\mathbb{Q}$-factorial, then also W is $\mathbb{Q}$-factorial and if g is not small, then $\rho(W/Y) = \rho(X/Y) - 1$.

Theorem 1.130 (Existence of flips) *[Fujino, Sato, 2004] Suppose that $f : X \to Z$ is a small toric morphism, D is a torus invariant $\mathbb{Q}$-Cartier divisor on X and $-D$ is f-ample and r is an integer with rD Cartier. Then there is a small projective toric morphism*

$$h : X^+ = \mathrm{Proj}_Z \left(\bigoplus_{m \geq 0} f_* \mathcal{O}_X (m \cdot r \cdot D) \right) \to Z$$

such that the proper transform D^+ of D on X^+ is an h-ample $\mathbb{Q}$-Cartier divisor. Then the birational map

$$\begin{array}{ccc} X & \longrightarrow & X^+ \\ f \searrow & & \swarrow h \\ & Z & \end{array}$$

is called the elementary transformation with respect to D. If X is $\mathbb{Q}$-factorial and $\rho(X/Z) = 1$, then X^+ is $\mathbb{Q}$-factorial and $\rho(X^+/Z) = 1$.

As the 1-skeleton of the fan is not changed by elementary transformations we have:

Theorem 1.131 (Termination of flips) *[Fujino, Sato, 2004] Let*

$$\begin{array}{ccccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ \searrow & & \swarrow \quad \searrow & & \swarrow \quad \searrow & & \\ & Z_0 & & Z_1 & & & \end{array}$$

be a sequence of elementary transformations with respect to the divisors D_i, $i = 0, 1, \ldots$ where D_{i+1} is the proper transform of D_i. Then this sequence terminates after finitely many steps.

Algorithm 1.132 (Toric minimal model program) *[Reid, 1983] Given a $\mathbb{Q}$-factorial toric variety, a projective toric morphism $f : X \to Y$ and a $\mathbb{Q}$-divisor D on X we have two possibilities:*

1. *D is f-nef, i.e., $D.C \geq 0$ for all curves C contracted by f.*

   *In this case the process stops, and we call X a **relative D-minimal model** over Y.*

2. *D is not f-nef:*

   In this case the Cone Theorem 1.128 gives the existence of an extremal ray R in $NE(X/Y)_{D<0}$ and the Contraction Theorem 1.129 yields the associated extremal contraction $g : X \to W$ over Y and we have

(a) *if* $\dim W < \dim X$*, then the process stops with a Mori fiber space.*

(b) *if* g *is birational and contracts a divisor, then* $\rho(W/Y) = \rho(X/Y) - 1$ *and* g *is called a* ***divisorial contraction****.*
Continue with $W \to Y$ *and the divisor* g_*D*.*

(c) *if* g *is small, then by Theorem 1.130 there an elementary transformation* $h : X \to X^+$ *with respect to* D*. The birational map* h *is also called a log-flip .*
Continue with $X^+ \to Y$ *and the divisor* h_*D*.*

This process stops, as $\rho(X/Y)$ *drops by divisorial contractions and any sequence of log-flips terminates by Theorem 1.131.*

In the standard Mori theory for toric varieties f is birational, Y is projective and $D = K_X$. Note that K_X is f-nef if and only if $K_X = f^*K_Y$ in the sense of $\mathbb{Q}$-Cartier divisors.

Theorem 1.133 *[Fujino, 2003] Let* X *be a* $\mathbb{Q}$*-factorial toric variety,* Y *a complete toric variety and* $f : X \to Y$ *a birational toric morphism. If* E *is a subset of the exceptional divisors of* f*, then* f *factors*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ g \searrow & & \nearrow h \\ & Y' & \end{array}$$

such that the birational map $g : X \longrightarrow Y'$

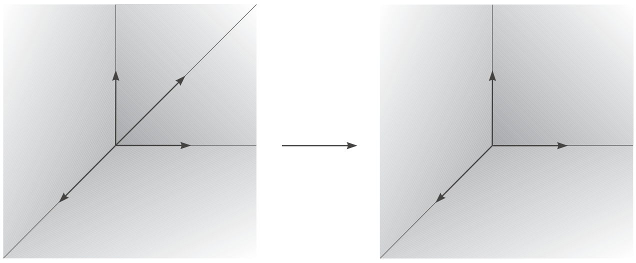
1. *contracts all divisors in* E*,*
2. *is a local isomorphism at every generic point of the divisors not in* E*,*
3. $g^{-1} : Y' \to X$ *contracts no divisor,*
4. Y' *is projective over* Y *and* $\mathbb{Q}$*-factorial.*

So if E *is the set of* f*-exceptional divisors, then* h *is a small projective* $\mathbb{Q}$*-factorialization.*

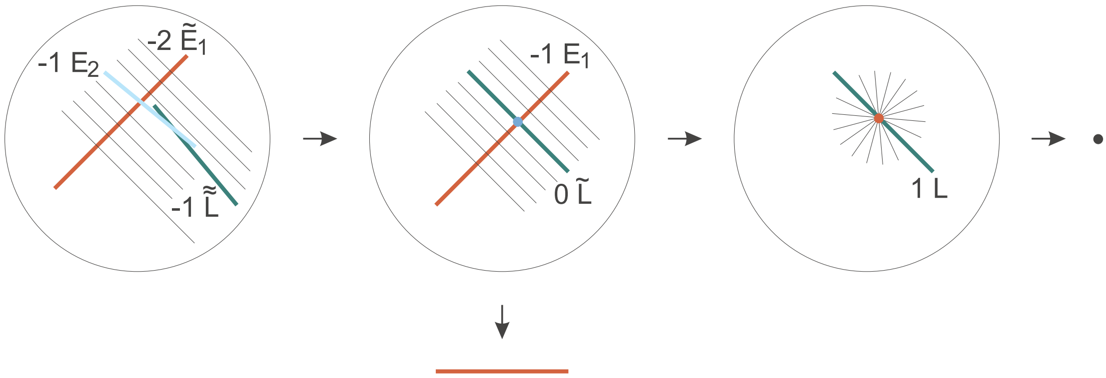
Proposition 1.134 *[Fujino, 2003] If* Y *is a complete toric variety and* $f_i : Y_i \to Y$*,* $i = 1, 2$ *are small projective* $\mathbb{Q}$*-factorializations, then there is a finite composition of elementary transformations* $Y_1 \to Y_2$*.*

Remark 1.135 *We illustrate in the following example the contraction process, the corresponding Mori cones and the linear forms* K_{X_i}*:*

- *Let* $f_1 : X_1 \to X_0 = \mathbb{P}^2$ *be the blowup of* X_0 *in a point* p_1 *and* E_1 *the exceptional.* $X_1 = \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\right)$ *has a fibration over* $\mathbb{P}^1$ *by the* 0*-curves. Denote by* H *a line in* X_0 *not meeting* p_1*. The map* f_1 *has a toric representation as a map of fans* $\Sigma_1 \to \Sigma$*.*



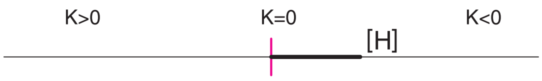
- *Further consider the blowup* $f_2 : X_2 \to X_1$ *of* X_1 *in a point* $p_2 \in E_1$ *with exceptional* E_2 *and denote* $\widetilde{E_1}$ *the strict transform of* E_1*. Denote by* L *the line through* p_1 *such that* $\widetilde{L}$ *meets* E_1 *in* p_2*.*



Any of the depicted maps corresponds to the contraction of a curve, whose class $[E]$ *generates an extremal ray of* $\overline{NE}(X_i)_{\mathbb{R}}$*. To see this, we calculate the Mori cones:*

1. $\overline{NE}(\mathbb{P}^2)_{\mathbb{R}}$*:*

   $N_1(\mathbb{P}^2) = \mathbb{R}$, $\overline{NE}(\mathbb{P}^2)_{\mathbb{R}} = \mathbb{R}_{\geq 0}$ *and* $K_{\mathbb{P}^2} = -3H$



From the toric point of view

$$\overline{NE}\left(\mathbb{P}^2\right)_{\mathbb{R}} = \text{hull}\,\{(1,1,1)\} \subset \langle(1,1,1)\rangle \subset \mathbb{R}^{\Sigma(1)}$$

2. $\overline{NE}\,(X_1)_{\mathbb{R}}$*:*

- $[\widetilde{H}]$ *and* $[E_1]$ *form a basis of* $N_1\,(X_1) \cong \mathbb{R}^2$*, and we choose*

$$[\widetilde{H}] = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad [E_1] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

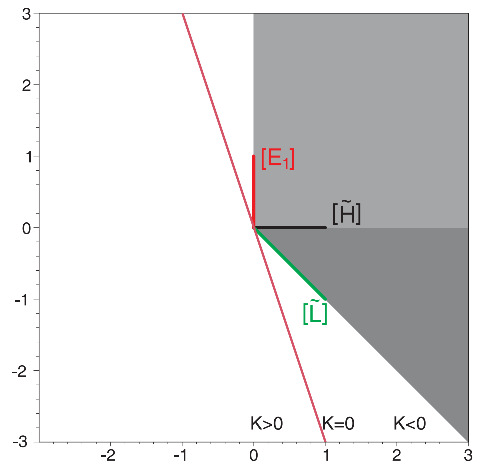
- $\widetilde{H}, E_1 \in N_1\,(X_1)$ *are determined by* $\widetilde{H}^2 = 1$*,* $\widetilde{H}.E_1 = 0$ *and* $E_1^2 = -1$*.*

- *From* $E_1^2 = -1$ *and* $\widetilde{L}^2 = 0$ *we know that* $[E_1]$ *generates an extremal ray of* $\overline{NE}\,(X_1)_{\mathbb{R}}$ *and* $[\widetilde{L}]$ *is on the boundary of* $\overline{NE\,(X_1)}$*, so we can conclude that they span* $\overline{NE}\,(X_1)_{\mathbb{R}} \subset \mathbb{R}^2$ *and* $[\widetilde{L}]$ *generates an extremal ray. By* $\widetilde{L}.\widetilde{H} = L.H - E_1.\widetilde{H} = 1$*,* $\widetilde{L}.E_1 = 1$ *we calculate*

$$[\widetilde{L}] = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- $K_{X_1} = -3\widetilde{H} + E_1$ *hence*

$$K_{X_1}.\left(x_1[\widetilde{H}] + x_2\,[E_1]\right) = -3x_1 - x_2$$



We observe that X_1 *is Fano.*
From the toric point of view $\overline{NE}\,(X_1)_{\mathbb{R}}$ *is a cone in the subvectorspace*

$$\ker\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\rangle$$

of $\mathbb{R}^{\Sigma_1(1)}$*:*

$$\overline{NE}\left(X_1\right)_{\mathbb{R}} = \text{hull}\left\{\begin{pmatrix}1\\1\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix},\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}\right\}$$

$$\begin{array}{ccccc} = \text{hull}\left\{\begin{pmatrix}1\\1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}\right\} & \subset & \left\langle\begin{pmatrix}1\\1\\1\\0\end{pmatrix},\begin{pmatrix}1\\1\\0\\-1\end{pmatrix}\right\rangle & \subset \mathbb{R}^{\Sigma_1(1)} \\ \downarrow & & \cong \downarrow pr_{1,4} & \\ \text{hull}\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\-1\end{pmatrix}\right\} & \subset & \mathbb{R}^2 & \end{array}$$

- *Finally we compute the coordinates of the classes* $[\widetilde{C}]$ *for all reduced irreducible curves* $C \subset \mathbb{P}^2$*:*
  We have $C \sim dH$ *with* $d = C.H$*, and* $\widetilde{C} = f_1^* C - m_{p_1}(C)\, E_1$ *with the multiplicity of* C *in* p_1*, hence*

  $\widetilde{H}.\widetilde{C} = H.C = d$ *and* $E_1.\widetilde{C} = f_{1*}\left(E_1\right).C - m_{p_1}(C)\, E_1^2 = m_{p_1}(C)$

  i.e.,

  $$[\widetilde{C}] = \begin{pmatrix} d \\ -m_{p_1}(C)\end{pmatrix}$$

  in particular the classes from $\mathbb{P}^2$ *all lie inside the cone generated by* $\widetilde{H}$ *and* L*, i.e., the dark grey area.*

3. $\overline{NE}\left(X_2\right)_{\mathbb{R}}$*:*

- $[\widetilde{\widetilde{H}}]$, $[\widetilde{E_1}]$ *and* $[E_2]$ *form a basis of* $N_1\left(X_2\right) = \mathbb{R}^3$*, and we choose coordinates*

  $$[\widetilde{H}] = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad [\widetilde{E_1}] = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad [E_2] = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

- $\widetilde{\widetilde{H}}, \widetilde{E_1}, E_2 \in N_1\left(X_2\right)$ *are determined by* $\widetilde{\widetilde{H}}^2 = 1$*,* $\widetilde{E_1}^2 = -2$*,* $E_2^2 = -1$*,* $\widetilde{\widetilde{H}}.\widetilde{E_1} = 0$*,* $\widetilde{\widetilde{H}}.E_2 = 0$ *and* $\widetilde{E_1}.E_2 = 1$*.*

- *From* $E_2^2 = -1$, $\widetilde{\widetilde{L}}^2 = -1$ *and* $\widetilde{E_1}^2 = -2$ *we know that* $[E_2], [\widetilde{E_1}]$ *and* $[\widetilde{\widetilde{L}}]$ *generate extremal rays of* $\overline{NE}(X_2)_{\mathbb{R}}$*. From*

$$\widetilde{\widetilde{L}}.\widetilde{\widetilde{H}} = \left(f_2^*\widetilde{L} - E_2\right).\widetilde{\widetilde{H}} = \widetilde{L}.\widetilde{H} = 1$$

$\widetilde{\widetilde{L}}.\widetilde{E_1} = 0$, $\widetilde{\widetilde{L}}.E_2 = 1$ *we get for the coordinates of* $\widetilde{\widetilde{L}} = x_1\widetilde{\widetilde{H}} + x_2\widetilde{E_1} + x_3E_2$ *that* $x_1 = 1$, $-2x_2 + x_3 = 0$, $x_2 - x_3 = 1$, *hence*

$$[\widetilde{\widetilde{L}}] = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$$

To check that $\overline{NE}(X_2)_{\mathbb{R}}$ *is really the cone generated by* $[\widetilde{E_1}], [E_2]$ *and* $[\widetilde{\widetilde{L}}]$*, we check that it contains all classes* $[\widetilde{C}]$ *for reduced irreducible curves* $C \subset \mathbb{P}^2$*:*
Let $d = C.H$, $m_{p_1}(C)$ *be the multiplicity of* C *in* p_1 *and* $m_{p_2}(\widetilde{C})$ *the tangency of* C *to the line* L*.*

$$\begin{aligned}
\widetilde{\widetilde{H}}.\widetilde{\widetilde{C}} &= \widetilde{H}.\widetilde{C} = H.C = d \\
\widetilde{E_1}.\widetilde{\widetilde{C}} &= (f_2^*E_1 - E_2).\left(f_2^*\widetilde{C} - m_{p_2}(\widetilde{C})E_2\right) = E_1.\widetilde{C} - m_{p_2}(\widetilde{C})E_2^2 \\
&= m_{p_1}(C) - m_{p_2}(\widetilde{C}) \\
E_2.\widetilde{\widetilde{C}} &= m_{p_2}(\widetilde{C})
\end{aligned}$$

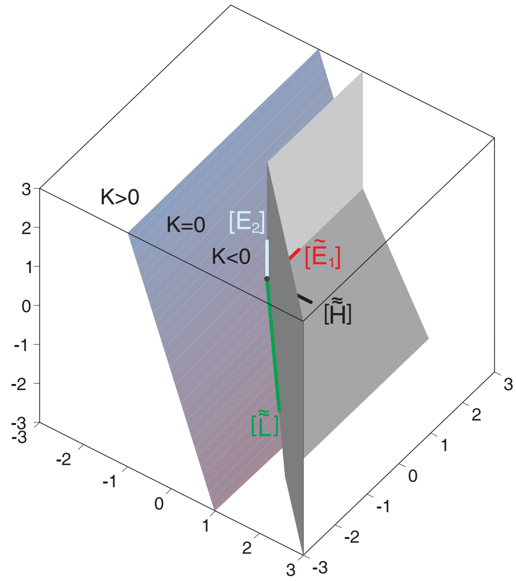
we get for the coordinates of $\widetilde{\widetilde{C}} = x_1\widetilde{\widetilde{H}} + x_2\widetilde{E_1} + x_3E_2$ *that* $x_1 = d$, $-2x_2 + x_3 = m_{p_1}(C) - m_{p_2}(\widetilde{C})$, $x_2 - x_3 = m_{p_2}(\widetilde{C})$, *hence*

$$[\widetilde{C}] = \begin{pmatrix} d \\ -m_{p_1}(C) \\ -m_{p_1}(C) - m_{p_2}(\widetilde{C}) \end{pmatrix}$$

All the classes from $\mathbb{P}^2$ *lie inside the cone spanned by*

$$\widetilde{\widetilde{L}} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \widetilde{\widetilde{H}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

i.e., the part of $\overline{NE}\left(X_2\right)_{\mathbb{R}}$ *with non-positive* $\widetilde{E_1}$ *and* E_2 *coordinate*



We see that X_2 *is no longer Fano.*

In the following for a given finite set $\mathcal{R}$ of 1-dimensional rational cones we describe the set of all Kähler cones of projective simplicial fans Σ with $\Sigma(1) \subset \mathcal{R}$. These Kähler cones fit together as the maximal cones of a fan. We will relate this fan to the birational geometry of the toric varieties given by projective fans Σ with $\Sigma(1) \subset \mathcal{R}$.

Given a toric variety $X(\Sigma)$, the Chow group of divisors $A_{n-1}(X(\Sigma))$ has the presentation

$$\begin{array}{ccccccc} 0 \rightarrow & M & \rightarrow & \mathbb{Z}^{\Sigma(1)} & \rightarrow & A_{n-1}(X(\Sigma)) & \rightarrow 0 \\ & m & \mapsto & \left(\langle m, \hat{r}\rangle\right)_{r\in\Sigma(1)} & & & \end{array}$$

hence depends only on the 1-skeleton $\mathcal{R} = \Sigma(1)$ of the fan Σ. So denote $A_{n-1}(X(\Sigma))$ by $A_{n-1}(\mathcal{R})$ and $A_{n-1}(X(\Sigma)) \otimes \mathbb{R}$ by $A_{n-1}(\mathcal{R})_{\mathbb{R}}$. If Σ is a projective simplicial fan with $\Sigma(1) = \mathcal{R}$, then the Kähler cone $K(X(\Sigma))$ canonically lies in $A_{n-1}(\mathcal{R})_{\mathbb{R}} \cong \mathbb{R}^{\mathcal{R}}/M_{\mathbb{R}}$.

So in the following let $\mathcal{R}$ be a finite set of 1-dimensional rational cones in $N_{\mathbb{R}}$, which is the set of rays of a complete fan in $N_{\mathbb{R}}$. For any projective simplicial fan Σ with $\Sigma(1) = \mathcal{R}$ we get the cone $\operatorname{cpl}(\Sigma)$ of dimension $|\mathcal{R}| - n$, which is the closure of the Kähler cone $K(X(\Sigma))$ and lies in the cone of effective divisor classes $A^+_{n-1}(\mathcal{R})_{\mathbb{R}}$. In the same way:

Lemma 1.136 *[Oda, Park, 1991] Let* Σ *be a projective fan with* $\Sigma(1) \subset \mathcal{R}$. *If* $a = \sum_{r\in\Sigma(1)} a_r [D_r] \in A^+_{n-1}(\mathcal{R})_{\mathbb{R}}$, *then for any* $\sigma \in \Sigma$ *there is an* $m_\sigma \in M_{\mathbb{R}}$

such that $\langle m_\sigma, \hat{r}\rangle = -a_r$ *for all rays* $r \subset \sigma$. *If* $\langle m_\sigma, \hat{r}\rangle \geq -a_r$ *for all* $r \in \mathcal{R}$, $r \not\subset \sigma$, *then* a *is called* Σ-***convex***. *The set of all* Σ-*convex* $a \in A_{n-1}^{+}(\mathcal{R})_{\mathbb{R}}$ *is an* $|\mathcal{R}| - n$ *dimensional convex cone, which we also denote by* $\mathrm{cpl}(\Sigma)$.

Theorem 1.137 *[Oda, Park, 1991] The set of all* $\mathrm{cpl}(\Sigma)$ *for projective fans* Σ *with* $\Sigma(1) \subset \mathcal{R}$ *form the set of the* $|\mathcal{R}| - n$ *dimensional cones of a fan with support* $A_{n-1}^{+}(\mathcal{R})_{\mathbb{R}}$. *It is called the Gelfand-Kapranov-Zelevinsky decomposition* $GKZ(\mathcal{R})$ *associated to* $\mathcal{R}$.

Non-simplicial fans Σ with $\Sigma(1) \subset \mathcal{R}$ correspond to cones in $GKZ(\mathcal{R})$ of dimension less than $|\mathcal{R}| - n$. Note that the converse is not true.

Proposition 1.138 *[Oda, Park, 1991], [Cox, Katz, 1999] Two cones* $\mathrm{cpl}(\Sigma)$ *and* $\mathrm{cpl}(\Sigma')$ *of dimension* $|\mathcal{R}| - n$ *of* $GKZ(\mathcal{R})$ *have a common facet if and only if the toric varieties* $X(\Sigma)$ *and* $X(\Sigma')$ *are related by a birational extremal contraction.*

The fan $GKZ(\mathcal{R})$ may be extended to a complete fan $\Sigma(\mathcal{R})$ in $A_{n-1}(\mathcal{R})_{\mathbb{R}}$:

Definition 1.139 *A* ***marked polytope*** *is a pair* (P, M) *where* $P \subset \mathbb{R}^n$ *is a convex polytope and* $M \subset P$ *is a finite subset with* $\mathrm{vertices}(P) \subset M$.

So we may view a marked polytope as just a finite set M of points in $\mathbb{R}^n$, and $P = \mathrm{convexhull}(M)$.

Definition 1.140 *A* ***polyhedral subdivision*** *of a marked polytope* (P, M) *in* $\mathbb{R}^n$ *is a set of marked polytopes* (P_i, M_i) *with* $\dim(P_i) = \dim(P)$ *such that*

$$\bigcup\nolimits_i P_i = P$$

and for all i, j *the intersection* $F = P_i \cap P_j$ *is a face of* P_i *and* P_j *(which may be empty) and*

$$M_i \cap F = M_j \cap F$$

i.e., $M_i \cap \mathrm{convexhull}(M_j) = M_j \cap \mathrm{convexhull}(M_i)$.

A polyhedral subdivision is called ***triangulation***, *if all* P_i *are simplices and* M_i *is the set of vertices of* P_i.

If $\{(P_i, M_i)\}$ *and* $\{(P_j', M_j')\}$ *are polyhedral subdivisions of* (P, M), *then* $\{(P_i, M_i)\}$ ***refines*** $\{(P_j', M_j')\}$, *if for all* j

$$\{(P_i, M_i) \mid P_i \subset P_j'\}$$

is a polyhedral subdivision of (P_j', M_j').

Hence the set of polyhedral subdivisions of (P, M) form a poset and the triangulations are the minimal elements.

Lemma 1.141 *[Gelfand, Kapranov, Zelevinsky, 1994, Sec. 7.2.] Let* (P, M) *be a marked polytope in* $\mathbb{R}^n$*. If* $f : M \to \mathbb{R}^n$*, i.e.,* $f \in \mathbb{R}^M$*, is a function let*

$$G_f = \text{convexhull}\left\{(x, y) \in \mathbb{R}^n \oplus \mathbb{R} \mid x \in M,\ y \in \mathbb{R} \text{ with } y \leq f(x)\right\}$$

Then

$$\begin{aligned} &g_f : P \to \mathbb{R} \\ &g_f(x) = \max\left\{y \in \mathbb{R} \mid (x, y) \in G_f\right\} \end{aligned}$$

is a piecewise linear function on P*. Denote by* P_i *the domains of linearity of* g_f *and let*

$$\begin{aligned} M_i &= \{x \in M \cap P_i \mid g_f(x) = f(x)\} \\ &= \{x \in M \cap P_i \mid (x, f(x)) \in \partial G_f\} \end{aligned}$$

Then $\{(P_i, M_i)\}$ *is a polyhedral subdivision of* (P, M) *denoted as* $\mathcal{S}(f)$*.*

Definition 1.142 *A polyhedral subdivision of a marked polytope* (P, M) *is called* ***coherent****, if it is of the form* $S(f)$ *for some* $f \in \mathbb{R}^M$*.*

Let $\mathcal{R}$ be a set of 1-dimensional rational cones in $N_{\mathbb{R}}$, denote by $\hat{r}$, $r \in \mathcal{R}$ the primitive lattice generators of the elements of $\mathcal{R}$. With

$$\mathcal{R}' = \{(\hat{r}, 1) \mid r \in \mathcal{R}\} \cup \{(0, 1)\} \subset N_{\mathbb{R}} \oplus \mathbb{R}$$

we have an exact sequence

$$0 \to M_{\mathbb{R}} \oplus \mathbb{R} \xrightarrow{A'} \mathbb{R}^{\mathcal{R}'} \xrightarrow{\pi} A_{n-1}(\mathcal{R})_{\mathbb{R}} \to 0$$

with the elements of $\mathcal{R}'$ forming the rows of A'. Consider the marked polytope $(P, M) = (\text{convexhull}(\mathcal{R}'), \mathcal{R}')$ in $N_{\mathbb{R}} \oplus \mathbb{R}$.

Definition 1.143 *If* $\mathcal{S} = \{(P_i, M_i)\}$ *is a coherent polyhedral subdivision of the marked polytope* (P, M)*, then let*

$$C(\mathcal{S}) = \left\{\pi(f) \mid f \in \mathbb{R}^{\mathcal{R}'},\ \mathcal{S} \text{ is a subdivision of } \mathcal{S}(f)\right\}$$

be the image under π *of the cone of those functions* $f \in \mathbb{R}^{\mathcal{R}'}$ *such that* $\mathcal{S}$ *is a subdivision of* $\mathcal{S}(f)$*.*

Proposition 1.144 *[Gelfand, Kapranov, Zelevinsky, 1994, Sec. 7.2.], [Cox, Katz, 1999, Sec. 3.4.] The cones* $C(\mathcal{S})$ *form a complete fan in* $A_{n-1}(\mathcal{R})_{\mathbb{R}}$*. This fan is called the* ***secondary fan*** $\Sigma(\mathcal{R})$ *of* $\mathcal{R}$*.*

Lemma 1.145 *[Cox, Katz, 1999, Sec. 3.4.] Let* $\mathcal{S} = \{(P_i, M_i)\}$ *be a coherent polyhedral subdivision of* (P, M)*. Then*

$$C(\mathcal{S}) = \bigcap\nolimits_i \operatorname{hull}\{\pi(e_{r'}) \mid r' \notin M_i\}$$

where $e_{r'} \in \mathbb{R}^{\mathcal{R}'}$ *denotes the standard basis vector corresponding to* r'*.*

If $C(\mathcal{S})$ is a coherent polyhedral subdivision involving $(0,1)$, then the cones over the polytopes of $\mathcal{S}$ form a complete fan in $N_{\mathbb{R}} \cong N_{\mathbb{R}} \times \{1\}$, and any projective fan in $N_{\mathbb{R}}$ with rays spanned by elements of $\mathcal{R}$ arises this way.

Lemma 1.146 *[Cox, Katz, 1999, Sec. 3.4.] The secondary fan* $\Sigma(\mathcal{R})$ *contains* $GKZ(\mathcal{R})$ *as a subfan. The cones of* $GKZ(\mathcal{R})$ *are those corresponding to coherent polyhedral subdivisions of* $(\operatorname{convexhull}(\mathcal{R}'), \mathcal{R}')$ *involving* $(0,1)$*.*

Example 1.147 *Consider the fan* Σ *given by the rays spanned by*

$$r_1 = (1,1),\ r_2 = (-1,1),\ r_3 = (1,-1),\ r_4 = (-1,-1) \in N = \mathbb{Z}^2$$

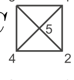
in $N_{\mathbb{R}} = \mathbb{R}^2$ *and denote* $r_5 = (0,0)$*. We choose a basis of* $\ker(\_ \circ A) = \langle(1,0,0,1,-2), (0,1,1,0,-2)\rangle$*, so we have the sequence*

$$\begin{array}{ccccccccc} 0 & \to & M_{\mathbb{R}} & \overset{A}{\to} & \mathbb{R}^{\Sigma(1)} & \to & A_{n-1}(\mathcal{R})_{\mathbb{R}} & \to 0 \\ & & & & & & \cong & \\ 0 & \to & M_{\mathbb{R}} \oplus \mathbb{R} & \overset{A'}{\to} & \mathbb{R}^{\Sigma(1)} \oplus \mathbb{R} & \overset{B}{\to} & \mathbb{R}^2 & \to 0 \end{array}$$

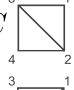
with

$$A' = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 1 & 0 & -2 \end{pmatrix}$$

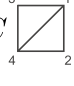
Hence considering the secondary fan as a subfan of $\mathbb{R}^2 \cong A_{n-1}(\mathcal{R})_{\mathbb{R}}$ *the cones corresponding to triangulations are*

C 

$= \operatorname{hull}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$

C 

$= \operatorname{hull}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix}\right)$

C 

$= \operatorname{hull}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix}\right)$

The secondary fan is shown in Figure 1.8.

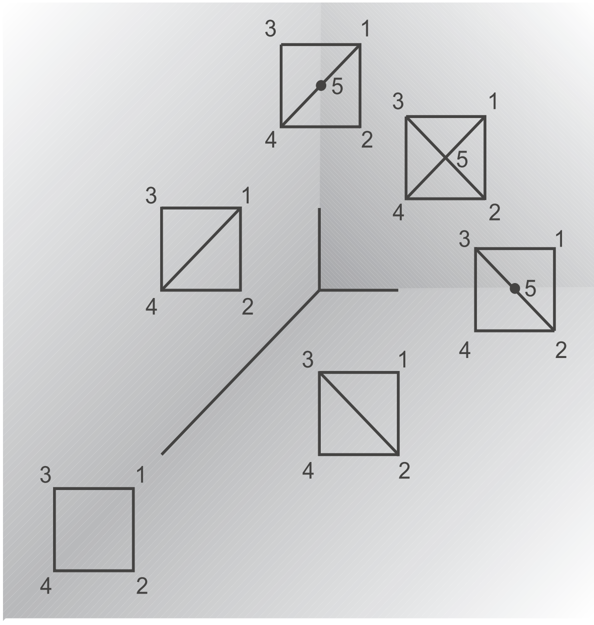


Figure 1.8: Secondary fan of $\mathbb{P}_1 \times \mathbb{P}_1$

Algorithm 1.148 *If $\mathcal{R} = \{r_1, ..., r_s\} \subset N_{\mathbb{R}} = \mathbb{Z}^n \otimes \mathbb{R}$ is a set of 1-dimensional rational cones, which are the rays of a projective fan, the following algorithm computes the secondary fan $\Sigma(\mathcal{R})$ of $\mathcal{R}$:*

1. *Let $\hat{\mathcal{R}} = \{\hat{r}_1, ..., \hat{r}_s, 0\}$ be the set of minimal lattice generators of the rays r_i together with 0.*

2. *Let $\Sigma(\mathcal{R}) = \{\}$.*

3. *Choose a random $f \in \mathbb{Z}_{\geq 0}^{\hat{\mathcal{R}}}$. Let*

$$G_f = \text{convexhull}\left\{\left(\hat{r}_j, f_{\hat{r}_j}\right), (\hat{r}_j, -1) \mid j = 1, ..., s\right\} \subset N_{\mathbb{R}} \oplus \mathbb{R}$$

   and compute the set $\mathcal{S}'$ of all faces of G_f which do not involve one of the vertices $(\hat{r}_j, -1)$.

4. *Compute the set $\mathcal{S}$ of projections of the faces of $\mathcal{S}'$ under $N_{\mathbb{R}} \oplus \mathbb{R} \to N_{\mathbb{R}}$.*

5. *Let*

$$\mathcal{R}' = \{(\hat{r}, 1) \mid r \in \mathcal{R}\} \cup \{(0, 1)\} \subset N_{\mathbb{R}} \oplus \mathbb{R}$$

   and A' be the linear map with rows $(\hat{r}, 1), (0, 1)$ and π the map

$$0 \to M_{\mathbb{R}} \oplus \mathbb{R} \xrightarrow{A'} \mathbb{R}^{\mathcal{R}'} \xrightarrow{\pi} A_{n-1}(\mathcal{R})_{\mathbb{R}} \to 0$$

Compute

$$C(\mathcal{S}) = \bigcap_{F \in \mathcal{S}'} \operatorname{hull}\{\pi(e_{r'}) \mid r' \notin F\}$$

6. *If* $\dim(C(\mathcal{S})) < s - n$, *then we found a non-maximal cone of the secondary fan (which we may remember), and we go back to* 3.

   Otherwise we set $\Sigma(\mathcal{R}) = \Sigma(\mathcal{R}) \cup \{C(\mathcal{S})\}$.

   If $\Sigma(\mathcal{R})$ *is not complete, then we go back to* 3.

   In order to get the fans associated to the cones in the GKZ decomposition, we may also remember for each cone $C(\mathcal{S})$ *the corresponding triangulation* $\mathcal{S}$.

Remark 1.149 *The Maple package* tropicalmirror *(see also Section 12.4) provides an implementation of this algorithm. Given a set* $\mathcal{R} = \{r_1, ..., r_s\}$ *of lattice vectors in* $\mathbb{Z}^n$, *which are the primitive lattice generators of the rays of a projective fan, the function* Triangulations *takes* $\mathcal{R}$ *as an argument and computes all triangulations of the marked polytope* $(\operatorname{convexhull}(\mathcal{R}), \mathcal{R})$. *Let* B *be a matrix such that the sequence*

$$0 \to \mathbb{R}^{n+1} \xrightarrow{A'} \mathbb{R}^{s+1} \xrightarrow{B} \mathbb{R}^{s-n} \to 0$$

with

$$A' = \begin{pmatrix} r_1 & 1 \\ \vdots & \vdots \\ r_s & 1 \\ 0 & 1 \end{pmatrix}$$

is exact, so choosing an isomorphism $A_{n-1}(\mathcal{R})_{\mathbb{R}} \cong \mathbb{R}^{s-n}$. *The function* SecondaryFan *takes as argument the list* $(r_1, ..., r_s)$ *and* B *and computes the secondary fan as a fan in* $\mathbb{R}^{s-n}$. *In the same way the function* GKZFan *takes the argument* $(r_1, ..., r_s)$ *and* B *and computes the GKZ decomposition of* $\mathcal{R}$.

1.4 Gröbner basics

To compute combinatorial objects in tropical geometry, we will use Gröbner basis techniques, so we recall some notation and make some remarks about their implementation in the Macaulay 2 library mora.m2 which is part of the computer algebra implementation of the mirror construction given here.

1.4.1 Semigroup orderings

Definition 1.150 *A* ***semigroup ordering*** *(monomial ordering) on the semigroup of monomials in the variables* $x_1, ..., x_n$ *is an ordering* $>$ *of the monomials with the following properties*

1. $>$ *is a total ordering*
2. $>$ *is compatible with multiplication, i.e.,* $x^\alpha > x^\beta \Rightarrow x^\alpha x^\gamma > x^\beta x^\gamma$.

Definition 1.151 *A* ***global ordering*** $>$ *is a semigroup ordering with the following equivalent properties*

1. $x_i > 1 \ \forall i$
2. $x^\alpha > 1$ *for all* $\alpha \neq 0$
3. $>$ *is a well ordering*
4. $\alpha \geq \beta$ *and* $\alpha \neq \beta \Rightarrow x^\alpha > x^\beta$

Definition 1.152 *A* ***local ordering*** $>$ *is a semigroup ordering with*

$$x_i < 1 \ \forall i$$

Local orderings are not well orderings. This leads to problems with the termination of normal form algorithms.

Remark 1.153 *In one variable all global (resp. local) orderings are equivalent.*

Definition 1.154 *A monomial ordering* $>$ *is called a* ***weighted degree ordering*** *if there is some* $w \in \mathbb{R}^n$ *with non-zero entries such that*

$$w\alpha > w\beta \Rightarrow x^\alpha > x^\beta$$

Example 1.155 *If* $>$ *is any monomial ordering and* $w \in \mathbb{R}^n$*, then* $>_w$ *given by*

$$x^\alpha >_w x^\beta \Leftrightarrow w\alpha > w\beta \text{ or } \left(w\alpha = w\beta \text{ and } x^\alpha > x^\beta\right)$$

is a monomial ordering. It is a weighted degree ordering, it is global if $w_i > 0$ *for all* i *and it is local if* $w_i < 0$ *for all* i.

Proposition 1.156 *[Greuel, Pfister, 2002, Sec. 1.2] Given any finite set of monomials M and any semigroup ordering $>$ there is some $w \in \mathbb{Z}^n$ such that*

$$x^\alpha > x^\beta \Leftrightarrow \sum w_i \alpha_i > \sum w_i \beta_i$$

for all $x^\alpha, x^\beta \in M$.
w can be chosen such that $w_i > 0$ if $x_i > 1$ and $w_i < 0$ if $x_i < 1$.
w is called a ***weight vector*** *inducing $>$ on M.*

Example 1.157 *The following orderings are semigroup orderings:*

- ***lexicographical*** *ordering* lp*:*

  $x^\alpha < x^\beta \Leftrightarrow \exists 1 \leq i \leq n : \alpha_1 = \beta_1, ..., \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i$.

- ***reverse lexicographical*** *ordering* rp*:*

  $x^\alpha < x^\beta \Leftrightarrow \exists 1 \leq i \leq n : \alpha_n = \beta_n, ..., \alpha_{i+1} = \beta_{i+1}, \alpha_i > \beta_i$.

- ***weighted reverse lexicographical*** *ordering* $\mathsf{wp}(w)$ *for $w \in \mathbb{R}^n$:*

  $x^\alpha < x^\beta \Leftrightarrow \sum w_i\alpha_i < \sum w_i\beta_i$ *or* $\sum w_i\alpha_i = \sum w_i\beta_i$ *and* $\exists 1 \leq i \leq n :$ $\alpha_n = \beta_n, ..., \alpha_{i+1} = \beta_{i+1}, \alpha_i > \beta_i$.

- ***weighted lexicographical*** *ordering* $\mathsf{Wp}(w)$ *for $w \in \mathbb{R}^n$:*

  $x^\alpha < x^\beta \Leftrightarrow \sum w_i\alpha_i < \sum w_i\beta_i$ *or* $\sum w_i\alpha_i = \sum w_i\beta_i$ *and* $\exists 1 \leq i \leq n :$ $\alpha_1 = \beta_1, ..., \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i$.

- ***degree reverse lexicographical*** *ordering* $\mathsf{dp} = \mathsf{wp}(1, ..., 1)$.

- ***negative lexicographical*** *ordering* ls*:*

  $x^\alpha < x^\beta \Leftrightarrow \exists 1 \leq i \leq n : \alpha_1 = \beta_1, ..., \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i$.

- ***matrix ordering*** *associated to*

  $$A = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \in Mat(m \times n, \mathbb{R})$$

  with $\operatorname{rank}(A) = n$*, is given by:*

  $x^\alpha < x^\beta \Leftrightarrow \exists 1 \leq i \leq m : a_1\alpha = a_1\beta, ..., a_{i-1}\alpha = a_{i-1}\beta, a_i\alpha < a_i\beta \Leftrightarrow$ $A\alpha <_{lex} A\beta$.

  Note:

- $\mathsf{lp} = \mathsf{Wp}(0)$ *and* $\mathsf{rp} = \mathsf{wp}(0)$.

- *If all weights are non-negative, then* wp *and* Wp *are global orderings.*
- ls *is a local ordering.*

Remark 1.158 $\mathsf{ws}(w) = \mathsf{wp}(-w)$ *is denoted as* ***local weighted reverse lexicographical*** *ordering,* $\mathsf{Ws}(w) = \mathsf{Wp}(-w)$ *is denoted as* ***local weighted lexicographical*** *ordering.*

The ***local degree reverse lexicographical*** *ordering is* $\mathsf{ds} = \mathsf{ws}(1, ..., 1)$.

Example 1.159 *On a finite set of monomials the ordering* lp *can be represented by the weight vector* $w = (v^{n-1}, ..., v, 1)$ *if all monomials are contained in a cube of side length* $\leq v$.

Remark 1.160 *The above monomial orderings are implemented in the Macaulay 2 package* mora.m2*. They are selected by the value of the global method* monord*, which can be given the values* lp*,* dp*,* wp*,* ls*,* Ws*,* ws*,* Wp *and* Mat *for matrix orderings. The weight vector, if needed, is represented by the global list* ww *and the matrix inducing above matrix ordering by the global Macaulay 2 type matrix* mm.

Remark 1.161 *Any matrix ordering can be represented by a matrix in* $Gl(n, \mathbb{R})$. *Note that one can add multiples of* a_i *to any lower row* a_j *with* $j > i$ *without changing the monomial order.*

Example 1.162 *The weight ordering* $Wp(w_1, ..., w_n)$ *can be represented by the matrix ordering given by*

$$\begin{pmatrix} w_1 & \cdots & \cdots & w_n \\ 1 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

If $w_j \neq 0$ *and* $w_{j+1} = 0, ..., w_n = 0$, *then this ordering is equivalent to*

$$\begin{pmatrix} w_1 & \cdots & w_{j-1} & w_j & 0 & \cdots & 0 \\ 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ w_1 & & w_{j-1} & 0 & 0 & \cdots & 0 \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

hence to

$$\begin{pmatrix} w_1 & \cdots & w_{j-1} & w_j & 0 & \cdots & 0 \\ 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ 0 & & 0 & 0 & 0 & \cdots & 0 \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

i.e., to the matrix ordering given by

$$\begin{pmatrix} w_1 & \cdots & w_{j-1} & w_j & 0 & \cdots & 0 \\ 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

Proposition 1.163 *[Robbiano, Sweedler, 1990] Every semigroup ordering is representable by a matrix ordering.*

Definition 1.164 *Let $>$ be a monomial ordering on the monomials of $K[x_1,...,x_n]$. For $f \in K[x_1,...,x_n]$, denote by $L(f)$ the **lead monomial**, i.e., the largest monomial with respect to $>$ appearing in f, by $LC(f)$ the **lead coefficient**, i.e., the coefficient of $L(f)$ in f, and by $LT(f) = LC(f)\,L(f)$ the **lead term** of f.*

1.4.2 Localizations

Remark 1.165 *The rings*

$$\mathbb{C}[x_1,...,x_n]_{\langle x_1,...,x_n\rangle} \subset \mathbb{C}\{x_1,...,x_n\} \subset \mathbb{C}[[x_1,...,x_n]]$$

correspond to looking at increasingly smaller neighborhoods of the origin:

1. *Elements of $\mathbb{C}[x_1,...,x_n]_{\langle x_1,...,x_n\rangle}$ are defined in the complement of an algebraic set, i.e., in a Zariski open neighborhood of the origin, e.g., $\frac{f}{g}$ is defined in the complement of $V(g)$.*

2. *Elements of $\mathbb{C}\{x_1,...,x_n\}$ are defined in a neighborhood of the origin in the analytic topology, which can be much smaller, e.g., the geometric series $\sum_{k=0}^{\infty} x^k$ is defined for $|x| < 1$.*

3. Elements of $\mathbb{C}[[x_1, ..., x_n]]$ are defined just at the origin in general.

Nevertheless, they all share the property of being local rings (for $\mathbb{C}[[x_1, ..., x_n]]$ this is shown by using the geometric series).

Remark 1.166 *Let K be a field. For any semigroup ordering $>$ on $K[x_1, ..., x_n]$*

$$L(gf) = L(g)L(f)$$
$$L(g+f) \leq \max\{L(g), L(f)\}$$

hence,

$$S_> = \{u \in K[x_1, ..., x_n] \setminus \{0\} \mid L(u) = 1\}$$

is multiplicatively closed.

$S_> = K^* \Leftrightarrow >$ *is global.*

$S_> = K[x_1, ..., x_n] \setminus \langle x_1, ..., x_n \rangle \Leftrightarrow >$ *is local.*

Definition 1.167 *Let $>$ be a semigroup ordering on $K[x_1, ..., x_n]$. The* ***localization of*** $K[x_1, ..., x_n]$ ***associated to*** $>$ *is*

$$K[x_1, ..., x_n]_> = S_>^{-1} K[x_1, ..., x_n] = \left\{ \frac{f}{u} \mid f, u \in K[x_1, ..., x_n], L(u) = 1 \right\}$$

Lemma 1.168 *[Decker, Schreyer, 2007], [Greuel, Pfister, 2002, Sec. 1.5] Given a semigroup ordering $>$ on the monomials of $K[x_1, ..., x_n]$, there is a natural extension of the leading data to $K[x_1, ..., x_n]_>$: If $f \in K[x_1, ..., x_n]_>$, then there is some $u \in K[x_1, ..., x_n]$ with $LT(u) = 1$ and $uf \in K[x_1, ..., x_n]$. The element $L(uf)$ is independent of the choice of u and is called $L(f)$, in the same way define $LT(f) := LT(uf)$ and $LC(f) := LC(uf)$.*

$LT(f)$ corresponds to a unique term in the power series expansion of f and subtracting this term gives the ***tail*** *of f denoted by $tail(f)$.*

Definition 1.169 *Given a semigroup ordering $>$ for any subset $G \subset K[x_1, ..., x_n]_>$ define the* ***lead ideal*** *of G as*

$$L(G) = {}_{K[x_1,...,x_n]}\langle L(g) \mid g \in G \setminus \{0\} \rangle$$

1.4.3 Normal forms

Fix a semigroup ordering $>$ on $K[x_1, ..., x_n]$ and let $R = K[x_1, ..., x_n]_>$.

Definition 1.170 *Let $\mathcal{G}$ be the set of all finite lists of elements in R. A map*

$$NF : R \times \mathcal{G} \to R$$

is called a ***weak normal form*** *on R if*

1. *$NF(0,G) = 0\ \forall G \in \mathcal{G}$*

2. *For all $G \in \mathcal{G}$ and $f \in R$*

$$NF(f,G) \neq 0 \Rightarrow L(NF(f,G)) \notin L(G)$$

3. *For all $G = \{g_1, ..., g_r\} \in \mathcal{G}$ and $f \in R$ there is a unit $u \in R^*$ with either*

   - *$uf = NF(f,G)$ or*
   - *$uf - NF(f,G) = \sum_{i=1}^r a_i g_i$ with $a_i \in R$ and for all i with $a_i g_i \neq 0$*

$$L(f) \geq L(a_i g_i)$$

Furthermore:

- *NF is called **normal form** if one can always take $u = 1$.*
- *NF is called **polynomial** if f and G are in $K[x_1, ..., x_n]$, then also u and a_i can be taken in $K[x_1, ..., x_n]$. A normal form is called **reduced** if no monomial of $NF(f,G)$ is divisible by some $L(g_i)$.*
- *If the above properties are satisfied for some fixed $G \in \mathcal{G}$ we call $NF(-,G) : R \to R$ a (weak, polynomial, reduced) **normal form with respect to** G.*

Remark 1.171 *If NF is polynomial, then $u \in R^* \cap K[x_1, ..., x_n] = S_>$.*

From any weak normal form NF, we can build a normal form by dividing by u, but the result will no longer be a polynomial.

Weak normal forms are introduced because they allow finite algorithmic computations in $R = K[x_1, ..., x_n]_>$ and in R a weak division expression $uf - NF(f,G) = \sum_{i=1}^r a_i g_i$ is as good as a division expression given by a normal form.

Definition 1.172 *If $f, g \in R \backslash \{0\}$, then their S**-polynomial** is*

$$\operatorname{SPolynomial}(f,g) = \frac{\operatorname{lcm}(L(f), L(g))}{L(f)} f - \frac{LC(f)}{LC(g)} \frac{\operatorname{lcm}(L(f), L(g))}{L(g)} g$$

Remark 1.173 *This is implemented in mora.m2 in the function* SPolynomial.

Algorithm 1.174 (Gröbner normal form) *[Decker, Schreyer, 2007], [Greuel, Pfister, 2002, Sec. 1.6] Let $>$ be a semigroup ordering and $G \in \mathcal{G}$.*

Let $\mathrm{divmon} := (h, G) \mapsto (g \in G \mid L(g) \text{ divides } L(h))$;

For any ordering of the sequences produced by divmon*, the following algorithm is a normal form $f \mapsto NFG(f, G) := h$ with respect to G.*

$h := f$;
while ($h \neq 0$ *and* $\mathrm{divmon}(h, G) \neq \emptyset$) do (
  $g := \mathrm{divmon}(h, G) \#0$;
  $h := \mathrm{SPolynomial}(h, g)$;
);
h;

This algorithm terminates if $>$ is a well ordering. Otherwise NFG may compute a power series convergent in the $\langle x_1, ..., x_n \rangle$-adic topology.

Remark 1.175 *This is implemented in* mora.m2 *in the function NFG.*

Algorithm 1.176 (Gröbner reduced normal form) *[Decker, Schreyer, 2007], [Greuel, Pfister, 2002, Sec. 1.6] Let $>$ be a semigroup ordering and $G \in \mathcal{G}$. The following algorithm is a reduced normal form $f \mapsto redNFG(f, G) := 1/LC(h) \cdot h$ with respect to G:*

$h := 0; g := f$;
while $g \neq 0$ do (
  $g := NFG(g, G)$;
  if $g \neq 0$ then (
    $h := h + LT(g)$;
    $g := g - LT(g)$;
  );
);
$1/LC(h) \cdot h$;

This algorithm terminates if $>$ is a well ordering.

Remark 1.177 *This is implemented in* mora.m2 *in the function $redNFG$.*

Remark 1.178 *If we apply NFG for the anti-degree order on $K[x]$, then dividing x by $x - x^2$, we get in $K[[x]]$*

$$x = \left(\sum_{k=0}^{\infty} x^k\right)\left(x - x^2\right) + 0$$

If we use a weak normal form, we can write

$$(1 - x) \cdot x = 1 \cdot \left(x - x^2\right) + 0$$

Algorithm 1.179 (Mora weak normal form) *[Mora, 1982], [Decker, Schreyer, 2007], [Greuel, Pfister, 2002, Sec. 1.6] Let $>$ be a semi-group ordering and $G \in \mathcal{G}$. For polynomial input and any ordering of the sequences produced by* mecart *the following algorithm $f \mapsto NFM(f, G) := h$ is a polynomial weak normal form with respect to G:*

Let

ecart $:= f \mapsto \deg(f) - \deg LM(f)$;

mecart $:= L \mapsto$ *the element of the sequence L with minimal ecart and minimal index;*

$h := f$;

$T := G$;

while ($h \neq 0$ *and* divmon$(h, T) \neq \emptyset$) do (

  $g :=$ mecart (divmon (h, T));

  if ecart $(g) >$ ecart (h) then $T :=$ append(T, h);

  $h :=$ SPolynomial(h, g);

);

h;

The algorithm terminates.

Remark 1.180 *This algorithm is implemented by the function NF of* mora.m2.

Remark 1.181 *The Mora algorithm allows reductions also by the results of previous reductions. In the above example*

$$x = 1 \cdot \left(x - x^2\right) + x^2$$

so we also allow reduction by x^2, i.e.,

$$x = 1 \cdot \left(x - x^2\right) + 1 \cdot x^2 + 0$$

which, as desired, can also be written

$$(1 - x) \cdot x = 1 \cdot \left(x - x^2\right) + 0$$

Remark 1.182 *For homogeneous input* ecart *is* 0*, hence NF and NFG agree (for the same choice of the ordering of the list produced by* divmon*).*

If $>$ is a well ordering, then any element h appended to T will not be used in further steps: If it would be used, then $L(h) \mid L(h_{new})$, hence $L(h) < L(h_{new})$ or $L(h) = L(h_{new})$, as $>$ is a well ordering. On the other hand the lead term of h was canceled in a previous step, so $L(h) > L(h_{new})$.

1.4.4 Standard bases

Let $R = K[x_1, ..., x_n]_>$ and fix a semigroup ordering $>$.

Definition 1.183 *Let $I \subset R$ be an ideal. A finite subset $G \subset I$ is called a **standard basis** (or **Gröbner basis** if $>$ is global) of I if $L(I) = L(G)$, equivalently, if for any $f \in I \setminus \{0\}$ there is some $g \in G$ with $L(g) \mid L(f)$.*

Proposition 1.184 *[Decker, Schreyer, 2007], [Greuel, Pfister, 2002, Sec. 1.6] Let $G \subset I \subset R$ be a standard basis of the ideal I and $NF(-, G)$ a weak normal form with respect to G, then:*

1. *For all $f \in R$ it holds*

$$f \in I \Leftrightarrow NF(f, G) = 0$$

2. *If $NF(-, G)$ is a reduced normal form, then it is uniquely determined by $>$ and I and denoted by $NF(-, I)$.*

Proposition 1.185 *[Decker, Schreyer, 2007], [Greuel, Pfister, 2002, Sec. 1.6] If $G \subset I \subset R$ is a standard basis of the ideal I and $NF(-, G)$ a weak normal form with respect to G, then it holds:*

1. *If $J \subset R$ is an ideal with $I \subset J$ and $L(I) = L(J)$, then $I = J$.*
2. *$I = \langle G \rangle$.*

Theorem 1.186 (Buchberger test) *[Decker, Schreyer, 2007], [Greuel, Pfister, 2002, Sec. 1.6] Let NF be a weak normal form, $G \in \mathcal{G}$ and $I \subset R$ an ideal. Then the following properties are equivalent:*

1. *G is a standard basis of I.*
2. *$I = \langle G \rangle$ and $NF(\text{SPolynomial}(g_i, g_j), G) = 0$ for all i, j.*
3. *$NF(f, G) = 0$ for all $f \in I$.*

This leads to the following algorithm:

Algorithm 1.187 (Gröbner basis, Standard basis) *[Mora, 1982], [Decker, Schreyer, 2007], [Greuel, Pfister, 2002, Sec. 1.6] Let NF be a weak normal form. Given $G \in \mathcal{G}$, the following algorithm computes a standard basis S of $\langle G \rangle \subset R$:*

$S := G;$

$P := \{(f, g) \mid f, g \in S, f \neq g\};$

while $P \neq \emptyset$ do (
  choose $(f,g) \in P$;
  $P := P \backslash \{(f,g)\}$;
  $h := NF(\text{SPolynomial}(f,g), S)$;
  if $h \neq 0$ then (
    $P := P \cup \{(h,f) \mid f \in S\}$;
    $S := S \cup \{h\}$;
  );
);
S;
This algorithm terminates.

Remark 1.188 *This algorithm is implemented in the library* mora.m2 *in the function* Std.

Remark 1.189 *Note that termination is only up to termination of NF. For a well-ordering NFG terminates, otherwise NFG may compute a power series convergent in the $\langle x_1,...,x_n\rangle$-adic topology. In this case we can use the Mora normal form instead, which for polynomial input will terminate with polynomial output, hence also the standard basis algorithm will.*

Given any semigroup ordering $>$ on the monomials of $K[x_1,...,x_n]$, the following ordering, introducing one additional variable s to homogenize the equations, can be used to compute standard bases via the Gröbner normal form.

Definition 1.190 *For $f \in K[x_1,...,x_n]$ of degree d define*

$$f^h = s^d f\left(\frac{x_1}{s},...,\frac{x_n}{s}\right) \in K[s,x_1,...,x_n]$$

to be its ***homogenization****.*

Definition 1.191 *Let $A \in GL(n,\mathbb{R})$ be the matrix associated to $>$ and the semigroup ordering on the monomials of $K[s,x_1,...,x_n]$ given by the matrix*

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}$$

i.e.,

$$s^a x^\alpha > s^b x^\beta \quad \Leftrightarrow \quad \begin{array}{l} a+|\alpha| > b+|\beta| \\ \textit{or} \\ a+|\alpha| = b+|\beta| \text{ and } x^\alpha > x^\beta \end{array}$$

Algorithm 1.192 (Lazard method) *[Greuel, Pfister, 2002, Sec. 1.7] Given polynomial* $G = \{g_1, ..., g_r\} \in \mathcal{G}$ *the following algorithm computes a standard basis* S *of* $\langle G \rangle \subset K[x_1, ..., x_n]$ *(note that we do not need* $R = K[x_1, ..., x_n]_>$ *coefficients):*

Using the Gröbner normal form NFG*, apply the standard basis algorithm to* $\left\{g_1^h, ..., g_r^h\right\}$ *with the induced monomial order from Definition 1.191 to compute* S *and put* $s = 1$.

Remark 1.193 *This algorithm is implemented in the library* mora.m2 *in the function* LStd.

Being a standard basis depends only on finitely many monomials.

Theorem 1.194 *[Greuel, Pfister, 2002, Sec. 1.7] For any ideal* $I \subset K[x_1, ..., x_n]$ *and standard basis* S *of* I *with respect to* $>$ *there is a finite set of monomials* F *(i.e., all monomials appearing in the Buchberger test computations) with the following property:*

For all monomial orders $>_1$ *identical to* $>$ *on* F

1. $L_>(g) = L_{>_1}(g)\ \forall g \in G$.

2. G *is also a standard basis with respect to* $>_1$.

Hence for computing standard bases any monomial order can be represented by an appropriate weight vector.

Now consider the question of uniqueness:

Definition 1.195 *A finite subset* $G \subset R$ *is called*

- ***interreduced*** *(or* **minimal***) if* $0 \notin G$ *and* $L(f) \nmid L(g)$ *for all* $f \neq g$.

- ***reduced*** *if it is interreduced and for all* f, g *no term of* $tail(g) \in K[[x_1, ..., x_n]]$ *is divisible by some* $L(f)$.

Remark 1.196 *If* $>$ *is global no term of* $tail(g)$ *is divisible by* $L(g)$*, hence* G *is reduced if for all* $f \neq g$ *no term of* g *is divisible by* $L(f)$.

By Proposition 1.185 the following algorithm computes an interreduced standard basis of I:

Algorithm 1.197 *Let* G *be a standard basis of* I. *Deleting successively all elements* g *with* $L(f) \mid L(g)$ *for some* $f \in G$, $f \neq g$ *leads to an interreduced standard basis of* I.

Remark 1.198 *This algorithm is implemented in* mora.m2 *in the function* MinimizeStd *and* MStd *computes an interreduced standard basis using Mora normal form and applying* MinimizeStd.

Algorithm 1.199 *If the input generators of I for the standard basis algorithm were reduced, and the standard basis algorithm used a reduced normal form NF, then after minimalization the output is also reduced. If the input was not reduced and $G = \{g_1, ..., g_n\}$ is the interreduced output of the standard basis algorithm using a reduced normal form NF, then*

$$M = \{NF_{>}(g_i, \{g_1, ..., g_{i-1}, g_{i+1}, ..., g_n\}) \mid i = 1, ..., n\}$$

is the reduced standard basis of I.

Remark 1.200 *For this setup $\{L(g_i) \mid i = 1, ..., n\}$ is the minimal generating set of $L(I)$ and*

$$M = \{NF_{>}(L(g_i), G) \mid i = 1, ..., n\}$$

Hence M is uniquely determined by I and $>$, as $NF_{>}(-, G)$ is.

Algorithm 1.201 *For this setup one may use as reduced normal form NF the reduced Gröbner normal form $redNFG_{>}$. If $>$ is global, then $redNFG_{>}$ terminates with an element in $K[x_1, ..., x_n]$, otherwise $redNFG_{>}$ computes an element in $K[[x_1, ..., x_n]]$ in general.*

If $G = \{g_1, ..., g_n\}$ is an interreduced standard basis of I with respect to $>$, computed using any weak normal form (e.g., Mora normal form), and NF is a reduced normal form (e.g., Gröbner normal form), then

$$\{L(g_i) + NF_{>}(tail(g_i), G) \mid i = 1, ..., n\} \subset K[[x_1, ..., x_n]]$$

is the unique reduced standard basis of I with respect $>$.

Remark 1.202 *This is implemented in* mora.m2 *in the function* ReduceGb. *For the non-global case the number of iterations can be limited by the global variable* iterlimit.

1.4.5 Localization in prime ideals

Proposition 1.203 *[Greuel, Pfister, 2002, Sec. 1.5] Let K be a field, $>$ a local ordering on the polynomial ring $K[x_1, ..., x_n]$. Then the localization of the polynomial ring $K[x_1, ..., x_n, y_1, ..., y_m]$ at the prime ideal $\langle x_1, ..., x_n\rangle$ is the localization of $K(y_1, ..., y_m)[x_1, ..., x_n]$ with respect to $>$*

$$K(y_1, ..., y_m)[x_1, ..., x_n]_{>} = K[x_1, ..., x_n, y_1, ..., y_m]_{\langle x_1, ..., x_n\rangle}$$

Recall that

$$\begin{aligned} K(y_1,...,y_m)[x_1,...,x_n]_> &= S_>^{-1}(K(y_1,...,y_m)[x_1,...,x_n]) \\ &= \left\{ \frac{f}{u} \mid f,u \in K(y_1,...,y_m)[x_1,...,x_n], L(u)=1 \right\} \end{aligned}$$

with

$$S_> = \{u \in K(y_1,...,y_m)[x_1,...,x_n] \setminus \{0\} \mid L(u)=1\}$$

This allows one to do Gröbner computations in localizations at prime ideals $\langle x_1,...,x_n\rangle$, e.g., at the ideals of the strata of a toric variety in the Cox ring.

2 Mirror constructions to generalize

2.1 Batyrev's construction for hypersurfaces in toric varieties

Let $Y = \mathbb{P}(\Delta)$ be a toric Fano variety of $\dim Y = n$ represented by the reflexive polytope $\Delta \subset M_{\mathbb{R}}$ and let $N = \mathrm{Hom}(M,\mathbb{Z})$.

Proposition 2.1 *[Batyrev, 1994], [Cox, Katz, 1999, Sec. 4.1.1], [Reid, 1980] A general element in $\left|-K_{\mathbb{P}(\Delta)}\right|$ is a Calabi Yau variety of dimension $n-1$.*

Theorem 2.2 *[Batyrev, 1994], [Batyrev, Borisov, 1996-I] For any reflexive Δ general elements X of $\left|-K_{\mathbb{P}(\Delta)}\right|$ and X° of $\left|-K_{\mathbb{P}(\Delta^*)}\right|$ are stringy topological mirror pairs (indeed mathematical mirror pairs), and there are explicit formulas computing $h_{st}^{d-1,1}(X)$ and $h_{st}^{1,1}(X)$ from the polytope:*

$$\begin{aligned} h_{st}^{d-1,1}(X) = |\Delta \cap M| - n - 1 - \sum_{Q \text{ facet of } \Delta} |\mathrm{int}_M(Q)| \\ + \sum_{\substack{Q \text{ face of } \Delta \\ \operatorname{codim} Q = 2}} |\mathrm{int}_M(Q)| \cdot |\mathrm{int}_N(Q^*)| \end{aligned} \tag{2.1}$$

$$\begin{aligned} h_{st}^{1,1}(X) = |\Delta^* \cap M| - n - 1 - \sum_{Q^* \text{ facet of } \Delta^*} |\mathrm{int}_N(Q^*)| \\ + \sum_{\substack{Q^* \text{ face of } \Delta^* \\ \operatorname{codim} Q^* = 2}} |\mathrm{int}_N(Q^*)| \cdot |\mathrm{int}_M(Q)| \end{aligned}$$

Here, $\mathrm{int}_M(Q)$ denotes the set of lattice points in the relative interior of the face Q with respect to the lattice M.

Example 2.3 *As discussed in Examples 1.68 and 1.88 the reflexive degree* 5 *Veronese polytope* Δ *of* $\mathbb{P}^4$ *and its dual yield* $\mathbb{P}(\Delta) = \mathbb{P}^4$ *and* $\mathbb{P}(\Delta^*) = \mathbb{P}^4/\mathbb{Z}_5^3$. *General anticanonical hypersurfaces* X *and* X° *inside satisfy*

$$h^{1,1}(X) = h_{st}^{2,1}(X^\circ) = (6-1) - 4 = 1$$
$$h^{2,1}(X) = h_{st}^{1,1}(X^\circ) = (126-1) - 24 = 101$$

As noticed in Example 1.101, X *is given by a general element in* $S_{\left[-K_{\mathbb{P}(\Delta)}\right]}$, *which is a general degree* 5 *polynomial in* $\mathbb{C}[x_1, ..., x_5]$ *and* X° *by a general element in* $S_{\left[-K_{\mathbb{P}(\Delta^*)}\right]}$, *which is*

$$\sum_{i=1}^{5} c_i y_i^5 + c_0 y_1 \cdot ... \cdot y_5$$

Modulo automorphisms of $\mathbb{P}(\Delta^*)$, *this is the one parameter family obtained from the Greene-Plesser construction, see Example 1.9.*

The singularities may be resolved crepantly via maximal projective subdivisions $\bar{\Sigma}$ of the fan Σ.

Definition 2.4 *Given a reflexive polytope* $\Delta \subset M_{\mathbb{R}}$, *a fan* $\bar{\Sigma}$ *in* $N_{\mathbb{R}}$ *is called a* ***projective subdivision*** *of the normal fan* Σ *of* Δ *if*

1. $\bar{\Sigma}$ *refines* Σ.
2. $\bar{\Sigma}(1) \subset \Delta^* \cap N - \{0\}$
3. $X\left(\bar{\Sigma}\right)$ *is projective and simplicial.*

$\bar{\Sigma}$ *is called* ***maximal*** *if* $\bar{\Sigma}(1) = \Delta^* \cap N - \{0\}$.

Proposition 2.5 *[Oda, Park, 1991], [Cox, Katz, 1999, 4.1.1], [Cox, Katz, 1999, 3.4] For any reflexive* Δ, *there exists a maximal projective subdivision of the normal fan* Σ *of* Δ.

Proposition 2.6 *[Batyrev, 1994], [Cox, Katz, 1999, 4.1.1] For any reflexive* Δ, *any projective subdivision* $\bar{\Sigma}$ *of the normal fan* Σ *of* Δ *gives a birational morphism*

$$f : X\left(\bar{\Sigma}\right) \to \mathbb{P}(\Delta)$$

and

1. $X\left(\bar{\Sigma}\right)$ *is a Gorenstein orbifold.*

2. *f is crepant, i.e., $f^* K_{\mathbb{P}(\Delta)} = K_{X(\bar{\Sigma})}$.*

3. *If the subdivision $\bar{\Sigma}$ is maximal, then $X\left(\bar{\Sigma}\right)$ has terminal singularities.*

Furthermore, for a general element $\bar{X}$ in $\left|-K_{X(\bar{\Sigma})}\right|$ it holds:

1. *$\bar{X}$ is a Calabi-Yau orbifold.*

2. *If the subdivision $\bar{\Sigma}$ is maximal, then $\bar{X}$ is called a **maximal projective crepant partial (MPCP) desingularization** of X and has the following properties:*

   (a) *$\bar{X}$ is a minimal Calabi-Yau orbifold in the sense of Mori theory.*

   (b) *$\bar{X}$ is the proper transform by f of a general element X in $\left|-K_{\mathbb{P}(\Delta)}\right|$. The induced map $f : \bar{X} \to X$ is crepant.*

   (c) *If $\dim X = 3$, then $\bar{X}$ is smooth, as Gorenstein orbifold terminal singularities are smooth in dimension 3.*

Example 2.7 *Consider the weighted projective space $\mathbb{P}(1,1,2,2,2)$ given by the fan Σ over*

$$\Delta^* = \operatorname{convexhull}((-1,-2,-2,-2),(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1))$$

hence, via the Cox ring it has the description

$$\mathbb{P}(1,1,2,2,2) \cong \frac{\mathbb{C}^5 - V(\langle y_0, ..., y_4 \rangle)}{\mathbb{C}^*}$$

with $\mathbb{C}^*$-action

$$\lambda(y_0, ..., y_4) = \left(\lambda y_0, \lambda y_1, \lambda^2 y_2, \lambda^2 y_3, \lambda^2 y_4\right)$$

A toric variety given by a fan Σ is smooth if and only if for every cone in Σ the minimal lattice generators are a subset of a $\mathbb{Z}$-basis of the lattice N. So $\operatorname{hull}((-1,-2,-2,-2),(1,0,0,0))$ *is singular. The only lattice point of Δ^*, which is not a vertex is*

$$\frac{1}{2}((-1,-2,-2,-2)+(1,0,0,0)) = (0,-1,-1,-1)$$

Hence, considering the fan $\bar{\Sigma}$ obtained by splitting each of the 3 maximal dimensional cones of Σ containing $(0,-1,-1,-1)$ into two cones via the new ray generated by $(0,-1,-1,-1)$, we obtain a MPCP desingularization *and indeed a smooth toric variety* $X\left(\bar{\Sigma}\right)$*, which is a blowup of* $\mathbb{P}(1,1,2,2,2)$*. Via its Cox ring,* $X\left(\bar{\Sigma}\right)$ *is given by*

$$X\left(\bar{\Sigma}\right) \cong \frac{\mathbb{C}^6 - V\left(\langle y_0, y_1\rangle \cap \langle y_2 ..., y_4, y_5\rangle\right)}{\left(\mathbb{C}^*\right)^2}$$

with $\left(\mathbb{C}^*\right)^2$*-action*

$$(\lambda, \mu)\left(y_0, ..., y_5\right) = \left(\lambda y_0, \lambda y_1, \mu y_2, \mu y_3, \mu y_4, \frac{\mu}{\lambda^2} y_5\right)$$

2.2 Batyrev's and Borisov's construction for complete intersections in toric varieties

Consider a toric Fano variety $Y = \mathbb{P}(\Delta)$ represented by the reflexive polytope $\Delta \subset M_{\mathbb{R}}$ with normal fan $\Sigma \subset N_{\mathbb{R}}$. A disjoint union

$$\Sigma(1) = I_1 \cup ... \cup I_c$$

is called a **nef partition** if all $E_j = \sum_{v \in I_j} D_v$ are Cartier, spanned by global sections. By $\sum_{j=1}^{c} E_j = -K_Y$, general sections of $\mathcal{O}(E_1), ..., \mathcal{O}(E_c)$ give a Calabi-Yau complete intersection $X \subset Y$.

Proposition 2.8 *[Batyrev, Borisov, 1996-II] The polytopes* $\Delta_j = \Delta_{E_j}$ *of sections of* E_j *are lattice polytopes (see Section 1.3.4), and it holds*

$$\Delta = \Delta_1 + ... + \Delta_c$$

Example 2.9 *Let*

$$\Delta = \text{convexhull}\left((-1,-1,-1), (3,-1,-1), (-1,3,-1), (-1,-1,3)\right)$$

be the reflexive degree 4 *Veronese polytope of* $\mathbb{P}^3$*. By the partition of the* 4 *vertices of* Δ^* *into* I_1 *and* I_2 *each with* 2 *elements*

$$\begin{aligned} I_1 &= \{(-1,-1,-1), (0,0,1)\} \\ I_2 &= \{(1,0,0), (0,1,0)\} \end{aligned}$$

a general $(2,2)$ *complete intersection elliptic curve in* $\mathbb{P}^3$ *is given and*

$$\begin{aligned} \Delta_1 &= \text{convexhull}\left((1,-1,0), (-1,-1,0), (-1,-1,2), (-1,1,0)\right) \\ \Delta_2 &= \text{convexhull}\left((0,0,-1), (0,0,1), (2,0,-1), (0,2,-1)\right) \end{aligned}$$

are degree 2 *Veronese polytopes, which add up to* $\Delta = \Delta_1 + \Delta_2$*, as shown in Figure 2.1.*

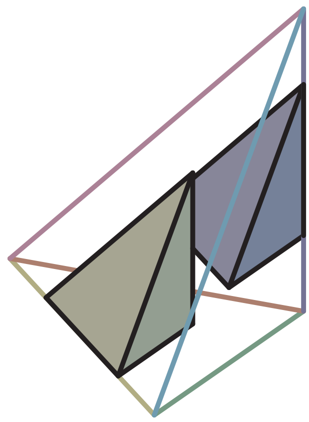


Figure 2.1: Batyrev-Borisov polytopes Δ_1, Δ_2 and their Minkowski sum Δ for the $(2,2)$ complete intersection elliptic curve in $\mathbb{P}^3$

Define the lattice polytopes

$$\nabla_j = \text{convexhull}\left\{\{0\} \cup I_j\right\}$$

and define ∇_{BB} by

$$\nabla_{BB}^* = \text{convexhull}\left(\Delta_1 \cup ... \cup \Delta_c\right)$$

Proposition 2.10 *[Batyrev, Borisov, 1996-II]* $\nabla_{BB} = \nabla_1 + ... + \nabla_c$.

In particular ∇_{BB} is a lattice polytope containing 0, hence:

Corollary 2.11 ∇_{BB} *is reflexive.*

Example 2.12 *In the above Example 2.9*

$$\begin{aligned} \nabla_1 &= \text{convexhull}\left\{(0,0,0), (-1,-1,-1), (0,0,1)\right\} \\ \nabla_2 &= \text{convexhull}\left\{(0,0,0), (1,0,0), (0,1,0)\right\} \end{aligned}$$

Figure 2.2 shows the polytopes ∇_1, ∇_1 *and their Minkowski sum* ∇_{BB}.

Let $Y^\circ = \mathbb{P}(\nabla_{BB})$ be the toric Fano variety associated to ∇_{BB}. Then

$$\sum_{j=1}^{c} D_{\nabla_j} = -K_{Y^\circ}$$

is a nef partition, and X° given by general sections of $\mathcal{O}(D_{\nabla_1}), ..., \mathcal{O}(D_{\nabla_c})$ is a Calabi-Yau complete intersection in Y°.

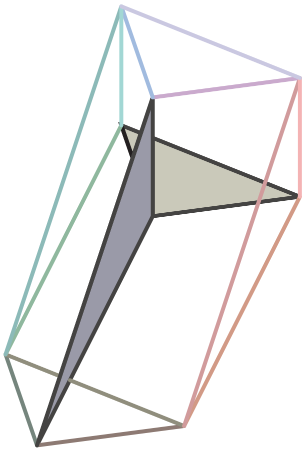


Figure 2.2: Batyrev-Borisov polytopes ∇_1, ∇_2 and their Minkowski sum ∇ for the mirror of the $(2,2)$ complete intersection elliptic curve in $\mathbb{P}^3$

Theorem 2.13 *[Batyrev, Borisov, 1996-I] X and X° form a stringy topological mirror pair.*

A maximal projective subdivision $\bar{\Sigma}$ of $\Sigma = \mathrm{NF}(\Delta)$ gives a maximal projective partial crepant desingularization

$$f : X\left(\bar{\Sigma}\right) \to \mathbb{P}(\Delta)$$

such that the T-divisors of the projective toric variety $X\left(\bar{\Sigma}\right)$ correspond to the lattice points of the boundary of Δ^*. Then f induces a resolution $\bar{X} \to X$ of the complete intersection $X \subset \mathbb{P}(\Delta)$ such that $\bar{X}$ is a complete intersection, has at most Gorenstein terminal abelian quotient singularities and $K_{\bar{X}} = \mathcal{O}_{\bar{X}}$, for a reference see [Batyrev, Borisov, 1996-II]. In particular, if $\dim\left(\bar{X}\right) \leq 3$, then $\bar{X}$ is smooth.

2.3 Rødland's orbifolding mirror construction for the degree 14 Pfaffian Calabi-Yau threefold in $\mathbb{P}^6$

Consider a 7-dimensional $\mathbb{C}$-vector space V and the trivial vector bundle $\mathcal{V}$ with fiber V on $\mathbb{P}\left(\bigwedge^2 V\right)$. Define M as the degeneracy locus of the universal skew symmetric linear map

$$\alpha : \mathcal{V}^*(-1) \to \mathcal{V}$$

i.e., as the locus $\mathrm{rank}\,\alpha \leq 4$.

M is given by the 6×6 Pfaffians of α, is locally Gorenstein of codimension 3 in $\mathbb{P}\left(\bigwedge^2 V\right)$ and has $K_M = \mathcal{O}_M(-14)$. Its singular locus is given by $\operatorname{rank} \alpha \leq 2$ and has codimension 7 in M.

Intersecting M with a general $\mathbb{P}^{d+3} \subset \mathbb{P}\left(\bigwedge^2 V\right)$ gives $X^d = M \cap \mathbb{P}^{3+d}$ of dimension d and $K_{X^d} = \mathcal{O}_{X^d}(3-d)$ and X^d is smooth for $d \leq 6$.

$X = X^3$ is a local complete intersection Calabi-Yau threefold with $h^{1,1}(X) = 1$ and $h^{1,2}(X) = 50$:

Remark 2.14 *By*

$$0 \to \mathcal{O}_{\mathbb{P}^6}(-7) \to \mathcal{O}_{\mathbb{P}^6}(-4)^{\oplus 7} \to \mathcal{O}_{\mathbb{P}^6}(-3)^{\oplus 7} \to \mathcal{O}_{\mathbb{P}^6} \to \mathcal{O}_X$$

we get

$$H^i(X, \mathcal{O}_X) \cong H^{i+3}\left(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(-7)\right) \cong \left\{ \begin{array}{cc} 0 & i = 1,2 \\ \mathbb{C} & i = 3 \end{array} \right\}$$

and using the resolution of $\mathcal{J}^2_{X_I}$, the Euler sequence, the definition of the conormal sheaf and the conormal sequence

$$\begin{array}{l} 0 \to \mathcal{O}_{\mathbb{P}^6}(-8)^{\oplus 21} \to \mathcal{O}_{\mathbb{P}^6}(-7)^{\oplus 48} \to \mathcal{O}_{\mathbb{P}^6}(-6)^{\oplus 28} \to \mathcal{J}^2_X \to 0 \\ 0 \to \Omega_{\mathbb{P}^6}\mid_X \to \mathcal{O}_X(-1)^{\oplus 7} \to \mathcal{O}_X \to 0 \\ 0 \to \mathcal{J}^2_X \to \mathcal{J}_X \to N^\vee_{X/\mathbb{P}^6} \to 0 \\ 0 \to N^\vee_{X/\mathbb{P}^6} \to \Omega_{\mathbb{P}^6}\mid_X \to \Omega_X \to 0 \end{array}$$

one computes

$$\begin{array}{l} h^{1,1}(X) = 1 \\ h^{1,2}(X) = 50 \end{array}$$

The mirror is constructed via Greene-Plesser orbifolding in an analogous way to [Greene, Plesser, 1992] and Example 1.9. As one expects for the mirror to hold $h^{1,2}(X^\circ) = h^{1,1}(X) = 1$, to apply Greene-Plesser orbifolding, one looks for a 1-parameter subfamily, i.e., a 1-parameter family of $\mathbb{P}^6 \subset \mathbb{P}\left(\bigwedge^2 V\right)$. Choosing a basis (e_i) of V Rødland considers the action on $\mathbb{P}\left(\bigwedge^2 V\right)$ of the group $G = \langle \sigma, \tau \rangle \subset \operatorname{Aut} \mathbb{P}\left(\bigwedge^2 V\right)$ of order 49 generated by

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\tau = \operatorname{diag}\left(\zeta^i\right)_{i=0,\ldots,6}$$

where ζ is a 7-th root of unity. Taking coordinates on $\mathbb{P}^6$, the subfamily X_y invariant under the action of G is given by the skew symmetric matrix

$$A_y = \begin{pmatrix} 0 & y_1x_1 & y_2x_2 & y_3x_3 & -y_3x_4 & -y_2x_5 & -y_1x_6 \\ -y_1x_1 & 0 & y_1x_3 & y_2x_4 & y_3x_5 & -y_3x_6 & -y_2x_0 \\ -y_2x_2 & -y_1x_3 & 0 & y_1x_5 & y_2x_6 & y_3x_0 & -y_3x_1 \\ -y_3x_3 & -y_2x_4 & -y_1x_5 & 0 & y_1x_0 & y_2x_1 & y_3x_2 \\ y_3x_4 & -y_3x_5 & -y_2x_6 & -y_1x_0 & 0 & y_1x_2 & y_2x_3 \\ y_2x_5 & y_3x_6 & -y_3x_0 & -y_2x_1 & -y_1x_2 & 0 & y_1x_4 \\ y_1x_6 & y_2x_0 & y_3x_1 & -y_3x_2 & -y_2x_3 & -y_1x_4 & 0 \end{pmatrix} \quad (2.2)$$

with $(y_1 : y_2 : y_2) \in \mathbb{P}^2$, and its general element has the 49 double points

$$G \cdot (0 : y_1 : y_2 : y_3 : -y_3 : -y_2 : -y_1)$$

The induced action on $\mathbb{P}^6$ is given by

$$\sigma(x_i) = x_{(i+2) \bmod 7} \qquad \tau(x_i) = \zeta^{2i} x_i$$

Let $H = \langle \tau \rangle$ and consider the $\mathbb{P}^1$-subfamily $X_s = X_{(0:1:s)}$. The general element of X_s has 56 double points and X_s degenerates for $s = 0, \infty$ into a configuration of 14 $\mathbb{P}^3$.

Theorem 2.15 *[Rødland, 1998] The quotient of the general X_s by H has a crepant resolution $\widetilde{X_s/H}$, and the Hodge numbers of $\widetilde{X_s/H}$ coincide with the mirrored Hodge numbers of the general X.*

Rødland conjectured [Rødland, 1998] and Tjøtta [Tjøtta, 2000] proved that the Picard-Fuchs equation of $\widetilde{X_s/H}$ at $s = \infty$ coincides with the A-model of the general degree 14 Pfaffian $X \subset \mathbb{P}^6$.

3 Degenerations and mirror symmetry

Degenerations to monomial ideals in toric varieties play an important role in almost all known mirror constructions. For the concept of flat families see Section 5.

3.1 Degenerations associated to complete intersections in toric varieties

We want to associate to any complete intersection inside a toric variety $\mathbb{P}(\Delta)$, given by a nef partition and represented by an ideal in the Cox ring S of $\mathbb{P}(\Delta)$, a monomial degeneration:

Suppose

$$\Sigma(1) = I_1 \cup \ldots \cup I_c$$

i.e.,

$$-K_Y = \sum_{j=1}^{c} \overbrace{\sum_{v \in I_j} D_v}^{E_j}$$

is a nef partition, i.e., all E_j are Cartier, spanned by global sections.

Example 3.1 *Consider $I \subset \mathbb{C}[t] \otimes S$ defined as*

$$\begin{aligned} m_j &= \prod_{v \in I_j} y_v \qquad (3.1) \\ I_0 &= \langle m_j \mid j = 1, \ldots, c \rangle \\ I &= \langle f_j = t \cdot g_j + m_j \mid j = 1, \ldots, c \rangle \subset \mathbb{C}[t] \otimes S \end{aligned}$$

where $g_j \in S_{[E_j]}$ corresponds to a general section of $\mathcal{O}(E_j)$, i.e., a general linear combination of the lattice points of Δ_{E_j} for $j = 1, \ldots, c$. Then I defines a flat degeneration $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[[t]]$ of a Calabi-Yau complete intersection in $Y = \mathbb{P}(\Delta)$, given by general sections of $\mathcal{O}(E_1), \ldots, \mathcal{O}(E_c)$, to the monomial special fiber given by I_0.

We may assume that the f_j are reduced with respect I_0 in the sense of Gröbner bases. Flatness of this family will be discussed in Section 8.1.

Example 3.2 *In particular, a degeneration $\mathfrak{X}$ of a Calabi-Yau hypersurface X in $Y = \mathbb{P}(\Delta)$, defined by a general section of $-K_Y$, to the monomial special fiber defined by $\left\langle \prod_{v \in \Sigma(1)} y_v \right\rangle$ is given by*

$$I = \left\langle t \cdot \langle Am \mid m \in \partial\Delta \rangle + \prod_{v \in \Sigma(1)} y_v \mid j = 1, \ldots, c \right\rangle \subset \mathbb{C}[t] \otimes S$$

Example 3.3 *The partitions for the above Example 2.9 induce degenerations given by the following ideals:*

1. *With variables $x_0, \ldots, x_3$ of the Cox ring S of $\mathbb{P}(\Delta) = \mathbb{P}^3$ corresponding to the vertices of Δ^* consider the ideal*

   $$I = \langle t \cdot g_1 + x_1 x_2,\ t \cdot g_2 + x_0 x_3 \rangle \subset \mathbb{C}[t] \otimes S$$

   where $g_1, g_2 \in \mathbb{C}[x_1, \ldots, x_4]_2$ are general not involving monomials in $I_0 = \langle x_1 x_2, x_0 x_3 \rangle$. The ideal I defines a flat degeneration $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[[t]]$ over $\operatorname{Spec} \mathbb{C}[[t]]$ of an elliptic curve X given as the complete intersection of two quadrics in $\mathbb{P}^3$ to the monomial special fiber defined by I_0.

2. With variables $y_1, ..., y_8$ of the Cox ring S° of $Y^\circ = \mathbb{P}(\nabla)$ corresponding to the vertices of ∇^* consider the ideal

$$\begin{aligned} I^\circ = \quad & \langle t \cdot (c_1 \cdot y_4^2 y_8^2 + c_2 \cdot y_3^2 y_6^2) + y_1 y_2 y_3 y_4, \\ & t \cdot (c_3 \cdot y_1^2 y_5^2 + c_4 \cdot y_2^2 y_7^2) + y_5 y_6 y_7 y_8 \rangle \quad \subset \mathbb{C}[t] \otimes S^\circ \end{aligned}$$

with general coefficients c_i. The ideal I° defines a flat degeneration $\mathfrak{X} \subset Y^\circ \times \operatorname{Spec} \mathbb{C}[[t]]$ of the mirror X° of X to the monomial ideal

$$I_0^\circ = \langle y_1 y_2 y_3 y_4, y_5 y_6 y_7 y_8 \rangle$$

Note that the subvariety of Y° defined by the ideal I_0° decomposes into 4 one-dimensional toric strata intersecting in 4 zero-dimensional strata.

The stratification of the vanishing locus of reduced monomial ideals in the Cox ring of a toric variety is explored in detail in Section 9.3.

3.2 Degenerations of Pfaffian Calabi-Yau manifolds

Flatness of the following Pfaffian degenerations, which is obtained from the structure theorem of Buchsbaum and Eisenbud [Buchsbaum, Eisenbud, 1977], is explored in Section 10.1.

Example 3.4 *Let S be the Cox ring of $\mathbb{P}^4$, i.e., the homogeneous coordinate ring of $\mathbb{P}^4$. By the 4×4 Pfaffians in $\mathbb{C}[t] \otimes S$ of the matrix*

$$A_t = t \cdot A + A_0$$

where

$$A_0 = \begin{pmatrix} 0 & 0 & x_1 & -x_4 & 0 \\ 0 & 0 & 0 & x_2 & -x_0 \\ -x_1 & 0 & 0 & 0 & x_3 \\ x_4 & -x_2 & 0 & 0 & 0 \\ 0 & x_0 & -x_3 & 0 & 0 \end{pmatrix}$$

and A is a general skew symmetric 5×5 matrix linear in $x_0, ..., x_4$, we obtain a flat degeneration $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[[t]]$ over $\operatorname{Spec} \mathbb{C}[[t]]$ of a generic Pfaffian elliptic curve in $\mathbb{P}^4$ to the monomial special fiber given by the 4×4 Pfaffians of A_0.

Recall that for a skew symmetric matrix A the determinants of the matrices A_j obtained by deleting the j-th row and column are squares, and the $\sqrt{\det A_j}$ are called the Pfaffians of A. For details on Pfaffian varieties see Section 10.1.

Example 3.5 *Let H be the group given in Section 2.3. The Cox ring of the quotient of $\mathbb{P}^6$ by H is the polynomial ring $S = \mathbb{C}[x_0, ..., x_6]$. By the 6×6 Pfaffians in $\mathbb{C}[t] \otimes S$ of*

$$A_t = \begin{pmatrix} 0 & tx_1 & x_2 & 0 & 0 & -x_5 & -tx_6 \\ -tx_1 & 0 & tx_3 & x_4 & 0 & 0 & -x_0 \\ -x_2 & -tx_3 & 0 & tx_4 & x_6 & 0 & 0 \\ 0 & -x_4 & -tx_4 & 0 & tx_0 & x_1 & 0 \\ 0 & 0 & -x_6 & -tx_0 & 0 & tx_2 & x_3 \\ x_5 & 0 & 0 & -x_1 & -tx_2 & 0 & tx_4 \\ tx_6 & x_0 & 0 & 0 & -x_3 & -tx_4 & 0 \end{pmatrix}$$

a flat degeneration $\mathfrak{X} \subset (\mathbb{P}^6/H) \times \operatorname{Spec} \mathbb{C}[[t]]$ over $\operatorname{Spec} \mathbb{C}[[t]]$ with monomial special fiber is given.

This is the one parameter family used to construct the mirror of a general degree 14 Pfaffian Calabi-Yau threefold in $\mathbb{P}^6$ via Greene-Plesser orbifolding by H.

Example 3.6 *Let S be the homogeneous coordinate ring of $\mathbb{P}^6$. By the 6×6 Pfaffians in $\mathbb{C}[t] \otimes S$ of*

$$A_t = t \cdot A + A_0$$

where

$$A_0 = \begin{pmatrix} 0 & 0 & x_2 & 0 & 0 & -x_5 & 0 \\ 0 & 0 & 0 & x_4 & 0 & 0 & -x_0 \\ -x_2 & 0 & 0 & 0 & x_6 & 0 & 0 \\ 0 & -x_4 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & -x_6 & 0 & 0 & 0 & x_3 \\ x_5 & 0 & 0 & -x_1 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 & -x_3 & 0 & 0 \end{pmatrix}$$

and A is a general skew symmetric 7×7 matrix linear in $x_0, ..., x_6$, one obtains a flat degeneration $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[[t]]$ over $\operatorname{Spec} \mathbb{C}[[t]]$ of a general degree 14 Pfaffian Calabi-Yau threefold in $\mathbb{P}^6$ to the monomial special fiber, given by the 6×6 Pfaffians of A_0.

Example 3.7 *Let S be the homogeneous coordinate ring of $\mathbb{P}^6$. The 5×5 Pfaffians in $\mathbb{C}[t] \otimes S$ of $A_t = t \cdot A + A_0$, where*

$$A_0 = \begin{pmatrix} 0 & 0 & x_3x_4 & -x_1x_2 & 0 \\ 0 & 0 & 0 & x_7 & x_6 \\ -x_3x_4 & 0 & 0 & 0 & -x_5 \\ x_1x_2 & -x_7 & 0 & 0 & 0 \\ 0 & -x_6 & x_5 & 0 & 0 \end{pmatrix}$$

and A *is a general skew symmetric map* $\mathcal{E}^*(-1) \to \mathcal{E}$ *with*

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^6}(1) \oplus \mathcal{O}_{\mathbb{P}^6}^4$$

one obtains a flat degeneration $\mathfrak{X} \subset Y \times \operatorname{Spec}\mathbb{C}[[t]]$ *over* $\operatorname{Spec}\mathbb{C}[[t]]$ *of a general degree* 13 *Pfaffian Calabi-Yau threefold in* $\mathbb{P}^6$ *with monomial special fiber given by the* 5×5 *Pfaffians of* A_0*.*

The special fiber X_0 *is obtained from a simplicial* 4*-polytope with* 7 *vertices given in [Grünbaum, Sreedharan, 1967].*

For a remark on more monomial Calabi-Yau ideals obtained in this way see Section 13.5.

4 Tropical geometry ingredients

Tropical geometry will be interpreted as a tool to explore one parameter degenerations inside toric varieties, as it associates to such a degeneration a combinatorial object.

4.1 Amoebas

Definition 4.1 *Let* Y *be a toric variety with torus* $(\mathbb{C}^*)^n$ *and* $V \subset Y$ *a subvariety. The* ***amoeba*** *of* V *is given as the image of* V *under*

$$\begin{array}{c} \log : (\mathbb{C}^*)^n \to \mathbb{R}^n \\ (z_1, ..., z_n) \mapsto (\log|z_1|, ..., \log|z_n|) \end{array}$$

Remark 4.2 *The amoeba can be considered as a subset of a lower half sphere via*

$$\begin{array}{ccl} \mathbb{R}^n & \to & S^n \cap \{w_t \leq 0\} \\ & & = \{(w_t, w_1, ..., w_n) \in \mathbb{R}^{n+1} \mid w_t^2 + w_1^2 + ... + w_n^2 = 1,\ w_t \leq 0\} \\ (w_1, ..., w_n) & \mapsto & \frac{1}{\|(-1, w_1, ..., w_n)\|}(-1, w_1, ..., w_n) \end{array}$$

We refer to the points on the equator of the sphere, i.e., the points with $w_t = 0$*, as the* ***points at infinity*** *of the amoeba.*

Example 4.3 *The amoeba of the line* $L = \{2x + y + 1\}$*, shown in Figure 4.1, is the image of*

$$\begin{array}{llllllll} p: & \mathbb{R}_{\geq 0} \times [0, 2\pi[& \to & \mathbb{C} & \to & \mathbb{C}^2 & & \to \\ & (r, \varphi) & \mapsto & re^{i\varphi} & \mapsto & (re^{i\varphi}, -1 - 2re^{i\varphi}) & & \mapsto \end{array}$$

$$\begin{array}{llll} \to & \mathbb{R}^2 & \to & S^2 \cap \{w_t \leq 0\} \\ \mapsto & (\log r, \log|1 + 2re^{i\varphi}|) & \mapsto & \frac{(-1, \log r, \log|1+2re^{i\varphi}|)}{\|(-1, \log r, \log|1+2re^{i\varphi}|)\|} \end{array}$$

Considered as a subset of a lower half sphere via the last map the points at infinity are

$$\lim_{r\to 0} p(r,\varphi) = (0,1,0)$$
$$\lim_{r\to \infty} p(r,\varphi) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$
$$\lim_{(r,\varphi)\to\left(\frac{1}{2},\pi\right)} p(r,\varphi) = (0,1,0)$$

The amoeba of the conic $\{x^2 + 2y^2 - 3xy + x + y - 1 = 0\}$ *is shown in Figure 4.2.*

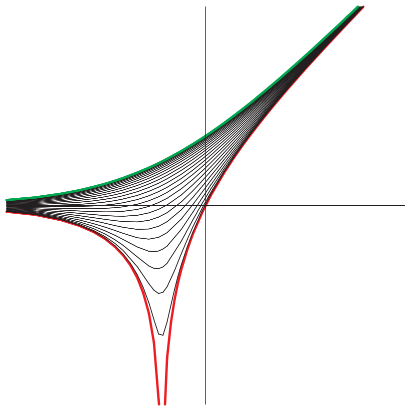


Figure 4.1: Amoeba of a line

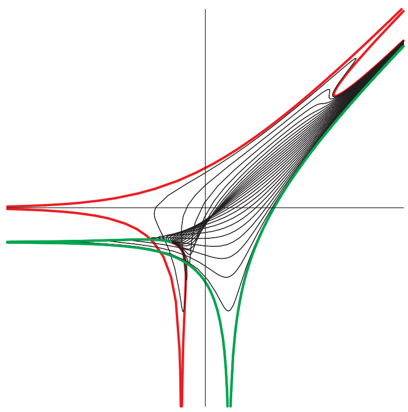


Figure 4.2: Amoeba of a conic

Example 4.4 *Replacing* log *by* $\log_t$ *the amoeba is rescaled, Figure 4.3 shows the limit* $t \to \infty$ *of both amoebas given in Example 4.3. For the conic one has to assign multiplicity* 2 *to each leg.*

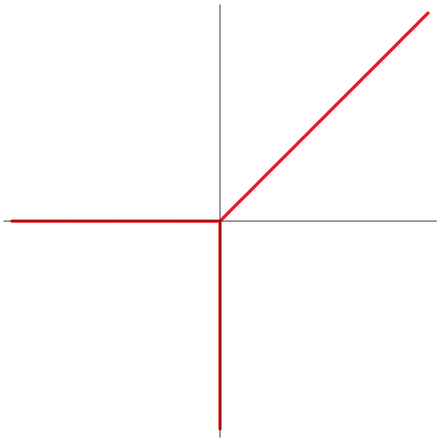


Figure 4.3: Limit amoeba

This limit process can be formalized in the following way:

4.2 Non-Archimedian amoebas

Consider the field of Puiseux series $\mathbb{C}\{\{t\}\}$, which is equipped with a valuation

$$val : \mathbb{C}\{\{t\}\} \to \mathbb{Q} \cup \{\infty\}$$
$$\sum_{j \in J} \alpha_j t^j \mapsto \min J$$

satisfying $val(f+g) \geq \min\{val(f), val(g)\}$, and with a norm $\|f\| = e^{-val(f)}$. Consider further the metric completion K of $\mathbb{C}\{\{t\}\}$ containing those elements $\sum_{j \in J} \alpha_j t^j$, which satisfy the condition that any subset of J has a minimum. Denote the corresponding valuation and norm on K again by val and $\|-\|$. K is a complete algebraically closed non-Archimedian field with surjective valuation

$$val : K \to \mathbb{R} \cup \{\infty\}$$

The term non-Archimedean means that the norm on K satisfies the inequality

$$\|f+g\| \leq \max\{\|f\|, \|g\|\}$$

for all $f, g \in K$. This in particular implies that the Archimedean axiom is not satisfied. If $f, g \in K$ with $\|f\| < \|g\|$, then for all natural number n we have $\|n \cdot f\| \leq \|f\| < \|g\|$, indeed $\|n \cdot f\| = \|f\|$.

Let I be an ideal in $K[x_1, ..., x_n]$ and $V_K(I)$ be the algebraic variety given by I in $(K^*)^n$.

As the norm is given by $\|-\| = e^{-val(-)}$, the corresponding amoeba map $\log\|-\|$ over K is given by the valuations

$$\begin{aligned} \mathrm{val}_- = \log\|-\| : (K^*)^n &\to \mathbb{R}^n \\ (z_1, ..., z_n) &\mapsto (-val(z_1), ..., -val(z_n)) \end{aligned}$$

Definition 4.5 *The* ***non-Archimedean amoeba*** *of* $V_K(I)$ *is* $\mathrm{val}_-(V_K(I))$.

A proof of the following theorem for hypersurfaces can be found in [Gelfand, Kapranov, Zelevinsky, 1994, Sec. 6.1], the general statement in terms of the Bergman fan (see Section 4.5 below) in [Sturmfels, 2002, Sec. 9.4].

Theorem 4.6 *The limit* $\lim_{t\to\infty} \log_t V(I_t)$ *exists as the limit in the Hausdorff metric on compacts, and*

$$\mathrm{val}_-(V_K(I)) = \lim_{t\to\infty} \log_t V(I_t)$$

Recall that the distance of two closed subsets of a metric space in the Hausdorff metric is given by

$$d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\}$$

so above convergence means that for any compact $D \subset \mathbb{R}^n$

$$\lim_{t\to\infty} d(D \cap \log_t V(I_t), D \cap \mathrm{val}\, V_K(I)) = 0$$

From the point of view of degenerations it will turn out to be more natural to consider the image

$$\mathrm{val}(V_K(I)) = -\mathrm{val}_-(V_K(I))$$

of $V_K(I)$ under the map

$$\begin{aligned} \mathrm{val} : (K^*)^n &\to \mathbb{R}^n \\ (z_1, ..., z_n) &\mapsto (val(z_1), ..., val(z_n)) \end{aligned}$$

associating to each component the minimal weight term.

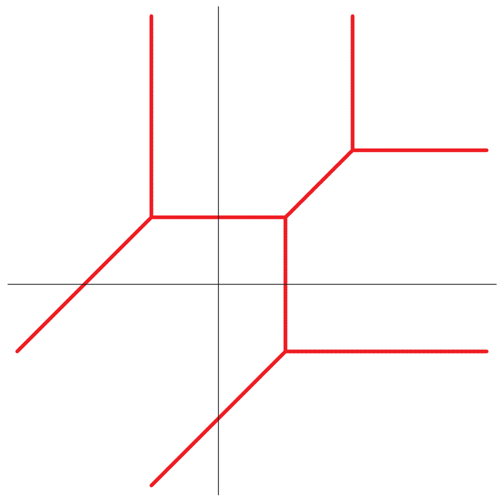


Figure 4.4: $\text{val}(V_K(I))$ for the ideal of a plane quadric with coefficients in K

Example 4.7 $\text{val}(V_K(I))$ *for the ideal of a plane quadric with coefficients in K is shown in Figure 4.4.*

Example 4.8 $\text{val}(V_K(I))$ *for the degeneration of general plane cubics*

$$\{x_0 x_1 x_2 + tf = 0\}$$

where f is a general element in $\mathbb{C}[x_0, x_1, x_2]_3$, is shown in Figure 4.5. Note that for an ideal I, homogeneous with respect to the grading $\deg x_i = 1\ \forall i$ on $K[x_1, ..., x_n]$, one can consider $\text{val}(V_K(I))$ as a subset of $\frac{\mathbb{R}^3}{\mathbb{R}(1,1,1)} \cong \mathbb{R}^2$, this will be explored in detail in Section 6.7.

Having made the geometric connection between degenerations and tropical geometry, we consider the algebraic connection:

4.3 Tropical varieties

Definition 4.9 *A **tropical variety** is a subset*

$$\text{tropvar}(I) = \text{val}_{-}(V_K(I)) \subset \mathbb{R}^n$$

where I is an ideal in $K[x_1, ..., x_n]$.

For $w \in \mathbb{R}^n$ the **initial form** $in_w(f)$ of $f \in K[x_1, ..., x_n]$ is the sum of the terms of maximal weight with respect to w and $weight(c) = -val(c)$ for $c \in K$. For any ideal $J \subset K[x_1, ..., x_n]$ its **initial ideal** is

$$in_w(J) = \langle in_w(f) \mid f \in J \rangle$$

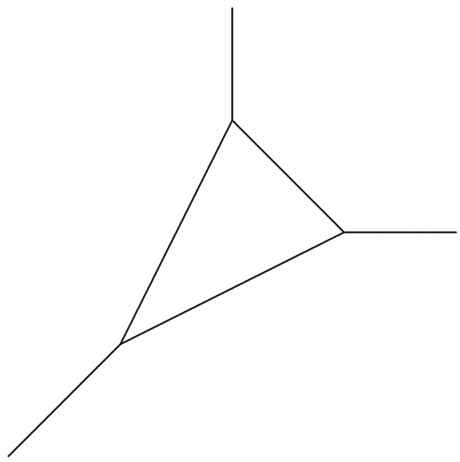


Figure 4.5: $\operatorname{val}(V_K(I))$ for the degeneration of plane cubics

Theorem 4.10 *[Sturmfels, 2002, Sec. 9.2], [Speyer, Sturmfels, 2004, Sec. 2], [Richter-Gebert, Sturmfels, Theobald, 2005, Sec. 2] Every ideal $I \subset K[x_1, ..., x_n]$ has a finite subset $\mathcal{G}$, called a* ***tropical basis*** *of I, such that*

1. *For all $w \in \operatorname{tropvar}(I)$ the set $\{in_w(g) \mid g \in \mathcal{G}\}$ generates $in_w(I)$.*
2. *For all $w \notin \operatorname{tropvar}(I)$ the set $\{in_w(g) \mid g \in \mathcal{G}\}$ contains a monomial.*

$\operatorname{tropvar}(I)$ *is a finite intersection of the tropical hypersurfaces* $\operatorname{tropvar}\langle (g)\rangle$ *for $g \in \mathcal{G}$, it is a polyhedral cell complex, its dimension is the Krull dimension of $K[x_1, ..., x_n]/I$, it is equidimensional if $V_K(I)$ is, and*

$$\operatorname{tropvar}(I) = \{w \in \mathbb{R}^n \mid in_w(I) \text{ contains no monomial}\}$$

Selecting the maximal weight term and defining the weight of a constant $c \in K$ as $weight(c) = -val(c)$ is the Gröbner basis point of view. With respect to degenerations it is more natural to look at the minimal weight term and take $weight(c) = val(c)$ for $c \in K$, i.e., to consider $\operatorname{val}(V_K(I)) = -\operatorname{val}_-(V_K(I))$.

Example 4.11 *The monomial initial ideals and the sets of weight vectors leading to them for the plane cubic case as in Example 4.8 are depicted in Figure 4.6. For $w \in \operatorname{tropvar}(I)$ the initial ideal is generated by a sum of the initial terms appearing in a neighborhood of w.*

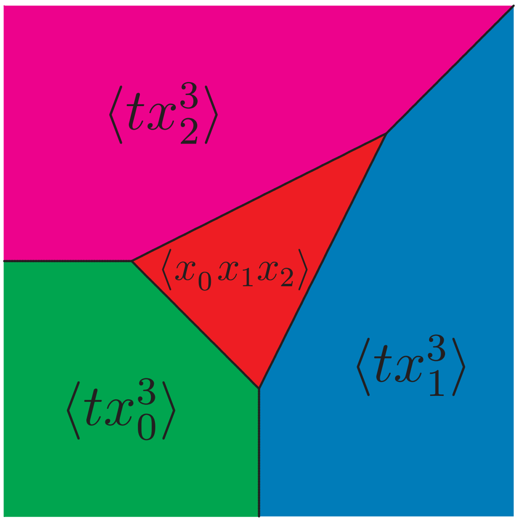


Figure 4.6: Tropical variety and initial ideals for the degeneration of plane cubics

4.4 Tropical prevarieties

Definition 4.12 *The* ***tropical semiring*** *is* $\mathbb{R} \cup \{-\infty\}$ *with* ***tropical addition*** *and* ***multiplication***

$$a \oplus b = \max(a, b)$$
$$a \odot b = a + b$$

The tropical semiring satisfies $(a \oplus b) \odot c = a \odot c \oplus b \odot c$, the additive unit is $-\infty$, the multiplicative unit is 0. In general there is no additive inverse in the tropical semiring.

Definition 4.13 *A* ***tropical polynomial*** *is a polynomial formed with* $\oplus$ *and* $\odot$*, i.e., a piecewise linear function*

$$F : \mathbb{R}^n \to \mathbb{R} \qquad F(x_1, ..., x_n) = \max\{a_{1j}x_1 + ... + a_{nj}x_n + c_j \mid j\}$$

Definition 4.14 *The* ***tropical prevariety*** $T(F)$ *of* F *is the set where the maximum is attained at least twice, and* $T(\mathcal{G}) = \bigcap_{g \in \mathcal{G}} T(g)$ *for any set of tropical polynomials* $\mathcal{G}$.

Definition 4.15 *For any polynomial* $f \in K[x_1, ..., x_n]$

$$f = \sum\nolimits_a b_a(t) \cdot x^a$$

*define its **tropicalization** as*

$$\operatorname{trop}(f) = \bigoplus_a -val\left(b_a(t)\right) \odot x^{\odot a}$$

So, consistent with the amoeba, the non-Archimedian amoeba and the tropical variety, we again adopt the Gröbner basis point of view, looking at the maximal weight term and take $weight(c) = -val(c)$ for $c \in K$.

Theorem 4.16 *[Sturmfels, 2002, Sec. 9.2], [Speyer, Sturmfels, 2004, Sec. 2], [Richter-Gebert, Sturmfels, Theobald, 2005, Sec. 2] Any tropical variety* $\operatorname{tropvar}(I)$, $I \subset K[x_1, ..., x_n]$ *is a tropical prevariety. For any ideal* $I \subset K[x_1, ..., x_n]$

$$\operatorname{tropvar}(I) = T(\operatorname{trop}(I))$$

Example 4.17 *For the general plane elliptic curve in Example 4.8 the tropical variety* $\operatorname{tropvar}\langle f\rangle$ *is the non-differentiability locus* $T(F)$ *of the piecewise linear function*

$$F = \max\{3x_1 - 1, 2x_1 + x_2 - 1, x_1 + 2x_2 - 1, 3x_2 - 1, ..., -1, x_1 + x_2\}$$

Figure 4.7 shows the graph of F.

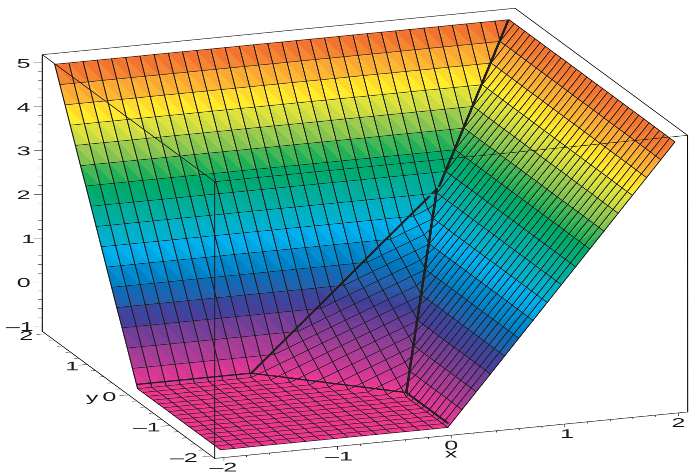


Figure 4.7: Piecewise linear function associated to the degeneration of plane cubics

Not every tropical prevariety is a tropical variety:

Example 4.18 *The intersection of the tropical lines* $L_1 = T(\operatorname{trop}(x + y + 1))$ *and* $L_2 = T(\operatorname{trop}(tx + y + 1))$, *as depicted in Figure 4.8, is a tropical prevariety, but not a tropical variety.*

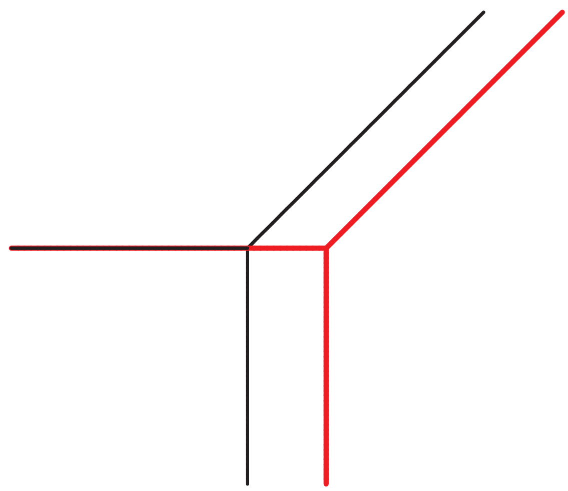


Figure 4.8: Intersecting two tropical lines

4.5 Tropical varieties and the Bergman fan

Let I be an ideal in $\mathbb{C}[t, x_1, ..., x_n]$.

Theorem 4.19 *[Bergman, 1971], [Bieri, Groves, 1984], [Sturmfels, 2002, Sec. 9.3] Define*

$$suppBC_-(I) = \left\{ p \in S^n \mid \begin{array}{r} \exists \text{ sequence } (p_j)_{j\in\mathbb{N}} \text{ with } p_j \in \log(V(I)) \cap jS^n \subset \mathbb{R}^{n+1} \\ \text{and } \lim_{j\to\infty} \frac{1}{j} p_j = p \end{array} \right\}$$

and

$$suppBF_-(I) = \left\{ p \in \mathbb{R}^{n+1} \backslash \{0\} \mid \frac{p}{\|p\|} \in suppBC_-(I) \right\} \cup \{0\}$$

If $V(I)$ is an irreducible subvariety of $(\mathbb{C}^*)^{n+1}$ of dimension $d+1$, then $suppBF_-(I)$ is a finite union $d+1$-dimensional convex polyhedral cones. The intersection of any two is a common face.

Denote by $BF_-(I)$ the corresponding fan and by $BC_-(I)$ the corresponding complex of dimension d.

Note:

- $V(I) \subset (\mathbb{C}^*)^{n+1}$
- These definitions are consistent with the Gröbner basis $(\max, +)$ point of view looking at the maximum weight terms.

- The definitions of $BF_{-}(I)$ and $BC_{-}(I)$ are symmetric in all variables $t, x_1, ..., x_n$.
- The complex $BC_{-}(I)$ and the fan $BF_{-}(I)$ are known in the literature as Bergman complex and Bergman fan respectively. However with degenerations in mind, i.e., the power series point of view, it is more natural to consider the reflection of these objects at the origin. So we use the following definition:

Definition 4.20 *Analogous to K denote by L the metric completion of the field $\mathbb{C}\{\{s\}\}$ of Puiseux series in a new variable s. If I is an ideal in $\mathbb{C}[t, x_1, ..., x_n]$, then* ***Bergman fan*** *$BF(I)$ of I is the image of the vanishing locus of $V_L(I)$ of I over L under the map*

$$\begin{aligned} (L^*)^{n+1} &\rightarrow \mathbb{R}^{n+1} \\ (t, x_1, ..., x_n) &\mapsto (val(t), val(x_1), ..., val(x_n)) \end{aligned}$$

The intersection of $BF(I)$ with the unit sphere S^n is called the ***Bergman complex*** *$BC(I)$ of I.*

Note that this non-Archimedean type definition has the advantage that it avoids problems with limit processes.

If you prefer the (max, +) point of view you may replace in this definiton val by $-val$.

Consider the stereographic projection π, visualized in Figure 4.9, of the upper half sphere

$$S^n \cap \{w_t > 0\} = \left\{(w_t, w_{x_1}, ..., w_{x_n}) \in \mathbb{R}^{n+1} \mid w_t^2 + w_{x_1}^2 + ... + w_{x_n}^2 = 1,\ w_t > 0\right\}$$

from 0 to $\mathbb{R}^n = \{w_t = 1\}$. Here we denote the coordinates of $\mathbb{R}^{n+1}$ corresponding to the variables of $\mathbb{C}[t, x_1, ..., x_n]$ by $w_t, w_{x_1}, ..., w_{x_n}$, as they are weights on the variables.

In the same way denote by π_{-} the stereographic projection of the lower half sphere from 0 to $\mathbb{R}^n = \{w_t = -1\}$.

Connecting the Bergman complex to the tropical variety via π (see [Sturmfels, 2002, Sec. 9.4]) and summarizing:

Theorem 4.21 *For any ideal I in $\mathbb{C}[t, x_1, ..., x_n]$ it holds*

$$\begin{aligned} \lim_{t\rightarrow\infty} (\log_t V(I_t)) &= \mathrm{val}_{-}(V_K(I)) = \mathrm{tropvar}(I) = T(\mathrm{trop}(I)) \\ &= \pi_{-}(BC_{-}(I) \cap \{w_t < 0\}) \subset \mathbb{R}^n \end{aligned}$$

If $I \subset \mathbb{C}[x_1, ..., x_n]$, then $BF_{-}(I) \subset \mathbb{R}^n$ coincides with the above when considering I as an ideal in $\mathbb{C}[t, x_1, ..., x_n]$.

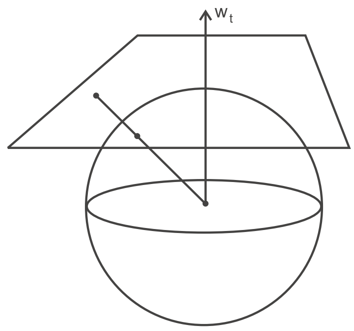


Figure 4.9: Stereographic projection relating the Bergman complex and the tropical variety

Remark 4.22 *Reflecting at the origin, we have*

$$\begin{aligned}\operatorname{val}(V_K(I)) &= \pi(BC(I) \cap \{w_t > 0\}) \\ &= -\lim_{t\to\infty}(\log_t V(I_t)) = -\operatorname{tropvar}(I) = -T(\operatorname{trop}(I))\end{aligned}$$

Our non-Archimedean definition of the Bergman fan relates to the limit definition by

$$\begin{aligned}BC(I) &= -BC_-(I) \\ BF(I) &= -BF_-(I)\end{aligned}$$

Passing from the Bergman complex $BC(I)$ to $\operatorname{val}(V_K(I))$, i.e., intersecting with the plane $\{w_t = 1\}$, amounts to the identification of the parameters s and t.

For the subset of $BC(I)$ lying inside the equator $\{w_t = 0\}$ of the sphere, we introduce the notation:

Definition 4.23 *$BC(I) \cap \{w_t = 0\}$ is called the* ***tropical variety at infinity****.*

Example 4.24 *For the plane elliptic curve in above Example 4.8 the Bergman fan $BF(I)$ is shown in Figure 4.10 (extending the depicted faces to infinity). Applying π to the $w_t > 0$ part of Figure 4.11, which is visualizing the Bergman complex $BC(I)$, gives $\operatorname{val} V_K(I)$.*

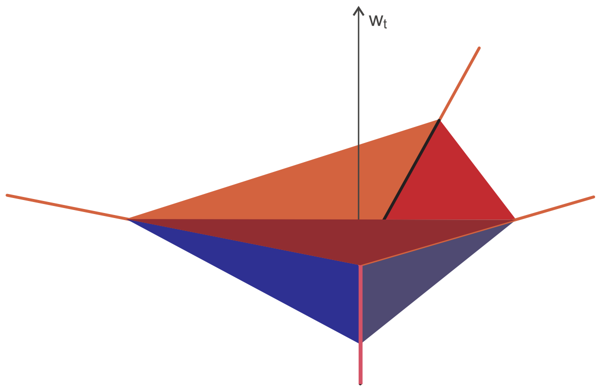


Figure 4.10: Bergman fan of the degeneration of plane elliptic curves

5 Flatness, Gröbner bases and the normal sheaf

5.1 Flatness

Definition 5.1 *Let A be a ring. An A-module M is called* ***flat*** *over A if for every injective homomorphism $N \to L$ the induced map $N \otimes_A M \to L \otimes_A M$ is injective.*

Proposition 5.2 *[Hartshorne, 1977, Ch. III.9.] Let A be a ring and M an A-module. M is flat over A if and only if for all finitely generated ideals $a \subset A$ the map*

$$a \otimes M \to M$$

is injective.

Definition 5.3 *Given a morphism of schemes $f : Z \to Y$, an $\mathcal{O}_Z$-module $\mathcal{F}$ is called flat over Y at $z \in Z$ if $\mathcal{F}_z$ is flat over $\mathcal{O}_{f(z),Y}$, which is considered as an $\mathcal{O}_{f(z),Y}$-module via the natural map $\mathcal{O}_{f(z),Y} \to \mathcal{O}_{z,Z}$. $\mathcal{F}$ is called flat over Y if it is flat over Y for all $z \in Z$.*

Z is called a ***flat family*** *over Y if $\mathcal{O}_Z$ is flat over Y.*

Proposition 5.4 *[Hartshorne, 1977, Ch. III.9.] Let $A_1 \to A_2$ be a ring homomorphism and*

$$f : \operatorname{Spec} A_2 \to \operatorname{Spec} A_1$$

the corresponding morphism of affine schemes. If M is an A_2-module, then $\tilde{M}$ is flat over $\operatorname{Spec} A_1$ if and only if M is flat over A_1.

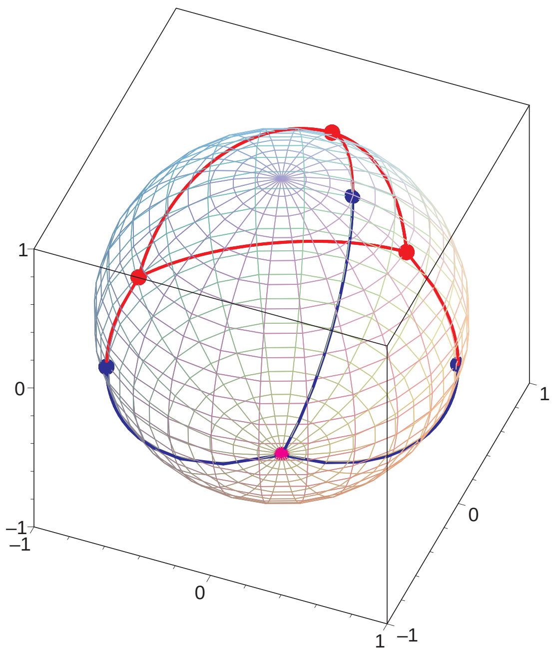


Figure 4.11: Bergman complex of the degeneration of plane elliptic curves

5.2 First order deformations and the normal sheaf

Definition 5.5 *If $X_0 \subset Y$ is a closed subscheme of a scheme Y over k, a* ***first order deformation*** *of X_0 in Y is a flat family $\mathfrak{X} \subset Y \times_k \operatorname{Spec} k[t] / \langle t^2 \rangle$ over $\operatorname{Spec} k[t] / \langle t^2 \rangle$ such that the fiber over $\operatorname{Spec} k \subset \operatorname{Spec} k[t] / \langle t^2 \rangle$ is X_0.*

The tangent space of the Hilbert scheme $\mathbb{H}_Y^P$ of subschemes with Hilbert polynomial P of the projective scheme Y at the point X_0 is the space of first order deformations of X_0 in Y.

We show that if $X_0 \subset Y$ is a closed subscheme of a scheme Y over k, the space of first order deformations of X_0 in Y coincides with the space of global sections of $N_{X_0/Y}$:

Suppose $\mathfrak{X} \subset Y \times \operatorname{Spec} k[t] / \langle t^2 \rangle$ is a subscheme such that X_0 is isomorphic to the fiber product

$$\begin{array}{ccc} \mathfrak{X} \times_{k[t]/\langle t^2 \rangle} \operatorname{Spec} k & \stackrel{\pi_1}{\to} & \mathfrak{X} \\ \pi_2 \downarrow & & \downarrow \\ \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} k[t] / \langle t^2 \rangle \end{array}$$

and fix an isomorphism of X_0 and $\mathfrak{X} \times_{k[t]/\langle t^2 \rangle} \operatorname{Spec} k$. Consider an affine open set $U \subset Y$:

Let $R = \mathcal{O}_Y(U)$ be the coordinate ring of U and $I(X_0 \cap U) \subset R$ the ideal of $X_0 \cap U$. Then $N_{X_0/Y} \mid_{X_0 \cap U}$ is the sheaf associated to

$$\mathrm{Hom}_R(I(X_0 \cap U), R/I(X_0 \cap U))$$

The coordinate ring of $U \times \operatorname{Spec} k[t]/\langle t^2 \rangle$ is $R \otimes k[t]/\langle t^2 \rangle$, so write the ideal of the intersection $\mathfrak{U}$ of $(U \times \operatorname{Spec} k[t]/\langle t^2 \rangle)$ and $\mathfrak{X}$ as

$$I(\mathfrak{U}) = \langle f_1 + t \cdot g_1, ..., f_r + t \cdot g_r \rangle$$

with $I(X_0 \cap U) = \langle f_1, ..., f_r \rangle$ and $g_i \in R$.

We give different characterizations of flatness of $\mathfrak{U}$ over $\operatorname{Spec} k[t]/\langle t^2 \rangle$:

$\mathfrak{U}$ is flat over $\operatorname{Spec} k[t]/\langle t^2 \rangle$ if and only if

$$\langle t \rangle \otimes \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \to \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$$

is injective, i.e., if and only if for all $f \in R$ it holds

$$tf \in I(\mathfrak{U}) \Rightarrow f \in I(X_0 \cap U)$$

- This is equivalent to the existence of a $\varphi_U \in \mathrm{Hom}_R(I(X_0 \cap U), R/I(X_0 \cap U))$ with $\varphi_U(f_i) = g_i$:

  First note that if $f \in R$ and $tf \in I(\mathfrak{U}) \subset R \otimes k[t]/\langle t^2 \rangle$, then there are $a_i, b_i \in R$ such that

  $$tf = \sum_i (a_i + tb_i)(f_i + tg_i) = \sum_i a_i f_i + t \sum_i (a_i g_i + b_i f_i)$$

  hence $\sum_i a_i f_i = 0$.

  $\Longleftarrow$: If there is a

  $$\varphi_U : I(X_0 \cap U) \to R/I(X_0 \cap U)$$

  with $\varphi_U(f_i) = g_i$, then

  $$\begin{aligned}\sum_i a_i g_i + I(X_0 \cap U) &= \sum_i a_i \varphi_U(f_i) = \varphi_U\left(\sum_i a_i f_i\right)\\ &= \varphi_U(0) = 0 \in R/I(X_0 \cap U)\end{aligned}$$

  hence

  $$f = \sum_i a_i g_i + \sum_i b_i f_i \in I(X_0 \cap U)$$

  i.e., for all $f \in R$ with $tf \in I(\mathfrak{U})$ we have $f \in I(X_0 \cap U)$.

$\Longrightarrow$: On the other hand if for all $f \in R$ with $tf \in I(\mathfrak{U})$ it holds $f \in I(X_0 \cap U)$, then the homomorphism

$$\varphi_U : I(X_0 \cap U) \to R/I(X_0 \cap U)$$

is given in the following way: If $f = \sum_i a_i f_i \in I(X_0 \cap U)$, define φ_U by

$$\varphi_U(f) = \sum_i a_i g_i + I(X_0 \cap U) \in R/I(X_0 \cap U)$$

This is well defined: If

$$\sum_i a_i f_i = 0$$

then

$$t \sum_i a_i g_i = \sum_i a_i (f_i + t g_i) \in I(\mathfrak{U})$$

hence $\sum_i a_i g_i \in I(X_0 \cap U)$.

- Existence of

$$\varphi_U \in \mathrm{Hom}_R(I(X_0 \cap U), R/I(X_0 \cap U))$$

with $\varphi_U(f_i) = g_i$ is equivalent to the condition that any syzygy between $f_1, ..., f_r$ can be lifted to a syzygy between

$$f_1 + t g_1, ..., f_r + t g_r$$

$\Longrightarrow$: Suppose $\sum_i a_i f_i = 0 \in R$ and there is φ_U with $\varphi_U(f_i) = g_i$, then as above

$$\sum_i a_i g_i + I(X_0 \cap U) = \varphi_U\left(\sum_i a_i f_i\right) = 0 \in R/I(X_0 \cap U)$$

i.e., $\sum_i a_i g_i \in I(X_0 \cap U)$, hence there are $b_i \in R$ such that $-\sum_i a_i g_i = \sum_i b_i f_i$. So

$$\sum_i a_i (f_i + t g_i) = -t \sum_i b_i f_i$$

hence

$$\sum_i (a_i + t b_i)(f_i + t g_i) = 0 \in R \otimes k[t] / \langle t^2 \rangle$$

$\Longleftarrow$: On the other hand if $f = \sum_i a_i f_i \in I(X_0 \cap U)$ and any syzygy lifts, define as above φ_U by

$$\varphi_U(f) = \sum_i a_i g_i + I(X_0 \cap U) \in R/I(X_0 \cap U)$$

This ist well defined: If $\sum_i a_i f_i = 0$, then there are $b_i \in R$ such that

$$t\left(\sum_i a_i g_i + \sum_i b_i f_i\right) = \sum_i (a_i + tb_i)(f_i + tg_i) = 0 \in R \otimes k[t] / \langle t^2 \rangle$$

hence $\sum_i a_i g_i \in I(X_0 \cap U)$.

Summarizing these statements:

Proposition 5.6 *[Eisenbud, Harris, 1992] Let $X_0 \subset Y$ be a closed subscheme of a scheme Y over k and $\mathfrak{X} \subset Y \times \operatorname{Spec} k[t] / \langle t^2 \rangle$ a subscheme such that $X_0 \cong \mathfrak{X} \times_{k[t]/\langle t^2 \rangle} \operatorname{Spec} k$. Consider an affine open $U \subset Y$, set $R = \mathcal{O}_Y(U)$ and write the ideal of the intersection $\mathfrak{U}$ of $\mathfrak{X}$ and $(U \times \operatorname{Spec} k[t] / \langle t^2 \rangle)$ as*

$$I(\mathfrak{U}) = \langle f_1 + t \cdot g_1, ..., f_r + t \cdot g_r \rangle$$

with $I(X_0 \cap U) = \langle f_1, ..., f_r \rangle$ and $g_i \in R$. Then the following statements are equivalent:

1. *$\mathfrak{U} \to \operatorname{Spec} k[t] / \langle t^2 \rangle$ is flat*
2. *$\langle t \rangle \otimes \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \to \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$ is injective*
3. *For all $f \in R$ it holds*

$$tf \in I(\mathfrak{U}) \Rightarrow f \in I(X_0 \cap U)$$

4. *There is a unique*

$$\varphi_U \in \operatorname{Hom}_R(I(X_0 \cap U), R/I(X_0 \cap U))$$

with

$$\varphi_U(f_i) = g_i$$

5. *Any syzygy between $f_1, ..., f_r$ lifts to a syzygy between $f_1 + tg_1, ..., f_r + tg_r$, i.e., if*

$$\sum_i a_i f_i = 0 \in R$$

with $a_i \in R$, there are $b_i \in R$ such that

$$\sum_i (a_i + tb_i)(f_i + tg_i) = 0 \in R \otimes k[t] / \langle t^2 \rangle$$

So if $\mathfrak{X} \to \operatorname{Spec} k[t] / \langle t^2 \rangle$ is flat, then for any affine open set U there is a unique φ_U, and the φ_U patch together to a section of $N_{X_0/Y}$. On the other hand, if φ is a global section of $N_{X_0/Y}$, then define the associated first order deformation $\mathfrak{X}$ by the local equations

$$\{f + t \cdot \varphi(f) = 0 \mid f \in I(X_0 \cap U)\}$$

on $U \times \operatorname{Spec} k[t] / \langle t^2 \rangle$ hence:

Theorem 5.7 *If $X_0 \subset Y$ is a closed subscheme of a scheme Y over k, the space of first order deformations of X_0 in Y equals the space of global sections of $N_{X_0/Y}$.*

5.3 Flatness over $k[[t]]$

The statement analogous to Proposition 5.6 for base $\operatorname{Spec} k[[t]]$ is given in the following.

Proposition 5.8 *Let $X_0 \subset Y$ be a closed subscheme of a scheme Y over k and $\mathfrak{X} \subset Y \times \operatorname{Spec} k[[t]]$ a subscheme such that $X_0 \cong \mathfrak{X} \times_{k[[t]]} \operatorname{Spec} k$. Consider an affine open $U \subset Y$, let $R = \mathcal{O}_Y(U)$ and write the ideal of the intersection $\mathfrak{U}$ of $\mathfrak{X}$ and $(U \times \operatorname{Spec} k[[t]])$ as*

$$I(\mathfrak{U}) = \langle f_1 + t \cdot g_1, ..., f_r + t \cdot g_r \rangle$$

with $I(X_0 \cap U) = \langle f_1, ..., f_r \rangle$ and $g_i \in R \otimes k[[t]]$. Then the following statements are equivalent:

1. *$\mathfrak{U} \to \operatorname{Spec} k[[t]]$ is flat*
2. *$\langle t \rangle \otimes \mathcal{O}_{\mathfrak{X}}(\mathfrak{U}) \to \mathcal{O}_{\mathfrak{X}}(\mathfrak{U})$ is injective*
3. *For all $f \in R$ it holds*

$$tf \in I(\mathfrak{U}) \Rightarrow f \in I(X_0 \cap U)$$

4. *Any syzygy between $f_1, ..., f_r$ lifts to a syzygy between $f_1 + tg_1, ..., f_r + tg_r$, i.e., if*

$$\sum_i a_i f_i = 0 \in R$$

with $a_i \in R$, there are $b_i \in R \otimes k[[t]]$ such that

$$\sum_i (a_i + tb_i)(f_i + tg_i) = 0 \in R \otimes k[[t]]$$

6 Gröbner fan, state polytope, Hilbert scheme and stability

6.1 Concept for computing the Bergman fan

Let I be an ideal in $\mathbb{C}[x_0, ..., x_n]$ and w a weight vector on the variables of $\mathbb{C}[x_0, ..., x_n]$ with $w_i \geq 0$ for all i. Given a monomial ordering $>$ we have $L_>(in_w(g)) = L_{>_w}(g)$ for every $g \in \mathbb{C}[x_0, ..., x_n]$, so the subsets of monomials in $L_>(in_w(I))$ and $L_{>_w}(I)$ are equal, hence:

Proposition 6.1 *If $>$ is any monomial ordering, then*

$$L_>(in_w(I)) = L_{>_w}(I)$$

Corollary 6.2 *If $g = (g_1, ..., g_r)$ is the reduced Gröbner basis of I with respect to $>_w$, then*

$$(in_w(g_i) \mid i = 1, ..., r)$$

is the reduced Gröbner basis of $in_w(I)$ with respect to $>$.

Proposition 6.3 *If $g = (g_1, ..., g_r)$ is the reduced Gröbner basis of I with respect to $>_w$, then* $\mathrm{in}_w(I)$ *contains a monomial if and only if*

$$(\langle in_w(g_i) \mid i = 1, ..., r\rangle : \langle x_0 \cdot ... \cdot x_n\rangle^\infty) = \langle 1\rangle$$

Remark 6.4 *To speed up computations, one first checks if any of the $in_w(g_i)$, $i = 1, ..., r$ is a monomial, before doing the saturation.*

Let I be an ideal in $\mathbb{C}[x_1, ..., x_n]$ and $J \subset \mathbb{C}[x_0, x_1, ..., x_n]$ be the ideal of the projective closure of $V(I)$.

Proposition 6.5 *[Sturmfels, 2002, Sec. 9.2]*

$$BF(I) = \left\{w \in \mathbb{R}^n \mid in_{(0,-w)}(J) \text{ does not contain a monomial}\right\}$$

This allows one to homogenize, so any monomial ordering will be equivalent to a global ordering, hence the reductions in Gröbner computations stay finite.

To compute $BF(I)$ we have to understand, which initial ideals can occur. This is described by the Gröbner fan.

6.2 Tropical representation of Gröbner cones

Although there are infinitely many global semigroup orderings on the monomials of $\mathbb{C}[x_1, ..., x_n]$, if we fix an ideal $J \subset \mathbb{C}[x_1, ..., x_n]$ and consider $>_1$ and $>_2$ equivalent if $L_{>_1}(J) = L_{>_2}(J)$, there are only finitely many equivalence classes of monomial orderings.

Definition 6.6 *Given a global ordering $>$ on the monomials of $\mathbb{C}[x_1, ..., x_n]$ and an ideal $J \subset \mathbb{C}[x_0, ..., x_n]$ define*

$$C_>(J) = \{w' \in \mathbb{R}^n \mid in_{w'}(J) = L_>(J)\}$$

If $J_0 = L_>(J)$ for some ordering $>$, then define $C_{J_0}(J) = \overline{C_>(J)}$.

By Proposition 1.156 and Proposition 1.194 we have:

Lemma 6.7 *Given a global ordering $>$ on the monomials of $\mathbb{C}[x_1, ..., x_n]$ and an ideal $J \subset \mathbb{C}[x_0, ..., x_n]$, there is some $w \in \mathbb{Z}^n$ with positive entries, such that $in_w(J) = L_>(J)$, hence, $C_>(J)$ is non-empty.*

Definition 6.8 *Given $w \in \mathbb{Z}^n$ with non-negative entries and $J \subset \mathbb{C}[x_0, ..., x_n]$ define*

$$C_w(J) = \{w' \in \mathbb{R}^n \mid in_{w'}(J) = in_w(J)\}$$

Consider some tie break ordering $>$ and the unique reduced Gröbner basis $\mathcal{G} = (g_1, ..., g_r)$ of J with respect to $>_w$, then

$$C_w(J) = \{w' \in \mathbb{R}^n \mid in_{w'}(g_i) = in_w(g_i)\, \forall i = 1, ..., r\} \tag{6.1}$$

To see this, suppose $w' \in \mathbb{R}^n$ with $in_{w'}(g_i) = in_w(g_i)\, \forall i$, then

$$in_w(J) = \langle in_{w'}(g_i) \mid i = 1, ..., r\rangle \subset in_{w'}(J)$$

so

$$L_{>_w}(J) = L_>(in_w(J)) \subset L_>(in_{w'}(J)) = L_{>_{w'}}(J)$$

therefore the lead ideals $L_{>_w}(J)$ and $L_{>_{w'}}(J)$ are equal, hence by Proposition 1.185

$$in_w(J) = in_{w'}(J)$$

i.e., $w' \in C_w(J)$.

On the other hand if $w' \in \mathbb{R}^n$ with

$$in_w(J) = in_{w'}(J)$$

then by Corollary 6.2 the reduced Gröbner basis of $in_{w'}(J)$ with respect to $>$ is given by $G = (in_w(g_1), ..., in_w(g_r))$. So for all $i = 1, ..., r$ we have $NF_>(in_{w'}(g_i), G) = 0$, hence $m_i = L_{>_w}(g_i)$ is a monomial of $in_{w'}(g_i)$, as by reducedness, m_i is the only monomial of g_i in $L_{>_w}(J)$. Write

$$\begin{aligned} in_w(g_i) &= m_i + h_i \\ in_{w'}(g_i) &= m_i + h_i' \end{aligned}$$

then h_i and h_i' do not involve monomials of $L_{>_w}(J)$. The first step of the division with remainder, calculating $NF_>(in_{w'}(g_i), G)$, gives $h_i' - h_i \in in_{w'}(J) = in_w(J)$. On the other hand, no term of $h_i' - h_i$ is in $L_{>_w}(J) = L_>(in_w(J))$, hence, $h_i' = h_i$, i.e., $in_{w'}(g_i) = in_w(g_i)$.

Suppose now $w \in \mathbb{R}^n$ is representing the monomial ordering $>$ and define

$$m_i = x^{a_i} = LT_>(g_i) = in_w(g_i)$$

and write $g_i = m_i + h_i$ with the tail h_i of g_i. By the description of $C_w(J)$ via Equation 6.1

$$\begin{aligned} C_w(J) &= \left\{w' \in \mathbb{R}^n \mid w'b_i < w'a_i \ \forall \text{ monomials } x^{b_i} \text{ of the tail } h_i \text{ and } \forall\ i = 1, ..., r\right\} \\ &= \{w' \in \mathbb{R}^n \mid \operatorname{trop}(g - in_w(g))(w') < \operatorname{trop}(in_w(g))\, \forall g \in \mathcal{G}\} \end{aligned}$$

so summarizing:

Theorem 6.9 *If $w \in \mathbb{Z}^n$ with non-negative entries, $J \subset \mathbb{C}[x_1, ..., x_n]$ and $>$ is some global ordering on the monomials of $\mathbb{C}[x_1, ..., x_n]$ and $\mathcal{G} = (g_1, ..., g_r)$ is the unique reduced Gröbner basis of J with respect to $>_w$, then*

$$C_w(J) = \{w' \in \mathbb{R}^n \mid in_{w'}(g_i) = in_w(g_i)\, \forall i = 1, ..., r\}$$

in particular $C_w(J)$ is a relatively open convex polyhedral cone.

If $in_w(J) = L_>(J)$, then

$$C_w(J) = \{w' \in \mathbb{R}^n \mid \operatorname{trop}(g - in_w(g))(w') < \operatorname{trop}(in_w(g))\, \forall g \in \mathcal{G}\} \quad (6.2)$$

Remark 6.10 *If $in_w(J)$ is not monomial, then $C_w(J)$ is given by these inequalities together with the equalities coming from the condition that for each $g \in \mathcal{G}$ all monomials of the initial form $in_w(g)$ have the same weight.*

Definition 6.11 *Let $J \subset \mathbb{C}[x_1, ..., x_n]$ be an ideal. The Gröbner region of J is*

$$GR(J) = \left\{w \in \mathbb{R}^n \mid \exists w' \in \mathbb{R}^n_{\geq 0} \text{ with } in_w(J) = in_{w'}(J)\right\}$$

Lemma 6.12 *[Sturmfels, 1996, Ch. 1] If $J \subset \mathbb{C}[x_1, ..., x_n]$ is homogeneous, then $GR(J) = \mathbb{R}^n$.*

Definition 6.13 *Let $J \subset \mathbb{C}[x_1, ..., x_n]$ and suppose $GR(J) = \mathbb{R}^n$. The* ***Gröbner fan*** *$GF(J)$ is the set of all closures $\overline{C_w(J)}$ of cones $C_w(J)$ for all $w \in \mathbb{R}^n$.*

If $f = \sum_\alpha c_\alpha x^\alpha$ is a Laurent polynomial in the variables $x_1, ..., x_n$, then its **Newton polytope** is

$$N(f) = \text{convexhull}\{\alpha \mid c_\alpha \neq 0\} \subset \mathbb{R}^{n+1}$$

Lemma 6.14 *[Gelfand, Kapranov, Zelevinsky, 1994, Section 6.1] The Newton polytope of f lies in the hyperplane $\{\alpha \in \mathbb{R}^{n+1} \mid w \cdot \alpha = a\}$ for some $a \in \mathbb{Z}$ and $w \in \mathbb{Z}^{n+1}$ if and only if f is w-homogeneous, i.e, $f(t^{w_0}x_1, ..., t^{w_n}x_n) = t^a f(x_1, ..., x_n)$.*

If f, g are Laurent polynomials, then $N(f \cdot g) = N(f) + N(g)$.

Lemma 6.15 *If $w \in \mathbb{Z}^n$ with non-negative entries, $I \subset \mathbb{C}[x_1, ..., x_n]$, $>$ is some global ordering on the monomials of $\mathbb{C}[x_0, ..., x_n]$ and $\mathcal{G}$ is the unique reduced Gröbner basis of J with respect to $>_w$, then*

$$C_w(J) = \sigma_Q(\text{face}_w(Q))$$

is the normal cone of the face $\text{face}_w(Q)$ of Q with

$$Q = N\left(\prod_{g \in \mathcal{G}} g\right) = \sum_{g \in \mathcal{G}} N(g)$$

Using this representation of $C_w(J)$ one can conclude:

Proposition 6.16 *[Sturmfels, 1996, Ch. 2] $GF(J)$ is a fan.*

6.3 Computing the Gröbner fan

Algorithm 6.17 *Given an ideal $J \subset k[x_1, ..., x_n]$ with $GR(J) = \mathbb{R}^n$ and a subfan $F \subset \mathbb{R}^n$ of the Gröbner fan of I the following algorithm* findRandomCone *computes some cone of the Gröbner fan which is not in F:*
Choose some random $w \in \mathbb{R}^n - \text{supp}(F)$;
$g := \text{redStd}_{Wp(w)}(J)$ *the reduced Gröbner basis of J with respect to $Wp(w)$;*
if $\text{in}_w(J) = \langle in_w(g_i) \mid i = 1, ..., r\rangle$ *is not monomial repeat with different w;*
Compute $\overline{C_w(J)}$ from g via Equation 6.2;
return $\left(\overline{C_w(J)}\right)$;

The following randomized algorithm computes the Gröbner fan:

Algorithm 6.18 *Given an ideal* $J \subset k[x_1, ..., x_n]$ *with* $GR(J) = \mathbb{R}^n$ *the following algorithm computes the Gröbner fan of* J*:*

Let F *be the empty fan in* $\mathbb{R}^n$*.*
while $\text{isComplete}(F) = false$ do
  $F :=$ *the fan generated by the cones of* F *and* $\text{findRandomCone}(J, F)$;
od;

The following algorithm avoids searching a weight vector in the complement of the support of a non-complete fan and it integrates the test for completeness:

Algorithm 6.19 *Given an ideal* $J \subset k[x_1, ..., x_n]$ *with* $GR(J) = \mathbb{R}^n$ *the following algorithm computes the Gröbner fan:*

$F :=$ *the fan generated by* $\text{findRandomCone}(J, F)$;
$remainingfacets := facets(maxcones(F)[1])$;
while $remainingfacets <> \{\}$ do
  $fc := remainingfacets[1]$;
  $outernormal := -\text{rays}(\text{dual}(fc))[1]$;
  (where $\text{dual}(fc)$ *is the face of* $C^\vee$ *dual to* fc
  if fc *was a face of the maximal cone* C*)*
  $internal := \text{sum}(\text{rays}(fc))$;
  $s := 1$;
  $w := s \cdot internal + outernormal$;
  while $w \in \text{support}(F)$ do
    $s := 10 \cdot s$;
    $w := s \cdot internal + outernormal$;
  od;
  $F :=$ *the fan given by the cones of* F *and all faces of* $\overline{C_w(J)}$;
  for all $fct \in facets\left(\overline{C_w(J)}\right)$ do
    if $fct \in remainingfacets$ then
      $remainingfacets := remainingfacets - \{fct\}$;
    else
      $remainingfacets := remainingfacets \cup \{fct\}$;
    fi;
  od;
od;

Note that this algorithm necessarily stops with a complete fan, as the set $remainingfacets$ *is empty if and only if all facets of maximal cones have appeared twice.*

Remark 6.20 *To compute the Bergman fan out of the Gröbner fan, it is computationally important to note that if a cone F of the Gröbner fan is not contained in the Bergman fan, then also all higher dimensional cones containing F are not in the Bergman fan.*

6.4 Hilbert scheme and state polytope: Projective setup

6.4.1 Gröbner fan and state polytopes

Let $I \subset S = \mathbb{C}[x_0, ..., x_n]$ be a homogeneous ideal. For $d \geq 1$ define

$$P_d(I) = \text{convexhull}\left\{ \prod_{x^\alpha \in M_d} x^\alpha \mid M = L_>(I), > \text{ a monomial ordering} \right\} \subset \mathbb{R}^{n+1}$$

If d_0 is the maximum degree appearing in a minimal universal Gröbner basis of I, define

$$P(I) = \sum_{d=1}^{d_0} P_d(I)$$

Definition 6.21 *A* ***state polytope for*** I *is a polytope* $P \subset M_{\mathbb{R}}$ *with* $GF(I) = \mathrm{NF}(P)$.

Proposition 6.22 *[Sturmfels, 1996] The Gröbner fan $GF(I)$ of I is the normal fan of $P(I)$*

$$GF(I) = \mathrm{NF}(P(I))$$

so $P(I)$ is a state polytope for I.

If $w = (w_0, ..., w_n) \in \mathbb{R}^{n+1}$, then

$$\text{face}_w(P_d(I)) = P_d(in_w(I))$$

and

$$\text{face}_w(P(I)) = \sum_{d=1}^{d_0} P_d(in_w(I))$$

If $>$ and $>'$ are monomial orderings, then $\prod_{x^\alpha \in L_>(I)_d} x^\alpha = \prod_{x^\alpha \in L_{>'}(I)_d} x^\alpha$ if and only if $L_>(I)_d = L_{>'}(I)_d$.

Proposition 6.23 *[Sturmfels, 1996] If $\mathcal{G}$ is a universal Gröbner basis of I which is reduced with respect to any monomial ordering, then*

$$\sum_{g \in \mathcal{G}} N(g)$$

is a state polytope for I.

6.4.2 State polytope and the Hilbert scheme

Suppose $V = {}_{\mathbb{C}}\langle x_0, ..., x_n\rangle = \mathbb{C}^n$, $S = \mathrm{Sym}(V) \cong \mathbb{C}[x_0, ..., x_n]$ and $I \subset S$ is a homogeneous ideal such that S/I has Hilbert polynomial $P = P_{S/I}$.

Lemma 6.24 *[Bayer, 1982] There is a degree d_0 such that for all $d \geq d_0$ and all homogeneous saturated ideals $J \subset S$ with Hilbert polynomial $P_{S/J} = P$*

- *J is determined by the degree d part J_d of J, i.e., $J = (\langle J_d\rangle : \langle x_0, ..., x_n\rangle^\infty)$*
- $\dim_{\mathbb{C}}(S_d/J_d) = P(d)$
- *For all semigroup orderings $>$*

$$in_>(J) = \langle in_>(f) \mid f \in J \text{ with } \deg(f) \leq d\rangle$$

Definition 6.25 *With above notation I_d is a point in the Grassmannian $\mathbb{G}(P(d), S_d)$ of subspaces with codimension $P(d)$. This point is denoted as the d-th **Hilbert point** $H(I)$ of I. The Hilbert point $H(I)$ determines $(I : \langle x_0, ..., x_n\rangle^\infty)$.*

*The set of all d-th Hilbert points $H(J)$ of homogeneous ideals $J \subset S$ with $P_{S/J} = P$ is a closed subscheme $\mathbb{H}_n^P \subset \mathbb{G}(P(d), S_d)$, the P-th **Hilbert scheme**.*

Remark 6.26 *Let $>$ be a total ordering of the monomials of degree d of S and $x^{\alpha_1}, ..., x^{\alpha_s}$ with $s = \binom{n+d-1}{d}$ a monomial basis of S_d ordered with respect to $>$. If $B = (f_1, ..., f_r)$ with $r = s - P(d)$ is a basis of I_d, then writing*

$$f_j = \sum_{i=1}^{s} a_{j,\alpha_i} x^{\alpha_i}$$

*we obtain the $>$-**Hilbert matrix** of I*

$$A = (a_{j,\alpha_i})_{j=1,...r,\ i=1,...,s} \in k^{r\times s}$$

representing $H(I)$ with respect to above bases

$$\begin{array}{ccc} I_d & \hookrightarrow & S_d \\ \uparrow\cong & & \uparrow\cong \\ \mathbb{C}^r & \overset{A^t}{\hookrightarrow} & \mathbb{C}^s \end{array}$$

If $B' = (f_1', ..., f_r')$ is another basis of I_d and A' the corresponding $>$-Hilbert matrix, then there is a $Q \in \mathrm{GL}(r, k)$ with $A' = QA$.

Let $s = \dim S_d$ and $r = s - P(d)$. The **Plücker embedding**

$$\begin{array}{rccc} \mathfrak{p}: & \mathbb{G}(P(d), S_d) & \to & \mathbb{P}(\bigwedge^r S_d) \\ & \langle f_1, ..., f_r \rangle & \mapsto & f_1 \wedge ... \wedge f_r \end{array}$$

of $\mathbb{G}(P(d), S_d)$ in the projective space $\mathbb{P}(W)$ with

$$W = \bigwedge^r S_d$$

is given by the positive line bundle $L = \det U^*$, where U is the universal subbundle $U \to \mathbb{G}(P(d), S_d)$ of $\mathbb{C}^s \times \mathbb{G}(P(d), S_d) \to \mathbb{G}(P(d), S_d)$ with fiber over a point of $\mathbb{G}(P(d), S_d)$ the corresponding subspace of $\mathbb{C}^s$.

Remark 6.27 *With respect to the basis*

$$x_B = x^{\alpha_{b_1}} \wedge ... \wedge x^{\alpha_{b_r}}$$

with $B = \{b_1, ..., b_r\} \subset \{1, ..., s\}$, $|B| = r$ *of* $\bigwedge^r S_d$ *the Plücker embedding is given by the* $r \times r$ *minors of the matrix representative*

$$A = (a_{j,\alpha_i})_{j=1,...r,\ i=1,...,s} \in \mathbb{C}^{r \times s}$$
$$f_j = \sum_{i=1}^{s} a_{j,\alpha_i} x^{\alpha_i}$$

Denoting by A_B *the matrix formed by the columns of* A *with indices* $b_1, ..., b_r$

$$\begin{array}{rccc} \mathfrak{p}: & \mathbb{G}(P(d), S_d) & \to & \mathbb{P}^{\binom{s}{r}-1} \\ & \langle f_1, ..., f_r \rangle & \mapsto & (\det A_B \mid |B| = r) \end{array}$$

Note that if $A' = UA$ *is another matrix representative, then* $\det A'_B = \det U \det A_B$ *hence the homogeneous coordinates are well defined.*

The action of $\mathrm{SL}(V)$ on V gives a representation of $\mathrm{SL}(V)$ on $S_d = \mathrm{Sym}_d(V)$ and on $W = \bigwedge^r S_d$ so inducing an action

$$\mathrm{SL}(V) \times \mathbb{P}(W) \to \mathbb{P}(W)$$

The Grassmannian $\mathbb{G}(P(d), S_d) \hookrightarrow \mathbb{P}(W)$ and the Hilbert scheme $\mathbb{H}_n^P \subset \mathbb{G}(P(d), S_d)$ are invariant under this action.

Let $T \subset \mathrm{SL}(V)$ be a maximal torus. For $\chi \in \widehat{T}$ define the subspace

$$W_\chi = \{v \in W \mid \Lambda v = \chi(\Lambda) v \ \forall \Lambda \in T\}$$

and

$$\mathrm{State}(W) = \left\{ \chi \in \widehat{T} \mid W_\chi \neq \{0\} \right\}$$

so

$$W = \bigoplus\nolimits_{\chi \in \mathrm{State}(W)} W_\chi$$

If $H(I)$ is the d-th Hilbert point of I and $h^* \in W$ is a representative of $\mathfrak{p}(H(I))$, then we get the corresponding decomposition

$$h^* = \sum\nolimits_{\chi \in \mathrm{State}(W)} h_\chi(I)$$

with $h_\chi(I) \in W_\chi$. The statements $h_\chi(I) = 0$ and $h_\chi(I) \neq 0$ are independent of the choice of h^*, as different representatives are $\mathbb{C}^*$ multiples of each other.

Definition 6.28 *The d**-th state polytope** of I is*

$$\mathrm{State}(I) = \mathrm{convexhull}\{\chi \in \mathrm{State}(W) \mid h_\chi(I) \neq 0\} \subset \widehat{T} \otimes_{\mathbb{Z}} \mathbb{R}$$

If the elements of $T \subset \mathrm{SL}(V)$ are diagonal with respect to the basis $x_0, ..., x_n$, then any one parameter subgroup $\lambda : \mathbb{C}^* \to T$ is of the form

$$\lambda(t) = \mathrm{diag}(t^{w_0}, ..., t^{w_n})$$

with weight vector $w = (w_0, ..., w_n) \in \mathbb{Z}^{n+1}$ and $\sum_{i=0}^n w_i = 0$. By the action of $\mathrm{SL}(V)$ a one parameter subgroup $\lambda : \mathbb{C}^* \to T \subset \mathrm{SL}(V)$ assigns a weight to any monomial of S and to any Plücker coordinate.

Definition 6.29 *A one parameter subgroup $\lambda : \mathbb{C}^* \to T$ is called d-generic if it induces a total ordering on the monomials of S of degree less or equal to d.*

Remark 6.30 *As seen in Section 1.4 for any d and any semigroup ordering $>$ there is a d-generic one parameter subgroup $\lambda : \mathbb{C}^* \to T$ representing $>$ on the monomials of degree at most d.*

Suppose $\lambda : \mathbb{C}^* \to T$, $\lambda(t) = \mathrm{diag}(t^{w_0}, ..., t^{w_n})$ is representing $>$ on the monomials of degree at most d and

$$A = (a_{j,\alpha_i})_{j=1,...r,\ i=1,...,s} \in k^{r \times s}$$

is the $>$-Hilbert matrix representing the d-th Hilbert point $H(I)$ of I with respect to the basis

$$f_j = \sum_{i=1}^{s} a_{j,\alpha_i} x^{\alpha_i},\ j = 1, ..., r$$

of I_d and the $>$-ordered basis $x^{\alpha_1}, ..., x^{\alpha_s}$ of S_d. Then

$$in_> (f_j) = a_{j,\alpha_{b_j}} x^{\alpha_{b_j}}$$

if and only if $a_{j,\alpha_l} = 0 \ \forall l = 1, ..., b_j - 1$ and $a_{j,\alpha_{b_j}} \neq 0$.

Suppose that f_j, $j = 1, ..., r$ is a basis of I_d such that A is in row echelon form

$$\begin{array}{c|ccccc} & x^{\alpha_1} & \cdots & & \cdots & x^{\alpha_s} \\ \hline f_1 & & a_{1,\alpha_{b_1}} & \cdots & & \\ & & & & & \\ \vdots & & & a_{j,\alpha_{b_j}} & \cdots & \\ & & & & & \\ f_r & & & & a_{r,\alpha_{b_r}} & \cdots \end{array} \tag{6.3}$$

Then the Plücker coordinate $\mathfrak{p}(H(I))_B \neq 0$ for

$$B = (b_1, ..., b_r)$$

Note that the Plücker coordinates are independent of the choice of the basis of I_d.

If $B' = (b'_1, ..., b'_r) \neq B$ is some other Plücker coordinate with $x^{\alpha_{b'_j}} > x^{\alpha_{b_j}}$ for some j, then $\mathfrak{p}(H(I))_{B'} = 0$, hence:

Lemma 6.31 *[Bayer, Morrison, 1988] Let $H(I) \in \mathbb{H}_n^P \subset \mathbb{G}(P(d), S_d)$ be the d-th Hilbert point of I. Fix a basis $V = {}_k\langle x_0, ..., x_n\rangle$ and let $\lambda : \mathbb{C}^* \to T$, $\lambda(t) = \operatorname{diag}(t^{w_0}, ..., t^{w_n})$, $w = (w_0, ..., w_n)$ be a d-generic one parameter subgroup of the torus T.*

Then there is a unique Plücker coordinate x_B with $\mathfrak{p}(H(I))_B \neq 0$ such that all Plücker coordinates $B' \neq B$ with $\mathfrak{p}(H(I))_{B'} \neq 0$ have smaller weight $w(x_{B'}) < w(x_B)$.

If $x^{\alpha_1}, ..., x^{\alpha_s}$ is a λ-ordered basis of S_d and $f_1, ..., f_r$ is a basis of I_d such that the corresponding Hilbert matrix of I is in row echelon form as in Equation 6.3, then $f_1, ..., f_r$ form a standard basis of $I_{\geq d}$ and

$$in_> (I_{\geq d}) = \langle x^{\alpha_{b_1}}, ..., x^{\alpha_{b_r}} \rangle$$

If $\lambda : \mathbb{C}^* \to T$ is a 1-parameter subgroup, then $\mathbb{C}^*$ acts on $\mathbb{G}(P(d), S_d)$ written as

$$\begin{array}{ccc} \mathbb{C}^* \times \mathbb{G}(P(d), S_d) & \to & \mathbb{G}(P(d), S_d) \\ (t, z) & \mapsto & \lambda(t) z \end{array}$$

Consider $\mathbb{C}^* \hookrightarrow \mathbb{A}^1$ via $\mathbb{C}[t] \hookrightarrow \mathbb{C}[t, t^{-1}]$, so $\mathbb{C}^* = \mathbb{A}^1 - \{0\}$. If $z \in \mathbb{G}(P(d), S_d)$ and $\mathbb{C}^* \to \mathbb{G}(P(d), S_d)$, $t \mapsto \lambda(t) z$ extends to a morphism $\mathbb{A}^1 \to \mathbb{G}(P(d), S_d)$, then call the image of $0 \in \mathbb{A}^1$ the limit of z under λ, written $\lim_{t \to 0} \lambda(t) z$.

Lemma 6.32 *Let* $H(I) \in \mathbb{H}_n^P \subset \mathbb{G}(P(d), S_d)$ *be the d-th Hilbert point of* I *and let* $\lambda : \mathbb{C}^* \to T$ *be a d-generic one parameter subgroup. With the action of* $\mathrm{SL}(V)$ *on* $\mathbb{G}(P(d), S_d)$

$$\lim_{t\to 0} \lambda(t) H(I) = H' \in \mathbb{H}_n^P \subset \mathbb{G}(P(d), S_d)$$

as $\mathbb{H}_n^P$ *is projective, so there is a homogeneous ideal* $I' \subset S$ *with* $P_{S/I'} = P$ *such that*

$$H' = H(I')$$

Lemma 6.33 *[Bayer, Morrison, 1988] With the setup of Lemma 6.32*

$$in_{>}(I) = I'$$

Proposition 6.34 *[Bayer, Morrison, 1988] The monomial initial ideals* $in_{>}(I)$ *for all semigroup orderings* $>$ *correspond to the vertices of the d-th state polytope* $\mathrm{State}(I)$.

Proposition 6.35 *The Gröbner fan* $GF(I)$ *considered as a fan in*

$$N_{\mathbb{R}} = \frac{\mathbb{R}^{n+1}}{\mathbb{R}(1, ..., 1)}$$

is the normal fan of the d-th state polytope $\mathrm{State}(I)$

$$GF(I) = \mathrm{NF}(\mathrm{State}(I))$$

6.4.3 State polytope and stability

Definition 6.36 *Suppose* $d \geq d_0$ *as in Lemma 6.24, let* $H(I) = I_d \in \mathbb{G}(P(d), S_d)$ *be the d-th Hilbert point of* I *and let* h^* *be a lift of* $\mathfrak{p}(H(I))$ *to*

$$W = \bigoplus\nolimits_{\chi\in\mathrm{State}(W)} W_\chi$$

The ideal I *is called* ***semi-stable*** *if* $0 \notin \overline{\mathrm{SL}(V)\, h^*}$, *otherwise it is called* ***unstable***.

Theorem 6.37 *[Bayer, Morrison, 1988] With the setup of the previous definition, the following conditions are equivalent:*

1. *I is semi-stable.*

2. *For any choice of a basis* $V = {}_k\langle x_0, ..., x_n\rangle$ *and any* 1*-parameter subgroup* $\lambda : \mathbb{C}^* \to D \subset \mathrm{SL}(V)$, $\lambda(t) = \mathrm{diag}(t^{w_0}, ..., t^{w_n})$, $w = (w_0, ..., w_n)$ *with* $\sum_{i=0}^n w_i = 0$ *there are Plücker coordinates* x_B *and* $x_{B'}$ *such that* $\mathfrak{p}(H(I))_B \neq 0$ *and* $\mathfrak{p}(H(I))_{B'} \neq 0$ *and for the corresponding weights it holds*
$$w(x_B) \leq 0 \leq w(x_{B'})$$
3. *For any choice of a basis* $V = {}_k\langle x_0, ..., x_n\rangle$ *the state polytope* $\mathrm{State}(I)$ *contains the origin.*

6.5 Hilbert scheme and state polytope: Polarized toric setup

6.5.1 Linearizations

Before reformulating and generalizing this setup, we need some general facts about linearizations of group actions on line bundles.

If G is an affine algebraic group over K acting rationally on an algebraic variety Y via
$$\sigma : G \times Y \to Y$$
then the pair (Y, σ) is called a G**-variety**.

Definition 6.38 *If* Y *is a* G*-variety by* $\sigma : G \times Y \to Y$ *and* L *is a line bundle on* Y*, then a* G***-linearization*** *of* L *is an action* $\overline{\sigma} : G \times L \to L$ *such that the diagram*
$$\begin{array}{cccccc} & G & \times & L & \overset{\overline{\sigma}}{\to} & L \\ id \times \pi & \downarrow & & & & \downarrow \pi \\ & G & \times & Y & \overset{\sigma}{\to} & Y \end{array}$$
is commutative and the action is linear on the fibers, i.e., for all $y \in Y$ *the maps* $\overline{\sigma}_y(g) : L_y \to L_{g \cdot y}$ *are linear.*

The pair $(L, \overline{\sigma})$ *is called a* G***-linearized line bundle****.*

Here $\pi : L \to Y$ denotes the projection of the total space of L to Y. If $g \in G$, then the group action induces an isomorphism
$$\begin{array}{cccc} \sigma(g) : & X & \to & X \\ & x & \mapsto & g \cdot x \end{array}$$
and for $y \in Y$ the maps $\overline{\sigma}_y(g) : L_y \to L_{g \cdot y}$ are isomorphisms of vector spaces giving an isomorphism of line bundles
$$\overline{\sigma}(g) : L \to g^* L$$

With

$$pr_2 : G \times Y \to Y$$

the isomorphisms of line bundles $\overline{\sigma}(g)$ for $g \in G$ form an isomorphism of line bundles

$$\Phi : pr_2^*(L) \to \sigma^*(L)$$

Indeed, also the converse is true:

Lemma 6.39 *[Kraft, Slodowy, Springer, 1989, Knop, Kraft, Luna and Vust, Sec. 4] If G is a connected affine algebraic group, Y is a G-variety and L is a line bundle on Y, then L has a G-linearization if and only if there is an isomorphism of line bundles*

$$\Phi : pr_2^*(L) \to \sigma^*(L)$$

The set of G-bundles on the G-variety X carries the structure of an abelian group, the **group of G-bundles** $\operatorname{Pic}^G(Y)$: If L and L' are G-bundles with linearizations given by the isomorphisms $\Phi : pr_2^*(L) \to \sigma^*(L)$ and $\Phi' : pr_2^*(L') \to \sigma^*(L')$, then on $L \otimes L'$ a G-linearization is given by the isomorphism

$$\begin{array}{cccc} \Phi \otimes \Phi' : & pr_2^*(L \otimes L') & \to & \sigma^*(L \otimes L') \\ & \parallel & & \parallel \\ & pr_2^*(L) \otimes pr_2^*(L') & \to & \sigma^*(L) \otimes \sigma^*(L') \end{array}$$

The neutral element of $\operatorname{Pic}^G(Y)$ is the line bundle $Y \times K \to Y$ with the G-linearization

$$\sigma \times id : G \times Y \times K \to Y \times K$$

If L is a G-bundle with linearization given by $\Phi : pr_2^*(L) \to \sigma^*(L)$, then its inverse is L^* with the linearization

$$(\Phi^*)^{-1} : pr_2^*(L^*) \to \sigma^*(L^*)$$

The map

$$\alpha : \operatorname{Pic}^G(Y) \to \operatorname{Pic}(Y)$$

forgetting the linearization is a homomorphism.

Proposition 6.40 *[Dolgachev, 2003, Sec. 7] If Y is connected and proper over $\mathbb{C}$, then*

$$\ker(\alpha) \cong \chi(G)$$

Lemma 6.41 *[Kraft, Slodowy, Springer, 1989, Knop, Kraft, Luna and Vust, Sec. 4] If G is a connected affine algebraic group, Y is a normal G-variety and E is a line bundle on $G \times Y$, then for all $y_0 \in Y$*

$$E \cong pr_1^* \left(E \mid_{G \times \{y_0\}}\right) \otimes pr_2^* \left(E \mid_{\{e\} \times Y}\right)$$

So if $y_0 \in Y$, define the homomorphism

$$\begin{array}{rccc} \delta : & \mathrm{Pic}\,(Y) & \rightarrow & \mathrm{Pic}\,(G) \\ & L & \mapsto & pr_2^* (L) \otimes \sigma^* (L^*) \mid_{G \times \{y_0\}} \end{array}$$

which has $\ker(\delta) = \mathrm{image}\,(\alpha)$.

Theorem 6.42 *[Dolgachev, 2003, Sec. 7] If G is a connected affine algebraic group and Y is a normal G-variety, then the sequence*

$$0 \rightarrow \ker(\alpha) \rightarrow \mathrm{Pic}^G (Y) \rightarrow \mathrm{Pic}\,(Y) \rightarrow \mathrm{Pic}\,(G)$$

is exact.

If G is a connected affine algebraic group, then $\mathrm{Pic}\,(G)$ is finite, see [Kraft, Slodowy, Springer, 1989, Knop, Kraft, Luna and Vust, Prop. 4.5], so:

Remark 6.43 $\mathrm{Pic}^G (Y)$ *has finite index in* $\mathrm{Pic}\,(Y)$*, hence for all $L \in \mathrm{Pic}\,(Y)$ there is an m such that $L^{\otimes m}$ is a G-bundle.*

If G is $\mathrm{GL}\,(n, \mathbb{C})$ or a torus $(\mathbb{C}^*)^n$ or $\mathrm{SL}\,(n, \mathbb{C})$, then $\mathrm{Pic}\,(G) = 0$.

Hence if $T = (\mathbb{C}^*)^n$ is a torus and Y is a T-variety, then we have an exact sequence

$$0 \rightarrow \widehat{T} \rightarrow \mathrm{Pic}^T (Y) \rightarrow \mathrm{Pic}\,(Y) \rightarrow 0$$

so any line bundle L on Y has a T-linearization and any two linearizations differ by a translation in the lattice $\widehat{T} \cong \mathbb{Z}^n$.

Remark 6.44 *If Y is a toric variety with torus T, then the sheaf of Zariski differential forms Ω_Y^p has a canonical linearization given by the pullback of differential forms with respect to the isomorphism*

$$\begin{array}{rccc} \sigma (g) : & X & \rightarrow & X \\ & x & \mapsto & g \cdot x \end{array}$$

6.5.2 Setup for subvarieties of a projective toric variety

Let $Y = X(\Sigma)$ be a simplicial toric variety of dimension n given by the fan $\Sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ with $N \cong \mathbb{Z}^n$ and let $L = \mathcal{O}_Y(D)$ be a very ample line bundle on Y. The lattice

$$M = \operatorname{Hom}(N, \mathbb{Z}) = \operatorname{Hom}(T, \mathbb{C}^*) = \widehat{T}$$

is the character group of the torus $T \subset Y$. Let $S = \mathbb{C}[y_v \mid v \in \Sigma(1)]$ be the Cox ring of Y,

$$0 \to M \xrightarrow{A} \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(Y) \to 0$$

the presentation of the Chow group and

$$\begin{gathered} 1 \to G(\Sigma) \to (\mathbb{C}^*)^{\Sigma(1)} \to T \to 1 \\ G(\Sigma) = \operatorname{Hom}_{\mathbb{Z}}(A_{n-1}(X(\Sigma)), \mathbb{C}^*) \end{gathered}$$

the corresponding sequence involving the tori $(\mathbb{C}^*)^{\Sigma(1)}$ and T.

By Section 6.5.1 there is a linearization of the action of T on Y on the line bundle L

$$\begin{array}{ccccccc} & T & \times & L & \xrightarrow{\overline{\sigma}} & L \\ id \times \pi & \downarrow & & & & \downarrow \pi \\ & T & \times & Y & \xrightarrow{\sigma} & Y \end{array}$$

With

$$V = H^0(Y, L) = S_{[D]}$$

the line bundle L defines an embedding

$$\begin{gathered} \phi_V : Y \to \mathbb{P}(V^*) \\ \phi_V(y) = \{s \in V \mid s(y) = 0\} \end{gathered}$$

identifying elements of $\mathbb{P}(V^*)$ with hyperplanes in V. The map ϕ_V is T-equivariant with respect to the action

$$\begin{gathered} T \times \mathbb{P}(V^*) \to \mathbb{P}(V^*) \\ g \cdot H = g^{-1}(H) \end{gathered}$$

The toric variety Y embedded by ϕ_V is isomorphic to the projective toric variety

$$\mathbb{P}(\Delta_D) = \operatorname{Proj} S(\Delta_D)$$

and the polytope ring $S(\Delta_D)$ is isomorphic to

$$S(\Delta_D) \cong \bigoplus_{d=0}^{\infty} S_{d[D]} \subset S$$

which is $\mathbb{Z}_{\geq 0}$-graded by d.

If $I \subset S$ is a homogeneous ideal, then it corresponds under the embedding ϕ_V of Y via $L = \mathcal{O}_Y(D)$ to the ideal

$$I_{\mathbb{N}[D]} = \bigoplus_{d=0}^{\infty} I_{d[D]} \subset S(\Delta_D)$$

The Hilbert function

$$h_{S/I}(k) = \dim_{\mathbb{C}} \left(S_{k[D]}/I_{k[D]}\right)$$

agrees for large k with a polynomial P, the Hilbert polynomial of S/I under the embedding of Y given by L.

Furthermore via this embedding there is a d_0 such that for any homogeneous ideal $J \subset S(\Delta_D)$ with Hilbert polynomial P under the embedding of Y given by L

$$\left(J_{\mathbb{N}[D]} : B(\Sigma)^{\infty}_{\mathbb{N}[D]}\right)_{\geq d_0}$$

is generated by the Hilbert point

$$J_{d_0[D]} \in \mathbb{G} = \mathbb{G}\left(P(d_0), S_{d_0[D]}\right)$$

and these points form the P-th Hilbert scheme

$$\mathbb{H}^P_L \subset \mathbb{G}$$

in the embedding of Y via L.

The action of T on $V = H^0(Y, L) = S_{[D]}$ induces an action of T on

$$W = \bigwedge^r S_{d_0[D]}$$

with $r = \dim S_{d_0[D]} - P(d_0)$, and the Plücker embedding

$$\mathfrak{p} : \mathbb{G} \to \mathbb{P}(W)$$

is T-equivariant. With

$$W_\chi = \{v \in W \mid \Lambda v = \chi(\Lambda)\, v \ \forall \Lambda \in T\}$$

for $\chi \in \widehat{T} = M$ and

$$\mathrm{State}_L(W) = \{\chi \in M \mid W_\chi \neq \{0\}\}$$

we get a decomposition

$$W = \bigoplus\nolimits_{\chi \in \mathrm{State}_L(W)} W_\chi$$

If $h^* \in W$ is a representative of $\mathfrak{p}\left(I_{d_0[D]}\right)$, there is the corresponding decomposition

$$h^* = \sum\nolimits_{\chi \in \mathrm{State}_L(W)} h_\chi$$

with $h_\chi \in W_\chi$. With

$$\mathrm{State}_L(h) = \{\chi \in M \mid h_\chi \neq 0\}$$

define the state polytope of I with respect to L as the convex hull

$$\mathrm{State}_L(I) = \mathrm{convexhull}\left(\mathrm{State}_L(h)\right) \subset \widehat{T} \otimes_{\mathbb{Z}} \mathbb{R} = M_{\mathbb{R}}$$

6.5.3 Hilbert-Mumford stability

Suppose G is a reductive group and Y is an irreducible G-variety.

Definition 6.45 *Let L be a G-bundle on Y and $y \in Y$.*

1. *y is called **semi-stable with respect to** L if there is an $a > 0$ and an $\alpha \in H^0(Y, L^a)^G$ such that*

   $$Y_\alpha = \{z \in Y \mid \alpha(z) \neq 0\}$$

   is affine and $y \in Y_\alpha$.

2. *y is called **unstable with respect to** L if it is not semi-stable with respect to L.*

3. *y is called **stable with respect to** L if the isotropy group G_y is finite and the G-orbits in Y_α are closed.*

Denote by $Y^{ss}(L), Y^{us}(L)$ and $Y^s(L)$ the set of semi-stable, unstable and stable points of Y, respectively.

Lemma 6.46 *[Dolgachev, 2003, Sec. 8] With the notation from above:*
The sets $Y^{ss}(L), Y^{us}(L)$ and $Y^s(L)$ do not change when replacing L by L^a for $a \in \mathbb{Z}_{>0}$.
If L is ample and Y is projective, then Y_α is always affine.

6.5.4 Stability on a variety with a torus action

Suppose X is a projective variety, the torus T acts on X and E is a very ample T-linearized line bundle on X. Let $W = H^0(X, E)$, $s = \dim_{\mathbb{C}} W$ and let $\phi_W : X \to \mathbb{P}(W)$ be the corresponding embedding. So T acts on X via a linear representation

$$T \to \mathrm{GL}(W^*)$$

If $x \in X$ and x^* is a representative of $\phi_W(x)$, then

$$x \in X^{us}(E) \Leftrightarrow 0 \in \overline{T \cdot x^*}$$

So if $0 \in \overline{\lambda(\mathbb{C}^*) \cdot x^*}$ for some one parameter subgroup $\lambda \in \widehat{T}^*$ of T

$$\lambda : \mathbb{C}^* \to T$$

then x is unstable.

Choosing a basis of W such that T acts via diagonal matrices write

$$x^* = (x_1, ..., x_s)$$

with respect to this basis. Then

$$\lambda(t)\, x^* = \left(t^{\beta_1} x_1, ..., t^{\beta_s} x_s\right)$$

with some β_i.

- If $\beta_i > 0$ for all i with $x_i \neq 0$, then

$$\begin{array}{cccc} \lambda_{x^*} : & \mathbb{A}^1 \backslash \{0\} & \to & \mathbb{A}^s \\ & t & \mapsto & \lambda(t)\, x^* \end{array}$$

  extends to a regular map

$$\begin{array}{ccl} \mathbb{A}^1 & \to & \mathbb{A}^s \\ t & \mapsto & \lambda(t)\, x^* \text{ for } t \neq 0 \\ 0 & \mapsto & 0 \end{array}$$

  so $0 \in \overline{\lambda(\mathbb{C}^*) \cdot x^*}$, hence x is unstable.

- If $\beta_i < 0$ for all i with $x_i \neq 0$, then above argument applied to λ^{-1} shows that x is unstable.

Define

$$\mu^E(x,\lambda) = \min\{\beta_i \mid x_i \neq 0\}$$

so if $\mu^E(x,\lambda) > 0$, then $x \in X^{us}(E)$, hence

$$x \in X^{ss}(E) \Rightarrow \mu^E(x,\lambda) \leq 0 \ \forall \lambda \in \widehat{T}^*$$

On the other hand if $\mu^E(x,\lambda) \leq 0 \ \forall \lambda \in \widehat{T}^*$ and there is a $\lambda \in \widehat{T}^*$ with $\mu^E(x,\lambda) = 0$, then $y^* = (y_i)$ with

$$y_i = \left\{ \begin{array}{ll} 0 & \text{if } x_i \neq 0 \text{ and } \beta_i > 0 \\ x_i & \text{otherwise} \end{array} \right\}$$

is in the closure of $\lambda(\mathbb{C}^*) \cdot x^*$, i.e.,

$$y^* \in \overline{\lambda(\mathbb{C}^*) \cdot x^*}$$

If x would be stable, then it would have to hold that $y^* \in \lambda(\mathbb{C}^*) \cdot x^*$, but this is not possible as

$$\lambda(t) \cdot y^* = y^* \text{ for all } t \in \mathbb{C}^*$$

hence

$$x \in X^s(E) \Rightarrow \mu^E(x,\lambda) < 0 \ \forall \lambda \in \widehat{T}^*$$

indeed both statements are characterizations of the semi-stable and stable points:

Theorem 6.47 *[Dolgachev, 2003] With the setup from above*

$$\begin{array}{lcl} x \in X^{ss}(E) & \Leftrightarrow & \mu^E(x,\lambda) \leq 0 \ \forall \lambda \in \widehat{T}^* \\ x \in X^s(E) & \Leftrightarrow & \mu^E(x,\lambda) < 0 \ \forall \lambda \in \widehat{T}^* \end{array}$$

6.5.5 State polytope and Stability

The bilinear pairing between characters and one parameter subgroups of T

$$\begin{array}{lcl} \widehat{T} \times \widehat{T}^* & \rightarrow & \widehat{\mathbb{C}^*} = \mathbb{Z} \\ (\chi,\lambda) & \mapsto & \langle \chi,\lambda \rangle = \chi \circ \lambda \end{array}$$

corresponds via the identification $\widehat{T} = M$ and $\widehat{T}^* = N$ to the canonical bilinear pairing

$$\langle -,- \rangle : M \times N \rightarrow \mathbb{Z}$$

Fix a T-invariant basis $x_0, ..., x_n$ of V and let

$$x_B = x_{b_1} \wedge ... \wedge x_{b_r}$$

with $B = \{b_1, ..., b_r\} \subset \{0, ..., n\}$, $|B| = r$ be the corresponding T-invariant basis of W, which is compatible with the decomposition

$$W = \bigoplus\nolimits_{\chi \in \mathrm{State}_L(W)} W_\chi$$

With respect to the basis (x_B) the representation

$$\rho : T \to \mathrm{GL}\,(W)$$

given by the action $T \times W \to W$ is of the form

$$\rho\,(x) = \mathrm{diag}\,(x^{m_1}, ..., x^{m_{\dim W}})$$

with $m_i \in M$.

If

$$\lambda : \mathbb{C}^* \to T$$
$$\lambda\,(t) = \mathrm{diag}\,(t^{w_1}, ..., t^{w_n})$$

is a one parameter subgroup of T, then the composition is

$$\begin{array}{rccl} \rho \circ \lambda : & \mathbb{C}^* & \to & \mathrm{GL}\,(W) \\ & t & \mapsto & \mathrm{diag}\left(t^{\langle w, m_1\rangle}, ..., t^{\langle w, m_{\dim W}\rangle}\right) \end{array}$$

Now suppose $h^* \in W$ is a representative of the image of the Hilbert point $I_{d_0[D]}$ under the Plücker embedding $\mathfrak{p}$, and write

$$h^* = \sum\nolimits_{\chi \in \mathrm{State}_L(W)} h_\chi$$

with $h_\chi \in W_\chi$. Write

$$h^* = (\alpha_1, ..., \alpha_{\dim W})$$

with respect to the basis (x_B). So

$$\lambda\,(t) \cdot h^* = \mathrm{diag}\left(t^{\langle w, m_1\rangle}\alpha_1, ..., t^{\langle w, m_{\dim W}\rangle}\alpha_{\dim W}\right)$$

hence with the line bundle

$$E = \overline{\mathfrak{p}}^*\left(\mathcal{O}_{\mathbb{P}(W)}\,(1)\right)$$

where $\overline{\mathfrak{p}} : \mathbb{H}_L^P \to \mathbb{P}\,(W)$ is the embedding of the Hilbert scheme induced by the Plücker embedding, we have

$$\mu^E\,(h, \lambda) = \min\left\{\langle w, m_i\rangle \mid \alpha_i \neq 0\right\} = \min_{\chi \in \mathrm{State}_L(h)} \langle \chi, \lambda\rangle$$

so by Theorem 6.47 we obtain:

Theorem 6.48 *Suppose $Y = X(\Sigma)$ is a toric variety given by the fan $\Sigma \subset N_{\mathbb{R}}$, $L = \mathcal{O}_Y(D)$ is a very ample T-line bundle on Y and S is the Cox ring of Y. If $I \subset S$ is homogeneous, then stability and semi-stability of the Hilbert point*

$$H_L(I) \in \mathbb{H} = \mathbb{H}_L^P$$

are characterized via the state polytope $\mathrm{State}_L(I) \subset M_{\mathbb{R}}$ *as follows:*

$$\begin{array}{lcl} H_L(I) \in \mathbb{H}^{ss}(E) & \Leftrightarrow & 0 \in \mathrm{State}_L(I) \\ H_L(I) \in \mathbb{H}^{s}(E) & \Leftrightarrow & 0 \in \mathrm{int}(\mathrm{State}_L(I)) \end{array}$$

6.6 Hilbert scheme and state polytope: Cox homogeneous setup

6.6.1 Grassmann functor

Let k be a commutative ring.

Definition 6.49 *Let N be a finitely generated k-module. The **Grassmann functor** $\mathbb{G}_N^r$:$\underline{k-Alg}\rightarrow\underline{Set}$ is defined as*

$$\mathbb{G}_N^r(R) = \{L \mid L \subset R \otimes N \textit{ submodule with } (R \otimes N)/L \textit{ locally free of rank } r\}$$

An R-module W is locally free of rank r if there are $f_1, ..., f_k \in R$ with $\langle f_1, ..., f_k\rangle = \langle 1 \rangle \subset R$ such that $W_{f_j} \cong R_{f_j}^r$ for all $j = 1, ..., k$.

The Grassmann functor $\mathbb{G}_N^r$ is represented by the **Grassmann scheme** $\mathbb{G}_N^r$ described in coordinates as follows:

- If $N = k^m$:

  Let $v_1, ..., v_m$ be a basis of N and let $B = \{v_{i_1}, ..., v_{i_r}\}$. The subfunctor $\mathbb{G}_{k^m\backslash B}^r$ of $\mathbb{G}_{k^m}^r$ is defined as

  $$\mathbb{G}_{k^m\backslash B}^r(R) = \{L \mid L \subset R^m \text{ submodule with } R^m/L \text{ free with basis } B\}$$

  It is represented by the affine space $\mathbb{A}^{r(m-r)}$ associating $L \in \mathbb{G}_{k^m\backslash B}^r(R)$ to $\left(\lambda_j^i\right)$ with

  $$R^m/L \ni \overline{v_i} = \sum_{j=1}^{r} \lambda_j^i \overline{v_{i_j}} \text{ for } i \notin \{i_1, ..., i_r\}$$

  Via the Plücker embedding the Grassmann functor $\mathbb{G}_{k^m}^r$ is represented by a projective scheme covered by affine open subsets representing $\mathbb{G}_{k^m\backslash B}^r$.

- If $N = k^m/J$ is a finitely generated k-module:

  Then $R \otimes N \cong R^m/RJ$ for any k-algebra R and

  $$\mathbb{G}^r_N(R) = \{L \in \mathbb{G}^r_{k^m}(R) \mid RJ \subset L\}$$

  If $v_1, ..., v_m$ is a basis of k^m and $B = \{v_{i_1}, ..., v_{i_r}\}$, then the subfunctor $\mathbb{G}^r_{k^m \backslash B} \cap \mathbb{G}^r_N$ of $\mathbb{G}^r_{k^m}$ is represented by the subscheme

  $$\left\{ \left(\lambda^i_j\right) \mid a^u_{i_j} + \sum_{i \notin \{i_1,...,i_r\}} a^u_i \lambda^i_j = 0 \ \forall u \in J \ \forall j = 1, ..., r \right\} \subset \mathbb{A}^{r(m-r)}$$

  where for $u \in J$ the $a^u_i \in k$ are defined by

  $$u = \sum_{i=1}^m a^u_i v_i$$

Proposition 6.50 *[Haiman, Sturmfels, 2004] If N is a finitely generated k-module, then the functor $\mathbb{G}^r_N$ is represented by a closed subscheme of the scheme representing $\mathbb{G}^r_{k^m}$.*

Let N be a finitely generated k-module and $M \subset N$ a submodule and consider the subfunctor $\mathbb{G}^r_{N\backslash M} \subset \mathbb{G}^r_N$

$$\begin{aligned}
\mathbb{G}^r_{N\backslash M}(R) &= \{L \in \mathbb{G}^r_N(R) \mid (R \otimes N)/L \text{ locally free with bases in } M\} \\
&= \left\{ L \in \mathbb{G}^r_N(R) \mid \begin{array}{l} \exists f_1, ..., f_k \in R \text{ with } \langle f_1, ..., f_k\rangle = \langle 1 \rangle \\ \text{such that } ((R \otimes N)/L)_{f_j} \text{ has a basis in } M \end{array} \right\} \\
&= \{L \in \mathbb{G}^r_N(R) \mid M \text{ generates } (R \otimes N)/L\}
\end{aligned}$$

Proposition 6.51 *[Haiman, Sturmfels, 2004] $\mathbb{G}^r_{N\backslash M}$ is represented by an open subscheme of the scheme representing $\mathbb{G}^r_N$, so by a quasiprojective scheme over k. It is called the* ***relative Grassmann functor of*** *$M \subset N$.*

If A is a finite set and $N = \bigoplus_{a \in A} N_a$ is a finitely generated graded k-module and $h : A \to \mathbb{N}$ is some function, then the **graded Grassmann functor** $\mathbb{G}^h_N$ is defined as

$$\mathbb{G}^h_N(R) = \left\{ L \mid \begin{array}{l} L \subset R \otimes N \text{ homogeneous submodule with} \\ (R \otimes N_a)/L_a \text{ locally free of rank } h(a) \ \forall a \in A \end{array} \right\}$$

and $\mathbb{G}^h_N$ is naturally isomorphic to $\prod_{a \in A} \mathbb{G}^{h(a)}_{N_a}$ hence is projective.

If $M \subset N$ a homogeneous submodule the **relative graded Grassmann functor of** $M \subset N$ is defined by

$$\mathbb{G}^h_{N \backslash M}(R) = \left\{L \in \mathbb{G}^h_N(R) \mid (R \otimes N_a)/L_a \text{ locally free with bases in } M \ \forall a \in A\right\}$$

and is represented by a quasiprojective scheme over k.

$\mathbb{G}^h_N$ and $\mathbb{G}^h_{N\backslash M}$ are subfunctors of $\mathbb{G}^r_N$ respectively $\mathbb{G}^r_{N\backslash M}$ with $r = \sum_{a \in A} h(a)$ and the corresponding morphisms of schemes are closed embeddings.

6.6.2 Hilbert functor

Let k be a commutative ring, A a set and let

$$S = \bigoplus_{a \in A} S_a$$

be a graded k-module. For all $a, b \in A$ let $F_{a,b} \subset \mathrm{Hom}_k(S_a, S_b)$ be a subset such that $F_{bc} \circ F_{a,b} \subset F_{a,c}$ $\forall a, b \in A$ and $id_{S_a} \in F_{a,a}$ $\forall a \in A$ and call $F = \bigcup_{a,b \in A} F_{a,b}$ a set of operators on S. So (S, F) is a small category of k-modules.

If R is a k-algebra, then

$$R \otimes S = \bigoplus_{a \in A} R \otimes S_a$$

is a graded R-module with operators

$$F^R_{a,b} = (1_R \otimes_k \_)(F_{a,b}) = \{1_R \otimes_k f \mid f \in F_{a,b}\}$$

A homogeneous submodule $L = \bigoplus_{a \in A} L_a \subset R \otimes S$ is called an **F-submodule** if $F^R_{a,b}(L_a) \subset L_b$ for all $a, b \in A$.

Definition 6.52 *If $h : A \to \mathbb{N}$ is a function and R is a k-algebra, then define*

$$\mathbb{H}^h_{(S,F)}(R) = \left\{ L \mid \begin{array}{l} L \subset R \otimes S \text{ is an } F\text{-submodule with} \\ (R \otimes S_a)/L_a \text{ locally free of rank } h(a) \ \forall a \in A \end{array} \right\}$$

If $\phi : R \to R'$ is a homomorphism of commutative rings and $L \in \mathbb{H}^h_{(S,F)}(R)$, then $L' = R' \otimes_R L$ is an F-submodule of $R' \otimes S$ and $(R' \otimes_k S_a)/L'_a$ is locally free of rank $h(a)$ for all $a \in A$, so define $\mathbb{H}^h_{(S,F)}(\phi) : \mathbb{H}^h_{(S,F)}(R) \to \mathbb{H}^h_{(S,F)}(R')$, $L \mapsto L'$. These assignments make $\mathbb{H}^h_{(S,F)}$ into a functor $\underline{k - Alg} \to \underline{Set}$, the ***Hilbert functor****.*

If $D \subset A$ is a subset the restriction (S_D, F_D) of (S,F) to degree D is defined by

$$S_D = \bigoplus_{a \in D} S_a \qquad F_D = \bigcup_{a,b \in D} F_{a,b}$$

It is a full subcategory of (S,F) and there is the natural restriction map

$$\begin{array}{ccc} \mathbb{H}^h_{(S,F)} & \to & \mathbb{H}^h_{(S_D,F_D)} \\ L & \mapsto & L_D = \bigoplus_{a \in D} L_a \end{array}$$

Lemma 6.53 *If $L' \subset R \otimes S_D$ is an F_D-submodule and $L \subset R \otimes S$ is the F-submodule generated by L', then $L_b = \sum_{a \in D} F_{a,b}(L'_a)$ for all $b \in A$, so $L_D = L'$.*

Theorem 6.54 *[Haiman, Sturmfels, 2004] Let k be a commutative ring, A a set, S an A-graded k-module with operators F and $h : A \to \mathbb{N}$ a function. If there are homogeneous k-submodules $M \subset N \subset S$ such that*

1. *N is a finitely generated k-module,*
2. *N generates S as an F-module,*
3. *for all fields $K \in \underline{k - Alg}$ and for all $L \in \mathbb{H}^h_{(S,F)}(K)$ the submodule $M \subset S$ spanns $(K \otimes S)/L$,*
4. *there is a subset $G \subset F$ which generates F as a category such that $GM \subset N$,*

then

- *N spans $(K \otimes S)/L$ so $\infty > \dim_K((K \otimes S)/L) = \sum_{a \in A} h(a)$ hence h has finite support,*
- *$\mathbb{H}^h_{(S,F)}$ is represented by a quasiprojective closed subscheme of $\mathbb{G}^h_{N \setminus M}$ over k, the Hilbert scheme.*

Corollary 6.55 *[Haiman, Sturmfels, 2004] If A is finite and S_a is a finitely generated k-module for all $a \in A$, then in above theorem one can choose $M = N = S$ and $G = F$, so $\mathbb{H}^h_{(S,F)}$ is represented by a closed subscheme of the projective Grassmann scheme $\mathbb{G}^h_{N \setminus M} = \mathbb{G}^h_N$, hence is projective.*

Theorem 6.56 *[Haiman, Sturmfels, 2004] Let k be a commutative ring, A a set, S an A-graded k-module with operators F and $h : A \to \mathbb{N}$ a function. Suppose $D \subset A$ such that*

1. $\mathbb{H}^h_{(S_D,F_D)}$ *is represented by a scheme over* k,

2. *for all* $a \in A$ *there is a finite set of operators* $E \subset \bigcup_{b\in D} F_{b,a}$ *such that* $S_a / \sum_{b\in D} E_{b,a}(S_b)$ *is a finitely generated* k*-module,*

3. *for all fields* $K \in \underline{k - Alg}$ *and for all* $L' \in \mathbb{H}^h_{(S_D,F_D)}(K)$

$$\dim\left(\left(K \otimes S_a\right)/L_a\right) \leq h(a)$$

for all $a \in A$, *where* $L \subset K \otimes S$ *is the* F*-submodule generated by* L'.

Then $\mathbb{H}^h_{(S,F)}$ *is a subfunctor of* $\mathbb{H}^h_{(S_D,F_D)}$ *via the natural restriction map* $L \mapsto L_D$ *and is represented by a closed subscheme of the Hilbert scheme representing* $\mathbb{H}^h_{(S_D,F_D)}$.

If D *is finite, then* $\mathbb{H}^h_{(S_D,F_D)}$ *is projective, hence* $\mathbb{H}^h_{(S,F)}$ *is projective.*

6.6.3 Example: Multigraded Hilbert schemes of admissible ideals

Let k be a commutative ring, A an abelian group and $S = k[x_1, ..., x_r]$ a polynomial ring graded by a homomorphism of semigroups $\deg : \mathbb{N}^n \to A$ via $\deg x^u = \deg u$, so

$$S = \bigoplus_{a\in A} S_a$$

and

$$S_a \cdot S_b \subset S_{a+b}$$

The ring S comes with operators $F = \bigcup_{a,b\in A} F_{a,b}$ where

$$F_{a,b} = \left\{ \left\{ \begin{array}{ccc} S_a & \to & S_b \\ f & \mapsto & m \cdot f \end{array} \right\} \in \operatorname{Hom}_k(S_a, S_b) \mid \begin{array}{l} m \in S \text{ a monomial with} \\ \deg m = b - a \end{array} \right\}$$

If $L \subset R \otimes S = \bigoplus_{a\in A} R \otimes S_a$ is an F-submodule, then L is a homogeneous ideal with respect to the grading of $R \otimes S$ by A.

A homogeneous ideal $I \subset S$ is called **admissible** if $(S/I)_a = S_a/I_a$ is a locally free k-module of finite rank for all $a \in A$. Denote by

$$\begin{array}{cccc} h_{S/I}: & A & \to & \mathbb{N} \\ & a & \mapsto & \operatorname{rank}_k\left((S/I)_a\right) \end{array}$$

the Hilbert function of S/I. Denote by $A_+ = \langle a_1, ..., a_r \rangle \subset A$ the subgroup generated by $a_i = \deg x_i$. The support of $h_{S/I}$ is contained in A_+.

If $h : A \to \mathbb{N}$ is a function with support on A_+, then for any $R \in \underline{k-Alg}$

$$\mathbb{H}^h_{(S,F)}(R) = \left\{ I \mid \begin{array}{l} I \subset R \otimes S \text{ homogeneous ideal such that} \\ (R \otimes S_a)/I_a \text{ locally free of rank } h(a)\ \forall a \in A \end{array} \right\}$$
$$= \left\{ I \mid I \subset R \otimes S \text{ admissible ideal with } h_{S/I} = h \right\}$$

consists of the admissible ideals in $R \otimes S$ with Hilbert function h.

An antichain of monomial ideals in S is a set C of monomial ideals such that for all $I_1, I_2 \in C$ it holds $I_1 \not\subset I_2$.

Lemma 6.57 *[Maclagan, 2001] If C is an antichain in S then C is finite.*

So if C is the set of all monomial ideals in S with Hilbert function h then C is finite.

Corollary 6.58 *[Haiman, Sturmfels, 2004] If $h : A \to \mathbb{N}$ is a function with support on A_+, then there is a finite set $D \subset A$ such that*

1. *any monomial ideal $I \subset S$ with $h_{S/I} = h$ is generated by monomials in degrees D,*
2. *any monomial ideal $I \subset S$ generated in degrees D satisfies: If $h_{S/I}(a) = h(a)$ for all $a \in D$, then $h_{S/I}(a) \leq h(a)$ for all $a \in A$.*

For any finite $D \subset A$ the assumptions of Theorem 6.54 hold for (S_D, F_D), hence $\mathbb{H}^h_{(S_D,F_D)}$ is represented by a quasiprojective scheme.

For D as given by Corollary 6.58 the assumptions of Theorem 6.56 are satisfied, hence:

Theorem 6.59 *[Haiman, Sturmfels, 2004] If $h : A \to \mathbb{N}$ is a function with support on A_+, then $\mathbb{H}^h_{(S,F)}$ is represented by a quasiprojective scheme.*

Note that this setup is not directly applicable to the Cox ring of a toric variety or the homogeneous coordinate ring of projective space.

6.6.4 Example: Classical Hilbert functor

The Grothendieck Hilbert scheme represents the functor $\mathbb{H}^P_n$ with

$$\mathbb{H}^P_n(R) = \{X \mid X \subset \mathbb{P}^n(R) \text{ flat family with Hilbert polynomial } P\}$$

for $R \in \underline{k-Alg}$. These X correspond to saturated homogeneous ideals $I \subset R[x_0, ..., x_n]$ with Hilbert polynomial P.

Given P and n there is a degree d_0, the maximum of the Castelnuovo-Mumford regularities of all saturated monomial ideals in $R[x_0, ..., x_n]$ with Hilbert polynomial P such that for all saturated homogeneous ideals $I \subset R[x_0, ..., x_n]$ with Hilbert polynomial P

$$h_{S/I}(a) = P(a) \text{ for all } a \geq d_0$$

Proposition 6.60 *[Haiman, Sturmfels, 2004] Consider $S = k[x_0, ..., x_n]$, let F be the multiplication by monomials, P some Hilbert polynomial and*

$$h(a) = \left\{ \begin{array}{ll} \binom{n+a-1}{a} & \text{for } a < d_0 \\ P(a) & \text{for } a \geq d_0 \end{array} \right\}$$

The Grothendieck Hilbert scheme representing $\mathbb{H}_n^P$ is isomorphic to the Hilbert scheme representing $\mathbb{H}_{(S,F)}^h$ via the bijection

$$\begin{array}{ccc} \mathbb{H}_n^P(R) & \rightleftarrows & \mathbb{H}_{(S,F)}^h(R) \\ I_{\geq a_0} & \leftarrow & I \\ J & \mapsto & (J : \langle x_0, ..., x_n \rangle^{\infty}) \end{array}$$

6.6.5 Tangent space and deformations

Let k be a field, A an abelian group, $S = k[x_1, ..., x_r]$ graded by $\deg : \mathbb{N}^n \to A$ and F the multiplication by monomials. Let $h : A \to \mathbb{N}$ be a function with support on A_+ and let $I \in \mathbb{H}_{(S,F)}^h(k)$. The S-module $\operatorname{Hom}_S(I, S/I)$ is graded by A and $\operatorname{Hom}_S(I, S/I)_a$ is a finite dimensional k-vector space for all $a \in A$.

Let $R = k[t] / \langle t^2 \rangle$ and $\phi : R \to k$, $s \mapsto s/\langle t \rangle$ so the map

$$\mathbb{H}_{(S,F)}^h(\phi) : \mathbb{H}_{(S,F)}^h(R) \to \mathbb{H}_{(S,F)}^h(k)$$

is given by $J \mapsto J/\langle t \rangle$. The Zariski tangent space of the scheme representing $\mathbb{H}_{(S,F)}^h$ at $I \in \mathbb{H}_{(S,F)}^h(k)$ is

$$\begin{array}{l} \left\{ J \in \mathbb{H}_{(S,F)}^h(R) \mid \mathbb{H}_{(S,F)}^h(\phi)(J) = I \right\} \\ = \left\{ J \mid \begin{array}{l} J \subset R \otimes S \text{ an } A\text{-homogeneous ideal with } J/\langle t \rangle = I \\ \text{such that } R[x_1, ..., x_r]/J \text{ is a free } R\text{-module} \end{array} \right\} \end{array}$$

and is isomorphic to $\operatorname{Hom}_S(I, S/I)_0$ by associating to J the homomorphism

$$S \xrightarrow{t\cdot} t \cdot R[x_1, ..., x_r] \to t \cdot R[x_1, ..., x_r] / (J \cap t \cdot R[x_1, ..., x_r]) \cong S/I$$

Proposition 6.61 *[Haiman, Sturmfels, 2004] The Zariski tangent space of the scheme representing $\mathbb{H}_{(S,F)}^h$ at $I \in \mathbb{H}_{(S,F)}^h(k)$ is canonically isomorphic to* $\operatorname{Hom}_S(I, S/I)_0$.

6.6.6 Stanley decompositions

Setup Let $Y = X(\Sigma)$ be a smooth complete toric variety of dimension n given by the fan $\Sigma \subset N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ with $N \cong \mathbb{Z}^n$. Let $S = \mathbb{C}[y_r \mid r \in \Sigma(1)]$ be the Cox ring of Y graded by $A_{n-1}(Y)$, and

$$B(\Sigma) = \left\langle y^{D_{\widehat{\sigma}}} \mid \sigma \in \Sigma \right\rangle \subset S$$
$$\text{with } D_{\widehat{\sigma}} = \sum_{r \in \Sigma(1),\ r \not\subset \sigma} D_r$$

the irrelevant ideal of Y. Write

$$0 \to M \xrightarrow{A} \mathrm{WDiv}_T(Y) \xrightarrow{\deg} A_{n-1}(Y) \to 0$$

for the presentation of the Chow group of Y and set $a_i = \deg D_i$. Denote by

$$Y' = \mathbb{A}^{\Sigma(1)} - V(B(\Sigma)) = X(\Sigma') \to Y$$

the Cox quotient presentation of Y as defined in Section 1.3.9 and set $Y'' = \mathbb{A}^{\Sigma(1)} = X(\Sigma'')$ with the fan $\Sigma'' \subset \mathbb{Z}^{\Sigma(1)}$ over the standard simplex. For

$$D \in \mathrm{WDiv}_T(Y'') \cong \mathrm{WDiv}_T(Y') \cong \mathrm{WDiv}_T(Y) \cong \mathbb{Z}^{\Sigma(1)}$$

denote by x^D the corresponding (Laurent-) monomial in the Cox ring S.

Denote by $\mathcal{K}$ the set of integral points in the closure of the Kähler cone

$$\mathrm{cpl}(\Sigma) \subset A^+_{n-1}(Y) \otimes \mathbb{R} \subset A_{n-1}(Y) \otimes \mathbb{R} \cong H^2(Y, \mathbb{R})$$

as described in Section 1.3.11.

Primary decompositions and Stanley decompositions of monomial ideals Consider first the vanishing locus of a monomial ideal in the affine space Y''.

Definition 6.62 *If $I \subset S$ is a monomial ideal, then a **Stanley decomposition** of I is a subset*

$$\mathcal{S} \subset \{(D, \sigma) \mid D \in \mathrm{WDiv}_T(Y''),\ D \text{ effective},\ \sigma \in \Sigma''\}$$

such that

$$S/I \cong \bigoplus_{(D,\sigma) \in \mathcal{S}} S_\sigma(-[D])$$

where $S_\sigma = \mathbb{C}[y_r \mid r \notin \sigma] \cong S/I(V_{Y''}(\sigma))$ is the Cox ring of $U_{Y''}(\sigma)$. Here $V_{Y''}(\sigma) \subset Y''$ is the torus orbit closure associated to $\sigma \in \Sigma''$ and $U_{Y''}(\sigma) = \mathrm{Spec}(\mathbb{C}[\check{\sigma} \cap M]) \subset Y''$.

Remark 6.63 *Note that for the Cox quotient representation $Y'' \supset Y' \to Y$ it holds*

$$\mathrm{WDiv}_T(Y'') \cong \mathrm{WDiv}_T(Y') \cong \mathrm{WDiv}_T(Y)$$

and

$$\Sigma'' \supset \Sigma' \supset \Sigma'(1) \overset{1:1}{\rightleftarrows} \Sigma(1)$$

so Σ can be considered as a subfan of Σ''. If $\sigma \in \Sigma$, then

$$\begin{array}{lcl} Y'' & \supset & V_{Y''}(\sigma) = \{y \in Y'' \mid y_r = 0 \ \forall r \in \Sigma(1), \ r \subset \sigma\} \\ \cup & & \cup \\ Y' = Y'' - V(B(\Sigma)) & \supset & V_{Y'}(\sigma) = \{y \in Y' \mid y_r = 0 \ \forall r \in \Sigma(1), \ r \subset \sigma\} \\ \downarrow & & \downarrow \\ Y & \supset & V_Y(\sigma) \end{array}$$

so the prime ideal

$$\langle y_r \mid r \in \Sigma(1), \ r \subset \sigma \rangle \subset S$$

corresponds to the torus orbit closure $V_Y(\sigma) \subset Y$.

Recall also that with

$$D_{\widehat{\sigma}} = \sum_{r \in \Sigma(1), \ r \not\subset \sigma} D_r$$

$$U_{Y'}(\sigma) = Y'' - V\left(y^{D_{\widehat{\sigma}}}\right)$$

we have

$$U_{Y'}(\sigma) / G(\Sigma) = U_Y(\sigma)$$

Any associated prime of I is of the form $\langle y_r \mid r \in \Sigma(1), r \subset \sigma \rangle$ for some $\sigma \in \Sigma''$.

Lemma 6.64 *[Maclagan, Smith, 2005] Let $I \subset S$ be a monomial ideal. Then I is $B(\Sigma)$-saturated if and only if all associated primes of I are of the form*

$$\langle y_r \mid r \in \Sigma(1), \ r \subset \sigma \rangle$$

for $\sigma \in \Sigma$.

A pair (D, σ) with $D \in \mathrm{WDiv}_T(Y'')$, D effective and $\sigma \in \Sigma''$ is called **admissible** if $\mathrm{supp}(D) \cap \mathrm{supp}(D_{\widehat{\sigma}}) = \emptyset$, i.e., if $D \subset U_{Y''}(\sigma)$.

A partial order on the set of admissible pairs is given by

$$\begin{array}{lcl} (D_1, \sigma_1) \leq (D_2, \sigma_2) & \Leftrightarrow & \begin{array}{l} D_2 - D_1 \geq 0 \text{ and} \\ \mathrm{supp}\left((D_2 - D_1) + D_{\widehat{\sigma_2}}\right) \subset \mathrm{supp}\left(D_{\widehat{\sigma_1}}\right) \end{array} \\ & \Leftrightarrow & \begin{array}{l} D_2 - D_1 \geq 0 \text{ and} \\ U_{Y''}(\sigma_1) \subset U_{Y''}(\sigma_2) \cap (Y'' - \mathrm{supp}(D_2 - D_1)) \end{array} \\ & \Leftrightarrow & y^{D_2} S_{\sigma_2} \subset y^{D_1} S_{\sigma_1} \end{array}$$

An admissible pair (D, σ) is called **standard** with respect to I if (D, σ) is minimal with respect to $\leq$ with the property $y^D S_\sigma \cap I = \{0\}$.

Lemma 6.65 *If $\mathcal{S}$ gives a Stanley decomposition of the monomial ideal $I \subset S$, then*

$$I = \bigcap_{(D,\sigma)\in\mathcal{S}} \left\langle y_r^{u_r+1} \mid r \in \Sigma(1),\ r \subset \sigma,\ D = \textstyle\sum_{r\in\Sigma(1)} u_r D_r \right\rangle$$

If $(D, \sigma) \in \mathcal{S}$ is a standard pair, then $\langle y_r \mid r \in \Sigma(1),\ r \subset \sigma\rangle = I(V_{Y''}(\sigma))$ is an associated prime of I.

Algorithm 6.66 *[Maclagan, Smith, 2005] The following algorithm computes a Stanley decomposition of the monomial ideal $I \subset S$:*

- *If I is a prime ideal and $I = \langle y_r \mid r \in \Sigma(1),\ r \subset \sigma\rangle$ with $\sigma \in \Sigma''$, then return $\{(0, \sigma)\}$.*
- *Otherwise, let $r \in \Sigma(1)$ such that $I \neq (I : \langle y_r\rangle) \neq \langle 1\rangle$.*

  Compute Stanley decompositions $\mathcal{S}_1$ of $S/(I + \langle y_r\rangle)$ and $\mathcal{S}_2$ of $S/(I : \langle y_r\rangle)$.

  Return

  $$\mathcal{S} = \{(D_1, \sigma_1) \mid (D_1, \sigma_1) \in \mathcal{S}_1\} \cup \{(D_2 + D_r, \sigma_2) \mid (D_2, \sigma_2) \in \mathcal{S}_2\}$$

Consider the following examples in projective space.

Example 6.67 *A Stanley decomposition of the reduced ideal*

$$I = \langle y_1 y_2 y_3\rangle \subset S = \mathbb{C}[y_0, y_1, y_2]$$

is given by

$$\begin{aligned} S/I = 1 \cdot \mathbb{C} \oplus &\\ & y_0 \cdot \mathbb{C}[y_0] \oplus y_1 \cdot \mathbb{C}[y_1] \oplus y_2 \cdot \mathbb{C}[y_2] \oplus \\ & y_0 y_1 \cdot \mathbb{C}[y_0, y_1] \oplus y_1 y_2 \cdot \mathbb{C}[y_1, y_2] \oplus y_0 y_2 \cdot \mathbb{C}[y_0, y_2] \end{aligned}$$

writing I as the intersection

$$I = \underbrace{\langle y_0, y_1, y_2\rangle}_{\text{irrelevant}} \cap \langle y_1, y_2\rangle \cap \langle y_0, y_2\rangle \cap \langle y_0, y_1\rangle \cap \underbrace{\langle y_2\rangle \cap \langle y_0\rangle \cap \langle y_1\rangle}_{\text{associated primes}}$$

corresponding to the toric stratification of I. The ideal $\langle y_0, y_1, y_2\rangle$ defines $0 \in Y''$ but does not correspond to a subvariety of Y' and Y, hence it is *irrelevant. The corresponding maximal cone σ of the fan Σ'' is not a cone of Σ'.*

The Stanley decomposition given by above algorithm is

$$\begin{aligned} S/I &= 1\cdot\mathbb{C}\left[y_1,y_2\right]\oplus y_0\cdot\mathbb{C}\left[y_0,y_2\right]\oplus y_0y_1\cdot\mathbb{C}\left[y_0,y_1\right] \\ I &= \langle y_0\rangle\cap\langle y_1\rangle\cap\langle y_2\rangle \end{aligned}$$

Note that there are ideals which do not admit a Stanley decomposition, where every factor corresponds to an associated prime, e.g.,

$$I = \langle y_1,y_2\rangle\cap\langle y_0,y_3\rangle\subset\mathbb{C}\left[y_0,...,y_3\right]$$

Example 6.68 *Consider the ideal $I = \langle y_1y_2, y_0y_3\rangle \subset S = \mathbb{C}\left[y_0,...,y_3\right]$. Then*

$$\begin{aligned} S/I = \;& 1\cdot\mathbb{C}\oplus \\ & y_0\cdot\mathbb{C}\left[y_0\right]\oplus y_1\cdot\mathbb{C}\left[y_1\right]\oplus y_2\cdot\mathbb{C}\left[y_2\right]\oplus y_3\cdot\mathbb{C}\left[y_3\right]\oplus \\ & y_0y_1\cdot\mathbb{C}\left[y_0,y_1\right]\oplus y_0y_2\cdot\mathbb{C}\left[y_0,y_2\right]\oplus y_1y_3\cdot\mathbb{C}\left[y_1,y_3\right]\oplus y_2y_3\cdot\mathbb{C}\left[y_2,y_3\right] \end{aligned}$$

is a Stanley decomposition of I representing the ideal as

$$\begin{aligned} I = \;& \overbrace{\langle y_0,y_1,y_2,y_3\rangle}^{irrelevant}\cap \\ & \langle y_1,y_2,y_3\rangle\cap\langle y_0,y_2,y_3\rangle\cap\langle y_0,y_1,y_3\rangle\cap\langle y_0,y_1,y_2\rangle\cap \\ & \underbrace{\langle y_0,y_1\rangle\cap\langle y_0,y_2\rangle\cap\langle y_1,y_3\rangle\cap\langle y_2,y_3\rangle}_{associated\ primes} \end{aligned}$$

The Stanley decomposition given by above algorithm is

$$\begin{aligned} S/I &= 1\cdot\mathbb{C}\left[y_2,y_3\right]\oplus y_1\cdot\mathbb{C}\left[y_1,y_3\right]\oplus y_0\cdot\mathbb{C}\left[y_0,y_2\right]\oplus y_0y_1\cdot\mathbb{C}\left[y_0,y_1\right] \\ I &= \langle y_0,y_1\rangle\cap\langle y_0,y_2\rangle\cap\langle y_1,y_3\rangle\cap\langle y_2,y_3\rangle \end{aligned}$$

The first decomposition is obtained from the second by further subdivision:

$$\begin{aligned} 1\cdot\mathbb{C}\left[y_2,y_3\right] &= 1\cdot\mathbb{C}\oplus y_2\cdot\mathbb{C}\left[y_2\right]\oplus y_3\cdot\mathbb{C}\left[y_3\right]\oplus y_2y_3\cdot\mathbb{C}\left[y_2,y_3\right] \\ y_0\cdot\mathbb{C}\left[y_0,y_2\right] &= y_0y_2\cdot\mathbb{C}\left[y_0,y_2\right]\oplus y_0\cdot\mathbb{C}\left[y_0\right] \\ y_1\cdot\mathbb{C}\left[y_1,y_3\right] &= y_1y_3\cdot\mathbb{C}\left[y_1,y_3\right]\oplus y_1\cdot\mathbb{C}\left[y_1\right] \end{aligned}$$

Stanley filtrations

Definition 6.69 *If $I\subset S$ is a monomial ideal, then a* ***Stanley filtration*** *is a Stanley decomposition with ordering of the elements*

$$\mathcal{S} = \left\{\left(D_1,\sigma_1\right),...,\left(D_s,\sigma_s\right)\right\}$$

such that for all $j = 1, ..., s$

$$\mathcal{S}_j = \{(D_1, \sigma_1), ..., (D_j, \sigma_j)\}$$

is a Stanley decomposition of

$$S / \left(I + \left\langle x^{D_{j+1}}, ..., x^{D_s} \right\rangle\right)$$

So a Stanley filtration gives Stanley decompositions

$$\begin{array}{lcl} S/\left(I + \left\langle x^{D_2}, ..., x^{D_s} \right\rangle\right) & \cong & S_{\sigma_1}(-[D_1]) \\ S/\left(I + \left\langle x^{D_3}, ..., x^{D_s} \right\rangle\right) & \cong & S_{\sigma_1}(-[D_1]) \oplus S_{\sigma_2}(-[D_2]) \\ \vdots & & \vdots \\ S/I & \cong & S_{\sigma_1}(-[D_1]) \oplus ... \oplus S_{\sigma_s}(-[D_s]) \end{array}$$

Algorithm 6.66 computes a Stanley filtration by ordering the leaves of the generated tree by listing the $(I + \langle y_r \rangle)$ child prior to the $(I : \langle y_r \rangle)$ child. This algorithm is implemented in the Macaulay2 library stanleyfiltration.m2.

Example 6.70 *For* $I = \langle y_1 y_2, y_0 y_3 \rangle \subset S = \mathbb{C}[y_0, ..., y_3]$ *above algorithm computes the Stanley filtration*

$$\begin{array}{rl} S/\langle y_1, y_0 \rangle = & 1 \cdot \mathbb{C}[y_2, y_3] \\ S/\langle y_1 y_2, y_0 \rangle = & 1 \cdot \mathbb{C}[y_2, y_3] \oplus y_1 \cdot \mathbb{C}[y_1, y_3] \\ S/\langle y_1 y_2, y_0 y_3, y_0 y_1 \rangle = & 1 \cdot \mathbb{C}[y_2, y_3] \oplus y_1 \cdot \mathbb{C}[y_1, y_3] \oplus y_0 \cdot \mathbb{C}[y_0, y_2] \\ S/\langle y_1 y_2, y_0 y_3 \rangle = & 1 \cdot \mathbb{C}[y_2, y_3] \oplus y_1 \cdot \mathbb{C}[y_1, y_3] \oplus y_0 \cdot \mathbb{C}[y_0, y_2] \oplus y_0 y_1 \cdot \mathbb{C}[y_0, y_1] \end{array}$$

corresponding to

$$\begin{array}{rl} \langle y_1, y_0 \rangle = & \langle y_0, y_1 \rangle \\ \langle y_1 y_2, y_0 \rangle = & \langle y_0, y_1 \rangle \cap \langle y_0, y_2 \rangle \\ \langle y_1 y_2, y_0 y_3, y_0 y_1 \rangle = & \langle y_0, y_1 \rangle \cap \langle y_0, y_2 \rangle \cap \langle y_1, y_3 \rangle \\ \langle y_1 y_2, y_0 y_3 \rangle = & \langle y_0, y_1 \rangle \cap \langle y_0, y_2 \rangle \cap \langle y_1, y_3 \rangle \cap \langle y_2, y_3 \rangle \end{array}$$

6.6.7 Multigraded regularity

Let $C = \{c_1, ..., c_e\} \subset A_{n-1}(Y)$ be a finite subset and $\mathbb{N}C \subset A_{n-1}(Y)$ the semigroup generated by C. A subset $D \subset A_{n-1}(Y)$ is called an $\mathbb{N}C$-module if $d + c \in D$ for all $d \in D$ and $c \in \mathbb{N}C$. If D is an $\mathbb{N}C$-module and $i \in \mathbb{Z}$, then

$$D[i] = \bigcup_{\substack{\lambda_1 + ... + \lambda_e = |i| \\ \lambda_j \in \mathbb{Z}_{\geq 0}}} \left(\operatorname{sign}(i) \cdot \textstyle\sum_{j=1}^{e} \lambda_j c_j + D\right) \subset A_{n-1}(Y)$$

is an $\mathbb{N}C$-module. For $m \in A_{n-1}(Y)$ it holds $(m + D)[i] = m + D[i]$ and $D[i+1] \subset D[i]$.

Definition 6.71 *Let M be a finitely generated $A_{n-1}(Y)$-graded S-module and let $m \in A_{n-1}(Y)$. Then M is called m-**regular** with respect to C if for all $i \geq 1$ and for all*

$$a \in m + \mathbb{N}C[1-i]$$

the local cohomology satisfies

$$H^i_{B(\Sigma)}(M)_a = 0$$

The regularity of M with respect to C is the subset

$$\operatorname{reg}_C(M) = \{m \in A_{n-1}(Y) \mid M \text{ is } m\text{-regular with respect to } C\}$$

Remark 6.72 *With $C = \{c_1, ..., c_e\}$ the module M is m-regular if and only if*

$$H^i_{B(\Sigma)}(M)_a = 0$$

holds for all $i \geq 1$ and all

$$a \in \bigcup_{\substack{\lambda_1+...+\lambda_e=i-1 \\ \lambda_j\in\mathbb{Z}_{\geq 0}}} \left(m - \textstyle\sum_{j=1}^{e} \lambda_j c_j + \mathbb{N}C\right)$$

and for $i = 0$ and all

$$a \in \bigcup_{j=1}^{e} (m + c_j + \mathbb{N}C)$$

Definition 6.73 *Let $m \in A_{n-1}(Y)$. A finitely generated $A_{n-1}(Y)$-graded S-module M is called m-**regular**, if it is m-regular with respect to the unique minimal Hilbert basis $C = \{c_1, ..., c_e\}$ of $\mathcal{K}$, giving any element of $\mathcal{K}$ as a $\mathbb{Z}_{\geq 0}$-linear combination, i.e., with $\mathbb{N}C = \mathcal{K}$. The **regularity** of M is*

$$\operatorname{reg}(M) = \{m \in A_{n-1}(Y) \mid M \text{ is } m\text{-regular}\}$$

The local cohomology groups may be computed in the following way:

Let Γ be a finite regular cell complex. A function $\varepsilon : \Gamma \times \Gamma \to \{-1, 0, 1\}$ is called an **incidence function** if

- $\varepsilon(F, G) \neq 0 \Leftrightarrow G$ is a face of F.
- $\varepsilon(F, \emptyset) = 1$ for all 0-cells $F \in \Gamma^0$.
- If $G \in \Gamma^{i-2}$ is a face of $F \in \Gamma^i$, then

  $$\varepsilon(F, H_1)\,\varepsilon(H_1, G) + \varepsilon(F, H_2)\,\varepsilon(H_2, G) = 0$$

  for the unique two faces $H_1, H_2 \in \Gamma^{i-1}$ of F such that G is a face of H_1 and H_2.

Lemma 6.74 *[Bruns, Herzog, 1993, Sec. 6.3] If Γ is a finite regular cell complex, then there is an incidence function on Γ determined by an orientation of the cells.*

Associated to a cell complex Γ of dimension n together with an incidence function ε there is the **augumented oriented chain complex**

$$\widetilde{\mathcal{C}}(\Gamma) : 0 \to \mathcal{C}_{n-1} \xrightarrow{\delta} \mathcal{C}_{n-2} \to ... \to \mathcal{C}_0 \xrightarrow{\delta} \mathcal{C}_{-1} \to 0$$

with coefficients in R where

$$\mathcal{C}_i = \bigoplus\nolimits_{F\in\Gamma^i} R \cdot F$$
$$\delta(F) = \sum_{G\in\Gamma^{i-1}} \varepsilon(F,G)\, G$$

for $F \in \Gamma^i$ and extended linearly. For different incidence functions the complexes $\widetilde{\mathcal{C}}(\Gamma)$ are isomorphic. Denote by

$$\widetilde{H}_i(\Gamma) = H_i\left(\widetilde{\mathcal{C}}(\Gamma)\right)$$

Theorem 6.75 *[Bruns, Herzog, 1993, Sec. 6.3] Let Γ be a cell complex and denote by $|\Gamma|$ the underlying topological space. Then*

$$\widetilde{H}_i(\Gamma) = \widetilde{H}_i(|\Gamma|)$$

is the reduced singular homology of $|\Gamma|$.

An algorithm computing $\widetilde{H}_i(|\Gamma|)$ via the augumented oriented chain complex is implemented in the Macaulay2 library homology.m2.

Consider the cell complex Γ given by the intersection of the fan Σ with a sphere of dimension $n-1$ together with the sphere as cell of dimension $n-1$ and $\emptyset$ as -1-cell. Let ε be an incidence function given by an orientation. By abuse of notation identify cones of Σ and cells of Γ. For $\sigma \in \Gamma$ denote by

$$S_{(\sigma)} = S_{x^{D_{\hat{\sigma}}}}$$

the localization of S in the multiplicatively closed set generated by $x^{D_{\hat{\sigma}}}$. This relates to the Cox quotient representation of $Y = X(\Sigma)$ as follows. We have $S_{(\sigma)} = \mathbb{C}\left[\check{\sigma}' \cap \mathbb{Z}^{\Sigma(1)}\right]$ where $\sigma' \in \Sigma'$ denotes the cone corresponding to σ, hence $U_{Y'}(\sigma') = \operatorname{Spec} S_{(\sigma)}$ and $U_Y(\sigma) = U_{Y'}(\sigma)/G(\Sigma)$. For the maximal cell $D_{\hat{\sigma}} = 0$ and $S_{(\sigma)} = S$.

Associate to Γ the canonical Čech complex

$$C^* : 0 \to C^0 \xrightarrow{\partial} C^1 \to ... \xrightarrow{\partial} C^n \to 0$$

with

$$C^i = \bigoplus\nolimits_{\sigma \in \Gamma^{n-i}} S_{(\sigma)}$$

and boundary map $\partial : C^{i-1} \to C^i$ given by the components

$$\partial : S_{(\sigma_1)} \to S_{(\sigma_2)}$$

defined as $\varepsilon(\sigma_1, \sigma_2)$ times the natural map $S_{(\sigma_1)} \to S_{(\sigma_2)}$ if σ_2 is a face of σ_1, and the 0-map otherwise.

Theorem 6.76 *[Bruns, Herzog, 1993, Sec. 6.3] If M is a finitely generated $A_{n-1}(Y)$-graded S-module, then*

$$H^i_{B(\Sigma)}(M) \cong H^i(M \otimes_S C^*)$$

Consider the natural $\mathbb{Z}^{\Sigma(1)}$-grading refining the $A_{n-1}(Y)$-grading. Recall that Σ'' is the fan over the simplex on $\Sigma(1)$ with $X(\Sigma'') = \mathbb{C}^{\Sigma(1)}$ and that Σ may be considered as a subfan of Σ''.

If $w \in \mathbb{Z}^{\Sigma(1)}$ define

$$G_w = \operatorname{hull}\{r \in \Sigma''(1) \,|\, w_r < 0\}$$

Then

$$\left(S_{(\tau)}\right)_w = \left\{ \begin{array}{ll} \mathbb{C} & \text{if } G_w \subset \hat{\tau} \\ 0 & \text{otherwise} \end{array} \right\}$$

For $G \in \Sigma''$ define

$$\Gamma_G = \{F \in \Gamma \mid F \subset G\}$$

Then

$$H^i(C^*_w) = \widetilde{H}_{n-i}\left(\Gamma_{\widehat{G_w}}\right)$$

If G_w is the maximal cone of Σ'', then $\Gamma_{G_w} = \Gamma$ is a sphere, hence

$$H^i(C^*_w) = \widetilde{H}_{n-i}(\emptyset) = \left\{ \begin{array}{ll} \mathbb{C} & i = n+1 \\ 0 & \text{otherwise} \end{array} \right\} = \widetilde{H}^{i-2}(\Gamma)$$

If G_w is the zero cone of Σ'', then

$$H^i(C^*_w) = \widetilde{H}_{n-i}(\Gamma) = 0 = \widetilde{H}^{i-2}(\emptyset) \text{ for } i \neq 1$$

If G_w lies between zero and maximal cone by Alexander duality

$$H^i(C^*_w) = \widetilde{H}_{n-i}\left(\Gamma_{\widehat{G_w}}\right) \cong \widetilde{H}^{i-2}\left(\Gamma \backslash \Gamma_{\widehat{G_w}}\right) \cong \widetilde{H}^{i-2}(\Gamma_{G_w})$$

hence:

Proposition 6.77 *For all $w \in \mathbb{Z}^{\Sigma(1)}$ and $i \neq 1$*

$$\left(H^i_{B(\Sigma)}(S)\right)_w \cong \widetilde{H}^{i-2}(\Gamma_{G_w})$$

and

$$\left(H^1_{B(\Sigma)}(S)\right)_w = 0$$

This allows one the computation of the local cohomology groups and regularity.

Example 6.78 *Let*

$$Y = X(\Sigma) = F_t = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(t))$$

be the Hirzebruch surface for $t \geq 0$ given by the fan with the rays

$$(1,0), (0,1), (-1,t), (0,-1)$$

let

$$0 \to M \xrightarrow{A} \mathbb{Z}^4 \to A_1(Y) \to 0$$

with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & t \\ 0 & -1 \end{pmatrix}$$

be the presentation of the Chow group of Y and, with respect to this numbering of the rays, let

$$S = \mathbb{C}[y_1, y_2, y_3, y_4]$$

be the Cox ring of Y, and

$$B(\Sigma) = \langle y_1, y_3 \rangle \cap \langle y_2, y_4 \rangle = \langle y_1y_2, y_2y_3, y_3y_4.y_4y_1 \rangle$$

the irrelevant ideal of Y. Fix an isomorphism

$$A_1(Y) \xrightarrow{B} \mathbb{Z}^2$$
$$B = \begin{pmatrix} 1 & -t & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Then

$$\operatorname{cpl}(\Sigma) = \operatorname{hull}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \subset \mathbb{Z}^2$$

and

$$\mathcal{K} = \operatorname{cpl}(\Sigma) \cap \mathbb{Z}^2 = \mathbb{N}C$$

with

$$C = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

If $t = 0, 1$, then the regularity of S is

$$\operatorname{reg}(S) = \mathcal{K} = \mathbb{Z}^2_{\geq 0}$$

and for $t \geq 2$

$$\operatorname{reg}(S) = \left(\begin{pmatrix} t-1 \\ 0 \end{pmatrix} + \mathcal{K} \right) \cup \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \mathcal{K} \right)$$

shown in Figure 6.1 for $t = 2$.

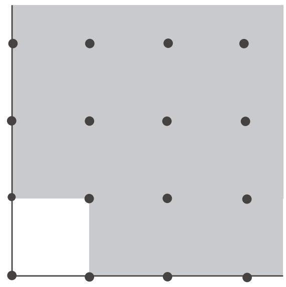


Figure 6.1: Regularity $\operatorname{reg}(S)$ for the Hirzebruch surface F_2

Proposition 6.79 *[Maclagan, Smith, 2004] Let M be a finitely generated $A_{n-1}(Y)$-graded S-module M. Then $\widetilde{M}$ is zero if and only if there is a $j > 0$ such that*

$$\left(B(\Sigma)^j M \right)_a = 0 \ \forall a \in \mathcal{K}$$

Definition 6.80 *Let $\mathcal{F}$ be a coherent $\mathcal{O}_Y$-module and $m \in A_{n-1}(Y)$. Then $\mathcal{F}$ is called m-**regular** with respect to C if*

$$H^i \left(Y, \mathcal{F} \otimes \widetilde{S(a)} \right) = 0$$

for all $i \geq 1$ and for all $a \in m + \mathbb{N}C[-i]$.

The regularity of $\mathcal{F}$ with respect to C is

$$\operatorname{reg}_C(\mathcal{F}) = \{ m \in A_{n-1}(Y) \mid \mathcal{F} \text{ is } m\text{-regular with respect to } C \}$$

*$\mathcal{F}$ is called m-**regular** if it is m-regular with respect to the minimal Hilbert basis of $\mathcal{K}$ and the regularity of $\mathcal{F}$ is*

$$\operatorname{reg}(\mathcal{F}) = \{ m \in A_{n-1}(Y) \mid \mathcal{F} \text{ is } m\text{-regular} \}$$

If $\mathcal{F}$ is m-regular with respect to C, then $\mathcal{F}$ is a-regular with respect to C for all $a \in m+\mathbb{N}C$. The regularity of a module M relates to the regularity of $\widetilde{M}$ as follows:

Proposition 6.81 *[Maclagan, Smith, 2004] If M is a finitely generated $A_{n-1}(Y)$-graded S-module and $m \in A_{n-1}(Y)$, then M is m-regular if and only if $\widetilde{M}$ is m-regular, the natural map*

$$M_a \to H^0\left(Y, \mathcal{F} \otimes \widetilde{S(a)}\right)$$

is surjective for all $a \in m+\mathbb{N}C$ and

$$\left(H^0_{B(\Sigma)}(S)\right)_a = 0$$

for all

$$a \in \bigcup_{j=1}^{e} (m + c_j + \mathbb{N}C)$$

Certain truncations do not change the sheafification:

Lemma 6.82 *[Maclagan, Smith, 2004] Let $C \subset \mathcal{K}$ such that the cone spanned by C has full dimension, let $m \in \mathbb{Z}\mathcal{K}$ and let M' be*

$$0 \to M \mid_{(m+\mathbb{N}C)} \to M \to M' \to 0$$

Then there is $j > 0$ such that

$$\left(B(\Sigma)^j M'\right)_a = 0 \ \forall a \in \mathbb{Z}\mathcal{K}$$

so $\widetilde{M'} = 0$, hence

$$\widetilde{M} = \widetilde{M \mid_{(m+\mathbb{N}C)}}$$

The following proposition allows to pass to initial ideals:

Proposition 6.83 *[Maclagan, Smith, 2004] If $>$ is a monomial ordering on S and $I \subset S$ is an ideal, then*

$$\operatorname{reg}(S/in_>(I)) \subset \operatorname{reg}(S/I)$$

If I is $B(\Sigma)$-saturated and $J = (in_>(I) : B(\Sigma)^\infty)$, then

$$\operatorname{reg}(S/J) \subset \operatorname{reg}(S/I)$$

The Hilbert function of S is $h_S(t) = \dim_{\mathbb{C}}(S_t)$ for $t \in \mathcal{K}$. Consider $\mathcal{K}$ as a subset of $\mathbb{Z}^a \cong A_{n-1}(Y)$. The Hilbert function of S is given by a polynomial:

Lemma 6.84 *[Maclagan, Smith, 2005] There is a polynomial $P_S \in \mathbb{Q}[t_1, ..., t_a]$ such that $h_S(t) = P_S(t)$ for all $t \in \mathcal{K}$.*

More generally if M is a module, then the Hilbert function is given by a polynomial for all $t \in \mathcal{K}$ sufficiently far from the boundary of $\mathcal{K}$.

Proposition 6.85 *[Maclagan, Smith, 2005] Let M be a finitely generated graded S-module. There is a polynomial $P_M \in \mathbb{Q}[t_1, ..., t_a]$ such that $h_M(t) = P_M(t)$ for all t in a finite intersection of translates of $\mathcal{K}$.*

Saturation does not change the Hilbert polynomial:

Lemma 6.86 *[Maclagan, Smith, 2005] Let M be a finitely generated graded S-module. Then*

$$P_M = P_{M/H^0_{B(\Sigma)}(M)}$$

Lemma 6.87 *[Maclagan, Smith, 2005] Let M be a finitely generated graded S-module. For all $t \in A_{n-1}(Y)$*

$$h_M(t) - P_M(t) = \sum_{i=0}^{n} (-1)^i \left(H^i_{B(\Sigma)}(M)\right)_t$$

If M is m-regular, then $\left(H^i_{B(\Sigma)}(M)\right)_t = 0$ for all $i = 0, ..., n$ and all $t \in m + \mathcal{K}$ with $t \neq m$, hence on the m-translate of $\mathcal{K}$ the Hilbert function of M agrees with its Hilbert polynomial:

Corollary 6.88 *[Maclagan, Smith, 2005] Let M be a finitely generated graded m-regular S-module. Then*

$$h_M(t) = P_M(t)$$

for all $t \in m + \mathcal{K}$ with $t \neq m$.

If $I \subset S$ is a monomial ideal, then a Stanley filtration of S/I gives a bound on the regularity of I:

Theorem 6.89 *[Maclagan, Smith, 2005] Let $I \subset S$ be a monomial ideal and let*

$$\mathcal{S} = \{(D_1, \sigma_1), ..., (D_s, \sigma_s)\}$$

with

$$S/\left(I + \left\langle x^{D_{j+1}}, ..., x^{D_s}\right\rangle\right) \cong S_{\sigma_1}(-[D_1]) \oplus ... \oplus S_{\sigma_j}(-[D_j])$$

for $j = 1, ..., s$ be a Stanley filtration of S/I. Then

$$\bigcap\nolimits_{(D,\sigma)\in\mathcal{S}} ([D] + \mathrm{reg}(S_\sigma)) \subset \mathrm{reg}(S/I)$$

Corollary 6.90 *Suppose I is $B(\Sigma)$-saturated. Let*

$$\overline{\mathcal{S}} = \{(D, \sigma) \in \mathcal{S} \mid \sigma \in \Sigma'\}$$

be the subset obtained by removing those Stanley pairs from $\mathcal{S}$, which correspond to irrelevant ideals in the intersection

$$I = \bigcap_{(D,\sigma)\in\mathcal{S}} \left\langle y_r^{u_r+1} \mid r \in \Sigma(1), \ r \subset \sigma, \ D = \textstyle\sum_{r\in\Sigma(1)} u_r D_r \right\rangle$$

in the sense that they define the empty subset of Y. Then

$$\bigcap\nolimits_{(D,\sigma)\in\overline{\mathcal{S}}} ([D] + \mathrm{reg}(S_\sigma)) \subset \mathrm{reg}(S/I)$$

Let $>$ be a monomial ordering on $\mathbb{Q}[t_1, ..., t_a]$ refining the degree ordering with $\deg t_i = 1$. By $>$ a partial ordering on the fan Σ is given via

$$\sigma_1 > \sigma_2 \Leftrightarrow \mathrm{in}_>\left(P_{S_{\sigma_1}}(t)\right) > \mathrm{in}_>\left(P_{S_{\sigma_2}}(t)\right)$$

refining the ordering of the cones of Σ by inclusion, i.e.,

$$V(\sigma_1) \subset V(\sigma_2) \Leftrightarrow \sigma_1 \supset \sigma_2 \Rightarrow \sigma_1 > \sigma_2$$

Algorithm 6.91 *[Maclagan, Smith, 2005] Let $>$ be a total ordering on Σ refining above partial ordering. The following algorithm computes a Stanley filtration*

$$\mathcal{S} = ((D_1, \sigma_1), ..., (D_s, \sigma_s))$$

of the monomial ideal $I \subset S$ such that if $\sigma_i \in \Sigma$ and $D_i \neq 0$ there is a $j < i$ with

$$\begin{aligned} &\sigma_j \in \Sigma \\ &\sigma_j < \sigma_i \\ &D_i = D_j + D_r \text{ with } r \in \Sigma(1) \text{ and } r \subset \sigma_j \end{aligned}$$

- *If I is a prime ideal and $I = \langle y_r \mid r \in \Sigma(1), r \subset \sigma \rangle$ with $\sigma \in \Sigma$, then return $((0, \sigma))$.*
- *Otherwise:*
  - *If $I \not\subset \langle y_r \mid r \in \Sigma(1), r \subset \sigma \rangle$ for all $\sigma \in \Sigma$ then choose $r \in \Sigma(1)$ such that $I \neq (I : \langle y_r \rangle) \neq \langle 1 \rangle$.*
  - *Otherwise: Choose $\sigma \in \Sigma$ minimal with respect to $>$ with the property*
  $$I \subsetneq \langle y_r \mid r \in \Sigma(1), r \subset \sigma \rangle$$
  and choose $r \subset \sigma$ such that $I \neq (I : \langle y_r \rangle) \neq \langle 1 \rangle$

  Compute Stanley decompositions $\mathcal{S}_1$ of $S/(I + \langle y_r \rangle)$ and
  $$\mathcal{S}_2 = ((D_1, \sigma_1), ..., (D_s, \sigma_s))$$
  of $S/(I : \langle y_r \rangle)$. Return
  $$\mathcal{S} = \mathcal{S}_1 \operatorname{join} ((D_1 + D_r, \sigma_1), ..., (D_s + D_r, \sigma_s))$$

With appropriate choice of the isomorphism $A_{n-1}(Y) \cong \mathbb{Z}^a$ we may assume that $\mathbb{R}^a_{\geq 0} \subset \operatorname{cpl}(\Sigma) \subset \mathbb{R}^a$. From this it follows that the lead coefficient of a Hilbert polynomial with respect to any graded ordering of the monomials of $\mathbb{Q}[t_1, ..., t_a]$ is positive.

Algorithm 6.92 *[Maclagan, Smith, 2005] Let $>$ be a total ordering on Σ induced by a graded ordering $>$ on $\mathbb{Q}[t_1, ..., t_a]$, suppose $\mathbb{R}^a_{\geq 0} \subset \operatorname{cpl}(\Sigma)$ and let $P(t) \in \mathbb{Q}[t_1, ..., t_a]$. The following algorithm returns all $B(\Sigma)$-saturated monomial ideals with Hilbert polynomial $P(t)$.*

- *Let* $\mathsf{finished} = \{\}$ *and* $\mathsf{todo} = \{(\emptyset, P(t))\}$.
- *Let* $(\mathcal{S}, Q(t)) \in \mathsf{todo}$.

  For all $\tau \in \Sigma$ and all $E \in \mathbb{Z}^{\Sigma(1)}$, $E \geq 0$ with the following properties

  1. *If $\mathcal{S} \neq \emptyset$ there is $(D, \sigma) \in \mathcal{S}$ with $\sigma \leq \tau$.*
  2. $\operatorname{in}_>(Q(t)) = \operatorname{in}_>(P_{S_\tau}(t))$
  3. *$LC_>(Q(t) - P_{S_\tau}(t))$ is positive.*
  4. *If $\mathcal{S} = \emptyset$, then $E = 0$.*
  5. *If $\mathcal{S} \neq \emptyset$, then there is an $r \in \Sigma(1)$ with $r \subset \tau$ such that $E = D + D_r$.*

if $Q(t) = P_{S_\tau}(t)$, *then*

$$\mathsf{finished} = \mathsf{finished} \cup \{\mathcal{S} \cup \{(E,\tau)\}\}$$

else

$$\mathsf{todo} = \mathsf{todo} \cup \{(\mathcal{S} \cup \{(E,\tau)\}, Q(t) - P_{S_\tau}(t))\}$$

- *Return all those monomial ideals*

$$\bigcap_{(D,\sigma)\in\mathcal{S}} \left\langle y_r^{u_r+1} \mid r \in \Sigma(1), \, r \subset \sigma, \; D = \sum\nolimits_{r\in\Sigma(1)} u_r D_r \right\rangle$$

for $\mathcal{S} \in \mathsf{finished}$, *which have Hilbert polynomial* $P(t)$.

The maximum of $|\mathcal{S}|$ *for* $\mathcal{S} \in \mathsf{finished}$ *is called the* ***Gotzmann number*** *of* $P(t)$.

Proposition 6.93 *For given* $P(t) \in \mathbb{Q}[t_1, ..., t_a]$ *there are only finitely many* $B(\Sigma)$*-saturated monomial ideals with Hilbert polynomial* $P(t)$.

Passing to the initial ideal we get:

Theorem 6.94 *[Maclagan, Smith, 2005] Let* $I \subset S$ *be an* $B(\Sigma)$*-saturated ideal,* m *the Gotzmann number of* $P_{S/I}(t)$ *and* $c \in \bigcap_{r\in\Sigma(1)} (\deg D_r + \mathcal{K})$, *then*

$$\bigcap\nolimits_{\sigma\in\Sigma} ((m-1)\,c + \operatorname{reg}(S_\sigma)) \subset \operatorname{reg}(S/I)$$

6.6.8 Multigraded Hilbert schemes

Consider the functor $\mathbb{H}_Y^P$ with

$$\mathbb{H}_Y^P(R) = \left\{ \mathcal{J} \mid \begin{array}{l} \mathcal{J} \text{ ideal sheaf of a family of subschemes } X \subset Y \times_{\mathbb{C}} \operatorname{Spec} R \to \operatorname{Spec} R \\ \text{with Hilbert polynomial } P \end{array} \right\}$$

for $R \in \underline{\mathbb{C} - Alg}$ and fixed multigraded Hilbert polynomial $P \in \mathbb{Q}[t_1, ..., t_s]$ with $s = |\Sigma(1)| - n$. By Section 1.3.10 there is a one-to-one correspondence

$$\begin{array}{ccc} \left\{\text{ideal sheaves in } \mathbb{H}_Y^P(R)\right\} & \leftrightarrows & \{B(\Sigma)\text{-saturated ideals } I \subset S \otimes_{\mathbb{C}} R\} \\ \widetilde{I} & \leftarrow & I \\ \mathcal{J} & \mapsto & \bigoplus_{a\in A_{n-1}(Y)} H^0(Y, \mathcal{J} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(a)) \end{array}$$

By Theorem 6.94 there is an $m \in \mathcal{K}$ such that all $B(\Sigma)$-saturated ideals are m-regular. With

$$I \mid_{m+\mathcal{K}} = S \cdot \left(\bigoplus_{a\in m+\mathcal{K}} I_a \right)$$

by Lemma 6.82 it holds

$$\widetilde{I\mid_{m+\mathcal{K}}} = \widetilde{I}$$

Define $h : A \cong \mathbb{Z}^s \to \mathbb{N}$ by $h(a) = P(a)$ and let F be the multiplication by monomials on S. Analogously to Corollary 6.58 there is a finite set $D \subset m+\mathcal{K}$ such that for all fields $K \in \underline{k-Alg}$ and for all $L' \in \mathbb{H}^h_{(S_D,F_D)}(K)$

$$\dim\left(\left(K \otimes S_a\right)/L_a\right) \leq h(a)$$

for all $a \in m+\mathcal{K}$, where $L \subset K \otimes S$ is the F-submodule generated by L'. Hence as D is finite by Theorem 6.56 the Hilbert functor $\mathbb{H}^h_{(S,F)}$ is a subfunctor of $\mathbb{H}^h_{(S_D,F_D)}$via the restriction map

$$\mathbb{H}^h_{(S,F)} \to \mathbb{H}^h_{(S_D,F_D)},\ L \mapsto L_D$$

and $\mathbb{H}^h_{(S,F)}$ is represented by a closed subscheme of the Hilbert scheme representing $\mathbb{H}^h_{(S_D,F_D)}$. As the Hilbert scheme representing $\mathbb{H}^h_{(S_D,F_D)}$ is a closed subscheme of the Grassmann scheme representing $\mathbb{G}^h_{S_D}$, the Hilbert scheme representing $\mathbb{H}^h_{(S,F)}$ is projective.

Theorem 6.95 *[Maclagan, Smith, 2005] If $P \in \mathbb{Q}[t_1,...,t_s]$ is a multigraded Hilbert polynomial, then $\mathbb{H}^P_Y$ is represented by a projective scheme over $\mathbb{C}$.*

Algorithm 6.96 *[Maclagan, Smith, 2005] The following algorithm computes a subset $D \subset m+\mathcal{K}$ such that for all fields $K \in \underline{k-Alg}$ and all $L' \in \mathbb{H}^h_{(S_D,F_D)}(K)$*

$$\dim\left(\left(K \otimes S_a\right)/L_a\right) \leq h(a)$$

for all $a \in m+\mathcal{K}$, where $L \subset K \otimes S$ is the F-submodule generated by L'.

1. *$D := \{m\}$*
2. *Compute by Algorithm 6.92 the finite set M of all monomial ideals I generated in degrees D with $h_{S/I}(t) = P(t)\ \forall t \in D$.*
3. *Suppose $I \in M$ with $h_{S/I}(t) \neq P(t)$ for some $t \in m+\mathcal{K}$, then $D := D \cup \{t\}$ and goto 2. otherwise return D.*

If $I \in M$, then $J = (I : B(\Sigma)^\infty)$ is m-regular and has Hilbert polynomial $P_{S/J} = P$. Hence $J\mid_{m+\mathcal{K}}$ is generated in degree m. As

$$h_{S/I}(m) = h_{S/J}(m) = P(m)$$

it holds $J_m = I_m$, so by $I \subset J$ we have $I\mid_{m+\mathcal{K}} = J\mid_{m+\mathcal{K}}$, hence

$$h_{S/J}(t) = P(t)\ \ \forall t \in m+\mathcal{K}$$

So in any step D satisfies the property required above.

Remark 6.97 *If Y is just simplicial, then one could replace S by the ring*

$$\bigoplus_{a\in \operatorname{Pic}(Y)} S_a$$

If Y is a non-simplicial toric variety, then one has to introduce an equivalence relation identifying different saturated ideals defining the same subscheme of Y.

6.6.9 State polytope

Let $I \subset S$ be a $B(\Sigma)$-saturated ideal with multigraded Hilbert polynomial $P(t)$ and define $h : A \cong \mathbb{Z}^s \to \mathbb{N}$ by $h(a) = P(a)$. Let $m \in \mathcal{K}$ such that all $B(\Sigma)$-saturated ideals are m-regular. Consider the finite set $D \subset m + \mathcal{K}$ as constructed in Section 6.6.8 such that $\mathbb{H}^h_{(S,F)}$ is a subfunctor of $\mathbb{H}^h_{(S_D,F_D)}$via the restriction map

$$\mathbb{H}^h_{(S,F)} \to \mathbb{H}^h_{(S_D,F_D)},\ L \mapsto L_D$$

and $\mathbb{H}^h_{(S,F)}$ is represented by a closed subscheme of the projective Hilbert scheme representing $\mathbb{H}^h_{(S_D,F_D)}$.

By Section 6.6.1 the functor $\mathbb{G}^h_{S_D}$ is a subfunctor of $\mathbb{G}^r_{S_D}$ with $r = \sum_{a\in D} h(a)$ and the corresponding morphism of schemes is a closed embedding. Consider the Plücker embedding of $\mathbb{G}^r_{S_D} \to \mathbb{P}(W)$ with $W = \bigwedge^{\dim S_D - r} V$ and $V = S_D$. So we have closed embeddings

$$\mathbb{H}^h_{(S,F)} \to \mathbb{H}^h_{(S_D,F_D)} \to \mathbb{G}^h_{S_D} \to \mathbb{G}^r_{S_D} \to \mathbb{P}(W)$$

Denote by T the torus of Y. With $\widehat{T} = M$ and $\widehat{T}^* = N$ the bilinear pairing between characters and one parameter subgroups of T

$$\begin{array}{ccc} \widehat{T} \times \widehat{T}^* & \to & \widehat{\mathbb{C}^*} = \mathbb{Z} \\ (\chi, \lambda) & \mapsto & \langle \chi, \lambda \rangle = \chi \circ \lambda \end{array}$$

corresponds to the canonical bilinear pairing

$$\langle -, - \rangle : M \times N \to \mathbb{Z}$$

Write the finite set $D = \{[D_1], ..., [D_p]\} \subset A_{n-1}(Y)$ and $L_i = \mathcal{O}_Y(D_i)$. Choose linearizations of the torus action $T \times Y \to Y$ on the L_i

$$\begin{array}{ccccc} T & \times & L_i & \overset{\overline{\sigma}}{\to} & L_i \\ id \times \pi & \downarrow & & & \downarrow \pi \\ T & \times & Y & \overset{\sigma}{\to} & Y \end{array}$$

which are unique up to translation in $\widehat{T} = M$. We will fix later particular linearizations.

The action of T on

$$S_D = \bigoplus_{i=1}^{p} H^0(Y, L_i) = \bigoplus_{i=1}^{p} S_{[D_i]}$$

induces an action of T on W and the Plücker embedding $\mathfrak{p} : \mathbb{G}^r_{S_D} \to \mathbb{P}(W)$ is T-equivariant.

For $\chi \in \widehat{T} = M$ let

$$W_\chi = \{v \in W \mid \Lambda v = \chi(\Lambda) v \ \forall \Lambda \in T\}$$

With

$$\operatorname{State}(W) = \{\chi \in M \mid W_\chi \neq \{0\}\}$$

there is a decomposition

$$W = \bigoplus_{\chi \in \operatorname{State}(W)} W_\chi$$

Denote by $H(I) \in \mathbb{H}^h_{(S,F)}$ the Hilbert point corresponding to I and let $h^* \in W$ be a representative of the image of $H(I)$ under the embedding $p : \mathbb{H}^h_{(S,F)} \to \mathbb{P}(W)$. Consider the decomposition of h^* corresponding to the decomposition of W

$$h^* = \sum_{\chi \in \operatorname{State}(W)} h_\chi$$

with $h_\chi \in W_\chi$. Define

$$\operatorname{State}(h) = \{\chi \in M \mid h_\chi \neq 0\}$$

and the state polytope of I as the convex hull

$$\operatorname{State}(I) = \operatorname{convexhull}(\operatorname{State}(h)) \subset \widehat{T} \otimes_{\mathbb{Z}} \mathbb{R} = M_{\mathbb{R}}$$

Let $x_0, ..., x_n$ be a T-invariant basis of V and

$$x_B = x_{b_1} \wedge ... \wedge x_{b_r}$$

the corresponding T-invariant basis of W, compatible with the decomposition of $W = \bigoplus_{\chi \in \operatorname{State}(W)} W_\chi$. With respect to the basis (x_B) the representation $\rho : T \to \operatorname{GL}(W)$ given by the action $T \times W \to W$ is of the form

$$\rho(x) = \operatorname{diag}(x^{m_1}, ..., x^{m_{\dim W}})$$

with $m_i \in M$.

Let

$$\lambda : \mathbb{C}^* \to T$$
$$\lambda(t) = \operatorname{diag}(t^{w_1}, ..., t^{w_n})$$

be a one parameter subgroup of T, then

$$\begin{array}{rccl} \rho \circ \lambda : & \mathbb{C}^* & \to & \operatorname{GL}(W) \\ & t & \mapsto & \operatorname{diag}\left(t^{\langle w, m_1 \rangle}, ..., t^{\langle w, m_{\dim W} \rangle}\right) \end{array}$$

With respect to the basis (x_B)

$$h^* = (\alpha_1, ..., \alpha_{\dim W})$$

and

$$\lambda(t) \cdot h^* = \operatorname{diag}\left(t^{\langle w, m_1 \rangle}\alpha_1, ..., t^{\langle w, m_{\dim W} \rangle}\alpha_{\dim W}\right)$$

Hence, with $p : \mathbb{H}^h_{(S,F)} \to \mathbb{P}(W)$ and the line bundle

$$E = p^*\left(\mathcal{O}_{\mathbb{P}(W)}(1)\right)$$

we have

$$\mu^E(h, \lambda) = \min\{\langle w, m_i \rangle \mid \alpha_i \neq 0\} = \min_{\chi \in \operatorname{State}(h)} \langle \chi, \lambda \rangle$$

so by Theorem 6.47 we obtain:

Theorem 6.98 *Suppose $Y = X(\Sigma)$ is a smooth toric variety given by the fan $\Sigma \subset N_{\mathbb{R}}$ and let S be the Cox ring of Y and $\mathcal{K} = \operatorname{cpl}(\Sigma) \cap A_{n-1}(Y)$.*

Let $I \subset S$ be a $B(\Sigma)$-saturated ideal with Hilbert polynomial $P(t)$, h the corresponding Hilbert function and $D \subset m + \mathcal{K}$ such that the restriction map gives a closed embedding $\mathbb{H}^h_{(S,F)} \to \mathbb{H}^h_{(S_D,F_D)}$. Fix linearizations of the T-action on Y on the elements of D.

Then stability and semi-stability of the Hilbert point $H(I) \in \mathbb{H} = \mathbb{H}^h_{(S,F)}$ are characterized as

$$\begin{array}{lcl} H(I) \in \mathbb{H}^{ss} & \Leftrightarrow & 0 \in \operatorname{State}(I) \\ H(I) \in \mathbb{H}^{s} & \Leftrightarrow & 0 \in \operatorname{int}(\operatorname{State}(I)) \end{array}$$

6.7 Toric homogeneous weight vectors and the Gröbner fan

In the same way as rational graded weight vectors on the coordinate ring of $\mathbb{P}^n$ are up to multiples parametrized by $\frac{\mathbb{Z}^{n+1}}{\mathbb{Z}(1,...,1)}$, we want to parametrize weight vectors, i.e., partial orderings given by weight vectors for the variables, on the graded pieces of the Cox ring in the general toric setting.

Let $Y = X(\Sigma)$ be a complete toric variety, $v_1, ..., v_r$ the minimal lattice generators of the rays of Σ forming the rows of the presentation matrix A of $A_{n-1}(X(\Sigma))$ in

$$0 \to M \xrightarrow{A} \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(X(\Sigma)) \to 0$$

Let $P = \operatorname{convexhull}(v_1, ..., v_r)$ and S be the Cox ring of Y. Any rational weight vector on S is representable by an element $w \in \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{\Sigma(1)}, \mathbb{Z}\right)$. Applying $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ to above sequence we get

$$\begin{array}{ccccccc} & & & 0 & \leftarrow & \operatorname{Ext}^1_{\mathbb{Z}}(A_{n-1}(X(\Sigma)), \mathbb{Z}) & \leftarrow \\ \leftarrow & \overbrace{\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Z})}^{=N} & \overset{\_\circ A}{\leftarrow} & \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{\Sigma(1)}, \mathbb{Z}\right) & \leftarrow & \operatorname{Hom}_{\mathbb{Z}}(A_{n-1}(X(\Sigma)), \mathbb{Z}) & \leftarrow & 0 \end{array}$$

hence

$$\frac{\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{\Sigma(1)}, \mathbb{Z}\right)}{\operatorname{Hom}_{\mathbb{Z}}(A_{n-1}(X(\Sigma)), \mathbb{Z})} \cong \operatorname{image}(\_ \circ A) \subset N$$

Now connect the left hand side to the weight vectors on the graded pieces of the Cox ring S:

Note that scaled weight vectors give the same ordering on the monomials. To take this into account, define the following equivalence relation: For

$$\overline{w_1}, \overline{w_2} \in \frac{\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{\Sigma(1)}, \mathbb{Z}\right)}{\operatorname{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{Z})}$$

let

$$\overline{w_1} \sim \overline{w_2} :\Leftrightarrow \exists \lambda_1, \lambda_2 \in \mathbb{Z}_{>0} : \lambda_1 \overline{w_1} = \lambda_2 \overline{w_2}$$

where $\lambda_1 \overline{w_1} = \overline{\lambda_1 w_1}$ is the induced $\mathbb{Z}$-module structure inherited from $\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{\Sigma(1)}, \mathbb{Z}\right)$. The map

$$\begin{array}{ccc} \frac{\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{\Sigma(1)}, \mathbb{Z}\right)}{\operatorname{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{Z})} & \xrightarrow{\psi} & \{\text{graded wt. vec. on } S\} \\ \overline{w} & \mapsto & \succ_w = \text{partial ordering given by } w \end{array}$$

is well defined, as

$$\begin{aligned}
&\overline{w_1} = \overline{w_2} \\
&\quad \Leftrightarrow (w_1 - w_2) \cdot \in \operatorname{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{Z}) \\
&\quad \Leftrightarrow (w_1 - w_2) \cdot \in \ker(_ \circ A) \\
&\quad \Leftrightarrow \operatorname{image}(A) \subset \ker((w_1 - w_2) \cdot) \\
&\quad \Leftrightarrow w_1 A = w_2 A \\
&\quad \Leftrightarrow w_1 \cdot = w_2 \cdot \text{ on } \operatorname{image}(A) \\
&\quad \Rightarrow (w_1 a > 0 \Leftrightarrow w_2 a > 0 \ \forall a \in \operatorname{image}(A)) \\
&\quad \Leftrightarrow (y^{m_1} >_{w_1} y^{m_2} \Leftrightarrow y^{m_1} >_{w_2} y^{m_2}) \ \forall \text{ Cox monomials } y^{m_1}, y^{m_2} \text{ with } \deg y^{m_1} = \deg y^{m_2} \\
&\quad \Leftrightarrow \succ_{w_1} = \succ_{w_2} \text{ on } S_{[D]} \ \forall [D] \in A_{n-1}(X)
\end{aligned}$$

Note that

$$\begin{aligned}
&\deg y^{m_1} = \deg y^{m_2} \\
&\qquad \Leftrightarrow \left[\sum_{v \in \mathbb{Z}^{\Sigma(1)}} m_{1v} D_v\right] = \left[\sum_{v \in \mathbb{Z}^{\Sigma(1)}} m_{2v} D_v\right] \in A_{n-1}(X) \\
&\qquad \Leftrightarrow m_1 = m_2 \operatorname{mod} \operatorname{image}(A) \\
&\qquad \Leftrightarrow m_1 - m_2 \in \operatorname{image}(A)
\end{aligned}$$

Surjectivity of ψ is obvious, and

$$\begin{aligned}
&\succ_{w_1} = \succ_{w_2} \text{ on } S_{[D]} \ \forall [D] \in A_{n-1}(X) \\
&\Leftrightarrow (w_1 a > 0 \Leftrightarrow w_2 a > 0 \ \forall a \in \operatorname{image}(A)) \\
&\Leftrightarrow \exists \lambda_1, \lambda_2 \in \mathbb{Z}_{>0} : \lambda_1 w_1 a = \lambda_2 w_2 a \ \forall a \in \operatorname{image}(A) \\
&\Leftrightarrow \exists \lambda_1, \lambda_2 \in \mathbb{Z}_{>0} : \lambda_1 w_1 \cdot = \lambda_2 w_2 \cdot \text{ on } \operatorname{image}(A) \\
&\Leftrightarrow (\lambda_1 w_1 - \lambda_2 w_2) A = 0 \\
&\Leftrightarrow \operatorname{image}(A) \subset \ker((\lambda_1 w_1 - \lambda_2 w_2) \cdot) \\
&\Leftrightarrow (\lambda_1 w_1 - \lambda_2 w_2) \cdot \in \ker(_ \circ A) \\
&\Leftrightarrow (\lambda_1 w_1 - \lambda_2 w_2) \cdot \in \operatorname{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{Z}) \\
&\Leftrightarrow \lambda_1 \overline{w_1} = \lambda_2 \overline{w_2} \\
&\Leftrightarrow \overline{w_1} \sim \overline{w_2}
\end{aligned}$$

hence:

Lemma 6.99 *The map*

$$\begin{array}{ccc}
\frac{\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{\Sigma(1)}, \mathbb{Z}\right)}{\operatorname{Hom}_{\mathbb{Z}}(A_{n-1}(X), \mathbb{Z})} & \xrightarrow{\psi} & \{\textit{graded weight vectors on } S\} \\
\overline{w} & \mapsto & \succ_w
\end{array}$$

is well defined, surjective and

$$\succ_{w_1}=\succ_{w_2}\Leftrightarrow \quad (\exists\lambda_1,\lambda_2\in\mathbb{Z}_{>0}:\lambda_1\overline{w_1}=\lambda_2\overline{w_2}) \quad \Leftrightarrow: \quad \overline{w_1}\sim\overline{w_2}$$

Proposition 6.100 *After tensoring with* $\mathbb{R}$*, the map*

$$\begin{array}{cccccc} N=\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Z}) & & & & \\ \cup & & & & \\ \operatorname{image}(\_\circ A) & \overset{\_\circ A}{\underset{\varphi}{\leftrightarrows}} & \frac{\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{\Sigma(1)},\mathbb{Z}\right)}{\operatorname{Hom}_{\mathbb{Z}}(A_{n-1}(X(\Sigma)),\mathbb{Z})} & \rightarrow & \{\textit{graded wt. vec. on } S\} \\ \sum_{i=1}^{r} w_i v_i\cdot & \mapsto & \overline{(w_1,...,w_r)} & \mapsto & \succ_w \end{array}$$

gives a one-to-one correspondence between half lines with origin 0 *in* $N_{\mathbb{R}}$ *and the real weight vectors on the graded parts of* S*.*

Lemma 6.101 *As* $0\in\operatorname{int}(P)$*, there are* $a_i>0$ *such that* $\sum_{i=1}^{r}a_iv_i=0$*, hence via translation by* $(a_1,...,a_r)$ *any weight ordering is equivalent to a global one.*

We extend the definition of the Gröbner fan to the general toric setting:

Definition 6.102 *The* ***Gröbner fan*** $GF(J)$ *of a homogeneous ideal* $J\subset S$ *is the complete polyhedral fan formed by the cones* $\varphi^{-1}\left(\overline{C_{\varphi(w)}(J)}\right)\subset N_{\mathbb{R}}$ *for* $w\in N_{\mathbb{R}}$*.*

Proposition 6.103 *If* Y *is a smooth toric variety and* $J\subset S$ *is a homogeneous ideal, then* $GF(J)=\operatorname{NF}(\operatorname{State}(J))$*.*

Note that the normal fan does not depend on translation of $\operatorname{State}(J)$ by choice of linearizations. Note also that the state polytope of J and of its saturation have the same normal fan.

7 $\mathbb{Q}$-Gorenstein varieties and Fano polytopes

7.1 Singularities of toric varieties

Let $N\cong\mathbb{Z}^n$, $M=\operatorname{Hom}(N,\mathbb{Z})$, let Y be an affine toric variety given by the rational polyhedral n-dimensional cone $\sigma\subset N_{\mathbb{R}}$ and let $v_1,...,v_s\in N$ be the minimal lattice generators of σ.

Lemma 7.1 *[Dais, 2002] The affine toric variety* Y *is* $\mathbb{Q}$*-Gorenstein if and only if there is an* $m\in M_{\mathbb{Q}}$ *with* $\langle m,v_i\rangle=-1$ $\forall i=1,...,s$*.*

Definition 7.2 *The minimal* $r \in \mathbb{Z}_{>0}$ *such that there is an* $m \in M$ *with* $\langle m, v_i \rangle = -r\ \forall i = 1, ..., s$ *is called the* ***index*** *of the singularity of* Y*. So* Y *is Gorenstein if and only if it has index* 1.

Lemma 7.3 *[Dais, 2002] Suppose* Y *is* $\mathbb{Q}$*-Gorenstein and* $m \in M_{\mathbb{Q}}$ *with* $\langle m, v_i \rangle = -1\ \forall i = 1, ..., s$*. Then* Y *is terminal if and only if*

$$\sigma \cap \{w \in N \mid \langle m, w \rangle \geq -1\} = \{0, v_1, ..., v_s\}$$

and Y *is canonical if and only if*

$$\sigma \cap \{w \in N \mid \langle m, w \rangle > -1\} = \{0\}$$

Proposition 7.4 *[Dais, 2002] If* Y *is* $\mathbb{Q}$*-Gorenstein, then it is log-terminal. If* Y *is Gorenstein, then it is canonical.*

Let Y be a normal $\mathbb{Q}$-Gorenstein toric variety of dimension n, given by the rational polyhedral fan $\Sigma \subset N_{\mathbb{R}}$. As Y is $\mathbb{Q}$-Gorenstein, there is a continuous function $\varphi_{K_Y} : N_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$ such that φ_{K_Y} is piecewise linear on the fan Σ and $\varphi_{K_Y}(\hat{r}) = 1$ for the minimal lattice generators $\hat{r}$ of all rays $r \in \Sigma(1)$.

Proposition 7.5 *[Kawamata, Matsuda, Matsuki, 1987] Suppose* Σ' *is a refinement of* Σ *inducing a resolution of singularities by the birational morphism* $f : X(\Sigma') \to X(\Sigma)$ *and denote by* $D_1, ..., D_r$ *the irreducible components of the exceptional divisor of* f*. Then* $D_1, ..., D_r$ *have only normal crossings,* $D_1, ..., D_r$ *correspond to the rays of* Σ' *not in* Σ*, and*

$$K_{X(\Sigma')} = f^* K_{X(\Sigma)} + \sum_{r \in \Sigma'(1) \setminus \Sigma(1)} a_r D_r$$

with

$$a_r = \varphi_{K_Y}(\hat{r}) - 1$$

In particular, f is crepant if and only if $\varphi_{K_Y}(\hat{r}) = 1$ for all $r \in \Sigma'(1) \setminus \Sigma(1)$.

7.2 Fano polytopes

Section 1.2 suggests to consider $\mathbb{Q}$-Gorenstein Fano varieties, so we generalize Definition 1.116 to the following:

Definition 7.6 *A normal variety* Y *is called* ***Fano*** *if some multiple of* $-K_Y$ *is an ample Cartier divisor.*

By $K_Y = -\sum_{v \in \Sigma(1)} D_v$ a toric variety Y is $\mathbb{Q}$-Gorenstein if and only if some multiple of $\sum_{v \in \Sigma(1)} D_v$ is Cartier.

Lemma 7.7 *If Y is a complete toric variety then it is Fano if and only if some multiple of $\sum_{v \in \Sigma(1)} D_v$ is Cartier and ample if and only if $Y \cong X(\mathrm{NF}(\Delta_{-K_Y}))$.*

Definition 7.8 *[Cox, 2002] A polytope $P \subset N_{\mathbb{R}} \cong \mathbb{R}^n$ of dimension n is called a* ***Fano polytope*** *if P is integral and 0 is the unique lattice point in the interior of P.*

If $P \subset N_{\mathbb{R}}$ is a Fano polytope, then P^* is cut out by the equations $\langle m, w_i \rangle \geq -1$ for the vertices $w_i \in N$, so if $m \in P^* \cap M$ is a lattice point in the interior of P^*, then $\langle m, w_i \rangle \in \mathbb{Z}$ and $\langle m, w_i \rangle > -1$ for all i, hence:

Lemma 7.9 *If $P \subset N_{\mathbb{R}}$ is a Fano polytope, then 0 is the unique interior lattice point of P^*.*

Definition 7.10 *Denote by $\Sigma(P)$ the* ***fan over the faces*** *of P.*

By the characterization of ample Cartier divisors in Section 1.3.4, we obtain:

Proposition 7.11 *If P is a Fano polytope, then $X(\Sigma(P))$ is a toric Fano variety. It is $\mathbb{Q}$-Gorenstein, hence it has log-terminal singularities by Proposition 7.4.*

Note that the vertices of P are the minimal lattice generators of the rays of $X(\Sigma)$. From Section 7.1 we also get:

Proposition 7.12 *If $P \subset N_{\mathbb{R}} \cong \mathbb{R}^n$ is a Fano polytope, then it holds:*

1. *If $P \cap N = \mathrm{vert}(P) \cup \{0\}$, i.e., all lattice points of ∂P are vertices, then $X(\Sigma(P))$ is terminal.*
2. *If all facets of P are of the form $P \cap \{w \in N_{\mathbb{R}} \mid \langle m, w \rangle = -1\}$ with integral $m \in M$, then $X(\Sigma(P))$ is Gorenstein.*

So the second condition is equivalent to $P^* = \Delta_{-K_Y}$ being integral: Writing all facets F of P as $F = P \cap \{w \in N_{\mathbb{R}} \mid \langle m_F, w \rangle = -1\}$ with $m_F \in M$, the m_F are the vertices of $P^* = \mathrm{convexhull}\{m_F \mid F \text{ facet of } P\}$.

Proposition 7.13 *A Fano polytope P is reflexive if and only if P^* is integral. Then $X(\Sigma(P))$ is a Gorenstein toric Fano variety, hence it has canonical singularities by Proposition 7.4.*

Proposition 7.14 *Suppose $P \subset N_{\mathbb{R}}$ is a Fano polytope, $\Sigma = \Sigma(P) \subset N_{\mathbb{R}}$ is the fan over the faces of P and $Y = X(\Sigma)$. As X is $\mathbb{Q}$-Gorenstein, there is a continuous function $\varphi_{K_Y} : N_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$ such that φ_{K_Y} is piecewise linear on the fan Σ and $\varphi_{K_Y}(v) = 1$ for all vertices of P, i.e., $P = \{w \in N_{\mathbb{R}} \mid \varphi_{K_Y}(w) \leq 1\}$. If Σ' is a refinement of Σ inducing a resolution of singularities via the birational morphism $f : X(\Sigma') \to X(\Sigma)$, then*

$$K_{X(\Sigma')} = f^* K_{X(\Sigma)} + \sum_{r \in \Sigma'(1) \setminus \Sigma(1)} (\varphi_{K_Y}(\hat{r}) - 1) D_r$$

Hence f is crepant if and only if the introduced rays $\Sigma'(1) \setminus \Sigma(1)$ have minimal lattice generators on the boundary of P.

8 The tropical mirror construction for complete intersections in toric varieties

In the following, we give a tropical mirror construction for complete intersections in toric varieties as defined in Section 2.2, and we show that the tropical mirror coincides with the Batyrev-Borisov mirror.

8.1 The degeneration for toric complete intersections

Consider the setup from Section 2.2, i.e., let $Y = \mathbb{P}(\Delta)$ be a Gorenstein toric Fano variety of dimension n, represented by the reflexive polytope $\Delta \subset M_{\mathbb{R}}$, with normal fan $\Sigma \subset N_{\mathbb{R}}$ and Cox ring S, and let $\Sigma(1) = I_1 \cup ... \cup I_c$ be a nef partition, so $E_j = \sum_{v \in I_j} D_v$ are Cartier, spanned by global sections and $\sum_{j=1}^{c} E_j = -K_Y$. Define $\Delta_j = \Delta_{E_j}$ as the polytope of sections of E_j and

$$\begin{aligned} \nabla_j &= \text{convexhull}\{\{0\} \cup I_j\} \\ \nabla_{BB}^* &= \text{convexhull}(\Delta_1 \cup ... \cup \Delta_c) \end{aligned}$$

so

$$\begin{aligned} \Delta &= \Delta_1 + ... + \Delta_c \\ \nabla_{BB} &= \nabla_1 + ... + \nabla_c \end{aligned}$$

Consider the monomial degeneration $\mathfrak{X}$ as defined in Section 3.1

$$\begin{aligned}
m_j &= \prod_{v\in I_j} y_v \text{ for } j=1,...,c \\
I_0 &= \langle m_j \mid j=1,...,c\rangle \\
I &= \langle f_j = t\cdot g_j + m_j \mid j=1,...,c\rangle \subset \mathbb{C}[t]\otimes S \\
g_j &\in S_{[E_j]},\ j=1,...,c \text{ general, reduced with respect to } I_0
\end{aligned}$$

of a complete intersection of dimension $d=n-c$ given by general sections of the Cartier divisors $E_1,...,E_c$ to the monomial ideal I_0.

The resolution of I_0 is given by the Koszul complex $K_\bullet$ on $m=(m_1,...,m_c)$, i.e., the complex of the simplex on $m_1,...,m_c$,

$$0 \to K_c \xrightarrow{\partial} ... \xrightarrow{\partial} K_1 \xrightarrow{\partial} K_0$$

with

$$\begin{aligned}
\mathcal{E} &= \mathcal{O}_Y(E_1)\oplus ... \oplus \mathcal{O}_Y(E_c) \\
K_0 &= \mathcal{O}_Y \\
K_p &= \bigwedge\nolimits^p \mathcal{E}^* \text{ for } p=1,...,c
\end{aligned}$$

and the maps ∂ are given by contraction with the section m of $\mathcal{E}$. With respect to the standard frame $e_{i_1...i_p}=e_{i_1}\wedge ... \wedge e_{i_p}$ for $1\le i_1<...<i_p\le c$ of K_p we can write more explicitly

$$0\to \mathcal{O}_Y(-E_1-...-E_c)\to ... \to \bigoplus_{1\le i_1<...<i_p\le c}\mathcal{O}_Y\left(-E_{i_1}-...-E_{i_p}\right)\to ...$$

$$... \to \bigoplus_{i=1}^{c}\mathcal{O}_Y(-E_i)\to \mathcal{O}_Y$$

and

$$\partial : K_p \to K_{p-1}$$

$$\partial\left(e_{i_1...i_p}\right) = \sum_{j=1}^{p}(-1)^{j-1} m_{i_j} e_{i_1...i_{j-1}i_{j+1}...i_p}$$

Denote by $\pi_1 : Y\times \operatorname{Spec}\mathbb{C}[[t]] \to Y$ the projection on the first component. So for above family $\mathfrak{X}$ defined by $I=\langle m_1+tg_1,...,m_c+tg_c\rangle$, the Koszul complex on $(m_1+tg_1,...,m_c+tg_c)$ considered as a section of $\pi_1^*\mathcal{E}$ gives a lift of all syzygies of $(m_1,...,m_c)$, so $\mathfrak{X}$ is flat.

In the same way any first order deformation over $\operatorname{Spec}(\mathbb{C}[t]/\langle t^2\rangle)$ gives a deformation over $\operatorname{Spec}(\mathbb{C}[[t]])$, hence:

Proposition 8.1 *The family $\mathfrak{X} \subset Y \times \operatorname{Spec}(\mathbb{C}[[t]])$ defined by I is a flat degeneration with fibers polarized in $Y = \mathbb{P}(\Delta)$ and monomial special fiber X_0 given by I_0. The fiber over the generic point of $\operatorname{Spec}(\mathbb{C}[[t]])$ is a Calabi-Yau complete intersection of codimension c in Y given by general sections of $\mathcal{O}(E_1), ..., \mathcal{O}(E_c)$.*

The deformations of I_0 are unobstructed and the base space is smooth.

Let $v_1, ..., v_p \in \operatorname{Hom}(I_0, S/I_0)_0$ be a basis of the tangent space of the Hilbert scheme of X_0. The degeneration $\mathfrak{X}$ is general in the sense that if v is the tangent vector of $\mathfrak{X}$ and $v = \sum_{i=1}^{p} \lambda_i v_i$, then we have $\lambda_i \neq 0$ $\forall i$.

The ideals of the maximal strata of X_0 are the ideals $\langle y_{j_1}, ..., y_{j_c} \rangle \subset S$ for $j_1 \in I_1, ..., j_c \in I_c$, hence are given by $c = \operatorname{codim}(X_t)$ equations.

Note that in a toric variety the ideal of a stratum of codimension c in the Cox ring S can have more than c generators and the face of Δ corresponding to the stratum may be contained in more than c facets, see Example 8.11.

8.2 The Gröbner cone associated to the special fiber and the polytope ∇

Fix a tie break ordering $>$ on $\mathbb{C}[t] \otimes S$ with t local and respecting the Chow grading on S, so $L_>(f_j) = m_j$. Denote by φ the map from $N_\mathbb{R}$ to the graded weight vectors on S as defined in Section 6.7. The special fiber Gröbner cone

$$C_{I_0}(I) = \left\{-(w_t, w_y) \in \mathbb{R} \oplus N_\mathbb{R} \mid L_{>_{(w_t, \varphi(w_y))}}(I) = I_0\right\}$$

is given by

$$C_{I_0}(I) = \{-(w_t, w_y) \in \mathbb{R} \oplus N_\mathbb{R} \mid \operatorname{trop}(g_j)(\varphi(w_y)) - w_t \leq \operatorname{trop}(m_j)(\varphi(w_y)) \ \forall j\}$$

Note that the equalities of the lead terms respectively the tropical inequalities are well defined by homogeneity.

As $\operatorname{convexhull}(\Delta_1 \cup ... \cup \Delta_c) = \nabla^*_{BB}$ contains 0 in its interior, for all $w \in N_\mathbb{R}$ there is j and a vertex $0 \neq \tilde{m}$ of Δ_j such that $\langle \tilde{m}, w \rangle > 0$. Then $m = m_j \cdot A\tilde{m}$ is a monomial of some $g_j$with $\varphi(w_y)\left(\frac{m}{m_j}\right) > 0$, i.e.,

$$\operatorname{trop}(m)(\varphi(w_y)) > \operatorname{trop} m_j(\varphi(w_y))$$

hence:

Lemma 8.2 *The special fiber Gröbner cone satisfies*

$$C_{I_0}(I) \cap \{w_t = 0\} = \{0\}$$

So I_0 cannot appear as lead ideal of the general fiber ideal I_{gen}.

Intersecting $C_{I_0}(I)$ with the hyperplane $\{w_t = 1\}$ we obtain the convex polytope

$$\nabla = C_{I_0}(I) \cap \{w_t = 1\} \subset N_{\mathbb{R}}$$

with

$$\nabla = \{-w_y \in N_{\mathbb{R}} \mid \operatorname{trop}(g_j)(\varphi(w_y)) - 1 \leq \operatorname{trop}(m_j)(\varphi(w_y)) \;\; \forall j = 1, ..., c\}$$

and $0 \in \operatorname{int} \nabla$.

As ∇ is given by integral linear equations corresponding to the deformations of I_0 appearing in I, the polytope ∇^* is integral.

Rewriting the tropical equations we have

$$\begin{aligned}\nabla &= \{-w_y \in N_{\mathbb{R}} \mid \operatorname{trop}(m)(\varphi(w_y)) - 1 \leq \operatorname{trop}(m_j)(\varphi(w_y)) \;\; \forall \text{ monomials } m \text{ of } g_j \; \forall j\} \\ &= \left\{ w_y \in N_{\mathbb{R}} \mid \varphi(w_y)\left(\frac{m}{m_j}\right) \geq -1 \; \forall \text{ monomials } m \text{ of } g_j \text{ and } \forall j \right\}\end{aligned}$$

The linear conditions defining ∇ do not change if we do not require the f_j to be reduced with respect to I_0. To see this, let $\frac{m}{m_j}$ be a degree 0 Cox Laurent monomial, i.e., $\frac{m}{m_j} \in \operatorname{image}(A)$. Any $\varphi(w_y)$ has a positive representative in $\mathbb{R}^{\Sigma(1)}$. If $m_j \mid m$, then

$$\varphi(w_y)\left(\frac{m}{m_j}\right) \geq 0 \geq -1$$

If $m \in S_{[E_j]}$ is divisible by some

$$m_i = \prod_{v \in I_i} y_v$$

for $j \neq i$, then $A^{-1}\left(\frac{m}{m_j}\right)$ is an interior point of a face F of Δ_j of dimension $\dim(F) \geq 1$, hence, the defining inequality of ∇

$$\left\langle A^{-1}\left(\frac{m}{m_j}\right), w_y \right\rangle \geq -1$$

given by m is redundant, so we get:

Proposition 8.3 *The polytope ∇ is given by*

$$\nabla = \left\{ w_y \in N_{\mathbb{R}} \mid \left\langle A^{-1}\left(\frac{m}{m_i}\right), w_y \right\rangle \geq -1 \; \forall \textit{ monomials } m \in S_{[E_j]} \; \forall j = 1, ..., c \right\}$$

and $0 \in \nabla^*$.

Reformulating this in terms of lattice monomials

$$\begin{aligned}\nabla &= \{w_y \in N_{\mathbb{R}} \mid \langle \widetilde{m}, w_y\rangle \geq -1 \ \forall \widetilde{m} \in \Delta_j \ \forall j\}\\ &= \{w_y \in N_{\mathbb{R}} \mid \langle \widetilde{m}, w_y\rangle \geq -1 \ \forall \widetilde{m} \in \nabla^*_{BB}\}\\ &= \nabla_{BB}\end{aligned}$$

by $\nabla^*_{BB} = \operatorname{convexhull}(\Delta_1 \cup ... \cup \Delta_c)$, hence:

Theorem 8.4 *The polytope* $\nabla = C_{I_0}(I) \cap \{w_t = 1\}$ *coincides with* ∇_{BB} *given in the mirror construction by Batyrev and Borisov*

$$\nabla = \nabla_{BB}$$

Corollary 8.5 ∇ *is reflexive, so it defines a Gorenstein toric Fano variety* $Y^\circ = \mathbb{P}(\nabla)$.

Denote by $S^\circ = \mathbb{C}[z_r \mid r \in \Sigma^\circ(1)]$ with $\Sigma^\circ = \operatorname{NF}(\nabla)$ the Cox ring of Y°.

Example 8.6 *Consider the monomial degeneration* $\mathfrak{X} \subset \mathbb{P}^3 \times \operatorname{Spec}\mathbb{C}[[t]]$ *over* $\operatorname{Spec}\mathbb{C}[[t]]$ *of an elliptic curve given as the complete intersection of two general quadrics in* $\mathbb{P}^3$ *as given in Example 3.3, i.e., by the ideal*

$$I = \langle t \cdot g_1 + x_1x_2,\ t \cdot g_2 + x_0x_3\rangle \subset \mathbb{C}[t] \otimes S$$

where $g_1, g_2 \in \mathbb{C}[x_0, ..., x_3]_2$ *are general, not involving monomials in* $I_0 = \langle x_1x_2, x_0x_3\rangle$. *Here* $S = \mathbb{C}[x_0, ..., x_3]$ *denotes the Cox ring of* $\mathbb{P}(\Delta) = \mathbb{P}^3$ *with variables* $x_0, ..., x_3$ *corresponding to the vertices of* Δ^* *and* Δ *is the degree* 4 *Veronese polytope of* $\mathbb{P}^3$.

For this example the reflexive polytope $\nabla = C_{I_0}(I) \cap \{w_t = 1\}$ *is depicted in Figure 8.1. As shown above it agrees with* ∇_{BB} *given in Example 2.9.*

8.3 The initial ideals of the faces of ∇

In the following we explicitly give the correspondence of lead ideals of I and faces of ∇^*.

Consider the notation from the last section. Note that all w in the interior of a face F of ∇ lead to the same initial ideal of I denoted by $\operatorname{in}_F(I)$. Let F be a face of ∇ and m a monomial of g_j. Then m is a monomial of $\operatorname{in}_F(g_j)$ if and only if $m \in S_{[E_j]}$ and

$$\varphi(w_y)(m) + 1 = \varphi(w_y)(m_j) \ \ \forall w_y \in F$$

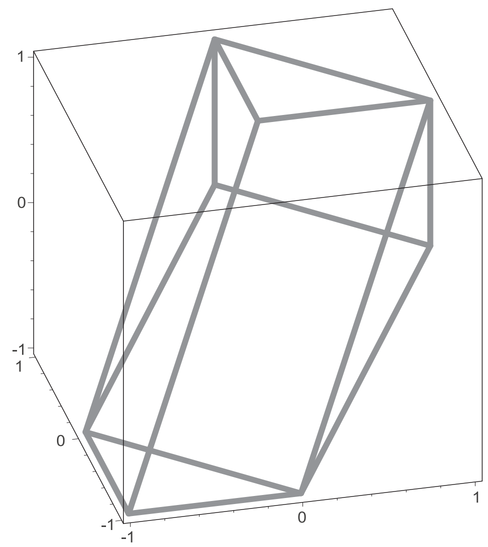


Figure 8.1: Reflexive supporting polyhedron of the special fiber Gröbner cone for the monomial degeneration of the complete intersection of two general quadrics in $\mathbb{P}^3$

so if and only if $A^{-1}\left(\frac{m}{m_i}\right) \in \Delta_j \cap M$ and

$$\left\langle A^{-1}\left(\frac{m}{m_i}\right), w_y\right\rangle = -1 \ \forall w_y \in F$$

i.e., if and only if there is an $\widetilde{m} \in \Delta_j \cap M$ with $m = m_j \cdot A(\widetilde{m})$ and

$$\langle \widetilde{m}, w_y\rangle = -1 \ \forall w_y \in F$$

hence:

Lemma 8.7 *The monomials appearing in* $\operatorname{in}_F(g_j)$ *are*

$$\{m_j\} \cup \{m_j \cdot A(\widetilde{m}) \mid \widetilde{m} \in \Delta_j \cap M \text{ with } \langle \widetilde{m}, w_y\rangle = -1 \ \forall w_y \in F \text{ and } m_j \cdot A(\widetilde{m}) \notin I_0\}$$

8.4 The dual complex of ∇

If F is a face of ∇ write

$$\operatorname{in}_F(f_j) = t \sum_{m \in G_j(F)} c_m m + m_j$$

for $j = 1, ..., c$.

Definition 8.8 *If F is a face of ∇, then define the* ***dual face*** *of F as the convex hull of all first order deformations appearing the initial ideal of I with respect to F*

$$\operatorname{dual}(F) = \operatorname{convexhull}\left(A^{-1}\left(\frac{m}{m_j}\right) \mid m \in G_j(F)\,,\ j = 1, ..., c\right) \subset M_{\mathbb{R}}$$

Then we have

$$\operatorname{dual}(F) = \operatorname{convexhull}\left(\bigcup_{j=1}^{c}\left\{A^{-1}\left(\frac{m}{m_j}\right) \mid m \text{ a monomial of } \operatorname{in}_F(g_j)\right\}\right)$$

so by Lemma 8.7

$$\begin{aligned}\operatorname{dual}(F) &= \operatorname{convexhull}\left(\bigcup_{j=1}^{c}\{\widetilde{m} \in \Delta_j \cap M \mid \langle \widetilde{m}, w_y\rangle = -1 \ \forall w_y \in F\}\right)\\ &= \operatorname{convexhull}\left(\left\{\widetilde{m} \in \bigcup_{j=1}^{c}\Delta_j \cap M \mid \langle \widetilde{m}, w_y\rangle = -1 \ \forall w_y \in F\right\}\right)\\ &= \{\widetilde{m} \in \nabla^* \mid \langle \widetilde{m}, w_y\rangle = -1 \ \forall w_y \in F\}\\ &= F^*\end{aligned}$$

hence:

Proposition 8.9 *If F is a face of ∇, then*

$$\operatorname{dual}(F) = F^* \subset \nabla^*$$

in particular $\operatorname{dual}(F)$ *is a face of ∇^*, so*

$$\operatorname{dual} : \operatorname{Poset}(\nabla) \to \operatorname{Poset}(\nabla^*)$$

is the inclusion reversing map from the face poset of ∇ to the face poset of ∇^* given by dualization of the face.

Example 8.10 *The complex of initial ideals* $\operatorname{dual}(\nabla)$ *for above Example 8.6 is visualized in Figure 8.2. Some faces of ∇ and their corresponding images under* dual *are highlighted.*

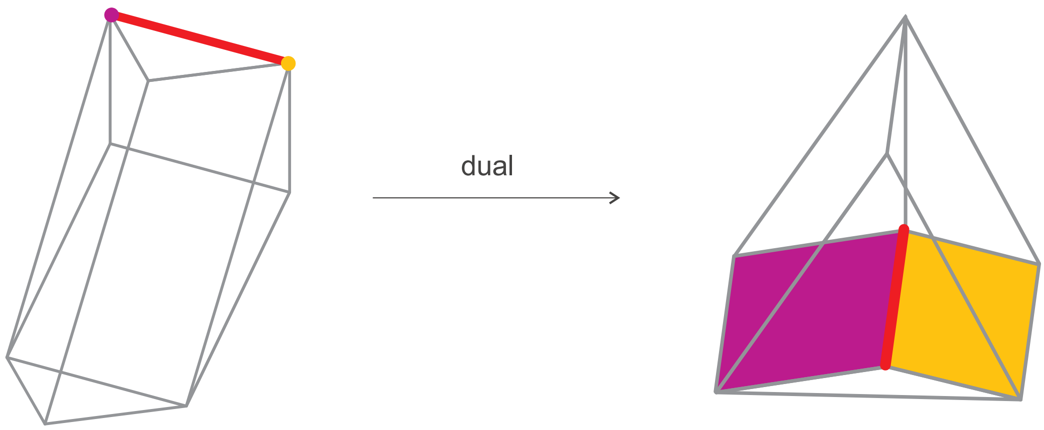


Figure 8.2: Faces of initial ideals for the monomial degeneration of the complete intersection of two general quadrics in $\mathbb{P}^3$

8.5 The Bergman subcomplex of ∇

Intersecting the Bergman complex with ∇, we obtain the following subcomplex of dimension d of the boundary complex of ∇

$$B(I) = BC_{I_0}(I) = (BF(I) \cap \operatorname{Poset}(C_{I_0}(I))) \cap \{w_t = 1\}$$

the Bergman subcomplex or tropical subcomplex of ∇. Here $\operatorname{Poset}(C_{I_0}(I))$ is the fan generated by the cone $C_{I_0}(I)$. The intersection of the fan $BF(I) \cap \operatorname{Poset}(C_{I_0}(I))$ with the hyperplane $\{w_t = 1\}$ is defined as the complex whose faces are the intersections of the cones of the fan with $\{w_t = 1\}$.

Example 8.11 *For above Example 8.6 the tropical subcomplex* $B(I) \subset \operatorname{Poset}(\nabla)$ *is shown in Figure 8.3.*

8.6 The mirror complex

If F is a face of $B(I)$ write

$$\operatorname{in}_F(f_j) = t \sum_{m \in G_j(F)} m + m_j$$

for $j = 1, ..., c$, then $G_j(F) \neq \emptyset \ \forall j$. Then we define the map

$$\mu : B(I) \to \operatorname{Poset}(\Delta)$$

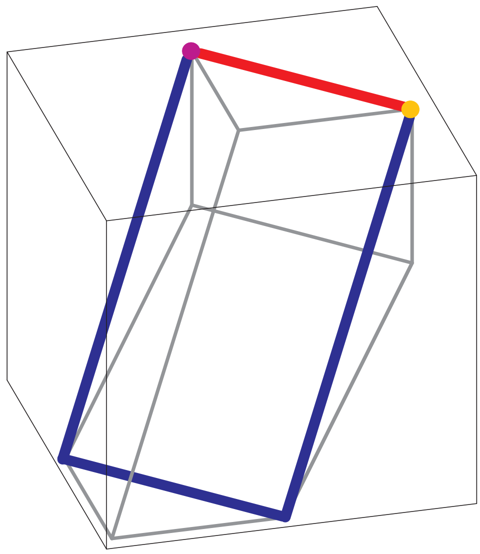


Figure 8.3: Bergman subcomplex for the monomial degeneration of the complete intersection of two general quadrics in $\mathbb{P}^3$

mapping a face of $B(I)$ to the Minkowski sum of the initial forms

$$\mu(F) = \sum_{j=1}^{c} \text{convexhull}\left(A^{-1}\left(\frac{m}{m_j}\right) \mid m \in G_j(F)\right)$$
$$= \sum_{j=1}^{c} \text{convexhull}\left(\tilde{m} \mid \tilde{m} \in \Delta_j, \langle w, \tilde{m}\rangle = -1 \forall w \in F\right)$$

Proposition 8.12 *$\mu(B(I))$ is a subcomplex of* Poset (Δ) *and μ induces an isomorphism of complexes*

$$B(I)^{\vee} \to \mu(B(I)) \subset \text{Poset}(\Delta)$$

If F is a face of $B(I)$, then

$$\dim(\mu(F)) = n - c - \dim(F) = d - \dim(F)$$

Example 8.13 *For above Example 8.6, the complexes $B(I) \subset \nabla$ and $\mu(B(I)) \subset \Delta$ are shown in Figure 8.4.*

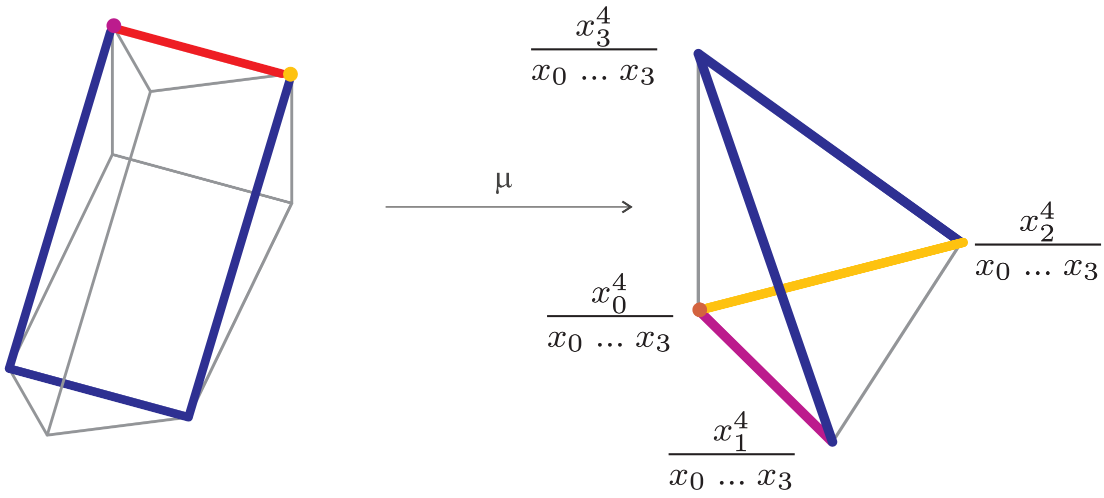


Figure 8.4: Mirror dual complex of the Bergman subcomplex for the monomial degeneration of the complete intersection of two general quadrics in $\mathbb{P}^3$

8.7 The dual complex of $B(I)$

Now consider the image of $B(I)$ under the map dual. By Lemma 8.4 the complex $\operatorname{dual}(B(I))$ is a subcomplex of ∇^*. If F is a face of $B(I)$, then

$$\dim(\operatorname{dual}(F)) = n - 1 - \dim F$$

Intersecting $\operatorname{dual}(F)$ with Δ_j, we can recover the initial monomials of the individual equations, as

$$\begin{aligned}\operatorname{dual}(F) \cap \Delta_j &= \operatorname{convexhull}\{\widetilde{m} \in \nabla^* \cap M \cap \Delta_j \mid \langle \widetilde{m}, w_y \rangle = -1 \ \forall w_y \in F\} \\ &= \operatorname{convexhull}(\widetilde{m} \in \Delta_j \cap M \mid \langle \widetilde{m}, w_y \rangle = -1 \ \forall w_y \in F) \\ &= \operatorname{convexhull}\left(A^{-1}\left(\frac{m}{m_j}\right) \mid m \text{ a monomial of } \operatorname{in}_F(g_j)\right)\end{aligned}$$

by Lemma 8.7, hence:

Lemma 8.14 *If F is a face of the special fiber Bergman complex $B(I)$, then the intersection of its dual face $\operatorname{dual}(F) \subset \nabla$ with $\Delta_j \subset \nabla$ is the face of Δ_j given as the convex hull of the deformations appearing in $\operatorname{in}_F(f_j)$, i.e.,*

$$\operatorname{dual}(F) \cap \Delta_j = \operatorname{convexhull}\left(A^{-1}\left(\frac{m}{m_j}\right) \mid m \textit{ a monomial of } \operatorname{in}_F(g_j)\right)$$

So we obtain:

Proposition 8.15 *If F is a face of $B(I)$, then $\mu(F)$ is the Minkowski sum*

$$\mu(F) = \sum_{j=1}^{c} \operatorname{dual}(F) \cap \Delta_j$$

Example 8.16 *For above Example 8.6 the complex* $\operatorname{dual}(B(I)) \subset \nabla^*$ *is shown in Figure 8.5.*

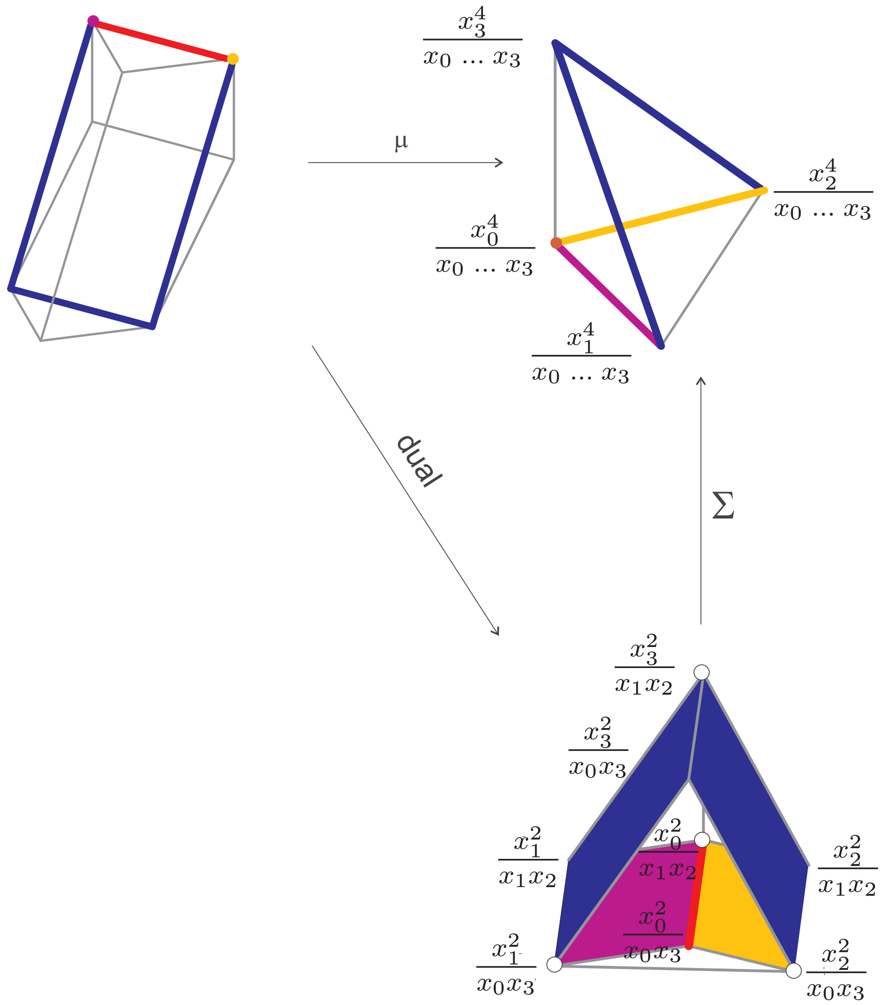


Figure 8.5: Complex of initial ideals for the monomial degeneration of the complete intersection of two general quadrics in $\mathbb{P}^3$

8.8 $B(I)$ and the complex of strata of X_0

Consider the map lim given by

$$\begin{array}{rccl} \lim: & B(I) & \to & \mathrm{Strata}(Y) \\ & F & \mapsto & \left\{\lim_{t\to 0} a(t) \mid a \in \mathrm{val}^{-1}(\mathrm{int}(F))\right\} \end{array}$$

where $\mathrm{Strata}(Y)$ denotes the poset of closures of the toric strata of the toric variety $Y = \frac{\mathbb{C}^{\Sigma(1)} - V(B(\Sigma))}{G(\Sigma)}$.

Proposition 8.17 *If F is a face of $B(I)$, then $\lim(F) = V((\mu(F))^*)$ and the complexes $\lim(B(I))$ and $\mu(B(I))$ are isomorphic.*

Note that the k-dimensional orbit closures $V(\sigma)$ correspond to the cones σ of Σ of dimension $n-k$ (i.e., faces of Δ^* of dimension $n-k-1$) .

Example 8.18 *In the above Example 8.6 for $w = (1,0,1) = (1,0,0) + (0,0,1)$ we have $\varphi(w) = (0,1,0,1) + \mathbb{Z}(1,1,1,1)$ and*

$$\lim(\{w\}) = V(x_1, x_3) \subset \frac{\mathbb{C}^4 - \{0\}}{\mathbb{C}^*}$$

Denote by $\mathrm{Strata}_\Delta(I_0)$ the complex of faces of Δ corresponding to the strata in Y of the reduced monomial ideal I_0.

Proposition 8.19 *The map*

$$\begin{array}{ccl} B(I)^\vee & \to & \mathrm{Strata}_\Delta(I_0) \\ F^\vee & \mapsto & \lim(F) \end{array}$$

is an isomorphism of complexes and

$$\dim(\lim(F)) = n - 1 - \dim(\mu(F))^* = \dim(\mu(F)) = d - \dim(F)$$

Example 8.20 *For above Example 8.6 the relation between the maps* lim *and μ is shown in Figure 8.6.*

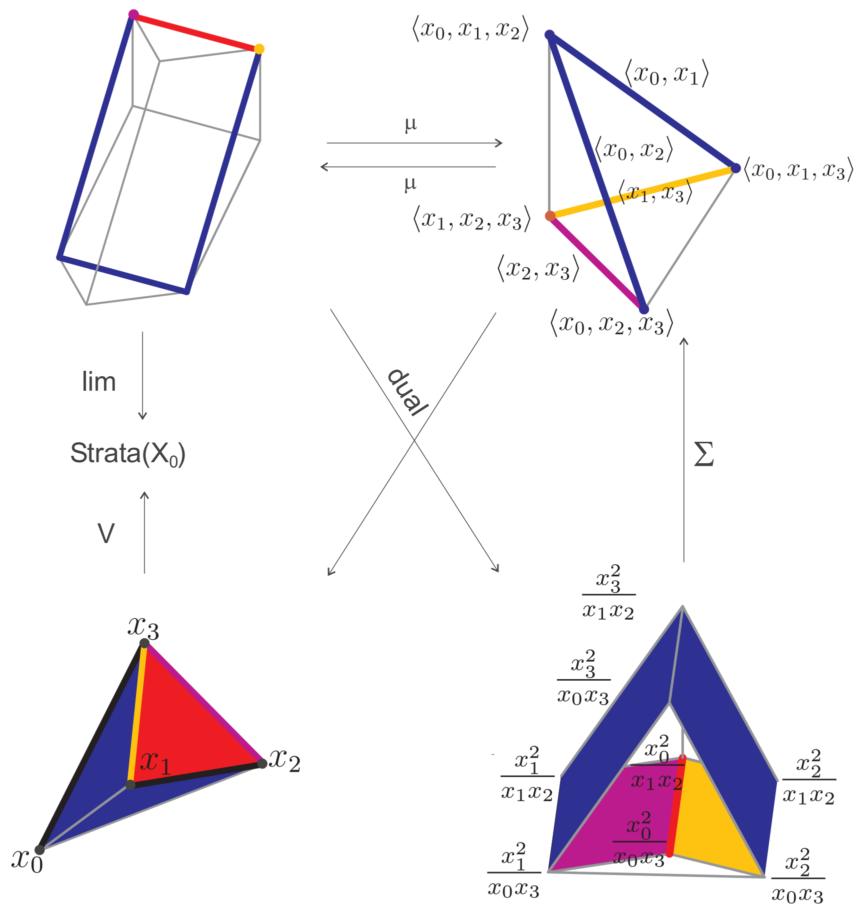


Figure 8.6: The image of lim for the monomial degeneration of the complete intersection of two general quadrics in $\mathbb{P}^3$

8.9 Remark on the topology of $B(I)$

In the case of a toric hypersurface $B(I)$ and $\lim(B(I)) \cong \mu(B(I))$ are the boundaries of the polytopes ∇ and Δ, respectively, hence are homeomorphic to spheres.

Remark 8.21 *Consider a degeneration obtained from a general complete intersection inside projective space $\mathbb{P}^n = \mathbb{P}(\Delta)$ given by the partition $\Sigma(1) = I_1 \cup ... \cup I_c$ and denote by $S(I_j)$ the simplex on I_j. Then $\lim(B(I)) \cong \mu(B(I)) \cong \mathrm{Strata}(X_0)$ is isomorphic to the join*

$$S(I_1) * ... * S(I_c) = S^{|I_1|-2} * ... * S^{|I_c|-2} \cong S^{n-2c+c} = S^d$$

via the complementary numbering

$$\mu(F) \mapsto \{r \in \Sigma(1) \mid r \not\subset \mathrm{hull}((\mu(F))^*)\}$$

Recall that the join of complexes C_1, C_2 is the complex

$$C_1 * C_2 = \{f \vee g \mid f \in C_1, g \in C_2\}$$

where $\vee$ denotes the disjoint union.

The complex $\text{Strata}(X_0)$ is homoemorphic to a sphere also in the general complete intersection setup given by a nef partition of Δ. So, as the dual cell complex of $\lim(B(I))$, also $B(I)$ is homeomorphic to a sphere. Note that, as we will see below, the complex $B(I)$ corresponds to the nef partition of ∇ dual to the nef partition of Δ.

8.10 Covering of $B(I)^{\vee}$ and reconstruction of I from the tropical data

The map

$$\begin{array}{ccc} \text{dual}(B(I)) & \longrightarrow & \mu(B(I)) \\ F^* & \mapsto & \sum_{j=1}^{c} F^* \cap \Delta_j = \mu(F) \end{array}$$

induces a $c:1$ covering of complexes:

Proposition 8.22 *The complex* $\text{dual}(B(I))$ *contains a trivial* $c:1$ *covering*

$$\bigcup_{j=1}^{c} \text{dual}(B(I)) \cap \Delta_j \xrightarrow{\pi} B(I)^{\vee}$$

with sheets $\text{dual}(B(I)) \cap \Delta_j$. *Here the intersection of the polytope* Δ_j *with the complex* $\text{dual}(B(I))$ *is defined as intersection of each face of* $\text{dual}(B(I))$ *with* Δ_j.

If $\text{dual}(F)$ *is a minimal face of* $\text{dual}(B(I))$, *i.e., has dimension* $n-1-d = c-1$, *then* F *has precisely* c *vertices (indeed precisely* c *lattice points).*

The union of the sheets should be related to the tropical subcomplex of infinity of the mirror degeneration, i.e.,

$$\bigcup_{j=1}^{c} \text{dual}(B(I)) \cap \Delta_j = BF(I^{\circ}) \cap \{w_t = 0\} \subset S^n \subset \mathbb{R}^{n+1}$$

Any face has c disjoint non-empty, but possibly degenerate, preimage faces. There is an algorithm, computing the above covering inductively from $\text{dual}(B(I))$ without using the polytopes Δ_j. It starts with associating to any face $\text{dual}(F)$ of lowest dimension $n-1-d$ the set of its c vertices. Inductively for growing dimension of $\text{dual}(F)$, associate to it the set of those of its faces, which intersect each previously computed set of sheet faces at most once:

Algorithm 8.23 *The following algorithm computes the above $c:1$ covering π of $B(I)^\vee$:*

- *If F is a face of $B(I)$ of* $\dim(F) = d$ *and $p_1, ..., p_c$ are the vertices of* $\operatorname{dual}(F)$ *then set*
$$\pi(p_j) = F^\vee$$
for $j = 1, ..., c$.

- *If $l > 0$ and F is a face of $B(I)$ of* $\dim(F) = d - l$ *then the faces of the covering π over $F^\vee$ are those faces H of* $\operatorname{dual}(F)$ *with*
  - *H intersects at most one of the elements of $\pi^{-1}(Q^\vee)$ for every face $Q^\vee \subsetneqq F^\vee$, i.e., for all faces Q of $B(I)$ with $F \subsetneqq Q$, and*
  - *$H \notin \pi^{-1}(Q^\vee)$ for all faces $Q^\vee \subsetneqq F^\vee$.*

Example 8.24 *In the case of the degeneration of the complete intersection of two general quadrics to the monomial ideal $\langle x_1x_2, x_0x_3\rangle$, as defined in Example 8.6, the two sheets of the covering inside* $\operatorname{dual}(B(I))$ *are shown in Figure 8.7. The sheets are formed by the initial terms of the defining equations $f_1 = x_1x_2 + tg_1$ and $f_2 = x_0x_3 + tg_2$ at the faces of the Bergman complex.*

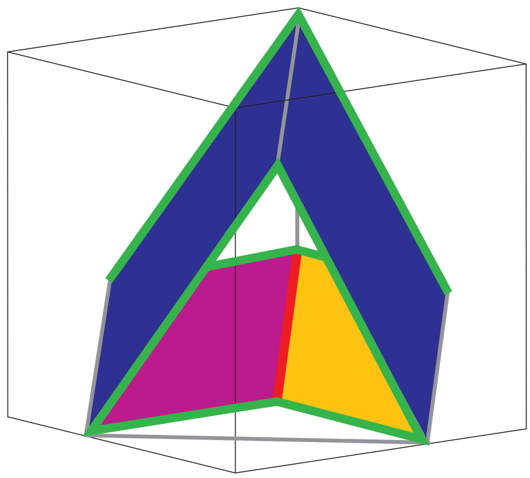


Figure 8.7: Covering of $B(I)^\vee$ given by the initial terms for the monomial degeneration of the complete intersection of two general quadrics in $\mathbb{P}^3$

Corollary 8.25 *Above covering allows to reconstruct the reduced Gröbner basis equations by clearing the denominators from*

$$f_j = t \cdot \sum_{\tilde{m} \in \operatorname{dual}(B(I)) \cap \Delta_j} A\tilde{m} + 1$$

8.11 Covering of $(\mu(B(I)))^*$, construction of I° from the tropical data and equivalence to the Batyrev-Borisov mirror

We now apply a similar procedure to construct the mirror family I°. The inclusion corresponding to $\operatorname{dual}(B(I)) = B(I)^* \subset \nabla^*$ on the mirror side should be $(\mu(B(I)))^* \subset \Delta^*$, indeed:

Lemma 8.26 *For any face F of $B(I)$ we have $F = \sum_{j=1}^{c} (\mu(F))^* \cap \nabla_j$.*
Applying the above algorithm yields a covering

$$\bigcup_{j=1}^{c} (\mu(B(I)))^* \cap \nabla_j \to (\mu(B(I)))^\vee$$

The union of the sheets should be related to the tropical subcomplex of infinity:

$$\bigcup_{j=1}^{c} (\mu(B(I)))^* \cap \nabla_j = BF(I) \cap \{w_t = 0\} \subset S^n \subset \mathbb{R}^{n+1}$$

Example 8.27 *In above Example 8.6 the faces of the covering inside the complex of mirror initial ideals is shown in Figure 8.8.*

As a corollary to Lemma 8.26 the sheets of the covering correspond to the equations defining the complete intersection special fiber $X_0 \subset Y$.

Corollary 8.28 *Denoting by $B_i = (\mu(B(I)))^* \cap \nabla_i$ the sheets of this covering, we have*

$$I_0 = \left\langle \prod_{\substack{r \in \Sigma(1) \\ \hat{r} \in \operatorname{supp}(B_i)}} y_r \mid i = 1, ..., c \right\rangle \subset S$$

In terms of the complex $\mu(B(I)) = \lim(B(I)) = \operatorname{Strata}_\Delta(I_0)$ we can define the ideal

$$\begin{aligned} I_0^\Sigma &= \bigcap\nolimits_{F \in \operatorname{Strata}_\Delta(I_0)_d} \langle y_{G^*} \mid G \text{ a facet of } \Delta \text{ with } F \subset G \rangle \\ &= \left\langle \prod_{v \in J} y_v \mid J \subset \Sigma(1) \text{ with } \operatorname{supp}(\mu(B(I))) \subset \bigcup_{v \in J} F_v \right\rangle \subset S \end{aligned}$$

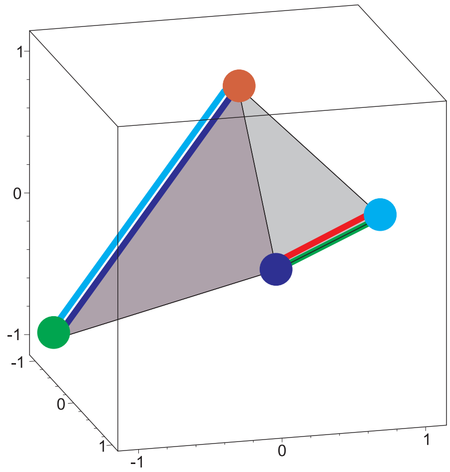


Figure 8.8: Covering of $\mu(B(I))^{\vee}$ inside Δ^{*} for the monomial degeneration of the complete intersection of two general quadrics in $\mathbb{P}^3$

Passing from I_0 to I_0^{Σ} is for reduced monomial ideals the non-simplicial Cox ring analogue of saturation in the irrelevant ideal.

Lemma 8.29 *The ideals I_0^{Σ} and I_0 in S both define the same subvariety X_0 of Y.*

Denote the map

$$0 \to M^{\circ} \xrightarrow{A^{\circ}} \mathbb{Z}^{\Sigma^{*}(1)} \to A_{n-1}(X(\Sigma^{\circ})) \to 0$$

by A°. We consider the ideal generated by equations corresponding to the sheets of the covering given in Proposition 8.26. The degeneration $\mathfrak{X}^{\circ} \subset \mathbb{P}(\nabla) \times \operatorname{Spec} \mathbb{C}[t]$ defined by

$$\left\langle t \cdot \sum_{\delta \in (\mu(B(I)))^{*} \cap \nabla_j \cap M} c_{\delta} \cdot A^{\circ}(\delta) + 1 \mid j = 1, ..., c \right\rangle$$

with generic coefficients c_{δ} coincides with the degeneration associated to the Batyrev-Borisov mirror. Clearing the denominators, the mirror degeneration is given by the $\operatorname{Pic}(Y^{\circ})$-generated ideal

$$I^{\circ} = \left\langle t \cdot \sum_{\substack{\delta \in (\mu(B(I)))^{*} \cap M \\ A^{\circ}(\delta) \cdot m_j^{\circ} \in S^{\circ}}} c_{\delta} \cdot A^{\circ}(\delta) \cdot m_j^{\circ} + m_j^{\circ} \mid j = 1, ..., c \right\rangle \subset S^{\circ} \otimes \mathbb{C}[t]$$

where m_j° is the least common multiple of the denominators of the Cox Laurent monomials $A^\circ(\delta)$ for lattice points δ in the sheet $B_j = (\mu(B(I)))^* \cap \nabla_j$. The above generators of I° coincide with the reduced Gröbner basis of the ideal of the degeneration associated to the Batyrev-Borisov mirror.

Theorem 8.30 *The mirror obtained from the tropical construction coincides with the Batyrev-Borisov mirror.*

Lemma 8.31 *Denoting by $B_i^\circ = \operatorname{dual}(B(I)) \cap \Delta_j$ the sheets of the covering given in Proposition 8.22, the special fiber of $\mathfrak{X}^\circ$ is given by*

$$I_0^\circ = \left\langle \prod_{\substack{r\in\Sigma(1)\\ \hat{r}\in\operatorname{supp}\left(B_i^\circ\right)}} z_r \mid i = 1, ..., c \right\rangle \subset S^\circ = \mathbb{C}\left[z_r \mid r \in \Sigma^\circ(1)\right]$$

In terms of the complex $B(I) = \operatorname{Strata}_\nabla(I_0^\circ)$ we have the ideal

$$\begin{aligned}(I_0^\circ)^{\Sigma^\circ} &= \bigcap\nolimits_{F\in B(I)_d} \langle z_{G^*} \mid G \text{ a facet of } \nabla \text{ with } F \subset G\rangle \\ &= \left\langle \prod_{v\in J} z_v \mid J \subset \Sigma^\circ(1) \text{ with } \operatorname{supp}(B(I)) \subset \bigcup_{v\in J} F_v \right\rangle \subset S^\circ\end{aligned}$$

Lemma 8.32 *The ideals $(I_0^\circ)^{\Sigma^\circ}$ and I_0° in S° both define the same subvariety X_0° of Y°.*

Indeed, from the point of view of saturation in the sense of removing the irrelevant components, we should associate to the special fiber X_0° of $\mathfrak{X}^\circ$ the ideal $(I_0^\circ)^\Sigma$, and to the degeneration $\mathfrak{X}^\circ$ the ideal

$$\left\langle t \cdot \sum_{\substack{\delta\in(\mu(B(I)))^*\cap\nabla_j\\ A^\circ(\delta)\cdot m_0\in S^\circ}} c_\delta \cdot A^\circ(\delta) \cdot m_0 + m_0 \mid m_0 \text{ a minimal generator of } (I_0^\circ)^{\Sigma^\circ} \right\rangle$$

in $S^\circ \otimes \mathbb{C}[t]$ with generic coefficients c_δ. The same holds true of course for $\mathfrak{X}$.

Note that passing to the saturated description does not change the objects involved in the tropical mirror construction, as the special fiber complex and the set of first order deformations does not change.

Example 8.33 *Figure 8.9 gives a summary of the tropical mirror construction for above monomial degeneration of the complete intersection of two general quadrics in $\mathbb{P}^3$.*

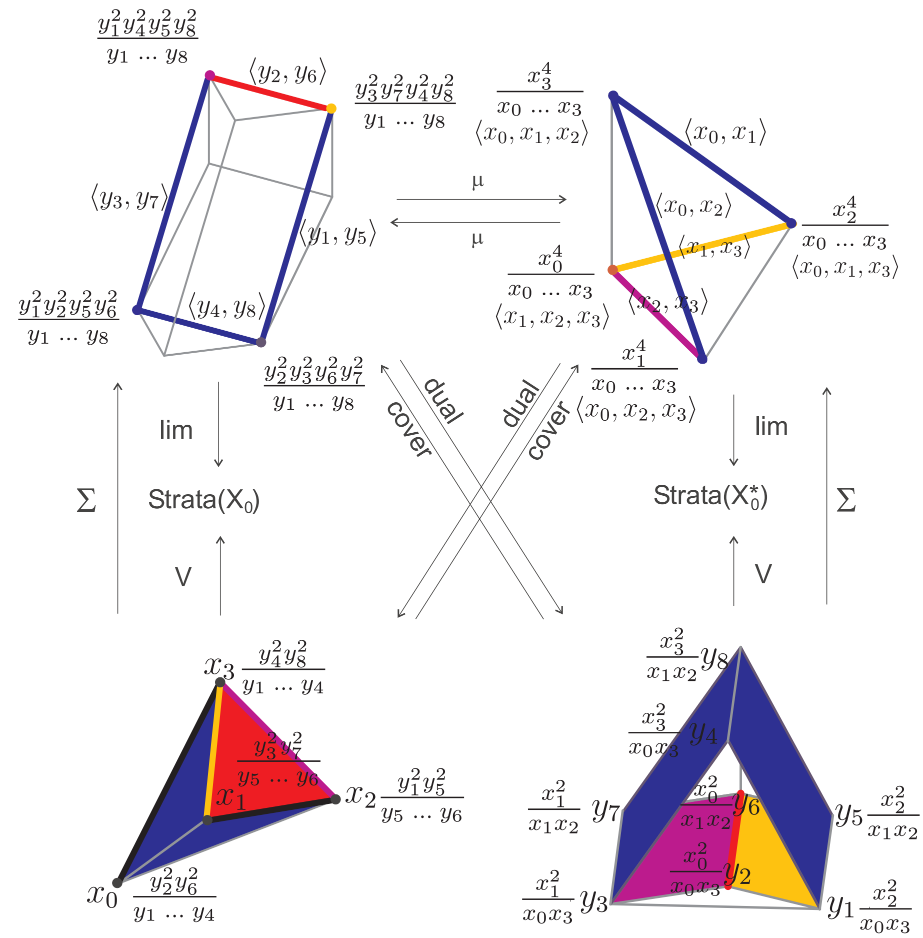


Figure 8.9: Summary of the tropical mirror construction for above monomial degeneration of the complete intersection of two general quadrics in $\mathbb{P}^3$

8.12 Examples

In the following we give the explicit computations for some simple examples. The text is computer generated from the output given by the Maple package tropicalmirror, which implements the tropical mirror construction. See also Section 12.4 for a short description of the tropicalmirror package.

8.12.1 The elliptic curve given as the complete intersection of two generic quadrics in $\mathbb{P}^3$

Setup Let $Y = \mathbb{P}^3 = X(\Sigma)$, $\Sigma = \Sigma(P) = NF(\Delta) \subset N_{\mathbb{R}}$ with the Fano polytope $P = \Delta^*$ given by

$$\Delta = \text{convexhull}\left(\begin{array}{lll} (3,-1,-1) & (-1,3,-1) & (-1,-1,3) \\ (-1,-1,-1) & & \end{array}\right) \subset M_{\mathbb{R}}$$

and let

$$S = \mathbb{C}[x_0, x_1, x_2, x_3]$$

be the Cox ring of Y with the variables

$$x_1 = x_{(1,0,0)} \quad x_2 = x_{(0,1,0)} \quad x_3 = x_{(0,0,1)} \quad x_0 = x_{(-1,-1,-1)}$$

associated to the rays of Σ. Consider the degeneration $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[t]$ of complete intersection elliptic curves of type $(2,2)$ with monomial special fiber

$$I_0 = \left\langle \; x_1 x_2 \quad x_0 x_3 \; \right\rangle$$

The degeneration is given by the ideal $I \subset S \otimes \mathbb{C}[t]$ with I_0-reduced generators

$$\left\{\begin{array}{l} x_1 x_2 + t(s_1 x_1^2 + s_2 x_1 x_3 + s_3 x_1 x_0 + s_4 x_2^2 + s_5 x_2 x_3 + s_6 x_2 x_0 + s_7 x_3^2 + s_8 x_0^2), \\ x_0 x_3 + t(s_9 x_1^2 + s_{10} x_1 x_3 + s_{11} x_1 x_0 + s_{12} x_2^2 + s_{13} x_2 x_3 + s_{14} x_2 x_0 + s_{15} x_3^2 + s_{16} x_0^2) \end{array}\right\}$$

Special fiber Gröbner cone The space of first order deformations of $\mathfrak{X}$ has dimension 16 and the deformations represented by the Cox Laurent monomials

$$\begin{array}{ccccccccccccc} \frac{x_0^2}{x_1 x_2} & \frac{x_3^2}{x_1 x_2} & \frac{x_1^2}{x_0 x_3} & \frac{x_2^2}{x_0 x_3} & \frac{x_0}{x_3} & \frac{x_3}{x_0} & \frac{x_2}{x_1} & \frac{x_1}{x_2} & \frac{x_0}{x_1} & \frac{x_1}{x_3} & \frac{x_0}{x_2} & \frac{x_2}{x_0} & \frac{x_3}{x_1} \\ \frac{x_1}{x_0} & \frac{x_3}{x_2} & \frac{x_2}{x_3} & & & & & & & & & & \end{array}$$

form a torus invariant basis. These deformations give linear inequalities defining the special fiber Gröbner cone, i.e.,

$$C_{I_0}(I) = \left\{(w_t, w_y) \in \mathbb{R} \oplus N_{\mathbb{R}} \mid \left\langle w, A^{-1}(m)\right\rangle \geq -w_t \forall m\right\}$$

for above monomials m and the presentation matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

of $A_2(Y)$. The vertices of the special fiber polytope

$$\nabla = C_{I_0}(I) \cap \{w_t = 1\}$$

and the number of faces, each vertex is contained in, are given by the table

3	$(0,0,1)$	$(-1,-1,-1)$	$(0,1,0)$	$(1,0,0)$
4	$(1,0,1)$	$(0,1,1)$	$(0,-1,-1)$	$(-1,0,-1)$

The number of faces of ∇ and their F-vectors are

Dimension	Number of faces	F-vector	
-1	1	$(0,0,0,0,0)$	
0	8	$(1,1,0,0,0)$	point
1	14	$(1,2,1,0,0)$	edge
2	4	$(1,3,3,1,0)$	triangle
2	4	$(1,4,4,1,0)$	quadrangle
3	1	$(1,8,14,8,1)$	

The dual $P^\circ = \nabla^*$ of ∇ is a Fano polytope with vertices

$(0,2,-1)$ $(0,0,-1)$ $(2,0,-1)$ $(0,0,1)$ $(-1,1,0)$ $(-1,-1,0)$
$(1,-1,0)$ $(-1,-1,2)$

and the F-vectors of the faces of P° are

Dimension	Number of faces	F-vector	
-1	1	$(0,0,0,0,0)$	
0	8	$(1,1,0,0,0)$	point
1	14	$(1,2,1,0,0)$	edge
2	4	$(1,3,3,1,0)$	triangle
2	4	$(1,4,4,1,0)$	quadrangle
3	1	$(1,8,14,8,1)$	

The polytopes ∇ and P° have 0 as the unique interior lattice point, and the Fano polytope P° defines the embedding toric Fano variety $Y^{\circ} = X(\Sigma(P^{\circ}))$ of the fibers of the mirror degeneration. The dimension of the automorphism group of Y° is

$$\dim(\operatorname{Aut}(Y^{\circ})) = 3$$

Let

$$S^{\circ} = \mathbb{C}[y_1, y_2, ..., y_8]$$

be the Cox ring of Y° with variables

$$\begin{array}{lll} y_1 = y_{(0,2,-1)} = \frac{x_2^2}{x_0\,x_3} & y_2 = y_{(0,0,-1)} = \frac{x_0}{x_3} & y_3 = y_{(2,0,-1)} = \frac{x_1^2}{x_0\,x_3} \\ y_4 = y_{(0,0,1)} = \frac{x_3}{x_0} & y_5 = y_{(-1,1,0)} = \frac{x_2}{x_1} & y_6 = y_{(-1,-1,0)} = \frac{x_0^2}{x_1\,x_2} \\ y_7 = y_{(1,-1,0)} = \frac{x_1}{x_2} & y_8 = y_{(-1,-1,2)} = \frac{x_3^2}{x_1\,x_2} & \end{array}$$

Bergman subcomplex Intersecting the tropical variety of I with the special fiber Gröbner cone $C_{I_0}(I)$ we obtain the special fiber Bergman subcomplex

$$B(I) = C_{I_0}(I) \cap BF(I) \cap \{w_t = 1\}$$

The following table chooses an indexing of the vertices of ∇ involved in $B(I)$ and gives for each vertex the numbers n_{∇} and n_B of faces of ∇ and faces of $B(I)$ it is contained in

n_{∇}	$n_{B(I)}$			
4	2	$1 = (1,0,1)$	$2 = (0,1,1)$	$3 = (0,-1,-1)$
		$4 = (-1,0,-1)$		

With this indexing the Bergman subcomplex $B(I)$ of $\operatorname{Poset}(\nabla)$ associated to the degeneration $\mathfrak{X}$ is

$$
\begin{array}{l}
[], \\
[[1],[2],[3],[4]], \\
[[1,3],[2,4],[3,4],[1,2]], \\
[], \\
[]
\end{array}
$$

$B(I)$ has the F-vector

Dimension	-1	0	1	2	3
Number of faces	0	4	4	0	0

and the F-vectors of its faces are

Dimension	Number of faces	F-vector	
0	4	$(1,1,0,0,0)$	point
1	4	$(1,2,1,0,0)$	edge

Dual complex The dual complex $\operatorname{dual}(B(I)) = (B(I))^*$ of deformations associated to $B(I)$ via initial ideals is given by

$$
\begin{array}{l}
[], \\
[], \\
[[1,3]^* = \left\langle \frac{x_2^2}{x_0 x_3}, \frac{x_2}{x_1} \right\rangle, [2,4]^* = \left\langle \frac{x_1^2}{x_0 x_3}, \frac{x_1}{x_2} \right\rangle, [3,4]^* = \left\langle \frac{x_3^2}{x_1 x_2}, \frac{x_3}{x_0} \right\rangle, [1,2]^* = \left\langle \frac{x_0^2}{x_1 x_2}, \frac{x_0}{x_3} \right\rangle], \\
[[1]^* = \left\langle \frac{x_0^2}{x_1 x_2}, \frac{x_2^2}{x_0 x_3}, \frac{x_0}{x_3}, \frac{x_2}{x_1} \right\rangle, [2]^* = \left\langle \frac{x_0^2}{x_1 x_2}, \frac{x_1^2}{x_0 x_3}, \frac{x_0}{x_3}, \frac{x_1}{x_2} \right\rangle, [3]^* = \left\langle \frac{x_3^2}{x_1 x_2}, \frac{x_2^2}{x_0 x_3}, \frac{x_3}{x_0}, \frac{x_2}{x_1} \right\rangle, \\
[4]^* = \left\langle \frac{x_3^2}{x_1 x_2}, \frac{x_1^2}{x_0 x_3}, \frac{x_3}{x_0}, \frac{x_1}{x_2} \right\rangle], \\
[]
\end{array}
$$

when writing the vertices of the faces as deformations of X_0. Note that the T-invariant basis of deformations associated to a face is given by all lattice points of the corresponding polytope in $M_{\mathbb{R}}$. When numbering the vertices of the faces of $\operatorname{dual}(B(I))$ by the Cox variables of the mirror toric Fano variety Y° the complex $\operatorname{dual}(B(I))$ is

$$[],$$

$$[],$$

$$[[1,3]^* = \langle y_1, y_5\rangle, [2,4]^* = \langle y_3, y_7\rangle, [3,4]^* = \langle y_8, y_4\rangle, [1,2]^* = \langle y_6, y_2\rangle],$$

$$[[1]^* = \langle y_6, y_1, y_2, y_5\rangle, [2]^* = \langle y_6, y_3, y_2, y_7\rangle, [3]^* = \langle y_8, y_1, y_4, y_5\rangle,$$
$$[4]^* = \langle y_8, y_3, y_4, y_7\rangle],$$

$$[]$$

The dual complex has the F-vector

Dimension	-1	0	1	2	3
Number of faces	0	0	4	4	0

and the F-vectors of the faces of $\operatorname{dual}(B(I))$ are

Dimension	Number of faces	F-vector	
1	4	$(1,2,1,0,0)$	edge
2	4	$(1,4,4,1,0)$	quadrangle

Recall that in this example the toric variety Y is projective space. The number of lattice points of the support of $\operatorname{dual}(B(I))$ relates to the dimension $h^{1,0}(X)$ of the complex moduli space of the generic fiber X of $\mathfrak{X}$ and to the dimension $h^{1,1}(\bar{X}^\circ)$ of the Kähler moduli space of the $MPCP$-blowup $\bar{X}^\circ$ of the generic fiber X° of the mirror degeneration

$$\begin{aligned}|\operatorname{supp}(\operatorname{dual}(B(I))) \cap M| &= 16 = 15 + 1 = \dim(\operatorname{Aut}(Y)) + h^{1,0}(X)\\ &= 12 + 3 + 1\\ &= |\operatorname{Roots}(Y)| + \dim(T_Y) + h^{1,1}(\bar{X}^\circ)\end{aligned}$$

There are

$$h^{1,0}(X) + \dim(T_{Y^\circ}) = 1 + 3$$

non-trivial toric polynomial deformations of X_0

$$\frac{x_2^2}{x_0\,x_3} \quad \frac{x_1^2}{x_0\,x_3} \quad \frac{x_3^2}{x_1\,x_2} \quad \frac{x_0^2}{x_1\,x_2}$$

They correspond to the toric divisors

$$D_{(0,2,-1)} \quad D_{(2,0,-1)} \quad D_{(-1,-1,2)} \quad D_{(-1,-1,0)}$$

on a MPCP-blowup of Y° inducing 1 non-zero toric divisor classes on the mirror X°. The following 12 toric divisors of Y induce the trivial divisor class on X°

$$\begin{array}{lllll} D_{(-1,1,0)} & D_{(1,-1,0)} & D_{(0,0,1)} & D_{(0,0,-1)} & D_{(0,1,-1)} \\ D_{(-1,0,0)} & D_{(1,0,-1)} & D_{(0,-1,0)} & D_{(0,1,0)} & D_{(-1,0,1)} \\ D_{(1,0,0)} & D_{(0,-1,1)} & & & \end{array}$$

Mirror special fiber The ideal I_0° of the monomial special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$ is generated by the following set of monomials in S°

$$\left\{ \begin{array}{lllllll} y_7\,y_2\,y_4\,y_5 & y_6\,y_8\,y_3\,y_5 & y_8\,y_2\,y_3\,y_5 & y_7\,y_1\,y_2\,y_4 & y_6\,y_7\,y_1\,y_4 & y_7\,y_8\,y_1\,y_2 & y_6\,y_7\,y_8\,y_1 \\ y_2\,y_3\,y_4\,y_5 & y_6\,y_3\,y_4\,y_5 & y_8\,y_1\,y_2\,y_3 & y_6\,y_8\,y_1\,y_3 & y_7\,y_8\,y_2\,y_5 & y_6\,y_7\,y_4\,y_5 & y_1\,y_2\,y_3\,y_4 \\ y_6\,y_7\,y_8\,y_5 & y_6\,y_1\,y_3\,y_4 & & & & & \end{array} \right\}$$

The $\mathrm{Pic}\,(Y^\circ)$-generated ideal

$$J_0^\circ = \left\langle \; y_6\,y_7\,y_8\,y_5 \quad y_1\,y_2\,y_3\,y_4 \; \right\rangle$$

defines the same subvariety X_0° of the toric variety Y°, and $J_0^{\circ\Sigma} = I_0^\circ$. Passing from J_0° to $J_0^{\circ\Sigma}$ is the non-simplicial toric analogue of saturation. The complex $B\,(I)^*$ labeled by the variables of the Cox ring S° of Y°, as written in the last section, is the complex $SP\,(I_0^\circ)$ of prime ideals of the toric strata of the special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$, i.e., the primary decomposition of I_0° is

$$I_0^\circ = \; \langle y_8, y_4 \rangle \cap \langle y_3, y_7 \rangle \cap \langle y_6, y_2 \rangle \cap \langle y_1, y_5 \rangle$$

Each facet $F \in B\,(I)$ corresponds to one of these ideals and this ideal is generated by the facets of ∇ containing F.

Covering structure in the deformation complex of the degeneration $\mathfrak{X}$ According to the local reduced Gröbner basis each face of the complex of deformations dual $(B(I))$ decomposes into 2 polytopes forming a 2 : 1 trivial covering of $B(I)$

$$[],$$

$$[],$$

$$[[\langle y_5\rangle, \langle y_1\rangle] \mapsto \langle y_1, y_5\rangle^{*\vee} = [1,3]^\vee, [\langle y_7\rangle, \langle y_3\rangle] \mapsto \langle y_3, y_7\rangle^{*\vee} = [2,4]^\vee,$$
$$[\langle y_8\rangle, \langle y_4\rangle] \mapsto \langle y_8, y_4\rangle^{*\vee} = [3,4]^\vee, [\langle y_6\rangle, \langle y_2\rangle] \mapsto \langle y_6, y_2\rangle^{*\vee} = [1,2]^\vee],$$

$$[[\langle y_6, y_5\rangle, \langle y_1, y_2\rangle] \mapsto \langle y_6, y_1, y_2, y_5\rangle^{*\vee} = [1]^\vee,$$
$$[\langle y_6, y_7\rangle, \langle y_3, y_2\rangle] \mapsto \langle y_6, y_3, y_2, y_7\rangle^{*\vee} = [2]^\vee,$$
$$[\langle y_8, y_5\rangle, \langle y_1, y_4\rangle] \mapsto \langle y_8, y_1, y_4, y_5\rangle^{*\vee} = [3]^\vee,$$
$$[\langle y_8, y_7\rangle, \langle y_3, y_4\rangle] \mapsto \langle y_8, y_3, y_4, y_7\rangle^{*\vee} = [4]^\vee],$$

$$[]$$

Here the faces $F \in B(I)$ are specified both via the vertices of F^* labeled by the variables of S° and by the numbering of the vertices of $B(I)$ chosen above. This covering has 2 sheets forming the complexes

$$[],$$

$$[],$$

$$[\langle y_1\rangle, \langle y_3\rangle, \langle y_4\rangle, \langle y_2\rangle],$$

$$[\langle y_1, y_2\rangle, \langle y_3, y_2\rangle, \langle y_1, y_4\rangle, \langle y_3, y_4\rangle],$$

$$[]$$

$$[],$$

$$[],$$

$$[\langle y_5\rangle, \langle y_7\rangle, \langle y_8\rangle, \langle y_6\rangle],$$

$$[\langle y_6, y_5\rangle, \langle y_6, y_7\rangle, \langle y_8, y_5\rangle, \langle y_8, y_7\rangle],$$

$$[]$$

with F-vector

Dimension	Number of faces	F-vector	
0	4	$(1,1,0,0,0)$	point
1	4	$(1,2,1,0,0)$	edge

Writing the vertices of the faces as deformations the covering is given by

$$
\begin{aligned}
&[],\\
&[],\\
&[[\left\langle \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^2}{x_0\,x_3} \right\rangle] \mapsto [1,3]^\vee, [\left\langle \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_1^2}{x_0\,x_3} \right\rangle] \mapsto [2,4]^\vee,\\
&[\left\langle \frac{x_3^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_3}{x_0} \right\rangle] \mapsto [3,4]^\vee, [\left\langle \frac{x_0^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_0}{x_3} \right\rangle] \mapsto [1,2]^\vee],\\
&[[\left\langle \frac{x_0^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^2}{x_0\,x_3}, \frac{x_0}{x_3} \right\rangle] \mapsto [1]^\vee,\\
&[\left\langle \frac{x_0^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_1^2}{x_0\,x_3}, \frac{x_0}{x_3} \right\rangle] \mapsto [2]^\vee,\\
&[\left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^2}{x_0\,x_3}, \frac{x_3}{x_0} \right\rangle] \mapsto [3]^\vee,\\
&[\left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_1^2}{x_0\,x_3}, \frac{x_3}{x_0} \right\rangle] \mapsto [4]^\vee],\\
&[]
\end{aligned}
$$

with the 2 sheets forming the complexes

$$
\begin{aligned}
&[],\\
&[],\\
&[\left\langle \frac{x_2^2}{x_0\,x_3} \right\rangle, \left\langle \frac{x_1^2}{x_0\,x_3} \right\rangle, \left\langle \frac{x_3}{x_0} \right\rangle, \left\langle \frac{x_0}{x_3} \right\rangle],\\
&[\left\langle \frac{x_2^2}{x_0\,x_3}, \frac{x_0}{x_3} \right\rangle, \left\langle \frac{x_1^2}{x_0\,x_3}, \frac{x_0}{x_3} \right\rangle, \left\langle \frac{x_2^2}{x_0\,x_3}, \frac{x_3}{x_0} \right\rangle, \left\langle \frac{x_1^2}{x_0\,x_3}, \frac{x_3}{x_0} \right\rangle],\\
&[]
\end{aligned}
$$

$$
\begin{aligned}
&[],\\
&[],\\
&[\left\langle \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_3^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_0^2}{x_1\,x_2} \right\rangle],\\
&[\left\langle \frac{x_0^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_0^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle],\\
&[]
\end{aligned}
$$

Note that the torus invariant basis of deformations corresponding to a Bergman face is given by the set of all lattice points of the polytope specified above.

Limit map The limit map $\lim : B(I) \to \text{Poset}(\Delta)$ associates to a face F of $B(I)$ the face of Δ formed by the limit points of arcs lying over the weight vectors $w \in F$, i.e., with lowest order term t^w. Labeling the faces of the Bergman complex $B(I) \subset \text{Poset}(\nabla)$ and the faces of $\text{Poset}(\Delta)$ by the corresponding dual faces of ∇^* and Δ^*, hence considering the limit map $\lim : B(I) \to \text{Poset}(\Delta)$ as a map $B(I)^* \to \text{Poset}(\Delta^*)$, the limit correspondence is given by

$$\begin{array}{l} [], \\ [\langle y_6, y_1, y_2, y_5\rangle \mapsto \langle x_1, x_3\rangle, \langle y_6, y_3, y_2, y_7\rangle \mapsto \langle x_2, x_3\rangle, \\ \langle y_8, y_1, y_4, y_5\rangle \mapsto \langle x_1, x_0\rangle, \langle y_8, y_3, y_4, y_7\rangle \mapsto \langle x_2, x_0\rangle], \\ [\langle y_1, y_5\rangle \mapsto \langle x_1, x_3, x_0\rangle, \langle y_3, y_7\rangle \mapsto \langle x_2, x_3, x_0\rangle, \\ \langle y_8, y_4\rangle \mapsto \langle x_1, x_2, x_0\rangle, \langle y_6, y_2\rangle \mapsto \langle x_1, x_2, x_3\rangle] \end{array}$$

The image of the limit map coincides with the image of μ and with the Bergman complex of the mirror, i.e., $\lim(B(I)) = \mu(B(I)) = B(I^\circ)$.

Mirror complex Numbering the vertices of the mirror complex $\mu(B(I))$ as

$$\begin{array}{lll} 1 = (3, -1, -1) & 2 = (-1, 3, -1) & 3 = (-1, -1, 3) \\ 4 = (-1, -1, -1) & & \end{array}$$

$\mu(B(I)) \subset \text{Poset}(\Delta)$ is

$$\begin{array}{l} [], \\ [[2], [1], [3], [4]], \\ [[2, 4], [1, 4], [2, 3], [1, 3]], \\ [], \\ [] \end{array}$$

The ordering of the faces of $\mu(B(I))$ is compatible with above ordering of the faces of $B(I)$. The F-vectors of the faces of $\mu(B(I))^*$ are

Dimension	Number of faces	F-vector	
1	4	$(1, 2, 1, 0, 0)$	edge
2	4	$(1, 3, 3, 1, 0)$	triangle

The first order deformations of the mirror special fiber X_0° correspond to the lattice points of the dual complex $(\mu(B(I)))^* \subset \mathrm{Poset}(\Delta^*)$ of the mirror complex. We label the first order deformations of X_0° corresponding to vertices of Δ^* by the homogeneous coordinates of Y

$$\begin{array}{ll} x_1 = x_{(1,0,0)} = \frac{y_3^2 y_7}{y_5\, y_6\, y_8} & x_2 = x_{(0,1,0)} = \frac{y_1^2 y_5}{y_6\, y_7\, y_8} \\ x_3 = x_{(0,0,1)} = \frac{y_4\, y_8{}^2}{y_1\, y_2\, y_3} & x_0 = x_{(-1,-1,-1)} = \frac{y_2\, y_6{}^2}{y_1\, y_3\, y_4} \end{array}$$

So writing the vertices of the faces of $(\mu(B(I)))^*$ as homogeneous coordinates of Y, the complex $(\mu(B(I)))^*$ is given by

$$\begin{array}{l} [], \\ [], \\ [\langle x_1, x_3\rangle, \langle x_2, x_3\rangle, \langle x_1, x_0\rangle, \langle x_2, x_0\rangle], \\ [\langle x_1, x_3, x_0\rangle, \langle x_2, x_3, x_0\rangle, \langle x_1, x_2, x_0\rangle, \langle x_1, x_2, x_3\rangle], \\ [] \end{array}$$

The complex $\mu(B(I))^*$ labeled by the variables of the Cox ring S gives the ideals of the toric strata of the special fiber X_0 of $\mathfrak{X}$, i.e., the complex $SP(I_0)$, so in particular the primary decomposition of I_0 is

$$I_0 = \langle x_2, x_0\rangle \cap \langle x_1, x_0\rangle \cap \langle x_1, x_3\rangle \cap \langle x_2, x_3\rangle$$

Labeling the vertices of the faces by the corresponding deformations the complex is given by

$$\begin{array}{l} [], \\ [], \\ \left[\left\langle \frac{y_3^2 y_7}{y_5\, y_6\, y_8}, \frac{y_4\, y_8{}^2}{y_1\, y_2\, y_3} \right\rangle, \left\langle \frac{y_1^2 y_5}{y_6\, y_7\, y_8}, \frac{y_4\, y_8{}^2}{y_1\, y_2\, y_3} \right\rangle, \left\langle \frac{y_2\, y_6{}^2}{y_1\, y_3\, y_4}, \frac{y_3{}^2 y_7}{y_5\, y_6\, y_8} \right\rangle,\right. \\ \left.\left\langle \frac{y_2\, y_6{}^2}{y_1\, y_3\, y_4}, \frac{y_1{}^2 y_5}{y_6\, y_7\, y_8} \right\rangle\right], \\ \left[\left\langle \frac{y_2\, y_6{}^2}{y_1\, y_3\, y_4}, \frac{y_3{}^2 y_7}{y_5\, y_6\, y_8}, \frac{y_4\, y_8{}^2}{y_1\, y_2\, y_3} \right\rangle, \left\langle \frac{y_2\, y_6{}^2}{y_1\, y_3\, y_4}, \frac{y_1{}^2 y_5}{y_6\, y_7\, y_8}, \frac{y_4\, y_8{}^2}{y_1\, y_2\, y_3} \right\rangle,\right. \\ \left.\left\langle \frac{y_2\, y_6{}^2}{y_1\, y_3\, y_4}, \frac{y_3{}^2 y_7}{y_5\, y_6\, y_8}, \frac{y_1{}^2 y_5}{y_6\, y_7\, y_8} \right\rangle, \left\langle \frac{y_3^2 y_7}{y_5\, y_6\, y_8}, \frac{y_1{}^2 y_5}{y_6\, y_7\, y_8}, \frac{y_4\, y_8{}^2}{y_1\, y_2\, y_3} \right\rangle\right], \\ [] \end{array}$$

Covering structure in the deformation complex of the mirror degeneration Each face of the complex of deformations $(\mu(B(I)))^*$ of the mirror special fiber X_0° decomposes as the convex hull of 2 polytopes forming a 2 : 1 trivial covering of $\mu(B(I))^\vee$

$$
\begin{array}{l}
[],\\
[],\\
[[\langle x_3\rangle, \langle x_1\rangle] \mapsto \langle x_1, x_3\rangle^{*\vee} = [2,4]^\vee, [\langle x_3\rangle, \langle x_2\rangle] \mapsto \langle x_2, x_3\rangle^{*\vee} = [1,4]^\vee,\\
[\langle x_0\rangle, \langle x_1\rangle] \mapsto \langle x_1, x_0\rangle^{*\vee} = [2,3]^\vee, [\langle x_0\rangle, \langle x_2\rangle] \mapsto \langle x_2, x_0\rangle^{*\vee} = [1,3]^\vee],\\
[[\langle x_3, x_0\rangle, \langle x_1\rangle] \mapsto \langle x_1, x_3, x_0\rangle^{*\vee} = [2]^\vee, [\langle x_3, x_0\rangle, \langle x_2\rangle] \mapsto \langle x_2, x_3, x_0\rangle^{*\vee} = [1]^\vee,\\
[\langle x_0\rangle, \langle x_1, x_2\rangle] \mapsto \langle x_1, x_2, x_0\rangle^{*\vee} = [3]^\vee, [\langle x_3\rangle, \langle x_1, x_2\rangle] \mapsto \langle x_1, x_2, x_3\rangle^{*\vee} = [4]^\vee],\\
[]
\end{array}
$$

Due to the singularities of Y° this covering involves degenerate faces, i.e., faces $G \mapsto F^\vee$ with $\dim(G) < \dim(F^\vee)$.

Writing the vertices of the faces as deformations of X_0° the covering is given by

$$
\begin{array}{l}
[],\\
[],\\
[[\left\langle \frac{y_4\, y_8{}^2}{y_1\, y_2\, y_3} \right\rangle, \left\langle \frac{y_3^2 y_7}{y_5\, y_6\, y_8} \right\rangle] \mapsto [2,4]^\vee, [\left\langle \frac{y_4\, y_8{}^2}{y_1\, y_2\, y_3} \right\rangle, \left\langle \frac{y_1^2 y_5}{y_6\, y_7\, y_8} \right\rangle] \mapsto [1,4]^\vee,\\
[\left\langle \frac{y_2\, y_6{}^2}{y_1\, y_3\, y_4} \right\rangle, \left\langle \frac{y_3^2 y_7}{y_5\, y_6\, y_8} \right\rangle] \mapsto [2,3]^\vee, [\left\langle \frac{y_2\, y_6{}^2}{y_1\, y_3\, y_4} \right\rangle, \left\langle \frac{y_1^2 y_5}{y_6\, y_7\, y_8} \right\rangle] \mapsto [1,3]^\vee],\\
[[\left\langle \frac{y_4\, y_8{}^2}{y_1\, y_2\, y_3}, \frac{y_2\, y_6{}^2}{y_1\, y_3\, y_4} \right\rangle, \left\langle \frac{y_3^2 y_7}{y_5\, y_6\, y_8} \right\rangle] \mapsto [2]^\vee, [\left\langle \frac{y_4\, y_8{}^2}{y_1\, y_2\, y_3}, \frac{y_2\, y_6{}^2}{y_1\, y_3\, y_4} \right\rangle, \left\langle \frac{y_1^2 y_5}{y_6\, y_7\, y_8} \right\rangle] \mapsto [1]^\vee,\\
[\left\langle \frac{y_2\, y_6{}^2}{y_1\, y_3\, y_4} \right\rangle, \left\langle \frac{y_3^2 y_7}{y_5\, y_6\, y_8}, \frac{y_1{}^2 y_5}{y_6\, y_7\, y_8} \right\rangle] \mapsto [3]^\vee, [\left\langle \frac{y_4\, y_8{}^2}{y_1\, y_2\, y_3} \right\rangle, \left\langle \frac{y_3^2 y_7}{y_5\, y_6\, y_8}, \frac{y_1{}^2 y_5}{y_6\, y_7\, y_8} \right\rangle] \mapsto [4]^\vee],\\
[]
\end{array}
$$

Mirror degeneration The space of first order deformations of X_0° in the mirror degeneration $\mathfrak{X}^\circ$ has dimension 4 and the deformations represented by the monomials

$$\left\{ \frac{y_1^2 y_5}{y_6\, y_7\, y_8} \quad \frac{y_3^2 y_7}{y_5\, y_6\, y_8} \quad \frac{y_4\, y_8{}^2}{y_1\, y_2\, y_3} \quad \frac{y_2\, y_6{}^2}{y_1\, y_3\, y_4} \right\}$$

form a torus invariant basis. The number of lattice points of the dual of the mirror complex of I relates to the dimension $h^{1,0}(X^\circ)$ of complex moduli space of the generic fiber X° of $\mathfrak{X}^\circ$ and to the dimension $h^{1,1}(X)$ of the Kähler moduli space of the generic fiber X of $\mathfrak{X}$ via

$$\begin{aligned} |\operatorname{supp}((\mu(B(I)))^*) \cap N| &= 4 = 3 + 1 \\ &= \dim(\operatorname{Aut}(Y^\circ)) + h^{1,0}(X^\circ) = \dim(T) + h^{1,1}(X) \end{aligned}$$

The mirror degeneration $\mathfrak{X}^\circ \subset Y^\circ \times \operatorname{Spec} \mathbb{C}[t]$ of $\mathfrak{X}$ is given by the ideal $I^\circ \subset S^\circ \otimes \mathbb{C}[t]$ generated by

$$t\,(s_1\, y_2^2 y_6^2 + s_3\, y_4^2 y_8^2) + y_1\, y_2\, y_3\, y_4 \quad t\,(s_2\, y_1^2 y_5^2 + s_4\, y_3^2 y_7^2) + y_6\, y_7\, y_8\, y_5$$
$$ts_3\, y_8^3 y_4 + y_8\, y_1\, y_2\, y_3 \quad ts_1\, y_6^3 y_2 + y_6\, y_1\, y_3\, y_4 \quad ts_4\, y_3^3 y_7 + y_6\, y_8\, y_3\, y_5$$
$$ts_2\, y_1^3 y_5 + y_6\, y_7\, y_8\, y_1$$

$$\begin{matrix} y_7\, y_8\, y_1\, y_2 & y_6\, y_3\, y_4\, y_5 & y_7\, y_1\, y_2\, y_4 & y_6\, y_7\, y_4\, y_5 & y_7\, y_8\, y_2\, y_5 \\ y_6\, y_8\, y_1\, y_3 & y_2\, y_3\, y_4\, y_5 & y_7\, y_2\, y_4\, y_5 & y_6\, y_7\, y_1\, y_4 & y_8\, y_2\, y_3\, y_5 \end{matrix}$$

The ideal J° which is $\operatorname{Pic}(Y^\circ)$-generated by

$$\left\{ \begin{array}{l} t(s_1\, y_2^2 y_6^2 + s_3\, y_4^2 y_8^2) + y_1\, y_2\, y_3\, y_4, \\ t(s_2\, y_1^2 y_5^2 + s_4\, y_3^2 y_7^2) + y_6\, y_7\, y_8\, y_5 \end{array} \right\}$$

defines a flat affine cone inducing $\mathfrak{X}^\circ$.

Contraction of the mirror degeneration In the following we give a birational map relating the degeneration $\mathfrak{X}^\circ$ to a Greene-Plesser type orbifolding mirror family by contracting divisors on Y°. See also Section 9.13 below. In order to contract the divisors

$$\begin{matrix} y_2 = y_{(0,0,-1)} = \frac{x_0}{x_3} & y_4 = y_{(0,0,1)} = \frac{x_3}{x_0} \\ y_5 = y_{(-1,1,0)} = \frac{x_2}{x_1} & y_7 = y_{(1,-1,0)} = \frac{x_1}{x_2} \end{matrix}$$

consider the $\mathbb{Q}$-factorial toric Fano variety $\hat{Y}^\circ = X(\hat{\Sigma}^\circ)$, where $\hat{\Sigma}^\circ = \Sigma(\hat{P}^\circ) \subset M_\mathbb{R}$ is the fan over the Fano polytope $\hat{P}^\circ \subset M_\mathbb{R}$ given as the convex hull of the remaining vertices of $P^\circ = \nabla^*$ corresponding to the Cox variables

$$\begin{array}{ll} y_1 = y_{(0,2,-1)} = \frac{x_2^2}{x_0\,x_3} & y_3 = y_{(2,0,-1)} = \frac{x_1^2}{x_0\,x_3} \\ y_8 = y_{(-1,-1,2)} = \frac{x_3^2}{x_1\,x_2} & y_6 = y_{(-1,-1,0)} = \frac{x_0^2}{x_1\,x_2} \end{array}$$

of the toric variety $\hat{Y}^\circ$ with Cox ring

$$\hat{S}^\circ = \mathbb{C}[y_1, y_3, y_8, y_6]$$

The Cox variables of $\hat{Y}^\circ$ correspond to the set of Fermat deformations of $\mathfrak{X}$. Let

$$Y^\circ = X\left(\Sigma^\circ\right) \to X(\hat{\Sigma}^\circ) = \hat{Y}^\circ$$

be a birational map from Y° to the toric Fano variety $\hat{Y}^\circ$, which contracts the divisors of the rays $\Sigma^\circ(1) - \hat{\Sigma}^\circ(1)$ corresponding to the Cox variables

$$y_2 \quad y_4 \quad y_5 \quad y_7$$

Representing $\hat{Y}^\circ$ as a quotient we have

$$\hat{Y}^\circ = \left(\mathbb{C}^4 - V\left(B(\hat{\Sigma}^\circ)\right)\right)/\hat{G}^\circ$$

with

$$\hat{G}^\circ = \mathbb{Z}_4 \times \left(\mathbb{C}^*\right)^1$$

acting via

$$\xi y = \left(u_1^3\,v_1 \cdot y_1,\; u_1\,v_1 \cdot y_3,\; u_1^2\,v_1 \cdot y_8,\; v_1 \cdot y_6\right)$$

for $\xi = (u_1, v_1) \in \hat{G}^\circ$ and $y \in \mathbb{C}^4 - V\left(B(\hat{\Sigma}^\circ)\right)$. Hence with the group

$$\hat{H}^\circ = \mathbb{Z}_4$$

of order 4 the toric variety $\hat{Y}^\circ$ is the quotient

$$\hat{Y}^\circ = \mathbb{P}^3/\hat{H}^\circ$$

of projective space $\mathbb{P}^3$. The mirror degeneration $\mathfrak{X}^\circ$ induces via $Y^\circ \to \hat{Y}^\circ$ a degeneration $\hat{\mathfrak{X}}^\circ \subset \hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]$ given by the ideal $\hat{I}^\circ \subset \langle y_1, y_3, y_8, y_6\rangle \subset \hat{S}^\circ \otimes \mathbb{C}[t]$ generated by the Fermat-type equations

$$\left\{\begin{array}{l} y_6\,y_8 + t(s_2\,y_1^2 + s_4\,y_3^2), \\ y_1\,y_3 + t(s_1\,y_6^2 + s_3\,y_8^2) \end{array}\right\}$$

The special fiber $\hat{X}_0^\circ \subset \hat{Y}^\circ$ of $\hat{\mathfrak{X}}^\circ$ is cut out by the monomial ideal

$$\hat{I}_0^\circ \subset B(\hat{\Sigma}^\circ) = \langle y_1, y_3, y_8, y_6\rangle \subset \hat{S}^\circ$$

generated by

$$\left\{\; y_1\,y_3 \quad y_6\,y_8 \;\right\}$$

The complex $SP(\hat{I}_0^\circ)$ of prime ideals of the toric strata of $\hat{X}_0^\circ$ is

$$
\begin{aligned}
&[],\\
&[],\\
&[\langle y_3, y_8\rangle, \langle y_1, y_8\rangle, \langle y_3, y_6\rangle, \langle y_1, y_6\rangle],\\
&[\langle y_3, y_8, y_6\rangle, \langle y_1, y_8, y_6\rangle, \langle y_1, y_3, y_6\rangle, \langle y_1, y_3, y_8\rangle],\\
&[]
\end{aligned}
$$

so $\hat{I}_0^\circ$ has the primary decomposition

$$\hat{I}_0^\circ = \langle y_1, y_6\rangle \cap \langle y_1, y_8\rangle \cap \langle y_3, y_8\rangle \cap \langle y_3, y_6\rangle$$

The Bergman subcomplex $B(I) \subset \mathrm{Poset}(\nabla)$ induces a subcomplex of $\mathrm{Poset}(\hat{\nabla})$, $\hat{\nabla} = (\hat{P}^\circ)^*$ corresponding to $\hat{I}_0^\circ$. Indexing of the vertices of $\hat{\nabla}$ by

$$
\begin{array}{ll}
1 = \left(-\frac{1}{2}, \frac{3}{2}, 0\right) & 2 = \left(\frac{3}{2}, -\frac{1}{2}, 0\right)\\
3 = \left(\frac{1}{2}, \frac{1}{2}, 2\right) & 4 = \left(-\frac{3}{2}, -\frac{3}{2}, -2\right)
\end{array}
$$

this complex is given by

$$
\left\{
\begin{array}{l}
[],\\
[[1],[2],[3],[4]],\\
[[1,4],[2,4],[1,3],[2,3]],\\
[],\\
[]
\end{array}
\right\}
$$

8.12.2 The $K3$ surface given as the complete intersection of a generic quadric and a generic cubic in $\mathbb{P}^4$

Setup Let $Y = \mathbb{P}^4 = X(\Sigma)$, $\Sigma = \Sigma(P) = NF(\Delta) \subset N_{\mathbb{R}}$ with the Fano polytope $P = \Delta^*$ given by

$$\Delta = \mathrm{convexhull}\left(\begin{array}{lll} (4,-1,-1,-1) & (-1,4,-1,-1) & (-1,-1,4,-1)\\ (-1,-1,-1,4) & (-1,-1,-1,-1) & \end{array}\right) \subset M_{\mathbb{R}}$$

and let

$$S = \mathbb{C}[x_0, x_1, x_2, x_3, x_4]$$

be the Cox ring of Y with the variables

$$
\begin{array}{ll}
x_1 = x_{(1,0,0,0)} & x_2 = x_{(0,1,0,0)}\\
x_3 = x_{(0,0,1,0)} & x_4 = x_{(0,0,0,1)}\\
x_0 = x_{(-1,-1,-1,-1)} &
\end{array}
$$

associated to the rays of Σ. Consider the degeneration $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[t]$ of complete intersection K3 surfaces of type $(2,3)$ with monomial special fiber

$$I_0 = \left\langle\ x_1\, x_2 \quad x_0\, x_3\, x_4\ \right\rangle$$

The degeneration is given by the ideal $I \subset S \otimes \mathbb{C}[t]$ with I_0-reduced generators

$$\left\{\begin{array}{l} x_1\, x_2 + t(s_1\, x_1^2 + \ldots + s_5\, x_2^2 + \ldots + s_9\, x_3^2 + \ldots + s_{12}\, x_4^2 + \ldots + s_{14}\, x_0^2), \\ x_0\, x_3\, x_4 + t(s_{15}\, x_1^3 + \ldots + s_{25}\, x_2^3 + \ldots + s_{35}\, x_3^3 + \ldots + s_{40}\, x_4^3 + \ldots + s_{43}\, x_0^3) \end{array}\right\}$$

Special fiber Gröbner cone The space of first order deformations of $\mathfrak{X}$ has dimension 42 and the deformations represented by the Cox Laurent monomials

$\frac{x_1^3}{x_0\, x_3\, x_4}$	$\frac{x_2^3}{x_0\, x_3\, x_4}$	$\frac{x_2\, x_3}{x_4\, x_0}$	$\frac{x_1\, x_0}{x_3\, x_4}$	$\frac{x_1\, x_4}{x_0\, x_3}$	$\frac{x_3\, x_4}{x_1\, x_2}$	$\frac{x_4\, x_0}{x_1\, x_2}$	$\frac{x_2\, x_0}{x_3\, x_4}$	$\frac{x_2\, x_4}{x_0\, x_3}$
$\frac{x_1\, x_3}{x_4\, x_0}$	$\frac{x_2^2}{x_3\, x_4}$	$\frac{x_1^2}{x_3\, x_4}$	$\frac{x_2^2}{x_4\, x_0}$	$\frac{x_3^2}{x_4\, x_0}$	$\frac{x_0^2}{x_3\, x_4}$	$\frac{x_4^2}{x_0\, x_3}$	$\frac{x_4^2}{x_1\, x_2}$	$\frac{x_0^2}{x_1\, x_2}$
$\frac{x_1^2}{x_4\, x_0}$	$\frac{x_3^2}{x_1\, x_2}$	$\frac{x_1^2}{x_0\, x_3}$	$\frac{x_2^2}{x_0\, x_3}$	$\frac{x_0}{x_3}$	$\frac{x_3}{x_0}$	$\frac{x_2}{x_1}$	$\frac{x_1}{x_2}$	$\frac{x_0}{x_1}$
$\frac{x_1}{x_3}$	$\frac{x_0}{x_2}$	$\frac{x_2}{x_0}$	$\frac{x_3}{x_1}$	$\frac{x_1}{x_0}$	$\frac{x_3}{x_2}$	$\frac{x_2}{x_3}$	$\frac{x_0}{x_4}$	$\frac{x_4}{x_2}$
$\frac{x_4}{x_0}$	$\frac{x_4}{x_3}$	$\frac{x_4}{x_1}$	$\frac{x_3}{x_4}$	$\frac{x_1}{x_4}$	$\frac{x_2}{x_4}$			

form a torus invariant basis. These deformations give linear inequalities defining the special fiber Gröbner cone, i.e.,

$$C_{I_0}(I) = \left\{(w_t, w_y) \in \mathbb{R} \oplus N_{\mathbb{R}} \mid \left\langle w, A^{-1}(m)\right\rangle \geq -w_t \forall m\right\}$$

for above monomials m and the presentation matrix

$$A = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{array}\right]$$

of $A_3(Y)$. The vertices of the special fiber polytope

$$\nabla = C_{I_0}(I) \cap \{w_t = 1\}$$

and the number of faces, each vertex is contained in, are given by the table

6	$(0,0,1,0)$ $(0,0,0,1)$ $(-1,-1,-1,-1)$ $(0,1,0,0)$ $(1,0,0,0)$
9	$(1,0,1,0)$ $(0,1,1,0)$ $(1,0,0,1)$ $(0,1,0,1)$ $(0,-1,-1,-1)$ $(-1,0,-1,-1)$

The number of faces of ∇ and their F-vectors are

Dimension	Number of faces	F-vector	
-1	1	$(0,0,0,0,0,0)$	
0	11	$(1,1,0,0,0,0)$	point
1	25	$(1,2,1,0,0,0)$	edge
2	12	$(1,4,4,1,0,0)$	quadrangle
2	12	$(1,3,3,1,0,0)$	triangle
3	2	$(1,4,6,4,1,0)$	tetrahedron
3	8	$(1,6,9,5,1,0)$	prism
4	1	$(1,11,25,24,10,1)$	

The dual $P^\circ = \nabla^*$ of ∇ is a Fano polytope with vertices

$(-1,-1,0,2)$	$(-1,-1,0,0)$	$(-1,1,0,0)$	$(1,-1,0,0)$	$(-1,-1,2,0)$
$(0,3,-1,-1)$	$(0,0,-1,2)$	$(0,0,-1,-1)$	$(3,0,-1,-1)$	$(0,0,2,-1)$

and the F-vectors of the faces of P° are

Dimension	Number of faces	F-vector	
-1	1	$(0,0,0,0,0,0)$	
0	10	$(1,1,0,0,0,0)$	point
1	24	$(1,2,1,0,0,0)$	edge
2	9	$(1,4,4,1,0,0)$	quadrangle
2	16	$(1,3,3,1,0,0)$	triangle
3	6	$(1,6,9,5,1,0)$	prism
3	5	$(1,4,6,4,1,0)$	tetrahedron
4	1	$(1,10,24,25,11,1)$	

The polytopes ∇ and P° have 0 as the unique interior lattice point, and the Fano polytope P° defines the embedding toric Fano variety $Y^{\circ} = X(\Sigma(P^{\circ}))$ of the fibers of the mirror degeneration. The dimension of the automorphism group of Y° is

$$\dim(\mathrm{Aut}(Y^{\circ})) = 4$$

Let

$$S^{\circ} = \mathbb{C}[y_1, y_2, ..., y_{10}]$$

be the Cox ring of Y° with variables

$$\begin{array}{lll} y_1 = y_{(-1,-1,0,2)} = \frac{x_4^2}{x_1 x_2} & y_2 = y_{(-1,-1,0,0)} = \frac{x_0^2}{x_1 x_2} & y_3 = y_{(-1,1,0,0)} = \frac{x_2}{x_1} \\ y_4 = y_{(1,-1,0,0)} = \frac{x_1}{x_2} & y_5 = y_{(-1,-1,2,0)} = \frac{x_3^2}{x_1 x_2} & y_6 = y_{(0,3,-1,-1)} = \frac{x_2^3}{x_0 x_3 x_4} \\ y_7 = y_{(0,0,-1,2)} = \frac{x_4^2}{x_0 x_3} & y_8 = y_{(0,0,-1,-1)} = \frac{x_0^2}{x_3 x_4} & y_9 = y_{(3,0,-1,-1)} = \frac{x_1^3}{x_0 x_3 x_4} \\ y_{10} = y_{(0,0,2,-1)} = \frac{x_3^2}{x_4 x_0} & & \end{array}$$

Bergman subcomplex Intersecting the tropical variety of I with the special fiber Gröbner cone $C_{I_0}(I)$ we obtain the special fiber Bergman subcomplex

$$B(I) = C_{I_0}(I) \cap BF(I) \cap \{w_t = 1\}$$

The following table chooses an indexing of the vertices of ∇ involved in $B(I)$ and gives for each vertex the numbers n_{∇} and n_B of faces of ∇ and faces of $B(I)$ it is contained in

n_{∇}	$n_{B(I)}$			
9	3	$1 = (1,0,1,0)$	$2 = (0,1,1,0)$	$3 = (1,0,0,1)$
		$4 = (0,1,0,1)$	$5 = (0,-1,-1,-1)$	$6 = (-1,0,-1,-1)$

With this indexing the Bergman subcomplex $B(I)$ of $\mathrm{Poset}(\nabla)$ associated to the degeneration $\mathfrak{X}$ is

$$\begin{array}{l} [], \\ [[1],[2],[3],[4],[5],[6]], \\ [[3,5],[3,4],[4,6],[1,5],[1,3],[1,2],[5,6],[2,6],[2,4]], \\ [[3,4,5,6],[1,3,5],[2,4,6],[1,2,5,6],[1,2,3,4]], \end{array}$$

[],
[]

$B(I)$ has the F-vector

Dimension	−1	0	1	2	3	4
Number of faces	0	6	9	5	0	0

and the F-vectors of its faces are

Dimension	Number of faces	F-vector	
0	6	$(1,1,0,0,0,0)$	point
1	9	$(1,2,1,0,0,0)$	edge
2	3	$(1,4,4,1,0,0)$	quadrangle
2	2	$(1,3,3,1,0,0)$	triangle

Dual complex The dual complex $\operatorname{dual}(B(I)) = (B(I))^*$ of deformations associated to $B(I)$ via initial ideals is given by

[],

[],

$[[3,4,5,6]^* = \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_3^2}{x_4\,x_0} \right\rangle, [1,3,5]^* = \left\langle \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{x_2}{x_1} \right\rangle, [2,4,6]^* = \left\langle \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{x_1}{x_2} \right\rangle,$
$[1,2,5,6]^* = \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_4^2}{x_0\,x_3} \right\rangle, [1,2,3,4]^* = \left\langle \frac{x_0^2}{x_1\,x_2}, \frac{x_0^2}{x_3\,x_4} \right\rangle],$

$[[3,5]^* = \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{x_3^2}{x_4\,x_0}, \frac{x_2}{x_1} \right\rangle, [3,4]^* = \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_0^2}{x_1\,x_2}, \frac{x_3^2}{x_4\,x_0}, \frac{x_0^2}{x_3\,x_4} \right\rangle,$
$[4,6]^* = \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{x_3^2}{x_4\,x_0}, \frac{x_1}{x_2} \right\rangle, [1,5]^* = \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{x_4^2}{x_0\,x_3}, \frac{x_2}{x_1} \right\rangle,$
$[1,3]^* = \left\langle \frac{x_0^2}{x_1\,x_2}, \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{x_0^2}{x_3\,x_4}, \frac{x_2}{x_1} \right\rangle, [1,2]^* = \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_0^2}{x_1\,x_2}, \frac{x_4^2}{x_0\,x_3}, \frac{x_0^2}{x_3\,x_4} \right\rangle,$
$[5,6]^* = \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_3^2}{x_1\,x_2}, \frac{x_4^2}{x_0\,x_3}, \frac{x_3^2}{x_4\,x_0} \right\rangle, [2,6]^* = \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{x_4^2}{x_0\,x_3}, \frac{x_1}{x_2} \right\rangle,$
$[2,4]^* = \left\langle \frac{x_0^2}{x_1\,x_2}, \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{x_0^2}{x_3\,x_4}, \frac{x_1}{x_2} \right\rangle],$

$[[1]^* = \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_0^2}{x_1\,x_2}, \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{x_4^2}{x_0\,x_3}, \frac{x_0^2}{x_3\,x_4}, \frac{x_2}{x_1} \right\rangle,$
$[2]^* = \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_0^2}{x_1\,x_2}, \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{x_4^2}{x_0\,x_3}, \frac{x_0^2}{x_3\,x_4}, \frac{x_1}{x_2} \right\rangle,$

$$[3]^* = \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_0^2}{x_1\,x_2}, \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{x_3^2}{x_4\,x_0}, \frac{x_0^2}{x_3\,x_4}, \frac{x_2}{x_1} \right\rangle,$$
$$[4]^* = \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_0^2}{x_1\,x_2}, \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{x_3^2}{x_4\,x_0}, \frac{x_0^2}{x_3\,x_4}, \frac{x_1}{x_2} \right\rangle,$$
$$[5]^* = \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_3^2}{x_1\,x_2}, \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{x_4^2}{x_0\,x_3}, \frac{x_3^2}{x_4\,x_0}, \frac{x_2}{x_1} \right\rangle,$$
$$[6]^* = \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_3^2}{x_1\,x_2}, \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{x_4^2}{x_0\,x_3}, \frac{x_3^2}{x_4\,x_0}, \frac{x_1}{x_2} \right\rangle],$$
$$[]$$

when writing the vertices of the faces as deformations of X_0. Note that the T-invariant basis of deformations associated to a face is given by all lattice points of the corresponding polytope in $M_{\mathbb{R}}$. When numbering the vertices of the faces of $\operatorname{dual}(B(I))$ by the Cox variables of the mirror toric Fano variety Y° the complex $\operatorname{dual}(B(I))$ is

$$[],$$
$$[],$$
$$[[3,4,5,6]^* = \langle y_5, y_{10}\rangle, [1,3,5]^* = \langle y_6, y_3\rangle, [2,4,6]^* = \langle y_9, y_4\rangle, [1,2,5,6]^* = \langle y_1, y_7\rangle,$$
$$[1,2,3,4]^* = \langle y_2, y_8\rangle],$$
$$[[3,5]^* = \langle y_5, y_6, y_{10}, y_3\rangle, [3,4]^* = \langle y_5, y_2, y_{10}, y_8\rangle, [4,6]^* = \langle y_5, y_9, y_{10}, y_4\rangle,$$
$$[1,5]^* = \langle y_1, y_6, y_7, y_3\rangle, [1,3]^* = \langle y_2, y_6, y_8, y_3\rangle, [1,2]^* = \langle y_1, y_2, y_7, y_8\rangle,$$
$$[5,6]^* = \langle y_1, y_5, y_7, y_{10}\rangle, [2,6]^* = \langle y_1, y_9, y_7, y_4\rangle, [2,4]^* = \langle y_2, y_9, y_8, y_4\rangle],$$
$$[[1]^* = \langle y_1, y_2, y_6, y_7, y_8, y_3\rangle, [2]^* = \langle y_1, y_2, y_9, y_7, y_8, y_4\rangle, [3]^* = \langle y_5, y_2, y_6, y_{10}, y_8, y_3\rangle,$$
$$[4]^* = \langle y_5, y_2, y_9, y_{10}, y_8, y_4\rangle, [5]^* = \langle y_1, y_5, y_6, y_7, y_{10}, y_3\rangle, [6]^* = \langle y_1, y_5, y_9, y_7, y_{10}, y_4\rangle],$$
$$[]$$

The dual complex has the F-vector

Dimension	-1	0	1	2	3	4
Number of faces	0	0	5	9	6	0

and the F-vectors of the faces of $\operatorname{dual}(B(I))$ are

Dimension	Number of faces	F-vector	
1	5	$(1,2,1,0,0,0)$	edge
2	9	$(1,4,4,1,0,0)$	quadrangle
3	6	$(1,6,9,5,1,0)$	prism

Recall that in this example the toric variety Y is projective space. The number of lattice points of the support of $\operatorname{dual}(B(I))$ relates to the dimension $h^{1,1}(X)$ of the complex moduli space of the generic fiber X of $\mathfrak{X}$ and to the dimension $h^{1,1}\left(\bar{X}^\circ\right)$ of the Kähler moduli space of the $MPCP$-blowup $\bar{X}^\circ$ of the generic fiber X° of the mirror degeneration

$$\begin{aligned} |\operatorname{supp}(\operatorname{dual}(B(I))) \cap M| &= 42 = 24 + 18 = \dim(\operatorname{Aut}(Y)) + h^{1,1}(X) \\ &= 20 + 4 + 18 \\ &= |\operatorname{Roots}(Y)| + \dim(T_Y) + h^{1,1}\left(\bar{X}^\circ\right) \end{aligned}$$

There are

$$h^{1,1}(X) + \dim(T_{Y^\circ}) = 18 + 4$$

non-trivial toric polynomial deformations of X_0

$$\begin{array}{cccccccccc} \frac{x_3^2}{x_4\,x_0} & \frac{x_3^2}{x_1\,x_2} & \frac{x_2^3}{x_0\,x_3\,x_4} & \frac{x_1^3}{x_0\,x_3\,x_4} & \frac{x_4^2}{x_0\,x_3} & \frac{x_4^2}{x_1\,x_2} & \frac{x_0^2}{x_3\,x_4} & \frac{x_0^2}{x_1\,x_2} & \frac{x_2^2}{x_4\,x_0} & \frac{x_2\,x_3}{x_4\,x_0} \\ \frac{x_1^2}{x_4\,x_0} & \frac{x_1\,x_3}{x_4\,x_0} & \frac{x_2^2}{x_0\,x_3} & \frac{x_2\,x_4}{x_0\,x_3} & \frac{x_2\,x_0}{x_3\,x_4} & \frac{x_2^2}{x_3\,x_4} & \frac{x_4\,x_0}{x_1\,x_2} & \frac{x_3\,x_4}{x_1\,x_2} & \frac{x_1^2}{x_0\,x_3} & \frac{x_1\,x_4}{x_0\,x_3} \\ \frac{x_1\,x_0}{x_3\,x_4} & \frac{x_1^2}{x_3\,x_4} & & & & & & & & \end{array}$$

They correspond to the toric divisors

$$\begin{array}{lllll} D_{(0,0,2,-1)} & D_{(-1,-1,2,0)} & D_{(0,3,-1,-1)} & D_{(3,0,-1,-1)} & D_{(0,0,-1,2)} \\ D_{(-1,-1,0,2)} & D_{(0,0,-1,-1)} & D_{(-1,-1,0,0)} & D_{(0,2,0,-1)} & D_{(0,1,1,-1)} \\ D_{(2,0,0,-1)} & D_{(1,0,1,-1)} & D_{(0,2,-1,0)} & D_{(0,1,-1,1)} & D_{(0,1,-1,-1)} \\ D_{(0,2,-1,-1)} & D_{(-1,-1,0,1)} & D_{(-1,-1,1,1)} & D_{(2,0,-1,0)} & D_{(1,0,-1,1)} \\ D_{(1,0,-1,-1)} & D_{(2,0,-1,-1)} & & & \end{array}$$

on a MPCP-blowup of Y° inducing 18 non-zero toric divisor classes on the mirror X°. The following 20 toric divisors of Y induce the trivial divisor class on X°

$$\begin{array}{lllll} D_{(-1,1,0,0)} & D_{(1,-1,0,0)} & D_{(-1,0,1,0)} & D_{(0,0,0,-1)} & D_{(0,0,1,-1)} \\ D_{(0,-1,1,0)} & D_{(-1,0,0,1)} & D_{(-1,0,0,0)} & D_{(0,0,-1,0)} & D_{(0,0,-1,1)} \\ D_{(0,0,1,0)} & D_{(0,0,0,1)} & D_{(0,-1,0,1)} & D_{(0,-1,0,0)} & D_{(0,1,-1,0)} \\ D_{(1,0,-1,0)} & D_{(0,1,0,-1)} & D_{(1,0,0,-1)} & D_{(0,1,0,0)} & D_{(1,0,0,0)} \end{array}$$

Mirror special fiber The ideal I_0° of the monomial special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$ is generated by the following set of monomials in S°

$$\left\{\begin{array}{lllll} y_6\,y_7\,y_2\,y_5\,y_9 & y_7\,y_8\,y_3\,y_{10}\,y_9 & y_8\,y_1\,y_3\,y_5\,y_9 & y_6\,y_1\,y_2\,y_4\,y_{10} & y_6\,y_8\,y_1\,y_4\,y_{10} \\ y_6\,y_8\,y_1\,y_{10}\,y_9 & y_6\,y_1\,y_2\,y_{10}\,y_9 & y_7\,y_2\,y_3\,y_{10}\,y_9 & y_6\,y_7\,y_2\,y_4\,y_{10} & y_6\,y_7\,y_8\,y_5\,y_9 \\ y_6\,y_8\,y_1\,y_4\,y_5 & y_6\,y_7\,y_8\,y_{10}\,y_9 & y_8\,y_1\,y_3\,y_4\,y_5 & y_6\,y_8\,y_1\,y_5\,y_9 & y_7\,y_8\,y_3\,y_4\,y_5 \\ y_1\,y_2\,y_3\,y_4\,y_5 & y_6\,y_7\,y_8\,y_4\,y_5 & y_6\,y_1\,y_2\,y_4\,y_5 & y_8\,y_1\,y_3\,y_{10}\,y_9 & y_6\,y_7\,y_2\,y_{10}\,y_9 \\ y_6\,y_7\,y_8\,y_4\,y_{10} & y_7\,y_2\,y_3\,y_5\,y_9 & y_8\,y_1\,y_3\,y_4\,y_{10} & y_1\,y_2\,y_3\,y_5\,y_9 & y_7\,y_2\,y_3\,y_4\,y_{10} \\ y_1\,y_2\,y_3\,y_4\,y_{10} & y_7\,y_8\,y_3\,y_5\,y_9 & y_7\,y_8\,y_3\,y_4\,y_{10} & y_7\,y_2\,y_3\,y_4\,y_5 & y_1\,y_2\,y_3\,y_{10}\,y_9 \\ y_6\,y_7\,y_2\,y_4\,y_5 & y_6\,y_1\,y_2\,y_5\,y_9 & & & \end{array}\right\}$$

The $\operatorname{Pic}(Y^\circ)$-generated ideal

$$J_0^\circ = \left\langle\ y_1\,y_2\,y_3\,y_4\,y_5 \quad y_6\,y_7\,y_8\,y_{10}\,y_9\ \right\rangle$$

defines the same subvariety X_0° of the toric variety Y°, and $J_0^{\circ\Sigma} = I_0^\circ$. Passing from J_0° to $J_0^{\circ\Sigma}$ is the non-simplicial toric analogue of saturation. The complex $B(I)^*$ labeled by the variables of the Cox ring S° of Y°, as written in the last section, is the complex $SP(I_0^\circ)$ of prime ideals of the toric strata of the special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$, i.e., the primary decomposition of I_0° is

$$I_0^\circ = \langle y_9, y_4\rangle \cap \langle y_5, y_{10}\rangle \cap \langle y_1, y_7\rangle \cap \langle y_2, y_8\rangle \cap \langle y_6, y_3\rangle$$

Each facet $F \in B(I)$ corresponds to one of these ideals and this ideal is generated by the facets of ∇ containing F.

Covering structure in the deformation complex of the degeneration $\mathfrak{X}$ According to the local reduced Gröbner basis each face of the complex of deformations $\operatorname{dual}(B(I))$ decomposes into 2 polytopes forming a $2:1$ trivial covering of $B(I)$

$[]$,

$[]$,

$[[\langle y_5\rangle, \langle y_{10}\rangle] \mapsto \langle y_5, y_{10}\rangle^{*\vee} = [3,4,5,6]^\vee, [\langle y_3\rangle, \langle y_6\rangle] \mapsto \langle y_6, y_3\rangle^{*\vee} = [1,3,5]^\vee,$
$[\langle y_4\rangle, \langle y_9\rangle] \mapsto \langle y_9, y_4\rangle^{*\vee} = [2,4,6]^\vee, [\langle y_1\rangle, \langle y_7\rangle] \mapsto \langle y_1, y_7\rangle^{*\vee} = [1,2,5,6]^\vee,$
$[\langle y_2\rangle, \langle y_8\rangle] \mapsto \langle y_2, y_8\rangle^{*\vee} = [1,2,3,4]^\vee],$

$[[\langle y_5, y_3\rangle, \langle y_6, y_{10}\rangle] \mapsto \langle y_5, y_6, y_{10}, y_3\rangle^{*\vee} = [3,5]^\vee,$
$[\langle y_5, y_2\rangle, \langle y_{10}, y_8\rangle] \mapsto \langle y_5, y_2, y_{10}, y_8\rangle^{*\vee} = [3,4]^\vee,$
$[\langle y_5, y_4\rangle, \langle y_9, y_{10}\rangle] \mapsto \langle y_5, y_9, y_{10}, y_4\rangle^{*\vee} = [4,6]^\vee,$

$[\langle y_1,y_3\rangle,\langle y_6,y_7\rangle]\mapsto\langle y_1,y_6,y_7,y_3\rangle^{*\vee}=[1,5]^{\vee},$
$[\langle y_2,y_3\rangle,\langle y_6,y_8\rangle]\mapsto\langle y_2,y_6,y_8,y_3\rangle^{*\vee}=[1,3]^{\vee},$
$[\langle y_1,y_2\rangle,\langle y_7,y_8\rangle]\mapsto\langle y_1,y_2,y_7,y_8\rangle^{*\vee}=[1,2]^{\vee},$
$[\langle y_1,y_5\rangle,\langle y_7,y_{10}\rangle]\mapsto\langle y_1,y_5,y_7,y_{10}\rangle^{*\vee}=[5,6]^{\vee},$
$[\langle y_1,y_4\rangle,\langle y_9,y_7\rangle]\mapsto\langle y_1,y_9,y_7,y_4\rangle^{*\vee}=[2,6]^{\vee},$
$[\langle y_2,y_4\rangle,\langle y_9,y_8\rangle]\mapsto\langle y_2,y_9,y_8,y_4\rangle^{*\vee}=[2,4]^{\vee}],$

$[[\langle y_1,y_2,y_3\rangle,\langle y_6,y_7,y_8\rangle]\mapsto\langle y_1,y_2,y_6,y_7,y_8,y_3\rangle^{*\vee}=[1]^{\vee},$
$[\langle y_1,y_2,y_4\rangle,\langle y_9,y_7,y_8\rangle]\mapsto\langle y_1,y_2,y_9,y_7,y_8,y_4\rangle^{*\vee}=[2]^{\vee},$
$[\langle y_5,y_2,y_3\rangle,\langle y_6,y_{10},y_8\rangle]\mapsto\langle y_5,y_2,y_6,y_{10},y_8,y_3\rangle^{*\vee}=[3]^{\vee},$
$[\langle y_5,y_2,y_4\rangle,\langle y_9,y_{10},y_8\rangle]\mapsto\langle y_5,y_2,y_9,y_{10},y_8,y_4\rangle^{*\vee}=[4]^{\vee},$
$[\langle y_1,y_5,y_3\rangle,\langle y_6,y_7,y_{10}\rangle]\mapsto\langle y_1,y_5,y_6,y_7,y_{10},y_3\rangle^{*\vee}=[5]^{\vee},$
$[\langle y_1,y_5,y_4\rangle,\langle y_9,y_7,y_{10}\rangle]\mapsto\langle y_1,y_5,y_9,y_7,y_{10},y_4\rangle^{*\vee}=[6]^{\vee}],$

[]

Here the faces $F\in B(I)$ are specified both via the vertices of F^* labeled by the variables of S° and by the numbering of the vertices of $B(I)$ chosen above. This covering has 2 sheets forming the complexes

[],

[],

$[\langle y_5\rangle,\langle y_3\rangle,\langle y_4\rangle,\langle y_1\rangle,\langle y_2\rangle],$

$[\langle y_5,y_3\rangle,\langle y_5,y_2\rangle,\langle y_5,y_4\rangle,\langle y_1,y_3\rangle,\langle y_2,y_3\rangle,\langle y_1,y_2\rangle,\langle y_1,y_5\rangle,\langle y_1,y_4\rangle,\langle y_2,y_4\rangle],$

$[\langle y_1,y_2,y_3\rangle,\langle y_1,y_2,y_4\rangle,\langle y_5,y_2,y_3\rangle,\langle y_5,y_2,y_4\rangle,\langle y_1,y_5,y_3\rangle,\langle y_1,y_5,y_4\rangle],$

[]

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[],

$[\langle y_{10}\rangle,\langle y_6\rangle,\langle y_9\rangle,\langle y_7\rangle,\langle y_8\rangle],$

$[\langle y_6,y_{10}\rangle,\langle y_{10},y_8\rangle,\langle y_9,y_{10}\rangle,\langle y_6,y_7\rangle,\langle y_6,y_8\rangle,\langle y_7,y_8\rangle,\langle y_7,y_{10}\rangle,\langle y_9,y_7\rangle,\langle y_9,y_8\rangle],$

$[\langle y_6,y_7,y_8\rangle,\langle y_9,y_7,y_8\rangle,\langle y_6,y_{10},y_8\rangle,\langle y_9,y_{10},y_8\rangle,\langle y_6,y_7,y_{10}\rangle,\langle y_9,y_7,y_{10}\rangle],$

[]

with F-vector

Dimension	Number of faces	F-vector	
0	5	$(1,1,0,0,0,0)$	point
1	9	$(1,2,1,0,0,0)$	edge
2	6	$(1,3,3,1,0,0)$	triangle

Writing the vertices of the faces as deformations the covering is given by

$$
\begin{aligned}
&[],\\
&[],\\
&[[\left\langle \frac{x_3^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_3^2}{x_4\,x_0} \right\rangle] \mapsto [3,4,5,6]^\vee, [\left\langle \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_3\,x_4} \right\rangle] \mapsto [1,3,5]^\vee,\\
&[\left\langle \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_1^3}{x_0\,x_3\,x_4} \right\rangle] \mapsto [2,4,6]^\vee, [\left\langle \frac{x_4^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_4^2}{x_0\,x_3} \right\rangle] \mapsto [1,2,5,6]^\vee,\\
&[\left\langle \frac{x_0^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_0^2}{x_3\,x_4} \right\rangle] \mapsto [1,2,3,4]^\vee],\\
&[[\left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{{x_3}^2}{x_4\,x_0} \right\rangle] \mapsto [3,5]^\vee, [\left\langle \frac{x_3^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_3^2}{x_4\,x_0}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle] \mapsto [3,4]^\vee,\\
&[\left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{{x_3}^2}{x_4\,x_0} \right\rangle] \mapsto [4,6]^\vee, [\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{{x_4}^2}{x_0\,x_3} \right\rangle] \mapsto [1,5]^\vee,\\
&[\left\langle \frac{x_0^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle] \mapsto [1,3]^\vee, [\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_4^2}{x_0\,x_3}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle] \mapsto [1,2]^\vee,\\
&[\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_3}^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_4^2}{x_0\,x_3}, \frac{{x_3}^2}{x_4\,x_0} \right\rangle] \mapsto [5,6]^\vee,\\
&[\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{{x_4}^2}{x_0\,x_3} \right\rangle] \mapsto [2,6]^\vee, [\left\langle \frac{x_0^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle] \mapsto [2,4]^\vee],\\
&[[\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{{x_4}^2}{x_0\,x_3}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle] \mapsto [1]^\vee,\\
&[\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{{x_4}^2}{x_0\,x_3}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle] \mapsto [2]^\vee,\\
&[\left\langle \frac{x_3^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{{x_3}^2}{x_4\,x_0}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle] \mapsto [3]^\vee,\\
&[\left\langle \frac{x_3^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{{x_3}^2}{x_4\,x_0}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle] \mapsto [4]^\vee,\\
&[\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_3}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{{x_4}^2}{x_0\,x_3}, \frac{{x_3}^2}{x_4\,x_0} \right\rangle] \mapsto [5]^\vee,\\
&[\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_3}^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{{x_4}^2}{x_0\,x_3}, \frac{{x_3}^2}{x_4\,x_0} \right\rangle] \mapsto [6]^\vee],\\
&[]
\end{aligned}
$$

with the 2 sheets forming the complexes

$[]$,

$[]$,

$$[\left\langle \frac{x_3^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_0^2}{x_1\,x_2} \right\rangle],$$

$$[\left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_0^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2} \right\rangle,$$
$$\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_3}^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_0^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle],$$

$$[\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle,$$
$$\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_3}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_3}^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle],$$

$[]$

$[]$,

$[]$,

$$[\left\langle \frac{x_3^2}{x_4\,x_0} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_3\,x_4} \right\rangle, \left\langle \frac{x_1^3}{x_0\,x_3\,x_4} \right\rangle, \left\langle \frac{x_4^2}{x_0\,x_3} \right\rangle, \left\langle \frac{x_0^2}{x_3\,x_4} \right\rangle],$$

$$[\left\langle \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{{x_3}^2}{x_4\,x_0} \right\rangle, \left\langle \frac{x_3^2}{x_4\,x_0}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle, \left\langle \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{{x_3}^2}{x_4\,x_0} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{{x_4}^2}{x_0\,x_3} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle,$$
$$\left\langle \frac{x_4^2}{x_0\,x_3}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle, \left\langle \frac{x_4^2}{x_0\,x_3}, \frac{{x_3}^2}{x_4\,x_0} \right\rangle, \left\langle \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{{x_4}^2}{x_0\,x_3} \right\rangle, \left\langle \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle],$$

$$[\left\langle \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{{x_4}^2}{x_0\,x_3}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle, \left\langle \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{{x_4}^2}{x_0\,x_3}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{{x_3}^2}{x_4\,x_0}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle,$$
$$\left\langle \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{{x_3}^2}{x_4\,x_0}, \frac{{x_0}^2}{x_3\,x_4} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_3\,x_4}, \frac{{x_4}^2}{x_0\,x_3}, \frac{{x_3}^2}{x_4\,x_0} \right\rangle, \left\langle \frac{x_1^3}{x_0\,x_3\,x_4}, \frac{{x_4}^2}{x_0\,x_3}, \frac{{x_3}^2}{x_4\,x_0} \right\rangle],$$

$[]$

Note that the torus invariant basis of deformations corresponding to a Bergman face is given by the set of all lattice points of the polytope specified above.

Limit map The limit map $\lim : B(I) \to \mathrm{Poset}(\Delta)$ associates to a face F of $B(I)$ the face of Δ formed by the limit points of arcs lying over the weight vectors $w \in F$, i.e., with lowest order term t^w. Labeling the faces of the Bergman complex $B(I) \subset \mathrm{Poset}(\nabla)$ and the faces of $\mathrm{Poset}(\Delta)$ by the corresponding dual faces of ∇^* and Δ^*, hence considering the limit map $\lim : B(I) \to \mathrm{Poset}(\Delta)$ as a map $B(I)^* \to \mathrm{Poset}(\Delta^*)$, the limit correspondence is given by

$[]$,

$$[\langle y_1, y_2, y_6, y_7, y_8, y_3\rangle \mapsto \langle x_1, x_3\rangle, \langle y_1, y_2, y_9, y_7, y_8, y_4\rangle \mapsto \langle x_2, x_3\rangle,$$
$$\langle y_5, y_2, y_6, y_{10}, y_8, y_3\rangle \mapsto \langle x_1, x_4\rangle, \langle y_5, y_2, y_9, y_{10}, y_8, y_4\rangle \mapsto \langle x_2, x_4\rangle,$$
$$\langle y_1, y_5, y_6, y_7, y_{10}, y_3\rangle \mapsto \langle x_1, x_0\rangle, \langle y_1, y_5, y_9, y_7, y_{10}, y_4\rangle \mapsto \langle x_2, x_0\rangle],$$

$$[\langle y_5, y_6, y_{10}, y_3\rangle \mapsto \langle x_1, x_4, x_0\rangle, \langle y_5, y_2, y_{10}, y_8\rangle \mapsto \langle x_1, x_2, x_4\rangle,$$
$$\langle y_5, y_9, y_{10}, y_4\rangle \mapsto \langle x_2, x_4, x_0\rangle, \langle y_1, y_6, y_7, y_3\rangle \mapsto \langle x_1, x_3, x_0\rangle,$$
$$\langle y_2, y_6, y_8, y_3\rangle \mapsto \langle x_1, x_3, x_4\rangle, \langle y_1, y_2, y_7, y_8\rangle \mapsto \langle x_1, x_2, x_3\rangle,$$
$$\langle y_1, y_5, y_7, y_{10}\rangle \mapsto \langle x_1, x_2, x_0\rangle, \langle y_1, y_9, y_7, y_4\rangle \mapsto \langle x_2, x_3, x_0\rangle,$$
$$\langle y_2, y_9, y_8, y_4\rangle \mapsto \langle x_2, x_3, x_4\rangle],$$

$$[\langle y_5, y_{10}\rangle \mapsto \langle x_1, x_2, x_4, x_0\rangle, \langle y_6, y_3\rangle \mapsto \langle x_1, x_3, x_4, x_0\rangle,$$
$$\langle y_9, y_4\rangle \mapsto \langle x_2, x_3, x_4, x_0\rangle, \langle y_1, y_7\rangle \mapsto \langle x_1, x_2, x_3, x_0\rangle,$$
$$\langle y_2, y_8\rangle \mapsto \langle x_1, x_2, x_3, x_4\rangle]$$

The image of the limit map coincides with the image of μ and with the Bergman complex of the mirror, i.e., $\lim(B(I)) = \mu(B(I)) = B(I^\circ)$.

Mirror complex Numbering the vertices of the mirror complex $\mu(B(I))$ as

$$\begin{array}{ll} 1 = (4, -1, -1, -1) & 2 = (-1, 4, -1, -1) \\ 3 = (-1, -1, 4, -1) & 4 = (-1, -1, -1, 4) \\ 5 = (-1, -1, -1, -1) & \end{array}$$

$\mu(B(I)) \subset \mathrm{Poset}(\Delta)$ is

[],

[[3], [2], [1], [4], [5]],

[[2, 3], [3, 5], [1, 3], [2, 4], [2, 5], [4, 5], [3, 4], [1, 4], [1, 5]],

[[2, 4, 5], [1, 4, 5], [2, 3, 5], [1, 3, 5], [2, 3, 4], [1, 3, 4]],

[],

[]

The ordering of the faces of $\mu(B(I))$ is compatible with above ordering of the faces of $B(I)$. The F-vectors of the faces of $\mu(B(I))^*$ are

Dimension	Number of faces	F-vector	
1	6	$(1,2,1,0,0,0)$	edge
2	9	$(1,3,3,1,0,0)$	triangle
3	5	$(1,4,6,4,1,0)$	tetrahedron

The first order deformations of the mirror special fiber X_0° correspond to the lattice points of the dual complex $(\mu(B(I)))^* \subset \mathrm{Poset}(\Delta^*)$ of the mirror complex. We label the first order deformations of X_0° corresponding to vertices of Δ^* by the homogeneous coordinates of Y

$$x_1 = x_{(1,0,0,0)} = \frac{y_4\, y_9{}^3}{y_1\, y_2\, y_3\, y_5} \qquad x_2 = x_{(0,1,0,0)} = \frac{y_3\, y_6{}^3}{y_1\, y_2\, y_4\, y_5}$$
$$x_3 = x_{(0,0,1,0)} = \frac{y_5^2 y_{10}{}^2}{y_6\, y_7\, y_8\, y_9} \qquad x_4 = x_{(0,0,0,1)} = \frac{y_1^2 y_7{}^2}{y_6\, y_8\, y_9\, y_{10}}$$
$$x_0 = x_{(-1,-1,-1,-1)} = \frac{y_2^2 y_8{}^2}{y_6\, y_7\, y_9\, y_{10}}$$

So writing the vertices of the faces of $(\mu(B(I)))^*$ as homogeneous coordinates of Y, the complex $(\mu(B(I)))^*$ is given by

$[]$,

$[]$,

$[\langle x_1, x_3\rangle, \langle x_2, x_3\rangle, \langle x_1, x_4\rangle, \langle x_2, x_4\rangle, \langle x_1, x_0\rangle, \langle x_2, x_0\rangle]$,

$[\langle x_1, x_4, x_0\rangle, \langle x_1, x_2, x_4\rangle, \langle x_2, x_4, x_0\rangle, \langle x_1, x_3, x_0\rangle, \langle x_1, x_3, x_4\rangle, \langle x_1, x_2, x_3\rangle,$
$\langle x_1, x_2, x_0\rangle, \langle x_2, x_3, x_0\rangle, \langle x_2, x_3, x_4\rangle]$,

$[\langle x_1, x_2, x_4, x_0\rangle, \langle x_1, x_3, x_4, x_0\rangle, \langle x_2, x_3, x_4, x_0\rangle, \langle x_1, x_2, x_3, x_0\rangle, \langle x_1, x_2, x_3, x_4\rangle]$,

$[]$

The complex $\mu(B(I))^*$ labeled by the variables of the Cox ring S gives the ideals of the toric strata of the special fiber X_0 of $\mathfrak{X}$, i.e., the complex $SP(I_0)$, so in particular the primary decomposition of I_0 is

$$I_0 = \langle x_2, x_0\rangle \cap \langle x_1, x_0\rangle \cap \langle x_1, x_3\rangle \cap \langle x_2, x_3\rangle \cap \langle x_1, x_4\rangle \cap \langle x_2, x_4\rangle$$

Labeling the vertices of the faces by the corresponding deformations the complex is given by

$[], [],$

$$[\left\langle \frac{y_4\, y_9{}^3}{y_1\, y_2\, y_3\, y_5}, \frac{y_5{}^2 y_{10}^2}{y_6\, y_7\, y_8\, y_9} \right\rangle, \left\langle \frac{y_3\, y_6{}^3}{y_1\, y_2\, y_4\, y_5}, \frac{y_5{}^2 y_{10}^2}{y_6\, y_7\, y_8\, y_9} \right\rangle, \left\langle \frac{y_4\, y_9{}^3}{y_1\, y_2\, y_3\, y_5}, \frac{y_1{}^2 y_7^2}{y_6\, y_8\, y_9\, y_{10}} \right\rangle,$$
$$\left\langle \frac{y_3\, y_6{}^3}{y_1\, y_2\, y_4\, y_5}, \frac{y_1{}^2 y_7^2}{y_6\, y_8\, y_9\, y_{10}} \right\rangle, \left\langle \frac{y_2^2 y_8{}^2}{y_6\, y_7\, y_9\, y_{10}}, \frac{y_4\, y_9{}^3}{y_1\, y_2\, y_3\, y_5} \right\rangle, \left\langle \frac{y_2^2 y_8{}^2}{y_6\, y_7\, y_9\, y_{10}}, \frac{y_3\, y_6{}^3}{y_1\, y_2\, y_4\, y_5} \right\rangle],$$

$$[\left\langle \frac{y_2^2 y_8{}^2}{y_6\, y_7\, y_9\, y_{10}}, \frac{y_4\, y_9{}^3}{y_1\, y_2\, y_3\, y_5}, \frac{y_1{}^2 y_7^2}{y_6\, y_8\, y_9\, y_{10}} \right\rangle, \left\langle \frac{y_4\, y_9{}^3}{y_1\, y_2\, y_3\, y_5}, \frac{y_3\, y_6{}^3}{y_1\, y_2\, y_4\, y_5}, \frac{y_1{}^2 y_7^2}{y_6\, y_8\, y_9\, y_{10}} \right\rangle,$$
$$\left\langle \frac{y_2^2 y_8{}^2}{y_6\, y_7\, y_9\, y_{10}}, \frac{y_3\, y_6{}^3}{y_1\, y_2\, y_4\, y_5}, \frac{y_1{}^2 y_7^2}{y_6\, y_8\, y_9\, y_{10}} \right\rangle, \left\langle \frac{y_2^2 y_8{}^2}{y_6\, y_7\, y_9\, y_{10}}, \frac{y_4\, y_9{}^3}{y_1\, y_2\, y_3\, y_5}, \frac{y_5{}^2 y_{10}^2}{y_6\, y_7\, y_8\, y_9} \right\rangle,$$
$$\left\langle \frac{y_4\, y_9{}^3}{y_1\, y_2\, y_3\, y_5}, \frac{y_5{}^2 y_{10}^2}{y_6\, y_7\, y_8\, y_9}, \frac{y_1{}^2 y_7^2}{y_6\, y_8\, y_9\, y_{10}} \right\rangle, \left\langle \frac{y_4\, y_9{}^3}{y_1\, y_2\, y_3\, y_5}, \frac{y_3\, y_6{}^3}{y_1\, y_2\, y_4\, y_5}, \frac{y_5{}^2 y_{10}^2}{y_6\, y_7\, y_8\, y_9} \right\rangle,$$
$$\left\langle \frac{y_2^2 y_8{}^2}{y_6\, y_7\, y_9\, y_{10}}, \frac{y_4\, y_9{}^3}{y_1\, y_2\, y_3\, y_5}, \frac{y_3\, y_6{}^3}{y_1\, y_2\, y_4\, y_5} \right\rangle, \left\langle \frac{y_2^2 y_8{}^2}{y_6\, y_7\, y_9\, y_{10}}, \frac{y_3\, y_6{}^3}{y_1\, y_2\, y_4\, y_5}, \frac{y_5{}^2 y_{10}^2}{y_6\, y_7\, y_8\, y_9} \right\rangle,$$
$$\left\langle \frac{y_3\, y_6{}^3}{y_1\, y_2\, y_4\, y_5}, \frac{y_5{}^2 y_{10}^2}{y_6\, y_7\, y_8\, y_9}, \frac{y_1{}^2 y_7^2}{y_6\, y_8\, y_9\, y_{10}} \right\rangle],$$

$$[\left\langle \frac{y_2^2 y_8{}^2}{y_6\, y_7\, y_9\, y_{10}}, \frac{y_4\, y_9{}^3}{y_1\, y_2\, y_3\, y_5}, \frac{y_3\, y_6{}^3}{y_1\, y_2\, y_4\, y_5}, \frac{y_1{}^2 y_7^2}{y_6\, y_8\, y_9\, y_{10}} \right\rangle,$$
$$\left\langle \frac{y_2^2 y_8{}^2}{y_6\, y_7\, y_9\, y_{10}}, \frac{y_4\, y_9{}^3}{y_1\, y_2\, y_3\, y_5}, \frac{y_5{}^2 y_{10}^2}{y_6\, y_7\, y_8\, y_9}, \frac{y_1{}^2 y_7^2}{y_6\, y_8\, y_9\, y_{10}} \right\rangle,$$
$$\left\langle \frac{y_2^2 y_8{}^2}{y_6\, y_7\, y_9\, y_{10}}, \frac{y_3\, y_6{}^3}{y_1\, y_2\, y_4\, y_5}, \frac{y_5{}^2 y_{10}^2}{y_6\, y_7\, y_8\, y_9}, \frac{y_1{}^2 y_7^2}{y_6\, y_8\, y_9\, y_{10}} \right\rangle,$$
$$\left\langle \frac{y_2^2 y_8{}^2}{y_6\, y_7\, y_9\, y_{10}}, \frac{y_4\, y_9{}^3}{y_1\, y_2\, y_3\, y_5}, \frac{y_3\, y_6{}^3}{y_1\, y_2\, y_4\, y_5}, \frac{y_5{}^2 y_{10}^2}{y_6\, y_7\, y_8\, y_9} \right\rangle,$$
$$\left\langle \frac{y_4\, y_9{}^3}{y_1\, y_2\, y_3\, y_5}, \frac{y_3\, y_6{}^3}{y_1\, y_2\, y_4\, y_5}, \frac{y_5{}^2 y_{10}^2}{y_6\, y_7\, y_8\, y_9}, \frac{y_1{}^2 y_7^2}{y_6\, y_8\, y_9\, y_{10}} \right\rangle],$$

$[]$

Covering structure in the deformation complex of the mirror degeneration Each face of the complex of deformations $(\mu(B(I)))^*$ of the mirror special fiber X_0° decomposes as the convex hull of 2 polytopes forming a 2 : 1 trivial covering of $\mu(B(I))^\vee$

$[],$

$[],$

$$[[\langle x_1\rangle, \langle x_3\rangle] \mapsto \langle x_1, x_3\rangle^{*\vee} = [2,4,5]^\vee, [\langle x_2\rangle, \langle x_3\rangle] \mapsto \langle x_2, x_3\rangle^{*\vee} = [1,4,5]^\vee,$$
$$[\langle x_1\rangle, \langle x_4\rangle] \mapsto \langle x_1, x_4\rangle^{*\vee} = [2,3,5]^\vee, [\langle x_2\rangle, \langle x_4\rangle] \mapsto \langle x_2, x_4\rangle^{*\vee} = [1,3,5]^\vee,$$
$$[\langle x_1\rangle, \langle x_0\rangle] \mapsto \langle x_1, x_0\rangle^{*\vee} = [2,3,4]^\vee, [\langle x_2\rangle, \langle x_0\rangle] \mapsto \langle x_2, x_0\rangle^{*\vee} = [1,3,4]^\vee],$$

$$[[\langle x_1\rangle, \langle x_4, x_0\rangle] \mapsto \langle x_1, x_4, x_0\rangle^{*\vee} = [2,3]^\vee, [\langle x_1, x_2\rangle, \langle x_4\rangle] \mapsto \langle x_1, x_2, x_4\rangle^{*\vee} = [3,5]^\vee,$$
$$[\langle x_2\rangle, \langle x_4, x_0\rangle] \mapsto \langle x_2, x_4, x_0\rangle^{*\vee} = [1,3]^\vee, [\langle x_1\rangle, \langle x_3, x_0\rangle] \mapsto \langle x_1, x_3, x_0\rangle^{*\vee} = [2,4]^\vee,$$
$$[\langle x_1\rangle, \langle x_3, x_4\rangle] \mapsto \langle x_1, x_3, x_4\rangle^{*\vee} = [2,5]^\vee, [\langle x_1, x_2\rangle, \langle x_3\rangle] \mapsto \langle x_1, x_2, x_3\rangle^{*\vee} = [4,5]^\vee,$$
$$[\langle x_1, x_2\rangle, \langle x_0\rangle] \mapsto \langle x_1, x_2, x_0\rangle^{*\vee} = [3,4]^\vee, [\langle x_2\rangle, \langle x_3, x_0\rangle] \mapsto \langle x_2, x_3, x_0\rangle^{*\vee} = [1,4]^\vee,$$
$$[\langle x_2\rangle, \langle x_3, x_4\rangle] \mapsto \langle x_2, x_3, x_4\rangle^{*\vee} = [1,5]^\vee],$$

$$
\begin{array}{l}
[[\langle x_1, x_2\rangle, \langle x_4, x_0\rangle] \mapsto \langle x_1, x_2, x_4, x_0\rangle^{*\vee} = [3]^\vee, \\
[\langle x_1\rangle, \langle x_3, x_4, x_0\rangle] \mapsto \langle x_1, x_3, x_4, x_0\rangle^{*\vee} = [2]^\vee, \\
[\langle x_2\rangle, \langle x_3, x_4, x_0\rangle] \mapsto \langle x_2, x_3, x_4, x_0\rangle^{*\vee} = [1]^\vee, \\
[\langle x_1, x_2\rangle, \langle x_3, x_0\rangle] \mapsto \langle x_1, x_2, x_3, x_0\rangle^{*\vee} = [4]^\vee, \\
[\langle x_1, x_2\rangle, \langle x_3, x_4\rangle] \mapsto \langle x_1, x_2, x_3, x_4\rangle^{*\vee} = [5]^\vee],
\end{array}
$$

$$
[]
$$

Due to the singularities of Y° this covering involves degenerate faces, i.e., faces $G \mapsto F^\vee$ with $\dim(G) < \dim(F^\vee)$.

Writing the vertices of the faces as deformations of X_0° the covering is given by

$$
[],
$$

$$
[],
$$

$$
\begin{array}{l}
[[\left\langle \frac{y_4 y_9^3}{y_1 y_2 y_3 y_5} \right\rangle, \left\langle \frac{y_5^2 y_{10}^2}{y_6 y_7 y_8 y_9} \right\rangle] \mapsto [2,4,5]^\vee, [\left\langle \frac{y_3 y_6^3}{y_1 y_2 y_4 y_5} \right\rangle, \left\langle \frac{y_5^2 y_{10}^2}{y_6 y_7 y_8 y_9} \right\rangle] \mapsto [1,4,5]^\vee, \\
[\left\langle \frac{y_4 y_9^3}{y_1 y_2 y_3 y_5} \right\rangle, \left\langle \frac{y_1^2 y_7^2}{y_6 y_8 y_9 y_{10}} \right\rangle] \mapsto [2,3,5]^\vee, [\left\langle \frac{y_3 y_6^3}{y_1 y_2 y_4 y_5} \right\rangle, \left\langle \frac{y_1^2 y_7^2}{y_6 y_8 y_9 y_{10}} \right\rangle] \mapsto [1,3,5]^\vee, \\
[\left\langle \frac{y_4 y_9^3}{y_1 y_2 y_3 y_5} \right\rangle, \left\langle \frac{y_2^2 y_8^2}{y_6 y_7 y_9 y_{10}} \right\rangle] \mapsto [2,3,4]^\vee, [\left\langle \frac{y_3 y_6^3}{y_1 y_2 y_4 y_5} \right\rangle, \left\langle \frac{y_2^2 y_8^2}{y_6 y_7 y_9 y_{10}} \right\rangle] \mapsto [1,3,4]^\vee],
\end{array}
$$

$$
\begin{array}{l}
[[\left\langle \frac{y_4 y_9^3}{y_1 y_2 y_3 y_5} \right\rangle, \left\langle \frac{y_1^2 y_7^2}{y_6 y_8 y_9 y_{10}}, \frac{y_2^2 y_8^2}{y_6 y_7 y_9 y_{10}} \right\rangle] \mapsto [2,3]^\vee, \\
[\left\langle \frac{y_4 y_9^3}{y_1 y_2 y_3 y_5}, \frac{y_3 y_6^3}{y_1 y_2 y_4 y_5} \right\rangle, \left\langle \frac{y_1^2 y_7^2}{y_6 y_8 y_9 y_{10}} \right\rangle] \mapsto [3,5]^\vee, \\
[\left\langle \frac{y_3 y_6^3}{y_1 y_2 y_4 y_5} \right\rangle, \left\langle \frac{y_1^2 y_7^2}{y_6 y_8 y_9 y_{10}}, \frac{y_2^2 y_8^2}{y_6 y_7 y_9 y_{10}} \right\rangle] \mapsto [1,3]^\vee, \\
[\left\langle \frac{y_4 y_9^3}{y_1 y_2 y_3 y_5} \right\rangle, \left\langle \frac{y_5^2 y_{10}^2}{y_6 y_7 y_8 y_9}, \frac{y_2^2 y_8^2}{y_6 y_7 y_9 y_{10}} \right\rangle] \mapsto [2,4]^\vee, \\
[\left\langle \frac{y_4 y_9^3}{y_1 y_2 y_3 y_5} \right\rangle, \left\langle \frac{y_5^2 y_{10}^2}{y_6 y_7 y_8 y_9}, \frac{y_1^2 y_7^2}{y_6 y_8 y_9 y_{10}} \right\rangle] \mapsto [2,5]^\vee, \\
[\left\langle \frac{y_4 y_9^3}{y_1 y_2 y_3 y_5}, \frac{y_3 y_6^3}{y_1 y_2 y_4 y_5} \right\rangle, \left\langle \frac{y_5^2 y_{10}^2}{y_6 y_7 y_8 y_9} \right\rangle] \mapsto [4,5]^\vee, \\
[\left\langle \frac{y_4 y_9^3}{y_1 y_2 y_3 y_5}, \frac{y_3 y_6^3}{y_1 y_2 y_4 y_5} \right\rangle, \left\langle \frac{y_2^2 y_8^2}{y_6 y_7 y_9 y_{10}} \right\rangle] \mapsto [3,4]^\vee, \\
[\left\langle \frac{y_3 y_6^3}{y_1 y_2 y_4 y_5} \right\rangle, \left\langle \frac{y_5^2 y_{10}^2}{y_6 y_7 y_8 y_9}, \frac{y_2^2 y_8^2}{y_6 y_7 y_9 y_{10}} \right\rangle] \mapsto [1,4]^\vee, \\
[\left\langle \frac{y_3 y_6^3}{y_1 y_2 y_4 y_5} \right\rangle, \left\langle \frac{y_5^2 y_{10}^2}{y_6 y_7 y_8 y_9}, \frac{y_1^2 y_7^2}{y_6 y_8 y_9 y_{10}} \right\rangle] \mapsto [1,5]^\vee],
\end{array}
$$

$$
\begin{array}{l}
[[\left\langle \frac{y_4 y_9^3}{y_1 y_2 y_3 y_5}, \frac{y_3 y_6^3}{y_1 y_2 y_4 y_5} \right\rangle, \left\langle \frac{y_1^2 y_7^2}{y_6 y_8 y_9 y_{10}}, \frac{y_2^2 y_8^2}{y_6 y_7 y_9 y_{10}} \right\rangle] \mapsto [3]^\vee, \\
[\left\langle \frac{y_4 y_9^3}{y_1 y_2 y_3 y_5} \right\rangle, \left\langle \frac{y_5^2 y_{10}^2}{y_6 y_7 y_8 y_9}, \frac{y_1^2 y_7^2}{y_6 y_8 y_9 y_{10}}, \frac{y_2^2 y_8^2}{y_6 y_7 y_9 y_{10}} \right\rangle] \mapsto [2]^\vee, \\
[\left\langle \frac{y_3 y_6^3}{y_1 y_2 y_4 y_5} \right\rangle, \left\langle \frac{y_5^2 y_{10}^2}{y_6 y_7 y_8 y_9}, \frac{y_1^2 y_7^2}{y_6 y_8 y_9 y_{10}}, \frac{y_2^2 y_8^2}{y_6 y_7 y_9 y_{10}} \right\rangle] \mapsto [1]^\vee, \\
[\left\langle \frac{y_4 y_9^3}{y_1 y_2 y_3 y_5}, \frac{y_3 y_6^3}{y_1 y_2 y_4 y_5} \right\rangle, \left\langle \frac{y_5^2 y_{10}^2}{y_6 y_7 y_8 y_9}, \frac{y_2^2 y_8^2}{y_6 y_7 y_9 y_{10}} \right\rangle] \mapsto [4]^\vee,
\end{array}
$$

$$[\left\langle \frac{y_4\,y_9{}^3}{y_1\,y_2\,y_3\,y_5}, \frac{y_3\,y_6{}^3}{y_1\,y_2\,y_4\,y_5} \right\rangle, \left\langle \frac{y_5^2 y_{10}{}^2}{y_6\,y_7\,y_8\,y_9}, \frac{y_1{}^2 y_7^2}{y_6\,y_8\,y_9\,y_{10}} \right\rangle] \mapsto [5]^\vee],$$

$$[]$$

Mirror degeneration The space of first order deformations of X_0° in the mirror degeneration $\mathfrak{X}^\circ$ has dimension 5 and the deformations represented by the monomials

$$\left\{ \frac{y_2^2 y_8{}^2}{y_6\,y_7\,y_9\,y_{10}} \quad \frac{y_1^2 y_7{}^2}{y_6\,y_8\,y_9\,y_{10}} \quad \frac{y_3\,y_6{}^3}{y_1\,y_2\,y_4\,y_5} \quad \frac{y_5^2 y_{10}{}^2}{y_6\,y_7\,y_8\,y_9} \quad \frac{y_4\,y_9{}^3}{y_1\,y_2\,y_3\,y_5} \right\}$$

form a torus invariant basis. The mirror degeneration $\mathfrak{X}^\circ \subset Y^\circ \times \operatorname{Spec} \mathbb{C}[t]$ of $\mathfrak{X}$ is given by the ideal $I^\circ \subset S^\circ \otimes \mathbb{C}[t]$ generated by

$$t\,(s_4\,y_1^2 y_7^3 + s_5\,y_2^2 y_8^3 + s_1\,y_5^2 y_{10}^3) + y_6\,y_7\,y_8\,y_{10}\,y_9$$
$$t\,(s_3\,y_3^2 y_6^3 + s_2\,y_4^2 y_9^3) + y_1\,y_2\,y_3\,y_4\,y_5$$
$$ts_3\,y_6^4 y_3 + y_6\,y_1\,y_2\,y_4\,y_5 \qquad ts_1\,y_5^3 y_{10}^2 + y_6\,y_7\,y_8\,y_5\,y_9 \qquad ts_4\,y_1^3 y_7^2 + y_6\,y_8\,y_1\,y_{10}\,y_9$$
$$ts_5\,y_2^3 y_8^2 + y_6\,y_7\,y_2\,y_{10}\,y_9 \qquad ts_2\,y_9^4 y_4 + y_1\,y_2\,y_3\,y_5\,y_9$$

and 25 monomials of degree 5

The ideal J° which is $\operatorname{Pic}(Y^\circ)$-generated by

$$\left\{ \begin{array}{l} t(s_3\,y_3^2 y_6^3 + s_2\,y_4^2 y_9^3) + y_1\,y_2\,y_3\,y_4\,y_5, \\ t(s_4\,y_1^2 y_7^3 + s_5\,y_2^2 y_8^3 + s_1\,y_5^2 y_{10}^3) + y_6\,y_7\,y_8\,y_{10}\,y_9 \end{array} \right\}$$

defines a flat affine cone inducing $\mathfrak{X}^\circ$.

Contraction of the mirror degeneration In the following we give a birational map relating the degeneration $\mathfrak{X}^\circ$ to a Greene-Plesser type orbifolding mirror family by contracting divisors on Y°. See also Section 9.13 below. In order to contract the divisors

$$\begin{array}{ll} y_3 = y_{(-1,1,0,0)} = \frac{x_2}{x_1} & y_4 = y_{(1,-1,0,0)} = \frac{x_1}{x_2} \\ y_7 = y_{(0,0,-1,2)} = \frac{x_4^2}{x_0\,x_3} & y_8 = y_{(0,0,-1,-1)} = \frac{x_0^2}{x_3\,x_4} \\ y_{10} = y_{(0,0,2,-1)} = \frac{x_3^2}{x_4\,x_0} & \end{array}$$

consider the $\mathbb{Q}$-factorial toric Fano variety $\hat{Y}^\circ = X(\hat{\Sigma}^\circ)$, where $\hat{\Sigma}^\circ = \Sigma(\hat{P}^\circ) \subset M_\mathbb{R}$ is the fan over the Fano polytope $\hat{P}^\circ \subset M_\mathbb{R}$ given as the convex hull of the remaining vertices of $P^\circ = \nabla^*$ corresponding to the Cox variables

$$\begin{array}{ll} y_5 = y_{(-1,-1,2,0)} = \frac{x_3^2}{x_1\,x_2} & y_6 = y_{(0,3,-1,-1)} = \frac{x_2^3}{x_0\,x_3\,x_4} \\ y_9 = y_{(3,0,-1,-1)} = \frac{x_1^3}{x_0\,x_3\,x_4} & y_1 = y_{(-1,-1,0,2)} = \frac{x_4^2}{x_1\,x_2} \\ y_2 = y_{(-1,-1,0,0)} = \frac{x_0^2}{x_1\,x_2} & \end{array}$$

of the toric variety $\hat{Y}^\circ$ with Cox ring

$$\hat{S}^\circ = \mathbb{C}[y_5, y_6, y_9, y_1, y_2]$$

The Cox variables of $\hat{Y}^\circ$ correspond to the set of Fermat deformations of $\mathfrak{X}$. Let

$$Y^\circ = X\left(\Sigma^\circ\right) \to X(\hat{\Sigma}^\circ) = \hat{Y}^\circ$$

be a birational map from Y° to the toric Fano variety $\hat{Y}^\circ$, which contracts the divisors of the rays $\Sigma^\circ(1) - \hat{\Sigma}^\circ(1)$ corresponding to the Cox variables

$$y_3 \quad y_4 \quad y_7 \quad y_8 \quad y_{10}$$

Representing $\hat{Y}^\circ$ as a quotient we have

$$\hat{Y}^\circ = \left(\mathbb{C}^5 - V\left(B(\hat{\Sigma}^\circ)\right)\right)/\hat{G}^\circ$$

with

$$\hat{G}^\circ = \mathbb{Z}_2 \times \mathbb{Z}_6 \times (\mathbb{C}^*)^1$$

acting via

$$\xi y = \left(u_1\,v_1 \cdot y_5,\; u_1\,u_2\,v_1 \cdot y_6,\; u_1\,u_2^5\,v_1 \cdot y_9,\; u_2^3\,v_1 \cdot y_1,\; v_1 \cdot y_2\right)$$

for $\xi = (u_1, u_2, v_1) \in \hat{G}^\circ$ and $y \in \mathbb{C}^5 - V\left(B(\hat{\Sigma}^\circ)\right)$. Hence with the group

$$\hat{H}^\circ = \mathbb{Z}_2 \times \mathbb{Z}_6$$

of order 12 the toric variety $\hat{Y}^\circ$ is the quotient

$$\hat{Y}^\circ = \mathbb{P}^4/\hat{H}^\circ$$

of projective space $\mathbb{P}^4$. The mirror degeneration $\mathfrak{X}^\circ$ induces via $Y^\circ \to \hat{Y}^\circ$ a degeneration $\hat{\mathfrak{X}}^\circ \subset \hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]$ given by the ideal $\hat{I}^\circ \subset \langle y_5, y_6, y_9, y_1, y_2\rangle \subset \hat{S}^\circ \otimes \mathbb{C}[t]$ generated by the Fermat-type equations

$$\left\{\begin{array}{l} y_1\,y_2\,y_5 + t(s_2\,y_6^3 + s_1\,y_9^3), \\ y_6\,y_9 + t(s_3\,y_1^2 + s_5\,y_2^2 + s_4\,y_5^2) \end{array}\right\}$$

The special fiber $\hat{X}_0^\circ \subset \hat{Y}^\circ$ of $\hat{\mathfrak{X}}^\circ$ is cut out by the monomial ideal

$$\hat{I}_0^\circ \subset B(\hat{\Sigma}^\circ) = \langle y_5, y_6, y_9, y_1, y_2 \rangle \subset \hat{S}^\circ$$

generated by

$$\left\{ \; y_1\, y_2\, y_5 \quad y_6\, y_9 \; \right\}$$

The complex $SP(\hat{I}_0^\circ)$ of prime ideals of the toric strata of $\hat{X}_0^\circ$ is

$$\begin{array}{l}
[], \\
[], \\
[\langle y_5, y_9\rangle , \langle y_5, y_6\rangle , \langle y_9, y_1\rangle , \langle y_9, y_2\rangle , \langle y_6, y_1\rangle , \langle y_6, y_2\rangle], \\
[\langle y_5, y_6, y_9\rangle , \langle y_6, y_9, y_1\rangle , \langle y_5, y_9, y_1\rangle , \langle y_5, y_6, y_1\rangle , \langle y_5, y_6, y_2\rangle , \\
\langle y_9, y_1, y_2\rangle , \langle y_6, y_1, y_2\rangle , \langle y_5, y_9, y_2\rangle , \langle y_6, y_9, y_2\rangle], \\
[\langle y_6, y_9, y_1, y_2\rangle , \langle y_5, y_9, y_1, y_2\rangle , \langle y_5, y_6, y_1, y_2\rangle , \langle y_5, y_6, y_9, y_2\rangle , \\
\langle y_5, y_6, y_9, y_1\rangle], \\
[]
\end{array}$$

so $\hat{I}_0^\circ$ has the primary decomposition

$$\hat{I}_0^\circ = \quad \langle y_6, y_2\rangle \cap \langle y_5, y_9\rangle \cap \langle y_6, y_1\rangle \cap \langle y_9, y_1\rangle \cap \langle y_9, y_2\rangle \cap \langle y_5, y_6\rangle$$

The Bergman subcomplex $B\,(I) \subset \text{Poset}\,(\nabla)$ induces a subcomplex of $\text{Poset}(\hat{\nabla})$, $\hat{\nabla} = (\hat{P}^\circ)^*$ corresponding to $\hat{I}_0^\circ$. Indexing of the vertices of $\hat{\nabla}$ by

$$\begin{array}{ll}
1 = \left(\frac{1}{2}, \frac{1}{2}, \frac{5}{2}, 0\right) & 2 = \left(-\frac{1}{3}, \frac{4}{3}, 0, 0\right) \\
3 = \left(\frac{4}{3}, -\frac{1}{3}, 0, 0\right) & 4 = \left(\frac{1}{2}, \frac{1}{2}, 0, \frac{5}{2}\right) \\
5 = \left(-2, -2, -\frac{5}{2}, -\frac{5}{2}\right) &
\end{array}$$

this complex is given by

$$\left\{ \begin{array}{l}
[], \\
[[1], [2], [3], [4], [5]], \\
[[4,5], [1,5], [2,5], [3,5], [3,4], [1,2], [1,3], [2,4], [1,4]], \\
[[2,4,5], [3,4,5], [1,2,5], [1,2,4], [1,3,5], [1,3,4]], \\
[], \\
[]
\end{array} \right\}$$

8.12.3 The Calabi-Yau threefold given as the complete intersection of a generic quadric and a generic quartic in $\mathbb{P}^5$

Setup Let $Y = \mathbb{P}^5 = X(\Sigma)$, $\Sigma = \Sigma(P) = NF(\Delta) \subset N_{\mathbb{R}}$ with the Fano polytope $P = \Delta^*$ given by

$$\Delta = \text{convexhull}\left(\begin{array}{cc} (5,-1,-1,-1,-1) & (-1,5,-1,-1,-1) \\ (-1,-1,5,-1,-1) & (-1,-1,-1,5,-1) \\ (-1,-1,-1,-1,5) & (-1,-1,-1,-1,-1) \end{array}\right) \subset M_{\mathbb{R}}$$

and let

$$S = \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]$$

be the Cox ring of Y with the variables

$$\begin{array}{ll} x_1 = x_{(1,0,0,0,0)} & x_2 = x_{(0,1,0,0,0)} \\ x_3 = x_{(0,0,1,0,0)} & x_4 = x_{(0,0,0,1,0)} \\ x_5 = x_{(0,0,0,0,1)} & x_0 = x_{(-1,-1,-1,-1,-1)} \end{array}$$

associated to the rays of Σ. Consider the degeneration $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[t]$ of complete intersection Calabi-Yau 3-folds of type $(2,4)$ with monomial special fiber

$$I_0 = \left\langle \; x_1 x_2 \quad x_0 x_3 x_4 x_5 \; \right\rangle$$

The degeneration is given by the ideal $I \subset S \otimes \mathbb{C}[t]$ with I_0-reduced generators

$$\left\{ \begin{array}{l} x_1 x_2 + t(s_1 x_1^2 + ... + s_6 x_2^2 + ... + s_{11} x_3^2 + ... + s_{15} x_4^2 + ... + s_{18} x_5^2 + ... + s_{20} x_0^2), \\ x_0 x_3 x_4 x_5 + t(s_{21} x_1^4 + ... + s_{56} x_2^4 + ... + s_{91} x_3^4 + ... + s_{110} x_4^4 + ... + s_{120} x_5^4 + ...+ \\ s_{124} x_0^4) \end{array} \right\}$$

Special fiber Gröbner cone The space of first order deformations of $\mathfrak{X}$ has dimension 124 and the deformations represented by the Cox Laurent monomials

$\frac{x_1^4}{x_0 x_3 x_4 x_5}$	$\frac{x_2^4}{x_0 x_3 x_4 x_5}$	$\frac{x_1^3}{x_0 x_3 x_4}$	$\frac{x_2^3}{x_0 x_3 x_4}$	$\frac{x_3^2 x_1}{x_4 x_0 x_5}$	$\frac{x_4 x_1^2}{x_0 x_3 x_5}$	$\frac{x_5^2 x_1}{x_0 x_3 x_4}$	$\frac{x_3 x_1^2}{x_4 x_0 x_5}$
$\frac{x_1^2 x_0}{x_3 x_4 x_5}$	$\frac{x_5 x_1^2}{x_0 x_3 x_4}$	$\frac{x_1 x_0^2}{x_3 x_4 x_5}$	$\frac{x_3 x_2^2}{x_4 x_0 x_5}$	$\frac{x_3^2 x_2}{x_4 x_0 x_5}$	$\frac{x_4 x_2^2}{x_0 x_3 x_5}$	$\frac{x_2^2 x_0}{x_3 x_4 x_5}$	$\frac{x_5^2 x_2}{x_0 x_3 x_4}$
$\frac{x_2 x_0^2}{x_3 x_4 x_5}$	$\frac{x_5 x_2^2}{x_0 x_3 x_4}$	$\frac{x_2^3}{x_0 x_3 x_5}$	$\frac{x_2^3}{x_3 x_4 x_5}$	$\frac{x_3^3}{x_4 x_0 x_5}$	$\frac{x_0^3}{x_3 x_4 x_5}$	$\frac{x_5^3}{x_0 x_3 x_4}$	$\frac{x_4^3}{x_0 x_3 x_5}$
$\frac{x_4^2 x_1}{x_0 x_3 x_5}$	$\frac{x_4^2 x_2}{x_0 x_3 x_5}$	$\frac{x_1^3}{x_0 x_3 x_5}$	$\frac{x_1^3}{x_4 x_0 x_5}$	$\frac{x_1^3}{x_3 x_4 x_5}$	$\frac{x_2^3}{x_4 x_0 x_5}$	$\frac{x_2 x_3}{x_4 x_0}$	$\frac{x_1 x_0}{x_3 x_4}$
$\frac{x_1 x_4}{x_0 x_3}$	$\frac{x_3 x_4}{x_1 x_2}$	$\frac{x_4 x_0}{x_1 x_2}$	$\frac{x_2 x_0}{x_3 x_4}$	$\frac{x_2 x_4}{x_0 x_3}$	$\frac{x_1 x_3}{x_4 x_0}$	$\frac{x_2^2}{x_3 x_4}$	$\frac{x_1^2}{x_3 x_4}$
$\frac{x_2^2}{x_4 x_0}$	$\frac{x_3^2}{x_4 x_0}$	$\frac{x_0^2}{x_3 x_4}$	$\frac{x_4^2}{x_0 x_3}$	$\frac{x_4^2}{x_1 x_2}$	$\frac{x_0^2}{x_1 x_2}$	$\frac{x_1^2}{x_4 x_0}$	$\frac{x_3^2}{x_1 x_2}$
$\frac{x_5^2}{x_3 x_4}$	$\frac{x_4 x_5}{x_1 x_2}$	$\frac{x_4 x_5}{x_0 x_3}$	$\frac{x_4 x_0}{x_3 x_5}$	$\frac{x_3 x_4}{x_0 x_5}$	$\frac{x_3 x_5}{x_1 x_2}$	$\frac{x_3 x_5}{x_4 x_0}$	$\frac{x_2^2}{x_0 x_5}$
$\frac{x_1 x_3}{x_4 x_5}$	$\frac{x_1 x_0}{x_4 x_5}$	$\frac{x_5 x_2}{x_4 x_0}$	$\frac{x_2 x_3}{x_4 x_5}$	$\frac{x_2 x_0}{x_4 x_5}$	$\frac{x_5 x_1}{x_0 x_3}$	$\frac{x_1 x_4}{x_3 x_5}$	$\frac{x_1 x_0}{x_3 x_5}$

$$
\begin{array}{cccccccc}
\frac{x_5 x_1}{x_3 x_4} & \frac{x_5 x_2}{x_0 x_3} & \frac{x_2 x_4}{x_3 x_5} & \frac{x_2 x_0}{x_3 x_5} & \frac{x_5 x_2}{x_3 x_4} & \frac{x_1 x_4}{x_0 x_5} & \frac{x_1 x_3}{x_0 x_5} & \frac{x_2 x_4}{x_0 x_5} \\
\frac{x_2 x_3}{x_0 x_5} & \frac{x_0 x_3}{x_1 x_2} & \frac{x_0 x_3}{x_4 x_5} & \frac{x_0 x_5}{x_1 x_2} & \frac{x_0 x_5}{x_3 x_4} & \frac{x_5 x_1}{x_4 x_0} & \frac{x_1^2}{x_4 x_5} & \frac{x_2^2}{x_4 x_5} \\
\frac{x_1^2}{x_3 x_5} & \frac{x_2^2}{x_3 x_5} & \frac{x_1^2}{x_0 x_5} & \frac{x_3^2}{x_4 x_5} & \frac{x_0^2}{x_4 x_5} & \frac{x_5^2}{x_0 x_3} & \frac{x_4^2}{x_3 x_5} & \frac{x_0^2}{x_3 x_5} \\
\frac{x_4^2}{x_0 x_5} & \frac{x_3^2}{x_0 x_5} & \frac{x_5^2}{x_4 x_0} & \frac{x_5^2}{x_1 x_2} & \frac{x_1^2}{x_0 x_3} & \frac{x_2^2}{x_0 x_3} & \frac{x_2}{x_5} & \frac{x_0}{x_3} \\
\frac{x_3}{x_0} & \frac{x_2}{x_1} & \frac{x_1}{x_2} & \frac{x_0}{x_1} & \frac{x_1}{x_3} & \frac{x_0}{x_2} & \frac{x_2}{x_0} & \frac{x_3}{x_1} \\
\frac{x_1}{x_0} & \frac{x_3}{x_2} & \frac{x_2}{x_3} & \frac{x_5}{x_2} & \frac{x_5}{x_1} & \frac{x_0}{x_4} & \frac{x_4}{x_2} & \frac{x_4}{x_0} \\
\frac{x_4}{x_3} & \frac{x_4}{x_1} & \frac{x_3}{x_4} & \frac{x_1}{x_4} & \frac{x_2}{x_4} & \frac{x_1}{x_5} & \frac{x_5}{x_4} & \frac{x_5}{x_3} \\
\frac{x_5}{x_0} & \frac{x_4}{x_5} & \frac{x_3}{x_5} & \frac{x_0}{x_5} & & & &
\end{array}
$$

form a torus invariant basis. These deformations give linear inequalities defining the special fiber Gröbner cone, i.e.,

$$C_{I_0}(I) = \left\{ (w_t, w_y) \in \mathbb{R} \oplus N_{\mathbb{R}} \mid \left\langle w, A^{-1}(m) \right\rangle \geq -w_t \forall m \right\}$$

for above monomials m and the presentation matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

of $A_4(Y)$. The vertices of the special fiber polytope

$$\nabla = C_{I_0}(I) \cap \{w_t = 1\}$$

and the number of faces, each vertex is contained in, are given by the table

10	$(0,0,1,0,0)$	$(0,0,0,1,0)$	$(0,0,0,0,1)$	$(-1,-1,-1,-1,-1)$
	$(0,1,0,0,0)$	$(1,0,0,0,0)$		
16	$(1,0,1,0,0)$	$(0,1,1,0,0)$	$(1,0,0,1,0)$	$(0,1,0,1,0)$
	$(1,0,0,0,1)$	$(0,1,0,0,1)$	$(0,-1,-1,-1,-1)$	$(-1,0,-1,-1,-1)$

The number of faces of ∇ and their F-vectors are

Dimension	Number of faces	F-vector	
-1	1	$(0,0,0,0,0,0,0)$	
0	14	$(1,1,0,0,0,0,0)$	point
1	39	$(1,2,1,0,0,0,0)$	edge
2	28	$(1,3,3,1,0,0,0)$	triangle
2	22	$(1,4,4,1,0,0,0)$	quadrangle
3	11	$(1,4,6,4,1,0,0)$	tetrahedron
3	24	$(1,6,9,5,1,0,0)$	prism
4	6	$(1,8,16,14,6,1,0)$	
4	4	$(1,9,18,15,6,1,0)$	
4	2	$(1,5,10,10,5,1,0)$	
5	1	$(1,14,39,50,35,12,1)$	

The dual $P^{\circ} = \nabla^{*}$ of ∇ is a Fano polytope with vertices

$(-1,-1,0,2,0)$	$(-1,-1,0,0,2)$	$(-1,-1,0,0,0)$	$(-1,1,0,0,0)$
$(1,-1,0,0,0)$	$(-1,-1,2,0,0)$	$(0,4,-1,-1,-1)$	$(0,0,-1,3,-1)$
$(0,0,-1,-1,3)$	$(0,0,-1,-1,-1)$	$(4,0,-1,-1,-1)$	$(0,0,3,-1,-1)$

and the F-vectors of the faces of P° are

Dimension	Number of faces	F-vector	
-1	1	$(0,0,0,0,0,0,0)$	
0	12	$(1,1,0,0,0,0,0)$	point
1	35	$(1,2,1,0,0,0,0)$	edge
2	14	$(1,4,4,1,0,0,0)$	quadrangle
2	36	$(1,3,3,1,0,0,0)$	triangle
3	23	$(1,4,6,4,1,0,0)$	tetrahedron
3	16	$(1,6,9,5,1,0,0)$	prism
4	8	$(1,8,16,14,6,1,0)$	
4	6	$(1,5,10,10,5,1,0)$	
5	1	$(1,12,35,50,39,14,1)$	

The polytopes ∇ and P° have 0 as the unique interior lattice point, and the Fano polytope P° defines the embedding toric Fano variety $Y^{\circ} = X(\Sigma(P^{\circ}))$

of the fibers of the mirror degeneration. The dimension of the automorphism group of Y° is

$$\dim\left(\operatorname{Aut}\left(Y^{\circ}\right)\right)=5$$

Let

$$S^{\circ}=\mathbb{C}[y_1,y_2,...,y_{12}]$$

be the Cox ring of Y° with variables

$$\begin{array}{lll}
y_1=y_{(-1,-1,0,2,0)}=\frac{x_4^2}{x_1\,x_2} & y_2=y_{(-1,-1,0,0,2)}=\frac{x_5^2}{x_1\,x_2} & y_3=y_{(-1,-1,0,0,0)}=\frac{x_0^2}{x_1\,x_2}\\
y_4=y_{(-1,1,0,0,0)}=\frac{x_2}{x_1} & y_5=y_{(1,-1,0,0,0)}=\frac{x_1}{x_2} & y_6=y_{(-1,-1,2,0,0)}=\frac{x_3^2}{x_1\,x_2}\\
y_7=y_{(0,4,-1,-1,-1)}=\frac{x_2^4}{x_0\,x_3\,x_4\,x_5} & y_8=y_{(0,0,-1,3,-1)}=\frac{x_4^3}{x_3\,x_5\,x_0} & y_9=y_{(0,0,-1,-1,3)}=\frac{x_5^3}{x_0\,x_3\,x_4}\\
y_{10}=y_{(0,0,-1,-1,-1)}=\frac{x_0^3}{x_3\,x_4\,x_5} & y_{11}=y_{(4,0,-1,-1,-1)}=\frac{x_1^4}{x_0\,x_3\,x_4\,x_5} & y_{12}=y_{(0,0,3,-1,-1)}=\frac{x_3^3}{x_4\,x_5\,x_0}
\end{array}$$

Bergman subcomplex Intersecting the tropical variety of I with the special fiber Gröbner cone $C_{I_0}(I)$ we obtain the special fiber Bergman subcomplex

$$B(I)=C_{I_0}(I)\cap BF(I)\cap\{w_t=1\}$$

The following table chooses an indexing of the vertices of ∇ involved in $B(I)$ and gives for each vertex the numbers n_{∇} and n_B of faces of ∇ and faces of $B(I)$ it is contained in

n_{∇}	$n_{B(I)}$		
16	4	$1=(1,0,1,0,0)$	$2=(0,1,1,0,0)$
		$3=(1,0,0,1,0)$	$4=(0,1,0,1,0)$
		$5=(1,0,0,0,1)$	$6=(0,1,0,0,1)$
		$7=(0,-1,-1,-1,-1)$	$8=(-1,0,-1,-1,-1)$

With this indexing the Bergman subcomplex $B(I)$ of $\operatorname{Poset}(\nabla)$ associated to the degeneration $\mathfrak{X}$ is

[],

[[1], [2], [3], [4], [5], [6], [7], [8]],

[[5, 7], [6, 8], [5, 6], [7, 8], [1, 3], [1, 5], [1, 7], [2, 4], [2, 6], [2, 8], [1, 2], [3, 4], [3, 5], [3, 7], [4, 6], [4, 8]], [[3, 4, 7, 8], [5, 6, 7, 8], [1, 2, 5, 6], [1, 3, 7], [1, 3, 5], [1, 2, 7, 8], [1, 5, 7], [3, 5, 7], [2, 4, 6], [2, 4, 8], [4, 6, 8], [2, 6, 8], [1, 2, 3, 4], [3, 4, 5, 6]],

[[3, 4, 5, 6, 7, 8], [2, 4, 6, 8], [1, 2, 3, 4, 7, 8], [1, 2, 5, 6, 7, 8], [1, 3, 5, 7], [1, 2, 3, 4, 5, 6]],

[],

[]

$B(I)$ has the F-vector

Dimension	−1	0	1	2	3	4	5
Number of faces	0	8	16	14	6	0	0

and the F-vectors of its faces are

Dimension	Number of faces	F-vector	
0	8	(1, 1, 0, 0, 0, 0, 0)	point
1	16	(1, 2, 1, 0, 0, 0, 0)	edge
2	8	(1, 3, 3, 1, 0, 0, 0)	triangle
2	6	(1, 4, 4, 1, 0, 0, 0)	quadrangle
3	4	(1, 6, 9, 5, 1, 0, 0)	prism
3	2	(1, 4, 6, 4, 1, 0, 0)	tetrahedron

Dual complex The dual complex $\operatorname{dual}(B(I)) = (B(I))^*$ of deformations associated to $B(I)$ via initial ideals is given by

[],

[],

$[[3,4,5,6,7,8]^* = \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_3^3}{x_4\,x_0\,x_5} \right\rangle, [2,4,6,8]^* = \left\langle \frac{x_1^4}{x_0\,x_3\,x_4\,x_5}, \frac{x_1}{x_2} \right\rangle, ...],$

$[[1,3,7]^* = \left\langle \frac{x_5^2}{x_1\,x_2}, \frac{x_2^4}{x_0\,x_3\,x_4\,x_5}, \frac{x_5^3}{x_0\,x_3\,x_4}, \frac{x_2}{x_1} \right\rangle, [3,4,7,8]^* = \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{x_5^2}{x_1\,x_2}, \frac{x_3^3}{x_4\,x_0\,x_5}, \frac{x_5^3}{x_0\,x_3\,x_4} \right\rangle,$
$...],$

$[[5,7]^* = \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_3^2}{x_1\,x_2}, \frac{x_2^4}{x_0\,x_3\,x_4\,x_5}, \frac{x_4^3}{x_0\,x_3\,x_5}, \frac{x_3^3}{x_4\,x_0\,x_5}, \frac{x_2}{x_1} \right\rangle,$
$...],$

$$[[1]^* = \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_5^2}{x_1\,x_2}, \frac{x_0^2}{x_1\,x_2}, \frac{x_2^4}{x_0\,x_3\,x_4\,x_5}, \frac{x_4^3}{x_0\,x_3\,x_5}, \frac{x_5^3}{x_0\,x_3\,x_4}, \frac{x_0^3}{x_3\,x_4\,x_5}, \frac{x_2}{x_1} \right\rangle,$$

$...],$

$[]$

when writing the vertices of the faces as deformations of X_0. Note that the T-invariant basis of deformations associated to a face is given by all lattice points of the corresponding polytope in $M_{\mathbb{R}}$. In order to compress the output we list one representative in any set of faces G with fixed F-vector of G and of G^*. When numbering the vertices of the faces of $\operatorname{dual}(B(I))$ by the Cox variables of the mirror toric Fano variety Y° the complex $\operatorname{dual}(B(I))$ is

$$
\begin{aligned}
&[],\\
&[],\\
&[[3,4,5,6,7,8]^* = \langle y_6, y_{12}\rangle, [2,4,6,8]^* = \langle y_{11}, y_5\rangle, ...],\\
&[[1,3,7]^* = \langle y_2, y_7, y_9, y_4\rangle, [3,4,7,8]^* = \langle y_6, y_2, y_{12}, y_9\rangle, ...],\\
&[[5,7]^* = \langle y_1, y_6, y_7, y_8, y_{12}, y_4\rangle, ...],\\
&[[1]^* = \langle y_1, y_2, y_3, y_7, y_8, y_9, y_{10}, y_4\rangle, ...],\\
&[]
\end{aligned}
$$

The dual complex has the F-vector

Dimension	-1	0	1	2	3	4	5
Number of faces	0	0	6	14	16	8	0

and the F-vectors of the faces of $\operatorname{dual}(B(I))$ are

Dimension	Number of faces	F-vector	
1	6	$(1,2,1,0,0,0,0)$	edge
2	14	$(1,4,4,1,0,0,0)$	quadrangle
3	16	$(1,6,9,5,1,0,0)$	prism
4	8	$(1,8,16,14,6,1,0)$	

Recall that in this example the toric variety Y is projective space. The number of lattice points of the support of dual $(B(I))$ relates to the dimension $h^{1,2}(X)$ of the complex moduli space of the generic fiber X of $\mathfrak{X}$ and to the dimension $h^{1,1}\left(\bar{X}^{\circ}\right)$ of the Kähler moduli space of the $MPCP$-blowup $\bar{X}^{\circ}$ of the generic fiber X° of the mirror degeneration

$$\begin{aligned} |\mathrm{supp}\,(\mathrm{dual}\,(B\,(I)))\cap M| &= 124 = 35+89 = \dim\left(\mathrm{Aut}\,(Y)\right) + h^{1,2}\,(X) \\ &= 30+5+89 \\ &= |\mathrm{Roots}\,(Y)| + \dim\left(T_Y\right) + h^{1,1}\left(\bar{X}^{\circ}\right) \end{aligned}$$

There are

$$h^{1,2}\,(X) + \dim\left(T_{Y^{\circ}}\right) = 89+5$$

non-trivial toric polynomial deformations of X_0

$\frac{x_3^3}{x_4 x_0 x_5}$	$\frac{x_3^2}{x_1 x_2}$	$\frac{x_1^4}{x_0 x_3 x_4 x_5}$	$\frac{x_5^2}{x_1 x_2}$	$\frac{x_5^3}{x_0 x_3 x_4}$	$\frac{x_4^3}{x_0 x_3 x_5}$	$\frac{x_4^2}{x_1 x_2}$	$\frac{x_2^4}{x_0 x_3 x_4 x_5}$	$\frac{x_0^3}{x_3 x_4 x_5}$
$\frac{x_0^2}{x_1 x_2}$	$\frac{x_3^2}{x_4 x_0}$	$\frac{x_5 x_3}{x_4 x_0}$	$\frac{x_5 x_3}{x_1 x_2}$	$\frac{x_5^2}{x_4 x_0}$	$\frac{x_3^2}{x_0 x_5}$	$\frac{x_3 x_4}{x_0 x_5}$	$\frac{x_4^2}{x_0 x_5}$	$\frac{x_3 x_4}{x_1 x_2}$
$\frac{x_0^2}{x_3 x_5}$	$\frac{x_4 x_0}{x_3 x_5}$	$\frac{x_4^2}{x_3 x_5}$	$\frac{x_4 x_0}{x_1 x_2}$	$\frac{x_2^3}{x_0 x_3 x_4}$	$\frac{x_5 {x_2}^2}{x_0 x_3 x_4}$	$\frac{x_5^2 x_2}{x_0 x_3 x_4}$	$\frac{x_2 {x_0}^2}{x_3 x_4 x_5}$	$\frac{x_2^2 x_0}{x_3 x_4 x_5}$
$\frac{x_2^3}{x_3 x_4 x_5}$	$\frac{x_4^2}{x_0 x_3}$	$\frac{x_4 x_5}{x_0 x_3}$	$\frac{x_4 x_5}{x_1 x_2}$	$\frac{x_5^2}{x_0 x_3}$	$\frac{x_2^3}{x_0 x_3 x_5}$	$\frac{x_4 {x_2}^2}{x_0 x_3 x_5}$	$\frac{x_4^2 x_2}{x_0 x_3 x_5}$	$\frac{x_2^3}{x_4 x_0 x_5}$
$\frac{x_3 {x_2}^2}{x_4 x_0 x_5}$	$\frac{x_3^2 x_2}{x_4 x_0 x_5}$	$\frac{x_1 {x_0}^2}{x_3 x_4 x_5}$	$\frac{x_1^2 x_0}{x_3 x_4 x_5}$	$\frac{x_1^3}{x_3 x_4 x_5}$	$\frac{x_1^3}{x_0 x_3 x_4}$	$\frac{x_5 {x_1}^2}{x_0 x_3 x_4}$	$\frac{x_5^2 x_1}{x_0 x_3 x_4}$	$\frac{x_1^3}{x_4 x_0 x_5}$
$\frac{x_3 {x_1}^2}{x_4 x_0 x_5}$	$\frac{x_3^2 x_1}{x_4 x_0 x_5}$	$\frac{x_1^3}{x_0 x_3 x_5}$	$\frac{x_4 {x_1}^2}{x_0 x_3 x_5}$	$\frac{x_4^2 x_1}{x_0 x_3 x_5}$	$\frac{x_0^2}{x_3 x_4}$	$\frac{x_5 x_0}{x_3 x_4}$	$\frac{x_5 x_0}{x_1 x_2}$	$\frac{x_5^2}{x_3 x_4}$
$\frac{x_0^2}{x_4 x_5}$	$\frac{x_0 x_3}{x_4 x_5}$	$\frac{x_3^2}{x_4 x_5}$	$\frac{x_0 x_3}{x_1 x_2}$	$\frac{x_2^2}{x_5 x_0}$	$\frac{x_2 x_3}{x_5 x_0}$	$\frac{x_2 x_4}{x_5 x_0}$	$\frac{x_1^2}{x_5 x_0}$	$\frac{x_1 x_3}{x_5 x_0}$
$\frac{x_1 x_4}{x_0 x_5}$	$\frac{x_2 x_0}{x_3 x_4}$	$\frac{x_2^2}{x_3 x_4}$	$\frac{x_5 x_2}{x_3 x_4}$	$\frac{x_2 x_0}{x_3 x_5}$	$\frac{x_2^2}{x_3 x_5}$	$\frac{x_2 x_4}{x_3 x_5}$	$\frac{x_2^2}{x_0 x_3}$	$\frac{x_2 x_4}{x_0 x_3}$
$\frac{x_5 x_2}{x_0 x_3}$	$\frac{x_1 x_0}{x_3 x_4}$	$\frac{x_1^2}{x_3 x_4}$	$\frac{x_5 x_1}{x_3 x_4}$	$\frac{x_1 x_0}{x_3 x_5}$	$\frac{x_1^2}{x_3 x_5}$	$\frac{x_1 x_4}{x_3 x_5}$	$\frac{x_1^2}{x_0 x_3}$	$\frac{x_1 x_4}{x_0 x_3}$
$\frac{x_5 x_1}{x_0 x_3}$	$\frac{x_2 x_0}{x_4 x_5}$	$\frac{x_2^2}{x_4 x_5}$	$\frac{x_2 x_3}{x_4 x_5}$	$\frac{x_2^2}{x_4 x_0}$	$\frac{x_2 x_3}{x_4 x_0}$	$\frac{x_5 x_2}{x_4 x_0}$	$\frac{x_1 x_0}{x_4 x_5}$	$\frac{x_1^2}{x_4 x_5}$
$\frac{x_1 x_3}{x_4 x_5}$	$\frac{x_1^2}{x_4 x_0}$	$\frac{x_1 x_3}{x_4 x_0}$	$\frac{x_5 x_1}{x_4 x_0}$					

They correspond to the toric divisors

$D_{(0,0,3,-1,-1)}$	$D_{(-1,-1,2,0,0)}$	$D_{(4,0,-1,-1,-1)}$	$D_{(-1,-1,0,0,2)}$
$D_{(0,0,-1,-1,3)}$	$D_{(0,0,-1,3,-1)}$	$D_{(-1,-1,0,2,0)}$	$D_{(0,4,-1,-1,-1)}$
$D_{(0,0,-1,-1,-1)}$	$D_{(-1,-1,0,0,0)}$	$D_{(0,0,2,-1,0)}$	$D_{(0,0,1,-1,1)}$
$D_{(-1,-1,1,0,1)}$	$D_{(0,0,0,-1,2)}$	$D_{(0,0,2,0,-1)}$	$D_{(0,0,1,1,-1)}$
$D_{(0,0,0,2,-1)}$	$D_{(-1,-1,1,1,0)}$	$D_{(0,0,-1,0,-1)}$	$D_{(0,0,-1,1,-1)}$
$D_{(0,0,-1,2,-1)}$	$D_{(-1,-1,0,1,0)}$	$D_{(0,3,-1,-1,0)}$	$D_{(0,2,-1,-1,1)}$
$D_{(0,1,-1,-1,2)}$	$D_{(0,1,-1,-1,-1)}$	$D_{(0,2,-1,-1,-1)}$	$D_{(0,3,-1,-1,-1)}$
$D_{(0,0,-1,2,0)}$	$D_{(0,0,-1,1,1)}$	$D_{(-1,-1,0,1,1)}$	$D_{(0,0,-1,0,2)}$

$D_{(0,3,-1,0,-1)}$	$D_{(0,2,-1,1,-1)}$	$D_{(0,1,-1,2,-1)}$	$D_{(0,3,0,-1,-1)}$
$D_{(0,2,1,-1,-1)}$	$D_{(0,1,2,-1,-1)}$	$D_{(1,0,-1,-1,-1)}$	$D_{(2,0,-1,-1,-1)}$
$D_{(3,0,-1,-1,-1)}$	$D_{(3,0,-1,-1,0)}$	$D_{(2,0,-1,-1,1)}$	$D_{(1,0,-1,-1,2)}$
$D_{(3,0,0,-1,-1)}$	$D_{(2,0,1,-1,-1)}$	$D_{(1,0,2,-1,-1)}$	$D_{(3,0,-1,0,-1)}$
$D_{(2,0,-1,1,-1)}$	$D_{(1,0,-1,2,-1)}$	$D_{(0,0,-1,-1,0)}$	$D_{(0,0,-1,-1,1)}$
$D_{(-1,-1,0,0,1)}$	$D_{(0,0,-1,-1,2)}$	$D_{(0,0,0,-1,-1)}$	$D_{(0,0,1,-1,-1)}$
$D_{(0,0,2,-1,-1)}$	$D_{(-1,-1,1,0,0)}$	$D_{(0,2,0,0,-1)}$	$D_{(0,1,1,0,-1)}$
$D_{(0,1,0,1,-1)}$	$D_{(2,0,0,0,-1)}$	$D_{(1,0,1,0,-1)}$	$D_{(1,0,0,1,-1)}$
$D_{(0,1,-1,-1,0)}$	$D_{(0,2,-1,-1,0)}$	$D_{(0,1,-1,-1,1)}$	$D_{(0,1,-1,0,-1)}$
$D_{(0,2,-1,0,-1)}$	$D_{(0,1,-1,1,-1)}$	$D_{(0,2,-1,0,0)}$	$D_{(0,1,-1,1,0)}$
$D_{(0,1,-1,0,1)}$	$D_{(1,0,-1,-1,0)}$	$D_{(2,0,-1,-1,0)}$	$D_{(1,0,-1,-1,1)}$
$D_{(1,0,-1,0,-1)}$	$D_{(2,0,-1,0,-1)}$	$D_{(1,0,-1,1,-1)}$	$D_{(2,0,-1,0,0)}$
$D_{(1,0,-1,1,0)}$	$D_{(1,0,-1,0,1)}$	$D_{(0,1,0,-1,-1)}$	$D_{(0,2,0,-1,-1)}$
$D_{(0,1,1,-1,-1)}$	$D_{(0,2,0,-1,0)}$	$D_{(0,1,1,-1,0)}$	$D_{(0,1,0,-1,1)}$
$D_{(1,0,0,-1,-1)}$	$D_{(2,0,0,-1,-1)}$	$D_{(1,0,1,-1,-1)}$	$D_{(2,0,0,-1,0)}$
$D_{(1,0,1,-1,0)}$	$D_{(1,0,0,-1,1)}$		

on a MPCP-blowup of Y° inducing 89 non-zero toric divisor classes on the mirror X°. The following 30 toric divisors of Y induce the trivial divisor class on X°

$D_{(1,-1,0,0,0)}$	$D_{(-1,1,0,0,0)}$	$D_{(-1,0,0,0,1)}$	$D_{(-1,0,0,0,0)}$	$D_{(-1,0,0,1,0)}$
$D_{(-1,0,1,0,0)}$	$D_{(0,-1,0,0,0)}$	$D_{(0,-1,0,0,1)}$	$D_{(0,-1,1,0,0)}$	$D_{(0,-1,0,1,0)}$
$D_{(0,0,0,0,-1)}$	$D_{(0,0,1,0,-1)}$	$D_{(0,0,0,1,-1)}$	$D_{(0,0,1,0,0)}$	$D_{(0,0,0,1,0)}$
$D_{(0,0,0,0,1)}$	$D_{(0,0,-1,0,0)}$	$D_{(0,0,-1,1,0)}$	$D_{(0,0,-1,0,1)}$	$D_{(0,0,0,-1,0)}$
$D_{(0,0,1,-1,0)}$	$D_{(0,0,0,-1,1)}$	$D_{(0,1,-1,0,0)}$	$D_{(1,0,-1,0,0)}$	$D_{(0,1,0,-1,0)}$
$D_{(1,0,0,-1,0)}$	$D_{(0,1,0,0,-1)}$	$D_{(1,0,0,0,-1)}$	$D_{(0,1,0,0,0)}$	$D_{(1,0,0,0,0)}$

Mirror special fiber The ideal I_0° of the monomial special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$ is generated by the following set of monomials in S°

$y_7\, y_1\, y_5\, y_{10}\, y_9\, y_{12}$	$y_6\, y_7\, y_8\, y_3\, y_5\, y_9$	$y_6\, y_7\, y_8\, y_5\, y_{10}\, y_9$
$y_1\, y_3\, y_4\, y_5\, y_9\, y_{12}$	$y_8\, y_3\, y_4\, y_5\, y_9\, y_{12}$	$y_8\, y_4\, y_5\, y_{10}\, y_9\, y_{12}$
$y_7\, y_1\, y_3\, y_5\, y_9\, y_{12}$	$y_6\, y_7\, y_1\, y_5\, y_{10}\, y_9$	$y_8\, y_2\, y_4\, y_5\, y_{10}\, y_{12}$
$y_7\, y_1\, y_2\, y_{11}\, y_{10}\, y_{12}$	$y_8\, y_2\, y_3\, y_4\, y_5\, y_{12}$	$y_7\, y_8\, y_2\, y_5\, y_{10}\, y_{12}$
$y_1\, y_4\, y_5\, y_{10}\, y_9\, y_{12}$	$y_7\, y_8\, y_2\, y_3\, y_5\, y_{12}$	$y_7\, y_1\, y_3\, y_{11}\, y_9\, y_{12}$
$y_6\, y_8\, y_2\, y_3\, y_4\, y_5$	$y_6\, y_7\, y_8\, y_2\, y_5\, y_{10}$	$y_6\, y_7\, y_1\, y_3\, y_5\, y_9$

$y_6\, y_7\, y_1\, y_2\, y_3\, y_5$	$y_8\, y_4\, y_{11}\, y_{10}\, y_9\, y_{12}$	$y_8\, y_3\, y_4\, y_{11}\, y_9\, y_{12}$
$y_7\, y_8\, y_3\, y_{11}\, y_9\, y_{12}$	$y_1\, y_2\, y_4\, y_5\, y_{10}\, y_{12}$	$y_1\, y_2\, y_3\, y_4\, y_5\, y_{12}$
$y_7\, y_1\, y_2\, y_5\, y_{10}\, y_{12}$	$y_7\, y_1\, y_2\, y_3\, y_5\, y_{12}$	$y_6\, y_8\, y_4\, y_5\, y_{10}\, y_9$
$y_6\, y_8\, y_4\, y_{11}\, y_{10}\, y_9$	$y_8\, y_2\, y_3\, y_4\, y_{11}\, y_{12}$	$y_6\, y_7\, y_1\, y_2\, y_{11}\, y_{10}$
$y_6\, y_8\, y_3\, y_4\, y_{11}\, y_9$	$y_6\, y_7\, y_8\, y_{11}\, y_{10}\, y_9$	$y_6\, y_7\, y_8\, y_3\, y_{11}\, y_9$
$y_7\, y_8\, y_2\, y_3\, y_{11}\, y_{12}$	$y_6\, y_7\, y_1\, y_2\, y_3\, y_{11}$	$y_6\, y_7\, y_1\, y_{11}\, y_{10}\, y_9$
$y_1\, y_4\, y_{11}\, y_{10}\, y_9\, y_{12}$	$y_6\, y_1\, y_4\, y_5\, y_{10}\, y_9$	$y_1\, y_3\, y_4\, y_{11}\, y_9\, y_{12}$
$y_6\, y_8\, y_2\, y_4\, y_{11}\, y_{10}$	$y_6\, y_8\, y_2\, y_3\, y_4\, y_{11}$	$y_1\, y_2\, y_3\, y_4\, y_{11}\, y_{12}$
$y_6\, y_7\, y_8\, y_2\, y_{11}\, y_{10}$	$y_6\, y_7\, y_8\, y_2\, y_3\, y_{11}$	$y_7\, y_1\, y_2\, y_3\, y_{11}\, y_{12}$
$y_6\, y_1\, y_3\, y_4\, y_5\, y_9$	$y_6\, y_8\, y_3\, y_4\, y_5\, y_9$	$y_6\, y_8\, y_2\, y_4\, y_5\, y_{10}$
$y_6\, y_1\, y_2\, y_3\, y_4\, y_5$	$y_8\, y_2\, y_4\, y_{11}\, y_{10}\, y_{12}$	$y_6\, y_7\, y_8\, y_2\, y_3\, y_5$
$y_6\, y_7\, y_1\, y_3\, y_{11}\, y_9$	$y_6\, y_7\, y_1\, y_2\, y_5\, y_{10}$	$y_7\, y_8\, y_5\, y_{10}\, y_9\, y_{12}$
$y_7\, y_8\, y_3\, y_5\, y_9\, y_{12}$	$y_6\, y_1\, y_2\, y_4\, y_{11}\, y_{10}$	$y_6\, y_1\, y_2\, y_3\, y_4\, y_{11}$
$y_7\, y_8\, y_2\, y_{11}\, y_{10}\, y_{12}$	$y_7\, y_8\, y_{11}\, y_{10}\, y_9\, y_{12}$	$y_6\, y_1\, y_2\, y_4\, y_5\, y_{10}$
$y_7\, y_1\, y_{11}\, y_{10}\, y_9\, y_{12}$	$y_6\, y_1\, y_4\, y_{11}\, y_{10}\, y_9$	$y_1\, y_2\, y_4\, y_{11}\, y_{10}\, y_{12}$
$y_6\, y_1\, y_3\, y_4\, y_{11}\, y_9$		

The Pic (Y°)-generated ideal

$$J_0^\circ = \left\langle \ y_6\, y_1\, y_2\, y_3\, y_4\, y_5 \quad y_7\, y_8\, y_{11}\, y_{10}\, y_9\, y_{12} \ \right\rangle$$

defines the same subvariety X_0° of the toric variety Y°, and $J_0^{\circ\Sigma} = I_0^\circ$. Passing from J_0° to $J_0^{\circ\Sigma}$ is the non-simplicial toric analogue of saturation. The complex $B(I)^*$ labeled by the variables of the Cox ring S° of Y°, as written in the last section, is the complex $SP(I_0^\circ)$ of prime ideals of the toric strata of the special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$, i.e., the primary decomposition of I_0° is

$$I_0^\circ = \ \langle y_{11}, y_5\rangle \cap \langle y_7, y_4\rangle \cap \langle y_3, y_{10}\rangle \cap \langle y_6, y_{12}\rangle \cap \langle y_1, y_8\rangle \cap \langle y_2, y_9\rangle$$

Each facet $F \in B(I)$ corresponds to one of these ideals and this ideal is generated by the facets of ∇ containing F.

Covering structure in the deformation complex of the degeneration $\mathfrak{X}$
According to the local reduced Gröbner basis each face of the complex of deformations dual $(B(I))$ decomposes into 2 polytopes forming a 2 : 1 trivial covering of $B(I)$

[],

[],

$$[[\langle y_6\rangle, \langle y_{12}\rangle] \mapsto [3,4,5,6,7,8]^\vee,$$
$$[\langle y_5\rangle, \langle y_{11}\rangle] \mapsto [2,4,6,8]^\vee, ...],$$
$$[[\langle y_2, y_4\rangle, \langle y_7, y_9\rangle] \mapsto [1,3,7]^\vee,$$
$$[\langle y_6, y_2\rangle, \langle y_{12}, y_9\rangle] \mapsto [3,4,7,8]^\vee,$$
$$...],$$
$$[[\langle y_1, y_6, y_4\rangle, \langle y_7, y_8, y_{12}\rangle] \mapsto [5,7]^\vee,$$
$$...],$$
$$[[\langle y_1, y_2, y_3, y_4\rangle, \langle y_7, y_8, y_9, y_{10}\rangle] \mapsto [1]^\vee,$$
$$...],$$
$$[]$$

Here the faces $F \in B(I)$ are specified both via the vertices of F^* labeled by the variables of S° and by the numbering of the vertices of $B(I)$ chosen above. This covering has 2 sheets forming the complexes

$$[],$$
$$[],$$
$$[\langle y_{12}\rangle, \langle y_{11}\rangle, ...],$$
$$[\langle y_7, y_9\rangle, \langle y_{12}, y_9\rangle, ...],$$
$$[\langle y_7, y_8, y_{12}\rangle, ...],$$
$$[\langle y_7, y_8, y_9, y_{10}\rangle, ...],$$
$$[]$$

$$[],$$
$$[],$$
$$[\langle y_6\rangle, \langle y_5\rangle, ...],$$
$$[\langle y_2, y_4\rangle, \langle y_6, y_2\rangle, ...],$$
$$[\langle y_1, y_6, y_4\rangle, ...],$$
$$[\langle y_1, y_2, y_3, y_4\rangle, ...],$$
$$[]$$

with F-vector

Dimension	Number of faces	F-vector

0	6	$(1,1,0,0,0,0,0)$	point
1	14	$(1,2,1,0,0,0,0)$	edge
2	16	$(1,3,3,1,0,0,0)$	triangle
3	8	$(1,4,6,4,1,0,0)$	tetrahedron

Writing the vertices of the faces as deformations the covering is given by

$$
\begin{aligned}
&[], [],\\
&[[\left\langle \frac{x_3^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_3^3}{x_4\,x_0\,x_5} \right\rangle] \mapsto [3,4,5,6,7,8]^\vee, [\left\langle \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_1^4}{x_0\,x_3\,x_4\,x_5} \right\rangle] \mapsto [2,4,6,8]^\vee,\\
&...],\\
&[[\left\langle \frac{x_5^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^4}{x_0\,x_3\,x_4\,x_5}, \frac{{x_5}^3}{x_0\,x_3\,x_4} \right\rangle] \mapsto [1,3,7]^\vee,\\
&[\left\langle \frac{x_3^2}{x_1\,x_2}, \frac{{x_5}^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_3^3}{x_4\,x_0\,x_5}, \frac{{x_5}^3}{x_0\,x_3\,x_4} \right\rangle] \mapsto [3,4,7,8]^\vee,\\
&...],\\
&[[\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_3}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^4}{x_0\,x_3\,x_4\,x_5}, \frac{{x_4}^3}{x_0\,x_3\,x_5}, \frac{{x_3}^3}{x_4\,x_0\,x_5} \right\rangle] \mapsto [5,7]^\vee,\\
&...],\\
&[[\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_5}^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^4}{x_0\,x_3\,x_4\,x_5}, \frac{{x_4}^3}{x_0\,x_3\,x_5}, \frac{{x_5}^3}{x_0\,x_3\,x_4}, \frac{{x_0}^3}{x_3\,x_4\,x_5} \right\rangle] \mapsto [1]^\vee,\\
&...],\\
&[]
\end{aligned}
$$

with the 2 sheets forming the complexes

$$
\begin{aligned}
&[],\\
&[],\\
&[\left\langle \frac{x_3^3}{x_4\,x_0\,x_5} \right\rangle, \left\langle \frac{x_1^4}{x_0\,x_3\,x_4\,x_5} \right\rangle, ...],\\
&[\left\langle \frac{x_2^4}{x_0\,x_3\,x_4\,x_5}, \frac{{x_5}^3}{x_0\,x_3\,x_4} \right\rangle, \left\langle \frac{x_3^3}{x_4\,x_0\,x_5}, \frac{{x_5}^3}{x_0\,x_3\,x_4} \right\rangle, ...],\\
&[\left\langle \frac{x_2^4}{x_0\,x_3\,x_4\,x_5}, \frac{{x_4}^3}{x_0\,x_3\,x_5}, \frac{{x_3}^3}{x_4\,x_0\,x_5} \right\rangle, ...],\\
&[\left\langle \frac{x_2^4}{x_0\,x_3\,x_4\,x_5}, \frac{{x_4}^3}{x_0\,x_3\,x_5}, \frac{{x_5}^3}{x_0\,x_3\,x_4}, \frac{{x_0}^3}{x_3\,x_4\,x_5} \right\rangle, ...],\\
&[]
\end{aligned}
$$

$$[],$$
$$[],$$
$$[\left\langle \frac{x_3^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_1}{x_2} \right\rangle, ...],$$
$$[\left\langle \frac{x_5^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_3^2}{x_1\,x_2}, \frac{{x_5}^2}{x_1\,x_2} \right\rangle, ...],$$
$$[\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_3}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, ...],$$
$$[\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{{x_5}^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, ...],$$
$$[]$$

Note that the torus invariant basis of deformations corresponding to a Bergman face is given by the set of all lattice points of the polytope specified above.

Limit map The limit map $\lim : B(I) \to \operatorname{Poset}(\Delta)$ associates to a face F of $B(I)$ the face of Δ formed by the limit points of arcs lying over the weight vectors $w \in F$, i.e., with lowest order term t^w. Labeling the faces of the Bergman complex $B(I) \subset \operatorname{Poset}(\nabla)$ and the faces of $\operatorname{Poset}(\Delta)$ by the corresponding dual faces of ∇^* and Δ^*, hence considering the limit map $\lim : B(I) \to \operatorname{Poset}(\Delta)$ as a map $B(I)^* \to \operatorname{Poset}(\Delta^*)$, the limit correspondence is given by

$$[],$$
$$[\langle y_1, y_2, y_3, y_7, y_8, y_9, y_{10}, y_4 \rangle \mapsto \langle x_1, x_3 \rangle, ...],$$
$$[\langle y_1, y_6, y_7, y_8, y_{12}, y_4 \rangle \mapsto \langle x_1, x_5, x_0 \rangle, ...],$$
$$[\langle y_2, y_7, y_9, y_4 \rangle \mapsto \langle x_1, x_3, x_4, x_0 \rangle, \langle y_6, y_2, y_{12}, y_9 \rangle \mapsto \langle x_1, x_2, x_4, x_0 \rangle, ...],$$
$$[\langle y_6, y_{12} \rangle \mapsto \langle x_1, x_2, x_4, x_5, x_0 \rangle, \langle y_{11}, y_5 \rangle \mapsto \langle x_2, x_3, x_4, x_5, x_0 \rangle, ...],$$
$$[],$$
$$[]$$

The image of the limit map coincides with the image of μ and with the Bergman complex of the mirror, i.e., $\lim(B(I)) = \mu(B(I)) = B(I^\circ)$.

Mirror complex Numbering the vertices of the mirror complex $\mu(B(I))$ as

$$\begin{array}{ll} 1 = (5,-1,-1,-1,-1) & 2 = (-1,5,-1,-1,-1) \\ 3 = (-1,-1,5,-1,-1) & 4 = (-1,-1,-1,5,-1) \\ 5 = (-1,-1,-1,-1,5) & 6 = (-1,-1,-1,-1,-1) \end{array}$$

$\mu(B(I)) \subset \mathrm{Poset}(\Delta)$ is

[],

[[3], [1], [5], [4], [2], [6]],

[[3, 5], [3, 4], [4, 6], [2, 5], [2, 6], [4, 5], [2, 4], [2, 3], [1, 6], [1, 5], [1, 3], [1, 4], [5, 6], [3, 6]],

[[2, 3, 4], [1, 3, 4], [3, 4, 6], [3, 4, 5], [2, 5, 6], [2, 4, 6], [2, 4, 5], [1, 5, 6], [1, 4, 6], [1, 4, 5], [4, 5, 6], [3, 5, 6], [2, 3, 6], [2, 3, 5], [1, 3, 6], [1, 3, 5]],

[[2, 4, 5, 6], [1, 4, 5, 6], [2, 3, 5, 6], [1, 3, 5, 6], [2, 3, 4, 6], [1, 3, 4, 6], [2, 3, 4, 5], [1, 3, 4, 5]],

[],

[]

The ordering of the faces of $\mu(B(I))$ is compatible with above ordering of the faces of $B(I)$. The F-vectors of the faces of $\mu(B(I))^*$ are

Dimension	Number of faces	F-vector	
1	8	(1, 2, 1, 0, 0, 0, 0)	edge
2	16	(1, 3, 3, 1, 0, 0, 0)	triangle
3	14	(1, 4, 6, 4, 1, 0, 0)	tetrahedron
4	6	(1, 5, 10, 10, 5, 1, 0)	

The first order deformations of the mirror special fiber X_0° correspond to the lattice points of the dual complex $(\mu(B(I)))^* \subset \mathrm{Poset}(\Delta^*)$ of the mirror complex. We label the first order deformations of X_0° corresponding to vertices of Δ^* by the homogeneous coordinates of Y

$$\begin{array}{ll} x_1 = x_{(1,0,0,0,0)} = \frac{y_5\, y_{11}{}^4}{y_1\, y_2\, y_3\, y_4\, y_6} & x_2 = x_{(0,1,0,0,0)} = \frac{y_4\, y_7{}^4}{y_1\, y_2\, y_3\, y_5\, y_6} \\ x_3 = x_{(0,0,1,0,0)} = \frac{y_6^2 y_{12}{}^3}{y_7\, y_8\, y_9\, y_{10}\, y_{11}} & x_4 = x_{(0,0,0,1,0)} = \frac{y_1^2 y_8{}^3}{y_7\, y_9\, y_{10}\, y_{11}\, y_{12}} \\ x_5 = x_{(0,0,0,0,1)} = \frac{y_2^2 y_9{}^3}{y_7\, y_8\, y_{10}\, y_{11}\, y_{12}} & x_0 = x_{(-1,-1,-1,-1,-1)} = \frac{y_3^2 y_{10}{}^3}{y_7\, y_8\, y_9\, y_{11}\, y_{12}} \end{array}$$

So writing the vertices of the faces of $(\mu(B(I)))^*$ as homogeneous coordinates of Y, the complex $(\mu(B(I)))^*$ is given by

$[]$,

$[]$,

$[\langle x_1, x_3\rangle, \langle x_2, x_3\rangle, \langle x_1, x_4\rangle, \langle x_2, x_4\rangle, \langle x_1, x_5\rangle, \langle x_2, x_5\rangle, \langle x_1, x_0\rangle,$
$\langle x_2, x_0\rangle]$,

$[\langle x_1, x_5, x_0\rangle, \langle x_2, x_5, x_0\rangle, \langle x_1, x_2, x_5\rangle, \langle x_1, x_2, x_0\rangle, \langle x_1, x_3, x_4\rangle, \langle x_1, x_3, x_5\rangle,$
$\langle x_1, x_3, x_0\rangle, \langle x_2, x_3, x_4\rangle, \langle x_2, x_3, x_5\rangle, \langle x_2, x_3, x_0\rangle, \langle x_1, x_2, x_3\rangle, \langle x_1, x_2, x_4\rangle,$
$\langle x_1, x_4, x_5\rangle, \langle x_1, x_4, x_0\rangle, \langle x_2, x_4, x_5\rangle, \langle x_2, x_4, x_0\rangle]$,

$[\langle x_1, x_2, x_4, x_0\rangle, \langle x_1, x_2, x_5, x_0\rangle, \langle x_1, x_2, x_3, x_5\rangle, \langle x_1, x_3, x_4, x_0\rangle, \langle x_1, x_3, x_4, x_5\rangle,$
$\langle x_1, x_2, x_3, x_0\rangle, \langle x_1, x_3, x_5, x_0\rangle, \langle x_1, x_4, x_5, x_0\rangle, \langle x_2, x_3, x_4, x_5\rangle, \langle x_2, x_3, x_4, x_0\rangle,$
$\langle x_2, x_4, x_5, x_0\rangle, \langle x_2, x_3, x_5, x_0\rangle, \langle x_1, x_2, x_3, x_4\rangle, \langle x_1, x_2, x_4, x_5\rangle]$,

$[\langle x_1, x_2, x_4, x_5, x_0\rangle, \langle x_2, x_3, x_4, x_5, x_0\rangle, \langle x_1, x_2, x_3, x_4, x_0\rangle, \langle x_1, x_2, x_3, x_5, x_0\rangle,$
$\langle x_1, x_3, x_4, x_5, x_0\rangle, \langle x_1, x_2, x_3, x_4, x_5\rangle]$,

$[]$

The complex $\mu(B(I))^*$ labeled by the variables of the Cox ring S gives the ideals of the toric strata of the special fiber X_0 of $\mathfrak{X}$, i.e., the complex $SP(I_0)$, so in particular the primary decomposition of I_0 is

$$I_0 = \begin{array}{l} \langle x_2, x_0\rangle \cap \langle x_1, x_0\rangle \cap \langle x_1, x_5\rangle \cap \langle x_2, x_5\rangle \cap \langle x_1, x_3\rangle \cap \langle x_2, x_3\rangle \cap \\ \cap \langle x_1, x_4\rangle \cap \langle x_2, x_4\rangle \end{array}$$

Covering structure in the deformation complex of the mirror degeneration Each face of the complex of deformations $(\mu(B(I)))^*$ of the mirror special fiber X_0° decomposes as the convex hull of 2 polytopes forming a $2:1$ trivial covering of $\mu(B(I))^\vee$

$[]$,

$[]$,

$[[\langle x_1\rangle, \langle x_3\rangle] \mapsto \langle x_1, x_3\rangle^{*\vee} = [2,4,5,6]^\vee$,
...],

$[[\langle x_1\rangle, \langle x_5, x_0\rangle] \mapsto \langle x_1, x_5, x_0\rangle^{*\vee} = [2,3,4]^\vee$,
...],

$[[\langle x_1\rangle, \langle x_3, x_4, x_0\rangle] \mapsto \langle x_1, x_3, x_4, x_0\rangle^{*\vee} = [2,5]^\vee$,
$[\langle x_1, x_2\rangle, \langle x_4, x_0\rangle] \mapsto \langle x_1, x_2, x_4, x_0\rangle^{*\vee} = [3,5]^\vee$,
...],

$$[[\langle x_1, x_2\rangle, \langle x_4, x_5, x_0\rangle] \mapsto \langle x_1, x_2, x_4, x_5, x_0\rangle^{*\vee} = [3]^\vee,$$
$$[\langle x_2\rangle, \langle x_3, x_4, x_5, x_0\rangle] \mapsto \langle x_2, x_3, x_4, x_5, x_0\rangle^{*\vee} = [1]^\vee,$$
$$...],$$
$$[]$$

Due to the singularities of Y° this covering involves degenerate faces, i.e., faces $G \mapsto F^\vee$ with $\dim(G) < \dim(F^\vee)$.

Mirror degeneration The space of first order deformations of X_0° in the mirror degeneration $\mathfrak{X}^\circ$ has dimension 6 and the deformations represented by the monomials

$$\left\{ \begin{array}{ccccc} \frac{y_4\, y_7{}^4}{y_1\, y_2\, y_3\, y_5\, y_6} & \frac{y_2^2 y_9{}^3}{y_7\, y_8\, y_{10}\, y_{11}\, y_{12}} & \frac{y_1^2 y_8{}^3}{y_7\, y_9\, y_{10}\, y_{11}\, y_{12}} & \frac{y_6^2 y_{12}{}^3}{y_7\, y_8\, y_9\, y_{10}\, y_{11}} & \frac{y_5\, y_{11}{}^4}{y_1\, y_2\, y_3\, y_4\, y_6} \\ \frac{y_3^2 y_{10}{}^3}{y_7\, y_8\, y_9\, y_{11}\, y_{12}} \end{array} \right\}$$

form a torus invariant basis. The number of lattice points of the dual of the mirror complex of I relates to the dimension $h^{1,2}(X^\circ)$ of complex moduli space of the generic fiber X° of $\mathfrak{X}^\circ$ and to the dimension $h^{1,1}(X)$ of the Kähler moduli space of the generic fiber X of $\mathfrak{X}$ via

$$\begin{aligned} |\mathrm{supp}\,((\mu(B(I)))^*) \cap N| &= 6 = 5 + 1 \\ &= \dim(\mathrm{Aut}(Y^\circ)) + h^{1,2}(X^\circ) = \dim(T) + h^{1,1}(X) \end{aligned}$$

The mirror degeneration $\mathfrak{X}^\circ \subset Y^\circ \times \mathrm{Spec}\,\mathbb{C}[t]$ of $\mathfrak{X}$ is given by the ideal $I^\circ \subset S^\circ \otimes \mathbb{C}[t]$ generated by

$$t\,(s_4\, y_1^2 y_8^4 + s_5\, y_2^2 y_9^4 + s_6\, y_3^2 y_{10}^4 + s_2\, y_6^2 y_{12}^4) + y_7\, y_8\, y_{11}\, y_{10}\, y_9\, y_{12}$$
$$t\,(s_1\, y_4^2 y_7^4 + s_3\, y_5^2 y_{11}^4) + y_6\, y_1\, y_2\, y_3\, y_4\, y_5$$
$$\begin{array}{ll} ts_4\, y_1^3 y_8^3 + y_7\, y_1\, y_{11}\, y_{10}\, y_9\, y_{12} & ts_5\, y_2^3 y_9^3 + y_7\, y_8\, y_2\, y_{11}\, y_{10}\, y_{12} \\ ts_1\, y_7^5 y_4 + y_6\, y_7\, y_1\, y_2\, y_3\, y_5 & ts_2\, y_6^3 y_{12}^3 + y_6\, y_7\, y_8\, y_{11}\, y_{10}\, y_9 \\ ts_6\, y_3^3 y_{10}^3 + y_7\, y_8\, y_3\, y_{11}\, y_9\, y_{12} & ts_3\, y_{11}^5 y_5 + y_6\, y_1\, y_2\, y_3\, y_4\, y_{11} \end{array}$$

and 56 monomials of degree 6

The ideal J° which is $\mathrm{Pic}(Y^\circ)$-generated by

$$\left\{ \begin{array}{l} t(s_1\, y_4^2 y_7^4 + s_3\, y_5^2 y_{11}^4) + y_6\, y_1\, y_2\, y_3\, y_4\, y_5, \\ t(s_4\, y_1^2 y_8^4 + s_5\, y_2^2 y_9^4 + s_6\, y_3^2 y_{10}^4 + s_2\, y_6^2 y_{12}^4) + y_7\, y_8\, y_{11}\, y_{10}\, y_9\, y_{12} \end{array} \right\}$$

defines a flat affine cone inducing $\mathfrak{X}^\circ$.

Contraction of the mirror degeneration In the following we give a birational map relating the degeneration $\mathfrak{X}^\circ$ to a Greene-Plesser type orbifolding mirror family by contracting divisors on Y°. See also Section 9.13 below. In order to contract the divisors

$$\begin{array}{ll} y_4 = y_{(-1,1,0,0,0)} = \frac{x_2}{x_1} & y_5 = y_{(1,-1,0,0,0)} = \frac{x_1}{x_2} \\ y_8 = y_{(0,0,-1,3,-1)} = \frac{x_4^3}{x_0\,x_3\,x_5} & y_9 = y_{(0,0,-1,-1,3)} = \frac{x_5^3}{x_0\,x_3\,x_4} \\ y_{10} = y_{(0,0,-1,-1,-1)} = \frac{x_0^3}{x_3\,x_4\,x_5} & y_{12} = y_{(0,0,3,-1,-1)} = \frac{x_3^3}{x_4\,x_0\,x_5} \end{array}$$

consider the $\mathbb{Q}$-factorial toric Fano variety $\hat{Y}^\circ = X(\hat{\Sigma}^\circ)$, where $\hat{\Sigma}^\circ = \Sigma(\hat{P}^\circ) \subset M_\mathbb{R}$ is the fan over the Fano polytope $\hat{P}^\circ \subset M_\mathbb{R}$ given as the convex hull of the remaining vertices of $P^\circ = \nabla^*$ corresponding to the Cox variables

$$\begin{array}{ll} y_6 = y_{(-1,-1,2,0,0)} = \frac{x_3^2}{x_1\,x_2} & y_{11} = y_{(4,0,-1,-1,-1)} = \frac{x_1^4}{x_0\,x_3\,x_4\,x_5} \\ y_2 = y_{(-1,-1,0,0,2)} = \frac{x_5^2}{x_1\,x_2} & y_1 = y_{(-1,-1,0,2,0)} = \frac{x_4^2}{x_1\,x_2} \\ y_7 = y_{(0,4,-1,-1,-1)} = \frac{x_2^4}{x_0\,x_3\,x_4\,x_5} & y_3 = y_{(-1,-1,0,0,0)} = \frac{x_0^2}{x_1\,x_2} \end{array}$$

of the toric variety $\hat{Y}^\circ$ with Cox ring

$$\hat{S}^\circ = \mathbb{C}[y_6, y_{11}, y_2, y_1, y_7, y_3]$$

The Cox variables of $\hat{Y}^\circ$ correspond to the set of Fermat deformations of $\mathfrak{X}$. Let

$$Y^\circ = X\left(\Sigma^\circ\right) \to X(\hat{\Sigma}^\circ) = \hat{Y}^\circ$$

be a birational map from Y° to the toric Fano variety $\hat{Y}^\circ$, which contracts the divisors of the rays $\Sigma^\circ(1) - \hat{\Sigma}^\circ(1)$ corresponding to the Cox variables

$$y_4 \quad y_5 \quad y_8 \quad y_9 \quad y_{10} \quad y_{12}$$

Representing $\hat{Y}^\circ$ as a quotient we have

$$\hat{Y}^\circ = \left(\mathbb{C}^6 - V\left(B(\hat{\Sigma}^\circ)\right)\right)/\hat{G}^\circ$$

with

$$\hat{G}^\circ = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8 \times (\mathbb{C}^*)^1$$

acting via

$$\xi y = \left(u_1\,v_1 \cdot y_6,\ u_1\,u_3^7\,v_1 \cdot y_{11},\ u_2\,v_1 \cdot y_2,\ u_1\,u_2\,u_3^4\,v_1 \cdot y_1,\ u_1\,u_3\,v_1 \cdot y_7,\ v_1 \cdot y_3\right)$$

for $\xi = (u_1, u_2, u_3, v_1) \in \hat{G}^\circ$ and $y \in \mathbb{C}^6 - V\left(B(\hat{\Sigma}^\circ)\right)$. Hence with the group

$$\hat{H}^\circ = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$$

of order 32 the toric variety $\hat{Y}^\circ$ is the quotient

$$\hat{Y}^\circ = \mathbb{P}^5 / \hat{H}^\circ$$

of projective space $\mathbb{P}^5$. The mirror degeneration $\mathfrak{X}^\circ$ induces via $Y^\circ \to \hat{Y}^\circ$ a degeneration $\hat{\mathfrak{X}}^\circ \subset \hat{Y}^\circ \times \operatorname{Spec} \mathbb{C}[t]$ given by the ideal $\hat{I}^\circ \subset \langle y_6, y_{11}, y_2, y_1, y_7, y_3 \rangle \subset \hat{S}^\circ \otimes \mathbb{C}[t]$ generated by the Fermat-type equations

$$\left\{ \begin{array}{l} y_1\, y_2\, y_3\, y_6 + t(s_6\, y_7^4 + s_5\, y_{11}^4), \\ y_7\, y_{11} + t(s_1\, y_1^2 + s_3\, y_2^2 + s_4\, y_3^2 + s_2\, y_6^2) \end{array} \right\}$$

The special fiber $\hat{X}_0^\circ \subset \hat{Y}^\circ$ of $\hat{\mathfrak{X}}^\circ$ is cut out by the monomial ideal

$$\hat{I}_0^\circ \subset B(\hat{\Sigma}^\circ) = \langle y_6, y_{11}, y_2, y_1, y_7, y_3 \rangle \subset \hat{S}^\circ$$

generated by

$$\left\{ \; y_1\, y_2\, y_3\, y_6 \quad y_7\, y_{11} \; \right\}$$

The complex $SP(\hat{I}_0^\circ)$ of prime ideals of the toric strata of $\hat{X}_0^\circ$ is

$[]$,

$[]$,

$[\langle y_1, y_7 \rangle, \langle y_6, y_{11} \rangle, \langle y_{11}, y_2 \rangle, \langle y_{11}, y_1 \rangle, \langle y_7, y_3 \rangle, \langle y_{11}, y_3 \rangle,$
$\langle y_6, y_7 \rangle, \langle y_2, y_7 \rangle]$,

$[\langle y_{11}, y_2, y_3 \rangle, \langle y_{11}, y_2, y_1 \rangle, \langle y_6, y_{11}, y_1 \rangle, \langle y_{11}, y_1, y_7 \rangle, \langle y_6, y_{11}, y_7 \rangle,$
$\langle y_6, y_2, y_7 \rangle, \langle y_{11}, y_1, y_3 \rangle, \langle y_1, y_7, y_3 \rangle, \langle y_6, y_1, y_7 \rangle, \langle y_2, y_1, y_7 \rangle,$
$\langle y_2, y_7, y_3 \rangle, \langle y_6, y_{11}, y_2 \rangle, \langle y_{11}, y_2, y_7 \rangle, \langle y_{11}, y_7, y_3 \rangle, \langle y_6, y_7, y_3 \rangle,$
$\langle y_6, y_{11}, y_3 \rangle]$,

$[\langle y_6, y_{11}, y_7, y_3 \rangle, \langle y_6, y_{11}, y_2, y_1 \rangle, \langle y_2, y_1, y_7, y_3 \rangle, \langle y_{11}, y_1, y_7, y_3 \rangle,$
$\langle y_{11}, y_2, y_1, y_3 \rangle, \langle y_6, y_2, y_7, y_3 \rangle, \langle y_6, y_1, y_7, y_3 \rangle, \langle y_6, y_{11}, y_2, y_7 \rangle,$
$\langle y_6, y_{11}, y_2, y_3 \rangle, \langle y_{11}, y_2, y_7, y_3 \rangle, \langle y_6, y_{11}, y_1, y_7 \rangle, \langle y_6, y_2, y_1, y_7 \rangle,$
$\langle y_{11}, y_2, y_1, y_7 \rangle, \langle y_6, y_{11}, y_1, y_3 \rangle]$,

$[\langle y_{11}, y_2, y_1, y_7, y_3 \rangle, \langle y_6, y_2, y_1, y_7, y_3 \rangle, \langle y_6, y_{11}, y_1, y_7, y_3 \rangle, \langle y_6, y_{11}, y_2, y_7, y_3 \rangle,$
$\langle y_6, y_{11}, y_2, y_1, y_3 \rangle, \langle y_6, y_{11}, y_2, y_1, y_7 \rangle]$,

$[]$

so $\hat{I}_0^\circ$ has the primary decomposition

$$\hat{I}_0^\circ = \begin{array}{l} \langle y_{11}, y_1 \rangle \cap \langle y_7, y_3 \rangle \cap \langle y_{11}, y_3 \rangle \cap \langle y_{11}, y_2 \rangle \cap \langle y_1, y_7 \rangle \cap \langle y_6, y_{11} \rangle \cap \\ \cap \langle y_6, y_7 \rangle \cap \langle y_2, y_7 \rangle \end{array}$$

The Bergman subcomplex $B(I) \subset \text{Poset}(\nabla)$ induces a subcomplex of $\text{Poset}(\hat{\nabla})$, $\hat{\nabla} = (\hat{P}^\circ)^*$ corresponding to $\hat{I}_0^\circ$. Indexing of the vertices of $\hat{\nabla}$ by

$$\begin{array}{ll} 1 = \left(\frac{1}{2}, \frac{1}{2}, 3, 0, 0\right) & 2 = \left(\frac{5}{4}, -\frac{1}{4}, 0, 0, 0\right) \\ 3 = \left(\frac{1}{2}, \frac{1}{2}, 0, 0, 3\right) & 4 = \left(\frac{1}{2}, \frac{1}{2}, 0, 3, 0\right) \\ 5 = \left(-\frac{1}{4}, \frac{5}{4}, 0, 0, 0\right) & 6 = \left(-\frac{5}{2}, -\frac{5}{2}, -3, -3, -3\right) \end{array}$$

this complex is given by

$$\left\{ \begin{array}{l} [], \\ [[1], [2], [3], [4], [5], [6]], \\ [[3,4], [5,6], [1,2], [1,3], [1,5], [2,4], [2,3], [4,6], [4,5], [1,4], [3,6], \\ [2,6], [1,6], [3,5]], \\ [[1,4,5], [1,5,6], [3,5,6], [1,3,6], [3,4,6], [2,4,6], [1,3,5], [1,2,3], \\ [2,3,6], [1,2,6], [1,2,4], [4,5,6], [1,4,6], [1,3,4], [2,3,4], [3,4,5]], \\ [[1,2,3,6], [3,4,5,6], [1,4,5,6], [1,3,5,6], [1,2,3,4], [1,3,4,5], \\ [2,3,4,6], [1,2,4,6]], \\ [], \\ [] \end{array} \right\}$$

8.12.4 The Calabi-Yau threefold given as the complete intersection of two generic cubics in $\mathbb{P}^5$

Setup Let $Y = \mathbb{P}^5 = X(\Sigma)$, $\Sigma = \Sigma(P) = NF(\Delta) \subset N_{\mathbb{R}}$ with the Fano polytope $P = \Delta^*$ given by

$$\Delta = \text{convexhull} \left(\begin{array}{ll} (5,-1,-1,-1,-1) & (-1,5,-1,-1,-1) \\ (-1,-1,5,-1,-1) & (-1,-1,-1,5,-1) \\ (-1,-1,-1,-1,5) & (-1,-1,-1,-1,-1) \end{array} \right) \subset M_{\mathbb{R}}$$

and let

$$S = \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]$$

be the Cox ring of Y with the variables

$$\begin{array}{ll} x_1 = x_{(1,0,0,0,0)} & x_2 = x_{(0,1,0,0,0)} \\ x_3 = x_{(0,0,1,0,0)} & x_4 = x_{(0,0,0,1,0)} \\ x_5 = x_{(0,0,0,0,1)} & x_0 = x_{(-1,-1,-1,-1,-1)} \end{array}$$

associated to the rays of Σ. Consider the degeneration $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[t]$ of complete intersection Calabi-Yau 3-folds of type $(3,3)$ with monomial special fiber

$$I_0 = \left\langle \; x_1 \, x_2 \, x_3 \quad x_0 \, x_4 \, x_5 \; \right\rangle$$

The degeneration is given by the ideal $I \subset S \otimes \mathbb{C}[t]$ with I_0-reduced generators

$$\left\{ \begin{array}{l} x_1 \, x_2 \, x_3 + t(s_1 \, x_1^3 + \ldots + s_{21} \, x_2^3 + \ldots + s_{36} \, x_3^3 + \ldots + s_{46} \, x_4^3 + \ldots + s_{51} \, x_5^3 + \\ \ldots + s_{54} \, x_0^3), \\ x_0 \, x_4 \, x_5 + t(s_{55} \, x_1^3 + \ldots + s_{75} \, x_2^3 + \ldots + s_{90} \, x_3^3 + \ldots + s_{100} \, x_4^3 + \ldots + s_{105} \, x_5^3 + \\ \ldots + s_{108} \, x_0^3) \end{array} \right\}$$

Special fiber Gröbner cone The space of first order deformations of $\mathfrak{X}$ has dimension 108 and the deformations represented by the Cox Laurent monomials

$\frac{x_0^3}{x_1 x_2 x_3}$	$\frac{x_5^3}{x_1 x_2 x_3}$	$\frac{x_4^3}{x_1 x_2 x_3}$	$\frac{x_3^2 x_1}{x_0 x_4 x_5}$	$\frac{x_3 x_1^2}{x_0 x_4 x_5}$	$\frac{x_3 x_2^2}{x_0 x_4 x_5}$	$\frac{x_3^2 x_2}{x_0 x_4 x_5}$	$\frac{x_5^2 x_0}{x_1 x_2 x_3}$
$\frac{x_2^2 x_1}{x_0 x_4 x_5}$	$\frac{x_5 x_0^2}{x_1 x_2 x_3}$	$\frac{x_2 x_1^2}{x_0 x_4 x_5}$	$\frac{x_4 x_5^2}{x_1 x_2 x_3}$	$\frac{x_4^2 x_5}{x_1 x_2 x_3}$	$\frac{x_4^2 x_0}{x_1 x_2 x_3}$	$\frac{x_4 x_0^2}{x_1 x_2 x_3}$	$\frac{x_3^3}{x_0 x_4 x_5}$
$\frac{x_1^3}{x_0 x_4 x_5}$	$\frac{x_2^3}{x_0 x_4 x_5}$	$\frac{x_2 x_3}{x_4 x_0}$	$\frac{x_3 x_4}{x_1 x_2}$	$\frac{x_4 x_0}{x_1 x_2}$	$\frac{x_1 x_3}{x_4 x_0}$	$\frac{x_5 x_2}{x_1 x_3}$	$\frac{x_2 x_0}{x_1 x_3}$
$\frac{x_1 x_4}{x_2 x_3}$	$\frac{x_2 x_4}{x_1 x_3}$	$\frac{x_4 x_0}{x_1 x_3}$	$\frac{x_4 x_5}{x_2 x_3}$	$\frac{x_4 x_5}{x_1 x_3}$	$\frac{x_1 x_2}{x_0 x_5}$	$\frac{x_1 x_2}{x_4 x_5}$	$\frac{x_1 x_0}{x_2 x_3}$
$\frac{x_5 x_1}{x_2 x_3}$	$\frac{x_2^2}{x_4 x_0}$	$\frac{x_3^2}{x_4 x_0}$	$\frac{x_4^2}{x_1 x_2}$	$\frac{x_0^2}{x_1 x_2}$	$\frac{x_0 x_5}{x_2 x_3}$	$\frac{x_0 x_5}{x_1 x_3}$	$\frac{x_4 x_0}{x_2 x_3}$
$\frac{x_1^2}{x_4 x_0}$	$\frac{x_1^2}{x_2 x_3}$	$\frac{x_2^2}{x_1 x_3}$	$\frac{x_3^2}{x_1 x_2}$	$\frac{x_0^2}{x_2 x_3}$	$\frac{x_5^2}{x_2 x_3}$	$\frac{x_4^2}{x_2 x_3}$	$\frac{x_4^2}{x_1 x_3}$
$\frac{x_5^2}{x_1 x_3}$	$\frac{x_0^2}{x_1 x_3}$	$\frac{x_4 x_5}{x_1 x_2}$	$\frac{x_3 x_4}{x_0 x_5}$	$\frac{x_5 x_3}{x_1 x_2}$	$\frac{x_5 x_3}{x_4 x_0}$	$\frac{x_2^2}{x_0 x_5}$	$\frac{x_1 x_3}{x_4 x_5}$
$\frac{x_1 x_0}{x_4 x_5}$	$\frac{x_5 x_2}{x_4 x_0}$	$\frac{x_2 x_3}{x_4 x_5}$	$\frac{x_2 x_0}{x_4 x_5}$	$\frac{x_1 x_4}{x_0 x_5}$	$\frac{x_1 x_3}{x_0 x_5}$	$\frac{x_2 x_4}{x_0 x_5}$	$\frac{x_2 x_3}{x_0 x_5}$
$\frac{x_3 x_0}{x_1 x_2}$	$\frac{x_3 x_0}{x_4 x_5}$	$\frac{x_0 x_5}{x_1 x_2}$	$\frac{x_5 x_1}{x_4 x_0}$	$\frac{x_1^2}{x_4 x_5}$	$\frac{x_2^2}{x_4 x_5}$	$\frac{x_1^2}{x_0 x_5}$	$\frac{x_3^2}{x_4 x_5}$
$\frac{x_0^2}{x_4 x_5}$	$\frac{x_4^2}{x_0 x_5}$	$\frac{x_3^2}{x_0 x_5}$	$\frac{x_5^2}{x_4 x_0}$	$\frac{x_5^2}{x_1 x_2}$	$\frac{x_1 x_2}{x_4 x_0}$	$\frac{x_2}{x_5}$	$\frac{x_0}{x_3}$
$\frac{x_3}{x_0}$	$\frac{x_2}{x_1}$	$\frac{x_1}{x_2}$	$\frac{x_0}{x_1}$	$\frac{x_1}{x_3}$	$\frac{x_0}{x_2}$	$\frac{x_2}{x_0}$	$\frac{x_3}{x_1}$
$\frac{x_1}{x_0}$	$\frac{x_3}{x_2}$	$\frac{x_2}{x_3}$	$\frac{x_5}{x_2}$	$\frac{x_5}{x_1}$	$\frac{x_0}{x_4}$	$\frac{x_4}{x_2}$	$\frac{x_4}{x_0}$
$\frac{x_4}{x_3}$	$\frac{x_4}{x_1}$	$\frac{x_3}{x_4}$	$\frac{x_1}{x_4}$	$\frac{x_2}{x_4}$	$\frac{x_1}{x_5}$	$\frac{x_5}{x_4}$	$\frac{x_5}{x_3}$
$\frac{x_5}{x_0}$	$\frac{x_4}{x_5}$	$\frac{x_3}{x_5}$	$\frac{x_0}{x_5}$				

form a torus invariant basis. These deformations give linear inequalities defining the special fiber Gröbner cone, i.e.,

$$C_{I_0}(I) = \left\{ (w_t, w_y) \in \mathbb{R} \oplus N_{\mathbb{R}} \mid \left\langle w, A^{-1}(m) \right\rangle \geq -w_t \forall m \right\}$$

for above monomials m and the presentation matrix

$$A=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

of $A_4(Y)$. The vertices of the special fiber polytope

$$\nabla = C_{I_0}(I) \cap \{w_t = 1\}$$

and the number of faces, each vertex is contained in, are given by the table

10	$(0,0,0,1,0)$	$(0,0,0,0,1)$	$(-1,-1,-1,-1,-1)$	$(1,0,0,0,0)$
	$(0,1,0,0,0)$	$(0,0,1,0,0)$		
16	$(1,0,0,1,0)$	$(0,1,0,1,0)$	$(0,0,1,1,0)$	$(1,0,0,0,1)$
	$(0,1,0,0,1)$	$(0,0,1,0,1)$	$(0,-1,-1,-1,-1)$	$(-1,0,-1,-1,-1)$
	$(-1,-1,0,-1,-1)$			

The number of faces of ∇ and their F-vectors are

Dimension	Number of faces	F-vector	
-1	1	$(0,0,0,0,0,0,0)$	
0	15	$(1,1,0,0,0,0,0)$	point
1	42	$(1,2,1,0,0,0,0)$	edge
2	26	$(1,3,3,1,0,0,0)$	triangle
2	27	$(1,4,4,1,0,0,0)$	quadrangle
3	30	$(1,6,9,5,1,0,0)$	prism
3	6	$(1,4,6,4,1,0,0)$	tetrahedron
4	6	$(1,8,16,14,6,1,0)$	
4	6	$(1,9,18,15,6,1,0)$	
5	1	$(1,15,42,53,36,12,1)$	

The dual $P^\circ = \nabla^*$ of ∇ is a Fano polytope with vertices

$$\begin{array}{llll} (-1,-1,-1,0,3) & (-1,-1,-1,0,0) & (-1,2,-1,0,0) & (-1,-1,2,0,0) \\ (2,-1,-1,0,0) & (-1,-1,-1,3,0) & (0,3,0,-1,-1) & (0,0,3,-1,-1) \\ (0,0,0,-1,2) & (0,0,0,-1,-1) & (3,0,0,-1,-1) & (0,0,0,2,-1) \end{array}$$

and the F-vectors of the faces of P° are

Dimension	Number of faces	F-vector	
-1	1	$(0,0,0,0,0,0,0)$	
0	12	$(1,1,0,0,0,0,0)$	point
1	36	$(1,2,1,0,0,0,0)$	edge
2	38	$(1,3,3,1,0,0,0)$	triangle
2	15	$(1,4,4,1,0,0,0)$	quadrangle
3	18	$(1,6,9,5,1,0,0)$	prism
3	24	$(1,4,6,4,1,0,0)$	tetrahedron
4	6	$(1,5,10,10,5,1,0)$	
4	9	$(1,8,16,14,6,1,0)$	
5	1	$(1,12,36,53,42,15,1)$	

The polytopes ∇ and P° have 0 as the unique interior lattice point, and the Fano polytope P° defines the embedding toric Fano variety $Y^\circ = X(\Sigma(P^\circ))$ of the fibers of the mirror degeneration. The dimension of the automorphism group of Y° is

$$\dim(\mathrm{Aut}(Y^\circ)) = 5$$

Let

$$S^\circ = \mathbb{C}[y_1, y_2, ..., y_{12}]$$

be the Cox ring of Y° with variables

$$\begin{array}{lll} y_1 = y_{(-1,-1,-1,0,3)} = \frac{x_5^3}{x_1\,x_2\,x_3} & y_2 = y_{(-1,-1,-1,0,0)} = \frac{x_0^3}{x_1\,x_2\,x_3} & y_3 = y_{(-1,2,-1,0,0)} = \frac{x_2^2}{x_1\,x_3} \\ y_4 = y_{(-1,-1,2,0,0)} = \frac{x_3^2}{x_1\,x_2} & y_5 = y_{(2,-1,-1,0,0)} = \frac{x_1^2}{x_2\,x_3} & y_6 = y_{(-1,-1,-1,3,0)} = \frac{x_4^3}{x_1\,x_2\,x_3} \\ y_7 = y_{(0,3,0,-1,-1)} = \frac{x_2^3}{x_0\,x_4\,x_5} & y_8 = y_{(0,0,3,-1,-1)} = \frac{x_3^3}{x_0\,x_4\,x_5} & y_9 = y_{(0,0,0,-1,2)} = \frac{x_5^2}{x_4\,x_0} \\ y_{10} = y_{(0,0,0,-1,-1)} = \frac{x_0^2}{x_4\,x_5} & y_{11} = y_{(3,0,0,-1,-1)} = \frac{x_1^3}{x_0\,x_4\,x_5} & y_{12} = y_{(0,0,0,2,-1)} = \frac{x_4^2}{x_0\,x_5} \end{array}$$

Bergman subcomplex Intersecting the tropical variety of I with the special fiber Gröbner cone $C_{I_0}(I)$ we obtain the special fiber Bergman subcomplex

$$B(I) = C_{I_0}(I) \cap BF(I) \cap \{w_t = 1\}$$

The following table chooses an indexing of the vertices of ∇ involved in $B(I)$ and gives for each vertex the numbers n_∇ and n_B of faces of ∇ and faces of $B(I)$ it is contained in

n_∇	$n_{B(I)}$		
		$1 = (1, 0, 0, 1, 0)$	$2 = (0, 1, 0, 1, 0)$
		$3 = (0, 0, 1, 1, 0)$	$4 = (1, 0, 0, 0, 1)$
16	4	$5 = (0, 1, 0, 0, 1)$	$6 = (0, 0, 1, 0, 1)$
		$7 = (0, -1, -1, -1, -1)$	$8 = (-1, 0, -1, -1, -1)$
		$9 = (-1, -1, 0, -1, -1)$	

With this indexing the Bergman subcomplex $B(I)$ of Poset (∇) associated to the degeneration $\mathfrak{X}$ is

[],

[[1], [2], [3], [4], [5], [6], [7], [8], [9]],

[[3, 9], [3, 6], [1, 4], [1, 7], [6, 9], [7, 9], [7, 8], [8, 9], [4, 7], [4, 6], [4, 5], [5, 6], [5, 8], [1, 2], [1, 3], [2, 8], [2, 3], [2, 5]],

[[1, 2, 4, 5], [1, 3, 4, 6], [1, 3, 7, 9], [4, 6, 7, 9], [4, 5, 7, 8], [4, 5, 6], [1, 4, 7], [1, 2, 7, 8], [5, 6, 8, 9], [7, 8, 9], [2, 5, 8], [3, 6, 9], [1, 2, 3], [2, 3, 8, 9], [2, 3, 5, 6]],

[[2, 3, 5, 6, 8, 9], [1, 2, 3, 4, 5, 6], [1, 2, 4, 5, 7, 8], [4, 5, 6, 7, 8, 9], [1, 2, 3, 7, 8, 9], [1, 3, 4, 6, 7, 9]],

[],

[]

$B(I)$ has the F-vector

Dimension	−1	0	1	2	3	4	5
Number of faces	0	9	18	15	6	0	0

and the F-vectors of its faces are

Dimension	Number of faces	F-vector	
0	9	$(1,1,0,0,0,0,0)$	point
1	18	$(1,2,1,0,0,0,0)$	edge
2	6	$(1,3,3,1,0,0,0)$	triangle
2	9	$(1,4,4,1,0,0,0)$	quadrangle
3	6	$(1,6,9,5,1,0,0)$	prism

Dual complex The dual complex $\operatorname{dual}(B(I)) = (B(I))^*$ of deformations associated to $B(I)$ via initial ideals is given by

$$
\begin{array}{l}
[], \\
[], \\
[[2,3,5,6,8,9]^* = \left\langle \frac{x_1^2}{x_2\,x_3}, \frac{x_1^3}{x_0\,x_4\,x_5} \right\rangle, ...], \\
[[4,5,6]^* = \left\langle \frac{x_4^3}{x_1\,x_2\,x_3}, \frac{x_0^3}{x_1\,x_2\,x_3}, \frac{x_4^2}{x_0\,x_5}, \frac{x_0^2}{x_4\,x_5} \right\rangle, [1,2,4,5]^* = \left\langle \frac{x_0^3}{x_1\,x_2\,x_3}, \frac{x_3^2}{x_1\,x_2}, \frac{x_3^3}{x_0\,x_4\,x_5}, \frac{x_0^2}{x_4\,x_5} \right\rangle, \\
...], \\
[[3,9]^* = \left\langle \frac{x_5^3}{x_1\,x_2\,x_3}, \frac{x_2^2}{x_1\,x_3}, \frac{x_1^2}{x_2\,x_3}, \frac{x_2^3}{x_0\,x_4\,x_5}, \frac{x_1^3}{x_0\,x_4\,x_5}, \frac{x_5^2}{x_4\,x_0} \right\rangle, \\
...], \\
[[1]^* = \left\langle \frac{x_5^3}{x_1\,x_2\,x_3}, \frac{x_0^3}{x_1\,x_2\,x_3}, \frac{x_2^2}{x_1\,x_3}, \frac{x_3^2}{x_1\,x_2}, \frac{x_2^3}{x_0\,x_4\,x_5}, \frac{x_3^3}{x_0\,x_4\,x_5}, \frac{x_5^2}{x_4\,x_0}, \frac{x_0^2}{x_4\,x_5} \right\rangle, \\
...], \\
[]
\end{array}
$$

when writing the vertices of the faces as deformations of X_0. Note that the T-invariant basis of deformations associated to a face is given by all lattice points of the corresponding polytope in $M_{\mathbb{R}}$. In order to compress the output we list one representative in any set of faces G with fixed F-vector of G and of G^*. When numbering the vertices of the faces of $\operatorname{dual}(B(I))$ by the Cox variables of the mirror toric Fano variety Y° the complex $\operatorname{dual}(B(I))$ is

$$
\begin{array}{l}
[], \\
[],
\end{array}
$$

$$[[2,3,5,6,8,9]^* = \langle y_5, y_{11}\rangle, ...],$$
$$[[4,5,6]^* = \langle y_6, y_2, y_{12}, y_{10}\rangle, [1,2,4,5]^* = \langle y_2, y_4, y_8, y_{10}\rangle, ...],$$
$$[[3,9]^* = \langle y_1, y_3, y_5, y_7, y_{11}, y_9\rangle, ...],$$
$$[[1]^* = \langle y_1, y_2, y_3, y_4, y_7, y_8, y_9, y_{10}\rangle, ...],$$
$$[]$$

The dual complex has the F-vector

Dimension	-1	0	1	2	3	4	5
Number of faces	0	0	6	15	18	9	0

and the F-vectors of the faces of $\operatorname{dual}(B(I))$ are

Dimension	Number of faces	F-vector	
1	6	$(1,2,1,0,0,0,0)$	edge
2	15	$(1,4,4,1,0,0,0)$	quadrangle
3	18	$(1,6,9,5,1,0,0)$	prism
4	9	$(1,8,16,14,6,1,0)$	

Recall that in this example the toric variety Y is projective space. The number of lattice points of the support of $\operatorname{dual}(B(I))$ relates to the dimension $h^{1,2}(X)$ of the complex moduli space of the generic fiber X of $\mathfrak{X}$ and to the dimension $h^{1,1}\left(\bar{X}^\circ\right)$ of the Kähler moduli space of the $MPCP$-blowup $\bar{X}^\circ$ of the generic fiber X° of the mirror degeneration

$$\begin{aligned} |\operatorname{supp}(\operatorname{dual}(B(I))) \cap M| &= 108 = 35 + 73 = \dim(\operatorname{Aut}(Y)) + h^{1,2}(X) \\ &= 30 + 5 + 73 \\ &= |\operatorname{Roots}(Y)| + \dim(T_Y) + h^{1,1}\left(\bar{X}^\circ\right) \end{aligned}$$

There are

$$h^{1,2}(X) + \dim(T_{Y^\circ}) = 73 + 5$$

non-trivial toric polynomial deformations of X_0

$$\frac{x_1^3}{x_0\,x_4\,x_5}\quad \frac{x_1^2}{x_2\,x_3}\quad \frac{x_0^2}{x_4\,x_5}\quad \frac{x_0^3}{x_1\,x_2\,x_3}\quad \frac{x_3^3}{x_0\,x_4\,x_5}\quad \frac{x_3^2}{x_1\,x_2}\quad \frac{x_4^2}{x_0\,x_5}\quad \frac{x_4^3}{x_1\,x_2\,x_3}\quad \frac{x_5^2}{x_4\,x_0}\quad \frac{x_5^3}{x_1\,x_2\,x_3}$$
$$\frac{x_2^3}{x_0\,x_4\,x_5}\quad \frac{x_2^2}{x_1\,x_3}\quad \frac{x_3\,x_0}{x_4\,x_5}\quad \frac{x_3^2}{x_4\,x_5}\quad \frac{x_0^2}{x_1\,x_2}\quad \frac{x_3\,x_0}{x_1\,x_2}\quad \frac{x_2\,x_0}{x_4\,x_5}\quad \frac{x_2^2}{x_4\,x_5}\quad \frac{x_0^2}{x_1\,x_3}\quad \frac{x_2\,x_0}{x_1\,x_3}$$

$\frac{x_2^2}{x_4\,x_0}$	$\frac{x_5\,x_2}{x_4\,x_0}$	$\frac{x_5\,x_2}{x_1\,x_3}$	$\frac{x_5^2}{x_1\,x_3}$	$\frac{x_2^2}{x_0\,x_5}$	$\frac{x_2\,x_4}{x_0\,x_5}$	$\frac{x_2\,x_4}{x_1\,x_3}$	$\frac{x_4^2}{x_1\,x_3}$	$\frac{x_3^2}{x_0\,x_5}$	$\frac{x_3\,x_4}{x_0\,x_5}$
$\frac{x_3\,x_4}{x_1\,x_2}$	$\frac{x_4^2}{x_1\,x_2}$	$\frac{x_4\,{x_0}^2}{x_1\,x_2\,x_3}$	$\frac{x_4^2 x_0}{x_1\,x_2\,x_3}$	$\frac{x_3\,{x_2}^2}{x_0\,x_4\,x_5}$	$\frac{x_3^2 x_2}{x_0\,x_4\,x_5}$	$\frac{x_3^2}{x_4\,x_0}$	$\frac{x_5\,x_3}{x_4\,x_0}$	$\frac{x_5\,x_3}{x_1\,x_2}$	$\frac{x_5^2}{x_1\,x_2}$
$\frac{x_1^2}{x_0\,x_5}$	$\frac{x_1\,x_4}{x_0\,x_5}$	$\frac{x_1\,x_4}{x_2\,x_3}$	$\frac{x_4^2}{x_2\,x_3}$	$\frac{x_4^2 x_5}{x_1\,x_2\,x_3}$	$\frac{x_4\,{x_5}^2}{x_1\,x_2\,x_3}$	$\frac{x_3\,{x_1}^2}{x_0\,x_4\,x_5}$	$\frac{x_3^2 x_1}{x_0\,x_4\,x_5}$	$\frac{x_2\,{x_1}^2}{x_0\,x_4\,x_5}$	$\frac{x_2^2 x_1}{x_0\,x_4\,x_5}$
$\frac{x_5\,{x_0}^2}{x_1\,x_2\,x_3}$	$\frac{x_5^2 x_0}{x_1\,x_2\,x_3}$	$\frac{x_1^2}{x_4\,x_0}$	$\frac{x_5\,x_1}{x_4\,x_0}$	$\frac{x_5\,x_1}{x_2\,x_3}$	$\frac{x_5^2}{x_2\,x_3}$	$\frac{x_1\,x_0}{x_4\,x_5}$	$\frac{x_1^2}{x_4\,x_5}$	$\frac{x_0^2}{x_2\,x_3}$	$\frac{x_1\,x_0}{x_2\,x_3}$
$\frac{x_1\,x_2}{x_4\,x_0}$	$\frac{x_1\,x_2}{x_4\,x_5}$	$\frac{x_2\,x_3}{x_4\,x_5}$	$\frac{x_2\,x_3}{x_4\,x_0}$	$\frac{x_1\,x_2}{x_0\,x_5}$	$\frac{x_4\,x_5}{x_1\,x_3}$	$\frac{x_4\,x_5}{x_1\,x_2}$	$\frac{x_4\,x_5}{x_2\,x_3}$	$\frac{x_2\,x_3}{x_0\,x_5}$	$\frac{x_4\,x_0}{x_1\,x_3}$
$\frac{x_4\,x_0}{x_1\,x_2}$	$\frac{x_4\,x_0}{x_2\,x_3}$	$\frac{x_1\,x_3}{x_0\,x_5}$	$\frac{x_0\,x_5}{x_1\,x_2}$	$\frac{x_0\,x_5}{x_1\,x_3}$	$\frac{x_1\,x_3}{x_4\,x_0}$	$\frac{x_0\,x_5}{x_2\,x_3}$	$\frac{x_1\,x_3}{x_4\,x_5}$		

They correspond to the toric divisors

$D_{(3,0,0,-1,-1)}$	$D_{(2,-1,-1,0,0)}$	$D_{(0,0,0,-1,-1)}$	$D_{(-1,-1,-1,0,0)}$
$D_{(0,0,3,-1,-1)}$	$D_{(-1,-1,2,0,0)}$	$D_{(0,0,0,2,-1)}$	$D_{(-1,-1,-1,3,0)}$
$D_{(0,0,0,-1,2)}$	$D_{(-1,-1,-1,0,3)}$	$D_{(0,3,0,-1,-1)}$	$D_{(-1,2,-1,0,0)}$
$D_{(0,0,1,-1,-1)}$	$D_{(0,0,2,-1,-1)}$	$D_{(-1,-1,0,0,0)}$	$D_{(-1,-1,1,0,0)}$
$D_{(0,1,0,-1,-1)}$	$D_{(0,2,0,-1,-1)}$	$D_{(-1,0,-1,0,0)}$	$D_{(-1,1,-1,0,0)}$
$D_{(0,2,0,-1,0)}$	$D_{(0,1,0,-1,1)}$	$D_{(-1,1,-1,0,1)}$	$D_{(-1,0,-1,0,2)}$
$D_{(0,2,0,0,-1)}$	$D_{(0,1,0,1,-1)}$	$D_{(-1,1,-1,1,0)}$	$D_{(-1,0,-1,2,0)}$
$D_{(0,0,2,0,-1)}$	$D_{(0,0,1,1,-1)}$	$D_{(-1,-1,1,1,0)}$	$D_{(-1,-1,0,2,0)}$
$D_{(-1,-1,-1,1,0)}$	$D_{(-1,-1,-1,2,0)}$	$D_{(0,2,1,-1,-1)}$	$D_{(0,1,2,-1,-1)}$
$D_{(0,0,2,-1,0)}$	$D_{(0,0,1,-1,1)}$	$D_{(-1,-1,1,0,1)}$	$D_{(-1,-1,0,0,2)}$
$D_{(2,0,0,0,-1)}$	$D_{(1,0,0,1,-1)}$	$D_{(1,-1,-1,1,0)}$	$D_{(0,-1,-1,2,0)}$
$D_{(-1,-1,-1,2,1)}$	$D_{(-1,-1,-1,1,2)}$	$D_{(2,0,1,-1,-1)}$	$D_{(1,0,2,-1,-1)}$
$D_{(2,1,0,-1,-1)}$	$D_{(1,2,0,-1,-1)}$	$D_{(-1,-1,-1,0,1)}$	$D_{(-1,-1,-1,0,2)}$
$D_{(2,0,0,-1,0)}$	$D_{(1,0,0,-1,1)}$	$D_{(1,-1,-1,0,1)}$	$D_{(0,-1,-1,0,2)}$
$D_{(1,0,0,-1,-1)}$	$D_{(2,0,0,-1,-1)}$	$D_{(0,-1,-1,0,0)}$	$D_{(1,-1,-1,0,0)}$
$D_{(1,1,0,-1,0)}$	$D_{(1,1,0,-1,-1)}$	$D_{(0,1,1,-1,-1)}$	$D_{(0,1,1,-1,0)}$
$D_{(1,1,0,0,-1)}$	$D_{(-1,0,-1,1,1)}$	$D_{(-1,-1,0,1,1)}$	$D_{(0,-1,-1,1,1)}$
$D_{(0,1,1,0,-1)}$	$D_{(-1,0,-1,1,0)}$	$D_{(-1,-1,0,1,0)}$	$D_{(0,-1,-1,1,0)}$
$D_{(1,0,1,0,-1)}$	$D_{(-1,-1,0,0,1)}$	$D_{(-1,0,-1,0,1)}$	$D_{(1,0,1,-1,0)}$
$D_{(0,-1,-1,0,1)}$	$D_{(1,0,1,-1,-1)}$		

on a MPCP-blowup of Y° inducing 73 non-zero toric divisor classes on the mirror X°. The following 30 toric divisors of Y induce the trivial divisor class on X°

$D_{(0,0,0,0,-1)}$	$D_{(0,0,0,1,-1)}$	$D_{(-1,1,0,0,0)}$	$D_{(-1,0,1,0,0)}$	$D_{(0,0,0,1,0)}$
$D_{(0,0,0,0,1)}$	$D_{(1,-1,0,0,0)}$	$D_{(0,-1,1,0,0)}$	$D_{(1,0,-1,0,0)}$	$D_{(0,1,-1,0,0)}$

$$
\begin{array}{lllll}
D_{(0,0,0,-1,0)} & D_{(0,0,0,-1,1)} & D_{(0,0,-1,0,1)} & D_{(0,0,-1,0,0)} & D_{(-1,0,0,0,0)} \\
D_{(-1,0,0,0,1)} & D_{(0,0,-1,1,0)} & D_{(0,1,0,0,0)} & D_{(0,0,1,0,0)} & D_{(1,0,0,0,0)} \\
D_{(-1,0,0,1,0)} & D_{(0,1,0,0,-1)} & D_{(0,0,1,0,-1)} & D_{(1,0,0,0,-1)} & D_{(0,-1,0,1,0)} \\
D_{(0,0,1,-1,0)} & D_{(0,1,0,-1,0)} & D_{(0,-1,0,0,1)} & D_{(1,0,0,-1,0)} & D_{(0,-1,0,0,0)}
\end{array}
$$

Mirror special fiber The ideal I_0° of the monomial special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$ is generated by the following set of monomials in S°

$$
\left\{
\begin{array}{lll}
y_6\,y_7\,y_8\,y_5\,y_{10}\,y_9 & y_1\,y_2\,y_3\,y_4\,y_5\,y_{12} & y_6\,y_7\,y_8\,y_{11}\,y_{10}\,y_9 \\
y_1\,y_2\,y_3\,y_4\,y_{11}\,y_{12} & y_6\,y_1\,y_2\,y_3\,y_4\,y_5 & y_7\,y_8\,y_5\,y_{10}\,y_9\,y_{12} \\
y_6\,y_1\,y_2\,y_3\,y_4\,y_{11} & y_7\,y_8\,y_{11}\,y_{10}\,y_9\,y_{12} & y_7\,y_8\,y_1\,y_2\,y_{11}\,y_{12} \\
y_6\,y_8\,y_2\,y_3\,y_{11}\,y_9 & y_7\,y_8\,y_2\,y_5\,y_9\,y_{12} & y_8\,y_2\,y_3\,y_5\,y_9\,y_{12} \\
y_6\,y_3\,y_4\,y_{11}\,y_{10}\,y_9 & y_7\,y_1\,y_2\,y_4\,y_5\,y_{12} & y_6\,y_7\,y_8\,y_1\,y_2\,y_5 \\
y_8\,y_1\,y_2\,y_3\,y_5\,y_{12} & y_7\,y_8\,y_1\,y_2\,y_5\,y_{12} & y_3\,y_4\,y_5\,y_{10}\,y_9\,y_{12} \\
y_6\,y_3\,y_4\,y_5\,y_{10}\,y_9 & y_1\,y_3\,y_4\,y_5\,y_{10}\,y_{12} & y_7\,y_4\,y_5\,y_{10}\,y_9\,y_{12} \\
y_7\,y_1\,y_4\,y_5\,y_{10}\,y_{12} & y_7\,y_4\,y_{11}\,y_{10}\,y_9\,y_{12} & y_2\,y_3\,y_4\,y_{11}\,y_9\,y_{12} \\
y_6\,y_2\,y_3\,y_4\,y_{11}\,y_9 & y_6\,y_7\,y_4\,y_5\,y_{10}\,y_9 & y_3\,y_4\,y_{11}\,y_{10}\,y_9\,y_{12} \\
y_7\,y_1\,y_4\,y_{11}\,y_{10}\,y_{12} & y_6\,y_2\,y_3\,y_4\,y_5\,y_9 & y_6\,y_1\,y_3\,y_4\,y_5\,y_{10} \\
y_6\,y_8\,y_2\,y_3\,y_5\,y_9 & y_1\,y_3\,y_4\,y_{11}\,y_{10}\,y_{12} & y_8\,y_1\,y_3\,y_5\,y_{10}\,y_{12} \\
y_6\,y_7\,y_4\,y_{11}\,y_{10}\,y_9 & y_2\,y_3\,y_4\,y_5\,y_9\,y_{12} & y_6\,y_7\,y_8\,y_1\,y_5\,y_{10} \\
y_6\,y_8\,y_1\,y_3\,y_5\,y_{10} & y_6\,y_7\,y_1\,y_4\,y_{11}\,y_{10} & y_6\,y_1\,y_3\,y_4\,y_{11}\,y_{10} \\
y_6\,y_8\,y_3\,y_5\,y_{10}\,y_9 & y_7\,y_2\,y_4\,y_5\,y_9\,y_{12} & y_6\,y_8\,y_3\,y_{11}\,y_{10}\,y_9 \\
y_6\,y_7\,y_8\,y_1\,y_2\,y_{11} & y_7\,y_8\,y_1\,y_{11}\,y_{10}\,y_{12} & y_6\,y_8\,y_1\,y_2\,y_3\,y_{11} \\
y_8\,y_1\,y_3\,y_{11}\,y_{10}\,y_{12} & y_8\,y_3\,y_{11}\,y_{10}\,y_9\,y_{12} & y_6\,y_7\,y_8\,y_2\,y_5\,y_9 \\
y_6\,y_7\,y_2\,y_4\,y_5\,y_9 & y_6\,y_7\,y_2\,y_4\,y_{11}\,y_9 & y_6\,y_7\,y_1\,y_4\,y_5\,y_{10} \\
y_7\,y_1\,y_2\,y_4\,y_{11}\,y_{12} & y_6\,y_7\,y_1\,y_2\,y_4\,y_{11} & y_6\,y_7\,y_8\,y_1\,y_{11}\,y_{10} \\
y_8\,y_3\,y_5\,y_{10}\,y_9\,y_{12} & y_6\,y_8\,y_1\,y_3\,y_{11}\,y_{10} & y_6\,y_7\,y_1\,y_2\,y_4\,y_5 \\
y_8\,y_2\,y_3\,y_{11}\,y_9\,y_{12} & y_7\,y_8\,y_1\,y_5\,y_{10}\,y_{12} & y_7\,y_8\,y_2\,y_{11}\,y_9\,y_{12} \\
y_6\,y_8\,y_1\,y_2\,y_3\,y_5 & y_6\,y_7\,y_8\,y_2\,y_{11}\,y_9 & y_8\,y_1\,y_2\,y_3\,y_{11}\,y_{12} \\
y_7\,y_2\,y_4\,y_{11}\,y_9\,y_{12} & &
\end{array}
\right\}
$$

The $\operatorname{Pic}(Y^\circ)$-generated ideal

$$
J_0^\circ = \left\langle \; y_6\,y_1\,y_2\,y_3\,y_4\,y_5 \quad y_7\,y_8\,y_{11}\,y_{10}\,y_9\,y_{12} \; \right\rangle
$$

defines the same subvariety X_0° of the toric variety Y°, and $J_0^{\circ\Sigma} = I_0^\circ$. Passing from J_0° to $J_0^{\circ\Sigma}$ is the non-simplicial toric analogue of saturation. The complex $B(I)^*$ labeled by the variables of the Cox ring S° of Y°, as written in the last section, is the complex $SP(I_0^\circ)$ of prime ideals of the toric strata of the special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$, i.e., the primary decomposition of I_0° is

$$I_0^\circ = \langle y_3, y_7\rangle \cap \langle y_5, y_{11}\rangle \cap \langle y_2, y_{10}\rangle \cap \langle y_4, y_8\rangle \cap \langle y_6, y_{12}\rangle \cap \langle y_1, y_9\rangle$$

Each facet $F \in B(I)$ corresponds to one of these ideals and this ideal is generated by the facets of ∇ containing F.

Covering structure in the deformation complex of the degeneration $\mathfrak{X}$ According to the local reduced Gröbner basis each face of the complex of deformations $\operatorname{dual}(B(I))$ decomposes into 2 polytopes forming a 2 : 1 trivial covering of $B(I)$

$$\begin{array}{l}
[],\\
[],\\
[[\langle y_5\rangle, \langle y_{11}\rangle] \mapsto [2,3,5,6,8,9]^\vee, ...],\\
[[\langle y_6, y_2\rangle, \langle y_{12}, y_{10}\rangle] \mapsto [4,5,6]^\vee, [\langle y_2, y_4\rangle, \langle y_8, y_{10}\rangle] \mapsto [1,2,4,5]^\vee,\\
...],\\
[[\langle y_1, y_3, y_5\rangle, \langle y_7, y_{11}, y_9\rangle] \mapsto [3,9]^\vee,\\
...],\\
[[\langle y_1, y_2, y_3, y_4\rangle, \langle y_7, y_8, y_9, y_{10}\rangle] \mapsto [1]^\vee,\\
...],\\
[]
\end{array}$$

Here the faces $F \in B(I)$ are specified both via the vertices of F^* labeled by the variables of S° and by the numbering of the vertices of $B(I)$ chosen above. This covering has 2 sheets forming the complexes

$$\begin{array}{l}
[],\\
[],\\
[\langle y_5\rangle, ...],\\
[\langle y_6, y_2\rangle, \langle y_2, y_4\rangle, ...],\\
[\langle y_1, y_3, y_5\rangle, ...],\\
[\langle y_1, y_2, y_3, y_4\rangle, ...],\\
[]
\end{array}$$

$$
\begin{aligned}
&[],\\
&[],\\
&[\langle y_{11}\rangle, ...],\\
&[\langle y_{12}, y_{10}\rangle, \langle y_8, y_{10}\rangle, ...],\\
&[\langle y_7, y_{11}, y_9\rangle, ...],\\
&[\langle y_7, y_8, y_9, y_{10}\rangle, ...],\\
&[]
\end{aligned}
$$

with F-vector

Dimension	Number of faces	F-vector	
0	6	$(1, 1, 0, 0, 0, 0, 0)$	point
1	15	$(1, 2, 1, 0, 0, 0, 0)$	edge
2	18	$(1, 3, 3, 1, 0, 0, 0)$	triangle
3	9	$(1, 4, 6, 4, 1, 0, 0)$	tetrahedron

Writing the vertices of the faces as deformations the covering is given by

$$
\begin{aligned}
&[],\\
&[],\\
&[[\left\langle \frac{x_1^2}{x_2\,x_3} \right\rangle, \left\langle \frac{x_1^3}{x_0\,x_4\,x_5} \right\rangle] \mapsto [2, 3, 5, 6, 8, 9]^{\vee}, ...],\\
&[[\left\langle \frac{x_4^3}{x_1\,x_2\,x_3}, \frac{{x_0}^3}{x_1\,x_2\,x_3} \right\rangle, \left\langle \frac{x_4^2}{x_0\,x_5}, \frac{{x_0}^2}{x_4\,x_5} \right\rangle] \mapsto [4, 5, 6]^{\vee},\\
&[\left\langle \frac{x_0^3}{x_1\,x_2\,x_3}, \frac{{x_3}^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_3^3}{x_0\,x_4\,x_5}, \frac{{x_0}^2}{x_4\,x_5} \right\rangle] \mapsto [1, 2, 4, 5]^{\vee},\\
&...],\\
&[[\left\langle \frac{x_5^3}{x_1\,x_2\,x_3}, \frac{{x_2}^2}{x_1\,x_3}, \frac{{x_1}^2}{x_2\,x_3} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_4\,x_5}, \frac{{x_1}^3}{x_0\,x_4\,x_5}, \frac{{x_5}^2}{x_4\,x_0} \right\rangle] \mapsto [3, 9]^{\vee},\\
&...],\\
&[[\left\langle \frac{x_5^3}{x_1\,x_2\,x_3}, \frac{{x_0}^3}{x_1\,x_2\,x_3}, \frac{{x_2}^2}{x_1\,x_3}, \frac{{x_3}^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_4\,x_5}, \frac{{x_3}^3}{x_0\,x_4\,x_5}, \frac{{x_5}^2}{x_4\,x_0}, \frac{{x_0}^2}{x_4\,x_5} \right\rangle] \mapsto [1]^{\vee},\\
&...],\\
&[]
\end{aligned}
$$

with the 2 sheets forming the complexes

$$[],$$
$$[],$$
$$[\left\langle \frac{x_1^2}{x_2\,x_3} \right\rangle, ...],$$
$$[\left\langle \frac{x_4^3}{x_1\,x_2\,x_3}, \frac{{x_0}^3}{x_1\,x_2\,x_3} \right\rangle, \left\langle \frac{x_0^3}{x_1\,x_2\,x_3}, \frac{{x_3}^2}{x_1\,x_2} \right\rangle, ...],$$
$$[\left\langle \frac{x_5^3}{x_1\,x_2\,x_3}, \frac{{x_2}^2}{x_1\,x_3}, \frac{{x_1}^2}{x_2\,x_3} \right\rangle, ...],$$
$$[\left\langle \frac{x_5^3}{x_1\,x_2\,x_3}, \frac{{x_0}^3}{x_1\,x_2\,x_3}, \frac{{x_2}^2}{x_1\,x_3}, \frac{{x_3}^2}{x_1\,x_2} \right\rangle, ...],$$
$$[]$$

$$[],$$
$$[],$$
$$[\left\langle \frac{x_1^3}{x_0\,x_4\,x_5} \right\rangle, ...],$$
$$[\left\langle \frac{x_4^2}{x_0\,x_5}, \frac{{x_0}^2}{x_4\,x_5} \right\rangle, \left\langle \frac{x_3^3}{x_0\,x_4\,x_5}, \frac{{x_0}^2}{x_4\,x_5} \right\rangle, ...],$$
$$[\left\langle \frac{x_2^3}{x_0\,x_4\,x_5}, \frac{{x_1}^3}{x_0\,x_4\,x_5}, \frac{{x_5}^2}{x_4\,x_0} \right\rangle, ...],$$
$$[\left\langle \frac{x_2^3}{x_0\,x_4\,x_5}, \frac{{x_3}^3}{x_0\,x_4\,x_5}, \frac{{x_5}^2}{x_4\,x_0}, \frac{{x_0}^2}{x_4\,x_5} \right\rangle, ...],$$
$$[]$$

Note that the torus invariant basis of deformations corresponding to a Bergman face is given by the set of all lattice points of the polytope specified above.

Limit map The limit map $\lim : B(I) \to \text{Poset}(\Delta)$ associates to a face F of $B(I)$ the face of Δ formed by the limit points of arcs lying over the weight vectors $w \in F$, i.e., with lowest order term t^w. Labeling the faces of the Bergman complex $B(I) \subset \text{Poset}(\nabla)$ and the faces of $\text{Poset}(\Delta)$ by the corresponding dual faces of ∇^* and Δ^*, hence considering the limit map $\lim : B(I) \to \text{Poset}(\Delta)$ as a map $B(I)^* \to \text{Poset}(\Delta^*)$, the limit correspondence is given by

$$[],$$
$$[\langle y_1, y_2, y_3, y_4, y_7, y_8, y_9, y_{10}\rangle \mapsto \langle x_1, x_4\rangle,$$
$$...],$$

$[\langle y_1, y_3, y_5, y_7, y_{11}, y_9 \rangle \mapsto \langle x_3, x_4, x_0 \rangle$,
$...]$,

$[\langle y_6, y_2, y_{12}, y_{10} \rangle \mapsto \langle x_1, x_2, x_3, x_5 \rangle$,
$\langle y_2, y_4, y_8, y_{10} \rangle \mapsto \langle x_1, x_2, x_4, x_5 \rangle$,
$...]$,

$[\langle y_5, y_{11} \rangle \mapsto \langle x_2, x_3, x_4, x_5, x_0 \rangle$,
$...]$,

[],

[]

The image of the limit map coincides with the image of μ and with the Bergman complex of the mirror, i.e., $\lim(B(I)) = \mu(B(I)) = B(I^\circ)$.

Mirror complex Numbering the vertices of the mirror complex $\mu(B(I))$ as

$$\begin{array}{ll} 1 = (5, -1, -1, -1, -1) & 2 = (-1, 5, -1, -1, -1) \\ 3 = (-1, -1, 5, -1, -1) & 4 = (-1, -1, -1, 5, -1) \\ 5 = (-1, -1, -1, -1, 5) & 6 = (-1, -1, -1, -1, -1) \end{array}$$

$\mu(B(I)) \subset \mathrm{Poset}(\Delta)$ is

[],

[[1], [6], [3], [4], [5], [2]],

[[3, 6], [2, 6], [2, 5], [2, 4], [3, 4], [4, 6], [2, 3], [3, 5], [1, 4], [4, 5], [1, 3], [1, 2], [5, 6], [1, 5], [1, 6]],

[[1, 2, 5], [1, 2, 6], [2, 3, 6], [2, 3, 5], [1, 2, 4], [2, 4, 5], [3, 4, 5], [1, 4, 5], [2, 3, 4], [2, 4, 6], [3, 4, 6], [1, 4, 6], [1, 3, 4], [3, 5, 6], [2, 5, 6], [1, 3, 5], [1, 5, 6], [1, 3, 6]],

[[2, 3, 5, 6], [1, 3, 5, 6], [1, 2, 5, 6], [2, 3, 4, 6], [1, 3, 4, 6], [1, 2, 4, 6], [2, 3, 4, 5], [1, 3, 4, 5], [1, 2, 4, 5]],

[],

[]

The ordering of the faces of $\mu(B(I))$ is compatible with above ordering of the faces of $B(I)$. The F-vectors of the faces of $\mu(B(I))^*$ are

Dimension	Number of faces	F-vector	
1	9	$(1,2,1,0,0,0,0)$	edge
2	18	$(1,3,3,1,0,0,0)$	triangle
3	15	$(1,4,6,4,1,0,0)$	tetrahedron
4	6	$(1,5,10,10,5,1,0)$	

The first order deformations of the mirror special fiber X_0° correspond to the lattice points of the dual complex $(\mu(B(I)))^* \subset \text{Poset}(\Delta^*)$ of the mirror complex. We label the first order deformations of X_0° corresponding to vertices of Δ^* by the homogeneous coordinates of Y

$$x_1 = x_{(1,0,0,0,0)} = \frac{y_5^2 {y_{11}}^3}{y_1\, y_2\, y_3\, y_4\, y_6} \qquad x_2 = x_{(0,1,0,0,0)} = \frac{y_3^2 {y_7}^3}{y_1\, y_2\, y_4\, y_5\, y_6}$$
$$x_3 = x_{(0,0,1,0,0)} = \frac{y_4^2 {y_8}^3}{y_1\, y_2\, y_3\, y_5\, y_6} \qquad x_4 = x_{(0,0,0,1,0)} = \frac{y_6^3 {y_{12}}^2}{y_7\, y_8\, y_9\, y_{10}\, y_{11}}$$
$$x_5 = x_{(0,0,0,0,1)} = \frac{y_1^3 {y_9}^2}{y_7\, y_8\, y_{10}\, y_{11}\, y_{12}} \qquad x_0 = x_{(-1,-1,-1,-1,-1)} = \frac{y_2^3 {y_{10}}^2}{y_7\, y_8\, y_9\, y_{11}\, y_{12}}$$

So writing the vertices of the faces of $(\mu(B(I)))^*$ as homogeneous coordinates of Y, the complex $(\mu(B(I)))^*$ is given by

[],

[],

$[\langle x_1,x_4\rangle, \langle x_2,x_4\rangle, \langle x_3,x_4\rangle, \langle x_1,x_5\rangle, \langle x_2,x_5\rangle, \langle x_3,x_5\rangle, \langle x_1,x_0\rangle,$
$\langle x_2,x_0\rangle, \langle x_3,x_0\rangle],$

$[\langle x_3,x_4,x_0\rangle, \langle x_3,x_4,x_5\rangle, \langle x_1,x_4,x_5\rangle, \langle x_1,x_4,x_0\rangle, \langle x_3,x_5,x_0\rangle, \langle x_1,x_3,x_0\rangle,$
$\langle x_1,x_2,x_0\rangle, \langle x_2,x_3,x_0\rangle, \langle x_1,x_5,x_0\rangle, \langle x_1,x_3,x_5\rangle, \langle x_1,x_2,x_5\rangle, \langle x_2,x_3,x_5\rangle,$
$\langle x_2,x_5,x_0\rangle, \langle x_1,x_2,x_4\rangle, \langle x_1,x_3,x_4\rangle, \langle x_2,x_4,x_0\rangle, \langle x_2,x_3,x_4\rangle, \langle x_2,x_4,x_5\rangle],$

$[\langle x_1,x_2,x_4,x_5\rangle, \langle x_1,x_3,x_4,x_5\rangle, \langle x_1,x_3,x_4,x_0\rangle, \langle x_1,x_3,x_5,x_0\rangle, \langle x_1,x_2,x_5,x_0\rangle,$
$\langle x_1,x_2,x_3,x_5\rangle, \langle x_1,x_4,x_5,x_0\rangle, \langle x_1,x_2,x_4,x_0\rangle, \langle x_2,x_3,x_5,x_0\rangle, \langle x_1,x_2,x_3,x_0\rangle,$
$\langle x_2,x_4,x_5,x_0\rangle, \langle x_3,x_4,x_5,x_0\rangle, \langle x_1,x_2,x_3,x_4\rangle, \langle x_2,x_3,x_4,x_0\rangle, \langle x_2,x_3,x_4,x_5\rangle],$

$[\langle x_2,x_3,x_4,x_5,x_0\rangle, \langle x_1,x_2,x_3,x_4,x_5\rangle, \langle x_1,x_2,x_4,x_5,x_0\rangle, \langle x_1,x_2,x_3,x_5,x_0\rangle,$
$\langle x_1,x_2,x_3,x_4,x_0\rangle, \langle x_1,x_3,x_4,x_5,x_0\rangle],$

[]

The complex $\mu(B(I))^*$ labeled by the variables of the Cox ring S gives the ideals of the toric strata of the special fiber X_0 of $\mathfrak{X}$, i.e., the complex $SP(I_0)$, so in particular the primary decomposition of I_0 is

$$I_0 = \begin{array}{l} \langle x_3, x_4\rangle \cap \langle x_2, x_0\rangle \cap \langle x_3, x_0\rangle \cap \langle x_1, x_0\rangle \cap \langle x_1, x_5\rangle \cap \langle x_2, x_5\rangle \cap \\ \cap \langle x_3, x_5\rangle \cap \langle x_1, x_4\rangle \cap \langle x_2, x_4\rangle \end{array}$$

Covering structure in the deformation complex of the mirror degeneration Each face of the complex of deformations $(\mu(B(I)))^*$ of the mirror special fiber X_0° decomposes as the convex hull of 2 polytopes forming a 2 : 1 trivial covering of $\mu(B(I))^\vee$

$$\begin{array}{l}
[], \\
[], \\
[[\langle x_1\rangle, \langle x_4\rangle] \mapsto \langle x_1, x_4\rangle^{*\vee} = [2,3,5,6]^\vee, \\
...], \\
[[\langle x_3\rangle, \langle x_4, x_0\rangle] \mapsto \langle x_3, x_4, x_0\rangle^{*\vee} = [1,2,5]^\vee, \\
...], \\
[[\langle x_1, x_2, x_3\rangle, \langle x_5\rangle] \mapsto \langle x_1, x_2, x_3, x_5\rangle^{*\vee} = [4,6]^\vee, \\
[\langle x_1, x_2\rangle, \langle x_4, x_5\rangle] \mapsto \langle x_1, x_2, x_4, x_5\rangle^{*\vee} = [3,6]^\vee, \\
...], \\
[[\langle x_2, x_3\rangle, \langle x_4, x_5, x_0\rangle] \mapsto \langle x_2, x_3, x_4, x_5, x_0\rangle^{*\vee} = [1]^\vee, \\
...], \\
[]
\end{array}$$

Due to the singularities of Y° this covering involves degenerate faces, i.e., faces $G \mapsto F^\vee$ with $\dim(G) < \dim(F^\vee)$.

Mirror degeneration The space of first order deformations of X_0° in the mirror degeneration $\mathfrak{X}^\circ$ has dimension 6 and the deformations represented by the monomials

$$\left\{ \begin{array}{lllll} \frac{y_2^3 y_{10}{}^2}{y_7\, y_8\, y_9\, y_{11}\, y_{12}} & \frac{y_1^3 y_9{}^2}{y_7\, y_8\, y_{10}\, y_{11}\, y_{12}} & \frac{y_4^2 y_8{}^3}{y_1\, y_2\, y_3\, y_5\, y_6} & \frac{y_3^2 y_7{}^3}{y_1\, y_2\, y_4\, y_5\, y_6} & \frac{y_6^3 y_{12}{}^2}{y_7\, y_8\, y_9\, y_{10}\, y_{11}} \\ \frac{y_5^2 y_{11}{}^3}{y_1\, y_2\, y_3\, y_4\, y_6} \end{array} \right\}$$

form a torus invariant basis. The number of lattice points of the dual of the mirror complex of I relates to the dimension $h^{1,2}(X^\circ)$ of complex moduli space of the generic fiber X° of $\mathfrak{X}^\circ$ and to the dimension $h^{1,1}(X)$ of the

Kähler moduli space of the generic fiber X of $\mathfrak{X}$ via

$$\begin{aligned}|\operatorname{supp}((\mu(B(I)))^*)\cap N| &= 6 = 5+1\\ &= \dim(\operatorname{Aut}(Y^\circ)) + h^{1,2}(X^\circ) = \dim(T) + h^{1,1}(X)\end{aligned}$$

The mirror degeneration $\mathfrak{X}^\circ \subset Y^\circ \times \operatorname{Spec}\mathbb{C}[t]$ of $\mathfrak{X}$ is given by the ideal $I^\circ \subset S^\circ \otimes \mathbb{C}[t]$ generated by

$$\begin{array}{l} t\,(s_4\,y_3^3y_7^3 + s_3\,y_4^3y_8^3 + s_6\,y_5^3y_{11}^3) + y_6\,y_1\,y_2\,y_3\,y_4\,y_5 \\ t\,(s_2\,y_1^3y_9^3 + s_1\,y_2^3y_{10}^3 + s_5\,y_6^3y_{12}^3) + y_7\,y_8\,y_{11}\,y_{10}\,y_9\,y_{12} \end{array}$$

$$\begin{array}{ll} ts_2\,y_1^4y_9^2 + y_7\,y_8\,y_1\,y_{11}\,y_{10}\,y_{12} & ts_4\,y_7^4y_3^2 + y_6\,y_7\,y_1\,y_2\,y_4\,y_5 \\ ts_1\,y_2^4y_{10}^2 + y_7\,y_8\,y_2\,y_{11}\,y_9\,y_{12} & ts_3\,y_8^4y_4^2 + y_6\,y_8\,y_1\,y_2\,y_3\,y_5 \\ ts_6\,y_{11}^4y_5^2 + y_6\,y_1\,y_2\,y_3\,y_4\,y_{11} & ts_5\,y_6^4y_{12}^2 + y_6\,y_7\,y_8\,y_{11}\,y_{10}\,y_9 \end{array}$$

and 56 monomials of degree 6

The ideal J° which is $\operatorname{Pic}(Y^\circ)$-generated by

$$\left\{\begin{array}{l} t(s_4\,y_3^3y_7^3 + s_3\,y_4^3y_8^3 + s_6\,y_5^3y_{11}^3) + y_6\,y_1\,y_2\,y_3\,y_4\,y_5, \\ t(s_2\,y_1^3y_9^3 + s_1\,y_2^3y_{10}^3 + s_5\,y_6^3y_{12}^3) + y_7\,y_8\,y_{11}\,y_{10}\,y_9\,y_{12} \end{array}\right\}$$

defines a flat affine cone inducing $\mathfrak{X}^\circ$.

Contraction of the mirror degeneration In the following we give a birational map relating the degeneration $\mathfrak{X}^\circ$ to a Greene-Plesser type orbifolding mirror family by contracting divisors on Y°. See also Section 9.13 below. In order to contract the divisors

$$\begin{array}{ll} y_3 = y_{(-1,2,-1,0,0)} = \frac{x_2^2}{x_1\,x_3} & y_4 = y_{(-1,-1,2,0,0)} = \frac{x_3^2}{x_1\,x_2} \\ y_5 = y_{(2,-1,-1,0,0)} = \frac{x_1^2}{x_2\,x_3} & y_9 = y_{(0,0,0,-1,2)} = \frac{x_5^2}{x_4\,x_0} \\ y_{10} = y_{(0,0,0,-1,-1)} = \frac{x_0^2}{x_4\,x_5} & y_{12} = y_{(0,0,0,2,-1)} = \frac{x_4^2}{x_0\,x_5} \end{array}$$

consider the $\mathbb{Q}$-factorial toric Fano variety $\hat{Y}^\circ = X(\hat{\Sigma}^\circ)$, where $\hat{\Sigma}^\circ = \Sigma(\hat{P}^\circ) \subset M_{\mathbb{R}}$ is the fan over the Fano polytope $\hat{P}^\circ \subset M_{\mathbb{R}}$ given as the convex hull of the remaining vertices of $P^\circ = \nabla^*$ corresponding to the Cox variables

$$\begin{array}{ll} y_{11} = y_{(3,0,0,-1,-1)} = \frac{x_1^3}{x_0\,x_4\,x_5} & y_2 = y_{(-1,-1,-1,0,0)} = \frac{x_0^3}{x_1\,x_2\,x_3} \\ y_8 = y_{(0,0,3,-1,-1)} = \frac{x_3^3}{x_0\,x_4\,x_5} & y_6 = y_{(-1,-1,-1,3,0)} = \frac{x_4^3}{x_1\,x_2\,x_3} \\ y_1 = y_{(-1,-1,-1,0,3)} = \frac{x_5^3}{x_1\,x_2\,x_3} & y_7 = y_{(0,3,0,-1,-1)} = \frac{x_2^3}{x_0\,x_4\,x_5} \end{array}$$

of the toric variety $\hat{Y}^{\circ}$ with Cox ring

$$\hat{S}^{\circ} = \mathbb{C}[y_{11}, y_2, y_8, y_6, y_1, y_7]$$

The Cox variables of $\hat{Y}^{\circ}$ correspond to the set of Fermat deformations of $\mathfrak{X}$. Let

$$Y^{\circ} = X\left(\Sigma^{\circ}\right) \to X(\hat{\Sigma}^{\circ}) = \hat{Y}^{\circ}$$

be a birational map from Y° to the toric Fano variety $\hat{Y}^{\circ}$, which contracts the divisors of the rays $\Sigma^{\circ}(1) - \hat{\Sigma}^{\circ}(1)$ corresponding to the Cox variables

$$y_3 \quad y_4 \quad y_5 \quad y_9 \quad y_{10} \quad y_{12}$$

Representing $\hat{Y}^{\circ}$ as a quotient we have

$$\hat{Y}^{\circ} = \left(\mathbb{C}^6 - V\left(B(\hat{\Sigma}^{\circ})\right)\right)/\hat{G}^{\circ}$$

with

$$\hat{G}^{\circ} = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9 \times \left(\mathbb{C}^*\right)^1$$

acting via

$$\xi y = \left(u_2^2 v_1 \cdot y_{11},\ u_1^2 u_2 u_3^4 v_1 \cdot y_2,\ u_2 u_3^3 v_1 \cdot y_8,\ u_2^2 u_3 v_1 \cdot y_6,\ u_1 u_3^4 v_1 \cdot y_1,\ v_1 \cdot y_7\right)$$

for $\xi = (u_1, u_2, u_3, v_1) \in \hat{G}^{\circ}$ and $y \in \mathbb{C}^6 - V\left(B(\hat{\Sigma}^{\circ})\right)$. Hence with the group

$$\hat{H}^{\circ} = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9$$

of order 81 the toric variety $\hat{Y}^{\circ}$ is the quotient

$$\hat{Y}^{\circ} = \mathbb{P}^5/\hat{H}^{\circ}$$

of projective space $\mathbb{P}^5$. The mirror degeneration $\mathfrak{X}^{\circ}$ induces via $Y^{\circ} \to \hat{Y}^{\circ}$ a degeneration $\hat{\mathfrak{X}}^{\circ} \subset \hat{Y}^{\circ} \times \operatorname{Spec}\mathbb{C}[t]$ given by the ideal $\hat{I}^{\circ} \subset \langle y_{11}, y_2, y_8, y_6, y_1, y_7\rangle \subset \hat{S}^{\circ} \otimes \mathbb{C}[t]$ generated by the Fermat-type equations

$$\left\{\begin{array}{l} y_2\, y_1\, y_6 + t(s_4\, y_7^3 + s_5\, y_8^3 + s_6\, y_{11}^3), \\ y_{11}\, y_7\, y_8 + t(s_3\, y_1^3 + s_2\, y_2^3 + s_1\, y_6^3) \end{array}\right\}$$

The special fiber $\hat{X}_0^{\circ} \subset \hat{Y}^{\circ}$ of $\hat{\mathfrak{X}}^{\circ}$ is cut out by the monomial ideal

$$\hat{I}_0^{\circ} \subset B(\hat{\Sigma}^{\circ}) = \langle y_{11}, y_2, y_8, y_6, y_1, y_7\rangle \subset \hat{S}^{\circ}$$

generated by

$$\left\{\ y_2\, y_1\, y_6 \quad y_{11}\, y_7\, y_8\ \right\}$$

The complex $SP(\hat{I}_0^{\circ})$ of prime ideals of the toric strata of $\hat{X}_0^{\circ}$ is

$[]$,

$[]$,

$[\langle y_{11}, y_6\rangle, \langle y_{11}, y_2\rangle, \langle y_2, y_8\rangle, \langle y_8, y_6\rangle, \langle y_6, y_7\rangle, \langle y_1, y_7\rangle,$
$\langle y_2, y_7\rangle, \langle y_{11}, y_1\rangle, \langle y_8, y_1\rangle]$,

$[\langle y_2, y_8, y_7\rangle, \langle y_2, y_8, y_6\rangle, \langle y_{11}, y_2, y_6\rangle, \langle y_{11}, y_2, y_1\rangle, \langle y_{11}, y_8, y_1\rangle,$
$\langle y_2, y_6, y_7\rangle, \langle y_6, y_1, y_7\rangle, \langle y_{11}, y_6, y_1\rangle, \langle y_{11}, y_8, y_6\rangle, \langle y_{11}, y_6, y_7\rangle,$
$\langle y_8, y_6, y_7\rangle, \langle y_8, y_6, y_1\rangle, \langle y_8, y_1, y_7\rangle, \langle y_{11}, y_2, y_8\rangle, \langle y_2, y_8, y_1\rangle,$
$\langle y_2, y_1, y_7\rangle, \langle y_{11}, y_1, y_7\rangle, \langle y_{11}, y_2, y_7\rangle]$,

$[\langle y_{11}, y_2, y_1, y_7\rangle, \langle y_{11}, y_2, y_8, y_6\rangle, \langle y_8, y_6, y_1, y_7\rangle, \langle y_2, y_6, y_1, y_7\rangle,$
$\langle y_2, y_8, y_6, y_7\rangle, \langle y_{11}, y_8, y_1, y_7\rangle, \langle y_{11}, y_6, y_1, y_7\rangle, \langle y_{11}, y_2, y_8, y_1\rangle,$
$\langle y_{11}, y_2, y_8, y_7\rangle, \langle y_2, y_8, y_1, y_7\rangle, \langle y_{11}, y_2, y_6, y_1\rangle, \langle y_{11}, y_8, y_6, y_7\rangle,$
$\langle y_{11}, y_8, y_6, y_1\rangle, \langle y_2, y_8, y_6, y_1\rangle, \langle y_{11}, y_2, y_6, y_7\rangle]$,

$[\langle y_2, y_8, y_6, y_1, y_7\rangle, \langle y_{11}, y_8, y_6, y_1, y_7\rangle, \langle y_{11}, y_2, y_6, y_1, y_7\rangle, \langle y_{11}, y_2, y_8, y_1, y_7\rangle,$
$\langle y_{11}, y_2, y_8, y_6, y_7\rangle, \langle y_{11}, y_2, y_8, y_6, y_1\rangle]$,

$[]$

so $\hat{I}_0^\circ$ has the primary decomposition

$$\begin{array}{rl}\hat{I}_0^\circ = & \langle y_8, y_1\rangle \cap \langle y_8, y_6\rangle \cap \langle y_2, y_8\rangle \cap \langle y_{11}, y_6\rangle \cap \langle y_{11}, y_1\rangle \cap \langle y_{11}, y_2\rangle \cap \\ & \cap \langle y_1, y_7\rangle \cap \langle y_6, y_7\rangle \cap \langle y_2, y_7\rangle\end{array}$$

The Bergman subcomplex $B(I) \subset \operatorname{Poset}(\nabla)$ induces a subcomplex of $\operatorname{Poset}(\hat{\nabla})$, $\hat{\nabla} = (\hat{P}^\circ)^*$ corresponding to $\hat{I}_0^\circ$. Indexing of the vertices of $\hat{\nabla}$ by

$$\begin{array}{ll} 1 = \left(\frac{5}{3}, -\frac{1}{3}, -\frac{1}{3}, 0, 0\right) & 2 = \left(-\frac{5}{3}, -\frac{5}{3}, -\frac{5}{3}, -2, -2\right) \\ 3 = \left(-\frac{1}{3}, -\frac{1}{3}, \frac{5}{3}, 0, 0\right) & 4 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 2, 0\right) \\ 5 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 2\right) & 6 = \left(-\frac{1}{3}, \frac{5}{3}, -\frac{1}{3}, 0, 0\right) \end{array}$$

this complex is given by

$$\left\{\begin{array}{l}[],\\ [[1],[2],[3],[4],[5],[6]],\\ [[3,4],[5,6],[1,2],[1,3],[1,5],[2,4],[2,3],[4,6],[4,5],[1,4],[3,6],\\ [2,5],[2,6],[1,6],[3,5]],\\ [[1,4,5],[1,5,6],[3,5,6],[3,4,6],[2,4,6],[1,3,5],[1,2,3],[2,3,6],\\ [2,5,6],[2,3,5],[1,2,5],[1,2,6],[1,2,4],[4,5,6],[1,4,6],[1,3,4],\\ [2,3,4],[3,4,5]],\\ [[2,3,5,6],[3,4,5,6],[1,4,5,6],[1,2,5,6],[1,2,3,5],[1,2,3,4],\\ [1,3,4,5],[2,3,4,6],[1,2,4,6]],\\ [],\\ []\end{array}\right\}$$

This is the one-parameter Greene-Plesser orbifolding mirror family of the generic complete intersection of two cubics in $\mathbb{P}^5$ given in [Libgober, Teitelbaum, 1993].

8.12.5 The Calabi-Yau threefold given as the Pfaffian complete intersection of two generic quadrics and a generic cubic in $\mathbb{P}^6$

Setup Let $Y = \mathbb{P}^6 = X(\Sigma)$, $\Sigma = \Sigma(P) = NF(\Delta) \subset N_{\mathbb{R}}$ with the Fano polytope $P = \Delta^*$ given by

$$\Delta = \text{convexhull}\begin{pmatrix} (6,-1,-1,-1,-1,-1) & (-1,6,-1,-1,-1,-1) \\ (-1,-1,6,-1,-1,-1) & (-1,-1,-1,6,-1,-1) \\ (-1,-1,-1,-1,6,-1) & (-1,-1,-1,-1,-1,6) \\ (-1,-1,-1,-1,-1,-1) & \end{pmatrix} \subset M_{\mathbb{R}}$$

and let

$$S = \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5, x_6]$$

be the Cox ring of Y with the variables

$$\begin{array}{ll} x_1 = x_{(1,0,0,0,0,0)} & x_2 = x_{(0,1,0,0,0,0)} \\ x_3 = x_{(0,0,1,0,0,0)} & x_4 = x_{(0,0,0,1,0,0)} \\ x_5 = x_{(0,0,0,0,1,0)} & x_6 = x_{(0,0,0,0,0,1)} \\ x_0 = x_{(-1,-1,-1,-1,-1,-1)} & \end{array}$$

associated to the rays of Σ. Consider the degeneration $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[t]$ of Pfaffian complete intersection Calabi-Yau 3-folds with Buchsbaum-Eisenbud resolution

$$0 \to \mathcal{O}_Y(-7) \to \mathcal{E}(-3) \xrightarrow{A_t} \mathcal{E}^*(-2) \to \mathcal{O}_Y \to \mathcal{O}_{X_t} \to 0$$

where

$$\mathcal{E} = \ 2\mathcal{O}(1) \oplus \mathcal{O}$$

$$A_t = A_0 + t \cdot A$$

$$A_0 = \begin{bmatrix} 0 & x_0\,x_5\,x_6 & -x_3\,x_4 \\ -x_0\,x_5\,x_6 & 0 & x_1\,x_2 \\ x_3\,x_4 & -x_1\,x_2 & 0 \end{bmatrix}$$

the monomial special fiber of $\mathfrak{X}$ is given by

$$I_0 = \left\langle \ x_1\,x_2 \quad x_3\,x_4 \quad x_0\,x_5\,x_6 \ \right\rangle$$

and generic $A \in \bigwedge^2 \mathcal{E}^*(1)$. The degeneration $\mathfrak{X}$ is given by the ideal $I \subset S \otimes \mathbb{C}[t]$ with I_0-reduced generators of degrees $2, 2, 3$

$$\left\{ \begin{array}{l} x_1\,x_2 + t(s_1\,x_1^2 + ... + s_7\,x_2^2 + ... + s_{13}\,x_3^2 + ... + s_{17}\,x_4^2 + ... + s_{21}\,x_5^2 + ...+ \\ s_{24}\,x_6^2 + ... + s_{26}\,x_0^2), \\ x_3\,x_4 + t(s_{27}\,x_1^2 + ... + s_{33}\,x_2^2 + ... + s_{39}\,x_3^2 + ... + s_{43}\,x_4^2 + ... + s_{47}\,x_5^2 + ...+ \\ s_{50}\,x_6^2 + ... + s_{52}\,x_0^2), \\ x_0\,x_5\,x_6 + t(s_{53}\,x_1^3 + ... + s_{73}\,x_2^3 + ... + s_{93}\,x_3^3 + ... + s_{103}\,x_4^3 + ... + s_{113}\,x_5^3 + \\ ... + s_{118}\,x_6^3 + ... + s_{121}\,x_0^3) \end{array} \right\}$$

Special fiber Gröbner cone The space of first order deformations of $\mathfrak{X}$ has dimension 121 and the deformations represented by the Cox Laurent monomials

$\frac{x_3^3}{x_0\,x_5\,x_6}$	$\frac{x_1^3}{x_0\,x_5\,x_6}$	$\frac{x_4^3}{x_0\,x_5\,x_6}$	$\frac{x_2^3}{x_0\,x_5\,x_6}$	$\frac{x_4^2 x_2}{x_0\,x_5\,x_6}$	$\frac{x_4\,{x_2}^2}{x_0\,x_5\,x_6}$	$\frac{x_3\,{x_2}^2}{x_0\,x_5\,x_6}$	$\frac{x_3^2 x_2}{x_0\,x_5\,x_6}$
$\frac{x_3^2 x_1}{x_0\,x_5\,x_6}$	$\frac{x_4^2 x_1}{x_0\,x_5\,x_6}$	$\frac{x_3\,{x_1}^2}{x_0\,x_5\,x_6}$	$\frac{x_4\,{x_1}^2}{x_0\,x_5\,x_6}$	$\frac{x_4\,x_0}{x_1\,x_2}$	$\frac{x_4^2}{x_1\,x_2}$	$\frac{x_0^2}{x_1\,x_2}$	$\frac{x_1^2}{x_0\,x_6}$
$\frac{x_3^2}{x_0\,x_6}$	$\frac{x_1^2}{x_5\,x_6}$	$\frac{x_2^2}{x_0\,x_6}$	$\frac{x_3^2}{x_1\,x_2}$	$\frac{x_6^2}{x_1\,x_2}$	$\frac{x_6^2}{x_3\,x_4}$	$\frac{x_2^2}{x_3\,x_4}$	$\frac{x_5^2}{x_0\,x_6}$
$\frac{x_4\,x_5}{x_1\,x_2}$	$\frac{x_5\,x_3}{x_1\,x_2}$	$\frac{x_2^2}{x_0\,x_5}$	$\frac{x_3\,x_0}{x_5\,x_6}$	$\frac{x_1\,x_0}{x_3\,x_4}$	$\frac{x_1\,x_0}{x_5\,x_6}$	$\frac{x_0\,x_6}{x_1\,x_2}$	$\frac{x_0\,x_6}{x_3\,x_4}$
$\frac{x_5\,x_3}{x_0\,x_6}$	$\frac{x_6\,x_3}{x_1\,x_2}$	$\frac{x_6\,x_3}{x_0\,x_5}$	$\frac{x_6\,x_1}{x_3\,x_4}$	$\frac{x_6\,x_1}{x_0\,x_5}$	$\frac{x_5\,x_1}{x_3\,x_4}$	$\frac{x_5\,x_1}{x_0\,x_6}$	$\frac{x_1\,x_4}{x_0\,x_5}$
$\frac{x_1\,x_3}{x_0\,x_5}$	$\frac{x_2\,x_4}{x_0\,x_5}$	$\frac{x_2\,x_3}{x_0\,x_5}$	$\frac{x_3\,x_0}{x_1\,x_2}$	$\frac{x_2\,x_0}{x_3\,x_4}$	$\frac{x_0\,x_5}{x_1\,x_2}$	$\frac{x_2\,x_0}{x_5\,x_6}$	$\frac{x_4\,x_0}{x_5\,x_6}$
$\frac{x_6\,x_2}{x_3\,x_4}$	$\frac{x_6\,x_2}{x_0\,x_5}$	$\frac{x_1\,x_3}{x_5\,x_6}$	$\frac{x_1\,x_3}{x_0\,x_6}$	$\frac{x_2\,x_3}{x_0\,x_6}$	$\frac{x_2\,x_3}{x_5\,x_6}$	$\frac{x_1\,x_4}{x_0\,x_6}$	$\frac{x_2\,x_4}{x_0\,x_6}$
$\frac{x_1\,x_4}{x_5\,x_6}$	$\frac{x_2\,x_4}{x_5\,x_6}$	$\frac{x_0\,x_5}{x_3\,x_4}$	$\frac{x_6\,x_4}{x_1\,x_2}$	$\frac{x_6\,x_4}{x_0\,x_5}$	$\frac{x_5\,x_4}{x_0\,x_6}$	$\frac{x_5\,x_2}{x_3\,x_4}$	$\frac{x_5\,x_2}{x_0\,x_6}$
$\frac{x_5\,x_6}{x_1\,x_2}$	$\frac{x_5\,x_6}{x_3\,x_4}$	$\frac{x_4^2}{x_5\,x_6}$	$\frac{x_3^2}{x_5\,x_6}$	$\frac{x_0^2}{x_3\,x_4}$	$\frac{x_6^2}{x_0\,x_5}$	$\frac{x_2^2}{x_5\,x_6}$	$\frac{x_4^2}{x_0\,x_6}$

$\frac{x_1^2}{x_3\,x_4}$	$\frac{x_5^2}{x_3\,x_4}$	$\frac{x_0^2}{x_5\,x_6}$	$\frac{x_1^2}{x_0\,x_5}$	$\frac{x_4^2}{x_0\,x_5}$	$\frac{x_3^2}{x_0\,x_5}$	$\frac{x_5^2}{x_1\,x_2}$	$\frac{x_2}{x_5}$
$\frac{x_1}{x_6}$	$\frac{x_2}{x_6}$	$\frac{x_4}{x_6}$	$\frac{x_3}{x_6}$	$\frac{x_0}{x_3}$	$\frac{x_3}{x_0}$	$\frac{x_2}{x_1}$	$\frac{x_1}{x_2}$
$\frac{x_0}{x_1}$	$\frac{x_1}{x_3}$	$\frac{x_0}{x_2}$	$\frac{x_2}{x_0}$	$\frac{x_3}{x_1}$	$\frac{x_1}{x_0}$	$\frac{x_3}{x_2}$	$\frac{x_2}{x_3}$
$\frac{x_6}{x_1}$	$\frac{x_6}{x_5}$	$\frac{x_5}{x_2}$	$\frac{x_6}{x_4}$	$\frac{x_6}{x_2}$	$\frac{x_5}{x_1}$	$\frac{x_5}{x_6}$	$\frac{x_0}{x_6}$
$\frac{x_6}{x_0}$	$\frac{x_0}{x_4}$	$\frac{x_6}{x_3}$	$\frac{x_4}{x_2}$	$\frac{x_4}{x_0}$	$\frac{x_4}{x_3}$	$\frac{x_4}{x_1}$	$\frac{x_3}{x_4}$
$\frac{x_1}{x_4}$	$\frac{x_2}{x_4}$	$\frac{x_1}{x_5}$	$\frac{x_5}{x_4}$	$\frac{x_5}{x_3}$	$\frac{x_5}{x_0}$	$\frac{x_4}{x_5}$	$\frac{x_3}{x_5}$
$\frac{x_0}{x_5}$							

form a torus invariant basis. These deformations give linear inequalities defining the special fiber Gröbner cone, i.e.,

$$C_{I_0}(I) = \left\{(w_t, w_y) \in \mathbb{R} \oplus N_{\mathbb{R}} \mid \left\langle w, A^{-1}(m)\right\rangle \geq -w_t \forall m\right\}$$

for above monomials m and the presentation matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

of $A_5(Y)$. The vertices of the special fiber polytope

$$\nabla = C_{I_0}(I) \cap \{w_t = 1\}$$

and the number of faces, each vertex is contained in, are given by the table

15	$(0,0,0,0,1,0)$	$(0,0,0,0,0,1)$	$(-1,-1,-1,-1,-1,-1)$
	$(0,0,1,0,0,0)$	$(0,0,0,1,0,0)$	$(0,1,0,0,0,0)$
	$(1,0,0,0,0,0)$		
25	$(1,0,0,0,1,0)$	$(0,1,0,0,1,0)$	$(0,0,1,0,1,0)$
	$(0,0,0,1,1,0)$	$(1,0,0,0,0,1)$	$(0,1,0,0,0,1)$
	$(0,0,1,0,0,1)$	$(0,0,0,1,0,1)$	$(0,-1,-1,-1,-1,-1)$
	$(-1,0,-1,-1,-1,-1)$	$(-1,-1,0,-1,-1,-1)$	$(-1,-1,-1,0,-1,-1)$
	$(1,0,1,0,0,0)$	$(0,1,1,0,0,0)$	$(1,0,0,1,0,0)$
	$(0,1,0,1,0,0)$		

30	$(1,0,1,0,1,0)$	$(0,1,1,0,1,0)$	$(1,0,0,1,1,0)$
	$(0,1,0,1,1,0)$	$(1,0,1,0,0,1)$	$(0,1,1,0,0,1)$
	$(1,0,0,1,0,1)$	$(0,1,0,1,0,1)$	$(0,-1,0,-1,-1,-1)$
	$(-1,0,0,-1,-1,-1)$	$(0,-1,-1,0,-1,-1)$	$(-1,0,-1,0,-1,-1)$

The number of faces of ∇ and their F-vectors are

Dimension	Number of faces	F-vector	
−1	1	$(0,0,0,0,0,0,0,0)$	
0	35	$(1,1,0,0,0,0,0,0)$	point
1	119	$(1,2,1,0,0,0,0,0)$	edge
2	55	$(1,3,3,1,0,0,0,0)$	triangle
2	128	$(1,4,4,1,0,0,0,0)$	quadrangle
3	110	$(1,6,9,5,1,0,0,0)$	prism
3	42	$(1,8,12,6,1,0,0,0)$	cube
3	8	$(1,4,6,4,1,0,0,0)$	tetrahedron
4	47	$(1,12,24,19,7,1,0,0)$	
4	21	$(1,9,18,15,6,1,0,0)$	
4	14	$(1,8,16,14,6,1,0,0)$	
5	4	$(1,16,40,44,26,8,1,0)$	
5	13	$(1,18,45,48,27,8,1,0)$	
5	4	$(1,12,30,34,21,7,1,0)$	
6	1	$(1,35,119,183,160,82,21,1)$	

The dual $P^\circ = \nabla^*$ of ∇ is a Fano polytope with vertices

$(-1,-1,0,0,0,2)$	$(-1,-1,0,0,0,0)$	$(-1,-1,0,2,0,0)$	$(-1,1,0,0,0,0)$
$(1,-1,0,0,0,0)$	$(-1,-1,2,0,0,0)$	$(-1,-1,0,0,2,0)$	$(0,0,-1,-1,0,2)$
$(0,0,-1,-1,0,0)$	$(0,2,-1,-1,0,0)$	$(0,0,-1,1,0,0)$	$(2,0,-1,-1,0,0)$
$(0,0,1,-1,0,0)$	$(0,0,-1,-1,2,0)$	$(0,3,0,0,-1,-1)$	$(0,0,0,3,-1,-1)$
$(0,0,0,0,-1,2)$	$(0,0,0,0,-1,-1)$	$(3,0,0,0,-1,-1)$	$(0,0,3,0,-1,-1)$
$(0,0,0,0,2,-1)$			

and the F-vectors of the faces of P° are

Dimension	Number of faces	F-vector	
−1	1	(0, 0, 0, 0, 0, 0, 0, 0)	
0	21	(1, 1, 0, 0, 0, 0, 0, 0)	point
1	82	(1, 2, 1, 0, 0, 0, 0, 0)	edge
2	59	(1, 4, 4, 1, 0, 0, 0, 0)	quadrangle
2	101	(1, 3, 3, 1, 0, 0, 0, 0)	triangle
3	102	(1, 6, 9, 5, 1, 0, 0, 0)	prism
3	81	(1, 4, 6, 4, 1, 0, 0, 0)	tetrahedron
4	37	(1, 5, 10, 10, 5, 1, 0, 0)	
4	24	(1, 9, 18, 15, 6, 1, 0, 0)	
4	58	(1, 8, 16, 14, 6, 1, 0, 0)	
5	7	(1, 6, 15, 20, 15, 6, 1, 0)	
5	12	(1, 12, 30, 34, 21, 7, 1, 0)	
5	16	(1, 10, 25, 30, 20, 7, 1, 0)	
6	1	(1, 21, 82, 160, 183, 119, 35, 1)	

The polytopes ∇ and P° have 0 as the unique interior lattice point, and the Fano polytope P° defines the embedding toric Fano variety $Y^{\circ} = X(\Sigma(P^{\circ}))$ of the fibers of the mirror degeneration. The dimension of the automorphism group of Y° is

$$\dim(\mathrm{Aut}(Y^{\circ})) = 6$$

Let

$$S^{\circ} = \mathbb{C}[y_1, y_2, ..., y_{21}]$$

be the Cox ring of Y° with variables

$y_1 = y_{(-1,-1,0,0,0,2)} = \frac{x_6^2}{x_1 x_2}$ $y_2 = y_{(-1,-1,0,0,0,0)} = \frac{x_0^2}{x_1 x_2}$ $y_3 = y_{(-1,-1,0,2,0,0)} = \frac{x_4^2}{x_1 x_2}$
$y_4 = y_{(-1,1,0,0,0,0)} = \frac{x_2}{x_1}$ $y_5 = y_{(1,-1,0,0,0,0)} = \frac{x_1}{x_2}$ $y_6 = y_{(-1,-1,2,0,0,0)} = \frac{x_3^2}{x_1 x_2}$
$y_7 = y_{(-1,-1,0,0,2,0)} = \frac{x_5^2}{x_1 x_2}$ $y_8 = y_{(0,0,-1,-1,0,2)} = \frac{x_6^2}{x_3 x_4}$ $y_9 = y_{(0,0,-1,-1,0,0)} = \frac{x_0^2}{x_3 x_4}$
$y_{10} = y_{(0,2,-1,-1,0,0)} = \frac{x_2^2}{x_3 x_4}$ $y_{11} = y_{(0,0,-1,1,0,0)} = \frac{x_4}{x_3}$ $y_{12} = y_{(2,0,-1,-1,0,0)} = \frac{x_1^2}{x_3 x_4}$
$y_{13} = y_{(0,0,1,-1,0,0)} = \frac{x_3}{x_4}$ $y_{14} = y_{(0,0,-1,-1,2,0)} = \frac{x_5^2}{x_3 x_4}$ $y_{15} = y_{(0,3,0,0,-1,-1)} = \frac{x_2^3}{x_0 x_5 x_6}$
$y_{16} = y_{(0,0,0,3,-1,-1)} = \frac{x_4^3}{x_0 x_5 x_6}$ $y_{17} = y_{(0,0,0,0,-1,2)} = \frac{x_6^2}{x_0 x_5}$ $y_{18} = y_{(0,0,0,0,-1,-1)} = \frac{x_0^2}{x_5 x_6}$
$y_{19} = y_{(3,0,0,0,-1,-1)} = \frac{x_1^3}{x_0 x_5 x_6}$ $y_{20} = y_{(0,0,3,0,-1,-1)} = \frac{x_3^3}{x_0 x_5 x_6}$ $y_{21} = y_{(0,0,0,0,2,-1)} = \frac{x_5^2}{x_6 x_0}$

Bergman subcomplex Intersecting the tropical variety of I with the special fiber Gröbner cone $C_{I_0}(I)$ we obtain the special fiber Bergman subcomplex

$$B(I) = C_{I_0}(I) \cap BF(I) \cap \{w_t = 1\}$$

The following table chooses an indexing of the vertices of ∇ involved in $B(I)$ and gives for each vertex the numbers n_∇ and n_B of faces of ∇ and faces of $B(I)$ it is contained in

n_∇	$n_{B(I)}$		
30	4	$1 = (1,0,1,0,1,0)$	$2 = (0,1,1,0,1,0)$
		$3 = (1,0,0,1,1,0)$	$4 = (0,1,0,1,1,0)$
		$5 = (1,0,1,0,0,1)$	$6 = (0,1,1,0,0,1)$
		$7 = (1,0,0,1,0,1)$	$8 = (0,1,0,1,0,1)$
		$9 = (0,-1,0,-1,-1,-1)$	$10 = (-1,0,0,-1,-1,-1)$
		$11 = (0,-1,-1,0,-1,-1)$	$12 = (-1,0,-1,0,-1,-1)$

With this indexing the Bergman subcomplex $B(I)$ of Poset(∇) associated to the degeneration $\mathfrak{X}$ is

[],

[[1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]],

[[9, 11], [9, 10], [10, 12], [1, 5], [1, 3], [1, 2], [1, 9], [11, 12], [2, 6], [2, 4], [2, 10], [5, 6], [5, 7], [5, 9], [6, 10], [6, 8], [3, 7], [3, 4], [3, 11], [7, 11], [7, 8], [8, 12], [4, 12], [4, 8]],

[[1, 3, 9, 11], [1, 2, 5, 6], [1, 3, 5, 7], [6, 8, 10, 12], [2, 4, 10, 12], [3, 4, 11, 12], [1, 5, 9], [7, 8, 11, 12], [1, 2, 3, 4], [2, 4, 6, 8], [2, 6, 10], [4, 8, 12], [3, 4, 7, 8], [9, 10, 11, 12], [5, 7, 9, 11], [5, 6, 9, 10], [1, 2, 9, 10], [3, 7, 11], [5, 6, 7, 8]],

[[1, 3, 5, 7, 9, 11], [1, 2, 3, 4, 9, 10, 11, 12], [1, 2, 3, 4, 5, 6, 7, 8], [1, 2, 5, 6, 9, 10], [5, 6, 7, 8, 9, 10, 11, 12], [2, 4, 6, 8, 10, 12], [3, 4, 7, 8, 11, 12]],

[],

[],

[]

$B(I)$ has the F-vector

Dimension	−1	0	1	2	3	4	5	6
Number of faces	0	12	24	19	7	0	0	0

and the F-vectors of its faces are

Dimension	Number of faces	F-vector	
0	12	$(1,1,0,0,0,0,0,0)$	point
1	24	$(1,2,1,0,0,0,0,0)$	edge
2	15	$(1,4,4,1,0,0,0,0)$	quadrangle
2	4	$(1,3,3,1,0,0,0,0)$	triangle
3	3	$(1,8,12,6,1,0,0,0)$	cube
3	4	$(1,6,9,5,1,0,0,0)$	prism

Dual complex The dual complex $\operatorname{dual}(B(I)) = (B(I))^*$ of deformations associated to $B(I)$ via initial ideals is given by

[],

[],

[],

$[[1,2,3,4,9,10,11,12]^* = \left\langle \frac{x_6^2}{x_1 x_2}, \frac{x_6^2}{x_3 x_4}, \frac{x_6^2}{x_0 x_5} \right\rangle, [1,3,5,7,9,11]^* = \left\langle \frac{x_2}{x_1}, \frac{x_2^2}{x_3 x_4}, \frac{x_2^3}{x_0 x_5 x_6} \right\rangle,$
...],

$[[1,3,9,11]^* = \left\langle \frac{x_6^2}{x_1 x_2}, \frac{x_2}{x_1}, \frac{x_6^2}{x_3 x_4}, \frac{x_2^2}{x_3 x_4}, \frac{x_2^3}{x_0 x_5 x_6}, \frac{x_6^2}{x_0 x_5} \right\rangle,$
$[1,5,9]^* = \left\langle \frac{x_4^2}{x_1 x_2}, \frac{x_2}{x_1}, \frac{x_2^2}{x_3 x_4}, \frac{x_4}{x_3}, \frac{x_2^3}{x_0 x_5 x_6}, \frac{x_4^3}{x_0 x_5 x_6} \right\rangle,$
...],

$[[9,11]^* = \left\langle \frac{x_5^2}{x_1 x_2}, \frac{x_6^2}{x_1 x_2}, \frac{x_2}{x_1}, \frac{x_5^2}{x_3 x_4}, \frac{x_6^2}{x_3 x_4}, \frac{x_2^2}{x_3 x_4}, \frac{x_2^3}{x_0 x_5 x_6}, \frac{x_5^2}{x_0 x_6}, \frac{x_6^2}{x_0 x_5} \right\rangle,$
...],

$[[1]^* = \left\langle \frac{x_6^2}{x_1 x_2}, \frac{x_0^2}{x_1 x_2}, \frac{x_4^2}{x_1 x_2}, \frac{x_2}{x_1}, \frac{x_6^2}{x_3 x_4}, \frac{x_0^2}{x_3 x_4}, \frac{x_2^2}{x_3 x_4}, \frac{x_4}{x_3}, \frac{x_2^3}{x_0 x_5 x_6}, \frac{x_4^3}{x_0 x_5 x_6}, \frac{x_6^2}{x_0 x_5}, \frac{x_0^2}{x_5 x_6} \right\rangle,$
...],

[]

when writing the vertices of the faces as deformations of X_0. Note that the T-invariant basis of deformations associated to a face is given by all lattice points of the corresponding polytope in $M_{\mathbb{R}}$. In order to compress the output we list one representative in any set of faces G with fixed F-vector of G and of G^*. When numbering the vertices of the faces of $\operatorname{dual}(B(I))$ by the Cox variables of the mirror toric Fano variety Y° the complex $\operatorname{dual}(B(I))$ is

$[]$,

$[]$,

$[]$,

$[[1,2,3,4,9,10,11,12]^* = \langle y_1, y_8, y_{17}\rangle, [1,3,5,7,9,11]^* = \langle y_4, y_{10}, y_{15}\rangle,$
$...]$,

$[[1,3,9,11]^* = \langle y_1, y_4, y_8, y_{10}, y_{15}, y_{17}\rangle,$
$[1,5,9]^* = \langle y_3, y_4, y_{10}, y_{11}, y_{15}, y_{16}\rangle, ...]$,

$[[9,11]^* = \langle y_7, y_1, y_4, y_{14}, y_8, y_{10}, y_{15}, y_{21}, y_{17}\rangle,$
$...]$,

$[[1]^* = \langle y_1, y_2, y_3, y_4, y_8, y_9, y_{10}, y_{11}, y_{15}, y_{16}, y_{17}, y_{18}\rangle,$
$...]$,

$[]$

The dual complex has the F-vector

Dimension	−1	0	1	2	3	4	5	6
Number of faces	0	0	0	7	19	24	12	0

and the F-vectors of the faces of $\operatorname{dual}(B(I))$ are

Dimension	Number of faces	F-vector	
2	7	$(1,3,3,1,0,0,0,0)$	triangle
3	19	$(1,6,9,5,1,0,0,0)$	prism
4	24	$(1,9,18,15,6,1,0,0)$	
5	12	$(1,12,30,34,21,7,1,0)$	

Recall that in this example the toric variety Y is projective space. The number of lattice points of the support of $\mathrm{dual}(B(I))$ relates to the dimension $h^{1,2}(X)$ of the complex moduli space of the generic fiber X of $\mathfrak{X}$ and to the dimension $h^{1,1}(\bar{X}^{\circ})$ of the Kähler moduli space of the $MPCP$-blowup $\bar{X}^{\circ}$ of the generic fiber X° of the mirror degeneration

$$\begin{aligned} |\mathrm{supp}\,(\mathrm{dual}\,(B\,(I)))\cap M| &= 121 = 48+73 = \dim(\mathrm{Aut}\,(Y)) + h^{1,2}(X) \\ &= 42+6+73 \\ &= |\mathrm{Roots}\,(Y)| + \dim(T_Y) + h^{1,1}\left(\bar{X}^{\circ}\right) \end{aligned}$$

There are

$$h^{1,2}(X) + \dim(T_{Y^{\circ}}) = 73+6$$

non-trivial toric polynomial deformations of X_0

$\frac{x_2^3}{x_0\,x_5\,x_6}$	$\frac{x_2^2}{x_3\,x_4}$	$\frac{x_6^2}{x_0\,x_5}$	$\frac{x_6^2}{x_3\,x_4}$	$\frac{x_6^2}{x_1\,x_2}$	$\frac{x_0^2}{x_5\,x_6}$	$\frac{x_0^2}{x_3\,x_4}$	$\frac{x_0^2}{x_1\,x_2}$	$\frac{x_4^3}{x_0\,x_5\,x_6}$	$\frac{x_4^2}{x_1\,x_2}$
$\frac{x_5^2}{x_0\,x_6}$	$\frac{x_5^2}{x_3\,x_4}$	$\frac{x_5^2}{x_1\,x_2}$	$\frac{x_1^3}{x_0\,x_5\,x_6}$	$\frac{x_1^2}{x_3\,x_4}$	$\frac{x_3^3}{x_0\,x_5\,x_6}$	$\frac{x_3^2}{x_1\,x_2}$	$\frac{x_2^2}{x_0\,x_5}$	$\frac{x_6\,x_2}{x_0\,x_5}$	$\frac{x_6\,x_2}{x_3\,x_4}$
$\frac{x_4\,x_0}{x_5\,x_6}$	$\frac{x_4^2}{x_5\,x_6}$	$\frac{x_4\,x_0}{x_1\,x_2}$	$\frac{x_2\,x_0}{x_5\,x_6}$	$\frac{x_2^2}{x_5\,x_6}$	$\frac{x_2\,x_0}{x_3\,x_4}$	$\frac{x_1^2}{x_0\,x_6}$	$\frac{x_5\,x_1}{x_0\,x_6}$	$\frac{x_5\,x_1}{x_3\,x_4}$	$\frac{x_1^2}{x_0\,x_5}$
$\frac{x_6\,x_1}{x_0\,x_5}$	$\frac{x_6\,x_1}{x_3\,x_4}$	$\frac{x_3^2}{x_0\,x_5}$	$\frac{x_6\,x_3}{x_0\,x_5}$	$\frac{x_6\,x_3}{x_1\,x_2}$	$\frac{x_4\,{x_2}^2}{x_0\,x_5\,x_6}$	$\frac{x_4^2x_2}{x_0\,x_5\,x_6}$	$\frac{x_3^2}{x_0\,x_6}$	$\frac{x_5\,x_3}{x_0\,x_6}$	$\frac{x_5\,x_3}{x_1\,x_2}$
$\frac{x_0\,x_6}{x_3\,x_4}$	$\frac{x_0\,x_6}{x_1\,x_2}$	$\frac{x_1\,x_0}{x_5\,x_6}$	$\frac{x_1^2}{x_5\,x_6}$	$\frac{x_1\,x_0}{x_3\,x_4}$	$\frac{x_4\,{x_1}^2}{x_0\,x_5\,x_6}$	$\frac{x_4^2x_1}{x_0\,x_5\,x_6}$	$\frac{x_3\,{x_1}^2}{x_0\,x_5\,x_6}$	$\frac{x_3^2x_1}{x_0\,x_5\,x_6}$	$\frac{x_3\,x_0}{x_5\,x_6}$
$\frac{x_3^2}{x_5\,x_6}$	$\frac{x_3\,x_0}{x_1\,x_2}$	$\frac{x_5\,x_6}{x_3\,x_4}$	$\frac{x_5\,x_6}{x_1\,x_2}$	$\frac{x_2^2}{x_0\,x_6}$	$\frac{x_5\,x_2}{x_0\,x_6}$	$\frac{x_5\,x_2}{x_3\,x_4}$	$\frac{x_4^2}{x_0\,x_6}$	$\frac{x_5\,x_4}{x_0\,x_6}$	$\frac{x_5\,x_4}{x_1\,x_2}$
$\frac{x_4^2}{x_0\,x_5}$	$\frac{x_6\,x_4}{x_0\,x_5}$	$\frac{x_6\,x_4}{x_1\,x_2}$	$\frac{x_3\,{x_2}^2}{x_0\,x_5\,x_6}$	$\frac{x_3^2x_2}{x_0\,x_5\,x_6}$	$\frac{x_0\,x_5}{x_3\,x_4}$	$\frac{x_0\,x_5}{x_1\,x_2}$	$\frac{x_2\,x_4}{x_5\,x_6}$	$\frac{x_2\,x_4}{x_0\,x_5}$	$\frac{x_1\,x_4}{x_5\,x_6}$
$\frac{x_1\,x_4}{x_0\,x_5}$	$\frac{x_2\,x_4}{x_0\,x_6}$	$\frac{x_1\,x_4}{x_0\,x_6}$	$\frac{x_2\,x_3}{x_5\,x_6}$	$\frac{x_2\,x_3}{x_0\,x_5}$	$\frac{x_2\,x_3}{x_0\,x_6}$	$\frac{x_1\,x_3}{x_0\,x_6}$	$\frac{x_1\,x_3}{x_0\,x_5}$	$\frac{x_1\,x_3}{x_5\,x_6}$	

They correspond to the toric divisors

$D_{(0,3,0,0,-1,-1)}$	$D_{(0,2,-1,-1,0,0)}$	$D_{(0,0,0,0,-1,2)}$	$D_{(0,0,-1,-1,0,2)}$
$D_{(-1,-1,0,0,0,2)}$	$D_{(0,0,0,0,-1,-1)}$	$D_{(0,0,-1,-1,0,0)}$	$D_{(-1,-1,0,0,0,0)}$
$D_{(0,0,0,3,-1,-1)}$	$D_{(-1,-1,0,2,0,0)}$	$D_{(0,0,0,0,2,-1)}$	$D_{(0,0,-1,-1,2,0)}$
$D_{(-1,-1,0,0,2,0)}$	$D_{(3,0,0,0,-1,-1)}$	$D_{(2,0,-1,-1,0,0)}$	$D_{(0,0,3,0,-1,-1)}$
$D_{(-1,-1,2,0,0,0)}$	$D_{(0,2,0,0,-1,0)}$	$D_{(0,1,0,0,-1,1)}$	$D_{(0,1,-1,-1,0,1)}$
$D_{(0,0,0,1,-1,-1)}$	$D_{(0,0,0,2,-1,-1)}$	$D_{(-1,-1,0,1,0,0)}$	$D_{(0,1,0,0,-1,-1)}$
$D_{(0,2,0,0,-1,-1)}$	$D_{(0,1,-1,-1,0,0)}$	$D_{(2,0,0,0,0,-1)}$	$D_{(1,0,0,0,1,-1)}$
$D_{(1,0,-1,-1,1,0)}$	$D_{(2,0,0,0,-1,0)}$	$D_{(1,0,0,0,-1,1)}$	$D_{(1,0,-1,-1,0,1)}$
$D_{(0,0,2,0,-1,0)}$	$D_{(0,0,1,0,-1,1)}$	$D_{(-1,-1,1,0,0,1)}$	$D_{(0,2,0,1,-1,-1)}$
$D_{(0,1,0,2,-1,-1)}$	$D_{(0,0,2,0,0,-1)}$	$D_{(0,0,1,0,1,-1)}$	$D_{(-1,-1,1,0,1,0)}$
$D_{(0,0,-1,-1,0,1)}$	$D_{(-1,-1,0,0,0,1)}$	$D_{(1,0,0,0,-1,-1)}$	$D_{(2,0,0,0,-1,-1)}$

$D_{(1,0,-1,-1,0,0)}$	$D_{(2,0,0,1,-1,-1)}$	$D_{(1,0,0,2,-1,-1)}$	$D_{(2,0,1,0,-1,-1)}$
$D_{(1,0,2,0,-1,-1)}$	$D_{(0,0,1,0,-1,-1)}$	$D_{(0,0,2,0,-1,-1)}$	$D_{(-1,-1,1,0,0,0)}$
$D_{(0,0,-1,-1,1,1)}$	$D_{(-1,-1,0,0,1,1)}$	$D_{(0,2,0,0,0,-1)}$	$D_{(0,1,0,0,1,-1)}$
$D_{(0,1,-1,-1,1,0)}$	$D_{(0,0,0,2,0,-1)}$	$D_{(0,0,0,1,1,-1)}$	$D_{(-1,-1,0,1,1,0)}$
$D_{(0,0,0,2,-1,0)}$	$D_{(0,0,0,1,-1,1)}$	$D_{(-1,-1,0,1,0,1)}$	$D_{(0,2,1,0,-1,-1)}$
$D_{(0,1,2,0,-1,-1)}$	$D_{(0,0,-1,-1,1,0)}$	$D_{(-1,-1,0,0,1,0)}$	$D_{(0,1,0,1,-1,-1)}$
$D_{(0,1,0,1,-1,0)}$	$D_{(1,0,0,1,-1,-1)}$	$D_{(1,0,0,1,-1,0)}$	$D_{(0,1,0,1,0,-1)}$
$D_{(1,0,0,1,0,-1)}$	$D_{(0,1,1,0,-1,-1)}$	$D_{(0,1,1,0,-1,0)}$	$D_{(0,1,1,0,0,-1)}$
$D_{(1,0,1,0,0,-1)}$	$D_{(1,0,1,0,-1,0)}$	$D_{(1,0,1,0,-1,-1)}$	

on a MPCP-blowup of Y° inducing 73 non-zero toric divisor classes on the mirror X°. The following 42 toric divisors of Y induce the trivial divisor class on X°

$D_{(-1,1,0,0,0,0)}$	$D_{(0,0,-1,1,0,0)}$	$D_{(1,-1,0,0,0,0)}$	$D_{(0,0,1,-1,0,0)}$
$D_{(-1,0,0,0,0,1)}$	$D_{(0,0,-1,0,0,0)}$	$D_{(-1,0,0,0,0,0)}$	$D_{(0,-1,0,0,1,0)}$
$D_{(0,-1,0,0,0,1)}$	$D_{(0,0,0,-1,0,1)}$	$D_{(0,1,-1,0,0,0)}$	$D_{(-1,0,0,1,0,0)}$
$D_{(0,0,0,-1,1,0)}$	$D_{(0,0,0,0,-1,0)}$	$D_{(0,0,0,0,-1,1)}$	$D_{(0,-1,0,0,0,0)}$
$D_{(1,0,-1,0,0,0)}$	$D_{(0,-1,0,1,0,0)}$	$D_{(1,0,0,-1,0,0)}$	$D_{(0,-1,1,0,0,0)}$
$D_{(0,0,0,-1,0,0)}$	$D_{(0,0,0,0,1,0)}$	$D_{(0,0,0,0,0,1)}$	$D_{(-1,0,0,0,1,0)}$
$D_{(0,0,-1,0,1,0)}$	$D_{(0,0,-1,0,0,1)}$	$D_{(0,1,0,-1,0,0)}$	$D_{(-1,0,1,0,0,0)}$
$D_{(0,0,0,0,0,-1)}$	$D_{(0,0,0,0,1,-1)}$	$D_{(0,1,0,0,0,0)}$	$D_{(0,0,0,1,0,0)}$
$D_{(1,0,0,0,0,0)}$	$D_{(0,1,0,0,-1,0)}$	$D_{(0,0,0,1,-1,0)}$	$D_{(0,0,1,0,0,0)}$
$D_{(1,0,0,0,-1,0)}$	$D_{(0,0,0,1,0,-1)}$	$D_{(0,1,0,0,0,-1)}$	$D_{(1,0,0,0,0,-1)}$
$D_{(0,0,1,0,-1,0)}$	$D_{(0,0,1,0,0,-1)}$		

Mirror special fiber The ideal I_0° of the monomial special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$ is generated by the following set of monomials in S°

$$\left\{\begin{array}{lll}
y_{15}\,y_{16}\,y_{18}\,y_{19}\,y_{20}\,y_{21}\,y_{17} & y_{8}\,y_{11}\,y_{10}\,y_{9}\,y_{12}\,y_{13}\,y_{14} & y_{6}\,y_{7}\,y_{1}\,y_{2}\,y_{3}\,y_{4}\,y_{5} \\
y_{15}\,y_{17}\,y_{2}\,y_{16}\,y_{21}\,y_{19}\,y_{20} & y_{1}\,y_{2}\,y_{3}\,y_{4}\,y_{7}\,y_{19}\,y_{6} & y_{15}\,y_{8}\,y_{18}\,y_{16}\,y_{21}\,y_{19}\,y_{20} \\
y_{10}\,y_{8}\,y_{9}\,y_{11}\,y_{14}\,y_{12}\,y_{6} & y_{15}\,y_{17}\,y_{18}\,y_{16}\,y_{7}\,y_{19}\,y_{20} & y_{10}\,y_{8}\,y_{9}\,y_{3}\,y_{14}\,y_{12}\,y_{13} \\
y_{10}\,y_{8}\,y_{9}\,y_{11}\,y_{14}\,y_{12}\,y_{20} & y_{10}\,y_{1}\,y_{2}\,y_{3}\,y_{7}\,y_{5}\,y_{6} & y_{10}\,y_{8}\,y_{9}\,y_{16}\,y_{14}\,y_{12}\,y_{13} \\
y_{1}\,y_{2}\,y_{3}\,y_{4}\,y_{7}\,y_{12}\,y_{6} & y_{15}\,y_{1}\,y_{2}\,y_{3}\,y_{7}\,y_{5}\,y_{6} & y_{15}\,y_{17}\,y_{9}\,y_{16}\,y_{21}\,y_{19}\,y_{20} \\
y_{15}\,y_{17}\,y_{18}\,y_{16}\,y_{14}\,y_{19}\,y_{20} & y_{15}\,y_{1}\,y_{18}\,y_{16}\,y_{21}\,y_{19}\,y_{20} & \\
\multicolumn{3}{l}{\text{and 2170 monomials of degree 7}}
\end{array}\right\}$$

The Pic (Y°)-generated ideal

$$J_0^\circ = \left\langle \; y_6\, y_7\, y_1\, y_2\, y_3\, y_4\, y_5 \quad y_8\, y_{11}\, y_{10}\, y_9\, y_{12}\, y_{13}\, y_{14} \quad y_{15}\, y_{16}\, y_{18}\, y_{19}\, y_{20}\, y_{21}\, y_{17} \; \right\rangle$$

defines the same subvariety X_0° of the toric variety Y°, and $J_0^{\circ\Sigma} = I_0^\circ$. Passing from J_0° to $J_0^{\circ\Sigma}$ is the non-simplicial toric analogue of saturation. The complex $B(I)^*$ labeled by the variables of the Cox ring S° of Y°, as written in the last section, is the complex $SP(I_0^\circ)$ of prime ideals of the toric strata of the special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$, i.e., the primary decomposition of I_0° is

$$I_0^\circ = \begin{array}{l} \langle y_4, y_{10}, y_{15}\rangle \cap \langle y_2, y_9, y_{18}\rangle \cap \langle y_3, y_{11}, y_{16}\rangle \cap \langle y_1, y_8, y_{17}\rangle \cap \langle y_7, y_{14}, y_{21}\rangle \cap \\ \cap \langle y_5, y_{12}, y_{19}\rangle \cap \langle y_6, y_{13}, y_{20}\rangle \end{array}$$

Each facet $F \in B(I)$ corresponds to one of these ideals and this ideal is generated by the facets of ∇ containing F.

Covering structure in the deformation complex of the degeneration $\mathfrak{X}$ According to the local reduced Gröbner basis each face of the complex of deformations dual $(B(I))$ decomposes into 3 polytopes forming a 3 : 1 trivial covering of $B(I)$

$[]$,

$[]$,

$[]$,

$[[\langle y_1\rangle, \langle y_8\rangle, \langle y_{17}\rangle] \mapsto [1,2,3,4,9,10,11,12]^\vee$,
$[\langle y_4\rangle, \langle y_{10}\rangle, \langle y_{15}\rangle] \mapsto [1,3,5,7,9,11]^\vee$,
...],

$[[\langle y_1, y_4\rangle, \langle y_8, y_{10}\rangle, \langle y_{15}, y_{17}\rangle] \mapsto [1,3,9,11]^\vee$,
$[\langle y_3, y_4\rangle, \langle y_{10}, y_{11}\rangle, \langle y_{15}, y_{16}\rangle] \mapsto [1,5,9]^\vee$,
...],

$[[\langle y_7, y_1, y_4\rangle, \langle y_{14}, y_8, y_{10}\rangle, \langle y_{15}, y_{21}, y_{17}\rangle] \mapsto [9,11]^\vee$,
...],

$[[\langle y_1, y_2, y_3, y_4\rangle, \langle y_8, y_9, y_{10}, y_{11}\rangle, \langle y_{15}, y_{16}, y_{17}, y_{18}\rangle] \mapsto [1]^\vee$,
...],

$[]$

Here the faces $F \in B(I)$ are specified both via the vertices of F^* labeled by the variables of S° and by the numbering of the vertices of $B(I)$ chosen above. This covering has 3 sheets forming the complexes

$$[],$$
$$[],$$
$$[],$$
$$[\langle y_1\rangle, \langle y_4\rangle, ...],$$
$$[\langle y_1, y_4\rangle, \langle y_3, y_4\rangle, ...],$$
$$[\langle y_7, y_1, y_4\rangle, ...],$$
$$[\langle y_1, y_2, y_3, y_4\rangle, ...],$$
$$[]$$
$$[],$$
$$[],$$
$$[],$$
$$[\langle y_8\rangle, \langle y_{10}\rangle, ...],$$
$$[\langle y_8, y_{10}\rangle, \langle y_{10}, y_{11}\rangle, ...],$$
$$[\langle y_{14}, y_8, y_{10}\rangle, ...],$$
$$[\langle y_8, y_9, y_{10}, y_{11}\rangle, ...],$$
$$[]$$
$$[],$$
$$[],$$
$$[],$$
$$[\langle y_{17}\rangle, \langle y_{15}\rangle, ...],$$
$$[\langle y_{15}, y_{17}\rangle, \langle y_{15}, y_{16}\rangle, ...],$$
$$[\langle y_{15}, y_{21}, y_{17}\rangle, ...],$$
$$[\langle y_{15}, y_{16}, y_{17}, y_{18}\rangle, ...],$$
$$[]$$

with F-vector

Dimension	Number of faces	F-vector

0	7	$(1,1,0,0,0,0,0,0)$	point
1	19	$(1,2,1,0,0,0,0,0)$	edge
2	24	$(1,3,3,1,0,0,0,0)$	triangle
3	12	$(1,4,6,4,1,0,0,0)$	tetrahedron

Writing the vertices of the faces as deformations the covering is given by

$$
\begin{array}{l}
[],\\
[],\\
[],\\
[[\left\langle \frac{x_6^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_6^2}{x_3\,x_4} \right\rangle, \left\langle \frac{x_6^2}{x_0\,x_5} \right\rangle] \mapsto [1,2,3,4,9,10,11,12]^\vee,\\
[\left\langle \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^2}{x_3\,x_4} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_5\,x_6} \right\rangle] \mapsto [1,3,5,7,9,11]^\vee,\\
...],\\
[[\left\langle \frac{x_6^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_6^2}{x_3\,x_4}, \frac{{x_2}^2}{x_3\,x_4} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_5\,x_6}, \frac{{x_6}^2}{x_0\,x_5} \right\rangle] \mapsto [1,3,9,11]^\vee,\\
[\left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^2}{x_3\,x_4}, \frac{x_4}{x_3} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_5\,x_6}, \frac{{x_4}^3}{x_0\,x_5\,x_6} \right\rangle] \mapsto [1,5,9]^\vee,\\
...],\\
[[\left\langle \frac{x_5^2}{x_1\,x_2}, \frac{{x_6}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_5^2}{x_3\,x_4}, \frac{{x_6}^2}{x_3\,x_4}, \frac{{x_2}^2}{x_3\,x_4} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_5\,x_6}, \frac{{x_5}^2}{x_0\,x_6}, \frac{{x_6}^2}{x_0\,x_5} \right\rangle] \mapsto [9,11]^\vee,\\
...],\\
[[\left\langle \frac{x_6^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2}, \frac{{x_4}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_6^2}{x_3\,x_4}, \frac{{x_0}^2}{x_3\,x_4}, \frac{{x_2}^2}{x_3\,x_4}, \frac{x_4}{x_3} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_5\,x_6}, \frac{{x_4}^3}{x_0\,x_5\,x_6}, \frac{{x_6}^2}{x_0\,x_5}, \frac{{x_0}^2}{x_5\,x_6} \right\rangle] \mapsto [1]^\vee,\\
...],\\
[]
\end{array}
$$

with the 3 sheets forming the complexes

$[],$

$[],$

$[],$

$[\left\langle \frac{x_6^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_2}{x_1} \right\rangle, ...],$

$[\left\langle \frac{x_6^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, ...],$

$[\left\langle \frac{x_5^2}{x_1\,x_2}, \frac{{x_6}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, ...],$

$[\left\langle \frac{x_6^2}{x_1\,x_2}, \frac{{x_0}^2}{x_1\,x_2}, \frac{{x_4}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, ...],$

$[]$

$[],$

$[],$

$[],$

$[\left\langle \frac{x_6^2}{x_3\,x_4} \right\rangle, \left\langle \frac{x_2^2}{x_3\,x_4} \right\rangle, ...],$

$[\left\langle \frac{x_6^2}{x_3\,x_4}, \frac{{x_2}^2}{x_3\,x_4} \right\rangle, \left\langle \frac{x_2^2}{x_3\,x_4}, \frac{x_4}{x_3} \right\rangle, ...],$

$[\left\langle \frac{x_5^2}{x_3\,x_4}, \frac{{x_6}^2}{x_3\,x_4}, \frac{{x_2}^2}{x_3\,x_4} \right\rangle, ...],$

$[\left\langle \frac{x_6^2}{x_3\,x_4}, \frac{{x_0}^2}{x_3\,x_4}, \frac{{x_2}^2}{x_3\,x_4}, \frac{x_4}{x_3} \right\rangle, ...],$

$[]$

$[],$

$[],$

$[],$

$[\left\langle \frac{x_6^2}{x_0\,x_5} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_5\,x_6} \right\rangle, ...],$

$[\left\langle \frac{x_2^3}{x_0\,x_5\,x_6}, \frac{{x_6}^2}{x_0\,x_5} \right\rangle, \left\langle \frac{x_2^3}{x_0\,x_5\,x_6}, \frac{{x_4}^3}{x_0\,x_5\,x_6} \right\rangle, ...],$

$[\left\langle \frac{x_2^3}{x_0\,x_5\,x_6}, \frac{{x_5}^2}{x_0\,x_6}, \frac{{x_6}^2}{x_0\,x_5} \right\rangle, ...],$

$[\left\langle \frac{x_2^3}{x_0\,x_5\,x_6}, \frac{{x_4}^3}{x_0\,x_5\,x_6}, \frac{{x_6}^2}{x_0\,x_5}, \frac{{x_0}^2}{x_5\,x_6} \right\rangle, ...],$

$[]$

Note that the torus invariant basis of deformations corresponding to a Bergman face is given by the set of all lattice points of the polytope specified above.

Limit map The limit map $\lim : B(I) \to \mathrm{Poset}(\Delta)$ associates to a face F of $B(I)$ the face of Δ formed by the limit points of arcs lying over the weight vectors $w \in F$, i.e., with lowest order term t^w. Labeling the faces of the Bergman complex $B(I) \subset \mathrm{Poset}(\nabla)$ and the faces of $\mathrm{Poset}(\Delta)$ by the corresponding dual faces of ∇^* and Δ^*, hence considering the limit map $\lim : B(I) \to \mathrm{Poset}(\Delta)$ as a map $B(I)^* \to \mathrm{Poset}(\Delta^*)$, the limit correspondence is given by

$$
\begin{array}{l}
[],\\
[\langle y_1, y_2, y_3, y_4, y_8, y_9, y_{10}, y_{11}, y_{15}, y_{16}, y_{17}, y_{18}\rangle \mapsto \langle x_1, x_3, x_5\rangle\,,\\
...],\\
[\langle y_7, y_1, y_4, y_{14}, y_8, y_{10}, y_{15}, y_{21}, y_{17}\rangle \mapsto \langle x_1, x_3, x_4, x_0\rangle\,,\\
...],\\
[\langle y_1, y_4, y_8, y_{10}, y_{15}, y_{17}\rangle \mapsto \langle x_1, x_3, x_4, x_5, x_0\rangle\,,\\
\langle y_3, y_4, y_{10}, y_{11}, y_{15}, y_{16}\rangle \mapsto \langle x_1, x_3, x_5, x_6, x_0\rangle\,,\\
...],\\
[\langle y_1, y_8, y_{17}\rangle \mapsto \langle x_1, x_2, x_3, x_4, x_5, x_0\rangle\,,\\
\langle y_4, y_{10}, y_{15}\rangle \mapsto \langle x_1, x_3, x_4, x_5, x_6, x_0\rangle\,,\\
...],\\
[],\\
[],\\
[]
\end{array}
$$

The image of the limit map coincides with the image of μ and with the Bergman complex of the mirror, i.e., $\lim(B(I)) = \mu(B(I)) = B(I^\circ)$.

Mirror complex Numbering the vertices of the mirror complex $\mu(B(I))$ as

$$
\begin{array}{ll}
1 = (6,-1,-1,-1,-1,-1) & 2 = (-1,6,-1,-1,-1,-1)\\
3 = (-1,-1,6,-1,-1,-1) & 4 = (-1,-1,-1,6,-1,-1)\\
5 = (-1,-1,-1,-1,6,-1) & 6 = (-1,-1,-1,-1,-1,6)\\
7 = (-1,-1,-1,-1,-1,-1) &
\end{array}
$$

$\mu(B(I)) \subset \mathrm{Poset}(\Delta)$ is

[],

[[2], [6], [7], [4], [5], [1], [3]],

[[2, 6], [4, 7], [2, 7], [1, 5], [1, 6], [3, 6], [2, 4], [3, 5], [6, 7], [1, 7], [1, 4], [1, 3], [3, 7], [5, 6], [2, 5], [4, 5], [4, 6], [2, 3], [5, 7]],

[[2, 5, 6], [4, 5, 6], [1, 5, 6], [2, 4, 7], [2, 6, 7], [4, 6, 7], [2, 4, 6], [3, 5, 6], [1, 4, 7], [1, 6, 7], [1, 4, 6], [4, 5, 7], [2, 5, 7], [2, 4, 5], [1, 4, 5], [1, 5, 7], [2, 3, 7], [3, 6, 7], [2, 3, 6], [2, 3, 5], [3, 5, 7], [1, 3, 5], [1, 3, 6], [1, 3, 7]],

[[2, 4, 6, 7], [1, 4, 6, 7], [2, 3, 6, 7], [1, 3, 6, 7], [2, 4, 5, 7], [1, 4, 5, 7], [2, 3, 5, 7], [1, 3, 5, 7], [2, 4, 5, 6], [1, 4, 5, 6], [2, 3, 5, 6], [1, 3, 5, 6]],

[],

[],

[]

The ordering of the faces of $\mu(B(I))$ is compatible with above ordering of the faces of $B(I)$. The F-vectors of the faces of $\mu(B(I))^*$ are

Dimension	Number of faces	F-vector	
2	12	(1, 3, 3, 1, 0, 0, 0, 0)	triangle
3	24	(1, 4, 6, 4, 1, 0, 0, 0)	tetrahedron
4	19	(1, 5, 10, 10, 5, 1, 0, 0)	
5	7	(1, 6, 15, 20, 15, 6, 1, 0)	

The first order deformations of the mirror special fiber X_0° correspond to the lattice points of the dual complex $(\mu(B(I)))^* \subset \mathrm{Poset}(\Delta^*)$ of the mirror complex. We label the first order deformations of X_0° corresponding to vertices of Δ^* by the homogeneous coordinates of Y

$$
\begin{aligned}
x_1 &= x_{(1,0,0,0,0,0)} = \frac{y_5\, {y_{12}}^2 y_{19}^3}{y_1\, y_2\, y_3\, y_4\, y_6\, y_7} \\
x_2 &= x_{(0,1,0,0,0,0)} = \frac{y_4\, {y_{10}}^2 y_{15}^3}{y_1\, y_2\, y_3\, y_5\, y_6\, y_7} \\
x_3 &= x_{(0,0,1,0,0,0)} = \frac{y_6^2 y_{13}\, {y_{20}}^3}{y_8\, y_9\, y_{10}\, y_{11}\, y_{12}\, y_{14}} \\
x_4 &= x_{(0,0,0,1,0,0)} = \frac{y_3^2 y_{11}\, {y_{16}}^3}{y_8\, y_9\, y_{10}\, y_{12}\, y_{13}\, y_{14}} \\
x_5 &= x_{(0,0,0,0,1,0)} = \frac{y_7^2 {y_{14}}^2 y_{21}^2}{y_{15}\, y_{16}\, y_{17}\, y_{18}\, y_{19}\, y_{20}} \\
x_6 &= x_{(0,0,0,0,0,1)} = \frac{y_1^2 {y_8}^2 y_{17}^2}{y_{15}\, y_{16}\, y_{18}\, y_{19}\, y_{20}\, y_{21}} \\
x_0 &= x_{(-1,-1,-1,-1,-1,-1)} = \frac{y_2^2 {y_9}^2 y_{18}^2}{y_{15}\, y_{16}\, y_{17}\, y_{19}\, y_{20}\, y_{21}}
\end{aligned}
$$

So writing the vertices of the faces of $(\mu(B(I)))^*$ as homogeneous coordinates of Y, the complex $(\mu(B(I)))^*$ is given by

$[]$,

$[]$,

$[]$,

$[\langle x_1, x_3, x_5\rangle, \langle x_2, x_3, x_5\rangle, \langle x_1, x_4, x_5\rangle, \langle x_2, x_4, x_5\rangle, \langle x_1, x_3, x_6\rangle,$
$\langle x_2, x_3, x_6\rangle, \langle x_1, x_4, x_6\rangle, \langle x_2, x_4, x_6\rangle, \langle x_1, x_3, x_0\rangle, \langle x_2, x_3, x_0\rangle,$
$\langle x_1, x_4, x_0\rangle, \langle x_2, x_4, x_0\rangle]$,

$[\langle x_1, x_3, x_4, x_0\rangle, \langle x_1, x_2, x_3, x_0\rangle, \langle x_2, x_3, x_4, x_0\rangle, \langle x_1, x_3, x_5, x_6\rangle,$
$\langle x_1, x_3, x_4, x_5\rangle, \langle x_1, x_2, x_3, x_5\rangle, \langle x_1, x_3, x_5, x_0\rangle, \langle x_1, x_2, x_4, x_0\rangle,$
$\langle x_2, x_3, x_5, x_6\rangle, \langle x_2, x_3, x_4, x_5\rangle, \langle x_2, x_3, x_5, x_0\rangle, \langle x_1, x_2, x_3, x_6\rangle,$
$\langle x_1, x_3, x_4, x_6\rangle, \langle x_1, x_3, x_6, x_0\rangle, \langle x_2, x_3, x_6, x_0\rangle, \langle x_2, x_3, x_4, x_6\rangle,$
$\langle x_1, x_4, x_5, x_6\rangle, \langle x_1, x_2, x_4, x_5\rangle, \langle x_1, x_4, x_5, x_0\rangle, \langle x_1, x_4, x_6, x_0\rangle,$
$\langle x_1, x_2, x_4, x_6\rangle, \langle x_2, x_4, x_6, x_0\rangle, \langle x_2, x_4, x_5, x_0\rangle, \langle x_2, x_4, x_5, x_6\rangle]$,

$[\langle x_1, x_3, x_4, x_5, x_0\rangle, \langle x_1, x_2, x_3, x_5, x_6\rangle, \langle x_1, x_3, x_4, x_5, x_6\rangle,$
$\langle x_2, x_3, x_4, x_6, x_0\rangle, \langle x_2, x_3, x_4, x_5, x_0\rangle, \langle x_1, x_2, x_4, x_5, x_0\rangle,$
$\langle x_1, x_3, x_5, x_6, x_0\rangle, \langle x_1, x_2, x_4, x_6, x_0\rangle, \langle x_1, x_2, x_3, x_4, x_5\rangle,$
$\langle x_2, x_3, x_4, x_5, x_6\rangle, \langle x_2, x_3, x_5, x_6, x_0\rangle, \langle x_2, x_4, x_5, x_6, x_0\rangle,$
$\langle x_1, x_2, x_4, x_5, x_6\rangle, \langle x_1, x_2, x_3, x_4, x_0\rangle, \langle x_1, x_3, x_4, x_6, x_0\rangle,$
$\langle x_1, x_2, x_3, x_6, x_0\rangle, \langle x_1, x_2, x_3, x_5, x_0\rangle, \langle x_1, x_4, x_5, x_6, x_0\rangle,$
$\langle x_1, x_2, x_3, x_4, x_6\rangle]$,

$[\langle x_1, x_3, x_4, x_5, x_6, x_0\rangle, \langle x_1, x_2, x_3, x_4, x_5, x_0\rangle, \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle,$
$\langle x_1, x_2, x_3, x_5, x_6, x_0\rangle, \langle x_1, x_2, x_3, x_4, x_6, x_0\rangle, \langle x_2, x_3, x_4, x_5, x_6, x_0\rangle,$
$\langle x_1, x_2, x_4, x_5, x_6, x_0\rangle]$,

$[]$

The complex $\mu(B(I))^*$ labeled by the variables of the Cox ring S gives the ideals of the toric strata of the special fiber X_0 of $\mathfrak{X}$, i.e., the complex $SP(I_0)$, so in particular the primary decomposition of I_0 is

$$\begin{array}{rl} & \langle x_2, x_4, x_0\rangle \cap \langle x_1, x_4, x_0\rangle \cap \langle x_1, x_4, x_5\rangle \cap \langle x_1, x_4, x_6\rangle \cap \langle x_2, x_4, x_6\rangle \cap \\ I_0 = & \cap \langle x_1, x_3, x_6\rangle \cap \langle x_2, x_3, x_6\rangle \cap \langle x_1, x_3, x_5\rangle \cap \langle x_2, x_3, x_0\rangle \cap \langle x_1, x_3, x_0\rangle \cap \\ & \cap \langle x_2, x_3, x_5\rangle \cap \langle x_2, x_4, x_5\rangle \end{array}$$

Covering structure in the deformation complex of the mirror degeneration Each face of the complex of deformations $(\mu(B(I)))^*$ of the mirror special fiber X_0° decomposes as the convex hull of 3 polytopes forming a $3:1$ trivial covering of $\mu(B(I))^\vee$

$[], [], []$

$[[\langle x_1\rangle, \langle x_3\rangle, \langle x_5\rangle] \mapsto \langle x_1, x_3, x_5\rangle^{*\vee} = [2,4,6,7]^{\vee},$
$...],$

$[[\langle x_1\rangle, \langle x_3, x_4\rangle, \langle x_0\rangle] \mapsto \langle x_1, x_3, x_4, x_0\rangle^{*\vee} = [2,5,6]^{\vee},$
$...],$

$[[\langle x_1\rangle, \langle x_3, x_4\rangle, \langle x_5, x_0\rangle] \mapsto \langle x_1, x_3, x_4, x_5, x_0\rangle^{*\vee} = [2,6]^{\vee},$
$[\langle x_1\rangle, \langle x_3\rangle, \langle x_5, x_6, x_0\rangle] \mapsto \langle x_1, x_3, x_5, x_6, x_0\rangle^{*\vee} = [2,4]^{\vee},$
$...],$

$[[\langle x_1, x_2\rangle, \langle x_3, x_4\rangle, \langle x_5, x_0\rangle] \mapsto \langle x_1, x_2, x_3, x_4, x_5, x_0\rangle^{*\vee} = [6]^{\vee},$
$[\langle x_1\rangle, \langle x_3, x_4\rangle, \langle x_5, x_6, x_0\rangle] \mapsto \langle x_1, x_3, x_4, x_5, x_6, x_0\rangle^{*\vee} = [2]^{\vee},$
$...],$

$[]$

Due to the singularities of Y° this covering involves degenerate faces, i.e., faces $G \mapsto F^\vee$ with $\dim(G) < \dim(F^\vee)$.

Mirror degeneration The space of first order deformations of X_0° in the mirror degeneration $\mathfrak{X}^\circ$ has dimension 7 and the deformations represented by the monomials

$$\left\{ \begin{array}{cccc} \frac{y_5\, y_{12}{}^2 y_{19}^3}{y_1\, y_2\, y_3\, y_4\, y_6\, y_7} & \frac{y_6^2 y_{13}\, y_{20}{}^3}{y_8\, y_9\, y_{10}\, y_{11}\, y_{12}\, y_{14}} & \frac{y_7^2 y_{14}{}^2 y_{21}^2}{y_{15}\, y_{16}\, y_{17}\, y_{18}\, y_{19}\, y_{20}} & \frac{y_4\, y_{10}{}^2 y_{15}^3}{y_1\, y_2\, y_3\, y_5\, y_6\, y_7} \\ \frac{y_3^2 y_{11}\, y_{16}{}^3}{y_8\, y_9\, y_{10}\, y_{12}\, y_{13}\, y_{14}} & \frac{y_2^2 y_9{}^2 y_{18}^2}{y_{15}\, y_{16}\, y_{17}\, y_{19}\, y_{20}\, y_{21}} & \frac{y_1^2 y_8{}^2 y_{17}^2}{y_{15}\, y_{16}\, y_{18}\, y_{19}\, y_{20}\, y_{21}} & \end{array} \right\}$$

form a torus invariant basis. The number of lattice points of the dual of the mirror complex of I relates to the dimension $h^{1,2}(X^\circ)$ of complex moduli space of the generic fiber X° of $\mathfrak{X}^\circ$ and to the dimension $h^{1,1}(X)$ of the Kähler moduli space of the generic fiber X of $\mathfrak{X}$ via

$$\begin{aligned} |\operatorname{supp}((\mu(B(I)))^*) \cap N| &= 7 = 6 + 1 \\ &= \dim(\operatorname{Aut}(Y^\circ)) + h^{1,2}(X^\circ) = \dim(T) + h^{1,1}(X) \end{aligned}$$

The mirror degeneration $\mathfrak{X}^\circ \subset Y^\circ \times \operatorname{Spec}\mathbb{C}[t]$ of $\mathfrak{X}$ is given by the ideal $I^\circ \subset S^\circ \otimes \mathbb{C}[t]$ generated by

$$t\,(s_7\, y_1^2 y_8^2 y_{17}^3 + s_1\, y_2^2 y_9^2 y_{18}^3 + s_2\, y_7^2 y_{14}^2 y_{21}^3) + y_{15}\, y_{16}\, y_{18}\, y_{19}\, y_{20}\, y_{21}\, y_{17}$$

$$t\,(s_6\, y_3^2 y_{11}^2 y_{16}^3 + s_3\, y_6^2 y_{13}^2 y_{20}^3) + y_8\, y_{11}\, y_{10}\, y_9\, y_{12}\, y_{13}\, y_{14}$$
$$t\,(s_5\, y_4^2 y_{10}^2 y_{15}^3 + s_4\, y_5^2 y_{12}^2 y_{19}^3) + y_6\, y_7\, y_1\, y_2\, y_3\, y_4\, y_5$$

$$\begin{array}{ll}
ts_1\, y_2^3 y_9^2 y_{18}^2 + y_{15}\, y_{17}\, y_2\, y_{16}\, y_{21}\, y_{19}\, y_{20} & ts_4\, y_{19}^4 y_5\, y_{12}^2 + y_1\, y_2\, y_3\, y_4\, y_7\, y_{19}\, y_6 \\
ts_7\, y_8^3 y_1^2 y_{17}^2 + y_{15}\, y_8\, y_{18}\, y_{16}\, y_{21}\, y_{19}\, y_{20} & ts_3\, y_6^3 y_{13}\, y_{20}^3 + y_{10}\, y_8\, y_9\, y_{11}\, y_{14}\, y_{12}\, y_6 \\
ts_2\, y_7^3 y_{14}^2 y_{21}^2 + y_{15}\, y_{17}\, y_{18}\, y_{16}\, y_7\, y_{19}\, y_{20} & ts_6\, y_3^3 y_{11}\, y_{16}^3 + y_{10}\, y_8\, y_9\, y_3\, y_{14}\, y_{12}\, y_{13} \\
ts_3\, y_{20}^4 y_6^2 y_{13} + y_{10}\, y_8\, y_9\, y_{11}\, y_{14}\, y_{12}\, y_{20} & ts_5\, y_{10}^3 y_4\, y_{15}^3 + y_{10}\, y_1\, y_2\, y_3\, y_7\, y_5\, y_6 \\
ts_6\, y_{16}^4 y_3^2 y_{11} + y_{10}\, y_8\, y_9\, y_{16}\, y_{14}\, y_{12}\, y_{13} & ts_4\, y_{12}^3 y_5\, y_{19}^3 + y_1\, y_2\, y_3\, y_4\, y_7\, y_{12}\, y_6 \\
ts_5\, y_{15}^4 y_4\, y_{10}^2 + y_{15}\, y_1\, y_2\, y_3\, y_7\, y_5\, y_6 & ts_1\, y_9^3 y_2^2 y_{18}^2 + y_{15}\, y_{17}\, y_9\, y_{16}\, y_{21}\, y_{19}\, y_{20} \\
ts_2\, y_{14}^3 y_7^2 y_{21}^2 + y_{15}\, y_{17}\, y_{18}\, y_{16}\, y_{14}\, y_{19}\, y_{20} & ts_7\, y_1^3 y_8^2 y_{17}^2 + y_{15}\, y_1\, y_{18}\, y_{16}\, y_{21}\, y_{19}\, y_{20}
\end{array}$$

and 2170 monomials of degree 7

The ideal J° which is $\operatorname{Pic}(Y^\circ)$-generated by

$$\left\{\begin{array}{l}
t(s_5\, y_4^2 y_{10}^2 y_{15}^3 + s_4\, y_5^2 y_{12}^2 y_{19}^3) + y_6\, y_7\, y_1\, y_2\, y_3\, y_4\, y_5, \\
t(s_6\, y_3^2 y_{11}^2 y_{16}^3 + s_3\, y_6^2 y_{13}^2 y_{20}^3) + y_8\, y_{11}\, y_{10}\, y_9\, y_{12}\, y_{13}\, y_{14}, \\
t(s_7\, y_1^2 y_8^2 y_{17}^3 + s_1\, y_2^2 y_9^2 y_{18}^3 + s_2\, y_7^2 y_{14}^2 y_{21}^3) + y_{15}\, y_{16}\, y_{18}\, y_{19}\, y_{20}\, y_{21}\, y_{17}
\end{array}\right\}$$

defines a flat affine cone inducing $\mathfrak{X}^\circ$.

Contraction of the mirror degeneration In the following we give a birational map relating the degeneration $\mathfrak{X}^\circ$ to a Greene-Plesser type orbifolding mirror family by contracting divisors on Y°. See also Section 9.13 below. In order to contract the divisors

$$\begin{array}{ll}
y_1 = y_{(-1,-1,0,0,0,2)} = \frac{x_6^2}{x_1\, x_2} & y_2 = y_{(-1,-1,0,0,0,0)} = \frac{x_0^2}{x_1\, x_2} \\
y_4 = y_{(-1,1,0,0,0,0)} = \frac{x_2}{x_1} & y_5 = y_{(1,-1,0,0,0,0)} = \frac{x_1}{x_2} \\
y_{10} = y_{(0,2,-1,-1,0,0)} = \frac{x_2^2}{x_3\, x_4} & y_{11} = y_{(0,0,-1,1,0,0)} = \frac{x_4}{x_3} \\
y_{12} = y_{(2,0,-1,-1,0,0)} = \frac{x_1^2}{x_3\, x_4} & y_{13} = y_{(0,0,1,-1,0,0)} = \frac{x_3}{x_4} \\
y_{14} = y_{(0,0,-1,-1,2,0)} = \frac{x_5^2}{x_3\, x_4} & y_{16} = y_{(0,0,0,3,-1,-1)} = \frac{x_4^3}{x_0\, x_5\, x_6} \\
y_{17} = y_{(0,0,0,0,-1,2)} = \frac{x_6^2}{x_0\, x_5} & y_{18} = y_{(0,0,0,0,-1,-1)} = \frac{x_0^2}{x_5\, x_6} \\
y_{20} = y_{(0,0,3,0,-1,-1)} = \frac{x_3^3}{x_0\, x_5\, x_6} & y_{21} = y_{(0,0,0,0,2,-1)} = \frac{x_5^2}{x_0\, x_6}
\end{array}$$

consider the $\mathbb{Q}$-factorial toric Fano variety $\hat{Y}^\circ = X(\hat{\Sigma}^\circ)$, where $\hat{\Sigma}^\circ = \Sigma(\hat{P}^\circ) \subset M_\mathbb{R}$ is the fan over the Fano polytope $\hat{P}^\circ \subset M_\mathbb{R}$ given as the convex hull of the remaining vertices of $P^\circ = \nabla^*$ corresponding to the Cox variables

$$\begin{array}{ll} y_{15} = y_{(0,3,0,0,-1,-1)} = \frac{x_2^3}{x_0\,x_5\,x_6} & y_8 = y_{(0,0,-1,-1,0,2)} = \frac{x_6^2}{x_3\,x_4} \\ y_9 = y_{(0,0,-1,-1,0,0)} = \frac{x_0^2}{x_3\,x_4} & y_3 = y_{(-1,-1,0,2,0,0)} = \frac{x_4^2}{x_1\,x_2} \\ y_7 = y_{(-1,-1,0,0,2,0)} = \frac{x_5^2}{x_1\,x_2} & y_{19} = y_{(3,0,0,0,-1,-1)} = \frac{x_1^3}{x_0\,x_5\,x_6} \\ y_6 = y_{(-1,-1,2,0,0,0)} = \frac{x_3^2}{x_1\,x_2} & \end{array}$$

of the toric variety $\hat{Y}^\circ$ with Cox ring

$$\hat{S}^\circ = \mathbb{C}[y_{15}, y_8, y_9, y_3, y_7, y_{19}, y_6]$$

The Cox variables of $\hat{Y}^\circ$ correspond to the set of Fermat deformations of $\mathfrak{X}$. Let

$$Y^\circ = X\left(\Sigma^\circ\right) \to X(\hat{\Sigma}^\circ) = \hat{Y}^\circ$$

be a birational map from Y° to the toric Fano variety $\hat{Y}^\circ$, which contracts the divisors of the rays $\Sigma^\circ(1) - \hat{\Sigma}^\circ(1)$ corresponding to the Cox variables

$$\begin{array}{cccccccc} y_1 & y_2 & y_4 & y_5 & y_{10} & y_{11} & y_{12} & y_{13} \\ y_{14} & y_{16} & y_{17} & y_{18} & y_{20} & y_{21} & & \end{array}$$

Representing $\hat{Y}^\circ$ as a quotient we have

$$\hat{Y}^\circ = \left(\mathbb{C}^7 - V\left(B(\hat{\Sigma}^\circ)\right)\right)/\hat{G}^\circ$$

with

$$\hat{G}^\circ = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12} \times (\mathbb{C}^*)^1$$

acting via

$$\xi y = \left(u_1\,u_3^{11}\,v_1 \cdot y_{15},\ u_2\,u_3^9\,v_1 \cdot y_8,\ u_2\,u_3^3\,v_1 \cdot y_9,\ u_2\,u_3^6\,v_1 \cdot y_3,\ u_1\,u_2\,u_3^3\,v_1 \cdot y_7,\ u_1\,u_3^7\,v_1 \cdot y_{19},\ v_1 \cdot y_6\right)$$

for $\xi = (u_1, u_2, u_3, v_1) \in \hat{G}^\circ$ and $y \in \mathbb{C}^7 - V\left(B(\hat{\Sigma}^\circ)\right)$. Hence with the group

$$\hat{H}^\circ = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}$$

of order 48 the toric variety $\hat{Y}^\circ$ is the quotient

$$\hat{Y}^\circ = \mathbb{P}^6/\hat{H}^\circ$$

of projective space $\mathbb{P}^6$. The mirror degeneration $\mathfrak{X}^\circ$ induces via $Y^\circ \to \hat{Y}^\circ$ a degeneration $\hat{\mathfrak{X}}^\circ \subset \hat{Y}^\circ \times \operatorname{Spec} \mathbb{C}[t]$ given by the ideal $\hat{I}^\circ \subset \langle y_{15}, y_8, y_9, y_3, y_7, y_{19}, y_6 \rangle \subset \hat{S}^\circ \otimes \mathbb{C}[t]$ generated by the Fermat-type equations

$$\left\{ \begin{array}{l} y_{15}\, y_{19} + t(s_3\, y_7^2 + s_2\, y_8^2 + s_1\, y_9^2), \\ y_8\, y_9 + t(s_6\, y_3^2 + s_4\, y_6^2), \\ y_7\, y_3\, y_6 + t(s_7\, y_{15}^3 + s_5\, y_{19}^3) \end{array} \right\}$$

The special fiber $\hat{X}_0^\circ \subset \hat{Y}^\circ$ of $\hat{\mathfrak{X}}^\circ$ is cut out by the monomial ideal

$$\hat{I}_0^\circ \subset B(\hat{\Sigma}^\circ) = \langle y_{15}, y_8, y_9, y_3, y_7, y_{19}, y_6 \rangle \subset \hat{S}^\circ$$

generated by

$$\left\{ \begin{array}{lll} y_7\, y_3\, y_6 & y_8\, y_9 & y_{15}\, y_{19} \end{array} \right\}$$

The complex $SP(\hat{I}_0^\circ)$ of prime ideals of the toric strata of $\hat{X}_0^\circ$ is

$[], [], []$

$[\langle y_{15}, y_9, y_6 \rangle, \langle y_{15}, y_8, y_3 \rangle, \langle y_{15}, y_8, y_7 \rangle, \langle y_{15}, y_9, y_7 \rangle, \langle y_8, y_3, y_{19} \rangle,$
$\langle y_9, y_{19}, y_6 \rangle, \langle y_8, y_{19}, y_6 \rangle, \langle y_{15}, y_9, y_3 \rangle, \langle y_9, y_3, y_{19} \rangle, \langle y_9, y_7, y_{19} \rangle,$
$\langle y_8, y_7, y_{19} \rangle, \langle y_{15}, y_8, y_6 \rangle],$

$[\langle y_{15}, y_9, y_7, y_6 \rangle, \langle y_{15}, y_8, y_7, y_{19} \rangle, \langle y_{15}, y_8, y_9, y_3 \rangle, \langle y_9, y_3, y_7, y_{19} \rangle,$
$\langle y_{15}, y_8, y_3, y_6 \rangle, \langle y_8, y_3, y_7, y_{19} \rangle, \langle y_8, y_9, y_3, y_{19} \rangle, \langle y_{15}, y_9, y_7, y_{19} \rangle,$
$\langle y_{15}, y_8, y_9, y_7 \rangle, \langle y_{15}, y_8, y_{19}, y_6 \rangle, \langle y_9, y_3, y_{19}, y_6 \rangle, \langle y_8, y_9, y_7, y_{19} \rangle,$
$\langle y_{15}, y_8, y_3, y_7 \rangle, \langle y_{15}, y_9, y_3, y_{19} \rangle, \langle y_{15}, y_8, y_7, y_6 \rangle, \langle y_{15}, y_9, y_{19}, y_6 \rangle,$
$\langle y_8, y_7, y_{19}, y_6 \rangle, \langle y_8, y_9, y_{19}, y_6 \rangle, \langle y_{15}, y_9, y_3, y_7 \rangle, \langle y_{15}, y_8, y_3, y_{19} \rangle,$
$\langle y_{15}, y_8, y_9, y_6 \rangle, \langle y_8, y_3, y_{19}, y_6 \rangle, \langle y_9, y_7, y_{19}, y_6 \rangle, \langle y_{15}, y_9, y_3, y_6 \rangle],$

$[\langle y_{15}, y_8, y_7, y_{19}, y_6 \rangle, \langle y_{15}, y_9, y_3, y_{19}, y_6 \rangle, \langle y_{15}, y_8, y_9, y_3, y_7 \rangle, \langle y_{15}, y_9, y_3, y_7, y_{19} \rangle,$
$\langle y_{15}, y_8, y_3, y_7, y_{19} \rangle, \langle y_{15}, y_8, y_9, y_7, y_{19} \rangle, \langle y_{15}, y_8, y_9, y_3, y_{19} \rangle, \langle y_8, y_9, y_3, y_7, y_{19} \rangle,$
$\langle y_8, y_9, y_7, y_{19}, y_6 \rangle, \langle y_{15}, y_8, y_3, y_7, y_6 \rangle, \langle y_{15}, y_9, y_7, y_{19}, y_6 \rangle, \langle y_{15}, y_8, y_9, y_3, y_6 \rangle,$
$\langle y_{15}, y_8, y_9, y_{19}, y_6 \rangle, \langle y_{15}, y_8, y_3, y_{19}, y_6 \rangle, \langle y_{15}, y_8, y_9, y_7, y_6 \rangle, \langle y_{15}, y_9, y_3, y_7, y_6 \rangle,$
$\langle y_8, y_9, y_3, y_{19}, y_6 \rangle, \langle y_8, y_3, y_7, y_{19}, y_6 \rangle, \langle y_9, y_3, y_7, y_{19}, y_6 \rangle],$

$[\langle y_8, y_9, y_3, y_7, y_{19}, y_6 \rangle, \langle y_{15}, y_9, y_3, y_7, y_{19}, y_6 \rangle, \langle y_{15}, y_8, y_3, y_7, y_{19}, y_6 \rangle,$
$\langle y_{15}, y_8, y_9, y_7, y_{19}, y_6 \rangle, \langle y_{15}, y_8, y_9, y_3, y_{19}, y_6 \rangle, \langle y_{15}, y_8, y_9, y_3, y_7, y_6 \rangle,$
$\langle y_{15}, y_8, y_9, y_3, y_7, y_{19} \rangle],$

$[]$

so $\hat{I}_0^\circ$ has the primary decomposition

$$\hat{I}_0^\circ = \begin{array}{l} \langle y_{15}, y_9, y_6\rangle \cap \langle y_{15}, y_8, y_3\rangle \cap \langle y_{15}, y_8, y_7\rangle \cap \langle y_{15}, y_9, y_7\rangle \cap \langle y_8, y_3, y_{19}\rangle \cap \\ \cap \langle y_9, y_{19}, y_6\rangle \cap \langle y_8, y_{19}, y_6\rangle \cap \langle y_{15}, y_9, y_3\rangle \cap \langle y_8, y_7, y_{19}\rangle \cap \langle y_{15}, y_8, y_6\rangle \cap \\ \cap \langle y_9, y_3, y_{19}\rangle \cap \langle y_9, y_7, y_{19}\rangle \end{array}$$

The Bergman subcomplex $B(I) \subset \mathrm{Poset}(\nabla)$ induces a subcomplex of $\mathrm{Poset}(\hat{\nabla})$, $\hat{\nabla} = (\hat{P}^\circ)^*$ corresponding to $\hat{I}_0^\circ$. Indexing of the vertices of $\hat{\nabla}$ by

$$\begin{array}{l} 1 = \left(-\frac{1}{6}, \frac{13}{6}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right) \\ 2 = \left(1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{7}{2}\right) \\ 3 = \left(-\frac{5}{2}, -\frac{5}{2}, -3, -3, -3, -\frac{7}{2}\right) \\ 4 = \left(-\frac{3}{4}, -\frac{3}{4}, -\frac{5}{4}, \frac{9}{4}, -\frac{5}{4}, 0\right) \\ 5 = \left(1, 1, \frac{1}{2}, \frac{1}{2}, 4, 0\right) \\ 6 = \left(\frac{13}{6}, -\frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right) \\ 7 = \left(-\frac{3}{4}, -\frac{3}{4}, \frac{9}{4}, -\frac{5}{4}, -\frac{5}{4}, 0\right) \end{array}$$

this complex is given by

[],

[[1], [2], [3], [4], [5], [6], [7]],

[[3, 4], [2, 5], [6, 7], [2, 7], [3, 7], [4, 7], [5, 7], [1, 7], [1, 4], [3, 6], [2, 4], [5, 6], [4, 5], [3, 5], [4, 6], [2, 6], [1, 5], [1, 3], [1, 2]],

[[2, 4, 6], [3, 4, 7], [5, 6, 7], [1, 2, 7], [3, 5, 6], [1, 3, 7], [1, 5, 7], [2, 4, 7], [4, 6, 7], [3, 4, 5], [1, 2, 5], [1, 4, 7], [3, 6, 7], [2, 5, 7], [3, 4, 6], [2, 4, 5], [1, 3, 4], [1, 4, 5], [2, 6, 7], [3, 5, 7], [4, 5, 6], [1, 3, 5], [1, 2, 4], [2, 5, 6]],

[[2, 4, 5, 6], [3, 5, 6, 7], [3, 4, 6, 7], [2, 4, 6, 7], [1, 3, 5, 7], [1, 2, 4, 5], [1, 3, 4, 5], [2, 5, 6, 7], [1, 2, 5, 7], [1, 2, 4, 7], [1, 3, 4, 7], [3, 4, 5, 6]],

[], [], []

9 The tropical mirror construction

9.1 Concept of the tropical mirror construction

In the following, the concepts involved in the tropical mirror construction are summarized, omitting detailed conditions and technicalities.

Let $N \cong \mathbb{Z}^n$ be a lattice, $M = \mathrm{Hom}(N, \mathbb{Z})$ the dual lattice and $Y = X(\Sigma)$ a toric Fano variety of dimension n given by a Fano polytope $P \subset N_{\mathbb{R}}$, i.e., $\Sigma = \Sigma(P)$ is the fan over the faces of P. Denote by $\Delta = P^*$ the dual polytope of P (which is not necessarily integral) and by S the Cox ring of Y.

Let $\mathfrak{X} \subset Y \times \mathrm{Spec}(\mathbb{C}[[t]])$ be a flat family of Calabi-Yau varieties of dimension d given by the ideal $I \subset \mathbb{C}[t] \otimes S$. Suppose that the special fiber of $\mathfrak{X}$ over the zero point $\mathrm{Spec}(\mathbb{C})$ is given by the reduced monomial ideal I_0. We require that the tangent vector of $\mathfrak{X}$ at X_0 is sufficiently general in the tangent space of the component of moduli space of X_0 containing the family $\mathfrak{X}$.

The goal is to associate to $\mathfrak{X}$ a degeneration $\mathfrak{X}^\circ$ of Calabi-Yau varieties with fibers in a toric Fano variety such that the general fibers of $\mathfrak{X}$ and $\mathfrak{X}^\circ$ form a mirror pair.

The presentation of the Chow group of Y

$$0 \to M \xrightarrow{A} \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(Y) \to 0$$

induces a correspondence of weight vectors on the Cox ring S and the elements of

$$\frac{\mathrm{Hom}_{\mathbb{R}}\left(\mathbb{R}^{\Sigma(1)}, \mathbb{R}\right)}{\mathrm{Hom}_{\mathbb{R}}\left(A_{n-1}(Y) \otimes_{\mathbb{Z}} \mathbb{R}, \mathbb{R}\right)} \cong N_{\mathbb{R}}$$

This vector space naturally contains the lattice

$$\mathrm{image}(\_ \circ A) \cong \mathbb{Z}^n$$

We associate to $\mathfrak{X}$ the special fiber Gröbner cone

$$C_{I_0}(I) \subset \mathbb{R} \oplus N_{\mathbb{R}}$$

defined as the closure of the set of weight vectors on $\mathbb{C}[t] \otimes S$ which select I_0 as initial ideal of I. It is a closed strongly convex rational polyhedral cone.

The cone $C_{I_0}(I)$ intersects the hyperplane of t -weight $w_t = 1$, which contains via stereographic projection the Bergman complex of I, in the polytope

$$\nabla = C_{I_0}(I) \cap \{w_t = 1\} \subset \{w_t = 1\} = N_{\mathbb{R}}$$

The dual polytope ∇^* is integral and contains 0 as unique interior lattice point, i.e., ∇^* is a Fano polytope, defining a toric Fano variety $Y^\circ = \mathbb{P}(\Sigma^\circ)$ by the fan $\Sigma^\circ = \Sigma(\nabla^*)$ over the faces of ∇^*.

The intersection of the Bergman fan with the special fiber Gröbner cone

$$B(I) = (BF(I) \cap \mathrm{Poset}(C_{I_0}(I))) \cap \{w_t = 1\} \subset \mathrm{Poset}(\nabla)$$

is a subcomplex of dimension d of the boundary of ∇. Here $\mathrm{Poset}(C_{I_0}(I))$ is the fan generated by the cone $C_{I_0}(I)$ and the intersection of a fan with $\{w_t = 1\}$ is defined as the intersection of all cones with $\{w_t = 1\}$.

As $B(I)$ is a subcomplex of $\mathrm{val}(V_K(I))$ for the metric completion K of the field of Puiseux series, we have a map of complexes

$$\begin{array}{rccl} \lim: & B(I) & \to & \mathrm{Strata}(Y) \cong \mathrm{Poset}(\Delta) \\ & F & \mapsto & \left\{\lim_{t\to 0} a(t) \mid a \in \mathrm{val}^{-1}(\mathrm{int}(F))\right\} \end{array}$$

Here val is the valuation map defined in Section 4.2 and $\mathrm{Strata}(Y)$ is the complex of all closures of toric strata of Y. Note that $\mathrm{Strata}(Y)$ is isomorphic to the complex of faces $\mathrm{Poset}(\Delta)$ of $\Delta \subset M_{\mathbb{R}}$, considered as a complex. The image of the map lim is the complex $\mathrm{Strata}_\Delta(I_0)$ of strata of X_0 considered as a subcomplex of $\mathrm{Poset}(\Delta)$, i.e.,

$$\lim(B(I)) = \mathrm{Strata}_\Delta(I_0) \subset \mathrm{Poset}(\Delta)$$

As a consequence we expect that $B(I) \subset \mathrm{Poset}(\nabla)$ is the complex of strata of the special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$, i.e.,

$$B(I) = \mathrm{Strata}_\Delta(I_0^\circ)$$

and its ideal I_0° is obtained as follows:

Any ray v of the normal fan $\Sigma^\circ = \mathrm{NF}(\nabla)$ corresponds to a facet F_v of ∇. Write $S^\circ = \mathbb{C}[z_v \mid v \in \Sigma^\circ(1)]$ for the Cox ring of Y°. The subcomplex $B(I) \subset \nabla$ defines a monomial ideal

$$I_0^\circ = \left\langle \prod_{v\in J} z_v \mid J \subset \Sigma^\circ(1) \text{ with } \mathrm{supp}(B(I)) \subset \bigcup_{v\in J} F_v \right\rangle \subset S^\circ$$

generated by the products of variables of S° such that the corresponding union of facets contains $\mathrm{supp}(B(I))$ as a subset. Here $\mathrm{supp}(B(I))$ denotes the underlying set of the subcomplex $B(I) \subset \mathrm{Poset}(\nabla)$. The special fiber of the mirror degeneration is expected to be given by

$$X_0^\circ = V(I_0^\circ) \subset Y^\circ$$

Interpreting N as the lattice of monomials of Y°, a general first order polynomial deformation of I_0°, including trivial deformations, is given as a general linear combination of the lattice points of

$$(\lim(B(I)))^* \subset \operatorname{Poset}(\Delta^*) = P \subset N_{\mathbb{R}}$$

which is the complex of faces $F^* \subset \Delta^*$ dual to the faces F of $\lim(B(I))$. These lattice points map to Cox Laurent monomials via the presentation of the Chow group of Y°

$$0 \to N \xrightarrow{A^{\circ}} \mathbb{Z}^{\Sigma^{\circ}(1)} \to A_{n-1}(Y^{\circ}) \to 0$$

Denote their image by

$$\Xi^{\circ} = A^{\circ}(N \cap \operatorname{supp}(\lim(B(I)))^*)$$

Define the first order deformation of X_0°

$$\mathfrak{X}^{1\circ} \subset Y^{\circ} \times \operatorname{Spec} \mathbb{C}[t] / \langle t^2 \rangle$$

by the ideal

$$I^{1\circ} = \left\langle u + t \cdot \sum_{\alpha \in \Xi^{\circ}} a_{\alpha} \cdot \alpha(u) \mid u \in I_0^{\circ} \right\rangle \subset \mathbb{C}[t] / \langle t^2 \rangle \otimes S^{\circ}$$

with general coefficients a_{α}. The family $\mathfrak{X}^{1\circ}$ is expected to be up to first order the mirror degeneration of $\mathfrak{X}$.

This construction is motivated by the following structure on the first order deformations of X_0: For any face F of $B(I)$ denote the associated initial ideal by $\operatorname{in}_F(I)$. For any tie break ordering $>$ inside $C_{I_0}(I)$ we have $L_>(\operatorname{in}_F(I)) = I_0$. Associated to F there is a first order deformation

$$\mathfrak{X}_F \subset Y \times \operatorname{Spec}\left(\mathbb{C}[t] / \langle t^2 \rangle\right)$$

defined by the image of $\operatorname{in}_F(I)$ under

$$\begin{array}{ccc} \mathbb{C}[t] \otimes S & \to & \mathbb{C}[t] / \langle t^2 \rangle \otimes S \\ \cup & & \cup \\ \operatorname{in}_F(I) & \to & \langle m_{i0} + t \sum a_{ij} m_{ij} \mid i \rangle \end{array}$$

where $m_{10}, ..., m_{r0}$ are minimal generators of I_0. By homogeneity the Cox Laurent monomials $\frac{m_{ij}}{m_{i0}}$ are in the image of A, and the image of the map

$$\begin{array}{rccc} \operatorname{dual}: & B(I) & \to & \operatorname{Poset}(\nabla^*) \\ & F & \mapsto & \operatorname{convexhull}\left(\left\{A^{-1}\left(\frac{m_{ij}}{m_{i0}}\right) \mid i, j\right\}\right) \end{array}$$

associating to each face the convex hull of the torus invariant first order deformations appearing in its initial ideal, carries the structure of a complex, indeed $\mathrm{dual}(F) = F^* \subset \nabla^*$. The lattice points

$$M \cap \mathrm{supp}(\mathrm{dual}(B(I)))$$

of the image form a torus invariant basis of the space of those first order polynomial deformations of X_0, which map modulo trivial deformations in the tangent space of the component of the moduli space of X_0, containing the family $\mathfrak{X}$.

The above construction depends on the degeneration $\mathfrak{X}$ only up to first order. Applying the construction to the first order mirror family $\mathfrak{X}^\circ$ recovers the original degeneration $\mathfrak{X}$ up to first order.

We summarize the key identifications, made in above construction, denoting the identifications by the symbols $\updownarrow$ and $\leftrightarrow$. To make a clear distinction between the two mirror partners and their embedding toric Fano varieties, denote the lattice of monomials of $Y^\circ = X(\Sigma(\nabla^*))$ by M° and its dual lattice by $N^\circ = \mathrm{Hom}(M^\circ, \mathbb{Z})$. Denote by X the general fiber of $\mathfrak{X}$. Write $SP(I_0^\circ)$ for the complex of ideals of toric strata of the mirror special fiber.

$$\begin{array}{rcccc}
 & & & & \text{containing} \\
\begin{array}{r}\text{weight vectors on}\\ \text{Cox ring } S \text{ of } X(\Sigma)\end{array} & = & \frac{\mathrm{Hom}_{\mathbb{R}}\left(\mathbb{R}^{\Sigma(1)},\mathbb{R}\right)}{\mathrm{Hom}_{\mathbb{R}}(A_{n-1}(Y)\otimes_{\mathbb{Z}}\mathbb{R},\mathbb{R})} & \supset & \mathrm{Poset}(\nabla) \supset B(I) \leftrightarrow SP(I_0^\circ) \\
 & & \cong\downarrow & & \begin{array}{c}\text{toric Kähler}\\ \text{classes on } X\\ \cap\end{array} \\
\begin{array}{r}\text{one parameter sub-}\\ \text{groups of } T \subset X(\Sigma)\end{array} & = & N \subset N_{\mathbb{R}} & \supset & P \supset N \cap (\lim(B(I)))^* \\
 & & \updownarrow & & \updownarrow \\
\begin{array}{r}\text{characters of}\\ T^\circ \subset X(\Sigma^\circ)\end{array} & = & M^\circ \subset M^\circ_{\mathbb{R}} & \supset & P \supset M^\circ \cap \mathrm{dual}(B(I^\circ)) \\
 & & \cap & & \updownarrow \\
\begin{array}{r}\text{Cox Laurent}\\ \text{monomials of } X(\Sigma^\circ)\end{array} & = & \mathbb{Z}^{\Sigma^\circ(1)} \subset \mathbb{R}^{\Sigma^\circ(1)} & \supset & \begin{array}{c}\text{first order}\\ \text{deformations of } X_0^\circ\end{array}
\end{array}$$

and the analogous mirrored diagram. The key connections between both diagramms are made by the maps lim relating the complexes of strata of I_0 and I_0°, and by the map dual, i.e., the correspondence between weights and initial ideals.

As discussed in [Aspinwall, Greene, Morrison, 1993] in the case of hypersurfaces, the identification of the lattice N and the lattice of monomials M° of $X(\Sigma^\circ)$ gives rise to a mirror map between complex and Kähler moduli.

In an analogous way, above tropical mirror construction allows interpretation of the vertices of the faces (or, via MPCP-blowup, of the lattice points) of dual $(B(I^\circ))$ as first order polynomial deformations of X_0° or as toric Kähler classes on X, which should induce a mirror correspondence between complex moduli and Kähler moduli.

9.2 First order deformations and degree 0 Cox Laurent monomials

Consider a toric variety $X(\Sigma)$ of dimension n with Cox ring S, and recall that the map deg in

$$0 \to M \overset{A}{\to} \mathbb{Z}^{\Sigma(1)} \overset{\deg}{\to} A_{n-1}(X(\Sigma)) \to 0$$

can be considered as the map associating to a Cox Laurent monomial its degree in the Chow group of divisors $A_{n-1}(X(\Sigma))$. Hence $\operatorname{image}(A) = \ker(\deg)$ is precisely the set of degree 0 Cox Laurent monomials. So there is an isomorphism

$$M \overset{A}{\rightleftarrows} \operatorname{image}(A) \subset \mathbb{Z}^{\Sigma(1)}$$

of M and the degree 0 Cox Laurent monomials, and $M_{\mathbb{R}} \subset \mathbb{R}^{\Sigma(1)}$ is a sub vector space containing the lattice $M \cong \mathbb{Z}^n$.

The characters of the big torus $(\mathbb{C}^*)^{\Sigma(1)} \cong \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*\right)$ are the Cox Laurent monomials, i.e., the elements of $\mathbb{Z}^{\Sigma(1)}$.

Let $I_0 \subset S$ be a monomial ideal. As I_0 is generated by finitely many elements and the space of elements of S of this degree is finite dimensional, the degree 0 homomorphisms in $\operatorname{Hom}(I_0, S/I_0)$ form a finite dimensional vector space denoted by $\operatorname{Hom}(I_0, S/I_0)_0$. The big torus $(\mathbb{C}^*)^{\Sigma(1)}$ acts by

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*\right) \times \mathbb{C}\left[\mathbb{Z}^{\Sigma(1)}\right] & \to & \mathbb{C}\left[\mathbb{Z}^{\Sigma(1)}\right] \\ (\lambda, m) & \mapsto & \lambda(m) \cdot m \end{array}$$

on $\mathbb{C}\left[\mathbb{Z}^{\Sigma(1)}\right]$ and on S. The induced action of the abelian group $\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*\right)$ on the vector space $\operatorname{Hom}(I_0, S/I_0)_0$ gives a representation

$$\operatorname{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{\Sigma(1)}, \mathbb{C}^*\right) \to \operatorname{GL}\left(\operatorname{Hom}(I_0, S/I_0)_0\right)$$

which decomposes into characters, as any irreducible representation of an abelian group over an algebraically closed field is 1-dimensional.

So, denoting first order deformations which are characters as $(\mathbb{C}^*)^{\Sigma(1)}$**-deformations**, the vector space $\operatorname{Hom}(I_0, S/I_0)_0$ has a basis of $(\mathbb{C}^*)^{\Sigma(1)}$-deformations . Any such homomorphism δ is represented by a degree 0 Cox

Laurent monomial, i.e., by a character of $(\mathbb{C}^*)^{\Sigma(1)}$. There are relatively prime monomials $q_0, q_1 \in S$ with $\frac{q_1}{q_0} \in \text{image}(A)$ such that for all minimal generators $m \in I_0$ with $\delta(m) \neq 0$ we have $\frac{\delta(m)}{m} = \frac{q_1}{q_0}$. So δ is the degree 0 homomorphism $I_0 \to S/I_0$ defined by

$$\delta(m) = \left\{ \begin{array}{ll} \frac{q_1}{q_0} \cdot m & \text{if } q_0 \mid m \\ 0 & \text{otherwise} \end{array} \right\}$$

for minimal generators $m \in I_0$.

Lemma 9.1 *If I_0 is a monomial ideal, then* $\text{Hom}(I_0, S/I_0)_0$ *has a basis of* $(\mathbb{C}^*)^{\Sigma(1)}$*-deformations represented by elements of* $\text{image}(A) \cong M$.

With respect to weights on I recall from Section 6.7 that there is a bijection

$$N \supset \text{image}(\_ \circ A) \underset{\varphi}{\overset{\_\circ A}{\leftrightarrows}} \frac{\text{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{\Sigma(1)}, \mathbb{Z}\right)}{\text{Hom}_{\mathbb{Z}}(A_{n-1}(X(\Sigma)), \mathbb{Z})} \quad \to \quad \{\text{graded wt. vec. on } S\}$$

and an isomorphism of vector spaces

$$N_{\mathbb{R}} \underset{\varphi_{\mathbb{R}}}{\leftrightarrows} \frac{\text{Hom}_{\mathbb{R}}\left(\mathbb{R}^{\Sigma(1)}, \mathbb{R}\right)}{\text{Hom}_{\mathbb{R}}\left(A_{n-1}\left(X\left(\Sigma\right)\right) \otimes_{\mathbb{Z}} \mathbb{R}, \mathbb{R}\right)}$$

i.e., $N_{\mathbb{R}}$ is a quotient of $\text{Hom}_{\mathbb{R}}\left(\mathbb{R}^{\Sigma(1)}, \mathbb{R}\right)$.

The mirror correspondence between Calabi-Yau degenerations with fibers polarized in toric Fano varieties $X(\Sigma)$ with lattices N and M respectively $X(\Sigma^\circ)$ with lattices N° and M° will be induced by the identification of

$$\frac{\text{Hom}_{\mathbb{R}}\left(\mathbb{R}^{\Sigma(1)}, \mathbb{R}\right)}{\text{Hom}_{\mathbb{R}}(A_{n-1}(X(\Sigma)) \otimes_{\mathbb{Z}} \mathbb{R}, \mathbb{R})} \quad \cong \quad N_{\mathbb{R}} \quad \text{with} \quad M^\circ_{\mathbb{R}} \quad \subset \quad \mathbb{R}^{\Sigma^\circ(1)}$$

and of

$$\mathbb{R}^{\Sigma(1)} \quad \supset \quad M_{\mathbb{R}} \quad \text{with} \quad N^\circ_{\mathbb{R}} \quad \cong \quad \frac{\text{Hom}_{\mathbb{R}}\left(\mathbb{R}^{\Sigma^\circ(1)}, \mathbb{R}\right)}{\text{Hom}_{\mathbb{R}}(A_{n-1}(X(\Sigma^\circ)) \otimes_{\mathbb{Z}} \mathbb{R}, \mathbb{R})}$$

and of the corresponding lattices.

9.3 Monomial ideals in the Cox ring and the stratified toric primary decomposition

Let $N \cong \mathbb{Z}^n$, let $M = \text{Hom}(N, \mathbb{Z})$ be the dual lattice, $\Sigma \subset N_{\mathbb{R}}$ a complete fan, $Y = X(\Sigma)$ the corresponding toric variety and $S = \mathbb{C}[y_r \mid r \in \Sigma(1)]$ the Cox ring of Y.

Definition 9.2 *Let $I_0 \subset S$ be a reduced monomial ideal. If $m \in I_0$ is a monomial, define*

$$\operatorname{rays}_m(\Sigma) = \{r \in \Sigma(1) \mid y_r \text{ divides } m\}$$

The ***stratified toric primary decomposition*** *$SP(I_0)$ of I_0 is the complex with faces of dimension s given by*

$$SP(I_0)_s = \left\{ \langle y_r \mid r \in \operatorname{rays}(\sigma)\rangle \;\middle|\; \begin{array}{l} \sigma \in \Sigma(n-s) \text{ with} \\ \operatorname{rays}(\sigma) \cap \operatorname{rays}_m(\Sigma) \neq \emptyset \\ \text{for all monomials } m \in I_0 \end{array} \right\}$$

Remark 9.3 *Suppose that all maximal faces $SP(I_0)$ have the same dimension, i.e., the vanishing locus of I_0 is equidimensional. The* ***intersection complex*** *$IS(I_0)$ of I_0 is the subcomplex of the simplex on the maximal faces $SP(I_0)$, containing the face F if the ideal*

$$\sum_{J \in F} J \in SP(I_0)$$

i.e., if the ideal $\sum_{J \in F} J$ is again a face of $SP(I_0)$. The complexes $SP(I_0)$ and $IS(I_0)$ are dual to each other.

Suppose D is a divisor on $Y = X(\Sigma)$ such that some multiple of D is ample Cartier, then $\Delta = \Delta_D$ is not necessarily integral, but combinatorially dual to Σ, i.e., $\Sigma = \operatorname{NF}(\Delta_D)$.

For example we could consider a Fano polytope $P \subset N_{\mathbb{R}}$, $\Sigma = \Sigma(P)$ the fan over P and $Y = X(\Sigma)$ the corresponding toric Fano variety and $\Delta = \Delta_{-K_Y} = P^* \subset M_{\mathbb{R}}$.

We can reformulate above notations in terms of a subcomplex of the polytope Δ and the dimensions of the faces are the geometric dimension of the corresponding faces of Δ:

If F is a face of Δ, define

$$\operatorname{facets}_F(\Delta) = \{G \mid G \text{ facet of } \Delta \text{ with } F \subset G\}$$

as the set of facets of Δ containing F. If $m \in I_0$ is a monomial, define

$$\operatorname{facets}_m(\Delta) = \{G \mid G \text{ facet of } \Delta \text{ with } y_{G^*} \mid m\}$$

as the set of those facets of Δ, which appear, considered as Cox variable, as a factor of m.

Definition 9.4 *The* ***complex of strata*** *of* I_0 *is the subcomplex* $\operatorname{Strata}_\Delta(I_0)$ *of the associated complex* $\operatorname{Poset}(\Delta)$ *of* Δ *with faces of* $\operatorname{Strata}_\Delta(I_0)$ *of dimension* s *given by*

$$\operatorname{Strata}_\Delta(I_0)_s = \left\{ F \mid \begin{array}{l} F \text{ a face of } \Delta \text{ of } \dim(F) = s \text{ with} \\ \operatorname{facets}_F(\Delta) \cap \operatorname{facets}_m(\Delta) \neq \emptyset \\ \text{for all monomials } m \in I_0 \end{array} \right\}$$

Lemma 9.5 *The faces of dimension* s *of the stratified toric primary decomposition* $SP(I_0)$ *of* I_0 *are given by*

$$SP(I_0)_s = \{\langle y_{G^*} \mid G \text{ a facet of } \Delta \text{ with } F \subset G\rangle \mid F \in \operatorname{Strata}_\Delta(I_0)_s\}$$

and $\operatorname{Strata}_\Delta(I_0) \cong SP(I_0)$.

These definitions may be generalized to the case of non-reduced monomial ideals, though this is not used in the following.

Proposition 9.6 *Let* $I_0 \subset S$ *be a reduced monomial ideal such that* $\operatorname{Strata}_\Delta(I_0)$ *is equidimensional of dimension* d. *Then there is a unique monomial ideal* $I_0^\Sigma \subset S$ *maximal with respect to inclusion such that* $\operatorname{Strata}_\Delta(I_0) = \operatorname{Strata}_\Delta\left(I_0^\Sigma\right)$. *It holds*

$$\begin{aligned} I_0^\Sigma &= \bigcap\nolimits_{F \in \operatorname{Strata}_\Delta(I_0)_d} \langle y_{G^*} \mid G \text{ a facet of } \Delta \text{ with } F \subset G\rangle \\ &= \left\langle \prod_{v \in J} y_v \mid J \subset \Sigma(1) \text{ with } \operatorname{supp}(\operatorname{Strata}_\Delta(I_0)) \subset \bigcup_{v \in J} F_v \right\rangle \subset S \end{aligned}$$

Definition 9.7 *We denote* I_0^Σ *as the* Σ***-saturation*** *of* I_0.

Remark 9.8 *If* Σ *is simplicial, then*

$$I_0^\Sigma = \left(I_0 : B(\Sigma)^\infty\right)$$

In the special case of $Y = \mathbb{P}^n$ the complex $\operatorname{Strata}_\Delta(I_0)$ is related to the representation of Stanley-Reisner ideals by the following remark (see also Section 13.6):

Remark 9.9 *Suppose* $Y = \mathbb{P}(\Delta) \cong \mathbb{P}^n$ *where* Δ *is the degree* $n+1$ *Veronese polytope and let* S *be the homogeneous coordinate ring of* Y. *So* Δ^* *(and of course also* Δ*) is a simplex and the faces of* Δ^* *correspond to the subsets of the set of vertices of* Δ^*. *The vertices of* Δ^* *generate the rays of* $\Sigma = \operatorname{NF}(\Delta)$, *the cones of* Σ *correspond to the subsets of* $\Sigma(1)$. *The rays of* Σ *correspond to the variables of* $S = \mathbb{C}[y_r \mid r \in \Sigma(1)]$.

Let Z be a simplicial subcomplex of $\operatorname{Poset}(\Sigma) \cong \operatorname{Poset}(\Delta^*)$, *where each face of Z is considered as a set of rays of Σ, and*

$$I_0 = \left\langle \prod_{r\in M} y_r \mid M \subset \Sigma(1) \text{ a non-face of } Z \right\rangle \subset S$$

the corresponding ***Stanley-Reisner ideal****.*

If $F \in \operatorname{Strata}_\Delta(I_0)$ is a face and $F^* \subset \Delta^*$ the dual face of F then denote the set of all rays of Σ in the complement of $\operatorname{hull}(F^*) \in \Sigma$ *by* $\operatorname{comp}(F)$*, so, e.g., if F is a vertex of Δ then the complement of $F^* \subset \Delta^*$ contains precisely one vertex of Δ^*. The map*

$$\begin{array}{cccccc} & \operatorname{Poset}(\Delta) & & \operatorname{Poset}(\Sigma) & \cong & \operatorname{Poset}(\Delta^*) \\ & \cup & & \cup & & \\ \operatorname{comp}: & \operatorname{Strata}_\Delta(I_0) & \overset{\cong}{\to} & Z & & \\ & F & \mapsto & \operatorname{comp}(F) & & \end{array}$$

is an isomorphism of complexes and

$$I_0 = \left\langle \prod_{v\in J} y_v \mid J \subset \Sigma(1) \text{ with } \operatorname{supp}(\operatorname{Strata}_\Delta(I_0)) \subset \bigcup_{v\in J} F_v \right\rangle$$

Example 9.10 *Let $Y = \mathbb{P}(\Delta) \cong \mathbb{P}^3$ with*

$$\Delta = \operatorname{convexhull}\{(-1,-1,-1),(3,-1,-1),(-1,3,-1),(-1,-1,3)\}$$

so

$$\Delta^* = \operatorname{convexhull}\{(-1,-1,-1),(1,0,0),(0,1,0),(0,0,1)\}$$

and write

$$\begin{array}{llll} r_0 = \operatorname{hull}\{(-1,-1,-1)\} & r_1 = \operatorname{hull}\{(1,0,0)\} & r_2 = \operatorname{hull}\{(0,1,0)\} & r_3 = \operatorname{hull}\{(0,0,1)\} \\ x_0 = y_{r_0} & x_1 = y_{r_1} & x_2 = y_{r_2} & x_3 = y_{r_3} \end{array}$$

Consider the complex

$$Z = \left(\begin{array}{c} \{\} \\ \{\{r_2\},\{r_0\},\{r_3\},\{r_1\}\} \\ \{\{r_2,r_0\},\{r_0,r_1\},\{r_3,r_2\},\{r_1,r_3\}\} \\ \{\} \\ \{\} \end{array} \right)$$

The Stanley-Reisner ideal associated to Z is the monomial ideal

$$\begin{aligned} I_0 &= \langle x_0x_3, x_1x_2, x_0x_1x_2, x_0x_1x_3, x_0x_2x_3, x_1x_2x_3, x_0x_1x_2x_3\rangle \\ &= \langle x_0x_3, x_1x_2\rangle \subset S = \mathbb{C}\left[x_0, x_1, x_2, x_3\right] \end{aligned}$$

The complex of strata associated to I_0 is

$$\mathrm{Strata}_{\Delta}\left(I_0\right) = \mathrm{convexhull}\left(\begin{array}{c} \{\} \\ \left\{\begin{array}{c} \{(-1,3,-1)\} \\ \{(-1,-1,-1)\} \\ \{(-1,-1,3)\} \\ \{(3,-1,-1)\} \end{array}\right\} \\ \left\{\begin{array}{c} \{(-1,3,-1),(-1,-1,-1)\} \\ \{(-1,-1,-1),(3,-1,-1)\} \\ \{(-1,-1,3),(-1,3,-1)\} \\ \{(3,-1,-1),(-1,-1,3)\} \end{array}\right\} \\ \{\} \\ \{\} \end{array}\right)$$

This notation is short for applying convexhull *to each face of the complex in the argument. The dual of* $\mathrm{Strata}_{\Delta}\left(I_0\right)$ *is*

$$\left(\mathrm{Strata}_{\Delta}\left(I_0\right)\right)^* = \mathrm{convexhull}\left(\begin{array}{c} \{\} \\ \{\} \\ \left\{\begin{array}{c} \{(1,0,0),(0,0,1)\} \\ \{(0,0,1),(0,1,0)\} \\ \{(1,0,0),(-1,-1,-1)\} \\ \{(-1,-1,-1),(0,1,0)\} \end{array}\right\} \\ \left\{\begin{array}{c} \{(1,0,0),(-1,-1,-1),(0,0,1)\}, \\ \{(0,0,1),(0,1,0),(1,0,0)\}, \\ \{(-1,-1,-1),(0,1,0),(1,0,0)\}, \\ \{(0,1,0),(-1,-1,-1),(0,0,1)\} \end{array}\right\} \\ \{\} \end{array}\right)$$

so

$$\operatorname{comp}\left(\left(\operatorname{Strata}_{\Delta}\left(I_0\right)\right)\right) = \left(\begin{array}{c} \{\} \\ \left\{\begin{array}{c} \{\operatorname{hull}(0,1,0)\} \\ \{\operatorname{hull}(-1,-1,-1)\} \\ \{\operatorname{hull}(0,0,1)\} \\ \{\operatorname{hull}(1,0,0)\} \end{array}\right\} \\ \left\{\begin{array}{c} \{\operatorname{hull}(-1,-1,-1), \operatorname{hull}(0,1,0)\} \\ \{\operatorname{hull}(-1,-1,-1), \operatorname{hull}(1,0,0)\} \\ \{\operatorname{hull}(0,0,1), \operatorname{hull}(0,1,0)\} \\ \{\operatorname{hull}(1,0,0), \operatorname{hull}(0,0,1)\} \end{array}\right\} \\ \{\} \\ \{\} \end{array}\right)$$
$$= Z$$

Figure 9.1 shows the complexes $\operatorname{Strata}_{\Delta}(I_0) \subset \operatorname{Poset}(\Delta)$*, Figure 9.2 the complex* $(\operatorname{Strata}_{\Delta}(I_0))^* \subset \operatorname{Poset}(\Delta^*)$ *and Figure 9.3 the corresponding Stanley-Reisner complex* Z *considered as a subcomplex of* $\operatorname{Poset}(\Delta^*)$*.*

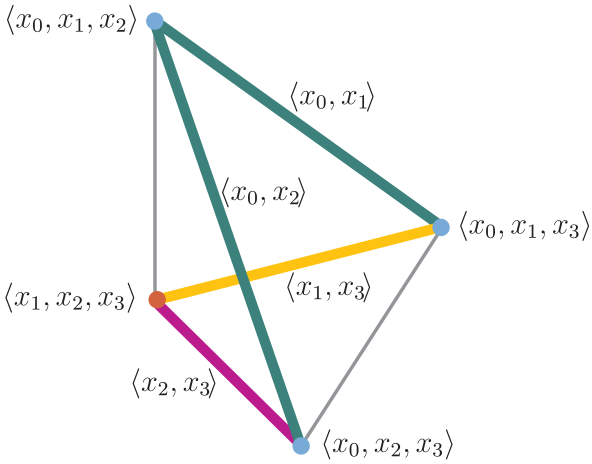

Figure 9.1: The complex $\operatorname{Strata}_{\Delta}(I_0) \subset \operatorname{Poset}(\Delta)$ for the ideal $I_0 = \langle x_0x_3, x_1x_2\rangle$

9.4 Locally relevant deformations

Let $N \cong \mathbb{Z}^n$ and $M = \operatorname{Hom}(N, \mathbb{Z})$, let $\Sigma \subset N_{\mathbb{R}}$ be a complete fan and $Y = X(\Sigma)$ the associated toric variety.

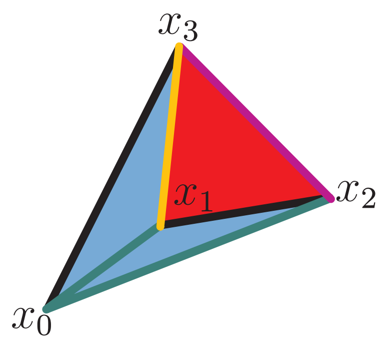

Figure 9.2: The complex $(\mathrm{Strata}_\Delta(I_0))^* \subset \mathrm{Poset}(\Delta^*)$ for the ideal $I_0 = \langle x_0x_3, x_1x_2\rangle$

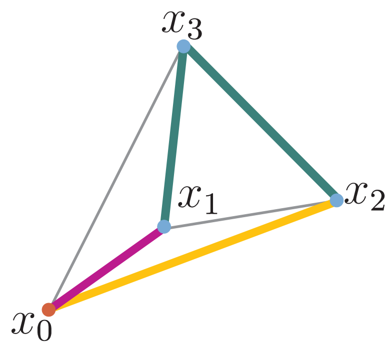

Figure 9.3: The subcomplex of $\mathrm{Poset}(\Delta^*)$ defining the Stanley-Reisner ideal $I_0 = \langle x_0x_3, x_1x_2\rangle$

Definition 9.11 *Let $X_0 \subset Y$ a subvariety which is a union of equidimensional strata, let $X_i \in \mathrm{Strata}(X_0)$ be a torus stratum of X_0 and consider a first order deformation $\mathfrak{X} \subset Y \times \mathrm{Spec}(\mathbb{C}[t] / \langle t^2\rangle)$ of X_0. Then $\mathfrak{X}$ is called* ***locally irrelevant*** *at the stratum X_i if there is a formal analytic open neighborhood $\tilde{U} \subset Y$ of X_i such that for the open neighborhood $U = \tilde{U} \cap X_0$ of X_i in X_0 there is an isomorphism*

$$U \times \mathrm{Spec}(\mathbb{C}[t] / \langle t^2\rangle) \cong \mathfrak{X} \cap \left(\tilde{U} \times \mathrm{Spec}(\mathbb{C}[t] / \langle t^2\rangle)\right)$$

which extends

$$X_i \times \mathrm{Spec}(\mathbb{C}[t] / \langle t^2\rangle) \subset \mathfrak{X}$$

Otherwise, $\mathfrak{X}$ is called ***locally relevant*** *at X_i. The deformation $\mathfrak{X}$ is called* ***strongly locally relevant*** *at X_i if $\mathfrak{X}$ is locally relevant at X_i and locally irrelevant for all strata $X_j \in \mathrm{Strata}(X_0)$ with $X_i \cap X_j = \emptyset$.*

Example 9.12 *Consider $X_0 \subset Y = \mathbb{P}^3$ given by the monomial ideal $I_0 = \langle x_0x_3, x_1x_2\rangle \subset S = \mathbb{C}[x_0, ..., x_3]$. The ideals of the strata of X_0 are shown in Figure 9.1. Consider the following torus invariant deformations given by Cox Laurent monomials:*

- *For $\frac{x_3^2}{x_1x_2}$ the deformation $\mathfrak{X}$ is given by*

  $$\left\langle x_0x_3, x_1x_2 + t \cdot x_3^2\right\rangle = \langle x_3, x_1\rangle \cap \langle x_3, x_2\rangle \cap \left\langle x_0, x_1x_2 + t \cdot x_3^2\right\rangle$$

  and is locally relevant at $\langle x_0, x_2\rangle, \langle x_0, x_1\rangle, \langle x_0, x_1, x_2\rangle$. It is strongly locally relevant at $\langle x_0, x_1, x_2\rangle$.

- *For $\frac{x_3}{x_0}$ the deformation $\mathfrak{X}$ is given by*

  $$\left\langle x_0x_3 + t \cdot x_3^2, x_1x_2\right\rangle = \langle x_3, x_1\rangle \cap \langle x_3, x_2\rangle \cap \langle x_1, x_0 + t \cdot x_3\rangle \cap \langle x_2, x_0 + t \cdot x_3\rangle$$

  and is locally relevant at $\langle x_0, x_2\rangle, \langle x_0, x_1\rangle, \langle x_0, x_1, x_2\rangle$. It is strongly locally relevant at $\langle x_0, x_1, x_2\rangle$.

- *For $\frac{x_1}{x_0}$ the deformation $\mathfrak{X}$ is given by*

  $$\langle x_0x_3 + t \cdot x_1x_3, x_1x_2\rangle = \langle x_3, x_1\rangle \cap \langle x_3, x_2\rangle \cap \langle x_1, x_0\rangle \cap \langle x_2, x_0 + t \cdot x_1\rangle$$

  and is locally relevant at $\langle x_0, x_1, x_2\rangle, \langle x_0, x_2, x_3\rangle, \langle x_0, x_2\rangle, \langle x_1, x_0\rangle, \langle x_3, x_2\rangle$. It is strongly locally relevant at $\langle x_0, x_2\rangle$.

The Figures 9.4, 9.5 and 9.6 visualize the strata of X_0 where these deformations are locally relevant or irrelevant.

See also Example 9.47 below.

9.5 Setup for the tropical mirror construction for monomial degenerations of Calabi-Yau varieties polarized in toric Fano varieties

Consider the following setup for the tropical mirror construction. The conditions on the degeneration may be subject to generalization and redundancy.

We begin with the following setup:

- Let $N \cong \mathbb{Z}^n$, let $M = \operatorname{Hom}(N, \mathbb{Z})$ be the dual lattice, P a Fano polytope, $\Sigma = \Sigma(P)$ and $Y = X(\Sigma)$ the corresponding toric Fano variety. Denote by $S = \mathbb{C}[y_r \mid r \in \Sigma(1)]$ the Cox ring of Y and by

  $$0 \to M \xrightarrow{A} \mathbb{Z}^{\Sigma(1)} \to A_{n-1}(Y) \to 0$$

  the presentation of the Chow group of divisors of Y.

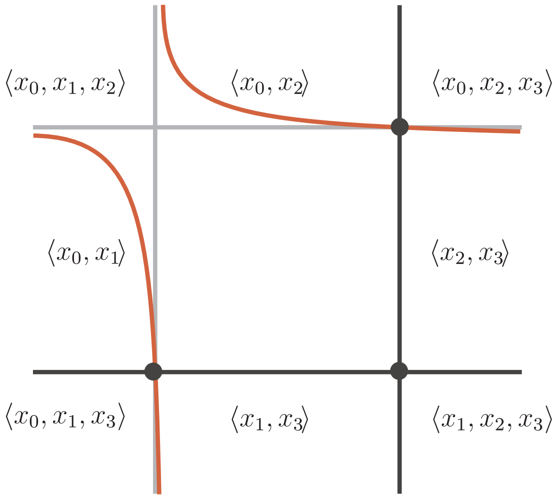

Figure 9.4: Visualization of the deformation $\frac{x_3^2}{x_1 x_2}$ of $I_0 = \langle x_0 x_3, x_1 x_2\rangle$

- Let $I_0 = \langle m_1, ..., m_r\rangle \subset S$ be an equidimensional reduced monomial ideal and $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[[t]]$ an irreducible flat family of Calabi-Yau varieties of dimension d given by the ideal $I = \langle f_j = m_j + t g_j \mid j = 1, ..., r\rangle \subset \mathbb{C}[t] \otimes S$.

  Assume that the underlying topological space of the cell complex $\operatorname{Strata}_{\Delta}(I_0)$ is homeomorphic to a sphere.

Define the following:

- Let $>$ be a monomial ordering on $\mathbb{C}[t] \otimes S$, which is respecting the Chow grading on S and is local in t. Let

  $$C_{I_0}(I) = \left\{ -(w_t, w_y) \in \mathbb{R} \oplus N_{\mathbb{R}} \mid L_{>_{(w_t, \varphi(w_y))}}(I) = I_0 \right\}$$

  be the Gröbner cone corresponding to the lead ideal I_0.

- Let

  $$BF_{I_0}(I) = BF(I) \cap \operatorname{Poset}(C_{I_0}(I))$$

We require $\mathfrak{X}$ to satisfy the following conditions:

1. $C_{I_0}(I) \cap \{w_t = 0\} = \{0\}$.
2. $C_{I_0}(I)$ is the cone defined by the half-space equations corresponding to the torus invariant first order deformations appearing in the reduced

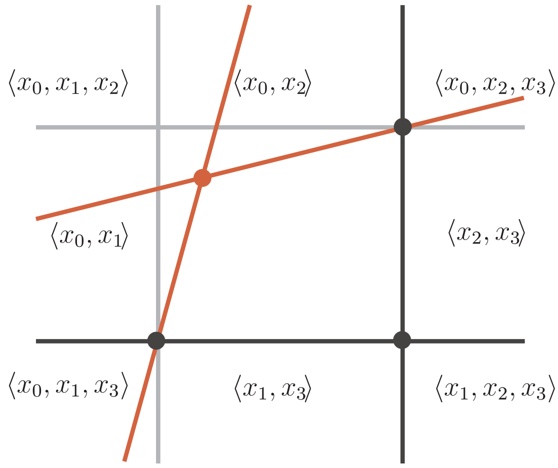

Figure 9.5: Visualization of the deformation $\frac{x_3}{x_0}$ of $I_0 = \langle x_0x_3, x_1x_2\rangle$

standard basis of I in $S\times\mathbb{C}[t]/\langle t^2\rangle$ with respect to a monomial ordering in the interior of $C_{I_0}(I)$.

All lattice points of F^* appear as deformations in I.

3. Any first order deformation appearing in I is also a non-zero deformation of the anticanonical Calabi-Yau hypersurface in Y.

4. Any facet of $\text{Strata}_\Delta(I_0)$ is contained in precisely c facets of $\Delta = P^*$.

5. Any facet of $BF_{I_0}(I)$ is contained in precisely c facets of $\text{Poset}(C_{I_0}(I))$.

In the following we give a geometric interpretation of these conditions.

1. We can satisfy requirement 1. via a condition on the position of the Hilbert point of I_0 with respect to the state polytope of the general fiber:

   Let $\mathcal{K} = \text{cpl}(\Sigma) \cap \text{Pic}(Y)$ and $I_{gen} \subset S$ be the saturated ideal of the general fiber of $\mathfrak{X}$. Let $P(t)$ be the Hilbert polynomial of I_{gen}, h the corresponding Hilbert function, $D \subset m + \mathcal{K}$ such that the restriction map gives a closed embedding $\mathbb{H}^h_{(S,F)} \to \mathbb{H}^h_{(S_D,F_D)}$. Fix linearizations of the action of T on the elements of D. We require that the Hilbert point
$$H(I_0) \in \text{int}(\text{State}(I_{gen})) \subset M_\mathbb{R}$$
   If we fix the linearizations such that $H(I_0)$ corresponds to $0 \in M_\mathbb{R}$, then by Theorem 6.98 this condition is equivalent to $H(I_{gen}) \in \mathbb{H}^s(E)$

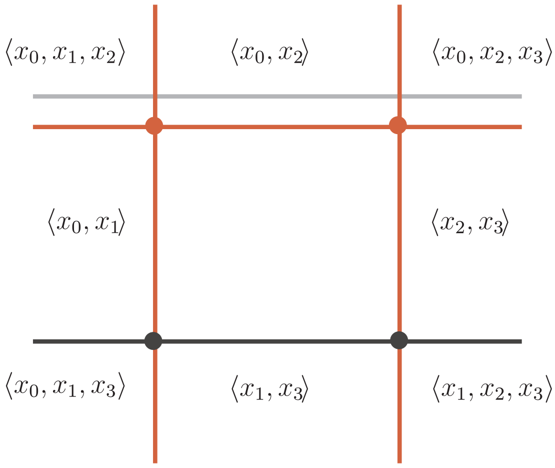

Figure 9.6: Visualization of the deformation $\frac{x_1}{x_0}$ of $I_0 = \langle x_0x_3, x_1x_2\rangle$

with $E = p^*\left(\mathcal{O}_{\mathbb{P}(W)}(1)\right)$, i.e., that the Hilbert point of I_{gen} is in the stable locus of the Hilbert scheme.

The construction of the Hilbert scheme in Section 6 assumes Y to be a smooth toric variety. With an appropriate definition of $\mathrm{State}(I_{gen})$ as discussed in Remark 6.97, this condition is expected to be stated in the same form for a general simplicial or even non-simplicial toric variety Y.

If the Gröbner cone $C_{I_0}(I)$ corresponding to I_0 intersects $BF(I) \cap \{w_t = 0\}$, the Hilbert point $H(I_0)$ of I_0 would lie on the boundary of $\mathrm{State}(I_{gen}) \subset M_{\mathbb{R}}$ contradicting $H(I_0) \in \mathrm{int}(\mathrm{State}(I_{gen}))$, hence

$$C_{I_0}(I) \cap \{w_t = 0\} = \{0\}$$

2. We can satisfy condition 2. via a genericity condition on the tangent vector with respect to the tangent space of the component of the Hilbert scheme containing $\mathfrak{X}$:

   Assume that $\mathfrak{X}$ lies in a smooth component of the complex moduli space $\mathcal{M}$ of X_0 (for example normal crossing at X_0).

   Let $v_1, ..., v_p \in \mathrm{Hom}(I_0, S/I_0)_0$ be a basis of the tangent space of that component of the Hilbert scheme at X_0, which contains the tangent vector v of $\mathfrak{X}$.

   Assume that $\mathfrak{X}$ is maximal in its component of the Hilbert scheme, i.e., writing $v = \sum_{i=1}^{p} \lambda_i v_i$ we have $\lambda_i \neq 0\ \forall i$. Consider the reduced standard basis of I with respect to a monomial ordering in the interior of $C_{I_0}(I)$. Then already the first order deformations appearing in this standard basis, i.e., the Cox Laurent monomials corresponding to t-linear non-special fiber terms, give the linear half-space equations defining $C_{I_0}(I)$.

3. We can satisfy condition 3. via a condition of the resolution of $\mathcal{O}_{X_0}$:

   Assume that $\mathcal{O}_{X_0}$ has a resolution

   $$0 \to \mathcal{O}_Y(-K_Y) \to ... \to \mathcal{F}_1 \to \mathcal{O}_Y \to \mathcal{O}_{X_0} \to 0$$

   with direct sums $\mathcal{F}_j = \bigoplus_i \mathcal{O}_Y(D_{ji})$ with divisors D_{ji}.

   Consider the reduced standard basis of I in $S \times \mathbb{C}[t] / \langle t^2 \rangle$ with respect to a monomial ordering in the interior of $C_{I_0}(I)$. Then any first order $(\mathbb{C}^*)^{\Sigma(1)}$-deformation δ appearing in the standard basis is represented by a Cox Laurent monomial such that the denominator divides $\prod_{r\in\Sigma(1)} y_r$. Hence δ is also a deformation of the anticanonical Calabi-Yau hypersurface in Y defined by $\left\langle \prod_{r\in\Sigma(1)} y_r \right\rangle$.

4. Denote by c the codimension of $X_0 \subset Y$. We interpret condition 4. as the condition that the Σ-saturated ideals defining the components (i.e., strata of maximal dimension d) of X_0 are generated by c variables of S, i.e., are of the form $\langle y_{r_1}, ..., y_{r_c} \rangle \subset S$.

5. We can satisfy condition 5. via a condition on the locally relevant deformations of $\mathfrak{X}$ at the zero dimensional strata of X_0:

   Let p be a zero dimensional stratum of X_0 and $\mathfrak{X}_p$ the flat family given by

   $$I_p = \left\langle m + t \cdot \sum\nolimits_{\delta} c_\delta \cdot \delta(m) \mid \ m \in I_0 \right\rangle$$

   where the sum with general coefficients c_δ goes over the deformations δ strongly locally relevant at p. For all zero dimensional strata p of X_0 we require: All initial ideals $\mathrm{in}_w I_p$ for $w \in C_{I_0}(I)$, which do not contain a monomial and are minimal with respect to the set of contributing deformations, involve precisely c first order deformations.

Remark 9.13 *Note that the condition* $H(I_0) \in \mathrm{int}(\mathrm{State}(I))$ *is independent of rescalation of* $\mathrm{State}(I)$ *by changing* D*, and independent of translation of* $\mathrm{State}(I)$ *by changing the linearizations.*

For hypersurfaces the requirement $H(I_0) \in \mathrm{int}(\mathrm{State}(I))$ *is equivalent to the condition that the special fiber of* $\mathfrak{X}$ *corresponds to the unique interior* *lattice point of the Batyrev polytope $\Delta = P^*$. We may fix a linearization of the torus action on $\mathcal{O}_Y(-K_Y)$ such that $0 \in M$ corresponds to the unique interior lattice point, i.e., we fix the element*

$$V\left(\prod_{r\in\Sigma(1)} y_r\right) \in |-K_Y|$$

of the linear system $|-K_Y|$.

The condition that all facets of $\mathrm{Strata}_\Delta(I_0)$ *are contained in precisely c facets of $\Delta = P^*$ says, by flatness of the family $\mathfrak{X}$, that the total space of $\mathfrak{X}$ is a local complete intersection at the generic points of the strata of maximal dimension d of X_0. So if $\langle y_{r_1}, ..., y_{r_c}\rangle$ is a stratum of maximal dimension, then I is given by c equations in the localization $S_{\langle y_{r_1},...,y_{r_c}\rangle} \otimes \mathbb{C}[t]$ at the prime ideal $\langle y_{r_1}, ..., y_{r_c}\rangle$. Note that*

$$S_{\langle y_{r_1},...,y_{r_c}\rangle} = \mathbb{C}(y_r \mid r \notin \{r_1, ..., r_c\})[y_{r_1}, ..., y_{r_c}]_>$$

for any local ordering $>$ on the monomials in the variables $y_{r_1}, ..., y_{r_c}$.

The condition on the locally relevant deformations at the 0*-dimensional strata of X_0, is a condition on the singularities of $\mathfrak{X}$ at these strata. But note that this condition is far away from requiring the total space of $\mathfrak{X}$ to be a local complete intersection there.*

9.6 The Gröbner cone associated to the special fiber

Consider the setup given in Section 9.5. Let $m_1, ..., m_r$ be minimal generators of the monomial ideal $I_0 \subset S$, let the flat family of Calabi-Yau varieties $\mathfrak{X} \subset Y \times \mathrm{Spec}(\mathbb{C}[[t]])$ be given by the ideal

$$I = \langle f_j = m_j + tg_j \mid j = 1, ..., r\rangle \subset \mathbb{C}[t] \otimes S$$

and suppose that the f_j are reduced with respect to I_0, i.e., no term of g_j is in I_0 considered as an ideal in $\mathbb{C}[t] \otimes S$.

Fix a tie break ordering $>$ on $\mathbb{C}[t] \otimes S$, which is respecting the Chow grading on S and is local in t, so $L_>(f_j) = m_j$.

Definition 9.14 *Let $C_{I_0}(I)$ be the cone of weight vectors selecting I_0 as lead ideal*

$$C_{I_0}(I) = \left\{-(w_t, w_y) \in \mathbb{R} \oplus N_\mathbb{R} \mid L_{>_{(w_t,\varphi(w_y))}}(I) = I_0\right\}$$

Consider $w = -(w_t, w_y) \in C_{I_0}(I)$ and the weight ordering $>_{(w_t,\varphi(w_y))}$ on $\mathbb{C}[t] \otimes S$ with tie break ordering $>$, so

$$L_{>_{(w_t,\varphi(w_y))}}(f_j) = m_j$$

As $\mathfrak{X}$ is flat, for every syzygy $s \in S^r$ of $m_1, ..., m_r$, i.e., with

$$(m_1, ..., m_r) \cdot s = 0$$

there is an $l \in (\mathbb{C}[[t]] \otimes S)^c$ such that

$$(f_1, ..., f_r) \cdot (s - t \cdot l) = 0$$

so

$$\frac{1}{t}(f_1, ..., f_r) \cdot s = (g_1, ..., g_r) \cdot s = (f_1, ..., f_r) \cdot l$$

i.e.,

$$\frac{1}{t}(f_1, ..., f_r) \cdot s \in \langle f_1, ..., f_r \rangle$$

and the Buchberger normal form in $\mathbb{C}[[t]] \otimes S$ yields

$$NF_{>_{(w_t, \varphi(w_y))}}((f_1, ..., f_r) \cdot s, I) = 0$$

so $f_1, ..., f_r$ form a minimal Gröbner basis of I with respect to $>_{(w_t,\varphi(w_y))}$.

As we have $f_1, ..., f_r$ assumed to be reduced, they form the reduced Gröbner basis of I with respect to $>_{(w_t,\varphi(w_y))}$ and hence the condition

$$L_{>_{(w_t,\varphi(w_y))}}(I) = I_0$$

is equivalent to

$$\operatorname{trop}(f_j - m_j)(w_t, \varphi(w_y)) \leq \operatorname{trop}(m_j)(\varphi(w_y)) \ \ \forall j$$

Here $\operatorname{trop}(f_j - m_j)$ denotes the corresponding piecewise linear function of $f_j - m_j \in \mathbb{C}[t] \otimes S$ and $\operatorname{trop}(m_j)$ the piecewise linear function of $m_j \in S$.

Lemma 9.15 *With the notation from above*

$$C_{I_0}(I) = \{-(w_t, w_y) \in \mathbb{R} \oplus N_{\mathbb{R}} \mid \operatorname{trop}(f_j - m_j)(w_t, \varphi(w_y)) \leq \operatorname{trop}(m_j)(\varphi(w_y)) \ \ \forall j\}$$

It is a closed polyhedral cone with $(1, 0, ..., 0) \in C_{I_0}(I)$.

The defining equations of $C_{I_0}(I)$ are given by the degree 0 Cox Laurent monomials appearing in I, which again correspond to lattice monomials in M, i.e., if $t^a m \neq m_j$ is a monomial of f_j, then

$$\left\langle w, A^{-1}\left(\frac{m}{m_j}\right)\right\rangle \geq -a \cdot w_t$$

is a defining equation of $C_{I_0}(I)$, hence:

Lemma 9.16 *The dual cone of $C_{I_0}(I)$ is spanned by the degree 0 Cox Laurent monomials appearing in I, i.e.,*

$$C_{I_0}(I)^* = \operatorname{hull}\left(\left\{(\tilde{m}_t, \tilde{m}) \in \mathbb{R} \oplus M_{\mathbb{R}} \mid \begin{array}{l} \exists j \text{ such that } t^{\tilde{m}_t} \cdot A(\tilde{m}) \cdot m_j \in \mathbb{C}[t] \otimes S \\ \text{and is a monomial of } f_j - m_j \end{array}\right\}\right)$$

and $(1, 0, ..., 0) \in C_{I_0}(I)^*$.

By assumption

$$C_{I_0}(I) \cap \{w_t = 0\} = \{0\}$$

hence:

Lemma 9.17 *The cone $C_{I_0}(I)$ minus the zero point is contained in the half-space $\{w_t > 0\}$.*

If $(1, 0, ..., 0) \in C_{I_0}(I)^*$ would lie on the boundary of $C_{I_0}(I)^*$, then $C_{I_0}(I)$ would contain a ray in $\{w_t = 0\}$, hence:

Lemma 9.18 *The monomial $(1, 0, ..., 0)$ lies in the interior of the dual cone $C_{I_0}(I)^*$, i.e.,*

$$(1, 0, ..., 0) \in \operatorname{int}(C_{I_0}(I)^*)$$

The flat family $\mathfrak{X} \subset Y \times \operatorname{Spec}(\mathbb{C}[[t]])$ induces a first order flat family

$$\mathfrak{X}^1 \subset Y \times \operatorname{Spec}\left(\mathbb{C}[t] / \langle t^2 \rangle\right)$$

given by

$$I^1 = \left\langle f_j^1 = m_j + t g_j^1 \mid j = 1, ..., r \right\rangle = \mathbb{C}[t] / \langle t^2 \rangle \otimes S$$

with $g_j^1 \in S$.

By above assumption the defining equations of $C_{I_0}(I)$ are given by first order deformations appearing in the reduced standard basis of I, so

$$C_{I_0}(I) = \left\{-(w_t, w_y) \in \mathbb{R} \oplus N_{\mathbb{R}} \mid \operatorname{trop}\left(g_j^1\right)(w_t, \varphi(w_y)) + w_t \leq \operatorname{trop}(m_j)(\varphi(w_y)) \;\; \forall j\right\}$$

Corollary 9.19 *Intersecting $C_{I_0}(I)$ with the hyperplane $\{w_t = 1\}$ we obtain the convex polytope*

$$\nabla = C_{I_0}(I) \cap \{w_t = 1\} \subset N_{\mathbb{R}}$$

with

$$\nabla = \{-w_y \in N_{\mathbb{R}} \mid \operatorname{trop}(g_j)(\varphi(w_y)) - 1 \leq \operatorname{trop}(m_j)(\varphi(w_y)) \;\; \forall j\}$$

and 0 in the interior of ∇.

Rewriting the tropical equations

$$\begin{aligned}\nabla &= \left\{-w_y \in N_{\mathbb{R}} \mid \operatorname{trop}(m)(\varphi(w_y)) - 1 \leq \operatorname{trop}(m_j)(\varphi(w_y)) \ \forall \text{ monomials } m \text{ of } g_j^1 \ \forall j\right\} \\ &= \left\{w_y \in N_{\mathbb{R}} \mid \varphi(w_y)\left(\frac{m}{m_j}\right) \geq -1 \ \forall \text{ monomials } m \text{ of } g_j^1 \text{ and } \forall j\right\} \\ &= \left\{w_y \in N_{\mathbb{R}} \mid \left\langle A^{-1}\left(\frac{m}{m_j}\right), w_y\right\rangle \geq -1 \ \forall \text{ monomials } m \text{ of } g_j^1 \text{ and } \forall j\right\}\end{aligned}$$

hence

$$\nabla^* = \operatorname{convexhull}\left\{A^{-1}\left(\frac{m}{m_j}\right) \in M_{\mathbb{R}} \mid \exists j \text{ such that } m \text{ is a monomial of } g_j^1\right\}$$

so it follows:

Lemma 9.20 ∇^* *is an integral polytope.*

Any first order deformation appearing in g_j^1 represented by a Cox Laurent monomial $\frac{m}{m_j}$ is also a deformation of the anticanonical Calabi-Yau hypersurface in Y defined by $\left\langle \prod_{r\in\Sigma(1)} y_r \right\rangle$, hence $A^{-1}\left(\frac{m}{m_i}\right) \in \Delta = \Delta_{-K_Y}$, i.e., $\nabla^* \subset \Delta$. As Δ is dual to a Fano polytope, it has 0 as unique interior lattice point by Lemma 7.9. Hence also ∇^* has no interior lattice point besides 0. As $(1,0,...,0) \in \operatorname{int}(C_{I_0}(I)^*)$, the polytope ∇^* contains 0 in its interior.

Lemma 9.21 ∇^* *contains* 0 *as unique interior lattice point.*

Theorem 9.22 ∇^* *is a Fano polytope, hence the fan* $\Sigma^\circ = \Sigma(\nabla^*)$ *over the faces of* ∇^* *defines a* $\mathbb{Q}$*-Gorenstein toric Fano variety* $Y^\circ = X(\Sigma^\circ)$*.*

9.7 The dual complex of initial ideals

Definition 9.23 *If* F *is a face of* ∇*, there is an associated* ***initial ideal of*** I ***with respect to the face*** F*: For all* w_1, w_2 *in the relative interior* $\operatorname{int}(F)$ *of* F *we have*

$$\operatorname{in}_{(1,\varphi(w_1))}(I) = \operatorname{in}_{(1,\varphi(w_2))}(I)$$

Denote this ideal by $\operatorname{in}_F(I)$*. For all* $w_1, w_2 \in \operatorname{int}(F)$ *and* $f \in I$

$$\operatorname{in}_{(1,\varphi(w_1))}(f) = \operatorname{in}_{(1,\varphi(w_2))}(f)$$

denote this initial term of f *by* $\operatorname{in}_F(f)$*.*

If F is a face of ∇, then

$$\mathrm{in}_F(I) = \langle \mathrm{in}_F(f_j) \mid j = 1, ..., r \rangle$$

as $f_1, ..., f_r$ form a Gröbner basis of I with respect to any weight vector in $C_{I_0}(I)$.

Recall that we wrote

$$I^1 = \left\langle f_j^1 = m_j + t g_j^1 \mid j = 1, ..., r \right\rangle$$

with $g_j^1 \in S$ for the ideal of the first order deformation $\mathfrak{X}^1$ associated to $\mathfrak{X}$.

For $j = 1, ..., r$ define $G_j(F)$ as

$$\mathrm{in}_F\left(f_j^1\right) = t \sum_{m \in G_j(F)} c_m \cdot m + m_j$$

Definition 9.24 *If F is a face of ∇, then define the* ***dual face*** *of F as*

$$\mathrm{dual}(F) = \mathrm{convexhull}\left(A^{-1}\left(\frac{m}{m_j}\right) \mid m \in G_j(F),\ j = 1, ..., r\right) \subset M_{\mathbb{R}}$$

the convex hull of the first order deformations appearing in the initial ideal with respect to F. The dual face is a lattice polytope in $M_{\mathbb{R}}$.

By the genericity condition on the tangent vector of $\mathfrak{X}$ we have:

Lemma 9.25

$$\mathrm{dual}(F) \cap M = \left\{A^{-1}\left(\frac{m}{m_j}\right) \mid m \in G_j(F),\ j = 1, ..., r\right\}$$

The dual of ∇ is the convex hull of the first order deformations appearing in I

$$\nabla^* = \mathrm{convexhull}\left(\left\{A^{-1}\left(\frac{m}{m_j}\right) \in M_{\mathbb{R}} \mid \exists j \text{ such that } m \text{ is a monomial of } g_j^1\right\}\right)$$

so for the dual face of F we have

$$\begin{aligned}
\mathrm{dual}(F) &= \mathrm{convexhull}\left(\left\{A^{-1}\left(\frac{m}{m_j}\right) \mid m \in G_j(F),\ j = 1, ..., r\right\}\right) \\
&= \mathrm{convexhull}\left(\bigcup_{j=1}^{r}\left\{A^{-1}\left(\frac{m}{m_j}\right) \mid m \in G_j(F)\right\}\right) \\
&= \mathrm{convexhull}\left(\bigcup_{j=1}^{r}\left\{A^{-1}\left(\frac{m}{m_j}\right) \mid \begin{array}{l} m \in g_j^1 \text{ with} \\ \left\langle A^{-1}\frac{m}{m_j}, w_y\right\rangle = -1\ \forall w_y \in F \end{array}\right\}\right) \\
&= \mathrm{convexhull}\left(\{\widetilde{m} \in \nabla^* \cap M \mid \langle \widetilde{m}, w_y \rangle = -1\ \forall w_y \in F\}\right) \\
&= \{\widetilde{m} \in \nabla^* \mid \langle \widetilde{m}, w_y \rangle = -1\ \forall w_y \in F\}
\end{aligned}$$

hence:

Proposition 9.26 *If F is a face of ∇, then*

$$\operatorname{dual}(F) = F^*$$

in particular $\operatorname{dual}(F)$ *is a face of* ∇^**,*

$$\begin{array}{rccc} \operatorname{dual}: & \operatorname{Poset}(\nabla) & \to & \operatorname{Poset}(\nabla^*) \\ & F & \mapsto & \operatorname{dual}(F) \end{array}$$

is an inclusion reversing map of complexes and

$$\dim(\operatorname{dual}(F)) = n - 1 - \dim(F)$$

The non-special fiber terms of $\operatorname{in}_F\left(f_j^1\right)$, $j = 1,...,r$, i.e., the elements of $G_j(F)$, $j = 1,...,r$ split into characters of the big torus $(\mathbb{C}^*)^{\Sigma(1)}$. These characters are the Cox Laurent monomials

$$\delta_F\left(I^1\right) = \left\{\frac{m}{m_j} \mid m \in G_j(F),\ j = 1,...,r\right\}$$

and represent the first order $(\mathbb{C}^*)^{\Sigma(1)}$-deformations contributing to the degeneration defined by $\operatorname{in}_F(I^1)$. Flat families defined by initial ideals are also called Gröbner deformations. Note that the syzygies of a monomial ideal are binomial and a syzygy between m_i and m_j is represented by the character of $(\mathbb{C}^*)^{\Sigma(1)}$ given by $\operatorname{lcm}(m_i, m_j)$. Note also that monomials $m \in G_i(F)$ and $m' \in G_j(F)$ with $\frac{m}{m_i} = \frac{m'}{m_j}$ appear with the same coefficient in the initial forms. On the other hand the elements of $\delta_F(I^1)$ correspond via A^{-1} to the lattice points of F^*, so:

Lemma 9.27 *The lattice points of F^* are in one-to-one correspondence to the first order deformations contributing to* $\operatorname{in}_F(I^1)$*. If $\delta \in F^*$ is a lattice point and $\frac{q_1}{q_0} = A(\delta)$ with relatively prime monomials $q_0, q_1 \in S$ then*

$$\delta(m) = \left\{\begin{array}{ll} \frac{q_1}{q_0} \cdot m & \text{if } q_0 \mid m \\ 0 & \text{otherwise} \end{array}\right\}$$

for minimal generators $m \in I_0$ defines the corresponding $(\mathbb{C}^*)^{\Sigma(1)}$-deformation in $\operatorname{Hom}(I_0, S/I_0)_0$.

9.8 Bergman subcomplex of ∇

Definition 9.28 *The* ***special fiber Bergman fan*** *is defined as the intersection of the fan* $\mathrm{Poset}(C_{I_0}(I))$ *over the special fiber Gröbner cone* $C_{I_0}(I)$ *with the Bergman fan* $BF(I)$ *of* I

$$BF_{I_0}(I) = BF(I) \cap \mathrm{Poset}(C_{I_0}(I))$$

The ***special fiber Bergman complex***

$$B(I) = BC_{I_0}(I) = (BF(I) \cap \mathrm{Poset}(C_{I_0}(I))) \cap \{w_t = 1\} \subset \mathrm{Poset}(\nabla)$$

is defined as the complex whose faces are the intersections of the hyperplane $\{w_t = 1\}$ *with the faces of the Bergman fan* $BF(I)$ *in* $C_{I_0}(I)$*.*

We also refer to $B(I)$ as the Bergman subcomplex or tropical subcomplex of ∇. By Theorem 4.10 we have:

Remark 9.29 *The complex* $B(I)$ *consists of those faces* F *of* ∇ *such that* $\mathrm{in}_F(I)$ *does not contain a monomial.*

Lemma 9.30 *The special fiber Bergman complex* $B(I)$ *is a polyhedral cell complex of dimension* d*, it is subcomplex of the boundary* $\partial\nabla$ *of* ∇*.*

9.9 Remarks on the covering structure in $\mathrm{dual}(B(I))$

Interpreting the lattice points of the faces of $\mathrm{dual}(B(I))$ as deformations of X_0 and associating them to the reduced standard basis equations f_i, $i = 1, ...r$ defining the total space we get a covering of $B(I)^{\vee}$.

If $F \in B(I)$ is a face, then denote by G_F a minimal standard basis of I in

$$S_{I_0(F)} \otimes \mathbb{C}[t] / \langle t^2 \rangle$$

with the localization

$$S_{I_0(F)} = \mathbb{C}(y_j \mid j \notin J)[y_j \mid j \in J]_{>}$$

where the prime ideal $I_0(F) = \langle y_j \mid j \in J \rangle \subset S$ denotes the face of $SP(I_0)$ corresponding to F and $>$ is a local ordering on the y_j, $j \in J$. The standard basis can be computed using Mora normal form. Let s be the maximum number of elements of the G_F over all faces $F \in B(I)$. Denote by $\tilde{G}_F$ the standard basis reduced via Gröbner normal form.

Lemma 9.31 *If $F \in B(I)$ is a face, the lattice points of* $\mathrm{dual}(F)$ *are the first order deformations appearing in the initial ideal of $\tilde{G}_F$ with respect to F.*

The complex $\mathrm{dual}(B(I))$ *contains an $s:1$ covering of faces: If G is a face over $F^{\vee} \in B(I)^{\vee}$, then the lattice points of G are the deformations appearing in the initial form of one of the equations of the reduced local standard basis $\tilde{G}_F$ of I considered as an ideal in $S_{I_0(F)} \otimes \mathbb{C}[t] / \langle t^2 \rangle$.*

In general this covering is branched and the number of faces over $F^{\vee} \in B(I)^{\vee}$ is the number of elements of the reduced local standard basis $\tilde{G}_F$ of I.

Note that this covering can have degenerate faces, i.e., faces G over $F^{\vee} \in B(I)^{\vee}$ with $\dim(G) < \dim(F^{\vee})$. It can be branched in the sense that if $\mathfrak{X}$ is not a local complete intersection, the number of faces G over a face of $B(I)^{\vee}$ may be larger than the codimension. Note that this number is bounded from below by the codimension.

If two first order deformations δ_1 and δ_2 lie in the same face of the covering, then there is an element $f_j = m_j + t g_j$ of the global reduced standard basis such that both δ_1 and δ_2 contribute in f_j, i.e., g_j involves the monomials $\delta_1(m_j)$ and $\delta_2(m_j)$. If two deformations contribute in the same element of the reduced global standard basis, they are connected by a chain of faces of the covering.

The set of faces over $B(I)^{\vee}$ can be totally disconnected, e.g., if every element of the global reduced Gröbner basis involves at most one of the first order deformations, then all fibers of the covering consist of points.

Removing all faces of the covering, which correspond to locally irrelevant equations, removing multiple faces, which correspond to locally equivalent equations, and keeping only faces, which involve only vertices of faces of the covering of smaller dimension, we obtain a covering π of $B(I)^{\vee}$ denoted as the **reduced covering**.

Remark 9.32 *If I is a complete intersection the reduced covering π is the $c:1$ covering given in Section 8.10. If I is a local complete intersection then π is also $c:1$.*

Algorithm 9.33 *The following algorithm computes the reduced covering π:*

- *If F is a face of $B(I)$ of* $\dim(F) = d$ *and $p_1, ..., p_c$ are the vertices of* $\mathrm{dual}(F)$ *then set*
$$\pi(p_j) = F^{\vee}$$
for $j = 1, ..., c$.
- *If $l > 0$ and F is a face of $B(I)$ of* $\dim(F) = d - l$ *then the faces of the covering π over $F^{\vee}$ are the convex hulls H of those subsets of the set of vertices of* $\mathrm{dual}(F)$ *with*

– *H involves only vertices of faces $\pi^{-1}(Q^\vee)$ with $Q^\vee \in B(I)^\vee$, $Q^\vee \subsetneqq F^\vee$, i.e., of faces of the covering lying in some lower dimensional* $\mathrm{dual}(F)$ *for $F \in B(I)$.*

– *H intersects at most one of the elements of $\pi^{-1}(Q^\vee)$ for all faces $Q^\vee \subsetneqq F^\vee$ of $B(I)^\vee$, i.e., for all faces Q of $B(I)$ with $F \subsetneqq Q$,*

– *$H \notin \pi^{-1}(Q^\vee)$ for all faces $Q^\vee \subsetneqq F^\vee$.*

9.10 Limit map

Recall that $N \cong \mathbb{Z}^n$, $M = \mathrm{Hom}(N, \mathbb{Z})$ is the dual lattice of N, P is a Fano polytope, $\Delta = P^*$, Σ is the fan over P and $Y = X(\Sigma)$ is the corresponding toric Fano variety with Cox ring $S = \mathbb{C}[y_r \mid r \in \Sigma(1)]$ and presentation

$$0 \to \quad M \quad \overset{A}{\to} \quad \mathrm{WDiv}_T(X(\Sigma)) \cong \mathbb{Z}^{\Sigma(1)} \quad \overset{\deg}{\to} \quad A_{n-1}(X(\Sigma)) \quad \to 0$$

of $A_{n-1}(X(\Sigma))$. By Section 1.3.9

$$G(\Sigma) = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X(\Sigma)), \mathbb{C}^*)$$

acts on $\mathbb{C}^{\Sigma(1)}$ and with the irrelevant ideal

$$B(\Sigma) = \left\langle \prod_{r \not\subset \sigma} y_r \mid \sigma \in \Sigma \right\rangle \subset S$$

in the Cox ring, $\mathbb{C}^{\Sigma(1)} - V(B(\Sigma))$ is invariant under $G(\Sigma)$ and we have

$$Y = \left(\mathbb{C}^{\Sigma(1)} - V(B(\Sigma))\right) // G(\Sigma)$$

Considering the setup from Section 9.5, recall that $I_0 \subset S$ is an equidimensional reduced monomial ideal and $\mathfrak{X} \subset Y \times \mathrm{Spec}\,\mathbb{C}[[t]]$ is a flat family of Calabi-Yau varieties of dimension d given by the ideal $I \subset \mathbb{C}[t] \otimes S$ and special fiber X_0, defined by I_0.

As defined in Section 4.2, we denote by K the metric completion of the field of Puiseux series $\mathbb{C}\{\{t\}\}$ and by

$$\begin{aligned} \mathrm{val} : (K^*)^n &\to \mathbb{R}^n \\ (f_1, ..., f_n) &\mapsto (val(f_1), ..., val(f_n)) \end{aligned}$$

the valuation map.

Applying $\mathrm{Hom}_{\mathbb{Z}}(-, K^*)$ to

$$0 \to \quad M \quad \overset{A}{\to} \quad \mathrm{WDiv}_T(X(\Sigma)) \quad \overset{\deg}{\to} \quad A_{n-1}(X(\Sigma)) \quad \to 0$$

we get an exact sequence

$$\begin{array}{rccl} 1 \to & \mathrm{Hom}_{\mathbb{Z}}\left(A_{n-1}\left(X\left(\Sigma\right)\right), K^{*}\right) & \to & \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{WDiv}_{T}\left(X\left(\Sigma\right)\right), K^{*}\right) & \to \\ & & \overset{\_\circ A}{\to} & \mathrm{Hom}_{\mathbb{Z}}\left(M, K^{*}\right) & \to & 1 \end{array}$$

The isomorphism

$$\begin{array}{ccc} \mathbb{Z}^{\Sigma(1)} & \to & \mathrm{WDiv}_{T}\left(X\left(\Sigma\right)\right) \\ \left(a_{r}\right)_{r\in\Sigma(1)} & \mapsto & \sum_{r\in\Sigma(1)} a_{r} D_{r} \end{array}$$

where D_r, $r \in \Sigma(1)$ denote the prime T-Weil divisors, gives an isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{WDiv}_{T}\left(X\left(\Sigma\right)\right), K^{*}\right) & \to & \left(K^{*}\right)^{\Sigma(1)} \\ g & \mapsto & \left(g\left(D_{r}\right)\right)_{r\in\Sigma(1)} \end{array}$$

and choosing a basis $e_1, ..., e_n$ of N, we have an isomorphism

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}}\left(M, K^{*}\right) & \to & \left(K^{*}\right)^{n} \\ h & \mapsto & \left(h\left(e_{j}^{*}\right)\right)_{j} \end{array}$$

Representing A by $(a_{rj})_{r\in\Sigma(1),\ j=1,\dots,n}$ with respect to these bases, we have

$$\begin{array}{cccccc} & 1 & \to & \mathrm{Hom}_{\mathbb{Z}}\left(A_{n-1}\left(X\left(\Sigma\right)\right), K^{*}\right) & \to & \\ \to & \mathrm{Hom}_{\mathbb{Z}}\left(\mathrm{WDiv}_{T}\left(X\left(\Sigma\right)\right), K^{*}\right) & \overset{\_\circ A}{\to} & \mathrm{Hom}_{\mathbb{Z}}\left(M, K^{*}\right) & \to & 1 \\ & \cong & & \cong & & \\ & \left(K^{*}\right)^{\Sigma(1)} & \overset{\pi}{\to} & \left(K^{*}\right)^{n} & & \\ & \left(c_{r}\right)_{r\in\Sigma(1)} & \mapsto & \left(\prod_{r\in\Sigma(1)} c_{r}^{a_{rj}}\right)_{j} & & \end{array}$$

Then $V_K(I) \subset (K^*)^n$ is the image of the vanishing locus of $I \subset \mathbb{C}[t] \otimes S$ in $(K^*)^{\Sigma(1)} / \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(X(\Sigma)), K^*)$ under the isomorphism induced by π

$$\left(K^{*}\right)^{\Sigma(1)} / \mathrm{Hom}_{\mathbb{Z}}\left(A_{n-1}\left(X\left(\Sigma\right)\right), K^{*}\right) \cong \left(K^{*}\right)^{n}$$

If F is a face of the special fiber Bergman complex

$$F \in B(I) \subset \mathrm{val}\left(V_K(I)\right) = -\,\mathrm{tropvar}(I)$$

then

$$\mathrm{val}^{-1}\left(\mathrm{int}(F)\right) \subset V_K(I) \subset \left(K^{*}\right)^{n}$$

is the set of arc solutions of I over the weight vectors in the relative interior of F. Hence if $w \in \mathrm{int}(F)$ there is an arc

$$a(t) = \left(a_i t^{w_i} + \mathrm{hot}\right)_{i=1,\dots,n} \in V_K(I) \subset \left(K^{*}\right)^{n}$$

with $a_i \in \mathbb{C}^*$. Using multi-index notation write

$$a(t) = (a_i t^{w_i} + \text{hot})_{i=1,\dots,n} = a_w \cdot t^w + \text{hot} \in (K^*)^n$$

with $a_w \in (\mathbb{C}^*)^n$. In the following we show that for all arcs $a(t) \in \text{val}^{-1}(\text{int}(F))$ the limit point $\lim_{t\to 0} a(t)$ lies in the same stratum of the fiber Y of $Y \times \operatorname{Spec}\mathbb{C}[[t]] \to \operatorname{Spec}\mathbb{C}[[t]]$ over $\operatorname{Spec}\mathbb{C}$. We identify the stratum and show that it is a stratum of X_0.

First suppose $a(t) = a_w \cdot t^w + \text{hot} \in (K^*)^n$ is any element of $(K^*)^n$, then approximating a real vector $w \in N_{\mathbb{R}} \cong \mathbb{R}^n$ by a sequence rational vectors (q_j) with $q_j \in \text{int}(F)$ and $\lim_{j\to\infty} q_j = w$, we may assume that $w \in \mathbb{Q}^n$. The limit of a power $\lim_{t\to 0} a(t)^b$ with $b \in \mathbb{Z}_{\geq 1}$ of the arc $a(t)$ exists if and only if $\lim_{t\to 0} a(t)$ exists and lies in the same stratum of Y. Taking the power of the arc multiplies $w \in \mathbb{Q}^n$ with b, hence we may assume that $w' = bw \in N$.

Recall from Section 1.3.2 that there is a one-to-one correspondence between lattice points of N and 1-parameter subgroups of $T = \text{Hom}(M, \mathbb{C}^*)$ given by

$$\begin{array}{ccc} N & \to & \text{Hom}(\mathbb{C}^*, T) \\ & & \begin{array}{cccc} \lambda_w : & \mathbb{C}^* & \to & \text{Hom}(M, \mathbb{C}^*) \\ & t & \mapsto & \begin{array}{cccc} \lambda_w(t) : & M & \to & \mathbb{C}^* \\ & m & \mapsto & t^{\langle m,w\rangle} \end{array} \end{array} \\ w & \mapsto & \end{array}$$

So if τ is a cone of Σ with $bw \in \text{int}(\tau)$ in the relative interior then, by Proposition 1.56,

$$\lim_{t\to 0} \lambda_{bw}(t) = x_\tau$$

where x_τ is the distinguished point

$$\begin{array}{cccc} x_\tau : & \check{\tau} \cap M & \to & \mathbb{C} \\ & m & \mapsto & \left\{ \begin{array}{ll} 1 & \text{if } m \in \tau^\perp \\ 0 & \text{otherwise} \end{array} \right\} \end{array}$$

As $\Sigma = \text{NF}(\Delta)$ is complete, $\lim_{t\to 0} a(t)$ exists in Y and lies in the unique stratum of Y containing $x_\tau = \lim_{t\to 0} \lambda_{bw}(t)$.

Lemma 9.34 *If $a(t) = a_w \cdot t^w + \text{hot} \in (K^*)^n \cong \text{Hom}_{\mathbb{Z}}(M, K^*)$, then $\lim_{t\to 0} a(t)$ exists in Y and lies in the unique stratum of Y containing x_τ where τ is the cone of Σ containing w in its relative interior. This stratum is $V(\tau)$.*

Recall also from Lemma 6.101 that any weight vector on the Cox ring of a complete toric variety has a non-negative representative.

Now suppose F is a face of the special fiber Bergman complex $B(I) \subset \text{val}(V_K(I))$ and $a(t) \in \text{val}^{-1}(\text{int}(F))$, so $a(t) = a_w \cdot t^w + \text{hot} \in (K^*)^n$ with $w \in \text{int}(F)$.

Lemma 9.35 *If F is a face of the special fiber Bergman complex $B(I)$ then there is a unique cone τ of Σ such that* $\operatorname{int}(F) \subset \operatorname{int}(\tau)$.

Definition 9.36 *Hence we can define the map*

$$\begin{array}{cccc} \mu: & B(I) & \to & \operatorname{Poset}(\Delta) \\ & F & \mapsto & G \end{array}$$

where G is the face of Δ with $\tau = \operatorname{hull}(G^*)$, *where τ the unique cone of Σ such that* $\operatorname{int}(F) \subset \operatorname{int}(\tau)$.

As X_0 is the special fiber of $\mathfrak{X}$ and intersecting the special fiber Bergman fan with $\{w_t = 1\}$ identifies the parameters of $\mathfrak{X}$ and of the power series solutions in $\operatorname{val}(V_K(I))$ of the total space of $\mathfrak{X}$, we have:

Lemma 9.37 *If F is a face of the special fiber Bergman complex and* $a(t) \in \operatorname{val}^{-1}(\operatorname{int}(F))$, *then* $\lim_{t\to 0} a(t) \in X_0$.

For any point $x_0 \in X_0$, by taking a hyperplane section of $\mathfrak{X} \subset Y \times \operatorname{Spec}\mathbb{C}[[t]]$ through x_0, there is an arc $a(t) \in V_K(I)$ such that $\lim_{t\to 0} a(t) = x_0$ and $\operatorname{val}(a(t)) \in B(I)$, hence:

Proposition 9.38 *If F is a face of $B(I)$, then*

$$\lim(F) = \left\{\lim_{t\to 0} a(t) \mid a \in \operatorname{val}^{-1}(\operatorname{int}(F))\right\}$$

is a closed stratum of X_0, called the ***limit stratum of*** *F.*

If τ is the unique cone of Σ such that $\operatorname{int}(F) \subset \operatorname{int}(\tau)$, *then*

$$\lim(F) = V(\tau) = V\left(\operatorname{hull}\left((\mu(F))^*\right)\right)$$

Associating to a face F of the special fiber Bergman subcomplex $B(I)$ its limit stratum, we obtain an inclusion reversing map of complexes

$$\begin{array}{cccc} \lim: & B(I) & \to & \operatorname{Strata}(X_0) \subset \operatorname{Strata}(Y) \\ & F & \mapsto & \left\{\lim_{t\to 0} a(t) \mid a \in \operatorname{val}^{-1}(\operatorname{int}(F))\right\} \end{array}$$

where $\operatorname{Strata}(Y)$ *denotes the poset of closures of toric strata of Y, and it holds*

$$\lim(B(I)) \cong SP(I_0) \cong \operatorname{Strata}_{\Delta}(I_0)$$

We have the following correspondence

$$\begin{array}{ccccccc} & & \mathrm{Strata}\,(Y) & \rightleftarrows & \mathrm{Poset}\,(\Delta) & & \\ & & \cup & & \cup & & \\ B\,(I) & \to & \mathrm{Strata}\,(X_0) & \rightleftarrows & \mathrm{Strata}_\Delta\,(I_0) & \rightleftarrows & SP\,(I_0) \\ F & \mapsto & \lim\,(F) = V\,(\tau) & \mapsto & H & \mapsto & \langle y_{G^*} \mid G \subset \Delta \textit{ facet with } H \subset G \rangle \\ & & & & & & \Vert \\ & & & & & & \langle y_r \mid r \in \Sigma\,(1)\,,\, r \subset \tau \rangle \end{array}$$

where $\tau \in \Sigma = \Sigma\,(\Delta^*)$ *is the cone with* $\mathrm{int}\,(F) \subset \mathrm{int}\,(\tau)$ *and* $H \subset \Delta$ *is the face dual to the supporting face of* τ*.*

Elements of $(K^*)^n \cong (K^*)^{\Sigma(1)} / \mathrm{Hom}_{\mathbb{Z}}\,(A_{n-1}\,(X\,(\Sigma))\,, K^*)$ are represented by **Cox arcs** in $(K^*)^{\Sigma(1)}$. Using multi-index notation we write $c\,(t) \in (K^*)^{\Sigma(1)}$ as

$$c\,(t) = \left(c_r t^{j_r} + \mathrm{hot}\right)_{r\in\Sigma(1)} = c_J \cdot t^J + \mathrm{hot}$$

Remark 9.39 *Cox arcs* $c_1\,(t) = c_{J_1} \cdot t^{J_1} + \mathrm{hot} \in (K^*)^{\Sigma(1)}$ *and* $c_2\,(t) = c_{J_2} \cdot t^{J_2} + \mathrm{hot} \in (K^*)^{\Sigma(1)}$ *represent the same arc in* $(K^*)^n$ *if and only if*

$$q\,(t) = c_1\,(t)\,c_2\,(t)^{-1}$$

satisfies

$$\left(\prod\nolimits_{r\in\Sigma(1)} q_r^{a_{rj}}\right)_{j=1,\ldots,n} = (1, ..., 1)$$

in particular for the lowest order exponents we have $A^t\,(J_1^t - J_2^t) = 0$*.*

As 0 is in the interior of

$$\Delta^* = \mathrm{convexhull}\,\{\hat{r} \mid r \in \Sigma\,(1)\}$$

there are $\lambda_r \in \mathbb{R}_{>0}$ such that

$$\sum\nolimits_{r\in\Sigma(1)} \lambda_r \hat{r} = 0$$

So denoting by $\mathbb{R}_{>0}^{\Sigma(1)} \subset \mathbb{R}^{\Sigma(1)}$ the positive orthant

$$(\lambda_r)_{r\in\Sigma(1)} \in \ker(A^t) \cap \mathrm{int}(\mathbb{R}_{>0}^{\Sigma(1)})$$

is in the interior of the positive orthant, hence there is a basis $w_1, ..., w_s \in \mathrm{int}(\mathbb{R}_{>0}^{\Sigma(1)})$ of $\ker\,(A^t)$. Denoting by $\mathrm{Poset}(\mathbb{R}_{>0}^{\Sigma(1)})$ the simplex of faces of $\mathbb{R}_{>0}^{\Sigma(1)}$ we have:

Lemma 9.40 *Suppose* $a(t) \in (K^*)^n$ *and* $c(t) \in (K^*)^{\Sigma(1)}$ *is a Cox arc representing* $a(t) = \pi(c(t))$*, and write* $c(t) = c_J \cdot t^J + \mathrm{hot}$ *with* $c_J \in (\mathbb{C}^*)^{\Sigma(1)}$*. The intersection of the affine space* $J^t + \ker(A^t)$ *with the elements of* $\mathrm{Poset}(\mathbb{R}^{\Sigma(1)}_{>0})$ *is a poset. There is a minimal* 0*-dimensional element* $(J')^t$ *and a Cox arc* $c'(t) = c_{J'} \cdot t^{J'} + \mathrm{hot}$ *with* $a(t) = \pi(c'(t))$ *such that*

$$\lim_{t\to 0} c'(t) \in \mathbb{C}^{\Sigma(1)} - V(B(\Sigma))$$

For any such minimal J' *and any Cox arc* $c'(t) = c_{J'} \cdot t^{J'} + \mathrm{hot}$ *with* $a(t) = \pi(c'(t))$ *the limit point* $\lim_{t\to 0} c'(t)$ *maps to*

$$\lim_{t\to 0} a(t) \in Y = \left(\mathbb{C}^{\Sigma(1)} - V(B(\Sigma))\right) // G(\Sigma)$$

The limit point $\lim_{t\to 0} a(t)$ *lies in the interior of the stratum of* Y *given by the ideal*

$$\langle y_r \mid r \in \Sigma(1) \text{ with } J'_r \neq 0\rangle \subset S$$

This allows us to compute an ideal in the Cox ring defining the limit of a Bergman face in terms of Cox arcs:

Remark 9.41 *Let* F *be a face of* $B(I)$*. Suppose* $a(t) \in \mathrm{val}^{-1}(\mathrm{int}(F))$ *and* $c(t) \in (K^*)^{\Sigma(1)}$ *is a Cox arc representing* $a(t) = \pi(c(t))$*. Write* $c(t) = c_J \cdot t^J + \mathrm{hot}$ *with* $c_J \in (\mathbb{C}^*)^{\Sigma(1)}$ *and let* $J_1^t, ..., J_q^t$ *be those* 0*-dimensional elements of the intersection of the affine space* $J^t + \ker(A^t)$ *with the elements of* $\mathrm{Poset}(\mathbb{R}^{\Sigma(1)}_{>0})$ *such that*

$$I_{F,i} = \langle y_r \mid r \in \Sigma(1) \text{ with } J_{i,r} \neq 0\rangle$$

satisfies

$$(I_{F,i} : B(\Sigma)^\infty) = I_{F,i}$$

Then $\lim(F) \subset Y$ *is given by any of the ideals*

$$\langle y_r \mid r \in \Sigma(1) \text{ with } J_{i,r} \neq 0\rangle \subset S$$

for $i = 1, ..., q$*, hence* $\lim(F)$ *is also the vanishing locus of the ideal*

$$\begin{aligned} I_F &= I_{F,1} + ... + I_{F,q} \\ &= \left\langle y_r \mid r \in \Sigma(1) \text{ such that } \exists i \text{ with } J'_{i,r} \neq 0\right\rangle \end{aligned}$$

Representing $\lim(F)$ by the ideal I_F has the advantage that the intersection $\lim(F_1) \cap \lim(F_2)$ of two Bergman faces is given by the sum $I_{F_1} + I_{F_2}$ of the corresponding ideals.

The saturated ideal defining the stratum $\lim(F)$ is unique if Y is simplicial.

Note that the 0-dimensional elements of the intersection of the affine space $J + \ker(A^t)$ with the elements of $\mathrm{Poset}(\mathbb{R}^{\Sigma(1)}_{>0})$ depend only on the 1-skeleton $\Sigma(1)$ of the fan Σ. The subset of admissible limit strata represented by $J_1, ..., J_q$ are given via the irrelevant ideal $B(\Sigma)$, i.e., by a subdivision of $\Sigma(1)$ to build a fan Σ.

For an example see Section 12.4.

Proposition 9.42 *By assumption the complex*

$$\lim(B(I)) \cong SP(I_0) \cong \mathrm{Strata}_{\Delta}(I_0)$$

is a polyhedral cell complex homeomorphic to a sphere. By the map lim *the complex $B(I)$ is a subdivision of the dual cell complex of $SP(I_0)$, hence $B(I)$ is homeomorphic to a sphere.*

In particular $B(I)$ is equidimensional, connected in codimension one and its dimension is the fiber dimension $d = \dim X_t$ of $\mathfrak{X}$.

Remark 9.43 *The primary decomposition of I_0 is given by*

$$I_0 = \bigcap\nolimits_{P \in SP(I_0)_d} P = \bigcap\nolimits_{H \in \mathrm{Strata}_{\Delta}(I_0)_d} \langle y_r \mid r \in \Sigma(1),\ r \subset \mathrm{hull}(H^*) \rangle$$

Remark 9.44 *We have the obvious representation of I_0 as the intersection of the prime ideals*

$$I_0 = \bigcap\nolimits_{P \in SP(I_0)_d} P = \bigcap\nolimits_{j=1}^{d} \bigcap\nolimits_{P \in SP(I_0)_j} P$$

This intersection corresponds to a Stanley decomposition of S/I_0

$$S/I_0 \cong \bigoplus\nolimits_{j=1}^{d} \bigoplus\nolimits_{P \in SP(I_0)_j} y^{D_P} \cdot \mathbb{C}[y_r \mid y_r \notin P]$$

with $D_P = \sum_{y_r \notin P} D_r$.

The dual complex relates to the locally relevant deformations.

Lemma 9.45 *The locally relevant deformations at the stratum X_i of X_0 are the lattice points*

$$\left\{ m \in M \mid \begin{array}{l} m \in \mathrm{dual}(G) \text{ for some } G \in B(I) \text{ with } X_i \subset \lim(G) \\ \text{and } m \notin \mathrm{dual}(G') \text{ for all } G \in B(I) \text{ with } X_i \not\subset \lim(G) \end{array} \right\}$$

i.e., the open star of the faces $\lim(F) \in \mathrm{dual}(B(I))$ with $\lim(F) = X_i$.

Let $\delta_1, ..., \delta_p \in \operatorname{Hom}(I_0, S/I_0)_0$ be a basis of the tangent space of the component of the Hilbert scheme at X_0, which contains the tangent vector v of $\mathfrak{X}$. Let δ_i be a first order deformation contributing to the tangent vector of the degeneration $\mathfrak{X}$ at X_0, i.e., writing $v = \sum_{i=1}^{p} \lambda_i \delta_i$ we have $\lambda_i \neq 0$. Then δ_i has to be locally relevant in at least one of the strata of X_0, hence:

Proposition 9.46 *All first order deformations contributing to the tangent space of $\mathfrak{X}$ at X_0 are among the lattice points of* $\operatorname{dual}(B(I))$.

Example 9.47 *Consider the degeneration $\mathfrak{X}$ of Pfaffian elliptic curves given by the ideal defined in Example 3.4. The first order deformations of X_0 appearing in $\mathfrak{X}$ fit together in the complex* $\operatorname{dual}(B(I))$ *consisting of* 5 *triangles and* 5 *prisms. The triangles have* 3 *lattice points forming their vertices and the prisms have* 7 *lattice points. Figure 9.7 visualizes the complex* $\operatorname{dual}(B(I))$. *The faces of this complex are in one-to-one correspondence to $SP(I_0)$, as also shown in Figure 9.7. For details see Section 10.3.*

For example, the torus invariant locally relevant deformations of X_0 at the stratum $(0:0:0:0:1)$ *given by* $\langle x_0, ...x_3 \rangle$ *are*

$$\frac{x_2}{x_0} \quad \frac{x_4}{x_1} \quad \frac{x_4}{x_3} \quad \frac{x_4}{x_0} \quad \frac{x_4^2}{x_1 x_2} \quad \frac{x_4}{x_2} \quad \frac{x_1}{x_3}$$

and $\frac{x_4}{x_3}, \frac{x_4}{x_0}, \frac{x_4^2}{x_1 x_2}$ *are the strongly locally relevant deformations.*

9.11 The special fiber X_0° of the mirror degeneration

In the same way the spherical subcomplex $\lim(B(I)) \cong \operatorname{Strata}_\Delta(I_0) \subset \Delta$ corresponds to the special fiber monomial ideal of the degeneration $\mathfrak{X}$, we expect the spherical subcomplex $B(I) \subset \nabla$ to correspond to the monomial special fiber of the mirror degeneration $\mathfrak{X}^\circ$.

By Theorem 9.22 the polytope ∇^* is a Fano polytope, so the fan $\Sigma^\circ = \Sigma(\nabla^*)$ over the faces of ∇^* defines a $\mathbb{Q}$-Gorenstein toric Fano variety $Y^\circ = X(\Sigma^\circ)$.

Denote by $S^\circ = \mathbb{C}[z_r \mid r \in \Sigma^\circ(1)]$ the Cox ring of Y°, so the variables of S° correspond to the vertices of the polytope ∇^* of first order deformations appearing in $\mathfrak{X}$.

Define the monomial ideal

$$I_0^\circ = \left\langle \prod_{v \in J} z_v \mid J \subset \Sigma^\circ(1) \text{ with } \operatorname{supp}(B(I)) \subset \bigcup_{v \in J} F_v \right\rangle \subset S^\circ$$

where F_v denotes the facet of ∇ corresponding to the ray v of the normal fan $\Sigma^\circ = \operatorname{NF}(\nabla)$ so:

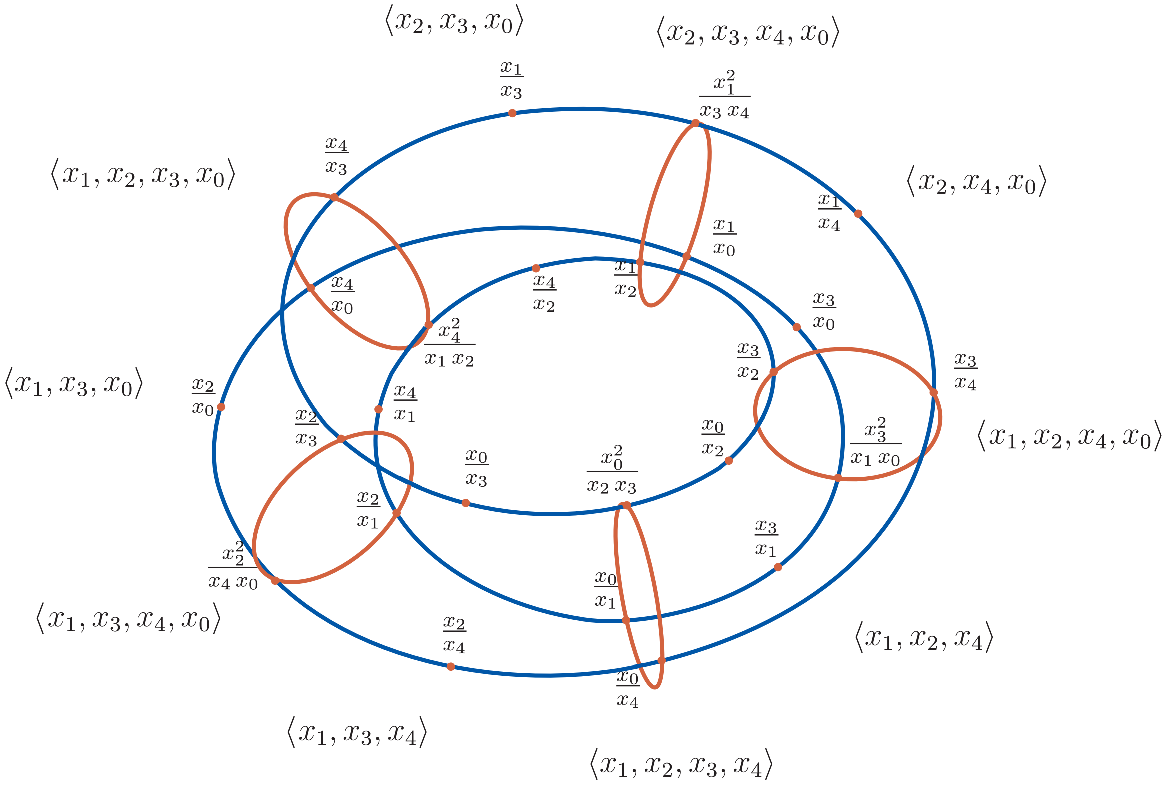

Figure 9.7: Complex of deformations for the Pfaffian elliptic curve

Proposition 9.48 *I° is a reduced monomial ideal. As $B(I)$ is a cell complex homeomorphic to a sphere, I° defines a Calabi-Yau variety $X_0^\circ \subset Y^\circ$, which is the union of toric strata of Y°*

$$\begin{aligned} I_0^\circ &= \bigcap\nolimits_{F \in B(I)_d} \langle z_{G^*} \mid G \text{ a facet of } \nabla \text{ with } F \subset G \rangle \\ &= \bigcap\nolimits_{H \in (\mathrm{dual}(B(I)))_{n-1-d}} \langle z_r \mid r \in \Sigma^\circ(1)\,,\ r \subset \mathrm{hull}\,(H) \in \Sigma^\circ \rangle \end{aligned}$$

9.12 First order mirror degeneration $\mathfrak{X}^\circ$ with special fiber X_0°

In the same way as the lattice points of $(B(I))^* \subset \nabla^*$ are the first order deformations of I_0 appearing in I, we want to consider the lattice points of $(\lim(B(I)))^* \subset \Delta^*$ as elements in $\mathrm{Hom}\,(I_0^\circ, S^\circ/I_0^\circ)_0$, i.e., as first order deformations of X_0°.

Note that the deformations of X_0° are represented by $(\lim(B(I)))^*$ independently of the embedding of X_0° in $Y^\circ = X(\Sigma^\circ)$. Indeed, the deformations of X_0° depend on the rays of the fan defining the embedding toric Fano variety Y of X_0.

The first order mirror degeneration $\mathfrak{X}^{1\circ} \subset Y^\circ \times \operatorname{Spec} \mathbb{C}[t] / \langle t^2 \rangle$ of $\mathfrak{X}$ is defined by the ideal

$$I^{1\circ} = \left\langle m + t \cdot \sum_{\alpha \in \operatorname{supp}((\lim(B(I)))^*) \cap N} c_\alpha \cdot \alpha(m) \mid m \text{ min. gen. of } I_0^\circ \right\rangle \subset \mathbb{C}[t] / \langle t^2 \rangle \otimes S^\circ$$

with general coefficients c_α.

Remark 9.49 *Suppose $\mathfrak{X}$ and $\mathfrak{X}'$ are degenerations as defined above with fibers in $Y = X(\Sigma(P))$ such that the reduced special fiber ideals I_0 and I_0' define the same subcomplex of P^* and $\mathfrak{X}_1$ and $\mathfrak{X}_2$ involve the same first order deformations, then the tropical mirror construction applied to $\mathfrak{X}$ respectively $\mathfrak{X}'$ will lead to the same result. Hence, e.g., passing from I_0 to I_0^Σ, i.e., the non-simplicial analogue of saturation, and from*

$$I = \left\langle m + t \cdot \sum_{\delta \in \operatorname{supp}(\operatorname{dual}(B(I))) \cap M} c_\delta \cdot \delta(m) \mid m \text{ min. gen. of } I_0 \right\rangle$$

to

$$I' = \left\langle m + t \cdot \sum_{\delta \in \operatorname{supp}(\operatorname{dual}(B(I))) \cap M} c_\delta \cdot \delta(m) \mid m \text{ min. gen. of } I_0^\Sigma \right\rangle$$

does not change the geometry of the degeneration and the objects involved in the tropical mirror construction.

In the following we give a representation of the mirror family by a flat affine cone in $\hat{Y}^\circ \times \operatorname{Spec} \mathbb{C}[t] / \langle t^2 \rangle$ for a simplicial toric variety $\hat{Y}^\circ = X(\hat{\Sigma}^\circ)$ given by a projective simplicial subdivision $\hat{\Sigma}^\circ$ of Σ°.

Consider the special fiber ideal $I_0^\circ \subset S^\circ$ and the set of first order deformations $\mathfrak{F} = \operatorname{supp}((\lim(B(I)))^*) \cap N$ as given above. Let $\hat{\Sigma}^\circ \subset M_\mathbb{R}$ be a projective simplicial subdivision of the fan Σ° and $\hat{Y}^\circ = X(\hat{\Sigma}^\circ) \to X(\Sigma^\circ) = Y^\circ$ the corresponding toric variety. Note that Y° and $\hat{Y}^\circ$ have the same Cox ring S°, as the presentation of the Chow group $A_{n-1}(\hat{Y}^\circ) = A_{n-1}(Y^\circ)$ depends only on the one-skeleton $\hat{\Sigma}^\circ(1) = \Sigma^\circ(1)$ of the fan.

The simplicial subdivision $\hat{\Sigma}^\circ$ of Σ° gives a subdivision

$$\operatorname{Poset}(\nabla^*)^\wedge = \left\{ \sigma \cap F \mid \sigma \in \hat{\Sigma}^\circ,\ F \in \operatorname{Poset}(\nabla^*) \right\}$$

of the complex $\operatorname{Poset}(\nabla^*)$ of faces of ∇^*. So it induces a subdivision

$$\operatorname{dual}(B(I))^\wedge = \left\{ G \in \operatorname{Poset}(\nabla^*)^\wedge \mid G \subset F,\ G \notin \partial F \text{ for some } F \in \operatorname{dual}(B(I)) \right\}$$

of the dual Bergman complex $\mathrm{dual}\,(B\,(I))$. Here ∂F denotes the boundary complex of the polytope F.

The faces of $\mathrm{dual}\,(B\,(I))^{\wedge}$ of minimal dimension $c = n-1-d$ correspond to the components of the subdivided special fiber ideal

$$\hat{I}_0 = \bigcap\nolimits_{F \in \mathrm{dual}(B(I))^{\wedge}_{c}} \langle y_r \mid \hat{r} \in F \rangle$$

We can represent the mirror family with fibers in $\hat{Y}^{\circ}$ using one of the three representations of Zariski closed subsets of simplicial toric varieties from Section 1.3.10.

Denote by

$$B(\hat{\Sigma}^{\circ}) = \left\langle \prod\nolimits_{r \in \Sigma^{\circ}(1),\; r \not\subset \sigma} y_r \mid \sigma \in \hat{\Sigma}^{\circ} \right\rangle \subset S^{\circ}$$

the irrelevant ideal of $\hat{Y}^{\circ}$. Recall from Proposition 1.103 that there is a one-to-one correspondence

$$\left\{\text{graded radical ideals } J \subset S^{\circ} \text{ with } J \subset B(\hat{\Sigma}^{\circ})\right\} \rightleftarrows \left\{\text{Zariski closed subsets of } \hat{Y}^{\circ}\right\}$$

by associating to the ideal J the Zariski closed subset of $\hat{Y}^{\circ}$ corresponding to the

$$G(\Sigma^{\circ}) = \mathrm{Hom}_{\mathbb{Z}}\left(A_{n-1}(Y^{\circ}), \mathbb{C}^*\right)$$

invariant subset

$$V\,(J) - V\left(B(\hat{\Sigma}^{\circ})\right) \subset \mathbb{C}^{\Sigma^{\circ}(1)} - V\left(B(\hat{\Sigma}^{\circ})\right)$$

The first order mirror degeneration of $\mathfrak{X}$ with fibers in $\hat{Y}^{\circ}$ is the Zariski closed subset $\hat{\mathfrak{X}}^{1\circ} \subset \hat{Y}^{\circ} \times \mathrm{Spec}\,\mathbb{C}\,[t]\,/\,\langle t^2 \rangle$ given by the ideal

$$\left\langle m + t \cdot \sum_{\alpha \in \mathfrak{F}} c_{\alpha} \cdot \alpha\,(m) \mid m \in \hat{I}_0 \cap B(\hat{\Sigma}^{\circ}) \right\rangle \cap B(\hat{\Sigma}^{\circ})$$

in $\mathbb{C}\,[t]\,/\,\langle t^2 \rangle \otimes S^{\circ}$.

For the second representation, recall that an ideal $J \subset S^{\circ}$ is called $\mathrm{Pic}(\hat{Y}^{\circ})$-generated, if it is generated by homogeneous elements $f \in S^{\circ}$ with $\deg\,(f) \in \mathrm{Pic}(\hat{Y}^{\circ})$ and it is called $\mathrm{Pic}(\hat{Y}^{\circ})$-saturated, if $J_{\alpha} = \left(J : B(\hat{\Sigma}^{\circ})^{\infty}\right)_{\alpha}$ for all $\alpha \in \mathrm{Pic}(\hat{Y}^{\circ})$. By Theorem 1.107 there is a one-to-one correspondence

$$\left\{\mathrm{Pic}(\hat{Y}^{\circ})\text{-gen. and } \mathrm{Pic}(\hat{Y}^{\circ})\text{-saturated ideals } J \subset S^{\circ}\right\} \rightleftarrows \left\{\text{closed subschemes of } \hat{Y}^{\circ}\right\}$$

Given a homogeneous ideal J, denote by

$$J^{\mathrm{Pic}} = \bigoplus\nolimits_{\alpha \in \mathrm{Pic}(\hat{Y}^{\circ})} S^{\circ} \cdot \left(J : B(\hat{\Sigma}^{\circ})\right)_{\alpha}$$

the $\mathrm{Pic}(\hat{Y}^{\circ})$-saturation of J. So we may also describe $\hat{\mathfrak{X}}^{1\circ}$ by the $\mathrm{Pic}(\hat{Y}^{\circ})$-saturation of

$$\left\langle m + t \cdot \sum_{\alpha \in \mathfrak{F}} c_{\alpha} \cdot \alpha\,(m) \mid m \in \left(\hat{I}_0^{\circ}\right)^{\mathrm{Pic}} \right\rangle$$

For the third representation, recall that the Picard-Cox ring of $\hat{Y}^{\circ}$ is

$$R^{\circ} = \bigoplus\nolimits_{\alpha \in \mathrm{Pic}(\hat{Y}^{\circ})} S_{\alpha}^{\circ}$$

and by Theorem 1.109 there is a one-to-one correspondence

$$\left\{\text{graded ideals } J \subset R^{\circ} \text{ saturated in } B(\hat{\Sigma}^{\circ}) \cap R^{\circ}\right\} \rightleftarrows \left\{\text{closed subschemes of } \hat{Y}^{\circ}\right\}$$

So we can describe $\hat{\mathfrak{X}}^{1\circ}$ by the $B(\hat{\Sigma}^{\circ}) \cap R^{\circ}$-saturation of

$$\left\langle m + t \cdot \sum_{\alpha \in \mathfrak{F}} c_{\alpha} \cdot \alpha\,(m) \mid m \in \left(\left(\hat{I}_0^{\circ} \cap R^{\circ}\right) : \left(B(\hat{\Sigma}^{\circ}) \cap R^{\circ}\right)\right) \right\rangle$$

Computationally the best choice is the description of Zariski closed subset of $\hat{Y}^{\circ}$ as graded radical ideals $J \subset S^{\circ}$ with $J \subset B(\hat{\Sigma}^{\circ})$. Standard basis calculations are done over the polynomial ring S° and in the examples of Sections 8.12 and 10 the intersection $\hat{I}_0 \cap B(\hat{\Sigma}^{\circ})$ has far less minimal generators than $\hat{I}_0$.

The representation by graded radical ideals contained in $B(\Sigma^{\circ})$ is particularly suitable, if Y° is given by a Fano simplex as in this case $B(\Sigma^{\circ}) = \langle y_r \mid r \in \Sigma^{\circ}\,(1)\rangle$. This applies for example to projective space and its orbifolds.

The description by $\mathrm{Pic}(\hat{Y}^{\circ})$-generated and $\mathrm{Pic}(\hat{Y}^{\circ})$-saturated ideals requires computations in the invariant ring $R^{\circ} = (S^{\circ})^{W}$ of

$$W = \mathrm{Hom}_{\mathbb{Z}}\left(A_{n-1}\,(Y^{\circ})\,/\,\mathrm{Pic}(\hat{Y}^{\circ}), \mathbb{C}^{*}\right)$$

This representation is particularly suitable for complete intersections defined by nef partitions, as the c monomials given the partition of $\Sigma^{\circ}\,(1)$ correspond to torus invariant Cartier divisors. Note that in standard basis calculations syzygies are computed over R°.

Flatness can be tested in any of these representations. The ideal in $\mathbb{C}[t] / \langle t^2 \rangle \otimes S^\circ$ respectively $\mathbb{C}[t] / \langle t^2 \rangle \otimes R^\circ$ generated by $m_i + t \cdot g_i$, $i = 1, ..., r$ defines a flat family, if for all syzygies $a_i \cdot m_i - a_j \cdot m_j = 0$ with $a_i, a_j \in S^\circ$ respectively R° it holds $a_i \cdot g_i - a_j \cdot g_j \in \langle m_i \mid i \rangle$.

For any choice of a projective crepant subdivision $\hat{\Sigma}^\circ$ of the fan Σ° we obtain a mirror family with fibers in the corresponding toric variety. Indeed, the mirror should be seen as the totality of all models of the tropical mirror family in some simplicial or non-simplicial toric variety $\hat{Y}^\circ = X(\hat{\Sigma}^\circ)$ with $\hat{\Sigma}^\circ(1) \subset \Sigma^\circ(1)$. Note that this is in line with the choice of a maximal projective partial desingularization in the constructions by Batyrev for hypersurfaces and Batyrev and Borisov for complete intersections. The fact that one obtains for every such subdivision a mirror Calabi-Yau is known as the multiple mirror phenomenon, which plays an important role in the global understanding of complex and Kähler moduli spaces.

Remark 9.50 *The function* ProjectiveSimplicialSubdivision *in the Maple package* tropicalmirror *(see also Section 12.4) returns a projective simplicial subdivision $\hat{\Sigma}^\circ$ of the fan Σ° using the ideas of Algorithm 1.148 computing the secondary fan.*

Given $\hat{\Sigma}^\circ$ and $\operatorname{dual}(B(I))$ *the function* SimplicialDualBergmanComplex *returns the induced subdivision* $\operatorname{dual}(B(I))^\wedge$ *of the dual Bergman complex and* SimplicialSpecialFiber *gives the corresponding special fiber ideal $\hat{I}_0^\circ \subset S^\circ$.*

The function SimplicialMirrorDegeneration *computes from $\hat{\Sigma}^\circ$,* $\operatorname{dual}(B(I))^\wedge$ *and* $(\lim(B(I)))^*$ *the ideal $\hat{I}^{1\circ} \subset \mathbb{C}[t] / \langle t^2 \rangle \otimes S^\circ$. It is represented by a standard basis with respect to a monomial ordering in the interior of the mirror special fiber cone* $\operatorname{hull}(\{1\} \times \Delta) \subset \mathbb{R} \oplus M_{\mathbb{R}}$*, i.e., selecting the special fiber monomials as lead term. Using this monomial ordering, the function* BuchbergerTest *checks flatness by computing all S-polynomials modulo the lead ideal.*

To avoid simplicial subdivision one can represent the mirror family with fibers in Y° by the radical of

$$\left\langle m + t \cdot \sum_{\alpha \in \mathfrak{F}} c_\alpha \cdot \alpha(m) \mid m \text{ min. gen. of } I_0^\circ \cap B(\Sigma^\circ) \right\rangle \cap I_0^\circ \cap B(\Sigma^\circ)$$

in $\mathbb{C}[t] / \langle t^2 \rangle \otimes S^\circ$.

Remark 9.51 *The special fiber Gröbner cone of $\mathfrak{X}^{1\circ}$ gives back $\Delta \subset M_{\mathbb{R}}$, i.e.,*

$$\Delta = C_{I_0^\circ}\left(I^{1\circ}\right) \cap \{w_t = 1\}$$

Let $I^\circ_{gen} \subset S^\circ$ be the ideal of the general fiber of $\mathfrak{X}^{1\circ}$. As $0 \in N$ is the unique interior point of the Fano polytope Δ^* the Hilbert point of I°_0 lies in the interior of the state polytope of the generic fiber I°_{gen}, i.e.,

$$H\left(I^\circ_0\right) \in \operatorname{int}\left(\operatorname{State}\left(I^\circ_{gen}\right)\right) \subset N_{\mathbb{R}}$$

Conjecture 9.52 *If $\mathfrak{X}^\circ \subset Y^\circ \times \operatorname{Spec}\mathbb{C}[t]$ is a flat family with tangent direction $\mathfrak{X}^{1\circ}$, then the general fiber of $\mathfrak{X}^\circ$ and the general fiber of $\mathfrak{X}$ form a mathematical mirror pair.*

9.13 Remarks on orbifolding mirror families

Suppose $Y = X(\Sigma)$ is projective space $\mathbb{P}^n$ given by the fan $\Sigma \subset N_{\mathbb{R}}$ with the rays generated by $(1,0,...,0), ..., (0,...,0,1), (-1,....,-1) \in N$. Let $(\delta_i)_i$ be the torus invariant basis of the space of first order deformations contributing to the tangent space of $\mathfrak{X}$ at X_0, so $\{\delta_i \mid i\} = \operatorname{dual}(B(I)) \cap M$. Denote by c the codimension of the fibers in Y.

Representing each deformation δ_i as a Cox Laurent monomial $\delta_i = \frac{a_i}{b_i}$ with relative prime Cox monomials a_i and b_i, there is a preordering on $\{\delta_i \mid i\}$ by divisibility of the denominators b_i, i.e., $\frac{a_i}{b_i} = \delta_i \leq \delta_j = \frac{a_j}{b_j} \Leftrightarrow b_i \mid b_j$.

If $\delta = \frac{a}{b}$ is deformation with relative prime Cox monomials a and b, then δ is called pure, if a has the form $a = y_s^d$ for some homogeneous variable y_s of Y and for some $d > 0$.

For each 0-dimensional stratum p of Y denote by

$$\mathfrak{D}_p = \{m \in M \mid m \in F^* \text{ with } \lim F = p\}$$

the set of all torus invariant first order deformations of X_0 in $\mathfrak{X}$ corresponding to p.

Definition 9.53 *Let $\mathfrak{F}$ be a set of non-trivial deformations of X_0 in $\mathfrak{X}$ corresponding to vertices of faces of* $\operatorname{dual}(B(I))$ *and denote by $\mathcal{R}$ the corresponding set of rays of Σ°. We call $\mathfrak{F}$ a set of* ***Fermat deformations*** *of $\mathfrak{X}$, if the following conditions are satisfied:*

- $|\mathcal{R} \cap \mathfrak{D}_p| = 1$ *for all 0-dimensional strata p of Y (in particular $|\mathcal{R}| = n+1$).*
- *The convexhull of $\{\hat{r} \mid r \in \mathcal{R}\}$ is a polytope of dimension $n = \dim(N_{\mathbb{R}})$ containing 0 in its interior, i.e.,$\mathcal{R}$ spans a projective fan $\hat{\Sigma}^\circ$ (note that this fan is uniquely determined by $\mathcal{R}$).*
- *The elements of $\mathfrak{F}$ are incomparable with respect to the preordering $\leq$.*

Let $\hat{Y}^\circ = X(\hat{\Sigma}^\circ)$ be the toric Fano variety defined by $\hat{\Sigma}^\circ$ with Cox ring $\hat{S}^\circ = \mathbb{C}[z_r \mid r \in \hat{\Sigma}^\circ(1)]$ and

$$0 \to M_{\mathbb{R}} \to \mathbb{Z}^{\mathcal{R}} \overset{\deg}{\to} A_{n-1}(\mathcal{R}) \to 0$$

the corresponding presentation of $A_{n-1}(\hat{Y}^\circ) = A_{n-1}(\mathcal{R}) \cong H \oplus \mathbb{Z}$ with finite H and let $(h_r, d_r) = \deg(z_r)$. As 0 is in the interior of the convex hull of $\left\{\hat{r} \mid r \in \hat{\Sigma}^\circ(1)\right\}$, we can assume that the d_r are positive integers. For all $w \in (\lim F)^* \cap N$, $F \in B(I)$ the Laurent monomials

$$\prod\nolimits_{r \in \hat{\Sigma}^\circ(1)} z_r^{\langle \hat{r}, w\rangle}$$

are of degree 0 with respect to the grading $\deg(z_r) = d_r$.

Note that for complete intersections the deformations corresponding to vertices of faces of $\operatorname{dual}(B(I))$ are pure. The set $\mathfrak{F}$ is not unique in general, see for example the complete intersection Calabi-Yau of degree 12 in $\mathbb{P}^6$. If $\mathfrak{X}$ is a degeneration of complete intersections of codimension $c = 2$, then there is a unique set of Fermat deformations $\mathfrak{F}$, which is the set of maximal elements of $\operatorname{dual}(B(I)) \cap M$ with respect to the preordering $\leq$ defined above.

Remark 9.54 *With the notations of the preceding sections, let $\mathfrak{F}$ be a set of Fermat deformations of $\mathfrak{X}$. Then*

$$\hat{P}^\circ = \operatorname{convexhull}\left\{A^{-1}(\delta) \mid \delta \in \mathfrak{F}\right\} \subset P^\circ = \nabla^*$$

is a Fano polytope. $\hat{Y}^\circ$ is an orbifold $\hat{Y}^\circ = \mathbb{P}(d_1, ..., d_{n+1})/G$ with the d_i defined as above. Let $Y^\circ = X(\Sigma^\circ) \to X(\hat{\Sigma}^\circ) = \hat{Y}^\circ$ be a birational map contracting all divisors of Y° corresponding to deformations not in $\mathfrak{F}$. Then the first order flat family $\hat{\mathfrak{X}}^{1\circ} \subset \hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]/\langle t^2\rangle$ induced by $\mathfrak{X}^{1\circ} \subset Y^\circ \times \operatorname{Spec}\mathbb{C}[t]/\langle t^2\rangle$ has special fiber given by

$$\hat{I}_0^\circ = \left\langle \hat{m} = \prod\nolimits_{r \in \hat{\Sigma}^\circ(1)} z_r \mid \exists \text{ minimal generator } m \text{ of } I_0^\circ \text{ divisible by } \hat{m} \right\rangle \subset \hat{S}^\circ$$

and involves the deformations

$$\left\{\prod\nolimits_{r \in \hat{\Sigma}^\circ(1)} z_r^{\langle \hat{r}, w\rangle} \mid w \in (\lim(B(I)))^* \cap N\right\}$$

10 Tropical mirror construction for the example of Pfaffian Calabi-Yau varieties

10.1 Pfaffian Calabi-Yau varieties

Definition 10.1 *Let K be a field. A subscheme X of $\mathbb{P}^n_K$ of codimension 3 is called **Pfaffian subscheme** if there is*

1. *a vector bundle E on $\mathbb{P}^n_K$ of rank $2k+1$ for some $k \in \mathbb{Z}_{\geq 0}$*

2. *and a skew symmetric map $\varphi : \mathcal{E}(-t) \to \mathcal{E}^*$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n_K}(E)$ such that*

   (a) *φ is generically of rank $2k$*

   (b) *φ degenerates to rank $2k-2$ in the expected codimension 3*

3. *X is scheme theoretically the degeneracy locus of φ.*

Theorem 10.2 (Buchsbaum-Eisenbud) *[Buchsbaum, Eisenbud, 1977], [Okonek, 1994], [Walter, 1996] A Pfaffian subscheme X of $\mathbb{P}^n_K$ has a locally free resolution*

$$0 \to \mathcal{O}_{\mathbb{P}^n_K}(-t-2s) \overset{\psi^*(-t-2s)}{\to} \mathcal{E}(-t-s) \overset{\varphi}{\to} \mathcal{E}^*(-s) \overset{\psi}{\to} \mathcal{O}_{\mathbb{P}^n_K} \to \mathcal{O}_X \to 0$$

where $s = c_1(\mathcal{E}) + kt$, and ψ is locally given by the Pfaffians of order $2k$ of φ.

Remark 10.3 *X is locally Gorenstein with*

$$\omega_X^\circ \cong \mathcal{O}_X(t+2s-n-1)$$

Thus $\omega_X^\circ \cong \mathcal{O}_X$ if and only if $t+2s = n+1$.

Theorem 10.4 *[Walter, 1996] Let K be a field with $\operatorname{char}(K) \neq 2$. If $X \subset \mathbb{P}^n_K$ is an equidimensional, locally Gorenstein subscheme of dimension $n-3$, which is subcanonical, i.e., $\omega_X^\circ \cong \mathcal{O}_X(l)$ for some integer l, then X is Pfaffian if and only if the following condition is satisfied*

$$n \equiv 0 \bmod 4 \text{ and } l = 2s \text{ even } \Rightarrow \chi(\mathcal{O}_X(s)) \text{ is even}$$

Corollary 10.5 *A codimension 3 subscheme of $\mathbb{P}^6$ is Pfaffian if and only if it is locally Gorenstein and subcanonical.*

Example 10.6 *Using this construction, we get the following projectively Gorenstein Pfaffian Calabi-Yau threefolds*

$\mathcal{E}^*$	$\operatorname{rank}(\mathcal{E})$	$\deg(X)$	$h^{1,2}(X)$	$h^{1,1}(X)$	$\chi(X)$
$2\mathcal{O}(1) \oplus \mathcal{O}$	3	12	73	1	-144
$\mathcal{O}(1) \oplus 4\mathcal{O}$	5	13	61	1	-120
$7\mathcal{O}$	7	14	50	1	-98

Example 10.7 *In [Tonoli, 2000] families of non-projectively Cohen-Macaulay Pfaffian Calabi-Yau threefolds with the following data were constructed and it is shown that generic elements of each family are smooth:*

$\mathcal{E}^*$	$\operatorname{rank}(\mathcal{E})$	$\deg(X)$	$h^{1,2}(X)$	$h^{1,1}(X)$	$\chi(X)$
$\Omega^1(1)\oplus 3\mathcal{O}$	9	15	40	1	-78
$Syz^1(M)$	11	16	31	1	-60
$Syz^1(M')$	13	17	23	1	-44

where M is a generic module of length 2 generated in degree -1 with Hilbert function $(2,1,0,...)$, and M' is a special module of length 2 generated in degree -1 with Hilbert function $(3,5,0,...)$ (the generic choice of M' gives a bundle $\mathcal{E} = Syz^1(M')$, which does not admit any alternating map $\mathcal{E}^*(-1) \to \mathcal{E}$). There are 3 unirational families of smooth Pfaffian Calabi-Yau threefolds of degree 17 and all 3 families have $h^{1,2}(X) = 23$. The Hodge numbers were obtained via computer algebra.

The Pfaffian given by $\mathcal{E} = 2\mathcal{O}(1)\oplus\mathcal{O}$ is a complete intersection and the mirror construction is given in Section 8. In the following, we will be concerned with the remaining two projectively Gorenstein examples.

10.2 Deformations of Pfaffian varieties

Let $Y = X(\Sigma)$ be a toric Fano variety given by the fan $\Sigma \subset N_{\mathbb{R}}$ over the Fano polytope $P \subset N_{\mathbb{R}}$ and denote its Cox ring by S.

Suppose that $I_0 = \langle m_1, ..., m_r\rangle \subset S$ is an ideal generated by monomials $m_i \in H^0(Y, \mathcal{O}_Y(E_i))$, $i = 1, ..., r$, which has a Pfaffian resolution

$$0 \to \mathcal{O}_Y(K_Y) \to \mathcal{F}(K_Y) \xrightarrow{\varphi^0} \mathcal{F}^* \xrightarrow{m} \mathcal{O}_Y$$

with

$$\begin{aligned}\mathcal{F} &= \mathcal{O}_Y(E_1)\oplus ...\oplus \mathcal{O}_Y(E_r)\\ m &= (m_1, ..., m_r)\end{aligned}$$

and

$$\varphi^0 \in \bigwedge^2 \mathcal{F}^*(-K_Y)$$

Suppose $\mathfrak{X}^1 \subset Y \times \operatorname{Spec}\mathbb{C}[t]/\langle t^2\rangle$ is a first order deformation of I_0 defined by

$$I^1 = \left\langle f_j^1 = t\cdot g_j + m_j \mid j = 1, ..., r\right\rangle \subset \mathbb{C}[t]/\left\langle t^2\right\rangle \otimes S$$

with $g_j \in S_{[E_j]}$.

Denote by $\overline{\pi}_1 : Y \times \operatorname{Spec} \mathbb{C}[t] / \langle t^2 \rangle \to Y$ the projection on the first component and by

$$K^1 = K_{Y \times \operatorname{Spec}(\mathbb{C}[t]/\langle t^2 \rangle)/\operatorname{Spec}(\mathbb{C}[t]/\langle t^2 \rangle)}$$

the relative canonical sheaf. Then flatness of $\mathfrak{X}^1$ gives a lift of the syzygies of m, hence a Pfaffian resolution of I^1 of the form

$$0 \to \mathcal{O}_{Y \times \operatorname{Spec} \mathbb{C}[t]/\langle t^2 \rangle} \left(K^1\right) \to \mathcal{E}^1 \left(K^1\right) \xrightarrow{\varphi^1} \left(\mathcal{E}^1\right)^* \xrightarrow{f^1} \mathcal{O}_{Y \times \operatorname{Spec} \mathbb{C}[t]/\langle t^2 \rangle}$$

with $f^1 = (f_1^1, ..., f_r^1)$ and

$$\mathcal{E}^1 = \overline{\pi}_1^* \mathcal{F}$$

and φ^1 is skew symmetric by the Theorem of Buchsbaum-Eisenbud, i.e.,

$$\varphi^1 \in \bigwedge^2 \mathcal{E}^{1*} \left(-K^1\right)$$

Denote by $\pi_1 : Y \times \operatorname{Spec} \mathbb{C}[[t]] \to Y$ the projection on the first component and by

$$K = K_{(Y \times \operatorname{Spec} \mathbb{C}[[t]])/\operatorname{Spec} \mathbb{C}[[t]]}$$

the relative canonical sheaf. Let

$$\mathcal{E} = \pi_1^* \mathcal{F}$$

and let $\varphi \in \bigwedge^2 \mathcal{E}^* (-K)$ be a representative of φ^1 of t-degree 1. Defining $f = (f_1, ..., f_r)$ as the Pfaffians of φ, one obtains a Pfaffian resolution of the ideal generated by $f_1, ..., f_r$

$$0 \to \mathcal{O}_{Y \times \operatorname{Spec} \mathbb{C}[[t]]} (K) \to \mathcal{E}(K) \xrightarrow{\varphi} \mathcal{E}^* \xrightarrow{f} \mathcal{O}_{Y \times \operatorname{Spec} \mathbb{C}[[t]]}$$

hence a lift of $\mathfrak{X}^1$ to a flat family $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[[t]]$, so:

Proposition 10.8 *The deformations of I_0 are unobstructed and the base space is smooth.*

By the same argument one obtains:

Proposition 10.9 *Let $Y = X(\Sigma)$ be a toric Fano variety given by the fan $\Sigma \subset N_{\mathbb{R}}$ over the Fano polytope $P \subset N_{\mathbb{R}}$. Denote the Cox ring of Y by S.*

Suppose that $X_0 \subset Y$ is defined by an ideal $I_0 = \langle m_1, ..., m_r \rangle \subset S$, which is generated by monomials $m_i \in H^0(Y, \mathcal{O}_Y(E_i))$, $i = 1, ..., r$ and has a Pfaffian resolution

$$0 \to \mathcal{O}_Y(K_Y) \to \mathcal{F}(K_Y) \xrightarrow{\varphi^0} \mathcal{F}^* \xrightarrow{m} \mathcal{O}_Y$$

with $m = (m_1, ..., m_r)$, $\mathcal{F} = \mathcal{O}_Y(E_1) \oplus ... \oplus \mathcal{O}_Y(E_r)$ *and* $\varphi^0 \in \bigwedge^2 \mathcal{F}^*(-K_Y)$.

Denote by $\pi_1 : Y \times \operatorname{Spec} \mathbb{C}[[t]] \to Y$ *the projection on the first component. Suppose that* $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[[t]]$ *is given by an ideal* $I \subset \mathbb{C}[[t]] \otimes S$*, which has a Pfaffian resolution*

$$0 \to \mathcal{O}_{Y \times \operatorname{Spec} \mathbb{C}[[t]]}(K) \to \mathcal{E}(K) \xrightarrow{\varphi} \mathcal{E}^* \to \mathcal{O}_{Y \times \operatorname{Spec} \mathbb{C}[[t]]}$$

with $\mathcal{E} = \pi_1^* \mathcal{F}$ *and* $K = K_{(Y \times \operatorname{Spec} \mathbb{C}[[t]])/\operatorname{Spec} \mathbb{C}[[t]]}$*, i.e.,* I *is generated by the Pfaffians of* $\varphi \in \bigwedge^2 \mathcal{E}^*(-K)$*. Suppose that* $X_0 \cong \mathfrak{X} \times_{k[[t]]} \operatorname{Spec} k$.

Then $\mathfrak{X}$ *is a flat degeneration of Pfaffian Calabi-Yau varieties with fibers polarized in* Y *and special fiber* X_0.

Corollary 10.10 *The families given in Example 3.4 (monomial degeneration of a general Pfaffian elliptic curve in* $\mathbb{P}^3$*), in Example 3.5 (*1*-parameter degeneration of a Pfaffian Calabi-Yau threefold in an orbifold of* $\mathbb{P}^6$*), in Example 3.6 (monomial degeneration of a general Pfaffian Calabi-Yau threefold of degree* 14 *in* $\mathbb{P}^6$*) and in Example 3.7 (monomial degeneration of a general Pfaffian Calabi-Yau threefold of degree* 13 *in* $\mathbb{P}^6$*) are flat and satisfy the genericity condition on the tangent direction, given in Section 9.5.*

10.3 Tropical mirror construction for the Pfaffian elliptic curve

10.3.1 Setup

Let $Y = \mathbb{P}^4 = X(\Sigma)$, $\Sigma = \Sigma(P) = NF(\Delta) \subset N_{\mathbb{R}}$ with the Fano polytope $P = \Delta^*$ given by

$$\Delta = \operatorname{convexhull}\begin{pmatrix} (4,-1,-1,-1) & (-1,4,-1,-1) & (-1,-1,4,-1) \\ (-1,-1,-1,4) & (-1,-1,-1,-1) & \end{pmatrix} \subset M_{\mathbb{R}}$$

and let

$$S = \mathbb{C}[x_0, x_1, x_2, x_3, x_4]$$

be the Cox ring of Y with the variables

$$\begin{array}{ll} x_1 = x_{(1,0,0,0)} & x_2 = x_{(0,1,0,0)} \\ x_3 = x_{(0,0,1,0)} & x_4 = x_{(0,0,0,1)} \\ x_0 = x_{(-1,-1,-1,-1)} & \end{array}$$

associated to the rays of Σ. Consider the degeneration $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[t]$ of Pfaffian elliptic curves with Buchsbaum-Eisenbud resolution

$$0 \to \mathcal{O}_Y(-5) \to \mathcal{E}(-3) \xrightarrow{A_t} \mathcal{E}^*(-2) \to \mathcal{O}_Y \to \mathcal{O}_{X_t} \to 0$$

where

$$\mathcal{E} = \ 5\mathcal{O}$$

$$A_t = A_0 + t \cdot A$$

$$A_0 = \begin{bmatrix} 0 & 0 & x_1 & -x_4 & 0 \\ 0 & 0 & 0 & x_2 & -x_0 \\ -x_1 & 0 & 0 & 0 & x_3 \\ x_4 & -x_2 & 0 & 0 & 0 \\ 0 & x_0 & -x_3 & 0 & 0 \end{bmatrix}$$

the monomial special fiber of $\mathfrak{X}$ is given by

$$I_0 = \left\langle \ -x_2\, x_3 \quad -x_3\, x_4 \quad -x_4\, x_0 \quad -x_1\, x_0 \quad -x_1\, x_2 \ \right\rangle$$

and generic $A \in \bigwedge^2 \mathcal{E}^* (1)$

$$A = \begin{bmatrix} 0 & w_1 & w_2 & w_3 & w_4 \\ -w_1 & 0 & w_5 & w_6 & w_7 \\ -w_2 & -w_5 & 0 & w_8 & w_9 \\ -w_3 & -w_6 & -w_8 & 0 & w_{10} \\ -w_4 & -w_7 & -w_9 & -w_{10} & 0 \end{bmatrix}$$

$$\begin{aligned}
w_1 &= s_1\, x_1 + s_2\, x_2 + s_3\, x_3 + s_4\, x_4 + s_5\, x_0 \\
w_2 &= s_6\, x_1 + s_7\, x_2 + s_8\, x_3 + s_9\, x_4 + s_{10}\, x_0 \\
w_3 &= s_{11}\, x_1 + s_{12}\, x_2 + s_{13}\, x_3 + s_{14}\, x_4 + s_{15}\, x_0 \\
w_4 &= s_{16}\, x_1 + s_{17}\, x_2 + s_{18}\, x_3 + s_{19}\, x_4 + s_{20}\, x_0 \\
w_5 &= s_{21}\, x_1 + s_{22}\, x_2 + s_{23}\, x_3 + s_{24}\, x_4 + s_{25}\, x_0 \\
w_6 &= s_{26}\, x_1 + s_{27}\, x_2 + s_{28}\, x_3 + s_{29}\, x_4 + s_{30}\, x_0 \\
w_7 &= s_{31}\, x_1 + s_{32}\, x_2 + s_{33}\, x_3 + s_{34}\, x_4 + s_{35}\, x_0 \\
w_8 &= s_{36}\, x_1 + s_{37}\, x_2 + s_{38}\, x_3 + s_{39}\, x_4 + s_{40}\, x_0 \\
w_9 &= s_{41}\, x_1 + s_{42}\, x_2 + s_{43}\, x_3 + s_{44}\, x_4 + s_{45}\, x_0 \\
w_{10} &= s_{46}\, x_1 + s_{47}\, x_2 + s_{48}\, x_3 + s_{49}\, x_4 + s_{50}\, x_0
\end{aligned}$$

The total space of the degeneration $\mathfrak{X}$ is a local complete intersection. The induced first order degeneration

$$\mathfrak{X}^1 \subset Y \times \operatorname{Spec} \mathbb{C}\,[t]\,/\,\langle t^2 \rangle$$

is given by the ideal $I \subset S \otimes \mathbb{C}[t] / \langle t^2 \rangle$ with I_0-reduced generators of degrees $2, 2, 2, 2, 2$

$$\left\{ \begin{array}{l} -x_2\,x_3 + t(c_7\,x_3\,x_0 + c_8\,x_2\,x_0 + c_{15}\,x_2\,x_4 + c_{16}\,x_1\,x_3 + c_{17}\,x_0^2 + c_{18}\,x_2^2 + c_{19}\,x_3^2), \\ -x_3\,x_4 + t(c_1\,x_3^2 + c_2\,x_3\,x_0 + c_6\,x_4\,x_1 + c_{15}\,x_4^2 + c_{18}\,x_2\,x_4 + c_{20}\,x_1^2 + c_{25}\,x_3\,x_1), \\ -x_4\,x_0 + t(c_1\,x_3\,x_0 + c_2\,x_0^2 + c_3\,x_4\,x_1 + c_4\,x_4^2 + c_5\,x_2^2 + c_9\,x_0\,x_2 + c_{10}\,x_4\,x_2), \\ -x_1\,x_0 + t(c_3\,x_1^2 + c_4\,x_4\,x_1 + c_{11}\,x_0\,x_3 + c_{12}\,x_1\,x_3 + c_{21}\,x_0^2 + c_{22}\,x_3^2 + c_{23}\,x_0\,x_2), \\ -x_1\,x_2 + t(c_{13}\,x_2\,x_4 + c_{14}\,x_1\,x_4 + c_{16}\,x_1^2 + c_{19}\,x_1\,x_3 + c_{21}\,x_0\,x_2 + c_{23}\,x_2^2 + c_{24}\,x_4^2) \end{array} \right\}$$

where

$c_1 = s_{13}$	$c_2 = s_{15}$	$c_3 = s_{31}$	$c_4 = s_{34}$
$c_5 = s_{17}$	$c_6 = -s_{41} - s_{49}$	$c_7 = -s_{38} - s_{30}$	$c_8 = -s_{37} - s_{45}$
$c_9 = s_{12} + s_{20}$	$c_{10} = s_{19} + s_{32}$	$c_{11} = -s_8 - s_5$	$c_{12} = -s_1 + s_{33}$
$c_{13} = -s_9 - s_{22}$	$c_{14} = -s_{29} - s_{21}$	$c_{15} = -s_{44}$	$c_{16} = -s_{26}$
$c_{17} = -s_{40}$	$c_{18} = -s_{42}$	$c_{19} = -s_{28}$	$c_{20} = -s_{46}$
$c_{21} = -s_{10}$	$c_{22} = -s_3$	$c_{23} = -s_7$	$c_{24} = -s_{24}$
$c_{25} = s_{11} - s_{48}$			

The corresponding syzygy matrix is given by

$w_1 = s_1\,x_1 + s_3\,x_3 + s_5\,x_0$	$w_2 = s_7\,x_2 + s_8\,x_3 + s_9\,x_4 + s_{10}\,x_0$
$w_3 = s_{11}\,x_1 + s_{12}\,x_2 + s_{13}\,x_3 + s_{15}\,x_0$	$w_4 = s_{17}\,x_2 + s_{19}\,x_4 + s_{20}\,x_0$
$w_5 = s_{21}\,x_1 + s_{22}\,x_2 + s_{24}\,x_4$	$w_6 = s_{26}\,x_1 + s_{28}\,x_3 + s_{29}\,x_4 + s_{30}\,x_0$
$w_7 = s_{31}\,x_1 + s_{32}\,x_2 + s_{33}\,x_3 + s_{34}\,x_4$	$w_8 = s_{37}\,x_2 + s_{38}\,x_3 + s_{40}\,x_0$
$w_9 = s_{41}\,x_1 + s_{42}\,x_2 + s_{44}\,x_4 + s_{45}\,x_0$	$w_{10} = s_{46}\,x_1 + s_{48}\,x_3 + s_{49}\,x_4$

10.3.2 Special fiber Gröbner cone

The space of first order deformations of $\mathfrak{X}$ has dimension 25 and the deformations represented by the Cox Laurent monomials

$\frac{x_0^2}{x_2\,x_3}$	$\frac{x_2^2}{x_4\,x_0}$	$\frac{x_3^2}{x_1\,x_0}$	$\frac{x_4^2}{x_1\,x_2}$	$\frac{x_1^2}{x_3\,x_4}$	$\frac{x_0}{x_3}$	$\frac{x_3}{x_0}$	$\frac{x_2}{x_1}$	$\frac{x_1}{x_2}$	$\frac{x_0}{x_1}$	$\frac{x_1}{x_3}$	$\frac{x_0}{x_2}$	$\frac{x_2}{x_0}$
$\frac{x_3}{x_1}$	$\frac{x_1}{x_0}$	$\frac{x_3}{x_2}$	$\frac{x_2}{x_3}$	$\frac{x_0}{x_4}$	$\frac{x_4}{x_2}$	$\frac{x_4}{x_0}$	$\frac{x_4}{x_3}$	$\frac{x_4}{x_1}$	$\frac{x_3}{x_4}$	$\frac{x_1}{x_4}$	$\frac{x_2}{x_4}$	

form a torus invariant basis. These deformations give linear inequalities defining the special fiber Gröbner cone, i.e.,

$$C_{I_0}(I) = \left\{ (w_t, w_y) \in \mathbb{R} \oplus N_{\mathbb{R}} \mid \left\langle w, A^{-1}(m) \right\rangle \geq -w_t \forall m \right\}$$

for above monomials m and the presentation matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{bmatrix}$$

of $A_3(Y)$. The vertices of the special fiber polytope

$$\nabla = C_{I_0}(I) \cap \{w_t = 1\}$$

and the number of faces, each vertex is contained in, are given by the table

6	$(1,0,0,0)$ $(0,1,0,0)$ $(-1,-1,-1,-1)$ $(0,0,0,1)$ $(0,0,1,0)$
8	$\left(0,-1,-\frac{1}{2},-1\right)$ $\left(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right)$ $\left(-1,-\frac{1}{2},-1,0\right)$ $\left(\frac{1}{2},0,1,1\right)$ $\left(1,1,0,\frac{1}{2}\right)$
9	$(0,-1,0,-1)$ $(-1,0,0,-1)$ $(1,0,1,1)$ $(1,1,0,1)$ $(-1,0,-1,0)$
10	$(1,0,1,0)$ $(-1,0,-1,-1)$ $(1,0,0,1)$ $(0,1,0,1)$ $(-1,-1,0,-1)$

The number of faces of ∇ and their F-vectors are

Dimension	Number of faces	F-vector	
-1	1	$(0,0,0,0,0,0)$	
0	20	$(1,1,0,0,0,0)$	point
1	50	$(1,2,1,0,0,0)$	edge
2	30	$(1,4,4,1,0,0)$	quadrangle
2	15	$(1,3,3,1,0,0)$	triangle
3	10	$(1,6,9,5,1,0)$	prism
3	5	$(1,9,15,8,1,0)$	
4	1	$(1,20,50,45,15,1)$	

The dual $P^\circ = \nabla^*$ of ∇ is a Fano polytope with vertices

$$\begin{array}{llll}
(-1,-1,0,2) & (2,0,-1,-1) & (0,-1,-1,0) & (0,2,0,-1) \\
(-1,0,2,0) & (-1,0,0,0) & (-1,1,0,0) & (0,1,-1,0) \\
(1,0,0,0) & (1,-1,0,0) & (0,-1,1,0) & (0,0,1,-1) \\
(0,0,0,1) & (0,0,0,-1) & (0,0,-1,1) &
\end{array}$$

and the F-vectors of the faces of P° are

Dimension	Number of faces	F-vector	
-1	1	$(0,0,0,0,0,0)$	
0	15	$(1,1,0,0,0,0)$	point
1	45	$(1,2,1,0,0,0)$	edge
2	15	$(1,4,4,1,0,0)$	quadrangle
2	35	$(1,3,3,1,0,0)$	triangle
3	5	$(1,5,8,5,1,0)$	pyramid
3	5	$(1,6,10,6,1,0)$	
3	5	$(1,4,6,4,1,0)$	tetrahedron
3	5	$(1,6,9,5,1,0)$	prism
4	1	$(1,15,45,50,20,1)$	

The polytopes ∇ and P° have 0 as the unique interior lattice point, and the Fano polytope P° defines the embedding toric Fano variety $Y^\circ = X(\Sigma(P^\circ))$ of the fibers of the mirror degeneration. The dimension of the automorphism group of Y° is

$$\dim(\mathrm{Aut}(Y^\circ)) = 4$$

Let

$$S^\circ = \mathbb{C}[y_1, y_2, ..., y_{15}]$$

be the Cox ring of Y° with variables

$$\begin{array}{lll}
y_1 = y_{(-1,-1,0,2)} = \frac{x_4^2}{x_1\,x_2} & y_2 = y_{(2,0,-1,-1)} = \frac{x_1^2}{x_3\,x_4} & y_3 = y_{(0,-1,-1,0)} = \frac{x_0^2}{x_2\,x_3} \\
y_4 = y_{(0,2,0,-1)} = \frac{x_2^2}{x_4\,x_0} & y_5 = y_{(-1,0,2,0)} = \frac{x_3^2}{x_1\,x_0} & y_6 = y_{(-1,0,0,0)} = \frac{x_0}{x_1} \\
y_7 = y_{(-1,1,0,0)} = \frac{x_2}{x_1} & y_8 = y_{(0,1,-1,0)} = \frac{x_2}{x_3} & y_9 = y_{(1,0,0,0)} = \frac{x_1}{x_0} \\
y_{10} = y_{(1,-1,0,0)} = \frac{x_1}{x_2} & y_{11} = y_{(0,-1,1,0)} = \frac{x_3}{x_2} & y_{12} = y_{(0,0,1,-1)} = \frac{x_3}{x_4} \\
y_{13} = y_{(0,0,0,1)} = \frac{x_4}{x_0} & y_{14} = y_{(0,0,0,-1)} = \frac{x_0}{x_4} & y_{15} = y_{(0,0,-1,1)} = \frac{x_4}{x_3}
\end{array}$$

10.3.3 Bergman subcomplex

Intersecting the tropical variety of I with the special fiber Gröbner cone $C_{I_0}(I)$ we obtain the special fiber Bergman subcomplex

$$B(I) = C_{I_0}(I) \cap BF(I) \cap \{w_t = 1\}$$

The following table chooses an indexing of the vertices of ∇ involved in $B(I)$ and gives for each vertex the numbers n_∇ and n_B of faces of ∇ and faces of $B(I)$ it is contained in

n_∇	$n_{B(I)}$		
9	2	$1 = (0,-1,0,-1)$	$2 = (-1,0,0,-1)$
		$3 = (1,0,1,1)$	$4 = (1,1,0,1)$
		$5 = (-1,0,-1,0)$	

With this indexing the Bergman subcomplex $B(I)$ of $\mathrm{Poset}(\nabla)$ associated to the degeneration $\mathfrak{X}$ is

$$\begin{array}{l} [], \\ [[1],[2],[3],[4],[5]], \\ [[1,3],[1,2],[3,4],[4,5],[2,5]], \\ [], \\ [], \\ [] \end{array}$$

$B(I)$ has the F-vector

Dimension	−1	0	1	2	3	4
Number of faces	0	5	5	0	0	0

and the F-vectors of its faces are

Dimension	Number of faces	F-vector

0	5	$(1,1,0,0,0,0)$	point
1	5	$(1,2,1,0,0,0)$	edge

10.3.4 Dual complex

The dual complex $\operatorname{dual}(B(I)) = (B(I))^*$ of deformations associated to $B(I)$ via initial ideals is given by

$$[],$$
$$[],$$
$$[],$$
$$\left[[1,3]^* = \left\langle \frac{x_2}{x_3}, \frac{x_2^2}{x_4\,x_0}, \frac{x_2}{x_1} \right\rangle, [1,2]^* = \left\langle \frac{x_4}{x_3}, \frac{x_4}{x_0}, \frac{x_4^2}{x_1\,x_2} \right\rangle, [3,4]^* = \left\langle \frac{x_0^2}{x_2\,x_3}, \frac{x_0}{x_4}, \frac{x_0}{x_1} \right\rangle,\right.$$
$$\left.[4,5]^* = \left\langle \frac{x_3}{x_4}, \frac{x_3^2}{x_1\,x_0}, \frac{x_3}{x_2} \right\rangle, [2,5]^* = \left\langle \frac{x_1^2}{x_3\,x_4}, \frac{x_1}{x_0}, \frac{x_1}{x_2} \right\rangle\right],$$
$$\left[[1]^* = \left\langle \frac{x_2}{x_3}, \frac{x_4}{x_3}, \frac{x_2^2}{x_4\,x_0}, \frac{x_4}{x_0}, \frac{x_4^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, [2]^* = \left\langle \frac{x_1^2}{x_3\,x_4}, \frac{x_4}{x_3}, \frac{x_1}{x_0}, \frac{x_4}{x_0}, \frac{x_4^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle,\right.$$
$$[3]^* = \left\langle \frac{x_0^2}{x_2\,x_3}, \frac{x_2}{x_3}, \frac{x_2^2}{x_4\,x_0}, \frac{x_0}{x_4}, \frac{x_0}{x_1}, \frac{x_2}{x_1} \right\rangle, [4]^* = \left\langle \frac{x_0^2}{x_2\,x_3}, \frac{x_0}{x_4}, \frac{x_3}{x_4}, \frac{x_3^2}{x_1\,x_0}, \frac{x_3}{x_2}, \frac{x_0}{x_1} \right\rangle,$$
$$\left.[5]^* = \left\langle \frac{x_1^2}{x_3\,x_4}, \frac{x_3}{x_4}, \frac{x_3^2}{x_1\,x_0}, \frac{x_1}{x_0}, \frac{x_3}{x_2}, \frac{x_1}{x_2} \right\rangle\right],$$
$$[]$$

when writing the vertices of the faces as deformations of X_0. Note that the T-invariant basis of deformations associated to a face is given by all lattice points of the corresponding polytope in $M_{\mathbb{R}}$. When numbering the vertices of the faces of $\operatorname{dual}(B(I))$ by the Cox variables of the mirror toric Fano variety Y° the complex $\operatorname{dual}(B(I))$ is

$$[],$$
$$[],$$
$$[],$$
$$\left[[1,3]^* = \langle y_8, y_4, y_7 \rangle, [1,2]^* = \langle y_{15}, y_{13}, y_1 \rangle, [3,4]^* = \langle y_3, y_{14}, y_6 \rangle,\right.$$
$$\left.[4,5]^* = \langle y_{12}, y_5, y_{11} \rangle, [2,5]^* = \langle y_2, y_9, y_{10} \rangle\right],$$
$$\left[[1]^* = \langle y_8, y_{15}, y_4, y_{13}, y_1, y_7 \rangle, [2]^* = \langle y_2, y_{15}, y_9, y_{13}, y_1, y_{10} \rangle,\right.$$
$$[3]^* = \langle y_3, y_8, y_4, y_{14}, y_6, y_7 \rangle, [4]^* = \langle y_3, y_{14}, y_{12}, y_5, y_{11}, y_6 \rangle,$$

$[5]^* = \langle y_2, y_{12}, y_5, y_9, y_{11}, y_{10} \rangle],$

$[]$

The dual complex has the F-vector

Dimension	-1	0	1	2	3	4
Number of faces	0	0	0	5	5	0

and the F-vectors of the faces of $\operatorname{dual}(B(I))$ are

Dimension	Number of faces	F-vector	
2	5	$(1, 3, 3, 1, 0, 0)$	triangle
3	5	$(1, 6, 9, 5, 1, 0)$	prism

Recall that in this example the toric variety Y is projective space. The number of lattice points of the support of $\operatorname{dual}(B(I))$ relates to the dimension $h^{1,0}(X)$ of the complex moduli space of the generic fiber X of $\mathfrak{X}$ and to the dimension $h^{1,1}\left(\bar{X}^\circ\right)$ of the Kähler moduli space of the $MPCP$-blowup $\bar{X}^\circ$ of the generic fiber X° of the mirror degeneration

$$\begin{aligned} |\operatorname{supp}(\operatorname{dual}(B(I))) \cap M| &= 25 = 24 + 1 = \dim(\operatorname{Aut}(Y)) + h^{1,0}(X) \\ &= 20 + 4 + 1 \\ &= |\operatorname{Roots}(Y)| + \dim(T_Y) + h^{1,1}\left(\bar{X}^\circ\right) \end{aligned}$$

There are

$$h^{1,0}(X) + \dim(T_{Y^\circ}) = 1 + 4$$

non-trivial toric polynomial deformations of X_0

$$\frac{x_2^2}{x_4\, x_0} \quad \frac{x_4^2}{x_1\, x_2} \quad \frac{x_0^2}{x_2\, x_3} \quad \frac{x_3^2}{x_1\, x_0} \quad \frac{x_1^2}{x_3\, x_4}$$

They correspond to the toric divisors

$$D_{(0,2,0,-1)} \quad D_{(-1,-1,0,2)} \quad D_{(0,-1,-1,0)} \quad D_{(-1,0,2,0)} \quad D_{(2,0,-1,-1)}$$

on a MPCP-blowup of Y° inducing 1 non-zero toric divisor classes on the mirror X°. The following 20 toric divisors of Y induce the trivial divisor class on X°

$$\begin{array}{lllll} D_{(0,1,-1,0)} & D_{(-1,1,0,0)} & D_{(0,0,-1,1)} & D_{(0,0,0,1)} & D_{(0,0,0,-1)} \\ D_{(-1,0,0,0)} & D_{(0,0,1,-1)} & D_{(0,-1,1,0)} & D_{(1,-1,0,0)} & D_{(1,0,0,0)} \\ D_{(0,1,0,0)} & D_{(-1,0,0,1)} & D_{(1,0,-1,0)} & D_{(0,-1,0,1)} & D_{(0,1,0,-1)} \\ D_{(0,0,-1,0)} & D_{(0,-1,0,0)} & D_{(-1,0,1,0)} & D_{(1,0,0,-1)} & D_{(0,0,1,0)} \end{array}$$

10.3.5 Mirror special fiber

The ideal I_0° of the monomial special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$ is generated by the following set of monomials in S°

$$\left\{ \begin{array}{llll} y_4\, y_5\, y_9\, y_{13}\, y_{14} & y_4\, y_{13}\, y_6\, y_5\, y_9 & y_{15}\, y_8\, y_3\, y_5\, y_2 & y_4\, y_1\, y_{14}\, y_{12}\, y_2 \\ y_2\, y_4\, y_{12}\, y_{15}\, y_{14} & y_1\, y_7\, y_6\, y_5\, y_{10} & y_7\, y_1\, y_3\, y_{11}\, y_{10} & y_{15}\, y_8\, y_3\, y_{12}\, y_2 \\ y_1\, y_2\, y_5\, y_6\, y_7 & y_4\, y_{13}\, y_3\, y_5\, y_9 & y_1\, y_3\, y_4\, y_{11}\, y_{10} & y_4\, y_{13}\, y_{14}\, y_{12}\, y_2 \\ y_8\, y_2\, y_3\, y_{11}\, y_{15} & y_6\, y_7\, y_1\, y_5\, y_9 & y_1\, y_3\, y_8\, y_{11}\, y_{10} & \\ \text{and 228 monomials of degree 5} & & & \end{array} \right\}$$

The Pic (Y°)-generated ideal

$$J_0^\circ = \left\langle \begin{array}{llll} y_2\, y_4\, y_{12}\, y_{15}\, y_{14} & y_7\, y_1\, y_3\, y_{11}\, y_{10} & y_4\, y_5\, y_9\, y_{13}\, y_{14} & y_6\, y_7\, y_1\, y_5\, y_9 \\ y_8\, y_2\, y_3\, y_{11}\, y_{15} & & & \end{array} \right\rangle$$

defines the same subvariety X_0° of the toric variety Y°, and $J_0^{\circ\Sigma} = I_0^\circ$. Passing from J_0° to $J_0^{\circ\Sigma}$ is the non-simplicial toric analogue of saturation. The complex $B(I)^*$ labeled by the variables of the Cox ring S° of Y°, as written in the last section, is the complex $SP(I_0^\circ)$ of prime ideals of the toric strata of the special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$, i.e., the primary decomposition of I_0° is

$$I_0^\circ = \langle y_3, y_{14}, y_6\rangle \cap \langle y_{15}, y_{13}, y_1\rangle \cap \langle y_8, y_4, y_7\rangle \cap \langle y_2, y_9, y_{10}\rangle \cap \langle y_{12}, y_5, y_{11}\rangle$$

Each facet $F \in B(I)$ corresponds to one of these ideals and this ideal is generated by the facets of ∇ containing F.

10.3.6 Covering structure in the deformation complex of the degeneration $\mathfrak{X}$

According to the local reduced Gröbner basis each face of the complex of deformations $\operatorname{dual}(B(I))$ decomposes into 3 polytopes forming a 3 : 1 unramified covering of $B(I)$

$$
\begin{array}{l}
[],\\
[],\\
[],\\
[[\langle y_8\rangle, \langle y_4\rangle, \langle y_7\rangle] \mapsto \langle y_8, y_4, y_7\rangle^{*\vee} = [1,3]^\vee,\\
[\langle y_{15}\rangle, \langle y_{13}\rangle, \langle y_1\rangle] \mapsto \langle y_{15}, y_{13}, y_1\rangle^{*\vee} = [1,2]^\vee,\\
[\langle y_3\rangle, \langle y_{14}\rangle, \langle y_6\rangle] \mapsto \langle y_3, y_{14}, y_6\rangle^{*\vee} = [3,4]^\vee,\\
[\langle y_{11}\rangle, \langle y_{12}\rangle, \langle y_5\rangle] \mapsto \langle y_{12}, y_5, y_{11}\rangle^{*\vee} = [4,5]^\vee,\\
[\langle y_2\rangle, \langle y_9\rangle, \langle y_{10}\rangle] \mapsto \langle y_2, y_9, y_{10}\rangle^{*\vee} = [2,5]^\vee],\\
[[\langle y_8, y_{15}\rangle, \langle y_4, y_{13}\rangle, \langle y_1, y_7\rangle] \mapsto \langle y_8, y_{15}, y_4, y_{13}, y_1, y_7\rangle^{*\vee} = [1]^\vee,\\
[\langle y_2, y_{15}\rangle, \langle y_9, y_{13}\rangle, \langle y_1, y_{10}\rangle] \mapsto \langle y_2, y_{15}, y_9, y_{13}, y_1, y_{10}\rangle^{*\vee} = [2]^\vee,\\
[\langle y_3, y_8\rangle, \langle y_4, y_{14}\rangle, \langle y_6, y_7\rangle] \mapsto \langle y_3, y_8, y_4, y_{14}, y_6, y_7\rangle^{*\vee} = [3]^\vee,\\
[\langle y_3, y_{11}\rangle, \langle y_{14}, y_{12}\rangle, \langle y_5, y_6\rangle] \mapsto \langle y_3, y_{14}, y_{12}, y_5, y_{11}, y_6\rangle^{*\vee} = [4]^\vee,\\
[\langle y_{11}, y_{10}\rangle, \langle y_2, y_{12}\rangle, \langle y_5, y_9\rangle] \mapsto \langle y_2, y_{12}, y_5, y_9, y_{11}, y_{10}\rangle^{*\vee} = [5]^\vee],\\
[]
\end{array}
$$

Here the faces $F \in B(I)$ are specified both via the vertices of F^* labeled by the variables of S° and by the numbering of the vertices of $B(I)$ chosen above. The numbers of faces of the covering in each face of $\operatorname{dual}(B(I))$, i.e., over each face of $B(I)^\vee$ are

Dimension	Number faces	number preimages
−1	0	0
0	0	0
1	0	0
2	5	3
3	5	3
4	0	0

This covering has one sheet forming the complex

$$[],$$

$$[],$$

$$[],$$

$$[\langle y_8\rangle, \langle y_4\rangle, \langle y_7\rangle, \langle y_{15}\rangle, \langle y_{13}\rangle, \langle y_1\rangle, \langle y_3\rangle, \langle y_{14}\rangle, \langle y_6\rangle, \langle y_{11}\rangle, \langle y_{12}\rangle, \langle y_5\rangle, \langle y_2\rangle, \langle y_9\rangle, \langle y_{10}\rangle],$$

$$[\langle y_8, y_{15}\rangle, \langle y_4, y_{13}\rangle, \langle y_1, y_7\rangle, \langle y_2, y_{15}\rangle, \langle y_9, y_{13}\rangle, \langle y_1, y_{10}\rangle, \langle y_3, y_8\rangle, \langle y_4, y_{14}\rangle, \langle y_6, y_7\rangle, \langle y_3, y_{11}\rangle, \langle y_{14}, y_{12}\rangle, \langle y_5, y_6\rangle, \langle y_{11}, y_{10}\rangle, \langle y_2, y_{12}\rangle, \langle y_5, y_9\rangle],$$

$$[]$$

with F-vector

Dimension	Number of faces	F-vector	
0	15	$(1,1,0,0,0,0)$	point
1	15	$(1,2,1,0,0,0)$	edge

Writing the vertices of the faces as deformations the covering is given by

$$[],$$

$$[],$$

$$[],$$

$$[[\left\langle \frac{x_2}{x_3}\right\rangle, \left\langle \frac{x_2^2}{x_4\,x_0}\right\rangle, \left\langle \frac{x_2}{x_1}\right\rangle] \mapsto [1,3]^\vee, [\left\langle \frac{x_4}{x_3}\right\rangle, \left\langle \frac{x_4}{x_0}\right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2}\right\rangle] \mapsto [1,2]^\vee, [\left\langle \frac{x_0^2}{x_2\,x_3}\right\rangle, \left\langle \frac{x_0}{x_4}\right\rangle, \left\langle \frac{x_0}{x_1}\right\rangle] \mapsto [3,4]^\vee, [\left\langle \frac{x_3}{x_2}\right\rangle, \left\langle \frac{x_3}{x_4}\right\rangle, \left\langle \frac{x_3^2}{x_1\,x_0}\right\rangle] \mapsto [4,5]^\vee, [\left\langle \frac{x_1^2}{x_3\,x_4}\right\rangle, \left\langle \frac{x_1}{x_0}\right\rangle, \left\langle \frac{x_1}{x_2}\right\rangle] \mapsto [2,5]^\vee],$$

$$[[\left\langle \frac{x_2}{x_3}, \frac{x_4}{x_3}\right\rangle, \left\langle \frac{x_2^2}{x_4\,x_0}, \frac{x_4}{x_0}\right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_2}{x_1}\right\rangle] \mapsto [1]^\vee, [\left\langle \frac{x_1^2}{x_3\,x_4}, \frac{x_4}{x_3}\right\rangle, \left\langle \frac{x_1}{x_0}, \frac{x_4}{x_0}\right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_1}{x_2}\right\rangle] \mapsto [2]^\vee, [\left\langle \frac{x_0^2}{x_2\,x_3}, \frac{x_2}{x_3}\right\rangle, \left\langle \frac{x_2^2}{x_4\,x_0}, \frac{x_0}{x_4}\right\rangle, \left\langle \frac{x_0}{x_1}, \frac{x_2}{x_1}\right\rangle] \mapsto [3]^\vee, [\left\langle \frac{x_0^2}{x_2\,x_3}, \frac{x_3}{x_2}\right\rangle, \left\langle \frac{x_0}{x_4}, \frac{x_3}{x_4}\right\rangle, \left\langle \frac{x_3^2}{x_1\,x_0}, \frac{x_0}{x_1}\right\rangle] \mapsto [4]^\vee, [\left\langle \frac{x_3}{x_2}, \frac{x_1}{x_2}\right\rangle, \left\langle \frac{x_1^2}{x_3\,x_4}, \frac{x_3}{x_4}\right\rangle, \left\langle \frac{x_3^2}{x_1\,x_0}, \frac{x_1}{x_0}\right\rangle] \mapsto [5]^\vee],$$

$$[]$$

with the one sheet forming the complex

[],

[],

[],

$[\left\langle \frac{x_2}{x_3} \right\rangle, \left\langle \frac{x_2^2}{x_4\,x_0} \right\rangle, \left\langle \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_4}{x_3} \right\rangle, \left\langle \frac{x_4}{x_0} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_0^2}{x_2\,x_3} \right\rangle, \left\langle \frac{x_0}{x_4} \right\rangle, \left\langle \frac{x_0}{x_1} \right\rangle, \left\langle \frac{x_3}{x_2} \right\rangle, \left\langle \frac{x_3}{x_4} \right\rangle, \left\langle \frac{x_3^2}{x_1\,x_0} \right\rangle, \left\langle \frac{x_1^2}{x_3\,x_4} \right\rangle, \left\langle \frac{x_1}{x_0} \right\rangle, \left\langle \frac{x_1}{x_2} \right\rangle],$

$[\left\langle \frac{x_2}{x_3}, \frac{x_4}{x_3} \right\rangle, \left\langle \frac{x_2^2}{x_4\,x_0}, \frac{x_4}{x_0} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_1^2}{x_3\,x_4}, \frac{x_4}{x_3} \right\rangle, \left\langle \frac{x_1}{x_0}, \frac{x_4}{x_0} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_0^2}{x_2\,x_3}, \frac{x_2}{x_3} \right\rangle, \left\langle \frac{x_2^2}{x_4\,x_0}, \frac{x_0}{x_4} \right\rangle, \left\langle \frac{x_0}{x_1}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_0^2}{x_2\,x_3}, \frac{x_3}{x_2} \right\rangle, \left\langle \frac{x_0}{x_4}, \frac{x_3}{x_4} \right\rangle, \left\langle \frac{x_3^2}{x_1\,x_0}, \frac{x_0}{x_1} \right\rangle, \left\langle \frac{x_3}{x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_1^2}{x_3\,x_4}, \frac{x_3}{x_4} \right\rangle, \left\langle \frac{x_3^2}{x_1\,x_0}, \frac{x_1}{x_0} \right\rangle],$

[]

Note that the torus invariant basis of deformations corresponding to a Bergman face is given by the set of all lattice points of the polytope specified above. See also Figure 9.7 for a visualization of the dual complex and the sheets of the covering. In general we have for local complete intersections:

Remark 10.11 *Let $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[t] / \langle t^2 \rangle$, $Y = X(\Sigma)$ be a degeneration with fibers of codimension c and special fiber X_0 given by I_0. If the total space $\mathfrak{X}$ is a local complete intersection, then all first order deformations of X_0 contribute precisely once in the local equations of $\mathfrak{X}$ at the strata of X_0:*

Suppose $F \in SP(I_0)$ is the prime ideal of a stratum of X_0 and

$$I_F \subset S_F \otimes \mathbb{C}[t] / \left\langle t^2 \right\rangle = \mathbb{C}(y_r \mid y_r \notin F)\,[y_r \mid y_r \in F]_> \otimes \mathbb{C}[t] / \left\langle t^2 \right\rangle$$

where $>$ is a local ordering on $y_r \in F$, $r \in \Sigma(1)$, is the localization of I at F. Then for all deformations $\delta \in \operatorname{dual}(B) \cap M$ any monomial $\delta(m_i)$ appears at most once in the minimal reduced standard basis of I_F.

The complex $\operatorname{dual}(B)$ *contains a $c:1$ unramified covering of $B^\vee$.*

10.3.7 Limit map

The limit map $\lim : B(I) \rightarrow \operatorname{Poset}(\Delta)$ associates to a face F of $B(I)$ the face of Δ formed by the limit points of arcs lying over the weight vectors $w \in F$, i.e., with lowest order term t^w. Labeling the faces of the Bergman complex $B(I) \subset \operatorname{Poset}(\nabla)$ and the faces of $\operatorname{Poset}(\Delta)$ by the corresponding dual faces of ∇^* and Δ^*, hence considering the limit map $\lim : B(I) \rightarrow \operatorname{Poset}(\Delta)$ as a map $B(I)^* \rightarrow \operatorname{Poset}(\Delta^*)$, the limit correspondence is given by

$$[],$$

$$[\langle y_8, y_{15}, y_4, y_{13}, y_1, y_7\rangle \mapsto \langle x_1, x_3, x_0\rangle,$$
$$\langle y_2, y_{15}, y_9, y_{13}, y_1, y_{10}\rangle \mapsto \langle x_2, x_3, x_0\rangle,$$
$$\langle y_3, y_8, y_4, y_{14}, y_6, y_7\rangle \mapsto \langle x_1, x_3, x_4\rangle,$$
$$\langle y_3, y_{14}, y_{12}, y_5, y_{11}, y_6\rangle \mapsto \langle x_1, x_2, x_4\rangle,$$
$$\langle y_2, y_{12}, y_5, y_9, y_{11}, y_{10}\rangle \mapsto \langle x_2, x_4, x_0\rangle],$$

$$[\langle y_8, y_4, y_7\rangle \mapsto \langle x_1, x_3, x_4, x_0\rangle, \langle y_{15}, y_{13}, y_1\rangle \mapsto \langle x_1, x_2, x_3, x_0\rangle,$$
$$\langle y_3, y_{14}, y_6\rangle \mapsto \langle x_1, x_2, x_3, x_4\rangle, \langle y_{12}, y_5, y_{11}\rangle \mapsto \langle x_1, x_2, x_4, x_0\rangle,$$
$$\langle y_2, y_9, y_{10}\rangle \mapsto \langle x_2, x_3, x_4, x_0\rangle]$$

The image of the limit map coincides with the image of μ and with the Bergman complex of the mirror, i.e., $\lim(B(I)) = \mu(B(I)) = B(I^\circ)$.

10.3.8 Mirror complex

Numbering the vertices of the mirror complex $\mu(B(I))$ as

$$\begin{array}{ll} 1 = (4, -1, -1, -1) & 2 = (-1, 4, -1, -1) \\ 3 = (-1, -1, 4, -1) & 4 = (-1, -1, -1, 4) \\ 5 = (-1, -1, -1, -1) & \end{array}$$

$\mu(B(I)) \subset \operatorname{Poset}(\Delta)$ is

$$[],$$
$$[[2], [4], [5], [3], [1]],$$
$$[[2, 4], [1, 4], [2, 5], [3, 5], [1, 3]],$$
$$[],$$
$$[],$$
$$[]$$

The ordering of the faces of $\mu(B(I))$ is compatible with above ordering of the faces of $B(I)$. The F-vectors of the faces of $\mu(B(I))^*$ are

Dimension	Number of faces	F-vector	
2	5	$(1, 3, 3, 1, 0, 0)$	triangle
3	5	$(1, 4, 6, 4, 1, 0)$	tetrahedron

The first order deformations of the mirror special fiber X_0° correspond to the lattice points of the dual complex $(\mu(B(I)))^* \subset \mathrm{Poset}(\Delta^*)$ of the mirror complex. We label the first order deformations of X_0° corresponding to vertices of Δ^* by the homogeneous coordinates of Y

$$\begin{array}{ll} x_1 = x_{(1,0,0,0)} = \frac{y_2^2 y_9 y_{10}}{y_1 y_5 y_6 y_7} & x_2 = x_{(0,1,0,0)} = \frac{y_4^2 y_7 y_8}{y_1 y_3 y_{10} y_{11}} \\ x_3 = x_{(0,0,1,0)} = \frac{y_5^2 y_{11} y_{12}}{y_2 y_3 y_8 y_{15}} & x_4 = x_{(0,0,0,1)} = \frac{y_1^2 y_{13} y_{15}}{y_2 y_4 y_{12} y_{14}} \\ x_0 = x_{(-1,-1,-1,-1)} = \frac{y_3^2 y_6 y_{14}}{y_4 y_5 y_9 y_{13}} & \end{array}$$

So writing the vertices of the faces of $(\mu(B(I)))^*$ as homogeneous coordinates of Y, the complex $(\mu(B(I)))^*$ is given by

$$\begin{array}{l} [], \\ [], \\ [], \\ [\langle x_1, x_3, x_0\rangle, \langle x_2, x_3, x_0\rangle, \langle x_1, x_3, x_4\rangle, \langle x_1, x_2, x_4\rangle, \langle x_2, x_4, x_0\rangle], \\ [\langle x_1, x_3, x_4, x_0\rangle, \langle x_1, x_2, x_3, x_0\rangle, \langle x_1, x_2, x_3, x_4\rangle, \langle x_1, x_2, x_4, x_0\rangle, \langle x_2, x_3, x_4, x_0\rangle], \\ [] \end{array}$$

The complex $\mu(B(I))^*$ labeled by the variables of the Cox ring S gives the ideals of the toric strata of the special fiber X_0 of $\mathfrak{X}$, i.e., the complex $SP(I_0)$, so in particular the primary decomposition of I_0 is

$$I_0 = \langle x_1, x_2, x_4\rangle \cap \langle x_1, x_3, x_4\rangle \cap \langle x_2, x_4, x_0\rangle \cap \langle x_2, x_3, x_0\rangle \cap \langle x_1, x_3, x_0\rangle$$

10.3.9 Covering structure in the deformation complex of the mirror degeneration

Each face of the complex of deformations $(\mu(B(I)))^*$ of the mirror special fiber X_0° decomposes as the convex hull of 3 respectively 4 polytopes forming a 4 : 1 ramified covering of $\mu(B(I))^\vee$

$$\begin{array}{l} [], \\ [], \\ [], \\ [[\langle x_0\rangle, \langle x_3\rangle, \langle x_1\rangle] \mapsto \langle x_1, x_3, x_0\rangle^{*\vee} = [2,4]^\vee, \end{array}$$

$$
\begin{aligned}
&[\langle x_0\rangle, \langle x_3\rangle, \langle x_2\rangle] \mapsto \langle x_2, x_3, x_0\rangle^{*\vee} = [1,4]^{\vee},\\
&[\langle x_4\rangle, \langle x_3\rangle, \langle x_1\rangle] \mapsto \langle x_1, x_3, x_4\rangle^{*\vee} = [2,5]^{\vee},\\
&[\langle x_4\rangle, \langle x_2\rangle, \langle x_1\rangle] \mapsto \langle x_1, x_2, x_4\rangle^{*\vee} = [3,5]^{\vee},\\
&[\langle x_0\rangle, \langle x_4\rangle, \langle x_2\rangle] \mapsto \langle x_2, x_4, x_0\rangle^{*\vee} = [1,3]^{\vee}],
\end{aligned}
$$

$$
\begin{aligned}
&[[\langle x_0\rangle, \langle x_4\rangle, \langle x_3\rangle, \langle x_1\rangle] \mapsto \langle x_1, x_3, x_4, x_0\rangle^{*\vee} = [2]^{\vee},\\
&[\langle x_0\rangle, \langle x_3\rangle, \langle x_2\rangle, \langle x_1\rangle] \mapsto \langle x_1, x_2, x_3, x_0\rangle^{*\vee} = [4]^{\vee},\\
&[\langle x_4\rangle, \langle x_3\rangle, \langle x_2\rangle, \langle x_1\rangle] \mapsto \langle x_1, x_2, x_3, x_4\rangle^{*\vee} = [5]^{\vee},\\
&[\langle x_0\rangle, \langle x_4\rangle, \langle x_2\rangle, \langle x_1\rangle] \mapsto \langle x_1, x_2, x_4, x_0\rangle^{*\vee} = [3]^{\vee},\\
&[\langle x_0\rangle, \langle x_4\rangle, \langle x_3\rangle, \langle x_2\rangle] \mapsto \langle x_2, x_3, x_4, x_0\rangle^{*\vee} = [1]^{\vee}],
\end{aligned}
$$

$$[]$$

Due to the singularities of Y° this covering involves degenerate faces, i.e., faces $G \mapsto F^\vee$ with $\dim(G) < \dim(F^\vee)$.

10.3.10 Mirror degeneration

The space of first order deformations of X_0° in the mirror degeneration $\mathfrak{X}^\circ$ has dimension 5 and the deformations represented by the monomials

$$\left\{ \frac{y_4^2 y_7\, y_8}{y_1\, y_3\, y_{10}\, y_{11}} \quad \frac{y_5^2 y_{11}\, y_{12}}{y_2\, y_3\, y_8\, y_{15}} \quad \frac{y_2^2 y_9\, y_{10}}{y_1\, y_5\, y_6\, y_7} \quad \frac{y_3^2 y_6\, y_{14}}{y_4\, y_5\, y_9\, y_{13}} \quad \frac{y_1^2 y_{13}\, y_{15}}{y_2\, y_4\, y_{12}\, y_{14}} \right\}$$

form a torus invariant basis. The number of lattice points of the dual of the mirror complex of I relates to the dimension $h^{1,0}(X^\circ)$ of complex moduli space of the generic fiber X° of $\mathfrak{X}^\circ$ and to the dimension $h^{1,1}(X)$ of the Kähler moduli space of the generic fiber X of $\mathfrak{X}$ via

$$
\begin{aligned}
|\mathrm{supp}\,((\mu(B(I)))^*) \cap N| &= 5 = 4+1\\
&= \dim(\mathrm{Aut}(Y^\circ)) + h^{1,0}(X^\circ) = \dim(T) + h^{1,1}(X)
\end{aligned}
$$

The conjectural first order mirror degeneration $\mathfrak{X}^{1\circ} \subset Y^\circ \times \mathrm{Spec}\,\mathbb{C}[t]/\langle t^2\rangle$ of $\mathfrak{X}$ is given by the ideal $I^{1\circ} \subset S^\circ \otimes \mathbb{C}[t]/\langle t^2\rangle$ generated by

$$
\begin{array}{ll}
ts_1\, y_{14}^2 y_3^2 y_6 + y_4\, y_5\, y_9\, y_{13}\, y_{14} & ts_1\, y_6^2 y_3^2 y_{14} + y_4\, y_{13}\, y_6\, y_5\, y_9\\
ts_4\, y_5^3 y_{11}\, y_{12} + y_{15}\, y_8\, y_3\, y_5\, y_2 & ts_5\, y_1^3 y_{13}\, y_{15} + y_4\, y_1\, y_{14}\, y_{12}\, y_2\\
ts_5\, y_{15}^2 y_1^2 y_{13} + y_2\, y_4\, y_{12}\, y_{15}\, y_{14} & ts_2\, y_{10}^2 y_2^2 y_9 + y_1\, y_7\, y_6\, y_5\, y_{10}\\
ts_3\, y_7^2 y_4^2 y_8 + y_7\, y_1\, y_3\, y_{11}\, y_{10} & ts_4\, y_{12}^2 y_5^2 y_{11} + y_{15}\, y_8\, y_3\, y_{12}\, y_2\\
ts_2\, y_2^3 y_9\, y_{10} + y_1\, y_2\, y_5\, y_6\, y_7 & ts_1\, y_3^3 y_6\, y_{14} + y_4\, y_{13}\, y_3\, y_5\, y_9\\
ts_3\, y_4^3 y_7\, y_8 + y_1\, y_3\, y_4\, y_{11}\, y_{10} & ts_5\, y_{13}^2 y_1^2 y_{15} + y_4\, y_{13}\, y_{14}\, y_{12}\, y_2\\
ts_4\, y_{11}^2 y_5^2 y_{12} + y_8\, y_2\, y_3\, y_{11}\, y_{15} & ts_2\, y_9^2 y_2^2 y_{10} + y_6\, y_7\, y_1\, y_5\, y_9\\
y_7\, y_8^2 ts_3\, y_4^2 + y_1\, y_3\, y_8\, y_{11}\, y_{10} &
\end{array}
$$

and 228 monomials of degree 5

Indeed already the ideal J° generated by

$$\left\{\begin{array}{l} ts_1\, y_{14}^2 y_3^2 y_6 + y_4\, y_5\, y_9\, y_{13}\, y_{14}, \\ ts_5\, y_{15}^2 y_1^2 y_{13} + y_2\, y_4\, y_{12}\, y_{15}\, y_{14}, \\ ts_4\, y_{11}^2 y_5^2 y_{12} + y_8\, y_2\, y_3\, y_{11}\, y_{15}, \\ ts_3\, y_7^2 y_4^2 y_8 + y_7\, y_1\, y_3\, y_{11}\, y_{10}, \\ ts_2\, y_9^2 y_2^2 y_{10} + y_6\, y_7\, y_1\, y_5\, y_9 \end{array}\right\}$$

defines $\mathfrak{X}^{1\circ}$.

10.3.11 Contraction of the mirror degeneration

In the following we give a birational map relating the degeneration $\mathfrak{X}^\circ$ to a Greene-Plesser type orbifolding mirror family by contracting divisors on Y°. In order to contract the divisors

$$\begin{array}{lll} y_6 = y_{(-1,0,0,0)} = \frac{x_0}{x_1} & y_7 = y_{(-1,1,0,0)} = \frac{x_2}{x_1} & y_8 = y_{(0,1,-1,0)} = \frac{x_2}{x_3} \\ y_9 = y_{(1,0,0,0)} = \frac{x_1}{x_0} & y_{10} = y_{(1,-1,0,0)} = \frac{x_1}{x_2} & y_{11} = y_{(0,-1,1,0)} = \frac{x_3}{x_2} \\ y_{12} = y_{(0,0,1,-1)} = \frac{x_3}{x_4} & y_{13} = y_{(0,0,0,1)} = \frac{x_4}{x_0} & y_{14} = y_{(0,0,0,-1)} = \frac{x_0}{x_4} \\ y_{15} = y_{(0,0,-1,1)} = \frac{x_4}{x_3} & & \end{array}$$

consider the $\mathbb{Q}$-factorial toric Fano variety $\hat{Y}^\circ = X(\hat{\Sigma}^\circ)$, where $\hat{\Sigma}^\circ = \Sigma(\hat{P}^\circ) \subset M_{\mathbb{R}}$ is the fan over the reflexive Fano polytope $\hat{P}^\circ \subset M_{\mathbb{R}}$ given as the convex hull of the remaining vertices of $P^\circ = \nabla^*$ corresponding to the Cox variables

$$\begin{array}{ll} y_4 = y_{(0,2,0,-1)} = \frac{x_2^2}{x_4\, x_0} & y_1 = y_{(-1,-1,0,2)} = \frac{x_4^2}{x_1\, x_2} \\ y_3 = y_{(0,-1,-1,0)} = \frac{x_0^2}{x_2\, x_3} & y_5 = y_{(-1,0,2,0)} = \frac{x_3^2}{x_1\, x_0} \\ y_2 = y_{(2,0,-1,-1)} = \frac{x_1^2}{x_3\, x_4} & \end{array}$$

of the toric variety $\hat{Y}^\circ$ with Cox ring

$$\hat{S}^\circ = \mathbb{C}[y_4, y_1, y_3, y_5, y_2]$$

The Cox variables of $\hat{Y}^\circ$ correspond to the set of Fermat deformations of $\mathfrak{X}$. Let

$$Y^\circ = X\left(\Sigma^\circ\right) \to X(\hat{\Sigma}^\circ) = \hat{Y}^\circ$$

be a birational map from Y° to the toric Fano variety $\hat{Y}^\circ$, which contracts the divisors of the rays $\Sigma^\circ(1) - \hat{\Sigma}^\circ(1)$ corresponding to the Cox variables

$$y_6 \quad y_7 \quad y_8 \quad y_9 \quad y_{10} \quad y_{11} \quad y_{12} \quad y_{13} \quad y_{14} \quad y_{15}$$

Representing $\hat{Y}^{\circ}$ as a quotient we have

$$\hat{Y}^{\circ} = \left(\mathbb{C}^5 - V\left(B(\hat{\Sigma}^{\circ})\right)\right)/\hat{G}^{\circ}$$

with

$$\hat{G}^{\circ} = \mathbb{Z}_5 \times (\mathbb{C}^*)^1$$

acting via

$$\xi y = \left(\, u_1^4 v_1 \cdot y_4,\, u_1^2 v_1 \cdot y_1,\, u_1 v_1 \cdot y_3,\, u_1^3 v_1 \cdot y_5,\, v_1 \cdot y_2\right)$$

for $\xi = (u_1, v_1) \in \hat{G}^{\circ}$ and $y \in \mathbb{C}^5 - V\left(B(\hat{\Sigma}^{\circ})\right)$. Hence with the group

$$\hat{H}^{\circ} = \mathbb{Z}_5$$

of order 5 the toric variety $\hat{Y}^{\circ}$ is the quotient

$$\hat{Y}^{\circ} = \mathbb{P}^4/\hat{H}^{\circ}$$

of projective space $\mathbb{P}^4$. The first order mirror degeneration $\mathfrak{X}^{1\circ}$ induces via $Y^{\circ} \to \hat{Y}^{\circ}$ a degeneration $\hat{\mathfrak{X}}^{1\circ} \subset \hat{Y}^{\circ} \times \operatorname{Spec} \mathbb{C}[t] / \langle t^2 \rangle$ given by the ideal $\hat{I}^{1\circ} \subset \langle y_4, y_1, y_3, y_5, y_2 \rangle \subset \hat{S}^{\circ} \otimes \mathbb{C}[t] / \langle t^2 \rangle$ generated by the Fermat-type equations

$$\left\{ \begin{array}{l} ts_1\, y_1^2 + y_2\, y_4, \\ ts_2\, y_3^2 + y_4\, y_5, \\ ts_5\, y_2^2 + y_1\, y_5, \\ s_4\, y_5^2 t + y_2\, y_3, \\ ts_3\, y_4^2 + y_1\, y_3 \end{array} \right\}$$

Note that

$$\left|\operatorname{supp}\left(\left(\mu\left(B\left(I\right)\right)\right)^*\right) \cap N - \operatorname{Roots}\left(\hat{Y}^{\circ}\right)\right| - \dim\left(T_{\hat{Y}^{\circ}}\right) = 5 - 4 = 1$$

so this family has one independent parameter. The special fiber $\hat{X}_0^{\circ} \subset \hat{Y}^{\circ}$ of $\hat{\mathfrak{X}}^{1\circ}$ is cut out by the monomial ideal

$$\hat{I}_0^{\circ} \subset B(\hat{\Sigma}^{\circ}) = \langle y_4, y_1, y_3, y_5, y_2 \rangle \subset \hat{S}^{\circ}$$

generated by

$$\left\{\; y_1\, y_3 \quad y_2\, y_4 \quad y_4\, y_5 \quad y_1\, y_5 \quad y_2\, y_3 \;\right\}$$

The complex $SP(\hat{I}_0^{\circ})$ of prime ideals of the toric strata of $\hat{X}_0^{\circ}$ is

$$
\begin{aligned}
&[],\\
&[],\\
&[],\\
&[\langle y_3,y_5,y_2\rangle,\langle y_4,y_1,y_3\rangle,\langle y_4,y_1,y_2\rangle,\langle y_4,y_3,y_5\rangle,\langle y_1,y_5,y_2\rangle],\\
&[\langle y_1,y_3,y_5,y_2\rangle,\langle y_4,y_3,y_5,y_2\rangle,\langle y_4,y_1,y_5,y_2\rangle,\langle y_4,y_1,y_3,y_2\rangle,\\
&\langle y_4,y_1,y_3,y_5\rangle],\\
&[]
\end{aligned}
$$

so $\hat{I}_0^\circ$ has the primary decomposition

$$\hat{I}_0^\circ = \quad \langle y_3,y_5,y_2\rangle \cap \langle y_4,y_1,y_3\rangle \cap \langle y_4,y_3,y_5\rangle \cap \langle y_1,y_5,y_2\rangle \cap \langle y_4,y_1,y_2\rangle$$

The Bergman subcomplex $B(I) \subset \text{Poset}(\nabla)$ induces a subcomplex of $\text{Poset}(\hat{\nabla})$, $\hat{\nabla} = (\hat{P}^\circ)^*$ corresponding to $\hat{I}_0^\circ$. Indexing the vertices of $\hat{\nabla}$ by

$$
\begin{array}{lll}
1 = (-1,2,-1,0) & 2 = (1,1,0,3) & 3 = (-3,-2,-2,-3)\\
4 = (0,-1,2,-1) & 5 = (3,0,1,1) &
\end{array}
$$

this complex is given by

$$
\left\{
\begin{array}{l}
[],\\
[[1],[2],[3],[4],[5]],\\
[[1,2],[4,5],[3,4],[2,5],[1,3]],\\
[],\\
[],\\
[]
\end{array}
\right\}
$$

The ideal $\hat{I}^{1\circ} \subset \hat{S}^\circ \otimes \mathbb{C}[t]/\langle t^2\rangle$ has a Pfaffian resolution

$$
\begin{gathered}
0 \to \mathcal{O}_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]/\langle t^2\rangle}\left(K^1\right) \to \mathcal{E}^1\left(K^1\right) \xrightarrow{\varphi^1} \left(\mathcal{E}^1\right)^* \xrightarrow{f^1} \mathcal{O}_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]/\langle t^2\rangle}\\
\text{where } \overline{\pi}_1 : \hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]/\left\langle t^2\right\rangle \to \hat{Y}^\circ \text{ and } \mathcal{E}^1 = \overline{\pi}_1^*\mathcal{F}
\end{gathered}
$$

with

$$
\begin{aligned}
\mathcal{F} = \quad & \mathcal{O}_{\hat{Y}^\circ}\left(D_{(0,2,0,-1)} + D_{(2,0,-1,-1)}\right) \oplus \mathcal{O}_{\hat{Y}^\circ}\left(D_{(0,2,0,-1)} + D_{(-1,0,2,0)}\right) \oplus\\
& \mathcal{O}_{\hat{Y}^\circ}\left(D_{(-1,-1,0,2)} + D_{(-1,0,2,0)}\right) \oplus \mathcal{O}_{\hat{Y}^\circ}\left(D_{(0,-1,-1,0)} + D_{(2,0,-1,-1)}\right) \oplus\\
& \mathcal{O}_{\hat{Y}^\circ}\left(D_{(-1,-1,0,2)} + D_{(0,-1,-1,0)}\right)
\end{aligned}
$$

and $K^1 = K_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]/\langle t^2\rangle / \operatorname{Spec}\mathbb{C}[t]/\langle t^2\rangle}$ and $\varphi^1 \in \bigwedge^2 \mathcal{E}^{1*}(-K^1)$ given by

$$\begin{bmatrix} 0 & ts_4\,y_5 & y_3 & -y_4 & -ts_5\,y_1 \\ -ts_4\,y_5 & 0 & ts_1\,y_4 & y_1 & -y_2 \\ -y_3 & -ts_1\,y_4 & 0 & ts_2\,y_2 & y_5 \\ y_4 & -y_1 & -ts_2\,y_2 & 0 & ts_3\,y_3 \\ ts_5\,y_1 & y_2 & -y_5 & -ts_3\,y_3 & 0 \end{bmatrix}$$

Hence via the Pfaffians of φ^1 we obtain a resolution

$$\begin{gathered} 0 \to \mathcal{O}_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]}(K) \to \mathcal{E}(K) \to \mathcal{E}^* \to \mathcal{O}_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]} \\ \text{where } \pi_1 : Y \times \operatorname{Spec}\mathbb{C}[t] \to Y \ , \ \mathcal{E} = \pi_1^*\mathcal{F} \\ \text{and } K = K_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t] / \operatorname{Spec}\mathbb{C}[t]} \end{gathered}$$

of the ideal $\hat{I}^\circ \subset \hat{S}^\circ \otimes \mathbb{C}[t]$ generated by

$$\left\{ \begin{array}{l} -y_1\,y_5 + t\,(-s_2\,y_2^2) + t^2\,(s_1\,s_3\,y_3\,y_4)\,, \\ -y_4\,y_5 + t\,(-s_3\,y_3^2) + t^2\,(s_5\,s_2\,y_1\,y_2)\,, \\ -y_2\,y_4 + t\,(-s_5\,y_1^2) + t^2\,(s_4\,s_3\,y_3\,y_5)\,, \\ -y_2\,y_3 + t\,(-s_4\,y_5^2) + t^2\,(s_5\,s_1\,y_1\,y_4)\,, \\ -y_1\,y_3 + t\,(-s_1\,y_4^2) + t^2\,(s_4\,s_2\,y_2\,y_5) \end{array} \right\}$$

which defines a flat family

$$\hat{\mathfrak{X}}^\circ \subset \hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]$$

10.4 Tropical mirror construction for the degree 14 Pfaffian Calabi-Yau threefold

10.4.1 Setup

Let $Y = \mathbb{P}^6 = X(\Sigma)$, $\Sigma = \Sigma(P) = NF(\Delta) \subset N_{\mathbb{R}}$ with the Fano polytope $P = \Delta^*$ given by

$$\Delta = \operatorname{convexhull}\begin{pmatrix} (6,-1,-1,-1,-1,-1) & (-1,6,-1,-1,-1,-1) \\ (-1,-1,6,-1,-1,-1) & (-1,-1,-1,6,-1,-1) \\ (-1,-1,-1,-1,6,-1) & (-1,-1,-1,-1,-1,6) \\ (-1,-1,-1,-1,-1,-1) & \end{pmatrix} \subset M_{\mathbb{R}}$$

and let

$$S = \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5, x_6]$$

be the Cox ring of Y with the variables

$$\begin{array}{ll} x_1 = x_{(1,0,0,0,0,0)} & x_2 = x_{(0,1,0,0,0,0)} \\ x_3 = x_{(0,0,1,0,0,0)} & x_4 = x_{(0,0,0,1,0,0)} \\ x_5 = x_{(0,0,0,0,1,0)} & x_6 = x_{(0,0,0,0,0,1)} \\ x_0 = x_{(-1,-1,-1,-1,-1,-1)} & \end{array}$$

associated to the rays of Σ. Consider the degeneration $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[t]$ of Pfaffian Calabi-Yau 3-folds with Buchsbaum-Eisenbud resolution

$$0 \to \mathcal{O}_Y(-7) \to \mathcal{E}(-4) \xrightarrow{A_t} \mathcal{E}^*(-3) \to \mathcal{O}_Y \to \mathcal{O}_{X_t} \to 0$$

where

$$\mathcal{E} = 7\mathcal{O}$$

$$A_t = A_0 + t \cdot A$$

$$A_0 = \begin{bmatrix} 0 & 0 & x_2 & 0 & 0 & -x_5 & 0 \\ 0 & 0 & 0 & x_4 & 0 & 0 & -x_0 \\ -x_2 & 0 & 0 & 0 & x_6 & 0 & 0 \\ 0 & -x_4 & 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & -x_6 & 0 & 0 & 0 & x_3 \\ x_5 & 0 & 0 & -x_1 & 0 & 0 & 0 \\ 0 & x_0 & 0 & 0 & -x_3 & 0 & 0 \end{bmatrix}$$

the monomial special fiber of $\mathfrak{X}$ is given by

$$I_0 = \left\langle \; x_0 x_6 x_1 \quad x_2 x_1 x_3 \quad x_5 x_4 x_3 \quad x_5 x_0 x_6 \quad x_2 x_0 x_1 \quad x_2 x_4 x_3 \quad x_5 x_4 x_6 \; \right\rangle$$

and generic $A \in \bigwedge^2 \mathcal{E}^*(1)$

$$A = \begin{bmatrix} 0 & w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \\ -w_1 & 0 & w_7 & w_8 & w_9 & w_{10} & w_{11} \\ -w_2 & -w_7 & 0 & w_{12} & w_{13} & w_{14} & w_{15} \\ -w_3 & -w_8 & -w_{12} & 0 & w_{16} & w_{17} & w_{18} \\ -w_4 & -w_9 & -w_{13} & -w_{16} & 0 & w_{19} & w_{20} \\ -w_5 & -w_{10} & -w_{14} & -w_{17} & -w_{19} & 0 & w_{21} \\ -w_6 & -w_{11} & -w_{15} & -w_{18} & -w_{20} & -w_{21} & 0 \end{bmatrix}$$

$$
\begin{aligned}
w_1 &= s_1\,x_1 + s_2\,x_2 + s_3\,x_3 + s_4\,x_4 + s_5\,x_5 + s_6\,x_6 + s_7\,x_0\\
w_2 &= s_8\,x_1 + s_9\,x_2 + s_{10}\,x_3 + s_{11}\,x_4 + s_{12}\,x_5 + s_{13}\,x_6 + s_{14}\,x_0\\
w_3 &= s_{15}\,x_1 + s_{16}\,x_2 + s_{17}\,x_3 + s_{18}\,x_4 + s_{19}\,x_5 + s_{20}\,x_6 + s_{21}\,x_0\\
w_4 &= s_{22}\,x_1 + s_{23}\,x_2 + s_{24}\,x_3 + s_{25}\,x_4 + s_{26}\,x_5 + s_{27}\,x_6 + s_{28}\,x_0\\
w_5 &= s_{29}\,x_1 + s_{30}\,x_2 + s_{31}\,x_3 + s_{32}\,x_4 + s_{33}\,x_5 + s_{34}\,x_6 + s_{35}\,x_0\\
w_6 &= s_{36}\,x_1 + s_{37}\,x_2 + s_{38}\,x_3 + s_{39}\,x_4 + s_{40}\,x_5 + s_{41}\,x_6 + s_{42}\,x_0\\
w_7 &= s_{43}\,x_1 + s_{44}\,x_2 + s_{45}\,x_3 + s_{46}\,x_4 + s_{47}\,x_5 + s_{48}\,x_6 + s_{49}\,x_0\\
w_8 &= s_{50}\,x_1 + s_{51}\,x_2 + s_{52}\,x_3 + s_{53}\,x_4 + s_{54}\,x_5 + s_{55}\,x_6 + s_{56}\,x_0\\
w_9 &= s_{57}\,x_1 + s_{58}\,x_2 + s_{59}\,x_3 + s_{60}\,x_4 + s_{61}\,x_5 + s_{62}\,x_6 + s_{63}\,x_0\\
w_{10} &= s_{64}\,x_1 + s_{65}\,x_2 + s_{66}\,x_3 + s_{67}\,x_4 + s_{68}\,x_5 + s_{69}\,x_6 + s_{70}\,x_0\\
w_{11} &= s_{71}\,x_1 + s_{72}\,x_2 + s_{73}\,x_3 + s_{74}\,x_4 + s_{75}\,x_5 + s_{76}\,x_6 + s_{77}\,x_0\\
w_{12} &= s_{78}\,x_1 + s_{79}\,x_2 + s_{80}\,x_3 + s_{81}\,x_4 + s_{82}\,x_5 + s_{83}\,x_6 + s_{84}\,x_0\\
w_{13} &= s_{85}\,x_1 + s_{86}\,x_2 + s_{87}\,x_3 + s_{88}\,x_4 + s_{89}\,x_5 + s_{90}\,x_6 + s_{91}\,x_0\\
w_{14} &= s_{92}\,x_1 + s_{93}\,x_2 + s_{94}\,x_3 + s_{95}\,x_4 + s_{96}\,x_5 + s_{97}\,x_6 + s_{98}\,x_0\\
w_{15} &= s_{99}\,x_1 + s_{100}\,x_2 + s_{101}\,x_3 + s_{102}\,x_4 + s_{103}\,x_5 + s_{104}\,x_6 + s_{105}\,x_0\\
w_{16} &= s_{106}\,x_1 + s_{107}\,x_2 + s_{108}\,x_3 + s_{109}\,x_4 + s_{110}\,x_5 + s_{111}\,x_6 + s_{112}\,x_0\\
w_{17} &= s_{113}\,x_1 + s_{114}\,x_2 + s_{115}\,x_3 + s_{116}\,x_4 + s_{117}\,x_5 + s_{118}\,x_6 + s_{119}\,x_0\\
w_{18} &= s_{120}\,x_1 + s_{121}\,x_2 + s_{122}\,x_3 + s_{123}\,x_4 + s_{124}\,x_5 + s_{125}\,x_6 + s_{126}\,x_0\\
w_{19} &= s_{127}\,x_1 + s_{128}\,x_2 + s_{129}\,x_3 + s_{130}\,x_4 + s_{131}\,x_5 + s_{132}\,x_6 + s_{133}\,x_0\\
w_{20} &= s_{134}\,x_1 + s_{135}\,x_2 + s_{136}\,x_3 + s_{137}\,x_4 + s_{138}\,x_5 + s_{139}\,x_6 + s_{140}\,x_0\\
w_{21} &= s_{141}\,x_1 + s_{142}\,x_2 + s_{143}\,x_3 + s_{144}\,x_4 + s_{145}\,x_5 + s_{146}\,x_6 + s_{147}\,x_0
\end{aligned}
$$

The degeneration

$$\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[t]$$

is given by the ideal $I \subset S \otimes \mathbb{C}[t]$ generated by the Pfaffians of $A_0 + t \cdot A$ of degrees $3, 3, 3, 3, 3, 3, 3$.

10.4.2 Special fiber Gröbner cone

The space of first order deformations of $\mathfrak{X}$ has dimension 98 and the deformations represented by the Cox Laurent monomials

$$
\begin{array}{ccccccc}
\frac{x_5\,x_6^2}{x_2\,x_1\,x_3} & \frac{x_5^2 x_4}{x_2\,x_0\,x_1} & \frac{x_0\,x_6^2}{x_2\,x_4\,x_3} & \frac{x_5\,x_4^2}{x_2\,x_0\,x_1} & \frac{x_2^2 x_3}{x_5\,x_0\,x_6} & \frac{x_2\,x_3^2}{x_5\,x_0\,x_6} & \frac{x_4^2 x_3}{x_0\,x_6\,x_1}\\
\frac{x_4\,x_3^2}{x_0\,x_6\,x_1} & \frac{x_5^2 x_6}{x_2\,x_1\,x_3} & \frac{x_1\,x_2^2}{x_5\,x_4\,x_6} & \frac{x_0^2 x_1}{x_5\,x_4\,x_3} & \frac{x_0\,x_1^2}{x_5\,x_4\,x_3} & \frac{x_0^2 x_6}{x_2\,x_4\,x_3} & \frac{x_1^2 x_2}{x_5\,x_4\,x_6}\\
\frac{x_1\,x_3}{x_0\,x_6} & \frac{x_5\,x_3}{x_0\,x_6} & \frac{x_3\,x_4}{x_6\,x_1} & \frac{x_4\,x_6}{x_1\,x_0} & \frac{x_2\,x_4}{x_1\,x_0} & \frac{x_2\,x_4}{x_5\,x_6} & \frac{x_2\,x_0}{x_5\,x_6}\\
\frac{x_5\,x_0}{x_3\,x_4} & \frac{x_5\,x_4}{x_2\,x_0} & \frac{x_2\,x_0}{x_3\,x_4} & \frac{x_1\,x_3}{x_5\,x_4} & \frac{x_6\,x_1}{x_5\,x_4} & \frac{x_1\,x_0}{x_5\,x_3} & \frac{x_5\,x_3}{x_1\,x_2}\\
\frac{x_1\,x_2}{x_4\,x_6} & \frac{x_0\,x_5}{x_1\,x_2} & \frac{x_5\,x_6}{x_1\,x_3} & \frac{x_1\,x_6}{x_2\,x_3} & \frac{x_4\,x_6}{x_2\,x_3} & \frac{x_0\,x_6}{x_2\,x_4} & \frac{x_2\,x_3}{x_0\,x_5}\\
\frac{x_1\,x_0}{x_3\,x_4} & \frac{x_2\,x_3}{x_5\,x_6} & \frac{x_1\,x_2}{x_5\,x_6} & \frac{x_3\,x_4}{x_1\,x_0} & \frac{x_5\,x_4}{x_1\,x_2} & \frac{x_0^2}{x_3\,x_4} & \frac{x_0\,x_6}{x_2\,x_3}\\
\frac{x_1\,x_0}{x_5\,x_4} & \frac{x_2\,x_3}{x_0\,x_6} & \frac{x_5\,x_6}{x_1\,x_2} & \frac{x_1\,x_2}{x_5\,x_4} & \frac{x_3\,x_4}{x_0\,x_6} & \frac{x_5\,x_6}{x_2\,x_3} & \frac{x_5\,x_4}{x_1\,x_0}\\
\frac{x_0\,x_6}{x_3\,x_4} & \frac{x_1^2}{x_5\,x_4} & \frac{x_5^2}{x_1\,x_2} & \frac{x_6^2}{x_2\,x_3} & \frac{x_3^2}{x_0\,x_6} & \frac{x_4^2}{x_1\,x_0} & \frac{x_2^2}{x_5\,x_6}
\end{array}
$$

$$
\begin{array}{ccccccc}
\frac{x_0}{x_3} & \frac{x_3}{x_0} & \frac{x_2}{x_1} & \frac{x_1}{x_2} & \frac{x_6}{x_3} & \frac{x_1}{x_5} & \frac{x_0}{x_1} \\
\frac{x_1}{x_3} & \frac{x_0}{x_2} & \frac{x_2}{x_0} & \frac{x_3}{x_1} & \frac{x_1}{x_0} & \frac{x_3}{x_2} & \frac{x_2}{x_3} \\
\frac{x_5}{x_2} & \frac{x_6}{x_2} & \frac{x_2}{x_6} & \frac{x_5}{x_1} & \frac{x_3}{x_6} & \frac{x_2}{x_5} & \frac{x_0}{x_4} \\
\frac{x_4}{x_2} & \frac{x_4}{x_0} & \frac{x_4}{x_3} & \frac{x_4}{x_1} & \frac{x_3}{x_4} & \frac{x_1}{x_4} & \frac{x_2}{x_4} \\
\frac{x_6}{x_4} & \frac{x_4}{x_5} & \frac{x_5}{x_4} & \frac{x_6}{x_5} & \frac{x_5}{x_3} & \frac{x_3}{x_5} & \frac{x_5}{x_0} \\
\frac{x_4}{x_6} & \frac{x_6}{x_1} & \frac{x_1}{x_6} & \frac{x_0}{x_6} & \frac{x_6}{x_0} & \frac{x_0}{x_5} & \frac{x_5}{x_6}
\end{array}
$$

form a torus invariant basis. These deformations give linear inequalities defining the special fiber Gröbner cone, i.e.,

$$C_{I_0}(I) = \left\{(w_t, w_y) \in \mathbb{R} \oplus N_{\mathbb{R}} \mid \left\langle w, A^{-1}(m)\right\rangle \geq -w_t \forall m\right\}$$

for above monomials m and the presentation matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

of $A_5(Y)$. The vertices of the special fiber polytope

$$\nabla = C_{I_0}(I) \cap \{w_t = 1\}$$

and the number of faces, each vertex is contained in, are given by the table

	$\left(-\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}, 0\right)$	$\left(0, -\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 0, 1, \frac{1}{2}, 1\right)$
35	$\left(\frac{1}{2}, -\frac{1}{2}, 0, 0, -\frac{1}{2}, \frac{1}{2}\right)$	$\left(-1, -\frac{1}{2}, -\frac{1}{2}, -1, 0, -\frac{1}{2}\right)$	$\left(1, \frac{1}{2}, 1, 0, \frac{1}{2}, \frac{1}{2}\right)$
	$\left(-\frac{1}{2}, 0, -1, -\frac{1}{2}, -\frac{1}{2}, -1\right)$		
	$(0, 0, 0, 1, 0, 0)$	$(0, 0, 0, 0, 1, 0)$	$(0, 0, 0, 0, 0, 1)$
65	$(0, 1, 0, 0, 0, 0)$	$(0, 0, 1, 0, 0, 0)$	$(1, 0, 0, 0, 0, 0)$
	$(-1, -1, -1, -1, -1, -1)$		
	$(0, 0, -1, 0, 0, -1)$	$(0, -1, 0, 0, 0, -1)$	$(0, -1, 0, 0, -1, 0)$
70	$(-1, 0, 0, 0, -1, 0)$	$(1, 1, 1, 0, 1, 1)$	$(1, 1, 0, 1, 1, 1)$
	$(-1, 0, 0, -1, 0, 0)$		

73	$(-1,-1,-1,0,-1,-1)$ $(0,1,0,0,1,0)$ $(-1,-1,0,-1,-1,-1)$	$(1,0,0,1,0,0)$ $(0,0,1,0,0,1)$	$(1,0,0,0,1,0)$ $(0,1,0,0,0,1)$
86	$(0,-1,-1,0,-1,-1)$ $(1,1,0,0,1,0)$ $(-1,-1,0,-1,-1,0)$	$(-1,-1,0,0,-1,-1)$ $(0,1,0,0,1,1)$	$(1,0,0,1,1,0)$ $(0,1,1,0,0,1)$
88	$(-1,0,-1,0,-1,-1)$ $(1,0,1,0,1,0)$ $(1,0,1,0,0,1)$	$(1,0,0,1,0,1)$ $(-1,0,-1,-1,0,-1)$	$(0,1,0,1,0,1)$ $(-1,-1,0,-1,0,-1)$

The number of faces of ∇ and their F-vectors are

Dimension	Number of faces	F-vector	
−1	1	$(0,0,0,0,0,0,0,0)$	
0	42	$(1,1,0,0,0,0,0,0)$	point
1	308	$(1,2,1,0,0,0,0,0)$	edge
2	98	$(1,4,4,1,0,0,0,0)$	quadrangle
2	693	$(1,3,3,1,0,0,0,0)$	triangle
3	497	$(1,4,6,4,1,0,0,0)$	tetrahedron
3	42	$(1,6,10,6,1,0,0,0)$	
3	280	$(1,5,8,5,1,0,0,0)$	pyramid
3	7	$(1,6,12,8,1,0,0,0)$	octahedron
3	7	$(1,8,12,6,1,0,0,0)$	cube
3	7	$(1,7,15,10,1,0,0,0)$	
3	42	$(1,6,9,5,1,0,0,0)$	prism
4	70	$(1,6,13,13,6,1,0,0)$	
4	21	$(1,8,21,22,9,1,0,0)$	
4	70	$(1,7,15,14,6,1,0,0)$	
4	7	$(1,6,14,16,8,1,0,0)$	
4	28	$(1,8,19,19,8,1,0,0)$	
4	21	$(1,9,25,27,11,1,0,0)$	
4	28	$(1,7,17,18,8,1,0,0)$	
4	14	$(1,8,21,23,10,1,0,0)$	
4	14	$(1,7,18,21,10,1,0,0)$	
4	14	$(1,10,24,23,9,1,0,0)$	
4	28	$(1,8,18,17,7,1,0,0)$	
4	7	$(1,9,20,18,7,1,0,0)$	
4	91	$(1,5,10,10,5,1,0,0)$	
4	14	$(1,7,16,16,7,1,0,0)$	

5	7	$(1, 14, 51, 77, 52, 14, 1, 0)$
5	14	$(1, 11, 38, 58, 41, 12, 1, 0)$
5	14	$(1, 12, 42, 61, 40, 11, 1, 0)$
5	14	$(1, 9, 28, 40, 28, 9, 1, 0)$
5	7	$(1, 11, 37, 54, 38, 12, 1, 0)$
5	14	$(1, 14, 53, 83, 58, 16, 1, 0)$
6	1	$(1, 42, 308, 791, 882, 427, 70, 1)$

The dual $P^\circ = \nabla^*$ of ∇ is a Fano polytope with vertices

$(-1, -1, -1, 0, 2, 1)$	$(-1, 0, 2, 1, 0, -1)$	$(-1, 0, 1, 2, 0, -1)$
$(0, 1, 2, 0, -1, -1)$	$(0, 2, 1, 0, -1, -1)$	$(-1, -1, 0, 2, 1, 0)$
$(-1, -1, 0, 1, 2, 0)$	$(0, -1, -1, -1, 0, 2)$	$(-1, -1, -1, 0, 1, 2)$
$(0, -1, -1, -1, 0, 1)$	$(2, 1, 0, -1, -1, -1)$	$(2, 0, -1, -1, -1, 0)$
$(1, 0, -1, -1, -1, 0)$	$(1, 2, 0, -1, -1, -1)$	$(0, 0, -1, -1, 1, 0)$
$(0, 0, -1, -1, 0, 0)$	$(0, 1, 0, 0, -1, -1)$	$(0, 2, 0, 0, -1, -1)$
$(0, 1, 0, 1, -1, -1)$	$(-1, 1, 0, 1, 0, 0)$	$(-1, 0, 0, 1, 0, 1)$
$(-1, 0, 0, 2, 0, 0)$	$(-1, 0, 1, 1, 0, -1)$	$(0, 0, 1, 0, 1, -1)$
$(1, 0, 1, 0, 0, -1)$	$(0, 1, 1, 0, -1, 0)$	$(0, 0, 2, 0, 0, -1)$
$(0, -1, 0, -1, 0, 1)$	$(0, -1, -1, 1, 0, 1)$	$(0, -1, -1, 0, 0, 2)$
$(1, -1, -1, 0, 0, 1)$	$(-1, 0, -1, 0, 1, 1)$	$(-1, -1, 0, 0, 1, 0)$
$(1, 1, 0, -1, 0, -1)$	$(-1, -1, 0, 0, 2, 0)$	$(-1, -1, 1, 0, 1, 0)$
$(1, 0, -1, 0, -1, 0)$	$(2, 0, 0, -1, -1, 0)$	$(1, 0, 0, -1, -1, 1)$
$(1, 0, 1, -1, -1, 0)$	$(0, 1, -1, -1, 0, 0)$	$(0, -1, 0, 1, 1, 0)$
$(0, 1, 0, 0, 0, 0)$	$(0, 0, 0, 0, 1, -1)$	$(0, 0, 0, 0, 1, 0)$
$(0, 0, 0, 1, 0, -1)$	$(-1, 0, 0, 0, 0, 1)$	$(1, 0, 0, 0, 0, -1)$
$(0, 0, 0, 0, 0, -1)$	$(0, 0, 0, 0, 0, 1)$	$(0, 0, -1, 1, 0, 0)$
$(0, 0, 0, 0, -1, 0)$	$(1, 0, 0, 0, 0, 0)$	$(0, 0, 1, -1, 0, 0)$
$(1, -1, 0, 0, 0, 0)$	$(0, -1, 1, 0, 0, 0)$	$(0, 1, -1, 0, 0, 0)$
$(-1, 1, 0, 0, 0, 0)$	$(-1, 0, 0, 0, 0, 0)$	$(0, -1, 0, 1, 0, 0)$
$(1, 0, -1, 0, 0, 0)$	$(-1, 0, 1, 0, 0, 0)$	$(0, 0, 0, 0, -1, 1)$
$(0, -1, 0, 0, 0, 0)$	$(0, 1, 0, -1, 0, 0)$	$(0, 0, -1, 0, 1, 0)$
$(0, 0, 1, 0, -1, 0)$	$(0, 0, 0, -1, 0, 1)$	$(0, 0, 0, 1, -1, 0)$
$(0, 0, 0, -1, 1, 0)$		

and the F-vectors of the faces of P° are

Dimension	Number of faces	F-vector	
-1	1	$(0,0,0,0,0,0,0,0)$	
0	70	$(1,1,0,0,0,0,0,0)$	point
1	427	$(1,2,1,0,0,0,0,0)$	edge
2	322	$(1,4,4,1,0,0,0,0)$	quadrangle
2	560	$(1,3,3,1,0,0,0,0)$	triangle
3	301	$(1,6,9,5,1,0,0,0)$	prism
3	28	$(1,6,10,6,1,0,0,0)$	
3	42	$(1,7,11,6,1,0,0,0)$	
3	14	$(1,8,13,7,1,0,0,0)$	
3	140	$(1,4,6,4,1,0,0,0)$	tetrahedron
3	35	$(1,6,12,8,1,0,0,0)$	octahedron
3	231	$(1,5,8,5,1,0,0,0)$	pyramid
4	35	$(1,12,30,28,10,1,0,0)$	
4	14	$(1,10,23,21,8,1,0,0)$	
4	14	$(1,6,13,13,6,1,0,0)$	
4	7	$(1,12,26,22,8,1,0,0)$	
4	21	$(1,7,17,18,8,1,0,0)$	
4	42	$(1,9,20,18,7,1,0,0)$	
4	7	$(1,9,18,15,6,1,0,0)$	
4	7	$(1,12,30,30,12,1,0,0)$	
4	28	$(1,7,15,14,6,1,0,0)$	
4	35	$(1,8,16,14,6,1,0,0)$	
4	21	$(1,12,31,30,11,1,0,0)$	
4	28	$(1,8,18,17,7,1,0,0)$	
4	14	$(1,8,19,19,8,1,0,0)$	
4	7	$(1,10,30,30,10,1,0,0)$	
4	14	$(1,12,31,31,12,1,0,0)$	
4	14	$(1,12,27,24,9,1,0,0)$	
5	7	$(1,18,65,93,58,14,1,0)$	
5	7	$(1,24,88,120,70,16,1,0)$	
5	7	$(1,11,35,49,33,10,1,0)$	
5	7	$(1,20,70,90,50,12,1,0)$	
5	7	$(1,24,86,117,69,16,1,0)$	
5	7	$(1,20,73,108,73,20,1,0)$	
6	1	$(1,70,427,882,791,308,42,1)$	

The polytopes ∇ and P° have 0 as the unique interior lattice point, and the Fano polytope P° defines the embedding toric Fano variety $Y^\circ = X(\Sigma(P^\circ))$

of the fibers of the mirror degeneration. The dimension of the automorphism group of Y° is

$$\dim\left(\mathrm{Aut}\left(Y^\circ\right)\right) = 6$$

Let

$$S^\circ = \mathbb{C}[y_1, y_2, ..., y_{70}]$$

be the Cox ring of Y° with variables

$$y_1 = y_{(-1,-1,-1,0,2,1)} = \frac{x_5^2 x_6}{x_2\, x_1\, x_3} \qquad y_2 = y_{(-1,0,2,1,0,-1)} = \frac{x_3^2 x_4}{x_0\, x_6\, x_1} \qquad y_3 = y_{(-1,0,1,2,0,-1)} = \frac{x_3\, {x_4}^2}{x_0\, x_6\, x_1}$$

$$y_4 = y_{(0,1,2,0,-1,-1)} = \frac{x_2\, {x_3}^2}{x_5\, x_0\, x_6} \qquad y_5 = y_{(0,2,1,0,-1,-1)} = \frac{x_2^2 x_3}{x_5\, x_0\, x_6} \qquad y_6 = y_{(-1,-1,0,2,1,0)} = \frac{x_4^2 x_5}{x_2\, x_0\, x_1}$$

$$y_7 = y_{(-1,-1,0,1,2,0)} = \frac{x_4\, {x_5}^2}{x_2\, x_0\, x_1} \qquad y_8 = y_{(0,-1,-1,-1,0,2)} = \frac{x_6^2 x_0}{x_2\, x_4\, x_3} \qquad y_9 = y_{(-1,-1,-1,0,1,2)} = \frac{x_5\, {x_6}^2}{x_2\, x_1\, x_3}$$

$$y_{10} = y_{(0,-1,-1,-1,0,1)} = \frac{x_6\, {x_0}^2}{x_2\, x_4\, x_3} \qquad y_{11} = y_{(2,1,0,-1,-1,-1)} = \frac{x_1^2 x_2}{x_5\, x_4\, x_6} \qquad y_{12} = y_{(2,0,-1,-1,-1,0)} = \frac{x_1^2 x_0}{x_5\, x_4\, x_3}$$

$$y_{13} = y_{(1,0,-1,-1,-1,0)} = \frac{x_1\, {x_0}^2}{x_5\, x_4\, x_3} \qquad y_{14} = y_{(1,2,0,-1,-1,-1)} = \frac{x_1\, {x_2}^2}{x_5\, x_4\, x_6} \qquad y_{15} = y_{(0,0,-1,-1,1,0)} = \frac{x_5\, x_0}{x_3\, x_4}$$

$$y_{16} = y_{(0,0,-1,-1,0,0)} = \frac{x_0^2}{x_3\, x_4} \qquad y_{17} = y_{(0,1,0,0,-1,-1)} = \frac{x_2\, x_0}{x_5\, x_6} \qquad y_{18} = y_{(0,2,0,0,-1,-1)} = \frac{x_2^2}{x_5\, x_6}$$

$$y_{19} = y_{(0,1,0,1,-1,-1)} = \frac{x_2\, x_4}{x_5\, x_6} \qquad y_{20} = y_{(-1,1,0,1,0,0)} = \frac{x_2\, x_4}{x_1\, x_0} \qquad y_{21} = y_{(-1,0,0,1,0,1)} = \frac{x_4\, x_6}{x_1\, x_0}$$

$$y_{22} = y_{(-1,0,0,2,0,0)} = \frac{x_4^2}{x_1\, x_0} \qquad y_{23} = y_{(-1,0,1,1,0,-1)} = \frac{x_3\, x_4}{x_1\, x_6} \qquad y_{24} = y_{(0,0,1,0,1,-1)} = \frac{x_3\, x_5}{x_6\, x_0}$$

$$y_{25} = y_{(1,0,1,0,0,-1)} = \frac{x_1\, x_3}{x_6\, x_0} \qquad y_{26} = y_{(0,1,1,0,-1,0)} = \frac{x_2\, x_3}{x_5\, x_0} \qquad y_{27} = y_{(0,0,2,0,0,-1)} = \frac{x_3^2}{x_6\, x_0}$$

$$y_{28} = y_{(0,-1,0,-1,0,1)} = \frac{x_6\, x_0}{x_2\, x_4} \qquad y_{29} = y_{(0,-1,-1,1,0,1)} = \frac{x_4\, x_6}{x_2\, x_3} \qquad y_{30} = y_{(0,-1,-1,0,0,2)} = \frac{x_6^2}{x_2\, x_3}$$

$$y_{31} = y_{(1,-1,-1,0,0,1)} = \frac{x_1\, x_6}{x_2\, x_3} \qquad y_{32} = y_{(-1,0,-1,0,1,1)} = \frac{x_5\, x_6}{x_1\, x_3} \qquad y_{33} = y_{(-1,-1,0,0,1,0)} = \frac{x_5\, x_0}{x_1\, x_2}$$

$$y_{34} = y_{(1,1,0,-1,0,-1)} = \frac{x_1\, x_2}{x_4\, x_6} \qquad y_{35} = y_{(-1,-1,0,0,2,0)} = \frac{x_5^2}{x_1\, x_2} \qquad y_{36} = y_{(-1,-1,1,0,1,0)} = \frac{x_3\, x_5}{x_1\, x_2}$$

$$y_{37} = y_{(1,0,-1,0,-1,0)} = \frac{x_1\, x_0}{x_3\, x_5} \qquad y_{38} = y_{(2,0,0,-1,-1,0)} = \frac{x_1^2}{x_4\, x_5} \qquad y_{39} = y_{(1,0,0,-1,-1,1)} = \frac{x_1\, x_6}{x_4\, x_5}$$

$$y_{40} = y_{(1,0,1,-1,-1,0)} = \frac{x_1\, x_3}{x_4\, x_5} \qquad y_{41} = y_{(0,1,-1,-1,0,0)} = \frac{x_2\, x_0}{x_3\, x_4} \qquad y_{42} = y_{(0,-1,0,1,1,0)} = \frac{x_4\, x_5}{x_2\, x_0}$$

$$y_{43} = y_{(0,1,0,0,0,0)} = \frac{x_2}{x_0} \qquad y_{44} = y_{(0,0,0,0,1,-1)} = \frac{x_5}{x_6} \qquad y_{45} = y_{(0,0,0,0,1,0)} = \frac{x_5}{x_0}$$

$$y_{46} = y_{(0,0,0,1,0,-1)} = \frac{x_4}{x_6} \qquad y_{47} = y_{(-1,0,0,0,0,1)} = \frac{x_6}{x_1} \qquad y_{48} = y_{(1,0,0,0,0,-1)} = \frac{x_1}{x_6}$$

$$y_{49} = y_{(0,0,0,0,0,-1)} = \frac{x_0}{x_6} \qquad y_{50} = y_{(0,0,0,0,0,1)} = \frac{x_6}{x_0} \qquad y_{51} = y_{(0,0,-1,1,0,0)} = \frac{x_4}{x_3}$$

$$y_{52} = y_{(0,0,0,0,-1,0)} = \frac{x_0}{x_5} \qquad y_{53} = y_{(1,0,0,0,0,0)} = \frac{x_1}{x_0} \qquad y_{54} = y_{(0,0,1,-1,0,0)} = \frac{x_3}{x_4}$$

$$y_{55} = y_{(1,-1,0,0,0,0)} = \frac{x_1}{x_2} \qquad y_{56} = y_{(0,-1,1,0,0,0)} = \frac{x_3}{x_2} \qquad y_{57} = y_{(0,1,-1,0,0,0)} = \frac{x_2}{x_3}$$

$$y_{58} = y_{(-1,1,0,0,0,0)} = \frac{x_2}{x_1} \qquad y_{59} = y_{(-1,0,0,0,0,0)} = \frac{x_0}{x_1} \qquad y_{60} = y_{(0,-1,0,1,0,0)} = \frac{x_4}{x_2}$$

$$y_{61} = y_{(1,0,-1,0,0,0)} = \frac{x_1}{x_3} \qquad y_{62} = y_{(-1,0,1,0,0,0)} = \frac{x_3}{x_1} \qquad y_{63} = y_{(0,0,0,0,-1,1)} = \frac{x_6}{x_5}$$

$$y_{64} = y_{(0,-1,0,0,0,0)} = \frac{x_0}{x_2} \qquad y_{65} = y_{(0,1,0,-1,0,0)} = \frac{x_2}{x_4} \qquad y_{66} = y_{(0,0,-1,0,1,0)} = \frac{x_5}{x_3}$$

$$y_{67} = y_{(0,0,1,0,-1,0)} = \frac{x_3}{x_5} \qquad y_{68} = y_{(0,0,0,-1,0,1)} = \frac{x_6}{x_4} \qquad y_{69} = y_{(0,0,0,1,-1,0)} = \frac{x_4}{x_5}$$

$$y_{70} = y_{(0,0,0,-1,1,0)} = \frac{x_5}{x_4}$$

10.4.3 Bergman subcomplex

Intersecting the tropical variety of I with the special fiber Gröbner cone $C_{I_0}(I)$ we obtain the special fiber Bergman subcomplex

$$B(I) = C_{I_0}(I) \cap BF(I) \cap \{w_t = 1\}$$

The following table chooses an indexing of the vertices of ∇ involved in $B(I)$ and gives for each vertex the numbers n_∇ and n_B of faces of ∇ and faces of $B(I)$ it is contained in

n_∇	$n_{B(I)}$		
70	10	$1 = (0,0,-1,0,0,-1)$	$2 = (0,-1,0,0,0,-1)$
		$3 = (0,-1,0,0,-1,0)$	$4 = (-1,0,0,0,-1,0)$
		$5 = (1,1,1,0,1,1)$	$6 = (1,1,0,1,1,1)$
		$7 = (-1,0,0,-1,0,0)$	
86	8	$8 = (0,-1,-1,0,-1,-1)$	$9 = (-1,-1,0,0,-1,-1)$
		$10 = (1,0,0,1,1,0)$	$11 = (1,1,0,0,1,0)$
		$12 = (0,1,0,0,1,1)$	$13 = (0,1,1,0,0,1)$
		$14 = (-1,-1,0,-1,-1,0)$	
88	8	$15 = (-1,0,-1,0,-1,-1)$	$16 = (1,0,0,1,0,1)$
		$17 = (0,1,0,1,0,1)$	$18 = (1,0,1,0,1,0)$
		$19 = (-1,0,-1,-1,0,-1)$	$20 = (-1,-1,0,-1,0,-1)$
		$21 = (1,0,1,0,0,1)$	

With this indexing the Bergman subcomplex $B(I)$ of $\text{Poset}(\nabla)$ associated to the degeneration $\mathfrak{X}$ is

[],

[[8], [15], [9], [10], [1], [2], [16], [17], [3], [4], [11], [18], [12], [5], [6], [19], [20], [21], [13], [14], [7]],

[[2, 9], [9, 20], [3, 9], [4, 9], [9, 14], [2, 10], [10, 16], [1, 10], [6, 10], [4, 15], [15, 17], [1, 15], [9, 15], [7, 19], [1, 2], [19, 20], [15, 19], [12, 17], [6, 17], [1, 11], [4, 17], [1, 19], [13, 17], [1, 6], [5, 6], [5, 13], [5, 21], [5, 7], [7, 20], [14, 20], [3, 21], [3, 14], [3, 4], [14, 21], [13, 21], [13, 14], [7, 13], [4, 13], [4, 7], [4, 14], [2, 3], [2, 20], [2, 18], [10, 11], [10, 18], [7, 14], [11, 18], [6, 11], [11, 19], [11, 12], [5, 11], [8, 9], [8, 10], [8, 16], [18, 20], [18, 21], [5, 18], [1, 8], [3, 8], [2, 8], [8, 15], [16, 21], [16, 17], [3, 16], [6, 16], [5, 12], [6, 12], [12, 19], [12, 13], [7, 12]],

[[1, 6, 8, 16], [2, 8, 10], [2, 10, 18], [2, 3, 10, 16], [1, 2, 10], [2, 3, 18, 21],
[3, 16, 21], [3, 14, 21], [3, 4, 13, 21], [6, 12, 17], [4, 7, 12, 17], [4, 13, 14], [12, 13, 17],
[5, 11, 18], [5, 7, 12], [5, 7, 18, 20], [5, 7, 13], [5, 13, 21], [5, 12, 13], [1, 6, 11],
[1, 11, 19], [3, 4, 14], [7, 14, 20], [6, 16, 17], [5, 6, 13, 17], [8, 10, 16], [7, 13, 14],
[6, 11, 12], [6, 10, 16], [5, 7, 14, 21], [5, 18, 21], [10, 16, 18, 21], [3, 4, 16, 17],
[2, 3, 8], [11, 18, 19, 20], [14, 18, 20, 21], [2, 3, 14, 20], [4, 13, 17], [4, 7, 15, 19],
[3, 4, 8, 15], [10, 11, 18], [11, 12, 19], [4, 15, 17], [5, 11, 12], [4, 7, 14], [8, 15, 16, 17],
[1, 6, 15, 17], [2, 9, 20], [2, 3, 9], [2, 8, 9], [1, 2, 9, 15], [12, 15, 17, 19],
[2, 18, 20], [1, 15, 19], [1, 8, 15], [4, 7, 9, 20], [9, 14, 20], [9, 15, 19, 20],
[8, 9, 15], [5, 6, 12], [5, 6, 11], [5, 6, 16, 21], [1, 6, 12, 19], [3, 9, 14], [3, 4, 9],
[3, 8, 9], [6, 10, 11], [5, 6, 10, 18], [3, 8, 16], [7, 12, 13], [4, 7, 13], [4, 9, 14],
[4, 9, 15], [1, 6, 10], [1, 8, 10], [1, 10, 11], [5, 7, 11, 19], [7, 12, 19], [7, 19, 20],
[13, 14, 21], [13, 16, 17, 21], [1, 2, 8], [1, 2, 11, 18], [1, 2, 19, 20]],

[[5, 7, 14, 18, 20, 21], [2, 3, 8, 10, 16], [1, 2, 8, 10], [3, 4, 8, 15, 16, 17],
[5, 6, 10, 11, 18], [2, 3, 10, 16, 18, 21], [1, 2, 10, 11, 18], [5, 6, 10, 16, 18, 21],
[5, 7, 11, 18, 19, 20], [1, 6, 12, 15, 17, 19], [4, 7, 9, 15, 19, 20], [5, 6, 12, 13, 17],
[1, 6, 8, 15, 16, 17], [2, 3, 8, 9], [5, 7, 11, 12, 19], [4, 7, 12, 15, 17, 19],
[5, 7, 12, 13], [5, 7, 13, 14, 21], [4, 7, 12, 13, 17], [5, 6, 13, 16, 17, 21],
[1, 2, 9, 15, 19, 20], [2, 3, 14, 18, 20, 21], [3, 4, 9, 14], [5, 6, 11, 12], [4, 7, 13, 14],
[3, 4, 13, 16, 17, 21], [4, 7, 9, 14, 20], [1, 6, 10, 11], [1, 6, 11, 12, 19],
[1, 2, 11, 18, 19, 20], [2, 3, 9, 14, 20], [3, 4, 13, 14, 21], [1, 2, 8, 9, 15],
[1, 6, 8, 10, 16], [3, 4, 8, 9, 15]],

[],

[],

[]

$B(I)$ has the F-vector

Dimension	−1	0	1	2	3	4	5	6
Number of faces	0	21	70	84	35	0	0	0

and the F-vectors of its faces are

Dimension	Number of faces	F-vector	
0	21	(1, 1, 0, 0, 0, 0, 0, 0)	point
1	70	(1, 2, 1, 0, 0, 0, 0, 0)	edge
2	28	(1, 4, 4, 1, 0, 0, 0, 0)	quadrangle
2	56	(1, 3, 3, 1, 0, 0, 0, 0)	triangle

3	7	$(1,4,6,4,1,0,0,0)$	tetrahedron
3	14	$(1,6,9,5,1,0,0,0)$	prism
3	14	$(1,5,8,5,1,0,0,0)$	pyramid

10.4.4 Dual complex

The dual complex $\operatorname{dual}(B(I)) = (B(I))^*$ of deformations associated to $B(I)$ via initial ideals is given by

$$[],$$

$$[],$$

$$[],$$

$$[[1,2,8,10]^* = \left\langle \frac{x_2\,x_3}{x_5\,x_0}, \frac{x_6}{x_1}, \frac{x_6}{x_4} \right\rangle, [5,7,14,18,20,21]^* = \left\langle \frac{x_4^2}{x_1\,x_0}, \frac{x_4}{x_3}, \frac{x_2\,x_4}{x_5\,x_6} \right\rangle,$$

$$[2,3,8,10,16]^* = \left\langle \frac{x_2^2 x_3}{x_5\,x_0\,x_6}, \frac{x_2}{x_1}, \frac{x_2}{x_4} \right\rangle, ...],$$

$$[[1,6,8,16]^* = \left\langle \frac{x_4\,{x_3}^2}{x_0\,x_6\,x_1}, \frac{x_3^2}{x_0\,x_6}, \frac{x_5\,x_3}{x_1\,x_2}, \frac{x_2\,{x_3}^2}{x_5\,x_0\,x_6}, \frac{x_3}{x_4}, \frac{x_3}{x_1} \right\rangle,$$

$$[10,16,18,21]^* = \left\langle \frac{x_0^2}{x_3\,x_4}, \frac{x_0}{x_1}, \frac{x_2^2}{x_5\,x_6}, \frac{x_2}{x_1}, \frac{x_2\,x_0}{x_5\,x_6}, \frac{x_2\,x_0}{x_3\,x_4} \right\rangle,$$

$$[1,2,10]^* = \left\langle \frac{x_2\,x_3}{x_5\,x_0}, \frac{x_0\,{x_6}^2}{x_2\,x_4\,x_3}, \frac{x_6\,x_1}{x_5\,x_4}, \frac{x_6}{x_5}, \frac{x_6}{x_1}, \frac{x_6}{x_4} \right\rangle,$$

$$[2,8,10]^* = \left\langle \frac{x_2^2 x_3}{x_5\,x_0\,x_6}, \frac{x_2\,x_3}{x_5\,x_0}, \frac{x_2}{x_1}, \frac{x_6}{x_1}, \frac{x_2}{x_4}, \frac{x_6}{x_4} \right\rangle,$$

$$...],$$

$$[[9,14]^* = \left\langle \frac{x_1^2 x_2}{x_5\,x_4\,x_6}, \frac{x_1}{x_0}, \frac{x_1\,{x_2}^2}{x_5\,x_4\,x_6}, \frac{x_1\,x_2}{x_4\,x_6}, \frac{x_2}{x_3}, \frac{x_5}{x_0}, \frac{x_2}{x_0}, \frac{x_1}{x_3}, \frac{x_5}{x_3} \right\rangle,$$

$$[2,9]^* = \left\langle \begin{array}{l} \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{x_6^2}{x_2\,x_3}, \frac{x_6}{x_0}, \frac{x_5\,x_6}{x_1\,x_3}, \frac{x_1\,{x_2}^2}{x_5\,x_4\,x_6}, \frac{x_0\,{x_6}^2}{x_2\,x_4\,x_3}, \frac{x_2\,x_0}{x_3\,x_4}, \frac{x_2}{x_3}, \frac{x_6\,x_1}{x_5\,x_4}, \\ \frac{x_2}{x_0}, \frac{x_2}{x_4}, \frac{x_6}{x_4} \end{array} \right\rangle,$$

$$[9,20]^* = \left\langle \begin{array}{l} \frac{x_6^2}{x_2\,x_3}, \frac{x_6}{x_0}, \frac{x_1^2 x_2}{x_5\,x_4\,x_6}, \frac{x_1^2}{x_5\,x_4}, \frac{x_1}{x_0}, \frac{x_1\,{x_2}^2}{x_5\,x_4\,x_6}, \frac{x_0\,{x_1}^2}{x_5\,x_4\,x_3}, \frac{x_2}{x_3}, \frac{x_6\,x_1}{x_5\,x_4}, \frac{x_6\,x_1}{x_2\,x_3}, \\ \frac{x_2}{x_0}, \frac{x_1}{x_3} \end{array} \right\rangle,$$

$$[1,2]^* = \left\langle \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{x_6^2}{x_2\,x_3}, \frac{x_4\,x_6}{x_1\,x_0}, \frac{x_6}{x_0}, \frac{x_2\,x_3}{x_5\,x_0}, \frac{x_0\,{x_6}^2}{x_2\,x_4\,x_3}, \frac{x_6\,x_1}{x_5\,x_4}, \frac{x_6}{x_5}, \frac{x_6}{x_1}, \frac{x_6}{x_4} \right\rangle,$$

$$...],$$

$$[[8]^* = \left\langle \begin{array}{l} \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{x_5^2 x_4}{x_2\,x_0\,x_1}, \frac{x_4\,{x_3}^2}{x_0\,x_6\,x_1}, \frac{x_3^2}{x_0\,x_6}, \frac{x_4\,x_6}{x_1\,x_0}, \frac{x_6}{x_0}, \frac{x_5\,x_3}{x_1\,x_2}, \frac{x_5^2 x_6}{x_2\,x_1\,x_3}, \\ \frac{x_5\,x_6}{x_1\,x_3}, \frac{x_2^2 x_3}{x_5\,x_0\,x_6}, \frac{x_2\,{x_3}^2}{x_5\,x_0\,x_6}, \frac{x_2\,x_3}{x_5\,x_0}, \frac{x_5^2}{x_1\,x_2}, \frac{x_2\,x_4}{x_1\,x_0}, \frac{x_5\,x_3}{x_0\,x_6}, \frac{x_2}{x_1}, \frac{x_3}{x_4}, \frac{x_5}{x_4}, \\ \frac{x_5}{x_0}, \frac{x_2}{x_0}, \frac{x_3}{x_1}, \frac{x_6}{x_1}, \frac{x_2}{x_4}, \frac{x_6}{x_4} \end{array} \right\rangle,$$

$$[15]^* = \left\langle \begin{array}{l} \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{x_5^2 x_4}{x_2\,x_0\,x_1}, \frac{x_3^2}{x_0\,x_6}, \frac{x_6^2}{x_2\,x_3}, \frac{x_5\,x_4}{x_2\,x_0}, \frac{x_6}{x_0}, \frac{x_5\,x_3}{x_1\,x_2}, \frac{x_3}{x_2}, \frac{x_5^2 x_6}{x_2\,x_1\,x_3}, \\ \frac{x_1^2}{x_5\,x_4}, \frac{x_1\,x_3}{x_0\,x_6}, \frac{x_1}{x_0}, \frac{x_5^2}{x_1\,x_2}, \frac{x_5\,x_3}{x_0\,x_6}, \frac{x_1\,x_3}{x_5\,x_4}, \frac{x_3}{x_4}, \frac{x_0\,{x_6}^2}{x_2\,x_4\,x_3}, \frac{x_6\,x_1}{x_5\,x_4}, \frac{x_6\,x_1}{x_2\,x_3}, \\ \frac{x_0\,x_6}{x_2\,x_4}, \frac{x_1}{x_2}, \frac{x_5}{x_4}, \frac{x_5}{x_0}, \frac{x_6}{x_4} \end{array} \right\rangle,$$

$$[1]^* = \left\langle \begin{array}{c} \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{x_4\,{x_3}^2}{x_0\,x_6\,x_1}, \frac{x_3^2}{x_0\,x_6}, \frac{x_6^2}{x_2\,x_3}, \frac{x_4\,x_6}{x_1\,x_0}, \frac{x_6}{x_0}, \frac{x_5\,x_3}{x_1\,x_2}, \frac{x_3}{x_2}, \frac{x_2\,{x_3}^2}{x_5\,x_0\,x_6}, \\ \frac{x_2\,x_3}{x_5\,x_0}, \frac{x_1\,x_3}{x_5\,x_4}, \frac{x_3}{x_4}, \frac{x_0\,{x_6}^2}{x_2\,x_4\,x_3}, \frac{x_6\,x_1}{x_5\,x_4}, \frac{x_6}{x_5}, \frac{x_0\,x_6}{x_2\,x_4}, \frac{x_3}{x_1}, \frac{x_6}{x_1}, \frac{x_3}{x_5}, \frac{x_6}{x_4} \end{array} \right\rangle,$$

...],

[]

when writing the vertices of the faces as deformations of X_0. Note that the T-invariant basis of deformations associated to a face is given by all lattice points of the corresponding polytope in $M_{\mathbb{R}}$. In order to compress the output we list one representative in any set of faces G with fixed F-vector of G and of G^*. When numbering the vertices of the faces of $\operatorname{dual}(B(I))$ by the Cox variables of the mirror toric Fano variety Y° the complex $\operatorname{dual}(B(I))$ is

[],

[],

[],

$[[1,2,8,10]^* = \langle y_{26}, y_{47}, y_{68}\rangle, [5,7,14,18,20,21]^* = \langle y_{22}, y_{51}, y_{19}\rangle,$
$[2,3,8,10,16]^* = \langle y_5, y_{58}, y_{65}\rangle, ...],$

$[[1,6,8,16]^* = \langle y_2, y_{27}, y_{36}, y_4, y_{54}, y_{62}\rangle, [10,16,18,21]^* = \langle y_{16}, y_{59}, y_{18}, y_{58}, y_{17}, y_{41}\rangle,$
$[1,2,10]^* = \langle y_{26}, y_8, y_{39}, y_{63}, y_{47}, y_{68}\rangle, [2,8,10]^* = \langle y_5, y_{26}, y_{58}, y_{47}, y_{65}, y_{68}\rangle,$
...],

$[[9,14]^* = \langle y_{11}, y_{53}, y_{14}, y_{34}, y_{57}, y_{45}, y_{43}, y_{61}, y_{66}\rangle,$
$[2,9]^* = \langle y_9, y_{30}, y_{50}, y_{32}, y_{14}, y_8, y_{41}, y_{57}, y_{39}, y_{43}, y_{65}, y_{68}\rangle,$
$[9,20]^* = \langle y_{30}, y_{50}, y_{11}, y_{38}, y_{53}, y_{14}, y_{12}, y_{57}, y_{39}, y_{31}, y_{43}, y_{61}\rangle,$
$[1,2]^* = \langle y_9, y_{30}, y_{21}, y_{50}, y_{26}, y_8, y_{39}, y_{63}, y_{47}, y_{68}\rangle,$
...],

$$[[8]^* = \left\langle \begin{array}{l} y_9, y_7, y_2, y_{27}, y_{21}, y_{50}, y_{36}, y_1, y_{32}, y_5, y_4, y_{26}, y_{35}, \\ y_{20}, y_{24}, y_{58}, y_{54}, y_{70}, y_{45}, y_{43}, y_{62}, y_{47}, y_{65}, y_{68} \end{array} \right\rangle,$$
$$[15]^* = \left\langle \begin{array}{l} y_9, y_7, y_{27}, y_{30}, y_{42}, y_{50}, y_{36}, y_{56}, y_1, y_{38}, y_{25}, y_{53}, y_{35}, \\ y_{24}, y_{40}, y_{54}, y_8, y_{39}, y_{31}, y_{28}, y_{55}, y_{70}, y_{45}, y_{68} \end{array} \right\rangle,$$
$$[1]^* = \left\langle \begin{array}{l} y_9, y_2, y_{27}, y_{30}, y_{21}, y_{50}, y_{36}, y_{56}, y_4, y_{26}, y_{40}, y_{54}, y_8, \\ y_{39}, y_{63}, y_{28}, y_{62}, y_{47}, y_{67}, y_{68} \end{array} \right\rangle,$$
...],

[]

The dual complex has the F-vector

Dimension	-1	0	1	2	3	4	5	6
Number of faces	0	0	0	35	84	70	21	0

and the F-vectors of the faces of $\mathrm{dual}\,(B\,(I))$ are

Dimension	Number of faces	F-vector	
2	35	$(1,3,3,1,0,0,0,0)$	triangle
3	35	$(1,6,12,8,1,0,0,0)$	octahedron
3	49	$(1,6,9,5,1,0,0,0)$	prism
4	7	$(1,9,18,15,6,1,0,0)$	
4	35	$(1,12,30,28,10,1,0,0)$	
4	21	$(1,12,31,30,11,1,0,0)$	
4	7	$(1,10,30,30,10,1,0,0)$	
5	7	$(1,24,86,117,69,16,1,0)$	
5	7	$(1,24,88,120,70,16,1,0)$	
5	7	$(1,20,70,90,50,12,1,0)$	

Recall that in this example the toric variety Y is projective space. The number of lattice points of the support of $\mathrm{dual}\,(B\,(I))$ relates to the dimension $h^{1,2}\,(X)$ of the complex moduli space of the generic fiber X of $\mathfrak{X}$ and to the dimension $h^{1,1}\left(\bar{X}^{\circ}\right)$ of the Kähler moduli space of the $MPCP$-blowup $\bar{X}^{\circ}$ of the generic fiber X° of the mirror degeneration

$$\begin{aligned}|\mathrm{supp}\,(\mathrm{dual}\,(B\,(I)))\cap M| &= 98 = 48+50 = \dim\,(\mathrm{Aut}\,(Y)) + h^{1,2}\,(X)\\ &= 42+6+50\\ &= |\mathrm{Roots}\,(Y)| + \dim\,(T_Y) + h^{1,1}\left(\bar{X}^{\circ}\right)\end{aligned}$$

There are

$$h^{1,2}\,(X) + \dim\,(T_{Y^{\circ}}) = 50+6$$

non-trivial toric polynomial deformations of X_0

$$\begin{array}{cccccccc}
\frac{x_2\,x_4}{x_5\,x_6} & \frac{x_4^2}{x_1\,x_0} & \frac{x_2^2 x_3}{x_5\,x_0\,x_6} & \frac{x_2\,x_3}{x_5\,x_0} & \frac{x_5\,x_3}{x_0\,x_6} & \frac{x_5^2}{x_1\,x_2} & \frac{x_0^2 x_6}{x_2\,x_4\,x_3} & \frac{x_2^2}{x_5\,x_6} \\
\frac{x_2\,x_0}{x_3\,x_4} & \frac{x_0\,{x_6}^2}{x_2\,x_4\,x_3} & \frac{x_2\,x_0}{x_5\,x_6} & \frac{x_0^2}{x_3\,x_4} & \frac{x_4\,x_6}{x_2\,x_3} & \frac{x_3^2}{x_0\,x_6} & \frac{x_1\,x_3}{x_5\,x_4} & \frac{x_1^2}{x_5\,x_4} \\
\frac{x_1\,x_6}{x_2\,x_3} & \frac{x_0^2 x_1}{x_5\,x_4\,x_3} & \frac{x_5\,x_3}{x_1\,x_2} & \frac{x_5\,x_6}{x_1\,x_3} & \frac{x_4^2 x_3}{x_0\,x_6\,x_1} & \frac{x_1\,x_3}{x_0\,x_6} & \frac{x_1\,x_0}{x_5\,x_3} & \frac{x_5\,{x_4}^2}{x_2\,x_0\,x_1} \\
\frac{x_0\,{x_1}^2}{x_5\,x_4\,x_3} & \frac{x_0\,x_5}{x_1\,x_2} & \frac{x_1\,x_6}{x_5\,x_4} & \frac{x_6^2}{x_2\,x_3} & \frac{x_2\,x_4}{x_1\,x_0} & \frac{x_1\,x_2}{x_4\,x_6} & \frac{x_3\,x_4}{x_6\,x_1} & \frac{x_5\,x_4}{x_2\,x_0} \\
\frac{x_5\,x_0}{x_3\,x_4} & \frac{x_1^2 x_2}{x_5\,x_4\,x_6} & \frac{x_0\,x_6}{x_2\,x_4} & \frac{x_4\,{x_3}^2}{x_0\,x_6\,x_1} & \frac{x_4\,x_6}{x_1\,x_0} & \frac{x_1\,{x_2}^2}{x_5\,x_4\,x_6} & \frac{x_5^2 x_4}{x_2\,x_0\,x_1} & \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}
\end{array}$$

$\frac{x_2\,x_3{}^2}{x_5\,x_0\,x_6}$	$\frac{x_5^2 x_6}{x_2\,x_1\,x_3}$	$\frac{x_0\,x_6}{x_3\,x_4}$	$\frac{x_5\,x_4}{x_1\,x_0}$	$\frac{x_1\,x_0}{x_5\,x_4}$	$\frac{x_0\,x_6}{x_2\,x_3}$	$\frac{x_5\,x_4}{x_1\,x_2}$	$\frac{x_3\,x_4}{x_1\,x_0}$
$\frac{x_1\,x_2}{x_5\,x_6}$	$\frac{x_2\,x_3}{x_5\,x_6}$	$\frac{x_1\,x_0}{x_3\,x_4}$	$\frac{x_1\,x_2}{x_5\,x_4}$	$\frac{x_5\,x_6}{x_1\,x_2}$	$\frac{x_2\,x_3}{x_0\,x_6}$	$\frac{x_5\,x_6}{x_2\,x_3}$	$\frac{x_3\,x_4}{x_0\,x_6}$

They correspond to the toric divisors

$D_{(0,1,0,1,-1,-1)}$	$D_{(-1,0,0,2,0,0)}$	$D_{(0,2,1,0,-1,-1)}$	$D_{(0,1,1,0,-1,0)}$
$D_{(0,0,1,0,1,-1)}$	$D_{(-1,-1,0,0,2,0)}$	$D_{(0,-1,-1,-1,0,1)}$	$D_{(0,2,0,0,-1,-1)}$
$D_{(0,1,-1,-1,0,0)}$	$D_{(0,-1,-1,-1,0,2)}$	$D_{(0,1,0,0,-1,-1)}$	$D_{(0,0,-1,-1,0,0)}$
$D_{(0,-1,-1,1,0,1)}$	$D_{(0,0,2,0,0,-1)}$	$D_{(1,0,1,-1,-1,0)}$	$D_{(2,0,0,-1,-1,0)}$
$D_{(1,-1,-1,0,0,1)}$	$D_{(1,0,-1,-1,-1,0)}$	$D_{(-1,-1,1,0,1,0)}$	$D_{(-1,0,-1,0,1,1)}$
$D_{(-1,0,1,2,0,-1)}$	$D_{(1,0,1,0,0,-1)}$	$D_{(1,0,-1,0,-1,0)}$	$D_{(-1,-1,0,2,1,0)}$
$D_{(2,0,-1,-1,-1,0)}$	$D_{(-1,-1,0,0,1,0)}$	$D_{(1,0,0,-1,-1,1)}$	$D_{(0,-1,-1,0,0,2)}$
$D_{(-1,1,0,1,0,0)}$	$D_{(1,1,0,-1,0,-1)}$	$D_{(-1,0,1,1,0,-1)}$	$D_{(0,-1,0,1,1,0)}$
$D_{(0,0,-1,-1,1,0)}$	$D_{(2,1,0,-1,-1,-1)}$	$D_{(0,-1,0,-1,0,1)}$	$D_{(-1,0,2,1,0,-1)}$
$D_{(-1,0,0,1,0,1)}$	$D_{(1,2,0,-1,-1,-1)}$	$D_{(-1,-1,0,1,2,0)}$	$D_{(-1,-1,-1,0,1,2)}$
$D_{(0,1,2,0,-1,-1)}$	$D_{(-1,-1,-1,0,2,1)}$	$D_{(0,0,-1,-1,0,1)}$	$D_{(-1,0,0,1,1,0)}$
$D_{(1,0,0,-1,-1,0)}$	$D_{(0,-1,-1,0,0,1)}$	$D_{(-1,-1,0,1,1,0)}$	$D_{(-1,0,1,1,0,0)}$
$D_{(1,1,0,0,-1,-1)}$	$D_{(0,1,1,0,-1,-1)}$	$D_{(1,0,-1,-1,0,0)}$	$D_{(1,1,0,-1,-1,0)}$
$D_{(-1,-1,0,0,1,1)}$	$D_{(0,1,1,0,0,-1)}$	$D_{(0,-1,-1,0,1,1)}$	$D_{(0,0,1,1,0,-1)}$

on a MPCP-blowup of Y° inducing 50 non-zero toric divisor classes on the mirror X°. The following 42 toric divisors of Y induce the trivial divisor class on X°

$D_{(0,0,-1,1,0,0)}$	$D_{(0,1,0,-1,0,0)}$	$D_{(-1,1,0,0,0,0)}$	$D_{(0,0,0,-1,0,1)}$
$D_{(-1,0,0,0,0,1)}$	$D_{(0,0,0,-1,1,0)}$	$D_{(0,0,0,0,-1,0)}$	$D_{(-1,0,0,0,0,0)}$
$D_{(0,0,0,0,-1,1)}$	$D_{(0,0,0,1,-1,0)}$	$D_{(0,-1,1,0,0,0)}$	$D_{(1,0,0,0,0,0)}$
$D_{(0,0,0,0,0,-1)}$	$D_{(0,-1,0,0,0,0)}$	$D_{(0,0,1,-1,0,0)}$	$D_{(0,1,0,0,0,0)}$
$D_{(0,-1,0,1,0,0)}$	$D_{(1,-1,0,0,0,0)}$	$D_{(0,0,0,1,0,-1)}$	$D_{(1,0,0,0,0,-1)}$
$D_{(0,0,0,0,0,1)}$	$D_{(0,1,-1,0,0,0)}$	$D_{(0,0,-1,0,1,0)}$	$D_{(0,0,0,0,1,0)}$
$D_{(1,0,-1,0,0,0)}$	$D_{(0,0,0,0,1,-1)}$	$D_{(0,0,1,0,-1,0)}$	$D_{(-1,0,1,0,0,0)}$
$D_{(0,1,0,0,-1,0)}$	$D_{(0,1,0,0,0,-1)}$	$D_{(-1,0,0,0,1,0)}$	$D_{(0,0,1,0,0,-1)}$
$D_{(-1,0,0,1,0,0)}$	$D_{(0,0,-1,0,0,0)}$	$D_{(0,-1,0,0,0,1)}$	$D_{(0,0,0,1,0,0)}$
$D_{(0,0,0,-1,0,0)}$	$D_{(0,-1,0,0,1,0)}$	$D_{(1,0,0,-1,0,0)}$	$D_{(0,0,-1,0,0,1)}$
$D_{(0,0,1,0,0,0)}$	$D_{(1,0,0,0,-1,0)}$		

10.4.5 Mirror special fiber

The complex $B(I)^*$ labeled by the variables of the Cox ring S° of Y°, as written in the last section, is the complex $SP(I_0^\circ)$ of prime ideals of the toric strata of the monomial special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$, i.e., the primary decomposition of I_0° is

$$I_0^\circ = \begin{array}{l} \langle y_{27}, y_{36}, y_{54}\rangle \cap \langle y_{32}, y_{43}, y_{65}\rangle \cap \langle y_3, y_{69}, y_{60}\rangle \cap \langle y_{38}, y_{25}, y_{55}\rangle \cap \\ \cap \langle y_{37}, y_{60}, y_{46}\rangle \cap \langle y_{27}, y_{56}, y_{40}\rangle \cap \langle y_{38}, y_{53}, y_{31}\rangle \cap \langle y_{13}, y_{49}, y_{64}\rangle \cap \\ \cap \langle y_{14}, y_{57}, y_{43}\rangle \cap \langle y_7, y_{44}, y_{66}\rangle \cap \langle y_9, y_{50}, y_{68}\rangle \cap \langle y_4, y_{54}, y_{62}\rangle \cap \\ \cap \langle y_1, y_{70}, y_{45}\rangle \cap \langle y_{35}, y_{44}, y_{15}\rangle \cap \langle y_{11}, y_{53}, y_{61}\rangle \cap \langle y_{28}, y_{62}, y_{67}\rangle \cap \\ \cap \langle y_2, y_{56}, y_{67}\rangle \cap \langle y_{30}, y_{21}, y_{63}\rangle \cap \langle y_6, y_{51}, y_{46}\rangle \cap \langle y_{16}, y_{33}, y_{49}\rangle \cap \\ \cap \langle y_{30}, y_{50}, y_{39}\rangle \cap \langle y_{18}, y_{20}, y_{57}\rangle \cap \langle y_{34}, y_{45}, y_{66}\rangle \cap \langle y_{23}, y_{64}, y_{52}\rangle \cap \\ \cap \langle y_{42}, y_{61}, y_{48}\rangle \cap \langle y_{12}, y_{55}, y_{48}\rangle \cap \langle y_5, y_{58}, y_{65}\rangle \cap \langle y_{35}, y_{24}, y_{70}\rangle \cap \\ \cap \langle y_{10}, y_{59}, y_{52}\rangle \cap \langle y_{26}, y_{47}, y_{68}\rangle \cap \langle y_{18}, y_{58}, y_{41}\rangle \cap \langle y_{16}, y_{59}, y_{17}\rangle \cap \\ \cap \langle y_{22}, y_{29}, y_{69}\rangle \cap \langle y_8, y_{63}, y_{47}\rangle \cap \langle y_{22}, y_{51}, y_{19}\rangle \end{array}$$

Each facet $F \in B(I)$ corresponds to one of these ideals and this ideal is generated by the facets of ∇ containing F.

10.4.6 Covering structure in the deformation complex of the degeneration $\mathfrak{X}$

According to the local reduced Gröbner basis each face of the complex of deformations dual $(B(I))$ decomposes into 3 respectively 5 polytopes forming a 5 : 1 ramified covering of $B(I)$

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$[[\langle y_{47}\rangle, \langle y_{26}\rangle, \langle y_{68}\rangle] \mapsto [1,2,8,10]^\vee, [\langle y_{22}\rangle, \langle y_{51}\rangle, \langle y_{19}\rangle] \mapsto [5,7,14,18,20,21]^\vee,$
$[\langle y_{58}\rangle, \langle y_5\rangle, \langle y_{65}\rangle] \mapsto [2,3,8,10,16]^\vee, ...],$

$[[\langle y_{36}, y_{62}\rangle, \langle y_{54}\rangle, \langle y_{27}, y_4\rangle] \mapsto [1,6,8,16]^\vee, [\langle y_{18}, y_{17}\rangle, \langle y_{59}, y_{58}\rangle, \langle y_{16}, y_{41}\rangle] \mapsto [10,16,18,21]^\vee,$
$[\langle y_{47}\rangle, \langle y_{26}, y_{63}\rangle, \langle y_8, y_{68}\rangle] \mapsto [1,2,10]^\vee, [\langle y_{58}, y_{47}\rangle, \langle y_5, y_{26}\rangle, \langle y_{65}, y_{68}\rangle] \mapsto [2,8,10]^\vee,$
$...],$

$[[\langle y_{57}, y_{61}, y_{66}\rangle, \langle y_{53}, y_{45}, y_{43}\rangle, \langle y_{11}, y_{14}, y_{34}\rangle] \mapsto [9,14]^\vee,$
$[\langle y_{50}, y_{43}\rangle, \langle y_9, y_{30}, y_{32}, y_{57}\rangle, \langle y_{14}, y_{39}, y_{65}, y_{68}\rangle] \mapsto [2,9]^\vee,$
$[\langle y_{50}, y_{53}, y_{43}\rangle, \langle y_{30}, y_{57}, y_{31}, y_{61}\rangle, \langle y_{11}, y_{38}, y_{14}, y_{39}\rangle] \mapsto [9,20]^\vee,$
$[\langle y_{21}, y_{50}, y_{47}\rangle, \langle y_9, y_{30}, y_{47}\rangle, \langle y_{50}, y_{26}, y_{63}\rangle, \langle y_{30}, y_8, y_{68}\rangle, \langle y_{39}, y_{63}, y_{68}\rangle] \mapsto [1,2]^\vee,$
$...],$

$[[\langle y_9, y_{36}, y_1, y_{32}, y_{35}, y_{58}, y_{62}, y_{47}\rangle, \langle y_{54}, y_{70}, y_{65}, y_{68}\rangle, \langle y_{27}, y_{50}, y_5, y_4, y_{26}, y_{24}, y_{45}, y_{43}\rangle] \mapsto [8]^\vee,$

$[\langle y_{27}, y_{50}, y_{25}, y_{53}, y_{24}, y_{45}\rangle, \langle y_9, y_{30}, y_{36}, y_{56}, y_1, y_{35}, y_{31}, y_{55}\rangle,$
$\langle y_{38}, y_{40}, y_{54}, y_{39}, y_{70}, y_{68}\rangle] \mapsto [15]^\vee, [\langle y_2, y_{27}, y_{21}, y_{50}, y_{62}, y_{47}\rangle, \langle y_9, y_{30}, y_{36}, y_{56}, y_{62}, y_{47}\rangle,$
$\langle y_{40}, y_{54}, y_{39}, y_{63}, y_{67}, y_{68}\rangle, \langle y_{27}, y_{50}, y_4, y_{26}, y_{63}, y_{67}\rangle, \langle y_{30}, y_{56}, y_{54}, y_8, y_{28}, y_{68}\rangle] \mapsto [1]^\vee,$
$...],$

$[]$

Here the faces $F \in B(I)$ are specified both via the vertices of F^* labeled by the variables of S° and by the numbering of the vertices of $B(I)$ chosen above. The numbers of faces of the covering in each face of $\operatorname{dual}(B(I))$, i.e., over each face of $B(I)^\vee$ are

Dimension	Number faces	number preimages
-1	0	0
0	0	0
1	0	0
2	35	3
3	84	3
4	63	3
4	7	5
5	14	3
5	7	5
6	0	0

This covering has one sheet forming the complex

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$[\langle y_{47}\rangle, \langle y_{26}\rangle, \langle y_{68}\rangle, \langle y_{22}\rangle, \langle y_{51}\rangle, \langle y_{19}\rangle, \langle y_{58}\rangle,$
$\langle y_{5}\rangle, \langle y_{65}\rangle, ...],$

$[\langle y_{36}, y_{62}\rangle, \langle y_{54}\rangle, \langle y_{27}, y_{4}\rangle, \langle y_{18}, y_{17}\rangle, \langle y_{59}, y_{58}\rangle, \langle y_{16}, y_{41}\rangle,$
$\langle y_{47}\rangle, \langle y_{26}, y_{63}\rangle, \langle y_{8}, y_{68}\rangle, \langle y_{58}, y_{47}\rangle, \langle y_{5}, y_{26}\rangle, \langle y_{65}, y_{68}\rangle, ...],$

$[\langle y_{57}, y_{61}, y_{66}\rangle, \langle y_{53}, y_{45}, y_{43}\rangle, \langle y_{11}, y_{14}, y_{34}\rangle, \langle y_{50}, y_{43}\rangle, \langle y_{9}, y_{30}, y_{32}, y_{57}\rangle,$
$\langle y_{14}, y_{39}, y_{65}, y_{68}\rangle, \langle y_{50}, y_{53}, y_{43}\rangle, \langle y_{30}, y_{57}, y_{31}, y_{61}\rangle, \langle y_{11}, y_{38}, y_{14}, y_{39}\rangle, \langle y_{21}, y_{50}, y_{47}\rangle,$
$\langle y_{9}, y_{30}, y_{47}\rangle, \langle y_{50}, y_{26}, y_{63}\rangle, \langle y_{30}, y_{8}, y_{68}\rangle, \langle y_{39}, y_{63}, y_{68}\rangle, ...],$

$[\langle y_{9}, y_{36}, y_{1}, y_{32}, y_{35}, y_{58}, y_{62}, y_{47}\rangle, \langle y_{54}, y_{70}, y_{65}, y_{68}\rangle, \langle y_{27}, y_{50}, y_{5}, y_{4}, y_{26}, y_{24}, y_{45}, y_{43}\rangle,$
$\langle y_{27}, y_{50}, y_{25}, y_{53}, y_{24}, y_{45}\rangle, \langle y_{9}, y_{30}, y_{36}, y_{56}, y_{1}, y_{35}, y_{31}, y_{55}\rangle, \langle y_{38}, y_{40}, y_{54}, y_{39}, y_{70}, y_{68}\rangle,$
$\langle y_{2}, y_{27}, y_{21}, y_{50}, y_{62}, y_{47}\rangle, \langle y_{9}, y_{30}, y_{36}, y_{56}, y_{62}, y_{47}\rangle, \langle y_{40}, y_{54}, y_{39}, y_{63}, y_{67}, y_{68}\rangle,$
$\langle y_{27}, y_{50}, y_{4}, y_{26}, y_{63}, y_{67}\rangle, \langle y_{30}, y_{56}, y_{54}, y_{8}, y_{28}, y_{68}\rangle, ...],$

[]

with F-vector

Dimension	Number of faces	F-vector	
0	70	$(1,1,0,0,0,0,0,0)$	point
1	182	$(1,2,1,0,0,0,0,0)$	edge
2	112	$(1,4,4,1,0,0,0,0)$	quadrangle
2	77	$(1,3,3,1,0,0,0,0)$	triangle
3	7	$(1,4,6,4,1,0,0,0)$	tetrahedron
3	14	$(1,8,13,7,1,0,0,0)$	
3	7	$(1,8,12,6,1,0,0,0)$	cube
3	49	$(1,6,9,5,1,0,0,0)$	prism

Writing the vertices of the faces as deformations the covering is given by

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$$[[\left\langle \frac{x_6}{x_1} \right\rangle, \left\langle \frac{x_2\,x_3}{x_5\,x_0} \right\rangle, \left\langle \frac{x_6}{x_4} \right\rangle] \mapsto [1, 2, 8, 10]^{\vee},$$
$$[\left\langle \frac{x_4^2}{x_1\,x_0} \right\rangle, \left\langle \frac{x_4}{x_3} \right\rangle, \left\langle \frac{x_2\,x_4}{x_5\,x_6} \right\rangle] \mapsto [5, 7, 14, 18, 20, 21]^{\vee},$$
$$[\left\langle \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^2 x_3}{x_5\,x_0\,x_6} \right\rangle, \left\langle \frac{x_2}{x_4} \right\rangle] \mapsto [2, 3, 8, 10, 16]^{\vee},$$
...],

$$[[\left\langle \frac{x_5\,x_3}{x_1\,x_2}, \frac{x_3}{x_1} \right\rangle, \left\langle \frac{x_3}{x_4} \right\rangle, \left\langle \frac{x_3^2}{x_0\,x_6}, \frac{x_2\,{x_3}^2}{x_5\,x_0\,x_6} \right\rangle] \mapsto [1, 6, 8, 16]^{\vee},$$
$$[\left\langle \frac{x_2^2}{x_5\,x_6}, \frac{x_2\,x_0}{x_5\,x_6} \right\rangle, \left\langle \frac{x_0}{x_1}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_0^2}{x_3\,x_4}, \frac{x_2\,x_0}{x_3\,x_4} \right\rangle] \mapsto [10, 16, 18, 21]^{\vee},$$
$$[\left\langle \frac{x_6}{x_1} \right\rangle, \left\langle \frac{x_2\,x_3}{x_5\,x_0}, \frac{x_6}{x_5} \right\rangle, \left\langle \frac{x_0\,{x_6}^2}{x_2\,x_4\,x_3}, \frac{x_6}{x_4} \right\rangle] \mapsto [1, 2, 10]^{\vee},$$
$$[\left\langle \frac{x_2}{x_1}, \frac{x_6}{x_1} \right\rangle, \left\langle \frac{x_2^2 x_3}{x_5\,x_0\,x_6}, \frac{x_2\,x_3}{x_5\,x_0} \right\rangle, \left\langle \frac{x_2}{x_4}, \frac{x_6}{x_4} \right\rangle] \mapsto [2, 8, 10]^{\vee},$$
...],

$$[[\left\langle \frac{x_2}{x_3}, \frac{x_1}{x_3}, \frac{x_5}{x_3} \right\rangle, \left\langle \frac{x_1}{x_0}, \frac{x_5}{x_0}, \frac{x_2}{x_0} \right\rangle, \left\langle \frac{x_1^2 x_2}{x_5\,x_4\,x_6}, \frac{x_1\,{x_2}^2}{x_5\,x_4\,x_6}, \frac{x_1\,x_2}{x_4\,x_6} \right\rangle] \mapsto [9, 14]^{\vee},$$
$$[\left\langle \frac{x_6}{x_0}, \frac{x_2}{x_0} \right\rangle, \left\langle \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{{x_6}^2}{x_2\,x_3}, \frac{x_5\,x_6}{x_1\,x_3}, \frac{x_2}{x_3} \right\rangle, \left\langle \frac{x_1\,{x_2}^2}{x_5\,x_4\,x_6}, \frac{x_6\,x_1}{x_5\,x_4}, \frac{x_2}{x_4}, \frac{x_6}{x_4} \right\rangle] \mapsto [2, 9]^{\vee},$$
$$[\left\langle \frac{x_6}{x_0}, \frac{x_1}{x_0}, \frac{x_2}{x_0} \right\rangle, \left\langle \frac{x_6^2}{x_2\,x_3}, \frac{x_2}{x_3}, \frac{x_1\,x_6}{x_2\,x_3}, \frac{x_1}{x_3} \right\rangle, \left\langle \frac{x_1^2 x_2}{x_5\,x_4\,x_6}, \frac{{x_1}^2}{x_5\,x_4}, \frac{x_1\,{x_2}^2}{x_5\,x_4\,x_6}, \frac{x_1\,x_6}{x_5\,x_4} \right\rangle] \mapsto [9, 20]^{\vee},$$
$$[\left\langle \frac{x_4\,x_6}{x_1\,x_0}, \frac{x_6}{x_0}, \frac{x_6}{x_1} \right\rangle, \left\langle \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{{x_6}^2}{x_2\,x_3}, \frac{x_6}{x_1} \right\rangle, \left\langle \frac{x_6}{x_0}, \frac{x_2\,x_3}{x_5\,x_0}, \frac{x_6}{x_5} \right\rangle,$$
$$\left\langle \frac{x_6^2}{x_2\,x_3}, \frac{x_0\,{x_6}^2}{x_2\,x_4\,x_3}, \frac{x_6}{x_4} \right\rangle, \left\langle \frac{x_6\,x_1}{x_5\,x_4}, \frac{x_6}{x_5}, \frac{x_6}{x_4} \right\rangle] \mapsto [1, 2]^{\vee},$$
...],

$$[[\left\langle \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{x_5\,x_3}{x_1\,x_2}, \frac{{x_5}^2 x_6}{x_2\,x_1\,x_3}, \frac{x_5\,x_6}{x_1\,x_3}, \frac{{x_5}^2}{x_1\,x_2}, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_6}{x_1} \right\rangle, \left\langle \frac{x_3}{x_4}, \frac{x_5}{x_4}, \frac{x_2}{x_4}, \frac{x_6}{x_4} \right\rangle,$$
$$\left\langle \frac{x_3^2}{x_0\,x_6}, \frac{x_6}{x_0}, \frac{{x_2}^2 x_3}{x_5\,x_0\,x_6}, \frac{x_2\,{x_3}^2}{x_5\,x_0\,x_6}, \frac{x_2\,x_3}{x_5\,x_0}, \frac{x_5\,x_3}{x_0\,x_6}, \frac{x_5}{x_0}, \frac{x_2}{x_0} \right\rangle] \mapsto [8]^{\vee},$$
$$[\left\langle \frac{x_3^2}{x_0\,x_6}, \frac{x_6}{x_0}, \frac{x_1\,x_3}{x_0\,x_6}, \frac{x_1}{x_0}, \frac{x_5\,x_3}{x_0\,x_6}, \frac{x_5}{x_0} \right\rangle, \left\langle \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{{x_6}^2}{x_2\,x_3}, \frac{x_5\,x_3}{x_1\,x_2}, \frac{x_3}{x_2}, \frac{{x_5}^2 x_6}{x_2\,x_1\,x_3}, \frac{{x_5}^2}{x_1\,x_2}, \frac{x_1\,x_6}{x_2\,x_3}, \frac{x_1}{x_2} \right\rangle,$$
$$\left\langle \frac{x_1^2}{x_5\,x_4}, \frac{x_1\,x_3}{x_5\,x_4}, \frac{x_3}{x_4}, \frac{x_6\,x_1}{x_5\,x_4}, \frac{x_5}{x_4}, \frac{x_6}{x_4} \right\rangle] \mapsto [15]^{\vee},$$
$$[\left\langle \frac{x_4\,{x_3}^2}{x_0\,x_6\,x_1}, \frac{{x_3}^2}{x_0\,x_6}, \frac{x_4\,x_6}{x_1\,x_0}, \frac{x_6}{x_0}, \frac{x_3}{x_1}, \frac{x_6}{x_1} \right\rangle, \left\langle \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{{x_6}^2}{x_2\,x_3}, \frac{x_5\,x_3}{x_1\,x_2}, \frac{x_3}{x_2}, \frac{x_3}{x_1}, \frac{x_6}{x_1} \right\rangle,$$
$$\left\langle \frac{x_1\,x_3}{x_5\,x_4}, \frac{x_3}{x_4}, \frac{x_6\,x_1}{x_5\,x_4}, \frac{x_6}{x_5}, \frac{x_3}{x_5}, \frac{x_6}{x_4} \right\rangle, \left\langle \frac{x_3^2}{x_0\,x_6}, \frac{x_6}{x_0}, \frac{x_2\,{x_3}^2}{x_5\,x_0\,x_6}, \frac{x_2\,x_3}{x_5\,x_0}, \frac{x_6}{x_5}, \frac{x_3}{x_5} \right\rangle,$$
$$\left\langle \frac{x_6^2}{x_2\,x_3}, \frac{x_3}{x_2}, \frac{x_3}{x_4}, \frac{x_0\,{x_6}^2}{x_2\,x_4\,x_3}, \frac{x_0\,x_6}{x_2\,x_4}, \frac{x_6}{x_4} \right\rangle] \mapsto [1]^{\vee},$$
...],

[]

with the one sheet forming the complex

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$$[],$$

$$[],$$

$$[\left\langle \frac{x_6}{x_1} \right\rangle, \left\langle \frac{x_2\,x_3}{x_5\,x_0} \right\rangle, \left\langle \frac{x_6}{x_4} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_0} \right\rangle, \left\langle \frac{x_4}{x_3} \right\rangle, \left\langle \frac{x_2\,x_4}{x_5\,x_6} \right\rangle, \left\langle \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_2^2 x_3}{x_5\,x_0\,x_6} \right\rangle,$$
$$\left\langle \frac{x_2}{x_4} \right\rangle, \ldots],$$

$$[\left\langle \frac{x_5\,x_3}{x_1\,x_2}, \frac{x_3}{x_1} \right\rangle, \left\langle \frac{x_3}{x_4} \right\rangle, \left\langle \frac{x_3^2}{x_0\,x_6}, \frac{x_2\,{x_3}^2}{x_5\,x_0\,x_6} \right\rangle, \left\langle \frac{x_2^2}{x_5\,x_6}, \frac{x_2\,x_0}{x_5\,x_6} \right\rangle, \left\langle \frac{x_0}{x_1}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_0^2}{x_3\,x_4}, \frac{x_2\,x_0}{x_3\,x_4} \right\rangle,$$
$$\left\langle \frac{x_6}{x_1} \right\rangle, \left\langle \frac{x_2\,x_3}{x_5\,x_0}, \frac{x_6}{x_5} \right\rangle, \left\langle \frac{x_0\,{x_6}^2}{x_2\,x_4\,x_3}, \frac{x_6}{x_4} \right\rangle, \left\langle \frac{x_2}{x_1}, \frac{x_6}{x_1} \right\rangle, \left\langle \frac{x_2^2 x_3}{x_5\,x_0\,x_6}, \frac{x_2\,x_3}{x_5\,x_0} \right\rangle,$$
$$\left\langle \frac{x_2}{x_4}, \frac{x_6}{x_4} \right\rangle, \ldots],$$

$$[\left\langle \frac{x_2}{x_3}, \frac{x_1}{x_3}, \frac{x_5}{x_3} \right\rangle, \left\langle \frac{x_1}{x_0}, \frac{x_5}{x_0}, \frac{x_2}{x_0} \right\rangle, \left\langle \frac{x_1^2 x_2}{x_5\,x_4\,x_6}, \frac{x_1\,{x_2}^2}{x_5\,x_4\,x_6}, \frac{x_1\,x_2}{x_4\,x_6} \right\rangle,$$
$$\left\langle \frac{x_6}{x_0}, \frac{x_2}{x_0} \right\rangle, \left\langle \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{{x_6}^2}{x_2\,x_3}, \frac{x_5\,x_6}{x_1\,x_3}, \frac{x_2}{x_3} \right\rangle, \left\langle \frac{x_1\,{x_2}^2}{x_5\,x_4\,x_6}, \frac{x_6\,x_1}{x_5\,x_4}, \frac{x_2}{x_4}, \frac{x_6}{x_4} \right\rangle,$$
$$\left\langle \frac{x_6}{x_0}, \frac{x_1}{x_0}, \frac{x_2}{x_0} \right\rangle, \left\langle \frac{x_6^2}{x_2\,x_3}, \frac{x_2}{x_3}, \frac{x_6\,x_1}{x_2\,x_3}, \frac{x_1}{x_3} \right\rangle, \left\langle \frac{x_1^2 x_2}{x_5\,x_4\,x_6}, \frac{{x_1}^2}{x_5\,x_4}, \frac{x_1\,{x_2}^2}{x_5\,x_4\,x_6}, \frac{x_6\,x_1}{x_5\,x_4} \right\rangle,$$
$$\left\langle \frac{x_4\,x_6}{x_1\,x_0}, \frac{x_6}{x_0}, \frac{x_6}{x_1} \right\rangle, \left\langle \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{{x_6}^2}{x_2\,x_3}, \frac{x_6}{x_1} \right\rangle, \left\langle \frac{x_6}{x_0}, \frac{x_2\,x_3}{x_5\,x_0}, \frac{x_6}{x_5} \right\rangle, \left\langle \frac{x_6^2}{x_2\,x_3}, \frac{x_0\,{x_6}^2}{x_2\,x_4\,x_3}, \frac{x_6}{x_4} \right\rangle,$$
$$\left\langle \frac{x_6\,x_1}{x_5\,x_4}, \frac{x_6}{x_5}, \frac{x_6}{x_4} \right\rangle, \ldots],$$

$$[\left\langle \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{x_5\,x_3}{x_1\,x_2}, \frac{{x_5}^2 x_6}{x_2\,x_1\,x_3}, \frac{x_5\,x_6}{x_1\,x_3}, \frac{{x_5}^2}{x_1\,x_2}, \frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_6}{x_1} \right\rangle,$$
$$\left\langle \frac{x_3}{x_4}, \frac{x_5}{x_4}, \frac{x_2}{x_4}, \frac{x_6}{x_4} \right\rangle, \left\langle \frac{x_3^2}{x_0\,x_6}, \frac{x_6}{x_0}, \frac{{x_2}^2 x_3}{x_5\,x_0\,x_6}, \frac{x_2\,{x_3}^2}{x_5\,x_0\,x_6}, \frac{x_2\,x_3}{x_5\,x_0}, \frac{x_5\,x_3}{x_0\,x_6}, \frac{x_5}{x_0}, \frac{x_2}{x_0} \right\rangle,$$
$$\left\langle \frac{x_3^2}{x_0\,x_6}, \frac{x_6}{x_0}, \frac{x_1\,x_3}{x_0\,x_6}, \frac{x_1}{x_0}, \frac{x_5\,x_3}{x_0\,x_6}, \frac{x_5}{x_0} \right\rangle, \left\langle \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{{x_6}^2}{x_2\,x_3}, \frac{x_5\,x_3}{x_1\,x_2}, \frac{x_3}{x_2}, \frac{{x_5}^2 x_6}{x_2\,x_1\,x_3}, \frac{{x_5}^2}{x_1\,x_2}, \frac{x_1\,x_6}{x_2\,x_3}, \frac{x_1}{x_2} \right\rangle,$$
$$\left\langle \frac{x_1^2}{x_5\,x_4}, \frac{x_1\,x_3}{x_5\,x_4}, \frac{x_3}{x_4}, \frac{x_1\,x_6}{x_5\,x_4}, \frac{x_5}{x_4}, \frac{x_6}{x_4} \right\rangle, \left\langle \frac{x_4\,{x_3}^2}{x_0\,x_6\,x_1}, \frac{{x_3}^2}{x_0\,x_6}, \frac{x_4\,x_6}{x_1\,x_0}, \frac{x_6}{x_0}, \frac{x_3}{x_1}, \frac{x_6}{x_1} \right\rangle,$$
$$\left\langle \frac{x_5\,{x_6}^2}{x_2\,x_1\,x_3}, \frac{{x_6}^2}{x_2\,x_3}, \frac{x_5\,x_3}{x_1\,x_2}, \frac{x_3}{x_2}, \frac{x_3}{x_1}, \frac{x_6}{x_1} \right\rangle, \left\langle \frac{x_1\,x_3}{x_5\,x_4}, \frac{x_3}{x_4}, \frac{x_6\,x_1}{x_5\,x_4}, \frac{x_6}{x_5}, \frac{x_3}{x_5}, \frac{x_6}{x_4} \right\rangle,$$
$$\left\langle \frac{x_3^2}{x_0\,x_6}, \frac{x_6}{x_0}, \frac{x_2\,{x_3}^2}{x_5\,x_0\,x_6}, \frac{x_2\,x_3}{x_5\,x_0}, \frac{x_6}{x_5}, \frac{x_3}{x_5} \right\rangle, \left\langle \frac{x_6^2}{x_2\,x_3}, \frac{x_3}{x_2}, \frac{x_3}{x_4}, \frac{x_0\,{x_6}^2}{x_2\,x_4\,x_3}, \frac{x_0\,x_6}{x_2\,x_4}, \frac{x_6}{x_4} \right\rangle, \ldots],$$

$$[]$$

Note that the torus invariant basis of deformations corresponding to a Bergman face is given by the set of all lattice points of the polytope specified above.

10.4.7 Limit map

The limit map $\lim : B(I) \to \text{Poset}(\Delta)$ associates to a face F of $B(I)$ the face of Δ formed by the limit points of arcs lying over the weight vectors $w \in F$, i.e., with lowest order term t^w. Labeling the faces of the Bergman complex $B(I) \subset \text{Poset}(\nabla)$ and the faces of $\text{Poset}(\Delta)$ by the corresponding dual faces of ∇^* and Δ^*, hence considering the limit map $\lim : B(I) \rightarrow \text{Poset}(\Delta)$ as a map $B(I)^* \rightarrow \text{Poset}(\Delta^*)$, the limit correspondence is given by

$$
\begin{array}{l}
[],\\
[\left\langle \begin{array}{l} y_9, y_7, y_2, y_{27}, y_{21}, y_{50}, y_{36}, y_1, y_{32}, y_5, y_4, y_{26}, y_{35},\\ y_{20}, y_{24}, y_{58}, y_{54}, y_{70}, y_{45}, y_{43}, y_{62}, y_{47}, y_{65}, y_{68} \end{array} \right\rangle \mapsto \langle x_1, x_4, x_0 \rangle ,\\
\left\langle \begin{array}{l} y_9, y_7, y_{27}, y_{30}, y_{42}, y_{50}, y_{36}, y_{56}, y_1, y_{38}, y_{25}, y_{53}, y_{35},\\ y_{24}, y_{40}, y_{54}, y_8, y_{39}, y_{31}, y_{28}, y_{55}, y_{70}, y_{45}, y_{68} \end{array} \right\rangle \mapsto \langle x_2, x_4, x_0 \rangle ,\\
\left\langle \begin{array}{l} y_9, y_2, y_{27}, y_{30}, y_{21}, y_{50}, y_{36}, y_{56}, y_4, y_{26}, y_{40}, y_{54}, y_8,\\ y_{39}, y_{63}, y_{28}, y_{62}, y_{47}, y_{67}, y_{68} \end{array} \right\rangle \mapsto \langle x_1, x_2, x_4, x_5, x_0 \rangle ,\\
...],\\
[\langle y_{11}, y_{53}, y_{14}, y_{34}, y_{57}, y_{45}, y_{43}, y_{61}, y_{66} \rangle \mapsto \langle x_3, x_4, x_6, x_0 \rangle ,\\
\langle y_9, y_{30}, y_{50}, y_{32}, y_{14}, y_8, y_{41}, y_{57}, y_{39}, y_{43}, y_{65}, y_{68} \rangle \mapsto \langle x_1, x_3, x_4, x_5, x_0 \rangle ,\\
\langle y_{30}, y_{50}, y_{11}, y_{38}, y_{53}, y_{14}, y_{12}, y_{57}, y_{39}, y_{31}, y_{43}, y_{61} \rangle \mapsto \langle x_3, x_4, x_5, x_0 \rangle ,\\
\langle y_9, y_{30}, y_{21}, y_{50}, y_{26}, y_8, y_{39}, y_{63}, y_{47}, y_{68} \rangle \mapsto \langle x_1, x_2, x_3, x_4, x_5, x_0 \rangle ,\\
...],\\
[\langle y_2, y_{27}, y_{36}, y_4, y_{54}, y_{62} \rangle \mapsto \langle x_1, x_2, x_4, x_5, x_6, x_0 \rangle ,\\
\langle y_{16}, y_{59}, y_{18}, y_{58}, y_{17}, y_{41} \rangle \mapsto \langle x_1, x_3, x_4, x_5, x_6 \rangle ,\\
\langle y_{26}, y_8, y_{39}, y_{63}, y_{47}, y_{68} \rangle \mapsto \langle x_1, x_2, x_3, x_4, x_5, x_0 \rangle ,\\
\langle y_5, y_{26}, y_{58}, y_{47}, y_{65}, y_{68} \rangle \mapsto \langle x_1, x_3, x_4, x_5, x_0 \rangle ,\\
...],\\
[\langle y_{26}, y_{47}, y_{68} \rangle \mapsto \langle x_1, x_2, x_3, x_4, x_5, x_0 \rangle ,\\
\langle y_{22}, y_{51}, y_{19} \rangle \mapsto \langle x_1, x_2, x_3, x_5, x_6, x_0 \rangle ,\\
\langle y_5, y_{58}, y_{65} \rangle \mapsto \langle x_1, x_3, x_4, x_5, x_6, x_0 \rangle ,\\
...],\\
[],\\
[],\\
[]
\end{array}
$$

Every zero dimensional stratum of $\mathbb{P}^6$ is the limit of 5 Bergman faces of dimension three, one tetrahedron, two pyramids and two prisms. Figure 10.1 shows a projection into 3-space of the set of these Bergman faces for one zero dimensional stratum. The union of the faces $F \in B(I)$ which have as limit $\lim(F) = p$ the same zero dimensional strata p of $\mathbb{P}^6$ is not convex.

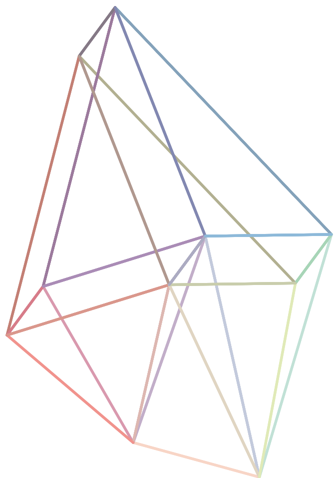


Figure 10.1: Projection of Bergman faces with same zero dimensional limit for the general degree 14 Pfaffian Calabi-Yau degeneration

10.4.8 Mirror complex

Numbering the vertices of the mirror complex $\mu(B(I))$ as

$$\begin{array}{ll} 1=(6,-1,-1,-1,-1,-1) & 2=(-1,6,-1,-1,-1,-1) \\ 3=(-1,-1,6,-1,-1,-1) & 4=(-1,-1,-1,6,-1,-1) \\ 5=(-1,-1,-1,-1,6,-1) & 6=(-1,-1,-1,-1,-1,6) \\ 7=(-1,-1,-1,-1,-1,-1) & \end{array}$$

$\mu(B(I)) \subset \operatorname{Poset}(\Delta)$ is

[],

[[4], [2], [6], [5], [7], [3], [1]],

[[2, 6], [2, 5], [3, 7], [1, 5], [1, 7], [4, 7], [3, 6], [1, 4], [2, 3], [2, 7], [4, 6], [2, 4], [6, 7], [3, 4], [3, 5], [1, 3], [1, 2], [1, 6], [5, 6], [4, 5], [5, 7]],

[[1, 2, 6], [1, 2, 5], [2, 3, 7], [1, 3, 5], [1, 5, 6], [1, 4, 6], [1, 3, 6], [1, 3, 7], [1, 5, 7], [1, 2, 4], [2, 4, 5], [4, 5, 7], [1, 4, 5], [3, 6, 7], [2, 6, 7], [4, 6, 7], [3, 4, 6], [3, 4, 7], [2, 5, 6], [2, 3, 6], [2, 3, 5], [2, 4, 6], [2, 4, 7], [3, 5, 6], [2, 5, 7], [3, 5, 7], [1, 3, 4], [1, 4, 7]],

[[2, 3, 5, 6], [1, 3, 5, 6], [1, 2, 5, 6], [2, 3, 6, 7], [2, 3, 5, 7], [1, 3, 5, 7], [3, 4, 6, 7], [2, 4, 6, 7], [1, 3, 4, 7], [1, 3, 4, 6], [1, 2, 4, 6], [2, 4, 5, 7], [1, 4, 5, 7], [1, 2, 4, 5]],

[],

[],

[]

The ordering of the faces of $\mu(B(I))$ is compatible with above ordering of the faces of $B(I)$. The F-vectors of the faces of $\mu(B(I))^*$ are

Dimension	Number of faces	F-vector	
2	14	$(1,3,3,1,0,0,0,0)$	triangle
3	28	$(1,4,6,4,1,0,0,0)$	tetrahedron
4	21	$(1,5,10,10,5,1,0,0)$	
5	7	$(1,6,15,20,15,6,1,0)$	

The first order deformations of the mirror special fiber X_0° correspond to the lattice points of the dual complex $(\mu(B(I)))^* \subset \operatorname{Poset}(\Delta^*)$ of the mirror complex. We label the first order deformations of X_0° corresponding to vertices of Δ^* by the homogeneous coordinates of Y

$$x_1 = x_{(1,0,0,0,0,0)} = \frac{y_{11}^2 y_{12}^2 y_{13}\, y_{14}\, y_{25}\, y_{31}\, y_{34}\, y_{37}\, y_{38}^2 y_{39}\, y_{40}\, y_{48}\, y_{53}\, y_{55}\, y_{61}}{y_1\, y_2\, y_3\, y_6\, y_7\, y_9\, y_{20}\, y_{21}\, y_{22}\, y_{23}\, y_{32}\, y_{33}\, y_{35}\, y_{36}\, y_{47}\, y_{58}\, y_{59}\, y_{62}}$$

$$x_2 = x_{(0,1,0,0,0,0)} = \frac{y_4\, y_5^2 y_{11}\, y_{14}^2 y_{17}\, y_{18}^2 y_{19}\, y_{20}\, y_{26}\, y_{34}\, y_{41}\, y_{43}\, y_{57}\, y_{58}\, y_{65}}{y_1\, y_6\, y_7\, y_8\, y_9\, y_{10}\, y_{28}\, y_{29}\, y_{30}\, y_{31}\, y_{33}\, y_{35}\, y_{36}\, y_{42}\, y_{55}\, y_{56}\, y_{60}\, y_{64}}$$

$$x_3 = x_{(0,0,1,0,0,0)} = \frac{y_2^2 y_3\, y_4^2 y_5\, y_{23}\, y_{24}\, y_{25}\, y_{26}\, y_{27}^2 y_{36}\, y_{40}\, y_{54}\, y_{56}\, y_{62}\, y_{67}}{y_1\, y_8\, y_9\, y_{10}\, y_{12}\, y_{13}\, y_{15}\, y_{16}\, y_{29}\, y_{30}\, y_{31}\, y_{32}\, y_{37}\, y_{41}\, y_{51}\, y_{57}\, y_{61}\, y_{66}}$$

$$x_4 = x_{(0,0,0,1,0,0)} = \frac{y_2\, y_3^2 y_6^2 y_7\, y_{19}\, y_{20}\, y_{21}\, y_{22}^2 y_{23}\, y_{29}\, y_{42}\, y_{46}\, y_{51}\, y_{60}\, y_{69}}{y_8\, y_{10}\, y_{11}\, y_{12}\, y_{13}\, y_{14}\, y_{15}\, y_{16}\, y_{28}\, y_{34}\, y_{38}\, y_{39}\, y_{40}\, y_{41}\, y_{54}\, y_{65}\, y_{68}\, y_{70}}$$

$$x_5 = x_{(0,0,0,0,1,0)} = \frac{y_1^2 y_6\, y_7^2 y_9\, y_{15}\, y_{24}\, y_{32}\, y_{33}\, y_{35}^2 y_{36}\, y_{42}\, y_{44}\, y_{45}\, y_{66}\, y_{70}}{y_4\, y_5\, y_{11}\, y_{12}\, y_{13}\, y_{14}\, y_{17}\, y_{18}\, y_{19}\, y_{26}\, y_{37}\, y_{38}\, y_{39}\, y_{40}\, y_{52}\, y_{63}\, y_{67}\, y_{69}}$$

$$x_6 = x_{(0,0,0,0,0,1)} = \frac{y_1\, y_8^2 y_9^2 y_{10}\, y_{21}\, y_{28}\, y_{29}\, y_{30}^2 y_{31}\, y_{32}\, y_{39}\, y_{47}\, y_{50}\, y_{63}\, y_{68}}{y_2\, y_3\, y_4\, y_5\, y_{11}\, y_{14}\, y_{17}\, y_{18}\, y_{19}\, y_{23}\, y_{24}\, y_{25}\, y_{27}\, y_{34}\, y_{44}\, y_{46}\, y_{48}\, y_{49}}$$

$$x_0 = x_{(-1,-1,-1,-1,-1,-1)} = \frac{y_8\, y_{10}^2 y_{12}\, y_{13}^2 y_{15}\, y_{16}^2 y_{17}\, y_{28}\, y_{33}\, y_{37}\, y_{41}\, y_{49}\, y_{52}\, y_{59}\, y_{64}}{y_2\, y_3\, y_4\, y_5\, y_6\, y_7\, y_{20}\, y_{21}\, y_{22}\, y_{24}\, y_{25}\, y_{26}\, y_{27}\, y_{42}\, y_{43}\, y_{45}\, y_{50}\, y_{53}}$$

So writing the vertices of the faces of $(\mu(B(I)))^*$ as homogeneous coordinates of Y, the complex $(\mu(B(I)))^*$ is given by

[],

[],

[],

$[\langle x_1, x_4, x_0\rangle, \langle x_2, x_4, x_0\rangle, \langle x_3, x_4, x_0\rangle, \langle x_1, x_4, x_5\rangle, \langle x_1, x_4, x_6\rangle,$
$\langle x_2, x_4, x_6\rangle, \langle x_1, x_2, x_5\rangle, \langle x_1, x_3, x_5\rangle, \langle x_2, x_5, x_6\rangle, \langle x_2, x_5, x_0\rangle,$
$\langle x_3, x_5, x_0\rangle, \langle x_1, x_3, x_6\rangle, \langle x_2, x_3, x_6\rangle, \langle x_3, x_6, x_0\rangle],$

$[\langle x_3, x_4, x_5, x_0\rangle, \langle x_3, x_4, x_6, x_0\rangle, \langle x_1, x_4, x_5, x_6\rangle, \langle x_2, x_4, x_6, x_0\rangle,$
$\langle x_2, x_3, x_4, x_0\rangle, \langle x_2, x_3, x_5, x_0\rangle, \langle x_2, x_4, x_5, x_0\rangle, \langle x_2, x_4, x_5, x_6\rangle,$

$\langle x_2, x_3, x_4, x_6\rangle, \langle x_3, x_5, x_6, x_0\rangle, \langle x_1, x_3, x_6, x_0\rangle, \langle x_1, x_2, x_3, x_6\rangle,$
$\langle x_2, x_3, x_6, x_0\rangle, \langle x_1, x_2, x_4, x_5\rangle, \langle x_1, x_3, x_4, x_5\rangle, \langle x_1, x_2, x_3, x_5\rangle,$
$\langle x_1, x_2, x_5, x_0\rangle, \langle x_1, x_2, x_5, x_6\rangle, \langle x_1, x_3, x_4, x_0\rangle, \langle x_1, x_4, x_5, x_0\rangle,$
$\langle x_1, x_4, x_6, x_0\rangle, \langle x_1, x_3, x_5, x_0\rangle, \langle x_1, x_3, x_5, x_6\rangle, \langle x_1, x_2, x_4, x_0\rangle,$
$\langle x_1, x_3, x_4, x_6\rangle, \langle x_1, x_2, x_4, x_6\rangle, \langle x_2, x_5, x_6, x_0\rangle, \langle x_2, x_3, x_5, x_6\rangle],$

$[\langle x_1, x_3, x_4, x_5, x_0\rangle, \langle x_1, x_3, x_4, x_6, x_0\rangle, \langle x_1, x_2, x_4, x_5, x_6\rangle, \langle x_2, x_3, x_4, x_6, x_0\rangle,$
$\langle x_2, x_3, x_4, x_5, x_6\rangle, \langle x_1, x_2, x_3, x_5, x_6\rangle, \langle x_1, x_2, x_4, x_5, x_0\rangle, \langle x_2, x_3, x_5, x_6, x_0\rangle,$
$\langle x_1, x_4, x_5, x_6, x_0\rangle, \langle x_1, x_3, x_4, x_5, x_6\rangle, \langle x_1, x_2, x_3, x_5, x_0\rangle, \langle x_1, x_3, x_5, x_6, x_0\rangle,$
$\langle x_1, x_2, x_3, x_4, x_5\rangle, \langle x_1, x_2, x_5, x_6, x_0\rangle, \langle x_1, x_2, x_4, x_6, x_0\rangle, \langle x_2, x_4, x_5, x_6, x_0\rangle,$
$\langle x_3, x_4, x_5, x_6, x_0\rangle, \langle x_2, x_3, x_4, x_5, x_0\rangle, \langle x_1, x_2, x_3, x_4, x_0\rangle, \langle x_1, x_2, x_3, x_6, x_0\rangle,$
$\langle x_1, x_2, x_3, x_4, x_6\rangle],$

$[\langle x_1, x_2, x_3, x_5, x_6, x_0\rangle, \langle x_1, x_3, x_4, x_5, x_6, x_0\rangle, \langle x_1, x_2, x_3, x_4, x_5, x_0\rangle,$
$\langle x_1, x_2, x_3, x_4, x_6, x_0\rangle, \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle, \langle x_1, x_2, x_4, x_5, x_6, x_0\rangle,$
$\langle x_2, x_3, x_4, x_5, x_6, x_0\rangle],$

$[]$

The complex $\mu(B(I))^*$ labeled by the variables of the Cox ring S gives the ideals of the toric strata of the special fiber X_0 of $\mathfrak{X}$, i.e., the complex $SP(I_0)$, so in particular the primary decomposition of I_0 is

$$I_0 = \begin{array}{l} \langle x_2, x_4, x_0\rangle \cap \langle x_3, x_4, x_0\rangle \cap \langle x_1, x_4, x_0\rangle \cap \langle x_1, x_4, x_5\rangle \cap \\ \cap \langle x_3, x_6, x_0\rangle \cap \langle x_2, x_5, x_0\rangle \cap \langle x_3, x_5, x_0\rangle \cap \langle x_1, x_3, x_6\rangle \cap \\ \cap \langle x_2, x_3, x_6\rangle \cap \langle x_2, x_5, x_6\rangle \cap \langle x_1, x_4, x_6\rangle \cap \langle x_2, x_4, x_6\rangle \cap \\ \cap \langle x_1, x_2, x_5\rangle \cap \langle x_1, x_3, x_5\rangle \end{array}$$

10.4.9 Covering structure in the deformation complex of the mirror degeneration

Each face of the complex of deformations $(\mu(B(I)))^*$ of the mirror special fiber X_0° decomposes as the convex hull of 3,4,5,6 respectively 7 polytopes forming a $7:1$ ramified covering of $\mu(B(I))^\vee$

$[],$

$[],$

$[],$

$[[\langle x_4\rangle, \langle x_0\rangle, \langle x_1, x_0\rangle, \langle x_1\rangle] \mapsto \langle x_1, x_4, x_0\rangle^{*\vee} = [2,3,5,6]^\vee,$
$[\langle x_4\rangle, \langle x_2\rangle, \langle x_0\rangle] \mapsto \langle x_2, x_4, x_0\rangle^{*\vee} = [1,3,5,6]^\vee,$

$[\langle x_4, x_5\rangle, \langle x_2\rangle, \langle x_5\rangle, \langle x_0\rangle, \langle x_1, x_0\rangle, \langle x_1, x_2\rangle, \langle x_4\rangle] \mapsto \langle x_1, x_2, x_4, x_5, x_0\rangle^{*\vee} = [3,6]^\vee$,
...],

$[[\langle x_4\rangle, \langle x_3\rangle, \langle x_6\rangle, \langle x_6, x_0\rangle, \langle x_0\rangle, \langle x_3, x_4\rangle] \mapsto \langle x_3, x_4, x_6, x_0\rangle^{*\vee} = [1,2,5]^\vee$,
$[\langle x_4, x_5\rangle, \langle x_3\rangle, \langle x_5\rangle, \langle x_0\rangle, \langle x_1, x_0\rangle, \langle x_1\rangle, \langle x_3, x_4\rangle] \mapsto \langle x_1, x_3, x_4, x_5, x_0\rangle^{*\vee} = [2,6]^\vee$,
$[\langle x_4, x_5\rangle, \langle x_3\rangle, \langle x_5\rangle, \langle x_0\rangle, \langle x_3, x_4\rangle] \mapsto \langle x_3, x_4, x_5, x_0\rangle^{*\vee} = [1,2,6]^\vee$,
$[\langle x_4, x_5\rangle, \langle x_2, x_3\rangle, \langle x_5\rangle, \langle x_0\rangle, \langle x_1, x_0\rangle, \langle x_1, x_2\rangle, \langle x_3, x_4\rangle] \mapsto \langle x_1, x_2, x_3, x_4, x_5, x_0\rangle^{*\vee} = [6]^\vee$,
...],

$[[\langle x_4, x_5\rangle, \langle x_2\rangle, \langle x_5, x_6\rangle, \langle x_6, x_0\rangle, \langle x_1, x_0\rangle, \langle x_1, x_2\rangle, \langle x_4\rangle] \mapsto \langle x_1, x_2, x_4, x_5, x_6, x_0\rangle^{*\vee} = [3]^\vee$,
$[\langle x_4, x_5\rangle, \langle x_3\rangle, \langle x_5, x_6\rangle, \langle x_6\rangle, \langle x_1\rangle, \langle x_3, x_4\rangle] \mapsto \langle x_1, x_3, x_4, x_5, x_6\rangle^{*\vee} = [2,7]^\vee$,
$[\langle x_4, x_5\rangle, \langle x_2, x_3\rangle, \langle x_5\rangle, \langle x_0\rangle, \langle x_1, x_0\rangle, \langle x_1, x_2\rangle, \langle x_3, x_4\rangle] \mapsto \langle x_1, x_2, x_3, x_4, x_5, x_0\rangle^{*\vee} = [6]^\vee$,
$[\langle x_4, x_5\rangle, \langle x_3\rangle, \langle x_5\rangle, \langle x_0\rangle, \langle x_1, x_0\rangle, \langle x_1\rangle, \langle x_3, x_4\rangle] \mapsto \langle x_1, x_3, x_4, x_5, x_0\rangle^{*\vee} = [2,6]^\vee$,
...],

$[[\langle x_4, x_5\rangle, \langle x_2, x_3\rangle, \langle x_5\rangle, \langle x_0\rangle, \langle x_1, x_0\rangle, \langle x_1, x_2\rangle, \langle x_3, x_4\rangle] \mapsto \langle x_1, x_2, x_3, x_4, x_5, x_0\rangle^{*\vee} = [6]^\vee$,
$[\langle x_5\rangle, \langle x_2, x_3\rangle, \langle x_5, x_6\rangle, \langle x_6, x_0\rangle, \langle x_1, x_0\rangle, \langle x_1, x_2\rangle, \langle x_3\rangle] \mapsto \langle x_1, x_2, x_3, x_5, x_6, x_0\rangle^{*\vee} = [4]^\vee$,
$[\langle x_4, x_5\rangle, \langle x_3\rangle, \langle x_5, x_6\rangle, \langle x_6, x_0\rangle, \langle x_1, x_0\rangle, \langle x_1\rangle, \langle x_3, x_4\rangle] \mapsto \langle x_1, x_3, x_4, x_5, x_6, x_0\rangle^{*\vee} = [2]^\vee$,
...],

[]

Due to the singularities of Y° this covering involves degenerate faces, i.e., faces $G \mapsto F^\vee$ with $\dim(G) < \dim(F^\vee)$.

10.4.10 Mirror degeneration

The space of first order deformations of X_0° in the mirror degeneration $\mathfrak{X}^\circ$ has dimension 7 and the deformations represented by the monomials

$$\left\{ \begin{array}{c} \frac{y_8\, y_{10}^2 y_{12}\, y_{13}^2 y_{15}\, y_{16}^2 y_{17}\, y_{28}\, y_{33}\, y_{37}\, y_{41}\, y_{49}\, y_{52}\, y_{59}\, y_{64}}{y_2\, y_3\, y_4\, y_5\, y_6\, y_7\, y_{20}\, y_{21}\, y_{22}\, y_{24}\, y_{25}\, y_{26}\, y_{27}\, y_{42}\, y_{43}\, y_{45}\, y_{50}\, y_{53}} \\ \frac{y_1\, y_8^2 y_9^2 y_{10}\, y_{21}\, y_{28}\, y_{29}\, y_{30}^2 y_{31}\, y_{32}\, y_{39}\, y_{47}\, y_{50}\, y_{63}\, y_{68}}{y_2\, y_3\, y_4\, y_5\, y_{11}\, y_{14}\, y_{17}\, y_{18}\, y_{19}\, y_{23}\, y_{24}\, y_{25}\, y_{27}\, y_{34}\, y_{44}\, y_{46}\, y_{48}\, y_{49}} \\ \frac{y_1^2 y_6\, y_7^2 y_9\, y_{15}\, y_{24}\, y_{32}\, y_{33}\, y_{35}^2 y_{36}\, y_{42}\, y_{44}\, y_{45}\, y_{66}\, y_{70}}{y_4\, y_5\, y_{11}\, y_{12}\, y_{13}\, y_{14}\, y_{17}\, y_{18}\, y_{19}\, y_{26}\, y_{37}\, y_{38}\, y_{39}\, y_{40}\, y_{52}\, y_{63}\, y_{67}\, y_{69}} \\ \frac{y_2^2 y_3\, y_4^2 y_5\, y_{23}\, y_{24}\, y_{25}\, y_{26}\, y_{27}^2 y_{36}\, y_{40}\, y_{54}\, y_{56}\, y_{62}\, y_{67}}{y_1\, y_8\, y_9\, y_{10}\, y_{12}\, y_{13}\, y_{15}\, y_{16}\, y_{29}\, y_{30}\, y_{31}\, y_{32}\, y_{37}\, y_{41}\, y_{51}\, y_{57}\, y_{61}\, y_{66}} \\ \frac{y_4\, y_5^2 y_{11}\, y_{14}^2 y_{17}\, y_{18}^2 y_{19}\, y_{20}\, y_{26}\, y_{34}\, y_{41}\, y_{43}\, y_{57}\, y_{58}\, y_{65}}{y_1\, y_6\, y_7\, y_8\, y_9\, y_{10}\, y_{28}\, y_{29}\, y_{30}\, y_{31}\, y_{33}\, y_{35}\, y_{36}\, y_{42}\, y_{55}\, y_{56}\, y_{60}\, y_{64}} \\ \frac{y_2\, y_3^2 y_6^2 y_7\, y_{19}\, y_{20}\, y_{21}\, y_{22}^2 y_{23}\, y_{29}\, y_{42}\, y_{46}\, y_{51}\, y_{60}\, y_{69}}{y_8\, y_{10}\, y_{11}\, y_{12}\, y_{13}\, y_{14}\, y_{15}\, y_{16}\, y_{28}\, y_{34}\, y_{38}\, y_{39}\, y_{40}\, y_{41}\, y_{54}\, y_{65}\, y_{68}\, y_{70}} \\ \frac{y_{11}^2 y_{12}^2 y_{13}\, y_{14}\, y_{25}\, y_{31}\, y_{34}\, y_{37}\, y_{38}^2 y_{39}\, y_{40}\, y_{48}\, y_{53}\, y_{55}\, y_{61}}{y_1\, y_2\, y_3\, y_6\, y_7\, y_9\, y_{20}\, y_{21}\, y_{22}\, y_{23}\, y_{32}\, y_{33}\, y_{35}\, y_{36}\, y_{47}\, y_{58}\, y_{59}\, y_{62}} \end{array} \right\}$$

form a torus invariant basis $\mathfrak{B}^\circ$. The number of lattice points of the dual of the mirror complex of I relates to the dimension $h^{1,2}(X^\circ)$ of complex moduli space of the generic fiber X° of $\mathfrak{X}^\circ$ and to the dimension $h^{1,1}(X)$ of the Kähler moduli space of the generic fiber X of $\mathfrak{X}$ via

$$\begin{aligned}|\mathrm{supp}\left(\left(\mu\left(B\left(I\right)\right)\right)^{*}\right)\cap N| &= 7 = 6+1\\ &= \dim\left(\mathrm{Aut}\left(Y^\circ\right)\right)+h^{1,2}\left(X^\circ\right)=\dim\left(T\right)+h^{1,1}\left(X\right)\end{aligned}$$

The conjectural first order mirror degeneration $\mathfrak{X}^{1\circ}\subset Y^\circ\times \mathrm{Spec}\,\mathbb{C}\left[t\right]/\langle t^2\rangle$ of $\mathfrak{X}$ is given by the ideal $I^{1\circ}\subset S^\circ\otimes\mathbb{C}\left[t\right]/\langle t^2\rangle$ generated by

$$\left\{m+\sum_{\delta\in\mathfrak{B}^\circ} t\cdot s_\delta\cdot\delta\left(m\right)\mid m\in I_0^\circ\right\}$$

10.4.11 Contraction of the mirror degeneration

In the following we give a birational map relating the degeneration $\mathfrak{X}^\circ$ to a Greene-Plesser type orbifolding mirror family by contracting divisors on Y°. In order to contract the divisors

$y_1=\frac{x_5^2x_6}{x_2\,x_1\,x_3}$	$y_2=\frac{x_4\,x_3{}^2}{x_0\,x_6\,x_1}$	$y_3=\frac{x_4^2x_3}{x_0\,x_6\,x_1}$	$y_4=\frac{x_2\,x_3{}^2}{x_5\,x_0\,x_6}$	$y_5=\frac{x_2^2x_3}{x_5\,x_0\,x_6}$
$y_6=\frac{x_5\,x_4{}^2}{x_2\,x_0\,x_1}$	$y_7=\frac{x_5^2x_4}{x_2\,x_0\,x_1}$	$y_8=\frac{x_0\,x_6{}^2}{x_2\,x_4\,x_3}$	$y_9=\frac{x_5\,x_6{}^2}{x_2\,x_1\,x_3}$	$y_{10}=\frac{x_0^2x_6}{x_2\,x_4\,x_3}$
$y_{11}=\frac{x_1^2x_2}{x_5\,x_4\,x_6}$	$y_{12}=\frac{x_0\,x_1{}^2}{x_5\,x_4\,x_3}$	$y_{13}=\frac{x_0^2x_1}{x_5\,x_4\,x_3}$	$y_{14}=\frac{x_1\,x_2{}^2}{x_5\,x_4\,x_6}$	$y_{15}=\frac{x_5\,x_0}{x_3\,x_4}$
$y_{17}=\frac{x_2\,x_0}{x_5\,x_6}$	$y_{19}=\frac{x_2\,x_4}{x_5\,x_6}$	$y_{20}=\frac{x_2\,x_4}{x_1\,x_0}$	$y_{21}=\frac{x_4\,x_6}{x_1\,x_0}$	$y_{23}=\frac{x_3\,x_4}{x_6\,x_1}$
$y_{24}=\frac{x_5\,x_3}{x_0\,x_6}$	$y_{25}=\frac{x_1\,x_3}{x_0\,x_6}$	$y_{26}=\frac{x_2\,x_3}{x_5\,x_0}$	$y_{28}=\frac{x_0\,x_6}{x_2\,x_4}$	$y_{29}=\frac{x_4\,x_6}{x_2\,x_3}$
$y_{31}=\frac{x_1\,x_6}{x_2\,x_3}$	$y_{32}=\frac{x_5\,x_6}{x_1\,x_3}$	$y_{33}=\frac{x_5\,x_0}{x_1\,x_2}$	$y_{34}=\frac{x_1\,x_2}{x_4\,x_6}$	$y_{36}=\frac{x_5\,x_3}{x_1\,x_2}$
$y_{37}=\frac{x_1\,x_0}{x_5\,x_3}$	$y_{39}=\frac{x_6\,x_1}{x_5\,x_4}$	$y_{40}=\frac{x_1\,x_3}{x_5\,x_4}$	$y_{41}=\frac{x_2\,x_0}{x_3\,x_4}$	$y_{42}=\frac{x_5\,x_4}{x_2\,x_0}$
$y_{43}=\frac{x_2}{x_0}$	$y_{44}=\frac{x_5}{x_6}$	$y_{45}=\frac{x_5}{x_0}$	$y_{46}=\frac{x_4}{x_6}$	$y_{47}=\frac{x_6}{x_1}$
$y_{48}=\frac{x_1}{x_6}$	$y_{49}=\frac{x_0}{x_6}$	$y_{50}=\frac{x_6}{x_0}$	$y_{51}=\frac{x_4}{x_3}$	$y_{52}=\frac{x_0}{x_5}$
$y_{53}=\frac{x_1}{x_0}$	$y_{54}=\frac{x_3}{x_4}$	$y_{55}=\frac{x_1}{x_2}$	$y_{56}=\frac{x_3}{x_2}$	$y_{57}=\frac{x_2}{x_3}$
$y_{58}=\frac{x_2}{x_1}$	$y_{59}=\frac{x_0}{x_1}$	$y_{60}=\frac{x_4}{x_2}$	$y_{61}=\frac{x_1}{x_3}$	$y_{62}=\frac{x_3}{x_1}$
$y_{63}=\frac{x_6}{x_5}$	$y_{64}=\frac{x_0}{x_2}$	$y_{65}=\frac{x_2}{x_4}$	$y_{66}=\frac{x_5}{x_3}$	$y_{67}=\frac{x_3}{x_5}$
$y_{68}=\frac{x_6}{x_4}$	$y_{69}=\frac{x_4}{x_5}$	$y_{70}=\frac{x_5}{x_4}$		

consider the $\mathbb{Q}$-factorial toric Fano variety $\hat{Y}^\circ=X(\hat{\Sigma}^\circ)$, where $\hat{\Sigma}^\circ=\Sigma(\hat{P}^\circ)\subset M_{\mathbb{R}}$ is the fan over the reflexive Fano polytope $\hat{P}^\circ\subset M_{\mathbb{R}}$ given as the convex hull of the remaining vertices of $P^\circ = \nabla^*$ corresponding to the Cox variables

$$\begin{array}{ll} y_{22} = y_{(-1,0,0,2,0,0)} = \frac{x_4^2}{x_1\,x_0} & y_{18} = y_{(0,2,0,0,-1,-1)} = \frac{x_2^2}{x_5\,x_6} \\ y_{30} = y_{(0,-1,-1,0,0,2)} = \frac{x_6^2}{x_2\,x_3} & y_{35} = y_{(-1,-1,0,0,2,0)} = \frac{x_5^2}{x_1\,x_2} \\ y_{16} = y_{(0,0,-1,-1,0,0)} = \frac{x_0^2}{x_3\,x_4} & y_{27} = y_{(0,0,2,0,0,-1)} = \frac{x_3^2}{x_0\,x_6} \\ y_{38} = y_{(2,0,0,-1,-1,0)} = \frac{x_1^2}{x_5\,x_4} & \end{array}$$

of the toric variety $\hat{Y}^\circ$ with Cox ring

$$\hat{S}^\circ = \mathbb{C}[y_{22}, y_{18}, y_{30}, y_{35}, y_{16}, y_{27}, y_{38}]$$

The Cox variables of $\hat{Y}^\circ$ correspond to the set of Fermat deformations of $\mathfrak{X}$. Let

$$Y^\circ = X\left(\Sigma^\circ\right) \to X(\hat{\Sigma}^\circ) = \hat{Y}^\circ$$

be a birational map from Y° to the toric Fano variety $\hat{Y}^\circ$, which contracts the divisors of the rays $\Sigma^\circ(1) - \hat{\Sigma}^\circ(1)$ corresponding to the Cox variables

$$\begin{array}{cccccccccccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 & y_9 & y_{10} & y_{11} & y_{12} & y_{13} & y_{14} & y_{15} & y_{17} \\ y_{19} & y_{20} & y_{21} & y_{23} & y_{24} & y_{25} & y_{26} & y_{28} & y_{29} & y_{31} & y_{32} & y_{33} & y_{34} & y_{36} & y_{37} & y_{39} \\ y_{40} & y_{41} & y_{42} & y_{43} & y_{44} & y_{45} & y_{46} & y_{47} & y_{48} & y_{49} & y_{50} & y_{51} & y_{52} & y_{53} & y_{54} & y_{55} \\ y_{56} & y_{57} & y_{58} & y_{59} & y_{60} & y_{61} & y_{62} & y_{63} & y_{64} & y_{65} & y_{66} & y_{67} & y_{68} & y_{69} & y_{70} & \end{array}$$

Representing $\hat{Y}^\circ$ as a quotient we have

$$\hat{Y}^\circ = \left(\mathbb{C}^7 - V\left(B(\hat{\Sigma}^\circ)\right)\right)/\hat{G}^\circ$$

with

$$\hat{G}^\circ = \mathbb{Z}_7 \times (\mathbb{C}^*)^1$$

acting via

$$\xi y = \left(u_1^5\,v_1 \cdot y_{22},\ u_1^4\,v_1 \cdot y_{18},\ u_1^6\,v_1 \cdot y_{30},\ u_1^2\,v_1 \cdot y_{35},\ u_1^3\,v_1 \cdot y_{16},\ u_1\,v_1 \cdot y_{27},\ v_1 \cdot y_{38}\right)$$

for $\xi = (u_1, v_1) \in \hat{G}^\circ$ and $y \in \mathbb{C}^7 - V\left(B(\hat{\Sigma}^\circ)\right)$. Hence with the group

$$\hat{H}^\circ = \mathbb{Z}_7$$

of order 7 the toric variety $\hat{Y}^\circ$ is the quotient

$$\hat{Y}^\circ = \mathbb{P}^6/\hat{H}^\circ$$

of projective space $\mathbb{P}^6$. The first order mirror degeneration $\mathfrak{X}^{1\circ}$ induces via $Y^\circ \to \hat{Y}^\circ$ a degeneration $\hat{\mathfrak{X}}^{1\circ} \subset \hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]/\langle t^2\rangle$ given by the ideal $\hat{I}^{1\circ} \subset$

$\langle y_{22}, y_{18}, y_{30}, y_{35}, y_{16}, y_{27}, y_{38}\rangle \subset \hat{S}^{\circ} \otimes \mathbb{C}[t] / \langle t^2\rangle$ generated by the Fermat-type equations

$$\left\{\begin{array}{l} y_{16}\, y_{18}\, y_{38} + t(s_3\, y_{16}\, y_{35}^2 + s_4\, y_{18}\, y_{22}^2), \\ y_{16}\, y_{30}\, y_{35} + t(s_2\, y_{16}\, y_{18}^2 + s_1\, y_{27}^2 y_{35}), \\ y_{18}\, y_{27}\, y_{38} + t(s_3\, y_{27}\, y_{35}^2 + s_5\, y_{30}^2 y_{38}), \\ y_{18}\, y_{22}\, y_{27} + t(s_6\, y_{16}^2 y_{18} + s_5\, y_{22}\, y_{30}^2), \\ y_{22}\, y_{27}\, y_{35} + t(s_6\, y_{16}^2 y_{35} + s_7\, y_{27}\, y_{38}^2), \\ y_{22}\, y_{30}\, y_{35} + t(s_2\, y_{18}^2 y_{22} + s_7\, y_{30}\, y_{38}^2), \\ y_{16}\, y_{30}\, y_{38} + t(s_4\, y_{22}^2 y_{30} + s_1\, y_{27}^2 y_{38}) \end{array}\right\}$$

Note that

$$\left|\operatorname{supp}\left(\left(\mu\left(B\left(I\right)\right)\right)^{*}\right) \cap N - \operatorname{Roots}\left(\hat{Y}^{\circ}\right)\right| - \dim\left(T_{\hat{Y}^{\circ}}\right) = 7 - 6 = 1$$

so this family has one independent parameter. The special fiber $\hat{X}_0^{\circ} \subset \hat{Y}^{\circ}$ of $\hat{\mathfrak{X}}^{1\circ}$ is cut out by the monomial ideal

$$\hat{I}_0^{\circ} \subset B(\hat{\Sigma}^{\circ}) = \langle y_{22}, y_{18}, y_{30}, y_{35}, y_{16}, y_{27}, y_{38}\rangle \subset \hat{S}^{\circ}$$

generated by

$$\left\{\begin{array}{llllll} y_{16}\, y_{18}\, y_{38} & y_{16}\, y_{30}\, y_{35} & y_{18}\, y_{22}\, y_{27} & y_{18}\, y_{27}\, y_{38} & y_{22}\, y_{30}\, y_{35} & y_{22}\, y_{27}\, y_{35} \\ y_{16}\, y_{30}\, y_{38} & & & & & \end{array}\right\}$$

The complex $SP(\hat{I}_0^{\circ})$ of prime ideals of the toric strata of $\hat{X}_0^{\circ}$ is

[],

[],

[],

$[\langle y_{35}, y_{16}, y_{27}\rangle, \langle y_{30}, y_{16}, y_{27}\rangle, \langle y_{22}, y_{30}, y_{38}\rangle, \langle y_{18}, y_{30}, y_{35}\rangle,$
$\langle y_{22}, y_{16}, y_{38}\rangle, \langle y_{18}, y_{35}, y_{38}\rangle, \langle y_{22}, y_{16}, y_{27}\rangle, \langle y_{22}, y_{18}, y_{30}\rangle,$
$\langle y_{18}, y_{35}, y_{16}\rangle, \langle y_{22}, y_{18}, y_{16}\rangle, \langle y_{30}, y_{27}, y_{38}\rangle, \langle y_{35}, y_{27}, y_{38}\rangle,$
$\langle y_{18}, y_{30}, y_{27}\rangle, \langle y_{22}, y_{35}, y_{38}\rangle],$

$[\langle y_{22}, y_{30}, y_{16}, y_{27}\rangle, \langle y_{18}, y_{30}, y_{16}, y_{27}\rangle, \langle y_{22}, y_{18}, y_{16}, y_{27}\rangle,$
$\langle y_{22}, y_{18}, y_{35}, y_{16}\rangle, \langle y_{22}, y_{18}, y_{30}, y_{16}\rangle, \langle y_{22}, y_{18}, y_{30}, y_{35}\rangle,$
$\langle y_{18}, y_{30}, y_{35}, y_{16}\rangle, \langle y_{18}, y_{35}, y_{16}, y_{27}\rangle, \langle y_{22}, y_{16}, y_{27}, y_{38}\rangle,$
$\langle y_{30}, y_{35}, y_{27}, y_{38}\rangle, \langle y_{30}, y_{35}, y_{16}, y_{27}\rangle, \langle y_{22}, y_{18}, y_{16}, y_{38}\rangle,$
$\langle y_{18}, y_{35}, y_{16}, y_{38}\rangle, \langle y_{22}, y_{18}, y_{35}, y_{38}\rangle, \langle y_{18}, y_{30}, y_{35}, y_{27}\rangle,$

$$\langle y_{22}, y_{35}, y_{16}, y_{27}\rangle, \langle y_{18}, y_{30}, y_{27}, y_{38}\rangle, \langle y_{35}, y_{16}, y_{27}, y_{38}\rangle,$$
$$\langle y_{22}, y_{18}, y_{30}, y_{27}\rangle, \langle y_{22}, y_{18}, y_{30}, y_{38}\rangle, \langle y_{22}, y_{30}, y_{27}, y_{38}\rangle,$$
$$\langle y_{18}, y_{35}, y_{27}, y_{38}\rangle, \langle y_{22}, y_{30}, y_{16}, y_{38}\rangle, \langle y_{22}, y_{35}, y_{16}, y_{38}\rangle,$$
$$\langle y_{30}, y_{16}, y_{27}, y_{38}\rangle, \langle y_{22}, y_{35}, y_{27}, y_{38}\rangle, \langle y_{22}, y_{30}, y_{35}, y_{38}\rangle,$$
$$\langle y_{18}, y_{30}, y_{35}, y_{38}\rangle],$$

$$[\langle y_{22}, y_{18}, y_{30}, y_{35}, y_{27}\rangle, \langle y_{22}, y_{18}, y_{30}, y_{16}, y_{27}\rangle, \langle y_{18}, y_{35}, y_{16}, y_{27}, y_{38}\rangle,$$
$$\langle y_{22}, y_{30}, y_{35}, y_{27}, y_{38}\rangle, \langle y_{22}, y_{18}, y_{35}, y_{16}, y_{38}\rangle, \langle y_{22}, y_{18}, y_{30}, y_{35}, y_{38}\rangle,$$
$$\langle y_{18}, y_{30}, y_{35}, y_{16}, y_{38}\rangle, \langle y_{22}, y_{18}, y_{35}, y_{16}, y_{27}\rangle, \langle y_{22}, y_{30}, y_{35}, y_{16}, y_{38}\rangle,$$
$$\langle y_{18}, y_{30}, y_{16}, y_{27}, y_{38}\rangle, \langle y_{22}, y_{18}, y_{35}, y_{27}, y_{38}\rangle, \langle y_{22}, y_{18}, y_{30}, y_{16}, y_{38}\rangle,$$
$$\langle y_{22}, y_{18}, y_{16}, y_{27}, y_{38}\rangle, \langle y_{30}, y_{35}, y_{16}, y_{27}, y_{38}\rangle, \langle y_{18}, y_{30}, y_{35}, y_{27}, y_{38}\rangle,$$
$$\langle y_{22}, y_{30}, y_{35}, y_{16}, y_{27}\rangle, \langle y_{22}, y_{18}, y_{30}, y_{35}, y_{16}\rangle, \langle y_{22}, y_{35}, y_{16}, y_{27}, y_{38}\rangle,$$
$$\langle y_{18}, y_{30}, y_{35}, y_{16}, y_{27}\rangle, \langle y_{22}, y_{30}, y_{16}, y_{27}, y_{38}\rangle, \langle y_{22}, y_{18}, y_{30}, y_{27}, y_{38}\rangle],$$

$$[\langle y_{18}, y_{30}, y_{35}, y_{16}, y_{27}, y_{38}\rangle, \langle y_{22}, y_{30}, y_{35}, y_{16}, y_{27}, y_{38}\rangle,$$
$$\langle y_{22}, y_{18}, y_{35}, y_{16}, y_{27}, y_{38}\rangle, \langle y_{22}, y_{18}, y_{30}, y_{16}, y_{27}, y_{38}\rangle,$$
$$\langle y_{22}, y_{18}, y_{30}, y_{35}, y_{27}, y_{38}\rangle, \langle y_{22}, y_{18}, y_{30}, y_{35}, y_{16}, y_{38}\rangle,$$
$$\langle y_{22}, y_{18}, y_{30}, y_{35}, y_{16}, y_{27}\rangle],$$

$$[]$$

so $\hat{I}_0^\circ$ has the primary decomposition

$$\hat{I}_0^\circ = \begin{array}{l} \langle y_{30}, y_{27}, y_{38}\rangle \cap \langle y_{22}, y_{16}, y_{27}\rangle \cap \langle y_{22}, y_{18}, y_{30}\rangle \cap \langle y_{18}, y_{35}, y_{16}\rangle \cap \\ \cap \langle y_{22}, y_{18}, y_{16}\rangle \cap \langle y_{35}, y_{27}, y_{38}\rangle \cap \langle y_{18}, y_{30}, y_{27}\rangle \cap \langle y_{22}, y_{35}, y_{38}\rangle \cap \\ \cap \langle y_{18}, y_{35}, y_{38}\rangle \cap \langle y_{35}, y_{16}, y_{27}\rangle \cap \langle y_{30}, y_{16}, y_{27}\rangle \cap \langle y_{22}, y_{30}, y_{38}\rangle \cap \\ \cap \langle y_{18}, y_{30}, y_{35}\rangle \cap \langle y_{22}, y_{16}, y_{38}\rangle \end{array}$$

The Bergman subcomplex $B(I) \subset \text{Poset}(\nabla)$ induces a subcomplex of $\text{Poset}(\hat{\nabla})$, $\hat{\nabla} = (\hat{P}^\circ)^*$ corresponding to $\hat{I}_0^\circ$. Indexing the vertices of $\hat{\nabla}$ by

$$\begin{array}{ll} 1 = (0,-3,-2,3,-2,-3) & 2 = (1,6,1,0,3,3) \\ 3 = (-1,2,2,-1,0,5) & 4 = (3,3,0,1,6,1) \\ 5 = (-5,-6,-3,-3,-6,-5) & 6 = (-3,-2,3,-2,-3,0) \\ 7 = (5,0,-1,2,2,-1) & \end{array}$$

this complex is given by

$$\left\{\begin{array}{l} [], \\ [[1],[2],[3],[4],[5],[6],[7]], \\ [[5,7],[4,7],[1,3],[2,5],[3,6],[5,6],[1,6],[3,7],[2,6],[1,4],[3,5], \\ [4,6],[3,4],[1,2],[1,5],[2,7],[6,7],[2,3],[1,7],[2,4],[4,5]], \\ [[2,4,7],[1,4,7],[3,4,7],[3,6,7],[4,6,7],[5,6,7],[1,6,7],[1,3,7], \\ [2,3,4],[1,2,5],[1,2,7],[3,4,6],[1,3,6],[3,5,6],[1,5,7],[2,3,7], \\ [1,4,5],[1,2,3],[4,5,7],[4,5,6],[2,4,5],[1,3,5],[2,4,6],[2,3,6], \\ [1,2,4],[2,3,5],[2,5,6],[1,5,6]], \\ [[1,2,3,7],[1,2,4,7],[2,4,5,6],[1,5,6,7],[2,3,4,6],[1,3,5,6], \\ [2,3,4,7],[4,5,6,7],[1,3,6,7],[3,4,6,7],[1,2,4,5],[1,2,3,5], \\ [1,4,5,7],[2,3,5,6]], \\ [], \\ [], \\ [] \end{array}\right\}$$

The ideal $\hat{I}^{1\circ} \subset \hat{S}^{\circ} \otimes \mathbb{C}[t] / \langle t^2 \rangle$ has a Pfaffian resolution

$$\begin{array}{c} 0 \to \mathcal{O}_{\hat{Y}^{\circ} \times \operatorname{Spec} \mathbb{C}[t]/\langle t^2\rangle}\left(K^1\right) \to \mathcal{E}^1\left(K^1\right) \xrightarrow{\varphi^1} \left(\mathcal{E}^1\right)^* \xrightarrow{f^1} \mathcal{O}_{\hat{Y}^{\circ} \times \operatorname{Spec} \mathbb{C}[t]/\langle t^2\rangle} \\ \text{where } \overline{\pi}_1 : \hat{Y}^{\circ} \times \operatorname{Spec} \mathbb{C}[t] / \left\langle t^2 \right\rangle \to \hat{Y}^{\circ} \text{ and } \mathcal{E}^1 = \overline{\pi}_1^* \mathcal{F} \end{array}$$

with

$$\mathcal{F} = \begin{array}{l} \mathcal{O}_{\hat{Y}^{\circ}}\left(D_{(0,2,0,0,-1,-1)} + D_{(0,0,-1,-1,0,0)} + D_{(2,0,0,-1,-1,0)}\right) \oplus \\ \mathcal{O}_{\hat{Y}^{\circ}}\left(D_{(0,-1,-1,0,0,2)} + D_{(-1,-1,0,0,2,0)} + D_{(0,0,-1,-1,0,0)}\right) \oplus \\ \mathcal{O}_{\hat{Y}^{\circ}}\left(D_{(0,2,0,0,-1,-1)} + D_{(0,0,2,0,0,-1)} + D_{(2,0,0,-1,-1,0)}\right) \oplus \\ \mathcal{O}_{\hat{Y}^{\circ}}\left(D_{(-1,0,0,2,0,0)} + D_{(0,2,0,0,-1,-1)} + D_{(0,0,2,0,0,-1)}\right) \oplus \\ \mathcal{O}_{\hat{Y}^{\circ}}\left(D_{(-1,0,0,2,0,0)} + D_{(-1,-1,0,0,2,0)} + D_{(0,0,2,0,0,-1)}\right) \oplus \\ \mathcal{O}_{\hat{Y}^{\circ}}\left(D_{(-1,0,0,2,0,0)} + D_{(0,-1,-1,0,0,2)} + D_{(-1,-1,0,0,2,0)}\right) \oplus \\ \mathcal{O}_{\hat{Y}^{\circ}}\left(D_{(0,-1,-1,0,0,2)} + D_{(0,0,-1,-1,0,0)} + D_{(2,0,0,-1,-1,0)}\right) \end{array}$$

and $K^1 = K_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]/\langle t^2\rangle / \operatorname{Spec}\mathbb{C}[t]/\langle t^2\rangle}$ and $\varphi^1 \in \bigwedge^2 \mathcal{E}^{1*}(-K^1)$ given by

$$\begin{bmatrix} 0 & -ty_{16}\,s_6 & y_{30} & 0 & 0 & -y_{27} & ty_{18}\,s_2 \\ ty_{16}\,s_6 & 0 & -ty_{35}\,s_3 & y_{22} & 0 & 0 & -y_{38} \\ -y_{30} & ty_{35}\,s_3 & 0 & -ty_{27}\,s_1 & y_{18} & 0 & 0 \\ 0 & -y_{22} & ty_{27}\,s_1 & 0 & -ty_{38}\,s_7 & y_{16} & 0 \\ 0 & 0 & -y_{18} & ty_{38}\,s_7 & 0 & -ty_{30}\,s_5 & y_{35} \\ y_{27} & 0 & 0 & -y_{16} & ty_{30}\,s_5 & 0 & -ty_{22}\,s_4 \\ -ty_{18}\,s_2 & y_{38} & 0 & 0 & -y_{35} & ty_{22}\,s_4 & 0 \end{bmatrix}$$

Hence via the Pfaffians of φ^1 we obtain a resolution

$$\begin{gathered} 0 \to \mathcal{O}_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]}(K) \to \mathcal{E}(K) \to \mathcal{E}^* \to \mathcal{O}_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]} \\ \text{where } \pi_1 : Y \times \operatorname{Spec}\mathbb{C}[t] \to Y \ ,\ \mathcal{E} = \pi_1^*\mathcal{F} \\ \text{and } K = K_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]/\operatorname{Spec}\mathbb{C}[t]} \end{gathered}$$

of the ideal $\hat{I}^\circ \subset \hat{S}^\circ \otimes \mathbb{C}[t]$ generated by

$$\left\{ \begin{array}{l} y_{16}\,y_{18}\,y_{38} + t\,(s_3\,y_{16}\,y_{35}^2 + s_4\,y_{18}\,y_{22}^2) + t^2\,(-s_1\,s_5\,y_{27}\,y_{30}\,y_{38}) + t^3\,(-s_3\,s_7\,s_4\,y_{22}\,y_{35}\,y_{38})\,, \\ y_{16}\,y_{30}\,y_{35} + t\,(s_2\,y_{16}\,y_{18}^2 + s_1\,y_{27}^2 y_{35}) + t^2\,(-s_7\,s_4\,y_{22}\,y_{30}\,y_{38}) + t^3\,(-s_2\,s_1\,s_5\,y_{18}\,y_{27}\,y_{30})\,, \\ y_{22}\,y_{27}\,y_{35} + t\,(s_6\,y_{16}^2 y_{35} + s_7\,y_{27}\,y_{38}^2) + t^2\,(-s_2\,s_5\,y_{18}\,y_{22}\,y_{30}) + t^3\,(-s_6\,s_7\,s_4\,y_{16}\,y_{22}\,y_{38})\,, \\ y_{18}\,y_{27}\,y_{38} + t\,(s_3\,y_{27}\,y_{35}^2 + s_5\,y_{30}^2 y_{38}) + t^2\,(-s_6\,s_4\,y_{16}\,y_{18}\,y_{22}) + t^3\,(-s_2\,s_3\,s_5\,y_{18}\,y_{30}\,y_{35})\,, \\ y_{16}\,y_{30}\,y_{38} + t\,(s_4\,y_{22}^2 y_{30} + s_1\,y_{27}^2 y_{38}) + t^2\,(-s_2\,s_3\,y_{16}\,y_{18}\,y_{35}) + t^3\,(-s_6\,s_1\,s_4\,y_{16}\,y_{22}\,y_{27})\,, \\ y_{22}\,y_{30}\,y_{35} + t\,(s_2\,y_{18}^2 y_{22} + s_7\,y_{30}\,y_{38}^2) + t^2\,(-s_6\,s_1\,y_{16}\,y_{27}\,y_{35}) + t^3\,(-s_2\,s_3\,s_7\,y_{18}\,y_{35}\,y_{38})\,, \\ y_{18}\,y_{22}\,y_{27} + t\,(s_6\,y_{16}^2 y_{18} + s_5\,y_{22}\,y_{30}^2) + t^2\,(-s_3\,s_7\,y_{27}\,y_{35}\,y_{38}) + t^3\,(-s_6\,s_1\,s_5\,y_{16}\,y_{27}\,y_{30}) \end{array} \right\}$$

which defines a flat family

$$\hat{\mathfrak{X}}^\circ \subset \hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]$$

This is the one parameter mirror family of the generic degree 14 Pfaffian Calabi-Yau threefold in $\mathbb{P}^6$, given in [Rødland, 1998].

10.5 Tropical mirror construction of the degree 13 Pfaffian Calabi-Yau threefold

10.5.1 Hodge numbers

The Hodge numbers of a general Pfaffian Calabi-Yau threefold X of degree 13 in $\mathbb{P}^6$, which is smooth as observed above, can be determined as follows.

The Pfaffian complex is

$$0 \to \mathcal{O}_{\mathbb{P}^6}(-7) \to \mathcal{O}_{\mathbb{P}^6}(-5) \oplus \mathcal{O}_{\mathbb{P}^6}(-4)^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^6}(-2) \oplus \mathcal{O}_{\mathbb{P}^6}(-3)^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^6} \to \mathcal{O}_X$$

so, decomposing into short exact sequences, one computes

$$H^i(X, \mathcal{O}_X) \cong H^{i+3}\left(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(-7)\right) \cong \left\{ \begin{array}{cc} 0 & i=1,2 \\ \mathbb{C} & i=3 \end{array} \right\}$$

Decomposing the resolution of $\mathcal{J}_X^2$

$$\begin{aligned} 0 &\to \mathcal{O}_{\mathbb{P}^6}(-8)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}^6}(-9)^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^6}(-6)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^6}(-7)^{\oplus 16} \oplus \mathcal{O}_{\mathbb{P}^6}(-8)^{\oplus 4} \to \\ &\to \mathcal{O}_{\mathbb{P}^6}(-4) \oplus \mathcal{O}_{\mathbb{P}^6}(-5)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^6}(-6)^{\oplus 10} \to \mathcal{J}_X^2 \to 0 \end{aligned}$$

into the short exact sequences

$$0 \to \mathcal{O}_{\mathbb{P}^6}(-8)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}^6}(-9)^{\oplus 4} \to \mathcal{O}_{\mathbb{P}^6}(-6)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^6}(-7)^{\oplus 16} \oplus \mathcal{O}_{\mathbb{P}^6}(-8)^{\oplus 4} \to \mathcal{K} \to 0$$

$$0 \to \mathcal{K} \to \mathcal{O}_{\mathbb{P}^6}(-4) \oplus \mathcal{O}_{\mathbb{P}^6}(-5)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^6}(-6)^{\oplus 10} \to \mathcal{J}_X^2 \to 0$$

the long exact cohomology sequences give

$$\begin{aligned} \ldots &\to H^i\left(\mathbb{P}^6, \mathcal{K}\right) \to 0 \to H^i\left(\mathbb{P}^6, \mathcal{J}_X^2\right) \to \\ &\to H^{i+1}\left(\mathbb{P}^6, \mathcal{K}\right) \to 0 \to H^{i+1}\left(\mathbb{P}^6, \mathcal{J}_X^2\right) \to \ldots \end{aligned}$$

for $i = 0, \ldots, 5$, so

$$H^i\left(\mathbb{P}^6, \mathcal{J}_X^2\right) \cong H^{i+1}\left(\mathbb{P}^6, \mathcal{K}\right) \text{ for } i = 0, \ldots, 5$$

and

$$\begin{aligned} 0 &\to 0 \to 0 \to H^0\left(\mathbb{P}^6, \mathcal{K}\right) \to \\ &\to 0 \to 0 \to H^1\left(\mathbb{P}^6, \mathcal{K}\right) \to \\ &\to 0 \to 0 \to H^2\left(\mathbb{P}^6, \mathcal{K}\right) \to \\ &\to 0 \to 0 \to H^3\left(\mathbb{P}^6, \mathcal{K}\right) \to \\ &\to 0 \to 0 \to H^4\left(\mathbb{P}^6, \mathcal{K}\right) \to \\ &\to 0 \to 0 \to H^5\left(\mathbb{P}^6, \mathcal{K}\right) \to \\ &\to H^6\left(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(-8)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}^6}(-9)^{\oplus 4}\right) \\ &\to H^6\left(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(-6)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^6}(-7)^{\oplus 16} \oplus \mathcal{O}_{\mathbb{P}^6}(-8)^{\oplus 4}\right) \\ &\to H^6\left(\mathbb{P}^6, \mathcal{K}\right) \to 0 \end{aligned}$$

hence

$$H^i\left(\mathbb{P}^6, \mathcal{K}\right) = 0 \text{ for } i = 0, ..., 4$$

and

$$h^5\left(\mathbb{P}^6, \mathcal{K}\right) - h^6\left(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(-8)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}^6}(-9)^{\oplus 4}\right)$$
$$+h^6\left(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(-6)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^6}(-7)^{\oplus 16} \oplus \mathcal{O}_{\mathbb{P}^6}(-8)^{\oplus 4}\right) - h^6\left(\mathbb{P}^6, \mathcal{K}\right) = 0$$

Using

$$h^6\left(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(-8)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}^6}(-9)^{\oplus 4}\right) = 6 \cdot 7 + 4 \cdot 28 = 154$$
$$h^6\left(\mathbb{P}^6, \mathcal{O}_{\mathbb{P}^6}(-6)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^6}(-7)^{\oplus 16} \oplus \mathcal{O}_{\mathbb{P}^6}(-8)^{\oplus 4}\right) = 16 + 4 \cdot 7 = 44$$

one has

$$h^5\left(\mathbb{P}^6, \mathcal{K}\right) - h^6\left(\mathbb{P}^6, \mathcal{K}\right) = 154 - 44 = 110$$

hence:

Lemma 10.12

$$H^i\left(\mathbb{P}^6, \mathcal{J}_X^2\right) = 0 \textit{ for } i = 0, ..., 3$$
$$h^4\left(\mathbb{P}^6, \mathcal{J}_X^2\right) - h^5\left(\mathbb{P}^6, \mathcal{J}_X^2\right) = 110$$

The long exact cohomology sequence of

$$0 \to \mathcal{J}_X^2 \to \mathcal{J}_X \to N_{X/\mathbb{P}^6}^{\vee} \to 0$$

reads

$$H^i\left(\mathbb{P}^6, N_{X/\mathbb{P}^6}^{\vee}\right) = 0 \text{ for } i = 0, 1, 2$$
$$H^4\left(\mathbb{P}^6, \mathcal{J}_X^2\right) / H^3\left(\mathbb{P}^6, N_{X/\mathbb{P}^6}^{\vee}\right) \cong \mathbb{C}$$
$$H^5\left(\mathbb{P}^6, \mathcal{J}_X^2\right) = 0$$

so $h^4\left(\mathbb{P}^6, \mathcal{J}_X^2\right) = 110$ and $h^3\left(\mathbb{P}^6, N_{X/\mathbb{P}^6}^{\vee}\right) = 109$. Hence the cohomology dimensions of $\mathcal{J}_X^2$ are

$$h^i\left(\mathbb{P}^6, \mathcal{J}_X^2\right) = 0 \text{ for } i = 0, ..., 3$$
$$h^4\left(\mathbb{P}^6, \mathcal{J}_X^2\right) = 110$$
$$h^5\left(\mathbb{P}^6, \mathcal{J}_X^2\right) = 0$$

and:

Lemma 10.13 *The cohomology dimensions of* $N^\vee_{X/\mathbb{P}^6}$ *are*

$$h^i\left(X, N^\vee_{X/\mathbb{P}^6}\right) = 0 \text{ for } i = 0,1,2$$
$$h^3\left(X, N^\vee_{X/\mathbb{P}^6}\right) = 109$$

Using the Euler sequence and conormal sequence

$$0 \to \Omega_{\mathbb{P}^6}\mid_X \to \mathcal{O}_X(-1)^{\oplus 7} \to \mathcal{O}_X \to 0$$
$$0 \to N^\vee_{X/\mathbb{P}^6} \to \Omega_{\mathbb{P}^6}\mid_X \to \Omega_X \to 0$$

as explained in Section 11.1, we get

$$h^1(X, \Omega_X) = 1$$
$$h^2(X, \Omega_X) = h^3\left(X, N^\vee_{X/\mathbb{P}^6}\right) - h^3\left(X, \Omega_{\mathbb{P}^6}\mid_X\right) = 109 - 48 = 61$$

Corollary 10.14 *The general degree* 13 *Pfaffian Calabi-Yau threefold* X *in* $\mathbb{P}^6$ *has*

$$h^{1,1}(X) = 1$$
$$h^{1,2}(X) = 61$$

10.5.2 Setup

Let $Y = \mathbb{P}^6 = X(\Sigma)$, $\Sigma = \Sigma(P) = NF(\Delta) \subset N_{\mathbb{R}}$ with the Fano polytope $P = \Delta^*$ given by

$$\Delta = \text{convexhull}\begin{pmatrix} (6,-1,-1,-1,-1,-1) & (-1,6,-1,-1,-1,-1) \\ (-1,-1,6,-1,-1,-1) & (-1,-1,-1,6,-1,-1) \\ (-1,-1,-1,-1,6,-1) & (-1,-1,-1,-1,-1,6) \\ (-1,-1,-1,-1,-1,-1) & \end{pmatrix} \subset M_{\mathbb{R}}$$

and let

$$S = \mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5, x_6]$$

be the Cox ring of Y with the variables

$$\begin{array}{ll} x_1 = x_{(1,0,0,0,0,0)} & x_2 = x_{(0,1,0,0,0,0)} \\ x_3 = x_{(0,0,1,0,0,0)} & x_4 = x_{(0,0,0,1,0,0)} \\ x_5 = x_{(0,0,0,0,1,0)} & x_6 = x_{(0,0,0,0,0,1)} \\ x_0 = x_{(-1,-1,-1,-1,-1,-1)} & \end{array}$$

associated to the rays of Σ. Consider the degeneration $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[t]$ of Pfaffian Calabi-Yau 3-folds with Buchsbaum-Eisenbud resolution

$$0 \to \mathcal{O}_Y(-7) \to \mathcal{E}(-3) \xrightarrow{A_t} \mathcal{E}^*(-2) \to \mathcal{O}_Y \to \mathcal{O}_{X_t} \to 0$$

where

$$\mathcal{E} = \mathcal{O}(1) \oplus 4\mathcal{O}$$

$$A_t = A_0 + t \cdot A$$

$$A_0 = \begin{bmatrix} 0 & 0 & x_3 x_4 & -x_1 x_2 & 0 \\ 0 & 0 & 0 & x_0 & x_6 \\ -x_3 x_4 & 0 & 0 & 0 & -x_5 \\ x_1 x_2 & -x_0 & 0 & 0 & 0 \\ 0 & -x_6 & x_5 & 0 & 0 \end{bmatrix}$$

the monomial special fiber of $\mathfrak{X}$ is given by

$$I_0 = \left\langle \; x_5 x_0 \quad x_1 x_2 x_5 \quad x_1 x_2 x_6 \quad x_3 x_4 x_6 \quad -x_3 x_4 x_0 \; \right\rangle$$

and generic $A \in \bigwedge^2 \mathcal{E}^*(1)$

$$A = \begin{bmatrix} 0 & w_1 & w_2 & w_3 & w_4 \\ -w_1 & 0 & w_5 & w_6 & w_7 \\ -w_2 & -w_5 & 0 & w_8 & w_9 \\ -w_3 & -w_6 & -w_8 & 0 & w_{10} \\ -w_4 & -w_7 & -w_9 & -w_{10} & 0 \end{bmatrix}$$

$$\begin{aligned}
w_1 &= s_1 x_1^2 + ... + s_8 x_2^2 + ... + s_{14} x_3^2 + ... + s_{19} x_4^2 + ... + s_{23} x_5^2 + ... + s_{26} x_6^2 + ... + s_{28} x_0^2 \\
w_2 &= s_{29} x_1^2 + ... + s_{36} x_2^2 + ... + s_{42} x_3^2 + ... + s_{47} x_4^2 + ... + s_{51} x_5^2 + ... + s_{54} x_6^2 + ... + s_{56} x_0^2 \\
w_3 &= s_{57} x_1^2 + ... + s_{64} x_2^2 + ... + s_{70} x_3^2 + ... + s_{75} x_4^2 + ... + s_{79} x_5^2 + ... + s_{82} x_6^2 + ... + s_{84} x_0^2 \\
w_4 &= s_{85} x_1^2 + ... + s_{92} x_2^2 + ... + s_{98} x_3^2 + ... + s_{103} x_4^2 + ... + s_{107} x_5^2 + ... + s_{110} x_6^2 + ... + s_{112} x_0^2 \\
w_5 &= s_{113} x_1 + s_{114} x_2 + s_{115} x_3 + s_{116} x_4 + s_{117} x_5 + s_{118} x_6 + s_{119} x_0 \\
w_6 &= s_{120} x_1 + s_{121} x_2 + s_{122} x_3 + s_{123} x_4 + s_{124} x_5 + s_{125} x_6 + s_{126} x_0 \\
w_7 &= s_{127} x_1 + s_{128} x_2 + s_{129} x_3 + s_{130} x_4 + s_{131} x_5 + s_{132} x_6 + s_{133} x_0 \\
w_8 &= s_{134} x_1 + s_{135} x_2 + s_{136} x_3 + s_{137} x_4 + s_{138} x_5 + s_{139} x_6 + s_{140} x_0 \\
w_9 &= s_{141} x_1 + s_{142} x_2 + s_{143} x_3 + s_{144} x_4 + s_{145} x_5 + s_{146} x_6 + s_{147} x_0 \\
w_{10} &= s_{148} x_1 + s_{149} x_2 + s_{150} x_3 + s_{151} x_4 + s_{152} x_5 + s_{153} x_6 + s_{154} x_0
\end{aligned}$$

The degeneration

$$\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[t]$$

is given by the ideal $I \subset S \otimes \mathbb{C}[t]$ generated by the Pfaffians of $A_0 + t \cdot A$ of degrees $2, 3, 3, 3, 3$.

10.5.3 Special fiber Gröbner cone

The space of first order deformations of $\mathfrak{X}$ has dimension 109 and the deformations represented by the Cox Laurent monomials

$\frac{x_3 x_4^2}{x_1 x_2 x_5}$	$\frac{x_3^2 x_4}{x_1 x_2 x_5}$	$\frac{x_1^2 x_2}{x_3 x_4 x_0}$	$\frac{x_1 x_2^2}{x_3 x_4 x_0}$	$\frac{x_2^2 x_5}{x_3 x_4 x_6}$	$\frac{x_4^2 x_0}{x_1 x_2 x_6}$	$\frac{x_1^2 x_5}{x_3 x_4 x_6}$	$\frac{x_3^2 x_0}{x_1 x_2 x_6}$
$\frac{x_1 x_5^2}{x_3 x_4 x_6}$	$\frac{x_0^2 x_3}{x_1 x_2 x_6}$	$\frac{x_0^3}{x_1 x_2 x_6}$	$\frac{x_5^3}{x_3 x_4 x_6}$	$\frac{x_0^2 x_4}{x_1 x_2 x_6}$	$\frac{x_2 x_5^2}{x_3 x_4 x_6}$	$\frac{x_2 x_0}{x_3 x_4}$	$\frac{x_5 x_3}{x_1 x_2}$
$\frac{x_1 x_0}{x_3 x_4}$	$\frac{x_5 x_4}{x_1 x_2}$	$\frac{x_0^2}{x_3 x_4}$	$\frac{x_5 x_6}{x_1 x_2}$	$\frac{x_0 x_6}{x_3 x_4}$	$\frac{x_5^2}{x_1 x_2}$	$\frac{x_5 x_4}{x_3 x_6}$	$\frac{x_5 x_3}{x_4 x_6}$
$\frac{x_3 x_4}{x_2 x_5}$	$\frac{x_3 x_4}{x_1 x_5}$	$\frac{x_1 x_0}{x_2 x_6}$	$\frac{x_2 x_0}{x_1 x_6}$	$\frac{x_1 x_6}{x_5 x_0}$	$\frac{x_4 x_6}{x_5 x_0}$	$\frac{x_6 x_3}{x_5 x_0}$	$\frac{x_6 x_2}{x_5 x_0}$
$\frac{x_1 x_2}{x_4 x_0}$	$\frac{x_1 x_2}{x_3 x_0}$	$\frac{x_3 x_0}{x_1 x_6}$	$\frac{x_6 x_4}{x_1 x_2}$	$\frac{x_4 x_0}{x_6 x_2}$	$\frac{x_0 x_6}{x_1 x_2}$	$\frac{x_1 x_6}{x_3 x_4}$	$\frac{x_6 x_2}{x_3 x_4}$
$\frac{x_6 x_3}{x_1 x_2}$	$\frac{x_5 x_1}{x_3 x_4}$	$\frac{x_4 x_0}{x_1 x_2}$	$\frac{x_5 x_2}{x_3 x_4}$	$\frac{x_1 x_2}{x_3 x_4}$	$\frac{x_3 x_0}{x_1 x_2}$	$\frac{x_5 x_6}{x_3 x_4}$	$\frac{x_4 x_0}{x_1 x_6}$
$\frac{x_2 x_5}{x_4 x_6}$	$\frac{x_1 x_5}{x_4 x_6}$	$\frac{x_6^2}{x_1 x_2}$	$\frac{x_3^2}{x_1 x_2}$	$\frac{x_4^2}{x_1 x_2}$	$\frac{x_6^2}{x_5 x_0}$	$\frac{x_2^2}{x_3 x_4}$	$\frac{x_1^2}{x_3 x_4}$
$\frac{x_6^2}{x_3 x_4}$	$\frac{x_5^2}{x_3 x_6}$	$\frac{x_5^2}{x_4 x_6}$	$\frac{x_0^2}{x_1 x_2}$	$\frac{x_0^2}{x_2 x_6}$	$\frac{x_2 x_5}{x_3 x_6}$	$\frac{x_0 x_3}{x_2 x_6}$	$\frac{x_5^2}{x_3 x_4}$
$\frac{x_0^2}{x_1 x_6}$	$\frac{x_1 x_5}{x_3 x_6}$	$\frac{x_3 x_4}{x_1 x_2}$	$\frac{x_0}{x_3}$	$\frac{x_3}{x_0}$	$\frac{x_2}{x_1}$	$\frac{x_1}{x_2}$	$\frac{x_6}{x_3}$
$\frac{x_1}{x_5}$	$\frac{x_0}{x_1}$	$\frac{x_1}{x_3}$	$\frac{x_0}{x_2}$	$\frac{x_2}{x_0}$	$\frac{x_3}{x_1}$	$\frac{x_1}{x_0}$	$\frac{x_3}{x_2}$
$\frac{x_2}{x_3}$	$\frac{x_5}{x_2}$	$\frac{x_6}{x_2}$	$\frac{x_2}{x_6}$	$\frac{x_5}{x_1}$	$\frac{x_3}{x_6}$	$\frac{x_2}{x_5}$	$\frac{x_0}{x_4}$
$\frac{x_4}{x_2}$	$\frac{x_4}{x_0}$	$\frac{x_4}{x_3}$	$\frac{x_4}{x_1}$	$\frac{x_3}{x_4}$	$\frac{x_1}{x_4}$	$\frac{x_2}{x_4}$	$\frac{x_6}{x_4}$
$\frac{x_4}{x_5}$	$\frac{x_5}{x_4}$	$\frac{x_6}{x_5}$	$\frac{x_5}{x_3}$	$\frac{x_3}{x_5}$	$\frac{x_5}{x_0}$	$\frac{x_4}{x_6}$	$\frac{x_6}{x_1}$
$\frac{x_1}{x_6}$	$\frac{x_0}{x_6}$	$\frac{x_6}{x_0}$	$\frac{x_0}{x_5}$	$\frac{x_5}{x_6}$			

form a torus invariant basis. These deformations give linear inequalities defining the special fiber Gröbner cone, i.e.,

$$C_{I_0}(I) = \left\{ (w_t, w_y) \in \mathbb{R} \oplus N_{\mathbb{R}} \mid \left\langle w, A^{-1}(m) \right\rangle \geq -w_t \forall m \right\}$$

for above monomials m and the presentation matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

of $A_5(Y)$. The vertices of special fiber polytope

$$\nabla = C_{I_0}(I) \cap \{w_t = 1\}$$

and the number of faces, each vertex is contained in, are given by the table

15	$\left(\frac{1}{5},\frac{1}{5},\frac{4}{5},\frac{1}{5},\frac{1}{5},\frac{3}{5}\right)$ $\left(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{4}{5},\frac{1}{5},\frac{3}{5}\right)$ $\left(0,\frac{3}{5},0,0,-\frac{1}{5},\frac{2}{5}\right)$ $\left(\frac{3}{5},0,0,0,-\frac{1}{5},\frac{2}{5}\right)$
19	$\left(-\frac{2}{3},-\frac{2}{3},-\frac{2}{3},-\frac{1}{3},-1,-1\right)$ $\left(-\frac{2}{3},-\frac{2}{3},-\frac{1}{3},-\frac{2}{3},-1,-1\right)$ $\left(\frac{2}{3},\frac{1}{3},\frac{1}{3},\frac{1}{3},1,0\right)$ $\left(\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{1}{3},1,0\right)$
38	$(0,0,1,0,0,0)$ $(0,0,0,1,0,0)$ $(1,0,0,0,0,0)$ $(0,1,0,0,0,0)$
45	$(-1,-1,-1,-1,-1,-1)$ $(0,0,0,0,0,1)$ $(0,0,0,0,1,0)$
46	$(1,0,1,0,0,0)$ $(0,1,1,0,0,0)$ $(1,0,0,1,0,0)$ $(0,1,0,1,0,0)$
51	$(0,0,1,0,1,0)$ $(0,0,0,1,1,0)$ $(0,-1,-1,-1,-1,-1)$ $(-1,0,-1,-1,-1,-1)$
59	$(1,0,1,0,1,0)$ $(0,1,1,0,1,0)$ $(1,0,0,1,1,0)$ $(0,1,0,1,1,0)$ $(0,-1,0,-1,-1,-1)$ $(-1,0,0,-1,-1,-1)$ $(0,-1,-1,0,-1,-1)$ $(-1,0,-1,0,-1,-1)$
62	$(-1,-1,-1,-1,-1,0)$ $(0,0,0,0,1,1)$
70	$(0,0,1,0,1,1)$ $(0,0,0,1,1,1)$ $(0,-1,-1,-1,-1,0)$ $(-1,0,-1,-1,-1,0)$ $(0,-1,0,-1,0,0)$ $(-1,0,0,-1,0,0)$ $(0,-1,-1,0,0,0)$ $(-1,0,-1,0,0,0)$
80	$(-1,-1,-1,-1,0,0)$

The number of faces of ∇ and their F-vectors are

Dimension	Number of faces	F-vector	
−1	1	$(0,0,0,0,0,0,0,0)$	
0	42	$(1,1,0,0,0,0,0,0)$	point
1	243	$(1,2,1,0,0,0,0,0)$	edge
2	417	$(1,3,3,1,0,0,0,0)$	triangle
2	118	$(1,4,4,1,0,0,0,0)$	quadrangle
3	116	$(1,6,9,5,1,0,0,0)$	prism
3	224	$(1,4,6,4,1,0,0,0)$	tetrahedron
3	4	$(1,7,15,10,1,0,0,0)$	
3	190	$(1,5,8,5,1,0,0,0)$	pyramid
3	11	$(1,8,12,6,1,0,0,0)$	cube

3	4	$(1,6,12,8,1,0,0,0)$	octahedron
3	8	$(1,7,14,9,1,0,0,0)$	
4	16	$(1,10,21,18,7,1,0,0)$	
4	26	$(1,8,18,17,7,1,0,0)$	
4	20	$(1,7,17,18,8,1,0,0)$	
4	5	$(1,9,22,23,10,1,0,0)$	
4	4	$(1,8,21,22,9,1,0,0)$	
4	3	$(1,12,24,19,7,1,0,0)$	
4	2	$(1,11,31,31,11,1,0,0)$	
4	4	$(1,11,29,28,10,1,0,0)$	
4	4	$(1,9,20,18,7,1,0,0)$	
4	4	$(1,10,27,27,10,1,0,0)$	
4	6	$(1,12,28,27,11,1,0,0)$	
4	38	$(1,5,10,10,5,1,0,0)$	
4	66	$(1,7,15,14,6,1,0,0)$	
4	4	$(1,9,24,24,9,1,0,0)$	
4	26	$(1,6,13,13,6,1,0,0)$	
4	8	$(1,9,25,27,11,1,0,0)$	
4	28	$(1,8,16,14,6,1,0,0)$	
4	8	$(1,9,24,25,10,1,0,0)$	
5	4	$(1,13,44,61,39,11,1,0)$	
5	1	$(1,18,65,94,61,16,1,0)$	
5	4	$(1,15,51,70,44,12,1,0)$	
5	2	$(1,18,53,68,43,12,1,0)$	
5	4	$(1,12,40,55,35,10,1,0)$	
5	2	$(1,12,38,54,37,11,1,0)$	
5	4	$(1,14,41,53,34,10,1,0)$	
5	4	$(1,15,53,75,48,13,1,0)$	
5	8	$(1,9,28,40,28,9,1,0)$	
5	2	$(1,16,46,59,39,12,1,0)$	
5	8	$(1,11,35,49,33,10,1,0)$	
5	4	$(1,11,37,54,38,12,1,0)$	
5	2	$(1,20,72,103,66,17,1,0)$	
6	1	$(1,42,243,535,557,272,49,1)$	

The dual $P^\circ = \nabla^*$ of ∇ is a Fano polytope with vertices

$$\begin{array}{lll} (-1,-1,2,0,0,-1) & (-1,-1,0,2,0,-1) & (2,0,-1,-1,1,-1) \\ (0,2,-1,-1,1,-1) & (1,2,-1,-1,0,0) & (2,1,-1,-1,0,0) \end{array}$$

$(0,0,-1,-1,3,-1)$	$(-1,-1,0,0,0,-1)$	$(-1,-1,1,2,-1,0)$
$(-1,-1,2,1,-1,0)$	$(0,0,1,-1,1,-1)$	$(0,0,-1,1,1,-1)$
$(0,0,-1,-1,0,2)$	$(0,0,-1,-1,0,0)$	$(2,0,-1,-1,0,0)$
$(0,2,-1,-1,0,0)$	$(1,1,-1,0,0,0)$	$(1,1,0,-1,0,0)$
$(0,1,0,0,-1,1)$	$(0,0,1,0,-1,1)$	$(0,0,0,1,-1,1)$
$(0,0,0,0,-1,2)$	$(1,0,0,0,-1,1)$	$(-1,-1,0,0,2,0)$
$(-1,-1,0,2,0,0)$	$(-1,-1,2,0,0,0)$	$(-1,-1,0,0,0,2)$
$(-1,1,0,0,0,-1)$	$(1,-1,0,0,0,-1)$	$(-1,0,1,1,-1,0)$
$(0,-1,1,1,-1,0)$	$(0,0,0,1,0,-1)$	$(0,0,1,0,0,-1)$
$(1,0,0,0,0,0)$	$(0,0,1,-1,0,0)$	$(0,0,-1,1,0,0)$
$(0,0,0,0,1,0)$	$(0,0,1,0,0,0)$	$(0,1,0,0,0,0)$
$(0,0,0,1,0,0)$	$(0,0,0,1,-1,0)$	$(0,0,1,0,-1,0)$
$(0,1,0,0,-1,0)$	$(1,0,0,0,-1,0)$	$(0,0,0,0,-1,0)$
$(1,-1,0,0,0,0)$	$(-1,1,0,0,0,0)$	$(0,1,0,0,0,-1)$
$(1,0,0,0,0,-1)$		

and the F-vectors of the faces of P° are

Dimension	Number of faces	F-vector	
-1	1	$(0,0,0,0,0,0,0,0)$	
0	49	$(1,1,0,0,0,0,0,0)$	point
1	272	$(1,2,1,0,0,0,0,0)$	edge
2	361	$(1,3,3,1,0,0,0,0)$	triangle
2	4	$(1,5,5,1,0,0,0,0)$	pentagon
2	192	$(1,4,4,1,0,0,0,0)$	quadrangle
3	222	$(1,6,9,5,1,0,0,0)$	prism
3	148	$(1,4,6,4,1,0,0,0)$	tetrahedron
3	24	$(1,7,11,6,1,0,0,0)$	
3	4	$(1,7,12,7,1,0,0,0)$	
3	108	$(1,5,8,5,1,0,0,0)$	pyramid
3	2	$(1,8,14,8,1,0,0,0)$	
3	3	$(1,6,10,6,1,0,0,0)$	
3	4	$(1,8,13,7,1,0,0,0)$	
3	20	$(1,6,12,8,1,0,0,0)$	octahedron
4	2	$(1,9,23,23,9,1,0,0)$	
4	39	$(1,5,10,10,5,1,0,0)$	
4	4	$(1,10,26,27,11,1,0,0)$	
4	20	$(1,12,30,28,10,1,0,0)$	
4	16	$(1,10,21,18,7,1,0,0)$	
4	8	$(1,10,23,21,8,1,0,0)$	

4	4	$(1, 10, 30, 30, 10, 1, 0, 0)$
4	4	$(1, 12, 28, 26, 10, 1, 0, 0)$
4	24	$(1, 8, 16, 14, 6, 1, 0, 0)$
4	44	$(1, 8, 18, 17, 7, 1, 0, 0)$
4	2	$(1, 9, 22, 22, 9, 1, 0, 0)$
4	8	$(1, 11, 24, 21, 8, 1, 0, 0)$
4	16	$(1, 7, 15, 14, 6, 1, 0, 0)$
4	8	$(1, 12, 31, 30, 11, 1, 0, 0)$
4	24	$(1, 9, 18, 15, 6, 1, 0, 0)$
4	2	$(1, 15, 41, 39, 13, 1, 0, 0)$
4	2	$(1, 12, 31, 31, 12, 1, 0, 0)$
4	2	$(1, 9, 19, 17, 7, 1, 0, 0)$
4	14	$(1, 6, 13, 13, 6, 1, 0, 0)$
5	1	$(1, 14, 45, 64, 45, 14, 1, 0)$
5	4	$(1, 7, 19, 26, 19, 7, 1, 0)$
5	4	$(1, 16, 51, 69, 45, 13, 1, 0)$
5	4	$(1, 14, 46, 64, 42, 12, 1, 0)$
5	2	$(1, 14, 45, 63, 43, 13, 1, 0)$
5	4	$(1, 21, 70, 91, 54, 14, 1, 0)$
5	8	$(1, 18, 59, 77, 46, 12, 1, 0)$
5	4	$(1, 12, 38, 51, 33, 10, 1, 0)$
5	4	$(1, 6, 15, 20, 15, 6, 1, 0)$
5	1	$(1, 24, 80, 106, 64, 16, 1, 0)$
5	4	$(1, 20, 70, 90, 50, 12, 1, 0)$
5	2	$(1, 18, 62, 91, 64, 19, 1, 0)$
6	1	$(1, 49, 272, 557, 535, 243, 42, 1)$

The polytopes ∇ and P° have 0 as the unique interior lattice point, and the Fano polytope P° defines the embedding toric Fano variety $Y^\circ = X(\Sigma(P^\circ))$ of the fibers of the mirror degeneration. The dimension of the automorphism group of Y° is

$$\dim(\mathrm{Aut}(Y^\circ)) = 6$$

Let

$$S^\circ = \mathbb{C}[y_1, y_2, ..., y_{49}]$$

be the Cox ring of Y° with variables

$$y_1 = y_{(-1,-1,2,0,0,-1)} = \frac{x_3^2 x_0}{x_1\, x_2\, x_6} \quad y_2 = y_{(-1,-1,0,2,0,-1)} = \frac{x_4^2 x_0}{x_1\, x_2\, x_6} \quad y_3 = y_{(2,0,-1,-1,1,-1)} = \frac{x_1^2 x_5}{x_3\, x_4\, x_6}$$
$$y_4 = y_{(0,2,-1,-1,1,-1)} = \frac{x_2^2 x_5}{x_3\, x_4\, x_6} \quad y_5 = y_{(1,2,-1,-1,0,0)} = \frac{x_1\, x_2{}^2}{x_3\, x_4\, x_0} \quad y_6 = y_{(2,1,-1,-1,0,0)} = \frac{x_1^2 x_2}{x_3\, x_4\, x_0}$$

$y_7 = y_{(0,0,-1,-1,3,-1)} = \frac{x_5^3}{x_3\,x_4\,x_6}$	$y_8 = y_{(-1,-1,0,0,0,-1)} = \frac{x_0^3}{x_1\,x_2\,x_6}$	$y_9 = y_{(-1,-1,1,2,-1,0)} = \frac{x_3\,x_4{}^2}{x_1\,x_2\,x_5}$
$y_{10} = y_{(-1,-1,2,1,-1,0)} = \frac{x_3^2 x_4}{x_1\,x_2\,x_5}$	$y_{11} = y_{(0,0,1,-1,1,-1)} = \frac{x_3\,x_5}{x_4\,x_6}$	$y_{12} = y_{(0,0,-1,1,1,-1)} = \frac{x_4\,x_5}{x_3\,x_6}$
$y_{13} = y_{(0,0,-1,-1,0,2)} = \frac{x_6^2}{x_3\,x_4}$	$y_{14} = y_{(0,0,-1,-1,0,0)} = \frac{x_0^2}{x_3\,x_4}$	$y_{15} = y_{(2,0,-1,-1,0,0)} = \frac{x_1^2}{x_3\,x_4}$
$y_{16} = y_{(0,2,-1,-1,0,0)} = \frac{x_2^2}{x_3\,x_4}$	$y_{17} = y_{(1,1,-1,0,0,0)} = \frac{x_1\,x_2}{x_3\,x_0}$	$y_{18} = y_{(1,1,0,-1,0,0)} = \frac{x_1\,x_2}{x_4\,x_0}$
$y_{19} = y_{(0,1,0,0,-1,1)} = \frac{x_6\,x_2}{x_5\,x_0}$	$y_{20} = y_{(0,0,1,0,-1,1)} = \frac{x_3\,x_6}{x_5\,x_0}$	$y_{21} = y_{(0,0,0,1,-1,1)} = \frac{x_4\,x_6}{x_5\,x_0}$
$y_{22} = y_{(0,0,0,0,-1,2)} = \frac{x_6^2}{x_5\,x_0}$	$y_{23} = y_{(1,0,0,0,-1,1)} = \frac{x_6\,x_1}{x_5\,x_0}$	$y_{24} = y_{(-1,-1,0,0,2,0)} = \frac{x_5^2}{x_1\,x_2}$
$y_{25} = y_{(-1,-1,0,2,0,0)} = \frac{x_4^2}{x_1\,x_2}$	$y_{26} = y_{(-1,-1,2,0,0,0)} = \frac{x_3^2}{x_1\,x_2}$	$y_{27} = y_{(-1,-1,0,0,0,2)} = \frac{x_6^2}{x_1\,x_2}$
$y_{28} = y_{(-1,1,0,0,0,-1)} = \frac{x_2\,x_0}{x_1\,x_6}$	$y_{29} = y_{(1,-1,0,0,0,-1)} = \frac{x_1\,x_0}{x_2\,x_6}$	$y_{30} = y_{(-1,0,1,1,-1,0)} = \frac{x_3\,x_4}{x_1\,x_5}$
$y_{31} = y_{(0,-1,1,1,-1,0)} = \frac{x_3\,x_4}{x_2\,x_5}$	$y_{32} = y_{(0,0,0,1,0,-1)} = \frac{x_4}{x_6}$	$y_{33} = y_{(0,0,1,0,0,-1)} = \frac{x_3}{x_6}$
$y_{34} = y_{(1,0,0,0,0,0)} = \frac{x_1}{x_0}$	$y_{35} = y_{(0,0,1,-1,0,0)} = \frac{x_3}{x_4}$	$y_{36} = y_{(0,0,-1,1,0,0)} = \frac{x_4}{x_3}$
$y_{37} = y_{(0,0,0,0,1,0)} = \frac{x_5}{x_0}$	$y_{38} = y_{(0,0,1,0,0,0)} = \frac{x_3}{x_0}$	$y_{39} = y_{(0,1,0,0,0,0)} = \frac{x_2}{x_0}$
$y_{40} = y_{(0,0,0,1,0,0)} = \frac{x_4}{x_0}$	$y_{41} = y_{(0,0,0,1,-1,0)} = \frac{x_4}{x_5}$	$y_{42} = y_{(0,0,1,0,-1,0)} = \frac{x_3}{x_5}$
$y_{43} = y_{(0,1,0,0,-1,0)} = \frac{x_2}{x_5}$	$y_{44} = y_{(1,0,0,0,-1,0)} = \frac{x_1}{x_5}$	$y_{45} = y_{(0,0,0,0,-1,0)} = \frac{x_0}{x_5}$
$y_{46} = y_{(1,-1,0,0,0,0)} = \frac{x_1}{x_2}$	$y_{47} = y_{(-1,1,0,0,0,0)} = \frac{x_2}{x_1}$	$y_{48} = y_{(0,1,0,0,0,-1)} = \frac{x_2}{x_6}$
$y_{49} = y_{(1,0,0,0,0,-1)} = \frac{x_1}{x_6}$		

10.5.4 Bergman subcomplex

Intersecting the tropical variety of I with the special fiber Gröbner cone $C_{I_0}(I)$ we obtain the special fiber Bergman subcomplex

$$B(I) = C_{I_0}(I) \cap BF(I) \cap \{w_t = 1\}$$

The following table chooses an indexing of the vertices of ∇ involved in $B(I)$ and gives for each vertex the numbers n_∇ and n_B of faces of ∇ and faces of $B(I)$ it is contained in

n_∇	$n_{B(I)}$		
59	6	$1 = (1,0,1,0,1,0)$	$2 = (0,1,1,0,1,0)$
		$3 = (1,0,0,1,1,0)$	$4 = (0,1,0,1,1,0)$
		$5 = (0,-1,0,-1,-1,-1)$	$6 = (-1,0,0,-1,-1,-1)$
		$7 = (0,-1,-1,0,-1,-1)$	$8 = (-1,0,-1,0,-1,-1)$
70	10	$9 = (0,0,1,0,1,1)$	$10 = (0,0,0,1,1,1)$
		$11 = (0,-1,-1,-1,-1,0)$	$12 = (-1,0,-1,-1,-1,0)$
		$13 = (0,-1,0,-1,0,0)$	$14 = (-1,0,0,-1,0,0)$
		$15 = (0,-1,-1,0,0,0)$	$16 = (-1,0,-1,0,0,0)$
80	8	$17 = (-1,-1,-1,-1,0,0)$	

With this indexing the Bergman subcomplex $B(I)$ of $\mathrm{Poset}(\nabla)$ associated to the degeneration $\mathfrak{X}$ is

[],

[[1], [2], [3], [4], [9], [10], [5], [6], [7], [8], [13], [14], [11], [12], [17], [15], [16]],

[[3, 7], [12, 17], [3, 15], [4, 8], [11, 15], [11, 17], [12, 16], [11, 12], [3, 10], [3, 4], [9, 10], [9, 14], [9, 13], [9, 17], [5, 7], [5, 13], [5, 11], [5, 6], [15, 16], [1, 2], [1, 13], [1, 3], [1, 9], [1, 5], [10, 17], [10, 15], [10, 16], [4, 16], [4, 10], [6, 12], [6, 14], [6, 8], [12, 14], [14, 17], [14, 16], [7, 15], [7, 8], [7, 11], [15, 17], [16, 17], [8, 12], [8, 16], [11, 13], [13, 17], [13, 15], [2, 4], [2, 6], [2, 9], [13, 14], [2, 14]],

[[13, 15, 17], [6, 12, 14], [6, 8, 12], [1, 3, 9, 10], [3, 4, 10], [1, 5, 13], [5, 11, 13], [5, 6, 13, 14], [10, 16, 17], [10, 15, 17], [4, 10, 16], [3, 4, 7, 8], [1, 3, 5, 7], [3, 7, 15], [12, 16, 17], [11, 12, 17], [12, 14, 17], [9, 10, 17], [9, 10, 13, 15], [9, 10, 14, 16], [2, 4, 9, 10], [2, 4, 6, 8], [4, 8, 16], [11, 13, 17], [2, 4, 14, 16], [1, 2, 3, 4], [11, 12, 13, 14], [7, 8, 11, 12], [5, 6, 11, 12], [2, 9, 14], [9, 14, 17], [1, 2, 5, 6], [9, 13, 14], [1, 2, 13, 14], [1, 2, 9], [1, 3, 13, 15], [3, 4, 15, 16], [3, 10, 15], [10, 15, 16], [7, 8, 15, 16], [15, 16, 17], [8, 12, 16], [12, 14, 16], [6, 8, 14, 16], [2, 6, 14], [11, 15, 17], [11, 12, 15, 16], [7, 11, 15], [11, 13, 15], [1, 9, 13], [5, 7, 13, 15], [5, 6, 7, 8], [5, 7, 11], [9, 13, 17], [13, 14, 17], [14, 16, 17]],

[[1, 2, 5, 6, 13, 14], [2, 4, 9, 10, 14, 16], [1, 3, 5, 7, 13, 15], [1, 2, 3, 4, 5, 6, 7, 8], [9, 13, 14, 17], [1, 2, 3, 4, 9, 10], [1, 3, 9, 10, 13, 15], [9, 10, 13, 15, 17], [2, 4, 6, 8, 14, 16], [5, 7, 11, 13, 15], [11, 13, 15, 17], [3, 4, 10, 15, 16], [11, 12, 15, 16, 17], [12, 14, 16, 17], [11, 12, 13, 14, 17], [7, 8, 11, 12, 15, 16], [3, 4, 7, 8, 15, 16], [6, 8, 12, 14, 16], [5, 6, 11, 12, 13, 14], [10, 15, 16, 17], [9, 10, 14, 16, 17], [5, 6, 7, 8, 11, 12], [1, 2, 9, 13, 14]],

[],

[],

[]

$B(I)$ has the F-vector

Dimension	−1	0	1	2	3	4	5	6
Number of faces	0	17	50	56	23	0	0	0

and the F-vectors of its faces are

Dimension	Number of faces	F-vector	
0	17	$(1, 1, 0, 0, 0, 0, 0, 0)$	point
1	50	$(1, 2, 1, 0, 0, 0, 0, 0)$	edge
2	22	$(1, 4, 4, 1, 0, 0, 0, 0)$	quadrangle
2	34	$(1, 3, 3, 1, 0, 0, 0, 0)$	triangle
3	4	$(1, 4, 6, 4, 1, 0, 0, 0)$	tetrahedron
3	10	$(1, 6, 9, 5, 1, 0, 0, 0)$	prism
3	8	$(1, 5, 8, 5, 1, 0, 0, 0)$	pyramid
3	1	$(1, 8, 12, 6, 1, 0, 0, 0)$	cube

10.5.5 Dual complex

The dual complex $\operatorname{dual}(B(I)) = (B(I))^*$ of deformations associated to $B(I)$ via initial ideals is given by

$$[],$$
$$[],$$
$$[],$$
$$[[9,13,14,17]^* = \left\langle \frac{x_1\,x_2}{x_3\,x_0}, \frac{x_4}{x_5}, \frac{x_4}{x_6} \right\rangle, [1,2,5,6,13,14]^* = \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_4}{x_3}, \frac{x_6\,x_4}{x_5\,x_0} \right\rangle,$$
$$[9,10,13,15,17]^* = \left\langle \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0}, \frac{x_2}{x_5}, \frac{x_2}{x_6} \right\rangle, [1,2,3,4,5,6,7,8]^* = \left\langle \frac{x_6^2}{x_5\,x_0}, \frac{x_6^2}{x_1\,x_2}, \frac{x_6^2}{x_3\,x_4} \right\rangle,$$
$$...],$$
$$[[5,6,13,14]^* = \left\langle \frac{x_4\,x_5}{x_3\,x_6}, \frac{x_1\,x_2}{x_3\,x_0}, \frac{x_4^2}{x_1\,x_2}, \frac{x_4}{x_3}, \frac{x_6\,x_4}{x_5\,x_0}, \frac{x_4}{x_0} \right\rangle,$$
$$[1,3,9,10]^* = \left\langle \frac{x_0^3}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_1\,x_6}, \frac{x_0^2}{x_3\,x_4}, \frac{x_0}{x_5}, \frac{x_2^2}{x_3\,x_4}, \frac{x_2}{x_5} \right\rangle,$$
$$[13,15,17]^* = \left\langle \frac{x_3\,x_4}{x_1\,x_5}, \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0}, \frac{x_6\,x_2}{x_5\,x_0}, \frac{x_2}{x_5}, \frac{x_2}{x_6}, \frac{x_2}{x_0} \right\rangle,$$
$$[6,12,14]^* = \left\langle \frac{x_1^2 x_5}{x_3\,x_4\,x_6}, \frac{x_4\,x_5}{x_3\,x_6}, \frac{x_4^2}{x_1\,x_2}, \frac{x_1}{x_2}, \frac{x_4}{x_0}, \frac{x_1}{x_0} \right\rangle,$$
$$...],$$
$$[[3,7]^* = \left\langle \frac{x_6^2}{x_5\,x_0}, \frac{x_6^2}{x_1\,x_2}, \frac{x_6^2}{x_3\,x_4}, \frac{x_3^2}{x_1\,x_2}, \frac{x_2}{x_1}, \frac{x_2^2}{x_3\,x_4}, \frac{x_3}{x_4}, \frac{x_6\,x_3}{x_5\,x_0}, \frac{x_6\,x_2}{x_5\,x_0} \right\rangle,$$
$$[3,15]^* = \left\langle \frac{x_3^2 x_0}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_1\,x_6}, \frac{x_3^2 x_4}{x_1\,x_2\,x_5}, \frac{x_3\,x_4}{x_1\,x_5}, \frac{x_3^2}{x_1\,x_2}, \frac{x_2}{x_1}, \frac{x_2^2}{x_3\,x_4}, \frac{x_3}{x_4}, \frac{x_6\,x_3}{x_5\,x_0}, \frac{x_6\,x_2}{x_5\,x_0}, \frac{x_2}{x_5}, \frac{x_3}{x_5} \right\rangle,$$
$$[11,12]^* = \left\langle \begin{array}{l} \frac{x_5^3}{x_3\,x_4\,x_6}, \frac{x_3^2 x_0}{x_1\,x_2\,x_6}, \frac{x_4^2 x_0}{x_1\,x_2\,x_6}, \frac{x_5^2}{x_1\,x_2}, \frac{x_3\,x_5}{x_4\,x_6}, \frac{x_4\,x_5}{x_3\,x_6}, \frac{x_3^2 x_4}{x_1\,x_2\,x_5}, \frac{x_3\,{x_4}^2}{x_1\,x_2\,x_5}, \frac{x_5}{x_0}, \frac{x_3^2}{x_1\,x_2}, \frac{x_4^2}{x_1\,x_2}, \\ \frac{x_3}{x_6}, \frac{x_4}{x_6}, \frac{x_3}{x_0}, \frac{x_4}{x_0} \end{array} \right\rangle,$$
$$[3,4]^* = \left\langle \frac{x_0^3}{x_1\,x_2\,x_6}, \frac{x_3^2 x_0}{x_1\,x_2\,x_6}, \frac{x_0^2}{x_3\,x_4}, \frac{x_6^2}{x_5\,x_0}, \frac{x_3^2 x_4}{x_1\,x_2\,x_5}, \frac{x_6^2}{x_1\,x_2}, \frac{x_0}{x_5}, \frac{x_6^2}{x_3\,x_4}, \frac{x_3^2}{x_1\,x_2}, \frac{x_3}{x_4}, \frac{x_6\,x_3}{x_5\,x_0}, \frac{x_3}{x_5} \right\rangle,$$
$$[15,16]^* = \left\langle \frac{x_3^2 x_0}{x_1\,x_2\,x_6}, \frac{x_3\,x_5}{x_6\,x_4}, \frac{x_3^2 x_4}{x_1\,x_2\,x_5}, \frac{x_1\,x_2}{x_4\,x_0}, \frac{x_3^2}{x_1\,x_2}, \frac{x_3}{x_4}, \frac{x_6\,x_3}{x_5\,x_0}, \frac{x_3}{x_5}, \frac{x_3}{x_6}, \frac{x_3}{x_0} \right\rangle,$$

...],

$$[[17]^* = \left\langle \begin{array}{c} \frac{x_3^2 x_4}{x_1\,x_2\,x_5}, \frac{x_3\,{x_4}^2}{x_1\,x_2\,x_5}, \frac{x_3\,x_4}{x_2\,x_5}, \frac{x_3\,x_4}{x_1\,x_5}, \frac{x_1^2 x_2}{x_3\,x_4\,x_0}, \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0}, \frac{x_1\,x_2}{x_3\,x_0}, \frac{x_1\,x_2}{x_4\,x_0}, \frac{x_6\,x_4}{x_5\,x_0}, \frac{x_6\,x_3}{x_5\,x_0}, \\ \frac{x_6\,x_1}{x_5\,x_0}, \frac{x_6\,x_2}{x_5\,x_0}, \frac{x_1}{x_5}, \frac{x_2}{x_5}, \frac{x_3}{x_5}, \frac{x_4}{x_5}, \frac{x_1}{x_6}, \frac{x_3}{x_6}, \frac{x_2}{x_6}, \frac{x_4}{x_6}, \frac{x_3}{x_0}, \frac{x_4}{x_0}, \frac{x_1}{x_0}, \frac{x_2}{x_0} \end{array} \right\rangle,$$

$$[1]^* = \left\langle \begin{array}{c} \frac{x_0^3}{x_1\,x_2\,x_6}, \frac{x_4^2 x_0}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_6\,x_1}, \frac{x_0^2}{x_3\,x_4}, \frac{x_6^2}{x_5\,x_0}, \frac{x_3\,{x_4}^2}{x_1\,x_2\,x_5}, \frac{x_6^2}{x_1\,x_2}, \frac{x_0}{x_5}, \frac{x_3\,x_4}{x_1\,x_5}, \frac{x_6^2}{x_3\,x_4}, \frac{x_4^2}{x_1\,x_2}, \frac{x_2}{x_1}, \\ \frac{x_2^2}{x_3\,x_4}, \frac{x_4}{x_3}, \frac{x_6\,x_4}{x_5\,x_0}, \frac{x_6\,x_2}{x_5\,x_0}, \frac{x_2}{x_5}, \frac{x_4}{x_5} \end{array} \right\rangle,$$

$$[9]^* = \left\langle \begin{array}{c} \frac{x_0^3}{x_1\,x_2\,x_6}, \frac{x_4^2 x_0}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_1\,x_6}, \frac{x_1\,x_0}{x_2\,x_6}, \frac{x_1^2 x_5}{x_3\,x_4\,x_6}, \frac{x_2^2 x_5}{x_3\,x_4\,x_6}, \frac{x_0^2}{x_3\,x_4}, \frac{x_4\,x_5}{x_3\,x_6}, \frac{x_0}{x_5}, \frac{x_1^2 x_2}{x_3\,x_4\,x_0}, \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0}, \\ \frac{x_1\,x_2}{x_3\,x_0}, \frac{x_1^2}{x_3\,x_4}, \frac{x_2^2}{x_3\,x_4}, \frac{x_4}{x_3}, \frac{x_1}{x_5}, \frac{x_2}{x_5}, \frac{x_4}{x_5}, \frac{x_1}{x_6}, \frac{x_2}{x_6}, \frac{x_4}{x_6} \end{array} \right\rangle,$$

$$[13]^* = \left\langle \begin{array}{c} \frac{x_4^2 x_0}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_6\,x_1}, \frac{x_2^2 x_5}{x_3\,x_4\,x_6}, \frac{x_4\,x_5}{x_3\,x_6}, \frac{x_3\,{x_4}^2}{x_1\,x_2\,x_5}, \frac{x_3\,x_4}{x_1\,x_5}, \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0}, \frac{x_1\,x_2}{x_3\,x_0}, \frac{x_4^2}{x_1\,x_2}, \frac{x_2}{x_1}, \frac{x_2^2}{x_3\,x_4}, \\ \frac{x_4}{x_3}, \frac{x_6\,x_4}{x_5\,x_0}, \frac{x_6\,x_2}{x_5\,x_0}, \frac{x_2}{x_5}, \frac{x_4}{x_5}, \frac{x_2}{x_6}, \frac{x_4}{x_6}, \frac{x_4}{x_0}, \frac{x_2}{x_0} \end{array} \right\rangle,$$

...],

[]

when writing the vertices of the faces as deformations of X_0. Note that the T-invariant basis of deformations associated to a face is given by all lattice points of the corresponding polytope in $M_{\mathbb{R}}$. In order to compress the output we list one representative in any set of faces G with fixed F-vector of G and of G^*. When numbering the vertices of the faces of $\operatorname{dual}(B(I))$ by the Cox variables of the mirror toric Fano variety Y° the complex $\operatorname{dual}(B(I))$ is

[],

[],

[],

$[[9,13,14,17]^* = \langle y_{17}, y_{41}, y_{32} \rangle, [1,2,5,6,13,14]^* = \langle y_{25}, y_{36}, y_{21} \rangle,$
$[9,10,13,15,17]^* = \langle y_5, y_{43}, y_{48} \rangle, [1,2,3,4,5,6,7,8]^* = \langle y_{22}, y_{27}, y_{13} \rangle,$
...],

$[[5,6,13,14]^* = \langle y_{12}, y_{17}, y_{25}, y_{36}, y_{21}, y_{40} \rangle,$
$[1,3,9,10]^* = \langle y_8, y_{28}, y_{14}, y_{45}, y_{16}, y_{43} \rangle,$
$[13,15,17]^* = \langle y_{30}, y_5, y_{19}, y_{43}, y_{48}, y_{39} \rangle,$
$[6,12,14]^* = \langle y_3, y_{12}, y_{25}, y_{46}, y_{40}, y_{34} \rangle,$
...],

$[[3,7]^* = \langle y_{22}, y_{27}, y_{13}, y_{26}, y_{47}, y_{16}, y_{35}, y_{20}, y_{19} \rangle,$
$[3,15]^* = \langle y_1, y_{28}, y_{10}, y_{30}, y_{26}, y_{47}, y_{16}, y_{35}, y_{20}, y_{19}, y_{43}, y_{42} \rangle,$
$[11,12]^* = \langle y_7, y_1, y_2, y_{24}, y_{11}, y_{12}, y_{10}, y_9, y_{37}, y_{26}, y_{25}, y_{33}, y_{32}, y_{38}, y_{40} \rangle,$
$[3,4]^* = \langle y_8, y_1, y_{14}, y_{22}, y_{10}, y_{27}, y_{45}, y_{13}, y_{26}, y_{35}, y_{20}, y_{42} \rangle,$

$[15,16]^* = \langle y_1, y_{11}, y_{10}, y_{18}, y_{26}, y_{35}, y_{20}, y_{42}, y_{33}, y_{38} \rangle$,
...],

$$[[17]^* = \left\langle \begin{array}{l} y_{10}, y_9, y_{31}, y_{30}, y_6, y_5, y_{17}, y_{18}, y_{21}, y_{20}, y_{23}, y_{19}, y_{44}, y_{43}, y_{42}, y_{41}, \\ y_{49}, y_{33}, y_{48}, y_{32}, y_{38}, y_{40}, y_{34}, y_{39} \end{array} \right\rangle,$$

$$[1]^* = \left\langle \begin{array}{l} y_8, y_2, y_{28}, y_{14}, y_{22}, y_9, y_{27}, y_{45}, y_{30}, y_{13}, y_{25}, y_{47}, y_{16}, y_{36}, y_{21}, y_{19}, \\ y_{43}, y_{41} \end{array} \right\rangle,$$

$$[9]^* = \left\langle \begin{array}{l} y_8, y_2, y_{28}, y_{29}, y_3, y_4, y_{14}, y_{12}, y_{45}, y_6, y_5, y_{17}, y_{15}, y_{16}, y_{36}, y_{44}, \\ y_{43}, y_{41}, y_{49}, y_{48}, y_{32} \end{array} \right\rangle,$$

$$[13]^* = \left\langle \begin{array}{l} y_2, y_{28}, y_4, y_{12}, y_9, y_{30}, y_5, y_{17}, y_{25}, y_{47}, y_{16}, y_{36}, y_{21}, y_{19}, y_{43}, y_{41}, \\ y_{48}, y_{32}, y_{40}, y_{39} \end{array} \right\rangle,$$

...],
[]

The dual complex has the F-vector

Dimension	−1	0	1	2	3	4	5	6
Number of faces	0	0	0	23	56	50	17	0

and the F-vectors of the faces of $\operatorname{dual}(B(I))$ are

Dimension	Number of faces	F-vector	
2	23	$(1,3,3,1,0,0,0,0)$	triangle
3	20	$(1,6,12,8,1,0,0,0)$	octahedron
3	36	$(1,6,9,5,1,0,0,0)$	prism
4	16	$(1,9,18,15,6,1,0,0)$	
4	20	$(1,12,30,28,10,1,0,0)$	
4	2	$(1,15,41,39,13,1,0,0)$	
4	8	$(1,12,31,30,11,1,0,0)$	
4	4	$(1,10,30,30,10,1,0,0)$	
5	1	$(1,24,80,106,64,16,1,0)$	
5	8	$(1,18,59,77,46,12,1,0)$	
5	4	$(1,21,70,91,54,14,1,0)$	
5	4	$(1,20,70,90,50,12,1,0)$	

Recall that in this example the toric variety Y is projective space. The number of lattice points of the support of $\operatorname{dual}(B(I))$ relates to the dimension $h^{1,2}(X)$ of the complex moduli space of the generic fiber X of $\mathfrak{X}$ and to the dimension $h^{1,1}\left(\bar{X}^\circ\right)$ of the Kähler moduli space of the $MPCP$-blowup $\bar{X}^\circ$

of the generic fiber X° of the mirror degeneration

$$\begin{aligned} |\mathrm{supp}\,(\mathrm{dual}\,(B\,(I))) \cap M| &= 109 = 48 + 61 = \dim\,(\mathrm{Aut}\,(Y)) + h^{1,2}\,(X) \\ &= 42 + 6 + 61 \\ &= |\mathrm{Roots}\,(Y)| + \dim\,(T_Y) + h^{1,1}\left(\bar{X}^{\circ}\right) \end{aligned}$$

There are

$$h^{1,2}\,(X) + \dim\,(T_{Y^{\circ}}) = 61 + 6$$

non-trivial toric polynomial deformations of X_0

$\frac{x_4^2}{x_1\,x_2}$	$\frac{x_6\,x_4}{x_5\,x_0}$	$\frac{x_1\,x_0}{x_2\,x_6}$	$\frac{x_1^2}{x_3\,x_4}$	$\frac{x_2^2}{x_3\,x_4}$	$\frac{x_2\,x_6}{x_5\,x_0}$	$\frac{x_6^2}{x_5\,x_0}$	$\frac{x_6^2}{x_3\,x_4}$	$\frac{x_6^2}{x_1\,x_2}$	$\frac{x_1\,x_2}{x_3\,x_0}$
$\frac{x_0^3}{x_1\,x_2\,x_6}$	$\frac{x_0^2}{x_3\,x_4}$	$\frac{x_2\,x_0}{x_1\,x_6}$	$\frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0}$	$\frac{x_1\,x_6}{x_5\,x_0}$	$\frac{x_2^2 x_5}{x_3\,x_4\,x_6}$	$\frac{x_3\,x_4}{x_1\,x_5}$	$\frac{x_3^2 x_0}{x_1\,x_2\,x_6}$	$\frac{x_3^2 x_4}{x_1\,x_2\,x_5}$	$\frac{x_3\,x_4}{x_2\,x_5}$
$\frac{x_3\,{x_4}^2}{x_1\,x_2\,x_5}$	$\frac{x_3\,x_5}{x_4\,x_6}$	$\frac{x_3^2}{x_1\,x_2}$	$\frac{x_6\,x_3}{x_5\,x_0}$	$\frac{x_1^2 x_5}{x_3\,x_4\,x_6}$	$\frac{x_4\,x_5}{x_3\,x_6}$	$\frac{x_1\,x_2}{x_4\,x_0}$	$\frac{x_1^2 x_2}{x_3\,x_4\,x_0}$	$\frac{x_5^3}{x_3\,x_4\,x_6}$	$\frac{x_5^2}{x_1\,x_2}$
$\frac{x_4^2 x_0}{x_1\,x_2\,x_6}$	$\frac{x_1\,x_5}{x_3\,x_6}$	$\frac{x_1\,{x_5}^2}{x_3\,x_4\,x_6}$	$\frac{x_0^2}{x_1\,x_6}$	$\frac{x_2\,x_0}{x_3\,x_4}$	$\frac{x_0^2 x_3}{x_1\,x_2\,x_6}$	$\frac{x_2\,x_5}{x_3\,x_6}$	$\frac{x_0\,x_3}{x_2\,x_6}$	$\frac{x_3\,x_6}{x_1\,x_2}$	$\frac{x_2\,x_6}{x_3\,x_4}$
$\frac{x_0^2}{x_2\,x_6}$	$\frac{x_1\,x_0}{x_3\,x_4}$	$\frac{x_6\,x_1}{x_3\,x_4}$	$\frac{x_0^2}{x_1\,x_2}$	$\frac{x_0\,x_6}{x_3\,x_4}$	$\frac{x_0\,x_6}{x_1\,x_2}$	$\frac{x_5^2}{x_4\,x_6}$	$\frac{x_5\,x_3}{x_1\,x_2}$	$\frac{x_5^2}{x_3\,x_6}$	$\frac{x_5\,x_4}{x_1\,x_2}$
$\frac{x_4\,x_0}{x_2\,x_6}$	$\frac{x_4\,x_6}{x_1\,x_2}$	$\frac{x_0^2 x_4}{x_1\,x_2\,x_6}$	$\frac{x_0\,x_3}{x_6\,x_1}$	$\frac{x_1\,x_5}{x_4\,x_6}$	$\frac{x_2\,x_5}{x_4\,x_6}$	$\frac{x_4\,x_0}{x_6\,x_1}$	$\frac{x_5^2}{x_3\,x_4}$	$\frac{x_5\,x_6}{x_3\,x_4}$	$\frac{x_5\,x_6}{x_1\,x_2}$
$\frac{x_2\,{x_5}^2}{x_3\,x_4\,x_6}$	$\frac{x_3\,x_4}{x_1\,x_2}$	$\frac{x_0\,x_3}{x_1\,x_2}$	$\frac{x_1\,x_2}{x_3\,x_4}$	$\frac{x_5\,x_2}{x_3\,x_4}$	$\frac{x_4\,x_0}{x_1\,x_2}$	$\frac{x_5\,x_1}{x_3\,x_4}$			

They correspond to the toric divisors

$D_{(-1,-1,0,2,0,0)}$	$D_{(0,0,0,1,-1,1)}$	$D_{(1,-1,0,0,0,-1)}$	$D_{(2,0,-1,-1,0,0)}$
$D_{(0,2,-1,-1,0,0)}$	$D_{(0,1,0,0,-1,1)}$	$D_{(0,0,0,0,-1,2)}$	$D_{(0,0,-1,-1,0,2)}$
$D_{(-1,-1,0,0,0,2)}$	$D_{(1,1,-1,0,0,0)}$	$D_{(-1,-1,0,0,0,-1)}$	$D_{(0,0,-1,-1,0,0)}$
$D_{(-1,1,0,0,0,-1)}$	$D_{(1,2,-1,-1,0,0)}$	$D_{(1,0,0,0,-1,1)}$	$D_{(0,2,-1,-1,1,-1)}$
$D_{(-1,0,1,1,-1,0)}$	$D_{(-1,-1,2,0,0,-1)}$	$D_{(-1,-1,2,1,-1,0)}$	$D_{(0,-1,1,1,-1,0)}$
$D_{(-1,-1,1,2,-1,0)}$	$D_{(0,0,1,-1,1,-1)}$	$D_{(-1,-1,2,0,0,0)}$	$D_{(0,0,1,0,-1,1)}$
$D_{(2,0,-1,-1,1,-1)}$	$D_{(0,0,-1,1,1,-1)}$	$D_{(1,1,0,-1,0,0)}$	$D_{(2,1,-1,-1,0,0)}$
$D_{(0,0,-1,-1,3,-1)}$	$D_{(-1,-1,0,0,2,0)}$	$D_{(-1,-1,0,2,0,-1)}$	$D_{(1,0,-1,0,1,-1)}$
$D_{(1,0,-1,-1,2,-1)}$	$D_{(-1,0,0,0,0,-1)}$	$D_{(0,1,-1,-1,0,0)}$	$D_{(-1,-1,1,0,0,-1)}$
$D_{(0,1,-1,0,1,-1)}$	$D_{(0,-1,1,0,0,-1)}$	$D_{(-1,-1,1,0,0,1)}$	$D_{(0,1,-1,-1,0,1)}$
$D_{(0,-1,0,0,0,-1)}$	$D_{(1,0,-1,-1,0,0)}$	$D_{(1,0,-1,-1,0,1)}$	$D_{(-1,-1,0,0,0,0)}$
$D_{(0,0,-1,-1,0,1)}$	$D_{(-1,-1,0,0,0,1)}$	$D_{(0,0,0,-1,2,-1)}$	$D_{(-1,-1,1,0,1,0)}$
$D_{(0,0,-1,0,2,-1)}$	$D_{(-1,-1,0,1,1,0)}$	$D_{(0,-1,0,1,0,-1)}$	$D_{(-1,-1,0,1,0,1)}$
$D_{(-1,-1,0,1,0,-1)}$	$D_{(-1,0,1,0,0,-1)}$	$D_{(1,0,0,-1,1,-1)}$	$D_{(0,1,0,-1,1,-1)}$
$D_{(-1,0,0,1,0,-1)}$	$D_{(0,0,-1,-1,2,0)}$	$D_{(0,0,-1,-1,1,1)}$	$D_{(-1,-1,0,0,1,1)}$
$D_{(0,1,-1,-1,2,-1)}$	$D_{(0,0,0,1,0,0)}$	$D_{(-1,-1,1,0,0,0)}$	$D_{(0,1,-1,-1,0,0)}$
$D_{(0,1,-1,-1,1,0)}$	$D_{(-1,-1,0,1,0,0)}$	$D_{(1,0,-1,-1,1,0)}$	

on a $MPCP$-blowup of Y° inducing 61 non-zero toric divisor classes on the mirror X°. The following 42 toric divisors of Y induce the trivial divisor class on X°

$$\begin{array}{llll}
D_{(0,0,-1,1,0,0)} & D_{(1,0,0,0,-1,0)} & D_{(-1,1,0,0,0,0)} & D_{(0,0,0,1,0,-1)} \\
D_{(0,0,0,1,-1,0)} & D_{(0,0,0,0,-1,0)} & D_{(0,1,0,0,-1,0)} & D_{(0,1,0,0,0,-1)} \\
D_{(1,-1,0,0,0,0)} & D_{(0,1,0,0,0,0)} & D_{(0,0,1,0,-1,0)} & D_{(0,0,1,-1,0,0)} \\
D_{(0,0,1,0,0,-1)} & D_{(0,0,1,0,0,0)} & D_{(1,0,0,0,0,-1)} & D_{(1,0,0,0,0,0)} \\
D_{(0,0,0,1,0,0)} & D_{(0,0,0,0,1,0)} & D_{(0,-1,0,1,0,0)} & D_{(0,-1,0,0,1,0)} \\
D_{(0,0,0,-1,0,0)} & D_{(0,1,-1,0,0,0)} & D_{(-1,0,0,1,0,0)} & D_{(1,0,0,-1,0,0)} \\
D_{(0,0,0,-1,0,1)} & D_{(-1,0,0,0,0,1)} & D_{(0,1,0,-1,0,0)} & D_{(-1,0,1,0,0,0)} \\
D_{(0,-1,0,0,0,1)} & D_{(0,-1,1,0,0,0)} & D_{(0,0,0,0,-1,1)} & D_{(1,0,-1,0,0,0)} \\
D_{(0,0,-1,0,0,1)} & D_{(0,0,-1,0,0,0)} & D_{(0,0,0,0,0,1)} & D_{(-1,0,0,0,1,0)} \\
D_{(0,0,1,-1,1,-1)} & D_{(0,0,0,0,0,-1)} & D_{(0,0,-1,0,1,0)} & D_{(-1,0,0,0,0,0)} \\
D_{(0,0,0,-1,1,0)} & D_{(0,-1,0,0,0,0)} & &
\end{array}$$

10.5.6 Mirror special fiber

The complex $B(I)^*$ labeled by the variables of the Cox ring S° of Y°, as written in the last section, is the complex $SP(I_0^\circ)$ of prime ideals of the toric strata of the monomial special fiber X_0° of the mirror degeneration $\mathfrak{X}^\circ$, i.e., the primary decomposition of I_0° is

$$I_0^\circ = \begin{array}{l}
\langle y_{17}, y_{41}, y_{32}\rangle \cap \langle y_{28}, y_{16}, y_{43}\rangle \cap \langle y_{5}, y_{43}, y_{48}\rangle \cap \langle y_{8}, y_{14}, y_{45}\rangle \cap \langle y_{26}, y_{35}, y_{20}\rangle \cap \\
\cap \langle y_{3}, y_{46}, y_{34}\rangle \cap \langle y_{12}, y_{25}, y_{40}\rangle \cap \langle y_{18}, y_{42}, y_{33}\rangle \cap \langle y_{6}, y_{44}, y_{49}\rangle \cap \langle y_{7}, y_{24}, y_{37}\rangle \cap \\
\cap \langle y_{2}, y_{36}, y_{41}\rangle \cap \langle y_{46}, y_{15}, y_{23}\rangle \cap \langle y_{30}, y_{48}, y_{39}\rangle \cap \langle y_{1}, y_{35}, y_{42}\rangle \cap \langle y_{10}, y_{33}, y_{38}\rangle \cap \\
\cap \langle y_{31}, y_{49}, y_{34}\rangle \cap \langle y_{9}, y_{32}, y_{40}\rangle \cap \langle y_{11}, y_{26}, y_{38}\rangle \cap \langle y_{4}, y_{47}, y_{39}\rangle \cap \langle y_{25}, y_{36}, y_{21}\rangle \cap \\
\cap \langle y_{29}, y_{15}, y_{44}\rangle \cap \langle y_{47}, y_{16}, y_{19}\rangle \cap \langle y_{22}, y_{27}, y_{13}\rangle
\end{array}$$

Each facet $F \in B(I)$ corresponds to one of these ideals and this ideal is generated by the facets of ∇ containing F.

10.5.7 Covering structure in the deformation complex of the degeneration $\mathfrak{X}$

According to the local reduced Gröbner basis each face of the complex of deformations dual $(B(I))$ decomposes into 3 respectively 5 polytopes forming a 5 : 1 ramified covering of $B(I)$

$[]$,

$[]$,

$[]$,

$[[\langle y_{41}\rangle, \langle y_{32}\rangle, \langle y_{17}\rangle] \mapsto [9,13,14,17]^\vee, [\langle y_{21}\rangle, \langle y_{25}\rangle, \langle y_{36}\rangle] \mapsto [1,2,5,6,13,14]^\vee,$
$[\langle y_{43}\rangle, \langle y_{48}\rangle, \langle y_{5}\rangle] \mapsto [9,10,13,15,17]^\vee, [\langle y_{22}\rangle, \langle y_{27}\rangle, \langle y_{13}\rangle] \mapsto [1,2,3,4,5,6,7,8]^\vee,$
$...]$,

$[[\langle y_{21}, y_{40}\rangle, \langle y_{25}\rangle, \langle y_{12}, y_{36}\rangle] \mapsto [5,6,13,14]^\vee, [\langle y_{45}, y_{43}\rangle, \langle y_{8}, y_{28}\rangle, \langle y_{14}, y_{16}\rangle] \mapsto [1,3,9,10]^\vee,$
$[\langle y_{30}, y_{43}\rangle, \langle y_{48}\rangle, \langle y_{5}, y_{39}\rangle] \mapsto [13,15,17]^\vee, [\langle y_{40}, y_{34}\rangle, \langle y_{25}, y_{46}\rangle, \langle y_{3}, y_{12}\rangle] \mapsto [6,12,14]^\vee,$
$...]$,

$[[\langle y_{22}, y_{20}, y_{19}\rangle, \langle y_{27}, y_{26}, y_{47}\rangle, \langle y_{13}, y_{16}, y_{35}\rangle] \mapsto [3,7]^\vee,$
$[\langle y_{20}, y_{19}, y_{43}, y_{42}\rangle, \langle y_{1}, y_{28}, y_{26}, y_{47}\rangle, \langle y_{16}, y_{35}\rangle] \mapsto [3,15]^\vee,$
$[\langle y_{37}, y_{38}, y_{40}\rangle, \langle y_{24}, y_{10}, y_{9}, y_{26}, y_{25}\rangle, \langle y_{7}, y_{11}, y_{12}, y_{33}, y_{32}\rangle] \mapsto [11,12]^\vee,$
$[\langle y_{22}, y_{45}, y_{20}, y_{42}\rangle, \langle y_{8}, y_{1}, y_{27}, y_{26}\rangle, \langle y_{14}, y_{13}, y_{35}\rangle] \mapsto [3,4]^\vee,$
$[\langle y_{20}, y_{42}, y_{38}\rangle, \langle y_{10}, y_{26}, y_{42}\rangle, \langle y_{1}, y_{26}, y_{33}\rangle, \langle y_{11}, y_{35}, y_{33}\rangle, \langle y_{18}, y_{35}, y_{38}\rangle] \mapsto [15,16]^\vee,$
$...]$,

$[[\langle y_{10}, y_{9}, y_{31}, y_{30}, y_{44}, y_{43}, y_{42}, y_{41}\rangle, \langle y_{49}, y_{33}, y_{48}, y_{32}\rangle, \langle y_{6}, y_{5}, y_{17}, y_{18}, y_{38}, y_{40}, y_{34}, y_{39}\rangle] \mapsto [17]^\vee,$
$[\langle y_{22}, y_{45}, y_{21}, y_{19}, y_{43}, y_{41}\rangle, \langle y_{8}, y_{2}, y_{28}, y_{27}, y_{25}, y_{47}\rangle, \langle y_{14}, y_{13}, y_{16}, y_{36}\rangle] \mapsto [1]^\vee,$
$[\langle y_{45}, y_{44}, y_{43}, y_{41}\rangle, \langle y_{8}, y_{2}, y_{28}, y_{29}, y_{49}, y_{48}, y_{32}\rangle, \langle y_{14}, y_{6}, y_{5}, y_{17}, y_{15}, y_{16}, y_{36}\rangle] \mapsto [9]^\vee,$
$[\langle y_{21}, y_{19}, y_{43}, y_{41}, y_{40}, y_{39}\rangle, \langle y_{9}, y_{30}, y_{25}, y_{47}, y_{43}, y_{41}\rangle, \langle y_{2}, y_{28}, y_{25}, y_{47}, y_{48}, y_{32}\rangle,$
$\langle y_{4}, y_{12}, y_{16}, y_{36}, y_{48}, y_{32}\rangle, \langle y_{5}, y_{17}, y_{16}, y_{36}, y_{40}, y_{39}\rangle] \mapsto [13]^\vee,$
$...]$,

$[]$

Here the faces $F \in B(I)$ are specified both via the vertices of F^* labeled by the variables of S° and by the numbering of the vertices of $B(I)$ chosen above. The numbers of faces of the covering in each face of $\mathrm{dual}(B(I))$, i.e., over each face of $B(I)^\vee$ are

Dimension	Number faces	number preimages
-1	0	0
0	0	0
1	0	0
2	23	3
3	56	3
4	46	3
4	4	5

5	13	3
5	4	5
6	0	0

This covering has one sheet forming the complex

$[],$

$[],$

$[],$

$[\langle y_{41}\rangle, \langle y_{32}\rangle, \langle y_{17}\rangle, \langle y_{21}\rangle, \langle y_{25}\rangle, \langle y_{36}\rangle, \langle y_{43}\rangle,$
$\langle y_{48}\rangle, \langle y_{5}\rangle, \langle y_{22}\rangle, \langle y_{27}\rangle, \langle y_{13}\rangle, \ldots],$

$[\langle y_{21}, y_{40}\rangle, \langle y_{25}\rangle, \langle y_{12}, y_{36}\rangle, \langle y_{45}, y_{43}\rangle, \langle y_{8}, y_{28}\rangle, \langle y_{14}, y_{16}\rangle,$
$\langle y_{30}, y_{43}\rangle, \langle y_{48}\rangle, \langle y_{5}, y_{39}\rangle, \langle y_{40}, y_{34}\rangle, \langle y_{25}, y_{46}\rangle, \langle y_{3}, y_{12}\rangle, \ldots],$

$[\langle y_{22}, y_{20}, y_{19}\rangle, \langle y_{27}, y_{26}, y_{47}\rangle, \langle y_{13}, y_{16}, y_{35}\rangle, \langle y_{20}, y_{19}, y_{43}, y_{42}\rangle,$
$\langle y_{1}, y_{28}, y_{26}, y_{47}\rangle, \langle y_{16}, y_{35}\rangle, \langle y_{37}, y_{38}, y_{40}\rangle, \langle y_{24}, y_{10}, y_{9}, y_{26}, y_{25}\rangle,$
$\langle y_{7}, y_{11}, y_{12}, y_{33}, y_{32}\rangle, \langle y_{22}, y_{45}, y_{20}, y_{42}\rangle, \langle y_{8}, y_{1}, y_{27}, y_{26}\rangle, \langle y_{14}, y_{13}, y_{35}\rangle,$
$\langle y_{20}, y_{42}, y_{38}\rangle, \langle y_{10}, y_{26}, y_{42}\rangle, \langle y_{1}, y_{26}, y_{33}\rangle, \langle y_{11}, y_{35}, y_{33}\rangle, \langle y_{18}, y_{35}, y_{38}\rangle, \ldots],$

$[\langle y_{10}, y_{9}, y_{31}, y_{30}, y_{44}, y_{43}, y_{42}, y_{41}\rangle, \langle y_{49}, y_{33}, y_{48}, y_{32}\rangle, \langle y_{6}, y_{5}, y_{17}, y_{18}, y_{38}, y_{40}, y_{34}, y_{39}\rangle,$
$\langle y_{22}, y_{45}, y_{21}, y_{19}, y_{43}, y_{41}\rangle, \langle y_{8}, y_{2}, y_{28}, y_{27}, y_{25}, y_{47}\rangle, \langle y_{14}, y_{13}, y_{16}, y_{36}\rangle, \langle y_{45}, y_{44}, y_{43}, y_{41}\rangle,$
$\langle y_{8}, y_{2}, y_{28}, y_{29}, y_{49}, y_{48}, y_{32}\rangle, \langle y_{14}, y_{6}, y_{5}, y_{17}, y_{15}, y_{16}, y_{36}\rangle, \langle y_{21}, y_{19}, y_{43}, y_{41}, y_{40}, y_{39}\rangle,$
$\langle y_{9}, y_{30}, y_{25}, y_{47}, y_{43}, y_{41}\rangle, \langle y_{2}, y_{28}, y_{25}, y_{47}, y_{48}, y_{32}\rangle, \langle y_{4}, y_{12}, y_{16}, y_{36}, y_{48}, y_{32}\rangle,$
$\langle y_{5}, y_{17}, y_{16}, y_{36}, y_{40}, y_{39}\rangle, \ldots],$

$[]$

with F-vector

Dimension	Number of faces	F-vector	
0	49	$(1, 1, 0, 0, 0, 0, 0, 0)$	point
1	128	$(1, 2, 1, 0, 0, 0, 0, 0)$	edge
2	56	$(1, 4, 4, 1, 0, 0, 0, 0)$	quadrangle
2	78	$(1, 3, 3, 1, 0, 0, 0, 0)$	triangle
2	4	$(1, 5, 5, 1, 0, 0, 0, 0)$	pentagon
3	13	$(1, 4, 6, 4, 1, 0, 0, 0)$	tetrahedron
3	36	$(1, 6, 9, 5, 1, 0, 0, 0)$	prism
3	8	$(1, 7, 11, 6, 1, 0, 0, 0)$	
3	2	$(1, 8, 14, 8, 1, 0, 0, 0)$	

Writing the vertices of the faces as deformations the covering is given by

$$[],$$

$$[],$$

$$[],$$

$$[[\left\langle \frac{x_4}{x_5} \right\rangle, \left\langle \frac{x_4}{x_6} \right\rangle, \left\langle \frac{x_1\,x_2}{x_3\,x_0} \right\rangle] \mapsto [9,13,14,17]^\vee,$$
$$[\left\langle \frac{x_6\,x_4}{x_5\,x_0} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_4}{x_3} \right\rangle] \mapsto [1,2,5,6,13,14]^\vee,$$
$$[\left\langle \frac{x_2}{x_5} \right\rangle, \left\langle \frac{x_2}{x_6} \right\rangle, \left\langle \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0} \right\rangle] \mapsto [9,10,13,15,17]^\vee,$$
$$[\left\langle \frac{x_6^2}{x_5\,x_0} \right\rangle, \left\langle \frac{x_6^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_6^2}{x_3\,x_4} \right\rangle] \mapsto [1,2,3,4,5,6,7,8]^\vee,$$
$$...],$$

$$[[\left\langle \frac{x_6\,x_4}{x_5\,x_0}, \frac{x_4}{x_0} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_4\,x_5}{x_3\,x_6}, \frac{x_4}{x_3} \right\rangle] \mapsto [5,6,13,14]^\vee,$$
$$[\left\langle \frac{x_0}{x_5}, \frac{x_2}{x_5} \right\rangle, \left\langle \frac{x_0^3}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_1\,x_6} \right\rangle, \left\langle \frac{x_0^2}{x_3\,x_4}, \frac{{x_2}^2}{x_3\,x_4} \right\rangle] \mapsto [1,3,9,10]^\vee,$$
$$[\left\langle \frac{x_3\,x_4}{x_1\,x_5}, \frac{x_2}{x_5} \right\rangle, \left\langle \frac{x_2}{x_6} \right\rangle, \left\langle \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0}, \frac{x_2}{x_0} \right\rangle] \mapsto [13,15,17]^\vee,$$
$$[\left\langle \frac{x_4}{x_0}, \frac{x_1}{x_0} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_1^2 x_5}{x_3\,x_4\,x_6}, \frac{x_4\,x_5}{x_3\,x_6} \right\rangle] \mapsto [6,12,14]^\vee,$$
$$...],$$

$$[[\left\langle \frac{x_6^2}{x_5\,x_0}, \frac{x_6\,x_3}{x_5\,x_0}, \frac{x_6\,x_2}{x_5\,x_0} \right\rangle, \left\langle \frac{x_6^2}{x_1\,x_2}, \frac{{x_3}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle,$$
$$\left\langle \frac{x_6^2}{x_3\,x_4}, \frac{{x_2}^2}{x_3\,x_4}, \frac{x_3}{x_4} \right\rangle] \mapsto [3,7]^\vee, [\left\langle \frac{x_6\,x_3}{x_5\,x_0}, \frac{x_6\,x_2}{x_5\,x_0}, \frac{x_2}{x_5}, \frac{x_3}{x_5} \right\rangle, \left\langle \frac{x_3^2 x_0}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_1\,x_6}, \frac{{x_3}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle,$$
$$\left\langle \frac{x_2^2}{x_3\,x_4}, \frac{x_3}{x_4} \right\rangle] \mapsto [3,15]^\vee, [\left\langle \frac{x_5}{x_0}, \frac{x_3}{x_0}, \frac{x_4}{x_0} \right\rangle, \left\langle \frac{x_5^2}{x_1\,x_2}, \frac{{x_3}^2 x_4}{x_1\,x_2\,x_5}, \frac{x_3\,{x_4}^2}{x_1\,x_2\,x_5}, \frac{{x_3}^2}{x_1\,x_2}, \frac{{x_4}^2}{x_1\,x_2} \right\rangle,$$
$$\left\langle \frac{x_5^3}{x_3\,x_4\,x_6}, \frac{x_3\,x_5}{x_4\,x_6}, \frac{x_4\,x_5}{x_3\,x_6}, \frac{x_3}{x_6}, \frac{x_4}{x_6} \right\rangle] \mapsto [11,12]^\vee,$$
$$[\left\langle \frac{x_6^2}{x_5\,x_0}, \frac{x_0}{x_5}, \frac{x_6\,x_3}{x_5\,x_0}, \frac{x_3}{x_5} \right\rangle, \left\langle \frac{x_0^3}{x_1\,x_2\,x_6}, \frac{{x_3}^2 x_0}{x_1\,x_2\,x_6}, \frac{{x_6}^2}{x_1\,x_2}, \frac{{x_3}^2}{x_1\,x_2} \right\rangle,$$
$$\left\langle \frac{x_0^2}{x_3\,x_4}, \frac{{x_6}^2}{x_3\,x_4}, \frac{x_3}{x_4} \right\rangle] \mapsto [3,4]^\vee, [\left\langle \frac{x_3\,x_6}{x_5\,x_0}, \frac{x_3}{x_5}, \frac{x_3}{x_0} \right\rangle, \left\langle \frac{x_3^2 x_4}{x_1\,x_2\,x_5}, \frac{{x_3}^2}{x_1\,x_2}, \frac{x_3}{x_5} \right\rangle,$$
$$\left\langle \frac{x_3^2 x_0}{x_1\,x_2\,x_6}, \frac{{x_3}^2}{x_1\,x_2}, \frac{x_3}{x_6} \right\rangle, \left\langle \frac{x_3\,x_5}{x_4\,x_6}, \frac{x_3}{x_4}, \frac{x_3}{x_6} \right\rangle,$$
$$\left\langle \frac{x_1\,x_2}{x_4\,x_0}, \frac{x_3}{x_4}, \frac{x_3}{x_0} \right\rangle] \mapsto [15,16]^\vee, ...],$$

$$[[\left\langle \frac{x_3^2 x_4}{x_1\,x_2\,x_5}, \frac{x_3\,{x_4}^2}{x_1\,x_2\,x_5}, \frac{x_3\,x_4}{x_2\,x_5}, \frac{x_3\,x_4}{x_1\,x_5}, \frac{x_1}{x_5}, \frac{x_2}{x_5}, \frac{x_3}{x_5}, \frac{x_4}{x_5} \right\rangle,$$
$$\left\langle \frac{x_1}{x_6}, \frac{x_3}{x_6}, \frac{x_2}{x_6}, \frac{x_4}{x_6} \right\rangle, \left\langle \frac{x_1^2 x_2}{x_3\,x_4\,x_0}, \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0}, \frac{x_1\,x_2}{x_3\,x_0}, \frac{x_1\,x_2}{x_4\,x_0}, \frac{x_3}{x_0}, \frac{x_4}{x_0}, \frac{x_1}{x_0}, \frac{x_2}{x_0} \right\rangle] \mapsto [17]^\vee,$$
$$[\left\langle \frac{x_6^2}{x_5\,x_0}, \frac{x_0}{x_5}, \frac{x_6\,x_4}{x_5\,x_0}, \frac{x_2\,x_6}{x_5\,x_0}, \frac{x_2}{x_5}, \frac{x_4}{x_5} \right\rangle, \left\langle \frac{x_0^3}{x_1\,x_2\,x_6}, \frac{{x_4}^2 x_0}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_1\,x_6}, \frac{{x_6}^2}{x_1\,x_2}, \frac{{x_4}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle,$$
$$\left\langle \frac{x_0^2}{x_3\,x_4}, \frac{{x_6}^2}{x_3\,x_4}, \frac{{x_2}^2}{x_3\,x_4}, \frac{x_4}{x_3} \right\rangle] \mapsto [1]^\vee, [\left\langle \frac{x_0}{x_5}, \frac{x_1}{x_5}, \frac{x_2}{x_5}, \frac{x_4}{x_5} \right\rangle, \left\langle \frac{x_0^3}{x_1\,x_2\,x_6}, \frac{{x_4}^2 x_0}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_1\,x_6}, \frac{x_1\,x_0}{x_2\,x_6}, \frac{x_1}{x_6}, \frac{x_2}{x_6}, \frac{x_4}{x_6} \right\rangle,$$

$$\left\langle \frac{x_0^2}{x_3\,x_4}, \frac{{x_1}^2x_2}{x_3\,x_4\,x_0}, \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0}, \frac{x_1\,x_2}{x_3\,x_0}, \frac{{x_1}^2}{x_3\,x_4}, \frac{{x_2}^2}{x_3\,x_4}, \frac{x_4}{x_3} \right\rangle] \mapsto [9]^\vee,$$

$$[\left\langle \frac{x_6\,x_4}{x_5\,x_0}, \frac{x_6\,x_2}{x_5\,x_0}, \frac{x_2}{x_5}, \frac{x_4}{x_5}, \frac{x_4}{x_0}, \frac{x_2}{x_0} \right\rangle, \left\langle \frac{x_3\,{x_4}^2}{x_1\,x_2\,x_5}, \frac{x_3\,x_4}{x_1\,x_5}, \frac{{x_4}^2}{x_1\,x_2}, \frac{x_2}{x_1}, \frac{x_2}{x_5}, \frac{x_4}{x_5} \right\rangle,$$

$$\left\langle \frac{x_4^2x_0}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_1\,x_6}, \frac{{x_4}^2}{x_1\,x_2}, \frac{x_2}{x_1}, \frac{x_2}{x_6}, \frac{x_4}{x_6} \right\rangle, \left\langle \frac{x_2^2x_5}{x_3\,x_4\,x_6}, \frac{x_4\,x_5}{x_3\,x_6}, \frac{{x_2}^2}{x_3\,x_4}, \frac{x_4}{x_3}, \frac{x_2}{x_6}, \frac{x_4}{x_6} \right\rangle,$$

$$\left\langle \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0}, \frac{x_1\,x_2}{x_3\,x_0}, \frac{{x_2}^2}{x_3\,x_4}, \frac{x_4}{x_3}, \frac{x_4}{x_0}, \frac{x_2}{x_0} \right\rangle] \mapsto [13]^\vee,$$

$$...],$$

$$[]$$

with the one sheet forming the complex

$$[],$$

$$[],$$

$$[],$$

$$[\left\langle \frac{x_4}{x_5} \right\rangle, \left\langle \frac{x_4}{x_6} \right\rangle, \left\langle \frac{x_1\,x_2}{x_3\,x_0} \right\rangle, \left\langle \frac{x_6\,x_4}{x_5\,x_0} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_4}{x_3} \right\rangle, \left\langle \frac{x_2}{x_5} \right\rangle,$$
$$\left\langle \frac{x_2}{x_6} \right\rangle, \left\langle \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0} \right\rangle, \left\langle \frac{x_6^2}{x_5\,x_0} \right\rangle, \left\langle \frac{x_6^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_6^2}{x_3\,x_4} \right\rangle, ...],$$

$$[\left\langle \frac{x_6\,x_4}{x_5\,x_0}, \frac{x_4}{x_0} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_4\,x_5}{x_3\,x_6}, \frac{x_4}{x_3} \right\rangle, \left\langle \frac{x_0}{x_5}, \frac{x_2}{x_5} \right\rangle, \left\langle \frac{x_0^3}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_1\,x_6} \right\rangle,$$
$$\left\langle \frac{x_0^2}{x_3\,x_4}, \frac{{x_2}^2}{x_3\,x_4} \right\rangle, \left\langle \frac{x_3\,x_4}{x_1\,x_5}, \frac{x_2}{x_5} \right\rangle, \left\langle \frac{x_2}{x_6} \right\rangle, \left\langle \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0}, \frac{x_2}{x_0} \right\rangle,$$
$$\left\langle \frac{x_4}{x_0}, \frac{x_1}{x_0} \right\rangle, \left\langle \frac{x_4^2}{x_1\,x_2}, \frac{x_1}{x_2} \right\rangle, \left\langle \frac{x_1^2x_5}{x_3\,x_4\,x_6}, \frac{x_4\,x_5}{x_3\,x_6} \right\rangle, ...],$$

$$[\left\langle \frac{x_6^2}{x_5\,x_0}, \frac{x_3\,x_6}{x_5\,x_0}, \frac{x_6\,x_2}{x_5\,x_0} \right\rangle, \left\langle \frac{x_6^2}{x_1\,x_2}, \frac{{x_3}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle, \left\langle \frac{x_6^2}{x_3\,x_4}, \frac{{x_2}^2}{x_3\,x_4}, \frac{x_3}{x_4} \right\rangle,$$
$$\left\langle \frac{x_6\,x_3}{x_5\,x_0}, \frac{x_6\,x_2}{x_5\,x_0}, \frac{x_2}{x_5}, \frac{x_3}{x_5} \right\rangle, \left\langle \frac{x_3^2x_0}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_1\,x_6}, \frac{{x_3}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle,$$
$$\left\langle \frac{x_2^2}{x_3\,x_4}, \frac{x_3}{x_4} \right\rangle, \left\langle \frac{x_5}{x_0}, \frac{x_3}{x_0}, \frac{x_4}{x_0} \right\rangle, \left\langle \frac{x_5^2}{x_1\,x_2}, \frac{{x_3}^2x_4}{x_1\,x_2\,x_5}, \frac{x_3\,{x_4}^2}{x_1\,x_2\,x_5}, \frac{{x_3}^2}{x_1\,x_2}, \frac{{x_4}^2}{x_1\,x_2} \right\rangle,$$
$$\left\langle \frac{x_5^3}{x_3\,x_4\,x_6}, \frac{x_3\,x_5}{x_4\,x_6}, \frac{x_4\,x_5}{x_3\,x_6}, \frac{x_3}{x_6}, \frac{x_4}{x_6} \right\rangle, \left\langle \frac{x_6^2}{x_5\,x_0}, \frac{x_0}{x_5}, \frac{x_3\,x_6}{x_5\,x_0}, \frac{x_3}{x_5} \right\rangle,$$
$$\left\langle \frac{x_0^3}{x_1\,x_2\,x_6}, \frac{{x_3}^2x_0}{x_1\,x_2\,x_6}, \frac{{x_6}^2}{x_1\,x_2}, \frac{{x_3}^2}{x_1\,x_2} \right\rangle, \left\langle \frac{x_0^2}{x_3\,x_4}, \frac{{x_6}^2}{x_3\,x_4}, \frac{x_3}{x_4} \right\rangle,$$
$$\left\langle \frac{x_6\,x_3}{x_5\,x_0}, \frac{x_3}{x_5}, \frac{x_3}{x_0} \right\rangle, \left\langle \frac{x_3^2x_4}{x_1\,x_2\,x_5}, \frac{{x_3}^2}{x_1\,x_2}, \frac{x_3}{x_5} \right\rangle, \left\langle \frac{x_3^2x_0}{x_1\,x_2\,x_6}, \frac{{x_3}^2}{x_1\,x_2}, \frac{x_3}{x_6} \right\rangle,$$
$$\left\langle \frac{x_3\,x_5}{x_4\,x_6}, \frac{x_3}{x_4}, \frac{x_3}{x_6} \right\rangle, \left\langle \frac{x_1\,x_2}{x_4\,x_0}, \frac{x_3}{x_4}, \frac{x_3}{x_0} \right\rangle, ...],$$

$$[\left\langle \frac{x_3^2x_4}{x_1\,x_2\,x_5}, \frac{x_3\,{x_4}^2}{x_1\,x_2\,x_5}, \frac{x_3\,x_4}{x_2\,x_5}, \frac{x_3\,x_4}{x_1\,x_5}, \frac{x_1}{x_5}, \frac{x_2}{x_5}, \frac{x_3}{x_5}, \frac{x_4}{x_5} \right\rangle,$$
$$\left\langle \frac{x_1}{x_6}, \frac{x_3}{x_6}, \frac{x_2}{x_6}, \frac{x_4}{x_6} \right\rangle, \left\langle \frac{x_1^2x_2}{x_3\,x_4\,x_0}, \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0}, \frac{x_1\,x_2}{x_3\,x_0}, \frac{x_1\,x_2}{x_4\,x_0}, \frac{x_3}{x_0}, \frac{x_4}{x_0}, \frac{x_1}{x_0}, \frac{x_2}{x_0} \right\rangle,$$
$$\left\langle \frac{x_6^2}{x_5\,x_0}, \frac{x_0}{x_5}, \frac{x_6\,x_4}{x_5\,x_0}, \frac{x_6\,x_2}{x_5\,x_0}, \frac{x_2}{x_5}, \frac{x_4}{x_5} \right\rangle,$$

$$\left\langle \frac{x_0^3}{x_1\,x_2\,x_6}, \frac{{x_4}^2x_0}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_1\,x_6}, \frac{{x_6}^2}{x_1\,x_2}, \frac{{x_4}^2}{x_1\,x_2}, \frac{x_2}{x_1} \right\rangle,$$
$$\left\langle \frac{x_0^2}{x_3\,x_4}, \frac{{x_6}^2}{x_3\,x_4}, \frac{{x_2}^2}{x_3\,x_4}, \frac{x_4}{x_3} \right\rangle, \left\langle \frac{x_0}{x_5}, \frac{x_1}{x_5}, \frac{x_2}{x_5}, \frac{x_4}{x_5} \right\rangle,$$
$$\left\langle \frac{x_0^3}{x_1\,x_2\,x_6}, \frac{{x_4}^2x_0}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_1\,x_6}, \frac{x_1\,x_0}{x_2\,x_6}, \frac{x_1}{x_6}, \frac{x_2}{x_6}, \frac{x_4}{x_6} \right\rangle,$$
$$\left\langle \frac{x_0^2}{x_3\,x_4}, \frac{{x_1}^2x_2}{x_3\,x_4\,x_0}, \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0}, \frac{x_1\,x_2}{x_3\,x_0}, \frac{{x_1}^2}{x_3\,x_4}, \frac{{x_2}^2}{x_3\,x_4}, \frac{x_4}{x_3} \right\rangle,$$
$$\left\langle \frac{x_6\,x_4}{x_5\,x_0}, \frac{x_6\,x_2}{x_5\,x_0}, \frac{x_2}{x_5}, \frac{x_4}{x_5}, \frac{x_4}{x_0}, \frac{x_2}{x_0} \right\rangle, \left\langle \frac{x_3\,{x_4}^2}{x_1\,x_2\,x_5}, \frac{x_3\,x_4}{x_1\,x_5}, \frac{{x_4}^2}{x_1\,x_2}, \frac{x_2}{x_1}, \frac{x_2}{x_5}, \frac{x_4}{x_5} \right\rangle,$$
$$\left\langle \frac{x_4^2x_0}{x_1\,x_2\,x_6}, \frac{x_2\,x_0}{x_1\,x_6}, \frac{{x_4}^2}{x_1\,x_2}, \frac{x_2}{x_1}, \frac{x_2}{x_6}, \frac{x_4}{x_6} \right\rangle,$$
$$\left\langle \frac{x_2^2x_5}{x_3\,x_4\,x_6}, \frac{x_4\,x_5}{x_3\,x_6}, \frac{{x_2}^2}{x_3\,x_4}, \frac{x_4}{x_3}, \frac{x_2}{x_6}, \frac{x_4}{x_6} \right\rangle,$$
$$\left\langle \frac{x_1\,{x_2}^2}{x_3\,x_4\,x_0}, \frac{x_1\,x_2}{x_3\,x_0}, \frac{{x_2}^2}{x_3\,x_4}, \frac{x_4}{x_3}, \frac{x_4}{x_0}, \frac{x_2}{x_0} \right\rangle, \ldots],$$

[]

Note that the torus invariant basis of deformations corresponding to a Bergman face is given by the set of all lattice points of the polytope specified above.

10.5.8 Limit map

The limit map $\lim : B(I) \to \mathrm{Poset}(\Delta)$ associates to a face F of $B(I)$ the face of Δ formed by the limit points of arcs lying over the weight vectors $w \in F$, i.e., with lowest order term t^w. Labeling the faces of the Bergman complex $B(I) \subset \mathrm{Poset}(\nabla)$ and the faces of $\mathrm{Poset}(\Delta)$ by the corresponding dual faces of ∇^* and Δ^*, hence considering the limit map $\lim : B(I) \to \mathrm{Poset}(\Delta)$ as a map $B(I)^* \to \mathrm{Poset}(\Delta^*)$, the limit correspondence is given by

[],

$$[\left\langle \begin{array}{l} y_{10}, y_9, y_{31}, y_{30}, y_6, y_5, y_{17}, y_{18}, y_{21}, y_{20}, y_{23}, y_{19}, y_{44}, y_{43}, y_{42}, y_{41}, \\ y_{49}, y_{33}, y_{48}, y_{32}, y_{38}, y_{40}, y_{34}, y_{39} \end{array} \right\rangle \mapsto \langle x_5, x_6, x_0 \rangle,$$
$$\left\langle \begin{array}{l} y_8, y_2, y_{28}, y_{14}, y_{22}, y_9, y_{27}, y_{45}, y_{30}, y_{13}, y_{25}, y_{47}, y_{16}, y_{36}, y_{21}, y_{19}, \\ y_{43}, y_{41} \end{array} \right\rangle \mapsto \langle x_1, x_3, x_5 \rangle,$$
$$\left\langle \begin{array}{l} y_8, y_2, y_{28}, y_{29}, y_3, y_4, y_{14}, y_{12}, y_{45}, y_6, y_5, y_{17}, y_{15}, y_{16}, y_{36}, y_{44}, \\ y_{43}, y_{41}, y_{49}, y_{48}, y_{32} \end{array} \right\rangle \mapsto \langle x_3, x_5, x_6 \rangle,$$
$$\left\langle \begin{array}{l} y_2, y_{28}, y_4, y_{12}, y_9, y_{30}, y_5, y_{17}, y_{25}, y_{47}, y_{16}, y_{36}, y_{21}, y_{19}, y_{43}, y_{41}, \\ y_{48}, y_{32}, y_{40}, y_{39} \end{array} \right\rangle \mapsto \langle x_1, x_3, x_5, x_6, x_0 \rangle,$$
$$\ldots],$$

$$[\langle y_{22}, y_{27}, y_{13}, y_{26}, y_{47}, y_{16}, y_{35}, y_{20}, y_{19} \rangle \mapsto \langle x_1, x_4, x_5, x_0 \rangle,$$
$$\langle y_1, y_{28}, y_{10}, y_{30}, y_{26}, y_{47}, y_{16}, y_{35}, y_{20}, y_{19}, y_{43}, y_{42} \rangle \mapsto \langle x_1, x_4, x_5, x_6, x_0 \rangle,$$
$$\langle y_7, y_1, y_2, y_{24}, y_{11}, y_{12}, y_{10}, y_9, y_{37}, y_{26}, y_{25}, y_{33}, y_{32}, y_{38}, y_{40} \rangle \mapsto \langle x_1, x_2, x_6, x_0 \rangle,$$

$\langle y_8, y_1, y_{14}, y_{22}, y_{10}, y_{27}, y_{45}, y_{13}, y_{26}, y_{35}, y_{20}, y_{42}\rangle \mapsto \langle x_1, x_2, x_4, x_5\rangle$,
$\langle y_1, y_{11}, y_{10}, y_{18}, y_{26}, y_{35}, y_{20}, y_{42}, y_{33}, y_{38}\rangle \mapsto \langle x_1, x_2, x_4, x_5, x_6, x_0\rangle$,
...],

[$\langle y_{12}, y_{17}, y_{25}, y_{36}, y_{21}, y_{40}\rangle \mapsto \langle x_1, x_2, x_3, x_5, x_6, x_0\rangle$,
$\langle y_8, y_{28}, y_{14}, y_{45}, y_{16}, y_{43}\rangle \mapsto \langle x_1, x_3, x_4, x_5, x_6\rangle$,
$\langle y_{30}, y_5, y_{19}, y_{43}, y_{48}, y_{39}\rangle \mapsto \langle x_1, x_3, x_4, x_5, x_6, x_0\rangle$,
$\langle y_3, y_{12}, y_{25}, y_{46}, y_{40}, y_{34}\rangle \mapsto \langle x_2, x_3, x_5, x_6, x_0\rangle$,
...],

[$\langle y_{17}, y_{41}, y_{32}\rangle \mapsto \langle x_1, x_2, x_3, x_5, x_6, x_0\rangle$,
$\langle y_{25}, y_{36}, y_{21}\rangle \mapsto \langle x_1, x_2, x_3, x_5, x_6, x_0\rangle$,
$\langle y_5, y_{43}, y_{48}\rangle \mapsto \langle x_1, x_3, x_4, x_5, x_6, x_0\rangle$,
$\langle y_{22}, y_{27}, y_{13}\rangle \mapsto \langle x_1, x_2, x_3, x_4, x_5, x_0\rangle$,
...],

[],

[],

[]

10.5.9 Mirror complex

Numbering the vertices of the mirror complex $\mu(B(I))$ as

$$\begin{array}{ll}
1 = (6, -1, -1, -1, -1, -1) & 2 = (-1, 6, -1, -1, -1, -1) \\
3 = (-1, -1, 6, -1, -1, -1) & 4 = (-1, -1, -1, 6, -1, -1) \\
5 = (-1, -1, -1, -1, 6, -1) & 6 = (-1, -1, -1, -1, -1, 6) \\
7 = (-1, -1, -1, -1, -1, -1) &
\end{array}$$

$\mu(B(I)) \subset \operatorname{Poset}(\Delta)$ is

[],

[[4], [1], [2], [6], [7], [3], [5]],

[[1, 4], [1, 5], [2, 7], [3, 7], [2, 4], [1, 3], [2, 3], [3, 6], [2, 6], [3, 4], [1, 2], [1, 7], [1, 6], [6, 7], [3, 5], [4, 5], [4, 6], [4, 7], [5, 6], [2, 5]],

[[2, 3, 6], [1, 3, 4], [1, 3, 6], [2, 3, 4], [3, 4, 5], [2, 3, 7], [3, 6, 7], [1, 2, 7], [1, 2, 4], [2, 5, 6], [2, 4, 5], [4, 5, 6], [4, 6, 7], [2, 6, 7], [2, 4, 7], [2, 4, 6], [1, 2, 3], [1, 3, 7], [1, 4, 5], [1, 5, 6], [3, 5, 6], [2, 3, 5], [1, 3, 5], [1, 6, 7], [1, 4, 6], [1, 4, 7]],

[[2, 4, 6, 7], [1, 4, 6, 7], [2, 3, 6, 7], [1, 3, 6, 7], [1, 2, 4, 7], [1, 2, 3, 7],
[2, 4, 5, 6], [1, 4, 5, 6], [2, 3, 5, 6], [1, 3, 5, 6], [2, 3, 4, 5], [1, 3, 4, 5],
[1, 2, 3, 4]],

[],

[],

[]

The ordering of the faces of $\mu(B(I))$ is compatible with above ordering of the faces of $B(I)$. The F-vectors of the faces of $\mu(B(I))^*$ are

Dimension	Number of faces	F-vector	
2	13	$(1,3,3,1,0,0,0,0)$	triangle
3	26	$(1,4,6,4,1,0,0,0)$	tetrahedron
4	20	$(1,5,10,10,5,1,0,0)$	
5	7	$(1,6,15,20,15,6,1,0)$	

The first order deformations of the mirror special fiber X_0° correspond to the lattice points of the dual complex $(\mu(B(I)))^* \subset \operatorname{Poset}(\Delta^*)$ of the mirror complex. We label the first order deformations of X_0° corresponding to vertices of Δ^* by the homogeneous coordinates of Y

$$x_1 = x_{(1,0,0,0,0,0)} = \frac{y_3^2 y_5 y_6^2 y_{15}^2 y_{17} y_{18} y_{23} y_{29} y_{34} y_{44} y_{46} y_{49}}{y_1 y_2 y_8 y_9 y_{10} y_{24} y_{25} y_{26} y_{27} y_{28} y_{30} y_{47}}$$
$$x_2 = x_{(0,1,0,0,0,0)} = \frac{y_4^2 y_5^2 y_6 y_{16}^2 y_{17} y_{18} y_{19} y_{28} y_{39} y_{43} y_{47} y_{48}}{y_1 y_2 y_8 y_9 y_{10} y_{24} y_{25} y_{26} y_{27} y_{29} y_{31} y_{46}}$$
$$x_3 = x_{(0,0,1,0,0,0)} = \frac{y_1^2 y_9 y_{10}^2 y_{11} y_{20} y_{26}^2 y_{30} y_{31} y_{33} y_{35} y_{38} y_{42}}{y_3 y_4 y_5 y_6 y_7 y_{12} y_{13} y_{14} y_{15} y_{16} y_{17} y_{36}}$$
$$x_4 = x_{(0,0,0,1,0,0)} = \frac{y_2^2 y_9^2 y_{10} y_{12} y_{21} y_{25}^2 y_{30} y_{31} y_{32} y_{36} y_{40} y_{41}}{y_3 y_4 y_5 y_6 y_7 y_{11} y_{13} y_{14} y_{15} y_{16} y_{18} y_{35}}$$
$$x_5 = x_{(0,0,0,0,1,0)} = \frac{y_3 y_4 y_7^3 y_{11} y_{12} y_{24}^2 y_{37}}{y_9 y_{10} y_{19} y_{20} y_{21} y_{22} y_{23} y_{30} y_{31} y_{41} y_{42} y_{43} y_{44} y_{45}}$$
$$x_6 = x_{(0,0,0,0,0,1)} = \frac{y_{13}^2 y_{19} y_{20} y_{21} y_{22}^2 y_{23} y_{27}^2}{y_1 y_2 y_3 y_4 y_7 y_8 y_{11} y_{12} y_{28} y_{29} y_{32} y_{33} y_{48} y_{49}}$$
$$x_0 = x_{(-1,-1,-1,-1,-1,-1)} = \frac{y_1 y_2 y_8^3 y_{14}^2 y_{28} y_{29} y_{45}}{y_5 y_6 y_{17} y_{18} y_{19} y_{20} y_{21} y_{22} y_{23} y_{34} y_{37} y_{38} y_{39} y_{40}}$$

So writing the vertices of the faces of $(\mu(B(I)))^*$ as homogeneous coordinates of Y, the complex $(\mu(B(I)))^*$ is given by

[],

[],

[],

$[\langle x_1, x_3, x_5\rangle, \langle x_2, x_3, x_5\rangle, \langle x_1, x_4, x_5\rangle, \langle x_2, x_4, x_5\rangle, \langle x_3, x_5, x_6\rangle,$

$\langle x_4, x_5, x_6\rangle, \langle x_1, x_3, x_0\rangle, \langle x_2, x_3, x_0\rangle, \langle x_1, x_4, x_0\rangle, \langle x_2, x_4, x_0\rangle,$
$\langle x_1, x_6, x_0\rangle, \langle x_2, x_6, x_0\rangle, \langle x_5, x_6, x_0\rangle],$

$[\langle x_1, x_4, x_5, x_0\rangle, \langle x_2, x_5, x_6, x_0\rangle, \langle x_2, x_4, x_5, x_0\rangle, \langle x_1, x_5, x_6, x_0\rangle,$
$\langle x_1, x_2, x_6, x_0\rangle, \langle x_1, x_4, x_5, x_6\rangle, \langle x_1, x_2, x_4, x_5\rangle, \langle x_3, x_4, x_5, x_6\rangle,$
$\langle x_3, x_5, x_6, x_0\rangle, \langle x_1, x_3, x_4, x_0\rangle, \langle x_1, x_3, x_6, x_0\rangle, \langle x_1, x_2, x_3, x_0\rangle,$
$\langle x_1, x_2, x_3, x_5\rangle, \langle x_1, x_3, x_4, x_5\rangle, \langle x_1, x_3, x_5, x_6\rangle, \langle x_1, x_3, x_5, x_0\rangle,$
$\langle x_4, x_5, x_6, x_0\rangle, \langle x_2, x_4, x_5, x_6\rangle, \langle x_2, x_3, x_6, x_0\rangle, \langle x_2, x_3, x_4, x_0\rangle,$
$\langle x_1, x_2, x_4, x_0\rangle, \langle x_1, x_4, x_6, x_0\rangle, \langle x_2, x_4, x_6, x_0\rangle, \langle x_2, x_3, x_4, x_5\rangle,$
$\langle x_2, x_3, x_5, x_0\rangle, \langle x_2, x_3, x_5, x_6\rangle],$

$[\langle x_2, x_3, x_5, x_6, x_0\rangle, \langle x_2, x_3, x_4, x_6, x_0\rangle, \langle x_1, x_3, x_4, x_5, x_6\rangle,$
$\langle x_1, x_2, x_4, x_5, x_6\rangle, \langle x_1, x_3, x_5, x_6, x_0\rangle, \langle x_2, x_4, x_5, x_6, x_0\rangle,$
$\langle x_1, x_4, x_5, x_6, x_0\rangle, \langle x_1, x_2, x_4, x_5, x_0\rangle, \langle x_1, x_3, x_4, x_5, x_0\rangle,$
$\langle x_1, x_2, x_5, x_6, x_0\rangle, \langle x_3, x_4, x_5, x_6, x_0\rangle, \langle x_2, x_3, x_4, x_5, x_6\rangle,$
$\langle x_2, x_3, x_4, x_5, x_0\rangle, \langle x_1, x_2, x_3, x_4, x_5\rangle, \langle x_1, x_2, x_4, x_6, x_0\rangle,$
$\langle x_1, x_2, x_3, x_6, x_0\rangle, \langle x_1, x_2, x_3, x_5, x_0\rangle, \langle x_1, x_2, x_3, x_5, x_6\rangle,$
$\langle x_1, x_2, x_3, x_4, x_0\rangle, \langle x_1, x_3, x_4, x_6, x_0\rangle],$

$[\langle x_1, x_2, x_3, x_5, x_6, x_0\rangle, \langle x_2, x_3, x_4, x_5, x_6, x_0\rangle, \langle x_1, x_3, x_4, x_5, x_6, x_0\rangle,$
$\langle x_1, x_2, x_3, x_4, x_5, x_0\rangle, \langle x_1, x_2, x_3, x_4, x_5, x_6\rangle, \langle x_1, x_2, x_4, x_5, x_6, x_0\rangle,$
$\langle x_1, x_2, x_3, x_4, x_6, x_0\rangle],$

$[]$

The complex $\mu(B(I))^*$ labeled by the variables of the Cox ring S gives the ideals of the toric strata of the special fiber X_0 of $\mathfrak{X}$, i.e., the complex $SP(I_0)$, so in particular the primary decomposition of I_0 is

$$I_0 = \begin{array}{l} \langle x_2, x_4, x_0\rangle \cap \langle x_1, x_4, x_0\rangle \cap \langle x_1, x_4, x_5\rangle \cap \langle x_1, x_3, x_5\rangle \cap \langle x_2, x_6, x_0\rangle \cap \\ \cap \langle x_5, x_6, x_0\rangle \cap \langle x_1, x_6, x_0\rangle \cap \langle x_2, x_4, x_5\rangle \cap \langle x_2, x_3, x_5\rangle \cap \langle x_3, x_5, x_6\rangle \cap \\ \cap \langle x_1, x_3, x_0\rangle \cap \langle x_2, x_3, x_0\rangle \cap \langle x_4, x_5, x_6\rangle \end{array}$$

10.5.10 Covering structure in the deformation complex of the mirror degeneration

Each face of the complex of deformations $(\mu(B(I)))^*$ of the mirror special fiber X_0° decomposes as the convex hull of 3,4 respectively 5 polytopes forming a $5:1$ ramified covering of $\mu(B(I))^\vee$

$[],$

$[],$

$$
\begin{aligned}
&[],\\
&[[\langle x_5\rangle, \langle x_6\rangle, \langle x_0\rangle] \mapsto \langle x_5, x_6, x_0\rangle^{*\vee} = [1,2,3,4]^\vee,\\
&[\langle x_3\rangle, \langle x_5\rangle, \langle x_1\rangle] \mapsto \langle x_1, x_3, x_5\rangle^{*\vee} = [2,4,6,7]^\vee,\\
&[\langle x_3\rangle, \langle x_5\rangle, \langle x_6\rangle] \mapsto \langle x_3, x_5, x_6\rangle^{*\vee} = [1,2,4,7]^\vee,\\
&[\langle x_3\rangle, \langle x_5\rangle, \langle x_1\rangle, \langle x_6\rangle, \langle x_0\rangle] \mapsto \langle x_1, x_3, x_5, x_6, x_0\rangle^{*\vee} = [2,4]^\vee,\\
&...],\\
&[[\langle x_4\rangle, \langle x_5\rangle, \langle x_1\rangle, \langle x_0\rangle] \mapsto \langle x_1, x_4, x_5, x_0\rangle^{*\vee} = [2,3,6]^\vee,\\
&[\langle x_4\rangle, \langle x_5\rangle, \langle x_1\rangle, \langle x_6\rangle, \langle x_0\rangle] \mapsto \langle x_1, x_4, x_5, x_6, x_0\rangle^{*\vee} = [2,3]^\vee,\\
&[\langle x_1, x_2\rangle, \langle x_6\rangle, \langle x_0\rangle] \mapsto \langle x_1, x_2, x_6, x_0\rangle^{*\vee} = [3,4,5]^\vee,\\
&[\langle x_4\rangle, \langle x_5\rangle, \langle x_1, x_2\rangle] \mapsto \langle x_1, x_2, x_4, x_5\rangle^{*\vee} = [3,6,7]^\vee,\\
&[\langle x_4\rangle, \langle x_5\rangle, \langle x_1, x_2\rangle, \langle x_6\rangle, \langle x_0\rangle] \mapsto \langle x_1, x_2, x_4, x_5, x_6, x_0\rangle^{*\vee} = [3]^\vee,\\
&...],\\
&[[\langle x_3\rangle, \langle x_5\rangle, \langle x_1, x_2\rangle, \langle x_6\rangle, \langle x_0\rangle] \mapsto \langle x_1, x_2, x_3, x_5, x_6, x_0\rangle^{*\vee} = [4]^\vee,\\
&[\langle x_3, x_4\rangle, \langle x_5\rangle, \langle x_1\rangle, \langle x_6\rangle] \mapsto \langle x_1, x_3, x_4, x_5, x_6\rangle^{*\vee} = [2,7]^\vee,\\
&[\langle x_3, x_4\rangle, \langle x_5\rangle, \langle x_1\rangle, \langle x_6\rangle, \langle x_0\rangle] \mapsto \langle x_1, x_3, x_4, x_5, x_6, x_0\rangle^{*\vee} = [2]^\vee,\\
&[\langle x_3\rangle, \langle x_5\rangle, \langle x_2\rangle, \langle x_6\rangle, \langle x_0\rangle] \mapsto \langle x_2, x_3, x_5, x_6, x_0\rangle^{*\vee} = [1,4]^\vee,\\
&...],\\
&[[\langle x_3\rangle, \langle x_5\rangle, \langle x_1, x_2\rangle, \langle x_6\rangle, \langle x_0\rangle] \mapsto \langle x_1, x_2, x_3, x_5, x_6, x_0\rangle^{*\vee} = [4]^\vee,\\
&[\langle x_3\rangle, \langle x_5\rangle, \langle x_1, x_2\rangle, \langle x_6\rangle, \langle x_0\rangle] \mapsto \langle x_1, x_2, x_3, x_5, x_6, x_0\rangle^{*\vee} = [4]^\vee,\\
&[\langle x_3, x_4\rangle, \langle x_5\rangle, \langle x_1\rangle, \langle x_6\rangle, \langle x_0\rangle] \mapsto \langle x_1, x_3, x_4, x_5, x_6, x_0\rangle^{*\vee} = [2]^\vee,\\
&[\langle x_3, x_4\rangle, \langle x_5\rangle, \langle x_1, x_2\rangle, \langle x_0\rangle] \mapsto \langle x_1, x_2, x_3, x_4, x_5, x_0\rangle^{*\vee} = [6]^\vee,\\
&...],\\
&[]
\end{aligned}
$$

Due to the singularities of Y° this covering involves degenerate faces, i.e., faces $G \mapsto F^\vee$ with $\dim(G) < \dim(F^\vee)$.

10.5.11 Mirror degeneration

The space of first order deformations of X_0° in the mirror degeneration $\mathfrak{X}^\circ$ has dimension 7 and the deformations represented by the monomials

$$
\left\{
\begin{array}{cc}
\frac{y_3^2 y_5 y_6^2 y_{15}^2 y_{17} y_{18} y_{23} y_{29} y_{34} y_{44} y_{46} y_{49}}{y_1 y_2 y_8 y_9 y_{10} y_{24} y_{25} y_{26} y_{27} y_{28} y_{30} y_{47}} & \frac{y_1^2 y_9 y_{10}^2 y_{11} y_{20} y_{26}^2 y_{30} y_{31} y_{33} y_{35} y_{38} y_{42}}{y_3 y_4 y_5 y_6 y_7 y_{12} y_{13} y_{14} y_{15} y_{16} y_{17} y_{36}} \\
\frac{y_3 y_4 y_7^3 y_{11} y_{12} y_{24}^2 y_{37}}{y_9 y_{10} y_{19} y_{20} y_{21} y_{22} y_{23} y_{30} y_{31} y_{41} y_{42} y_{43} y_{44} y_{45}} & \frac{y_1 y_2 y_8^3 y_{14}^2 y_{28} y_{29} y_{45}}{y_5 y_6 y_{17} y_{18} y_{19} y_{20} y_{21} y_{22} y_{23} y_{34} y_{37} y_{38} y_{39} y_{40}} \\
\frac{y_{13}^2 y_{19} y_{20} y_{21} y_{22}^2 y_{23} y_{27}^2}{y_1 y_2 y_3 y_4 y_7 y_8 y_{11} y_{12} y_{28} y_{29} y_{32} y_{33} y_{48} y_{49}} & \frac{y_4^2 y_5^2 y_6 y_{16}^2 y_{17} y_{18} y_{19} y_{28} y_{39} y_{43} y_{47} y_{48}}{y_1 y_2 y_8 y_9 y_{10} y_{24} y_{25} y_{26} y_{27} y_{29} y_{31} y_{46}} \\
\frac{y_2^2 y_9^2 y_{10} y_{12} y_{21} y_{25}^2 y_{30} y_{31} y_{32} y_{36} y_{40} y_{41}}{y_3 y_4 y_5 y_6 y_7 y_{11} y_{13} y_{14} y_{15} y_{16} y_{18} y_{35}} &
\end{array}
\right\}
$$

form a torus invariant basis $\mathfrak{B}^{\circ}$. The number of lattice points of the dual of the mirror complex of I relates to the dimension $h^{1,2}(X^{\circ})$ of complex moduli space of the generic fiber X° of $\mathfrak{X}^{\circ}$ and to the dimension $h^{1,1}(X)$ of the Kähler moduli space of the generic fiber X of $\mathfrak{X}$ via

$$\begin{aligned} |\operatorname{supp}((\mu(B(I)))^{*}) \cap N| &= 7 = 6 + 1 \\ &= \dim(\operatorname{Aut}(Y^{\circ})) + h^{1,2}(X^{\circ}) = \dim(T) + h^{1,1}(X) \end{aligned}$$

The conjectural first order mirror degeneration $\mathfrak{X}^{1\circ} \subset Y^{\circ} \times \operatorname{Spec} \mathbb{C}[t] / \langle t^{2} \rangle$ of $\mathfrak{X}$ is given by the ideal $I^{1\circ} \subset S^{\circ} \otimes \mathbb{C}[t] / \langle t^{2} \rangle$ generated by

$$\left\{ m + \sum_{\delta \in \mathfrak{B}^{\circ}} t \cdot s_{\delta} \cdot \delta(m) \mid m \in I_{0}^{\circ} \right\}$$

10.5.12 Contraction of the mirror degeneration

In the following we give a birational map relating the degeneration $\mathfrak{X}^{\circ}$ to a Greene-Plesser type orbifolding mirror family by contracting divisors on Y°. In order to contract the divisors

$$\begin{array}{lllll}
y_{1} = \frac{x_{3}^{2} x_{0}}{x_{1}\, x_{2}\, x_{6}} & y_{2} = \frac{x_{4}^{2} x_{0}}{x_{1}\, x_{2}\, x_{6}} & y_{3} = \frac{x_{1}^{2} x_{5}}{x_{3}\, x_{4}\, x_{6}} & y_{4} = \frac{x_{2}^{2} x_{5}}{x_{3}\, x_{4}\, x_{6}} & y_{11} = \frac{x_{3}\, x_{5}}{x_{4}\, x_{6}} \\
y_{12} = \frac{x_{4}\, x_{5}}{x_{3}\, x_{6}} & y_{13} = \frac{x_{6}^{2}}{x_{3}\, x_{4}} & y_{14} = \frac{x_{0}^{2}}{x_{3}\, x_{4}} & y_{15} = \frac{x_{1}^{2}}{x_{3}\, x_{4}} & y_{16} = \frac{x_{2}^{2}}{x_{3}\, x_{4}} \\
y_{17} = \frac{x_{1}\, x_{2}}{x_{3}\, x_{0}} & y_{18} = \frac{x_{1}\, x_{2}}{x_{4}\, x_{0}} & y_{19} = \frac{x_{6}\, x_{2}}{x_{5}\, x_{0}} & y_{20} = \frac{x_{3}\, x_{6}}{x_{5}\, x_{0}} & y_{21} = \frac{x_{6}\, x_{4}}{x_{5}\, x_{0}} \\
y_{23} = \frac{x_{6}\, x_{1}}{x_{5}\, x_{0}} & y_{24} = \frac{x_{5}^{2}}{x_{1}\, x_{2}} & y_{25} = \frac{x_{4}^{2}}{x_{1}\, x_{2}} & y_{26} = \frac{x_{3}^{2}}{x_{1}\, x_{2}} & y_{27} = \frac{x_{6}^{2}}{x_{1}\, x_{2}} \\
y_{28} = \frac{x_{2}\, x_{0}}{x_{6}\, x_{1}} & y_{29} = \frac{x_{1}\, x_{0}}{x_{2}\, x_{6}} & y_{30} = \frac{x_{3}\, x_{4}}{x_{1}\, x_{5}} & y_{31} = \frac{x_{3}\, x_{4}}{x_{2}\, x_{5}} & y_{32} = \frac{x_{4}}{x_{6}} \\
y_{33} = \frac{x_{3}}{x_{6}} & y_{34} = \frac{x_{1}}{x_{0}} & y_{35} = \frac{x_{3}}{x_{4}} & y_{36} = \frac{x_{4}}{x_{3}} & y_{37} = \frac{x_{5}}{x_{0}} \\
y_{38} = \frac{x_{3}}{x_{0}} & y_{39} = \frac{x_{2}}{x_{0}} & y_{40} = \frac{x_{4}}{x_{0}} & y_{41} = \frac{x_{4}}{x_{5}} & y_{42} = \frac{x_{3}}{x_{5}} \\
y_{43} = \frac{x_{2}}{x_{5}} & y_{44} = \frac{x_{1}}{x_{5}} & y_{45} = \frac{x_{0}}{x_{5}} & y_{46} = \frac{x_{1}}{x_{2}} & y_{47} = \frac{x_{2}}{x_{1}} \\
y_{48} = \frac{x_{2}}{x_{6}} & y_{49} = \frac{x_{1}}{x_{6}} & & &
\end{array}$$

consider the $\mathbb{Q}$-factorial toric Fano variety $\hat{Y}^{\circ} = X(\hat{\Sigma}^{\circ})$, where $\hat{\Sigma}^{\circ} = \Sigma(\hat{P}^{\circ}) \subset M_{\mathbb{R}}$ is the fan over the Fano polytope $\hat{P}^{\circ} \subset M_{\mathbb{R}}$ given as the convex hull of the remaining vertices of $P^{\circ} = \nabla^{*}$ corresponding to the Cox variables

$$\begin{array}{ll}
y_{9} = y_{(-1,-1,1,2,-1,0)} = \frac{x_{3}\, {x_{4}}^{2}}{x_{1}\, x_{2}\, x_{5}} & y_{6} = y_{(2,1,-1,-1,0,0)} = \frac{x_{1}^{2} x_{2}}{x_{3}\, x_{4}\, x_{0}} \\
y_{5} = y_{(1,2,-1,-1,0,0)} = \frac{x_{1}\, {x_{2}}^{2}}{x_{3}\, x_{4}\, x_{0}} & y_{22} = y_{(0,0,0,0,-1,2)} = \frac{x_{6}^{2}}{x_{5}\, x_{0}} \\
y_{8} = y_{(-1,-1,0,0,0,-1)} = \frac{x_{0}^{3}}{x_{1}\, x_{2}\, x_{6}} & y_{10} = y_{(-1,-1,2,1,-1,0)} = \frac{x_{3}^{2} x_{4}}{x_{1}\, x_{2}\, x_{5}} \\
y_{7} = y_{(0,0,-1,-1,3,-1)} = \frac{x_{5}^{3}}{x_{3}\, x_{4}\, x_{6}} &
\end{array}$$

of the toric variety $\hat{Y}^\circ$ with Cox ring

$$\hat{S}^\circ = \mathbb{C}[y_9, y_6, y_5, y_{22}, y_8, y_{10}, y_7]$$

The Cox variables of $\hat{Y}^\circ$ correspond to the set of Fermat deformations of $\mathfrak{X}$. Let

$$Y^\circ = X(\Sigma^\circ) \to X(\hat{\Sigma}^\circ) = \hat{Y}^\circ$$

be a birational map from Y° to the toric Fano variety $\hat{Y}^\circ$, which contracts the divisors of the rays $\Sigma^\circ(1) - \hat{\Sigma}^\circ(1)$ corresponding to the Cox variables

y_1	y_2	y_3	y_4	y_{11}	y_{12}	y_{13}	y_{14}	y_{15}	y_{16}	y_{17}	y_{18}	y_{19}	y_{20}	y_{21}
y_{23}	y_{24}	y_{25}	y_{26}	y_{27}	y_{28}	y_{29}	y_{30}	y_{31}	y_{32}	y_{33}	y_{34}	y_{35}	y_{36}	y_{37}
y_{38}	y_{39}	y_{40}	y_{41}	y_{42}	y_{43}	y_{44}	y_{45}	y_{46}	y_{47}	y_{48}	y_{49}			

Representing $\hat{Y}^\circ$ as a quotient we have

$$\hat{Y}^\circ = \left(\mathbb{C}^7 - V\left(B(\hat{\Sigma}^\circ)\right)\right) / \hat{G}^\circ$$

with

$$\hat{G}^\circ = \mathbb{Z}_{13} \times (\mathbb{C}^*)^1$$

acting via

$$\xi y = \left(u_1^{11} v_1 \cdot y_9,\ u_1^{10} v_1 \cdot y_6,\ u_1^{10} v_1 \cdot y_5,\ u_1^4 v_1 \cdot y_{22},\ u_1^8 v_1 \cdot y_8,\ u_1^{11} v_1 \cdot y_{10},\ v_1 \cdot y_7\right)$$

for $\xi = (u_1, v_1) \in \hat{G}^\circ$ and $y \in \mathbb{C}^7 - V\left(B(\hat{\Sigma}^\circ)\right)$. Hence with the group

$$\hat{H}^\circ = \mathbb{Z}_{13}$$

of order 13 the toric variety $\hat{Y}^\circ$ is the quotient

$$\hat{Y}^\circ = \mathbb{P}^6 / \hat{H}^\circ$$

of projective space $\mathbb{P}^6$. The first order mirror degeneration $\mathfrak{X}^{1\circ}$ induces via $Y^\circ \to \hat{Y}^\circ$ a degeneration $\hat{\mathfrak{X}}^{1\circ} \subset \hat{Y}^\circ \times \operatorname{Spec} \mathbb{C}[t] / \langle t^2 \rangle$ given by the ideal $\hat{I}^{1\circ} \subset \langle y_9, y_6, y_5, y_{22}, y_8, y_{10}, y_7 \rangle \subset \hat{S}^\circ \otimes \mathbb{C}[t] / \langle t^2 \rangle$ generated by the Fermat-type equations

$$\left\{\begin{array}{l} y_5\, y_6\, y_7 + t(s_4\, y_9^2 y_{10} + s_3\, y_9\, y_{10}^2), \\ s_1\, y_7^3 t + y_9\, y_{10}\, y_{22}, \\ y_8\, y_{10}\, y_9 + t(s_6\, y_5^2 y_6 + s_5\, y_5\, y_6^2), \\ s_2\, y_{22}^2 t + y_7\, y_8, \\ s_7\, y_8^3 t + y_5\, y_6\, y_{22} \end{array}\right\}$$

Note that

$$\left|\mathrm{supp}\left(\left(\mu\left(B\left(I\right)\right)\right)^{*}\right)\cap N-\mathrm{Roots}\left(\hat{Y}^{\circ}\right)\right|-\dim\left(T_{\hat{Y}^{\circ}}\right)=7-6=1$$

so this family has one independent parameter. The special fiber $\hat{X}_0^\circ \subset \hat{Y}^\circ$ of $\hat{\mathfrak{X}}^{1\circ}$ is cut out by the monomial ideal

$$\hat{I}_0^\circ \subset B(\hat{\Sigma}^\circ) = \langle y_9, y_6, y_5, y_{22}, y_8, y_{10}, y_7\rangle \subset \hat{S}^\circ$$

generated by

$$\left\{\ y_9\,y_{10}\,y_{22} \quad y_8\,y_{10}\,y_9 \quad y_5\,y_6\,y_{22} \quad y_5\,y_6\,y_7 \quad y_7\,y_8\ \right\}$$

The complex $SP(\hat{I}_0^\circ)$ of prime ideals of the toric strata of $\hat{X}_0^\circ$ is

$[],$

$[],$

$[],$

$[\langle y_5, y_{22}, y_8\rangle, \langle y_5, y_8, y_{10}\rangle, \langle y_6, y_{10}, y_7\rangle, \langle y_9, y_{22}, y_7\rangle, \langle y_6, y_{22}, y_8\rangle,$
$\langle y_9, y_6, y_8\rangle, \langle y_9, y_5, y_8\rangle, \langle y_9, y_6, y_7\rangle, \langle y_6, y_8, y_{10}\rangle, \langle y_9, y_5, y_7\rangle,$
$\langle y_5, y_{10}, y_7\rangle, \langle y_{22}, y_{10}, y_7\rangle, \langle y_{22}, y_8, y_7\rangle],$

$[\langle y_6, y_{22}, y_8, y_7\rangle, \langle y_9, y_5, y_8, y_7\rangle, \langle y_5, y_{22}, y_{10}, y_7\rangle, \langle y_6, y_{22}, y_8, y_{10}\rangle,$
$\langle y_9, y_5, y_8, y_{10}\rangle, \langle y_9, y_6, y_{22}, y_8\rangle, \langle y_6, y_5, y_8, y_{10}\rangle, \langle y_9, y_5, y_{22}, y_7\rangle,$
$\langle y_9, y_5, y_{10}, y_7\rangle, \langle y_6, y_8, y_{10}, y_7\rangle, \langle y_5, y_{22}, y_8, y_7\rangle, \langle y_6, y_5, y_{10}, y_7\rangle,$
$\langle y_9, y_6, y_5, y_8\rangle, \langle y_9, y_6, y_8, y_{10}\rangle, \langle y_9, y_5, y_{22}, y_8\rangle, \langle y_5, y_{22}, y_8, y_{10}\rangle,$
$\langle y_9, y_6, y_{22}, y_7\rangle, \langle y_{22}, y_8, y_{10}, y_7\rangle, \langle y_9, y_6, y_5, y_7\rangle, \langle y_9, y_{22}, y_8, y_7\rangle,$
$\langle y_6, y_{22}, y_{10}, y_7\rangle, \langle y_9, y_6, y_8, y_7\rangle, \langle y_9, y_{22}, y_{10}, y_7\rangle, \langle y_5, y_8, y_{10}, y_7\rangle,$
$\langle y_6, y_5, y_{22}, y_8\rangle, \langle y_9, y_6, y_{10}, y_7\rangle],$

$[\langle y_9, y_6, y_5, y_{22}, y_7\rangle, \langle y_9, y_6, y_{22}, y_8, y_7\rangle, \langle y_9, y_5, y_8, y_{10}, y_7\rangle, \langle y_9, y_6, y_{22}, y_{10}, y_7\rangle,$
$\langle y_9, y_6, y_5, y_8, y_7\rangle, \langle y_9, y_6, y_5, y_8, y_{10}\rangle, \langle y_9, y_5, y_{22}, y_8, y_7\rangle, \langle y_6, y_5, y_8, y_{10}, y_7\rangle,$
$\langle y_9, y_5, y_{22}, y_{10}, y_7\rangle, \langle y_9, y_5, y_{22}, y_8, y_{10}\rangle, \langle y_5, y_{22}, y_8, y_{10}, y_7\rangle, \langle y_9, y_6, y_8, y_{10}, y_7\rangle,$
$\langle y_6, y_5, y_{22}, y_8, y_{10}\rangle, \langle y_6, y_{22}, y_8, y_{10}, y_7\rangle, \langle y_9, y_6, y_5, y_{22}, y_8\rangle, \langle y_9, y_6, y_{22}, y_8, y_{10}\rangle,$
$\langle y_9, y_{22}, y_8, y_{10}, y_7\rangle, \langle y_9, y_6, y_5, y_{10}, y_7\rangle, \langle y_6, y_5, y_{22}, y_8, y_7\rangle, \langle y_6, y_5, y_{22}, y_{10}, y_7\rangle],$

$[\langle y_6, y_5, y_{22}, y_8, y_{10}, y_7\rangle, \langle y_9, y_5, y_{22}, y_8, y_{10}, y_7\rangle, \langle y_9, y_6, y_{22}, y_8, y_{10}, y_7\rangle,$
$\langle y_9, y_6, y_5, y_8, y_{10}, y_7\rangle, \langle y_9, y_6, y_5, y_{22}, y_{10}, y_7\rangle, \langle y_9, y_6, y_5, y_{22}, y_8, y_7\rangle,$
$\langle y_9, y_6, y_5, y_{22}, y_8, y_{10}\rangle],$

$[]$

so $\hat{I}_0^\circ$ has the primary decomposition

$$\hat{I}_0^\circ = \begin{array}{l} \langle y_5, y_{22}, y_8\rangle \cap \langle y_5, y_8, y_{10}\rangle \cap \langle y_6, y_{10}, y_7\rangle \cap \langle y_9, y_{22}, y_7\rangle \cap \langle y_6, y_{22}, y_8\rangle \cap \\ \cap \langle y_9, y_6, y_8\rangle \cap \langle y_9, y_5, y_8\rangle \cap \langle y_9, y_6, y_7\rangle \cap \langle y_6, y_8, y_{10}\rangle \cap \langle y_9, y_5, y_7\rangle \cap \\ \cap \langle y_5, y_{10}, y_7\rangle \cap \langle y_{22}, y_{10}, y_7\rangle \cap \langle y_{22}, y_8, y_7\rangle \end{array}$$

The Bergman subcomplex $B(I) \subset \text{Poset}(\nabla)$ induces a subcomplex of $\text{Poset}(\hat{\nabla})$, $\hat{\nabla} = (\hat{P}^\circ)^*$ corresponding to $\hat{I}_0^\circ$. Indexing of the vertices of $\hat{\nabla}$ by

$$\begin{array}{l} 1 = \left(\frac{8}{13}, \frac{8}{13}, -\frac{27}{13}, \frac{64}{13}, \frac{7}{13}, -\frac{3}{13}\right) \\ 2 = \left(\frac{57}{13}, -\frac{34}{13}, \frac{1}{13}, \frac{1}{13}, -\frac{7}{13}, -\frac{10}{13}\right) \\ 3 = \left(-\frac{34}{13}, \frac{57}{13}, \frac{1}{13}, \frac{1}{13}, -\frac{7}{13}, -\frac{10}{13}\right) \\ 4 = (-1, -1, -1, -1, 0, 3) \\ 5 = \left(-\frac{27}{13}, -\frac{27}{13}, -\frac{34}{13}, -\frac{34}{13}, -\frac{35}{13}, -\frac{24}{13}\right) \\ 6 = \left(\frac{8}{13}, \frac{8}{13}, \frac{64}{13}, -\frac{27}{13}, \frac{7}{13}, -\frac{3}{13}\right) \\ 7 = \left(\frac{1}{13}, \frac{1}{13}, \frac{8}{13}, \frac{8}{13}, \frac{35}{13}, \frac{11}{13}\right) \end{array}$$

this complex is given by

$$\left\{ \begin{array}{l} [], \\ [[1], [2], [3], [4], [5], [6], [7]], \\ [[5, 6], [3, 6], [2, 4], [3, 5], [4, 6], [4, 7], [2, 6], [1, 4], [2, 5], [2, 7], [1, 2], \\ [3, 4], [1, 7], [1, 3], [6, 7], [3, 7], [2, 3], [4, 5], [1, 6], [1, 5]], \\ [[1, 3, 6], [2, 4, 6], [1, 2, 5], [1, 3, 7], [2, 4, 7], [3, 6, 7], [1, 4, 7], [2, 5, 6], \\ [2, 4, 5], [1, 3, 4], [1, 2, 6], [1, 4, 5], [4, 6, 7], [3, 4, 7], [2, 6, 7], [1, 2, 7], \\ [3, 5, 6], [1, 2, 3], [4, 5, 6], [2, 3, 6], [1, 3, 5], [3, 4, 6], [2, 3, 5], [1, 2, 4], \\ [1, 6, 7], [3, 4, 5]], \\ [[1, 2, 6, 7], [1, 2, 4, 7], [1, 3, 4, 5], [2, 3, 5, 6], [1, 3, 6, 7], [3, 4, 6, 7], \\ [2, 4, 6, 7], [3, 4, 5, 6], [1, 3, 4, 7], [2, 4, 5, 6], [1, 2, 4, 5], [1, 2, 3, 5], \\ [1, 2, 3, 6]], \\ [], \\ [], \\ [] \end{array} \right\}$$

The ideal $\hat{I}^{1\circ} \subset \hat{S}^\circ \otimes \mathbb{C}[t] / \langle t^2 \rangle$ has a Pfaffian resolution

$$0 \to \mathcal{O}_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]/\langle t^2\rangle}\left(K^1\right) \to \mathcal{E}^1\left(K^1\right) \xrightarrow{\varphi^1} \left(\mathcal{E}^1\right)^* \xrightarrow{f^1} \mathcal{O}_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]/\langle t^2\rangle}$$
$$\text{where } \overline{\pi}_1 : \hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t] / \left\langle t^2 \right\rangle \to \hat{Y}^\circ \text{ and } \mathcal{E}^1 = \overline{\pi}_1^* \mathcal{F}$$

with

$$\mathcal{F} = \begin{array}{l} \mathcal{O}_{\hat{Y}^\circ}\left(D_{(2,1,-1,-1,0,0)} + D_{(1,2,-1,-1,0,0)} + D_{(0,0,-1,-1,3,-1)}\right) \oplus \\ \mathcal{O}_{\hat{Y}^\circ}\left(D_{(-1,-1,1,2,-1,0)} + D_{(0,0,0,0,-1,2)} + D_{(-1,-1,2,1,-1,0)}\right) \oplus \\ \mathcal{O}_{\hat{Y}^\circ}\left(D_{(-1,-1,1,2,-1,0)} + D_{(-1,-1,0,0,0,-1)} + D_{(-1,-1,2,1,-1,0)}\right) \oplus \\ \mathcal{O}_{\hat{Y}^\circ}\left(D_{(-1,-1,0,0,0,-1)} + D_{(0,0,-1,-1,3,-1)}\right) \oplus \\ \mathcal{O}_{\hat{Y}^\circ}\left(D_{(2,1,-1,-1,0,0)} + D_{(1,2,-1,-1,0,0)} + D_{(0,0,0,0,-1,2)}\right) \end{array}$$

and $K^1 = K_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]/\langle t^2\rangle / \operatorname{Spec}\mathbb{C}[t]/\langle t^2\rangle}$ and $\varphi^1 \in \bigwedge^2 \mathcal{E}^{1*}(-K^1)$ given by

$$\begin{bmatrix} 0 & ts_7\, {y_8}^2 & y_5\, y_6 & -y_9\, y_{10} & ts_1\, {y_7}^2 \\ -ts_7\, {y_8}^2 & 0 & t\,(s_4\, y_9 + s_3\, y_{10}) & y_7 & y_{22} \\ -y_5\, y_6 & -t\,(s_4\, y_9 + s_3\, y_{10}) & 0 & ts_2\, y_{22} & -y_8 \\ y_9\, y_{10} & -y_7 & -ts_2\, y_{22} & 0 & t\,(-s_6\, y_5 - s_5\, y_6) \\ -ts_1\, {y_7}^2 & -y_{22} & y_8 & -t\,(-s_6\, y_5 - s_5\, y_6) & 0 \end{bmatrix}$$

Hence via the Pfaffians of φ^1 we obtain a resolution

$$\begin{array}{c} 0 \to \mathcal{O}_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]}(K) \to \mathcal{E}(K) \to \mathcal{E}^* \to \mathcal{O}_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]} \\ \text{where } \pi_1 : Y \times \operatorname{Spec}\mathbb{C}[t] \to Y \ ,\ \mathcal{E} = \pi_1^*\mathcal{F} \\ \text{and } K = K_{\hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t] / \operatorname{Spec}\mathbb{C}[t]} \end{array}$$

of the ideal $\hat{I}^\circ \subset \hat{S}^\circ \otimes \mathbb{C}[t]$ generated by

$$\left\{ \begin{array}{l} y_7\, y_8 + t\,(s_2\, y_{22}^2) + t^2\,(-s_4\, y_9\, s_6\, y_5 - s_4\, y_9\, s_5\, y_6 - s_3\, y_{10}\, s_6\, y_5 - s_3\, y_{10}\, s_5\, y_6)\,, \\ y_8\, y_{10}\, y_9 + t\,(s_6\, y_5^2 y_6 + s_5\, y_5\, y_6^2) + t^2\,(-s_1\, s_2\, y_7^2 y_{22})\,, \\ y_9\, y_{10}\, y_{22} + t\,(s_1\, y_7^3) + t^2\,(-s_7\, y_8^2 s_6\, y_5 - s_7\, y_8^2 s_5\, y_6)\,, \\ y_5\, y_6\, y_{22} + t\,(s_7\, y_8^3) + t^2\,(-s_1\, y_7^2 s_4\, y_9 - s_1\, y_7^2 s_3\, y_{10})\,, \\ -y_5\, y_6\, y_7 + t\,(-s_4\, y_9^2 y_{10} - s_3\, y_9\, y_{10}^2) + t^2\,(s_7\, s_2\, y_8^2 y_{22}) \end{array} \right\}$$

which defines a flat family

$$\hat{\mathfrak{X}}^\circ \subset \hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]$$

11 Remarks on a tropical computation of the stringy E-function

Suppose we are given the setup of the tropical mirror construction via a degeneration $\mathfrak{X}$ given by the ideal I. In the following we make some remarks on the computation of Hodge numbers and the stringy E-function of the general fiber from the tropical data, i.e., from the polytopes Δ and ∇ and the complexes $B(I) \subset \operatorname{Poset}(\nabla)$ and $\lim(B(I)) \subset \operatorname{Poset}(\Delta)$.

We recall in Sections 11.4.3 and 11.4.4 the formulas by Batyrev and Borisov for the stringy E-function of a general Calabi-Yau hypersurface inside a Gorenstein toric Fano variety and for complete intersections given by nef partitions. These formulas give evidence that it should be possible to compute the stringy E-function from the tropical data via a formula analogous to those for hypersurfaces. Note also that stringy E-functions and tropical geometry share the concept of formal arcs. Furthermore the special fiber X_0 of $\mathfrak{X}$ is a union of toric varieties and, as noted in Proposition 11.35 below, the stringy E-function respects stratifications.

As this gives the general direction, we begin by recalling in Section 11.1 the relation of $h^{d-1,1}(X)$, $h^0\left(X, N_{X/\mathbb{P}^n}\right)$ and $\operatorname{Aut}(\mathbb{P}^n)$ for Calabi-Yau manifolds of dimension d in projective space $\mathbb{P}^n$.

11.1 Hodge numbers of Calabi-Yau manifolds in $\mathbb{P}^n$ and the relation between $h^{d-1,1}(X)$, $h^0\left(X, N_{X/\mathbb{P}^n}\right)$ and $\operatorname{Aut}(\mathbb{P}^n)$

Let $X \subset \mathbb{P}^n$ be a Calabi-Yau d-fold for $d \geq 3$.

- Note that for a Calabi-Yau d-fold

$$T_X = \wedge^1 \Omega_X^{1^*} \cong \wedge^d \Omega_X^{1^*} \otimes \wedge^{d-1} \Omega_X^1 = \left(\wedge^d \Omega_X^1\right)^* \otimes \wedge^{n-1} \Omega_X^1 = \Omega_X^{d-1}$$

- Tensoring the Euler sequence with $\mathcal{O}_X$ gives

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(1)^{n+1} \to T_{\mathbb{P}^n}|_X \to 0$$

hence the long exact sequence

$$\begin{array}{lllllll}
0 \to & H^0(X, \mathcal{O}_X) & \to & H^0\left(X, \mathcal{O}_X(1)^{n+1}\right) & \to & H^0(X, T_{\mathbb{P}^n}|_X) & \to \\
\to & H^1(X, \mathcal{O}_X) = 0 & \to & H^1\left(X, \mathcal{O}_X(1)^{n+1}\right) & \to & H^1(X, T_{\mathbb{P}^n}|_X) & \to \\
\to & H^2(X, \mathcal{O}_X) = 0 & & & & &
\end{array}$$

so

$$H^0(X, T_{\mathbb{P}^n}|_X) = \frac{H^0\left(X, \mathcal{O}_X(1)^{n+1}\right)}{H^0(X, \mathcal{O}_X)}$$

and

$$H^1(X, T_{\mathbb{P}^n}|_X) = H^1\left(X, \mathcal{O}_X(1)^{n+1}\right) = H^1(X, \mathcal{O}_X(1))^{n+1}$$

By Kodaira vanishing, as $\mathcal{O}_X(1)$ is positive and $\Omega^3_X = \mathcal{O}_X$ we get

$$H^i(X, \mathcal{O}_X(1)) = H^i\left(X, \mathcal{O}_X(1) \otimes \Omega^3_X\right) = 0 \text{ for } i > 0$$

hence

$$H^1(X, T_{\mathbb{P}^n}|_X) = 0$$

- The normal bundle sequence

$$0 \to T_X \to T_{\mathbb{P}^n}|_X \to N_{X/\mathbb{P}^n} \to 0$$

gives the long exact sequence

$$\begin{array}{ccccccc} 0 = H^0(X, T_X) & \to & H^0(X, T_{\mathbb{P}^n}|_X) & \to & H^0\left(X, N_{X/\mathbb{P}^n}\right) & \to \\ \to \quad H^1(X, T_X) & \to & H^1(X, T_{\mathbb{P}^n}|_X) = 0 & & & \end{array}$$

hence

$$H^1\left(X, \Omega^{d-1}_X\right) \cong H^1(X, T_X) \cong \frac{H^0\left(X, N_{X/\mathbb{P}^n}\right)}{H^0(X, T_{\mathbb{P}^n}|_X)}$$

- The long exact sequence for

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

reads

$$\begin{array}{cccccccc} 0 = & H^0(\mathbb{P}^n, \mathcal{I}_X) & \to & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = \mathbb{C} & \to & H^0(\mathbb{P}^n, \iota_*\mathcal{O}_X) = \mathbb{C} & \to \\ \to & H^1(\mathbb{P}^n, \mathcal{I}_X) & \to & H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0 & \to & H^1(\mathbb{P}^n, \iota_*\mathcal{O}_X) = 0 & \\ \to & H^2(\mathbb{P}^n, \mathcal{I}_X) & \to & H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0 & \to & & \end{array}$$

so

$$H^1(\mathbb{P}^n, \mathcal{I}_X) = 0$$
$$H^2(\mathbb{P}^n, \mathcal{I}_X) = 0$$

- The long exact sequence for

$$0 \to \mathcal{I}_X(1) \to \mathcal{O}_{\mathbb{P}^n}(1) \to \iota_* \mathcal{O}_X(1) \to 0$$

gives

$$\begin{array}{lllll} & 0 = H^0(\mathbb{P}^n, \mathcal{I}_X(1)) & \to & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) & \to & H^0(\mathbb{P}^n, \iota_* \mathcal{O}_X(1)) & \to \\ \to & H^1(\mathbb{P}^n, \mathcal{I}_X(1)) & \to & H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = 0 & & & \end{array}$$

hence $H^1(\mathbb{P}^n, \mathcal{I}_X(1)) = 0$ is equivalent to

$$H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(\mathbb{P}^n, \iota_* \mathcal{O}_X(1))$$

being surjective, i.e., to X being embedded by a complete linear system.

- Tensoring the Euler sequence with $\mathcal{I}_X$ gives the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_X(1)^{n+1} \to T_{\mathbb{P}^n} \otimes \mathcal{I}_X \to 0$$

hence the long exact sequence

$$\begin{array}{lllllll} & H^0(\mathbb{P}^n, \mathcal{I}_X) = 0 & \to & H^0\left(\mathbb{P}^n, \mathcal{I}_X(1)^{n+1}\right) & \to & H^0(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \mathcal{I}_X) & \to \\ \to & H^1(\mathbb{P}^n, \mathcal{I}_X) = 0 & \to & H^1\left(\mathbb{P}^n, \mathcal{I}_X(1)^{n+1}\right) & \to & H^1(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \mathcal{I}_X) & \to \\ \to & H^2(\mathbb{P}^n, \mathcal{I}_X) = 0 & \to & \ldots & & & \end{array}$$

If X does not lie in a hyperplane, $H^0(\mathbb{P}^n, \mathcal{I}_X(1)) = 0$, so

$$H^0(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \mathcal{I}_X) = 0$$

If $H^1(\mathbb{P}^n, \mathcal{I}_X(1)) = 0$, then

$$H^1(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \mathcal{I}_X) = 0$$

- Tensoring

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_X \to 0$$

with $T_{\mathbb{P}^n}$ gives the exact sequence

$$0 \to \mathcal{I}_X \otimes T_{\mathbb{P}^n} \to T_{\mathbb{P}^n} \to T_{\mathbb{P}^n}|_X \to 0$$

and the long exact sequence

$$\begin{array}{lllllll} 0 & \to & H^0(\mathbb{P}^n, \mathcal{I}_X \otimes T_{\mathbb{P}^n}) & \to & H^0(\mathbb{P}^n, T_{\mathbb{P}^n}) & \to & H^0(\mathbb{P}^n, T_{\mathbb{P}^n}|_X) & \to \\ & \to & H^1(\mathbb{P}^n, \mathcal{I}_X \otimes T_{\mathbb{P}^n}) & \to & \ldots & & & \end{array}$$

so

$$H^0(\mathbb{P}^n, T_{\mathbb{P}^n}|_X) = H^0(\mathbb{P}^n, T_{\mathbb{P}^n}) = \frac{H^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{n+1}\right)}{H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})} = \frac{H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))^{n+1}}{H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n})}$$

hence $h^0(\mathbb{P}^n, T_{\mathbb{P}^n}|_X) = (n+1)^2 - 1$. Note that any element in $H^0(\mathbb{P}^n, T_{\mathbb{P}^n})$ can be considered as a generator of an element in $\mathrm{Aut}(\mathbb{P}^n)$, so $h^0(\mathbb{P}^n, T_{\mathbb{P}^n}) = \dim \mathrm{Aut}(\mathbb{P}^n)$.

Summarizing these observations:

Proposition 11.1 *For any Calabi-Yau d-fold* $X \subset \mathbb{P}^n$ *with* $d \geq 3$ *and not in a hyperplane and with* $H^1(X, \mathcal{I}_X(1)) = 0$

$$H^1\left(X, \Omega_X^{d-1}\right) \cong H^1(X, T_X) \cong \frac{H^0\left(X, N_{X/\mathbb{P}^n}\right)}{H^0\left(X, T_{\mathbb{P}^n}|_X\right)}$$

and

$$H^0\left(X, T_{\mathbb{P}^n}|_X\right) \cong H^0\left(\mathbb{P}^n, T_{\mathbb{P}^n}\right)$$

in particular

$$h^{d-1,1}(X) = h^0\left(X, N_{X/\mathbb{P}^n}\right) - \dim\left(\mathrm{Aut}\left(\mathbb{P}^n\right)\right)$$

Remark 11.2 *Note that* $H^1(X, \mathcal{I}_X(1)) = 0$ *if* X *is projectively Cohen-Macaulay. But* $H^1(X, \mathcal{I}_X(1)) = 0$ *is also true for the Pfaffian examples of degree* $15, 16$ *and* 17 *given in [Tonoli, 2000] (see Section 10.1), which are not projectively Cohen-Macaulay.* $H^1(X, \mathcal{I}_X(1)) = 0$ *is equivalent to* X *being embedded by a complete linear system.*

Although for $K3$ surfaces and elliptic curves we know that $h^{1,1}(X) = 20$, respectively $h^{1,0}(X) = 1$, it is interesting to see how the calculation behaves:

Remark 11.3 *Recall that there are no algebraic families of dimension more than* 19*, whereas all* $K3$ *form a* $20 = h^{1,1}(X)$*-dimensional differentiable family.*

For $K3$ *surfaces* $T_X \cong \Omega_X^1$ *hence* $H^0(X, T_X) = H^0\left(X, \Omega_X^1\right) = 0$*, but* $H^2(X, \mathcal{O}_X) = 1$*, so from the Euler sequence tensored by* $\mathcal{O}_X$

$$\begin{array}{ccccccc} & H^1(X, \mathcal{O}_X) = 0 & \to & H^1\left(X, \mathcal{O}_X(1)^{n+1}\right) & \to & H^1\left(X, T_{\mathbb{P}^n}|_X\right) & \to \\ \to & H^2(X, \mathcal{O}_X) & \to & H^2\left(X, \mathcal{O}_X(1)^{n+1}\right) = 0 & & & \end{array}$$

where $H^2(X, \mathcal{O}_X(1)) = 0$ *by Kodaira vanishing, so*

$$H^1\left(X, T_{\mathbb{P}^n}|_X\right) = H^2(X, \mathcal{O}_X) \cong \mathbb{C}$$

From the normal bundle sequence

$$\begin{array}{ccccccc} & 0 = H^0(X, T_X) & \to & H^0\left(X, T_{\mathbb{P}^n}|_X\right) & \to & H^0\left(X, N_{X/\mathbb{P}^n}\right) & \to \\ \to & H^1(X, T_X) & \to & H^1\left(X, T_{\mathbb{P}^n}|_X\right) & \to & H^1\left(X, N_{X/\mathbb{P}^n}\right) & \to \\ \to & H^2(X, T_X) = 0 & & & & & \end{array}$$

and the fact that $h^1(X, T_X) = h^{1,1}(X) = 20$*, but the image in* $H^1(X, T_X)$ *is at most* 19 *dimensional, we have* $H^1\left(X, N_{X/\mathbb{P}^n}\right) = 0$*, hence*

$$h^{1,1}(X) = h^1(X, T_X) = h^0\left(X, N_{X/\mathbb{P}^n}\right) - h^0\left(X, T_{\mathbb{P}^n}|_X\right) + 1$$

Furthermore

$$H^1(\mathbb{P}^n, \mathcal{I}_X) = 0$$
$$H^2(\mathbb{P}^n, \mathcal{I}_X) = 0$$

so if $H^j(\mathbb{P}^n, \mathcal{I}_X(1)) = 0$ *for* $j = 0, 1$*, i.e.,* X *does not lie in a hyperplane and is embedded by a complete linear system, then also*

$$H^j(X, T_{\mathbb{P}^n} \otimes \mathcal{I}_X) = 0$$

for $j = 0, 1$*, hence*

$$H^0(\mathbb{P}^n, T_{\mathbb{P}^n}|_X) = H^0(\mathbb{P}^n, T_{\mathbb{P}^n})$$

and we get

$$h^{1,1}(X) = h^0\left(X, N_{X/\mathbb{P}^n}\right) - \left((n+1)^2 - 1\right) + 1$$

Remark 11.4 *For elliptic curves* $T_X \cong \mathcal{O}_X$*, hence* $H^0(X, T_X) \cong \mathbb{C}$*. So from the Euler sequence tensored by* $\mathcal{O}_X$

$$\ldots \to \quad \mathbb{C} \quad \to \quad H^1\left(X, \mathcal{O}_X(1)^{n+1}\right) = 0 \quad \to \quad H^1(X, T_{\mathbb{P}^n}|_X) \quad \to 0$$

hence

$$H^1(X, T_{\mathbb{P}^n}|_X) = 0$$

From the normal bundle sequence

$$\begin{array}{lllll} 0 \to & \mathbb{C} \cong H^0(X, T_X) & \to & H^0(X, T_{\mathbb{P}^n}|_X) & \to & H^0\left(X, N_{X/\mathbb{P}^n}\right) & \to \\ \to & H^1(X, T_X) & \to & 0 & & & \end{array}$$

hence

$$h^{0,1}(X) = h^1(X, \mathcal{O}_X) = h^1(X, T_X) = h^0\left(X, N_{X/\mathbb{P}^n}\right) - h^0(X, T_{\mathbb{P}^n}|_X) + 1$$

From

$$\begin{array}{lllllll} 0 = & H^0(\mathbb{P}^n, \mathcal{I}_X) & \to & H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \cong \mathbb{C} & \to & H^0(\mathbb{P}^n, \iota_*\mathcal{O}_X) \cong \mathbb{C} & \to \\ \to & H^1(\mathbb{P}^n, \mathcal{I}_X) & \to & H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0 & \to & H^1(\mathbb{P}^n, \iota_*\mathcal{O}_X) \cong \mathbb{C} & \\ \to & H^2(\mathbb{P}^n, \mathcal{I}_X) & \to & H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) = 0 & & & \end{array}$$

we get

$$H^1(\mathbb{P}^n, \mathcal{I}_X) = 0$$
$$H^2(\mathbb{P}^n, \mathcal{I}_X) \cong H^1(\mathbb{P}^n, \iota_*\mathcal{O}_X) \cong H^1(X, \mathcal{O}_X) \cong \mathbb{C}$$

By

$$
\begin{array}{lllll}
 & & H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = 0 & \to & H^1(\mathbb{P}^n, \iota_* \mathcal{O}_X(1)) = 0 \quad \to \\
\to & H^2(\mathbb{P}^n, \mathcal{I}_X(1)) \quad \to & H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) = 0 & \to & H^2(\mathbb{P}^n, \iota_* \mathcal{O}_X(1))
\end{array}
$$

one has

$$H^2(\mathbb{P}^n, \mathcal{I}_X(1)) = 0$$

so if X *is not contained in a hyperplane and* $H^1(\mathbb{P}^n, \mathcal{I}_X(1)) = 0$ *(i.e* X *embedded by a complete linear system), then from*

$$
\begin{array}{llllll}
 & H^0(\mathbb{P}^n, \mathcal{I}_X) = 0 & \to & H^0\left(\mathbb{P}^n, \mathcal{I}_X(1)^{n+1}\right) = 0 & \to & H^0(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \mathcal{I}_X) \quad \to \\
\to & H^1(\mathbb{P}^n, \mathcal{I}_X) = 0 & \to & H^1\left(\mathbb{P}^n, \mathcal{I}_X(1)^{n+1}\right) = 0 & \to & H^1(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \mathcal{I}_X) \quad \to \\
\to & H^2(\mathbb{P}^n, \mathcal{I}_X) \cong \mathbb{C} & \to & H^2\left(\mathbb{P}^n, \mathcal{I}_X(1)^{n+1}\right) = 0 & &
\end{array}
$$

we get

$$
\begin{array}{l}
H^0(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \mathcal{I}_X) = 0 \\
H^1(\mathbb{P}^n, T_{\mathbb{P}^n} \otimes \mathcal{I}_X) \cong H^2(\mathbb{P}^n, \mathcal{I}_X) \cong H^1(X, \mathcal{O}_X) \cong \mathbb{C}
\end{array}
$$

hence by

$$
\begin{array}{llllll}
0 \to & H^0(\mathbb{P}^n, \mathcal{I}_X \otimes T_{\mathbb{P}^n}) & \to & H^0(\mathbb{P}^n, T_{\mathbb{P}^n}) & \to & H^0(\mathbb{P}^n, T_{\mathbb{P}^n}|_X) \quad \to \\
\to & H^1(\mathbb{P}^n, \mathcal{I}_X \otimes T_{\mathbb{P}^n}) \cong \mathbb{C} & \to & H^1(\mathbb{P}^n, T_{\mathbb{P}^n}) = 0 & &
\end{array}
$$

it follows

$$h^0(\mathbb{P}^n, T_{\mathbb{P}^n}|_X) = h^0(\mathbb{P}^n, T_{\mathbb{P}^n}) + 1$$

so

$$h^{1,0}(X) = h^0\left(X, N_{X/\mathbb{P}^n}\right) - \left((n+1)^2 - 1\right)$$

11.2 Batyrev's Hodge formula

Let $\Delta \subset M_{\mathbb{R}}$ be a reflexive polytope and X a general anticanonical hypersurface in $Y = \mathbb{P}(\Delta)$. To given an idea on the proof of the Equations 2.1

$$
\begin{aligned}
h^{d-1,1}\left(\bar{X}\right) &= |\Delta \cap M| - n - 1 - \sum_{Q \text{ facet of } \Delta} |\mathrm{int}_M(Q)| \\
&\quad + \sum_{\substack{Q \text{ face of } \Delta \\ \mathrm{codim}\, Q = 2}} |\mathrm{int}_M(Q)| \cdot |\mathrm{int}_N(Q^*)| \\
h^{1,1}\left(\bar{X}\right) &= |\Delta^* \cap M| - n - 1 - \sum_{Q^* \text{ facet of } \Delta^*} |\mathrm{int}_N(Q^*)| \\
&\quad + \sum_{\substack{Q^* \text{ face of } \Delta^* \\ \mathrm{codim}\, Q^* = 2}} |\mathrm{int}_N(Q^*)| \cdot |\mathrm{int}_M(Q)|
\end{aligned}
$$

via MPCP desingularizations, suppose $\bar{\Sigma}$ is a maximal projective subdivision of the normal fan $\mathrm{NF}(\Delta) \subset N_{\mathbb{R}}$ of the reflexive polytope $\Delta \subset M_{\mathbb{R}}$, let

$$f : X\left(\bar{\Sigma}\right) \to \mathbb{P}(\Delta)$$

be the corresponding birational morphism inducing a crepant morphism $\bar{X} \to X$, and write D_w with $w \in \bar{\Sigma}(1)$ for the prime T-Weil divisors on $X\left(\bar{\Sigma}\right)$.

11.2.1 Toric divisor classes

Restriction of divisors from $X\left(\bar{\Sigma}\right)$ to $\bar{X}$ gives

$$\begin{array}{ccccccccc} 0 \to & M & \to & \mathrm{WDiv}_T\left(X\left(\bar{\Sigma}\right)\right) & \to & & & A_{n-1}\left(X\left(\bar{\Sigma}\right)\right) & \to 0 \\ & \shortparallel & & \downarrow & & & & \downarrow & \\ 0 \to & M & \to & \mathrm{WDiv}_T\left(\bar{X}\right) & \to & A_{d-1}\left(\bar{X}\right)_{toric} & \subset & A_{d-1}\left(\bar{X}\right) & \end{array} \tag{11.1}$$

The image of the toric Weil divisors $\mathrm{WDiv}_T\left(\bar{X}\right)$ in $A_{d-1}\left(\bar{X}\right)$ is not surjective in general, so denote the image by $A_{d-1}\left(\bar{X}\right)_{toric}$ and its complexification, i.e., the subspace of $H^{1,1}\left(\bar{X}\right)$ of toric divisor classes of $\bar{X}$, by

$$H^{1,1}_{toric}\left(\bar{X}\right) = A_{d-1}\left(\bar{X}\right)_{toric} \otimes \mathbb{C}$$

A divisor has trivial restriction if and only if its support is disjoint from the general hypersurface $\bar{X}$. If $w \in \bar{\Sigma}(1)$ is a lattice point in the relative interior of a facet of $\Delta^* \subset N_{\mathbb{R}}$, i.e., if

$$w \in \bigcup_{\mathrm{codim}(Q^*)=1} \mathrm{int}\left(Q^*\right)$$

then $f(D_w)$ is a point, so D_w is disjoint from any general element $\bar{X}$ of $\left|-K_{X(\bar{\Sigma})}\right|$. If $w \in \bar{\Sigma}(1)$ is not in the relative interior of a facet, then $\dim\left(f\left(D_w\right)\right) > 0$ so $f(D_w)$ meets X. Hence with

$$\Xi_0^* = \Delta^* \cap N - \bigcup_{\mathrm{codim}\, Q^* \leq 1} \mathrm{int}_N\left(Q^*\right)$$

we have

$$\mathrm{WDiv}_T\left(\bar{X}\right) \cong \mathbb{Z}^{\Xi_0^*}$$

and as cokernel of $M \to \mathrm{WDiv}_T\left(\bar{X}\right)$

$$A_{d-1}\left(\bar{X}\right)_{toric} \cong \mathbb{Z}^{\Xi_0^*}/M$$

so

$$H^{1,1}_{toric}\left(\bar{X}\right) \cong \mathbb{Z}^{\Xi_0^*}/M$$

with dimension

$$h^{1,1}_{toric}\left(\bar{X}\right) = \left|\Delta^* \cap N\right| - 1 - \sum_{\mathrm{codim}\, Q^*=1} \left|\mathrm{int}\left(Q^*\right)\right| - n$$

11.2.2 Polynomial deformations and complex moduli space

Define the subspace of polynomial first order deformations

$$H^{d-1,1}_{poly}\left(\bar{X}\right) \subset H^{d-1,1}\left(\bar{X}\right) \cong H^1\left(\bar{X}, T_{\bar{X}}\right)$$

as the subspace determined by $\left|-K_{X(\bar{\Sigma})}\right|$. Any element is given by a linear combination of the lattice monomials $\Delta \cap M$. Multiplication of the equation by a constant does not affect the zero set and the automorphism group of $X\left(\bar{\Sigma}\right)$ has dimension

$$\dim\left(\operatorname{Aut}\left(X\left(\bar{\Sigma}\right)\right)\right) = n + \sum_{\operatorname{codim} Q=1} \left|\operatorname{int}_M(Q)\right|$$

hence

$$h^{d-1,1}_{poly}\left(\bar{X}\right) = |\Delta \cap M| - 1 - n - \sum_{\operatorname{codim} Q=1} \left|\operatorname{int}_M(Q)\right|$$

The tangent space to the space of polynomial deformations is

$$H^{d-1,1}_{poly}\left(\bar{X}\right) \cong \left(\mathbb{Z}^{\Xi_0}/N\right) \otimes \mathbb{C}$$

with

$$\Xi_0 = \Delta \cap M - \bigcup_{\operatorname{codim} Q \leq 1} \operatorname{int}_M(Q)$$

and $\mathbb{Z}^{\Xi_0}/N$ given as the cokernel of the lower row in

$$\begin{array}{cccccccc}
0 \to & N & \to & WDiv_T\left(X\left(\bar{\Sigma}^\circ\right)\right) & \to & & A_{n-1}\left(X\left(\bar{\Sigma}^\circ\right)\right) & \to 0 \\
& \parallel & & \downarrow & & & \downarrow & \\
0 \to & N & \to & WDiv_T\left(\bar{X}^\circ\right) = \mathbb{Z}^{\Xi_0} & \to & A_{d-1}\left(\bar{X}^\circ\right)_{toric} & \subset A_{d-1}\left(\bar{X}^\circ\right) &
\end{array}$$

For a description of the non-toric divisor classes and non-polynomial deformations see, e.g., [Cox, Katz, 1999, Sec. 4.1].

11.3 First approximation of a tropical Hodge formula

Let $Y = \mathbb{P}(\Delta) = \mathbb{P}^n_{\mathbb{C}}$ for the degree $n+1$ reflexive Veronese simplex Δ and denote by S the homogeneous coordinate ring of Y. Consider the setup of Section 9: So let $\mathfrak{X} \subset Y \times \operatorname{Spec} \mathbb{C}[t]$ be a degeneration of projective Calabi-Yau varieties defined by the ideal $I \subset \mathbb{C}[t] \otimes S$ with monomial special fiber given by $I_0 \subset S$, general fiber X and satisfying the conditions given in Section 9.5. So X has only unobstructed polynomial deformations and as Y is assumed to be projective space I_0 is a Stanley-Reisner ideal.

Proposition 11.5 *A T-invariant basis of $H^0\left(X, N_{X_0/\mathbb{P}^n}\right)$ is given by*

$$A\left(\operatorname{supp}\left(\operatorname{dual}\left(B\left(I\right)\right)\right) \cap M\right)$$

and

$$\begin{array}{ccccc} M & & \mathbb{Z}^{\Sigma(1)} & & \operatorname{Hom}\left(I_0, S/I_0\right)_0 \\ \cup & & \cup & & \cup \\ \operatorname{supp}\left(\operatorname{dual}\left(B\left(I\right)\right)\right) \cap M & \underset{1:1}{\overset{A}{\rightleftarrows}} & A\left(\operatorname{supp}\left(\operatorname{dual}\left(B\left(I\right)\right)\right) \cap M\right) & \subset & H^0\left(X_0, N_{X_0/\mathbb{P}^n}\right) \end{array}$$

in particular $h^0\left(X_0, N_{X_0/\mathbb{P}^n}\right) = \left|\operatorname{supp}\left(\operatorname{dual}\left(B\left(I\right)\right)\right) \cap M\right|$ is the number of lattice points of $\operatorname{dual}\left(B\left(I\right)\right)$.

Corollary 11.6 *If X is a Calabi-Yau manifold, then*

$$h^{1,\dim X-1}\left(X\right) = \left|\operatorname{supp}\left(\operatorname{dual}\left(B\left(I\right)\right)\right) \cap M - \operatorname{Roots}\left(\mathbb{P}\left(\Delta\right)\right)\right| - \overset{=n}{\dim\left(T_Y\right)} + \overset{K3}{1}$$

Example 11.7 *For the elliptic curve given as a complete intersection of two quadrics in $\mathbb{P}^3$, as considered in Example 8.6, the dual complex* $\operatorname{dual}\left(B\left(I\right)\right)$ *together with the monomials corresponding to vertices of ∇^* is shown in Figure 11.1. The* 4 *lattice points of* $\operatorname{dual}\left(B\left(I\right)\right)$*, marked with dots, form a basis of $T^1_{X_0}$, the remaining* 12 *lattice points are roots, i.e., homomorphism of the form $x_i\frac{\partial}{\partial x_j} \in \operatorname{Hom}\left(I_0, S/I_0\right)_0$ for $i \neq j$, of $\mathbb{P}\left(\Delta\right) = \mathbb{P}^n$. So with the torus T of $\mathbb{P}\left(\Delta\right)$ we have*

$$\begin{aligned} \dim\left(T^1_{X_0}\right) + \left|\operatorname{Roots}\left(\mathbb{P}\left(\Delta\right)\right)\right| &= \\ h^{1,0}\left(X\right) + \dim\left(T\right) + \left|\operatorname{Roots}\left(\mathbb{P}\left(\Delta\right)\right)\right| &= \\ h^{1,0}\left(X\right) + \dim\left(\operatorname{Aut}\left(\mathbb{P}\left(\Delta\right)\right)\right) &= h^0\left(X_0, N_{X_0/\mathbb{P}^n}\right) \end{aligned}$$

The $h^{1,0}\left(X\right) = 1$-dimensional tangent space to the moduli space of X is a quotient of the 4*-dimensional $T^1_{X_0}$ by the* 3*-dimensional torus T of Y.*

Remark 11.8 *On the other hand lattice points of* $\operatorname{dual}\left(B\left(I\right)\right)$ *correspond to rays of the MPCP-desingularization of the toric variety Y° containing the Batyrev-Borisov mirror X° of X, hence rays correspond to toric divisor classes of X°, so we also have an interpretation of the formula*

$$h^{1,1}_{toric}\left(X^\circ\right) = \left|\operatorname{supp}\left(\operatorname{dual}\left(B\left(I\right)\right)\right) \cap M - \operatorname{Roots}\left(\mathbb{P}\left(\Delta\right)\right)\right| - \overset{=n}{\dim\left(T_Y\right)}$$

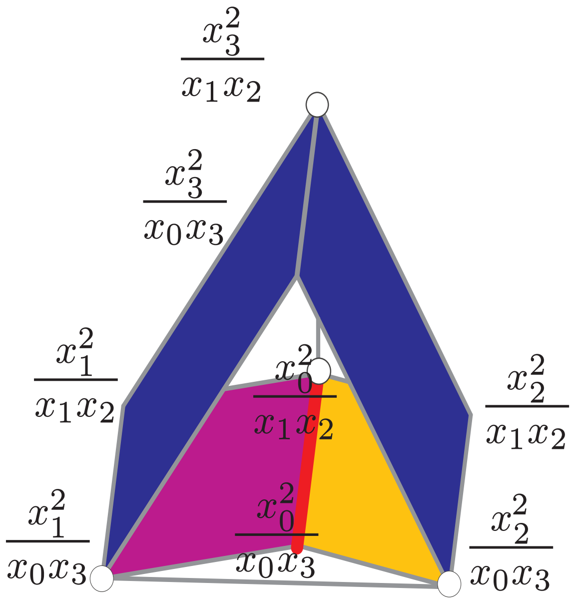


Figure 11.1: The dual complex and the monomials corresponding to the vertices of ∇^* associated to the degeneration of the general complete intersection elliptic curve in $\mathbb{P}^3$

Note that above formula agrees with the toric Batyrev formula for hypersurfaces

$$h^{1,\dim(X)-1}(X) = |\Delta \cap M| - n - 1 - \sum_{\Gamma \text{ facet of } \Delta} |\mathrm{int}_M(\Gamma)|$$

as for any reflexive polytope Δ

$$|\partial\Delta \cap M| = |\Delta \cap M| - 1$$

for any simplicial polytope

$$\dim(\mathrm{Aut}(\mathbb{P}(\Delta))) = n + \sum_{\Gamma \text{ facet of } \Delta} |\mathrm{int}_M(\Gamma)|$$

and the faces of the dual Δ^* of a Veronese polytope do not contain any interior lattice points, hence

$$\sum_{\substack{Q \text{ face of } \Delta \\ \operatorname{codim} Q = 2}} |\mathrm{int}_M(Q)| \cdot |\mathrm{int}_N(Q^*)| = 0$$

Remark 11.9 *The lattice points of* $\text{dual}(B(I))$ *corresponding to roots of* $\mathbb{P}(\Delta)$ *are the lattice points of* $\text{supp}(\text{dual}(B(I))) \subset \nabla^* \subset \Delta$ *in the relative interior of the facets of* Δ. *The complex* $\text{dual}(B(I)) \subset \text{dual}(\text{Poset}(\nabla))$ *and* Δ *are shown in Figure 11.2.*

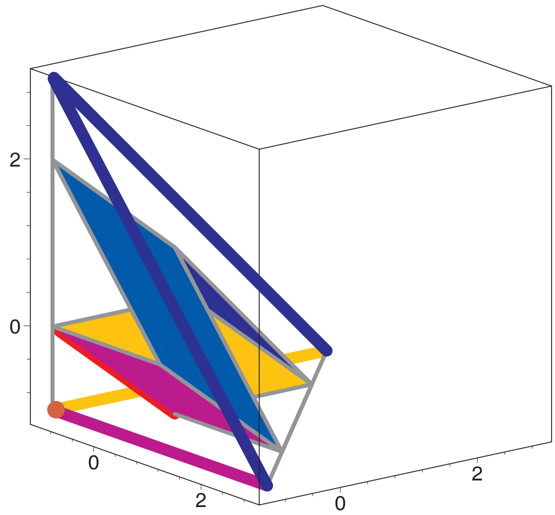


Figure 11.2: The complexes $\text{dual}(B(I)) \subset \text{dual}(\text{Poset}(\nabla))$ and Δ for the degeneration of the complete intersection of two general quadrics in $\mathbb{P}^3$

11.4 String cohomology

11.4.1 Stringy E-function for toric varieties

Let X be a normal $\mathbb{Q}$-Gorenstein toric variety of dimension n, given by the rational polyhedral fan $\Sigma \subset N_{\mathbb{R}}$ and let $\varphi_{K_Y} : N_{\mathbb{R}} \to \mathbb{R}_{\geq 0}$ be the continuous piecewise linear function with $\varphi_{K_Y}(\hat{r}) = 1$ for the minimal lattice generators $\hat{r}$ of all rays $r \in \Sigma(1)$.

Theorem 11.10 *[Batyrev, 1998] The stringy* E*-function of the normal* $\mathbb{Q}$*-Gorenstein toric variety* X *of dimension* n *is given by*

$$E_{st}(X, u, v) = (uv - 1)^n \sum_{\sigma \in \Sigma} \sum_{n \in N \cap \text{int}(\sigma)} (uv)^{-\varphi_{K_Y(n)}}$$

Recall that $\text{int}(\sigma)$ denotes the relative interior of σ. For the 0-cone we define $\text{int}(0) = \{0\}$.

11.4.2 The combinatorics of posets

Recall that a **poset** P is a finite partially ordered set, i.e., a finite set P with a reflexive, antisymmetric ($x \leq y$ and $y \leq x$ implies $x = y$) and transitive relation $\leq$.

Lemma 11.11 *[Batyrev, Borisov, 1996-I, Sec. 2] There is a unique function*

$$\mu_P : P \times P \to \mathbb{Z}$$

*called **Möbius function**, such that for every function $f : P \to A$ to some abelian group A and*

$$g(y) = \sum_{x \leq y} f(x)$$

it holds

$$f(y) = \sum_{x \leq y} \mu_P(x, y)\, g(x)$$

Definition 11.12 *Suppose that P has a unique minimal element $\min(P)$ and maximal element $\max(P)$ and that any maximal chain in P has the same length d. If $x \leq y$, then define*

$$[x, y] = \{z \in P \mid x \leq z \leq y\}$$

*The **rank function** $\rho : P \to \{0, ..., d\}$ associates to any $x \in P$ the length of any maximal chain in $[\min(P), x]$.*

Definition 11.13 *A poset P with above properties is called **Eulerian** if its Möbius function satisfies*

$$\mu_P(x, y) = (-1)^{\rho(y) - \rho(x)}$$

for all $x \leq y$.

Lemma 11.14 *If P is an Eulerian poset and $[x, y] \subset P$, then also $[x, y]$ is in an Eulerian poset with rank function*

$$\begin{array}{ccc} [x, y] & \to & \{0, ..., \rho(y) - \rho(x)\} \\ z & \mapsto & \rho(z) - \rho(x) \end{array}$$

Lemma 11.15 *Reversing the partial order, every Eulerian poset P has a **dual poset** P^*, which is again Eulerian with rank function*

$$\rho^*(x) = \rho(P) - \rho(x)$$

Example 11.16 *For any n-dimensional strongly convex rational polyhedral cone $C \subset N_{\mathbb{R}}$, the set of faces of C, together with inclusion, forms an Eulerian poset* $\mathrm{Poset}(C)$ *with rank function*

$$\begin{array}{rlcl} \rho: & \mathrm{Poset}(C) & \to & \{0, ..., \dim(C)\} \\ & F & \mapsto & \dim(F) \end{array}$$

and with minimal respectively maximal element

$$\min(P) = \{0\}$$
$$\max(P) = C$$

The dual poset of P is the poset of the faces of the dual cone $\check{C} \subset M_{\mathbb{R}}$.

Define the **truncation operator** by

$$\tau_{<s} : \mathbb{Z}[t] \to \mathbb{Z}[t]$$
$$\tau_{<s}\left(\sum_{i=0}^{d} a_i t^i\right) = \sum_{\substack{i=0 \\ i<s}}^{d} a_i t^i$$

Definition 11.17 *If P is an Eulerian poset of rank d, then define the polynomials $G(P,t), H(P,t) \in \mathbb{Z}[t]$ recursively by*

$$G(P,t) = 1$$
$$H(P,t) = 1$$

for $d = 0$ and

$$H(P,t) = \sum_{\substack{x \in P \\ x > \min(P)}} (t-1)^{\rho(x)-1} G([x,P], t)$$
$$G(P,t) = \tau_{<\frac{d}{2}}((1-t) H(P,t))$$

for $d > 0$.

Example 11.18 *Suppose P is the poset of the faces of a cone over the degree 5 Veronese simplex of $\mathbb{P}^4$, then*

$$H(P,t) = 1 + t + t^2 + t^3 + t^4$$
$$G(P,t) = 1$$

indeed, for any boolean algebra P of rank n, we have $H(P,t) = 1+t+...+t^{n-1}$ and $G(P,t) = 1$.

11.4.3 String-theoretic Hodge formula for hypersurfaces

Definition 11.19 *Suppose $N = \mathbb{Z}^n$ and $M = \operatorname{Hom}(N, \mathbb{Z})$. A cone C of dimension $d \geq 1$ in $M_{\mathbb{R}}$ is called **Gorenstein cone** if there is a $w \in N$ with $\langle m, w \rangle > 0$ for all $0 \neq m \in C$ and*

$$\{m \in C \mid \langle m, w \rangle = 1\}$$

*is an integral convex polytope, called the **supporting polytope** of C.*

Remark 11.20 *Consider the setup of Section 9, so let Y be a toric Fano variety, $\mathfrak{X}$ a degeneration of Calabi-Yau varieties, given by the ideal $I \subset \mathbb{C}[t] \otimes S$ and with monomial special fiber $I_0 \subset S$. Applying the tropical mirror construction, we obtain the strongly convex polyhedral cone*

$$\begin{array}{ccc} C_{I_0}(I) & \subset & N_{\mathbb{R}} \oplus \mathbb{R} \\ & & \cup \\ & & N \oplus \mathbb{Z} \end{array}$$

which is the closure of the set of weight vectors selecting I_0 as initial ideal of I. Then the dual cone $C_{I_0}(I)^{\vee}$ of $C_{I_0}(I)$ is a Gorenstein cone with supporting polytope $\nabla^* \subset M_{\mathbb{R}}$.

If $\mathfrak{X}$ is a degeneration of complete intersections in a Gorenstein toric Fano $Y = \mathbb{P}(\Delta)$, then also $C_{I_0}(I)$ is a Gorenstein cone with reflexive supporting polytope $\nabla = C_{I_0}(I) \cap \{w_t = 1\}$ and

$$Y^{\circ} = \mathbb{P}(\nabla) = \operatorname{Proj} \mathbb{C}[C_{I_0}(I) \cap (N \oplus \mathbb{Z})]$$

with the natural grading on $\mathbb{C}[C_{I_0}(I) \cap (N \oplus \mathbb{Z})]$.

Example 11.21 *The cone*

$$C = \{(\lambda, \lambda m) \in (\mathbb{Z} \oplus M)_{\mathbb{R}} \mid \lambda \in \mathbb{R}_{\geq 0},\ m \in \Delta\}$$

where $\Delta \subset M_{\mathbb{R}}$ is the degree 5 Veronese polytope, is a Gorenstein cone over the reflexive polytope Δ.

Definition 11.22 *Let C be a Gorenstein cone in $M_{\mathbb{R}}$ and Δ its supporting polytope. The **Ehrhart power series** of Δ is*

$$P_{\Delta}(t) = \sum_{k=0}^{\infty} |k\Delta \cap M| \cdot t^k$$

Lemma 11.23 *[Batyrev, 1994] Let C be a Gorenstein cone of dimension d in $M_{\mathbb{R}}$ and Δ its supporting polyhedron. Then there are $\psi_0, ..., \psi_d \in \mathbb{Z}_{\geq 0}$ such that*

$$P_\Delta(t) = \frac{\psi_0 + \psi_1 \cdot t + ... + \psi_{d-1} \cdot t^{d-1}}{(1-t)^d}$$

Define

$$S(C,t) = \psi_0 + \psi_1 \cdot t + ... + \psi_{d-1} \cdot t^{d-1}$$

Remark 11.24 *Note that*

$$S(C,t) = \psi_0 + \psi_1 \cdot t + ... + \psi_{d-1} \cdot t^{d-1} = (1-t)^d \cdot \sum_{k=0}^{\infty} |k\Delta \cap M| \cdot t^k$$

depends only on the values $|k\Delta \cap M|$ for $k = 0, ..., d-1$, because of the recursion relation $(1-t)^d$.

Example 11.25 *For the Gorenstein cone C over the degree 5 Veronese polyhedron Δ as defined in Example 11.21 we have*

k	0	1	2	3	4
$\|k\Delta \cap M\|$	1	126	1001	3876	10626

hence

$$S(C,t) = 1 + 121t + 381t^2 + 121t^3 + t^4$$

Definition 11.26 *If C is a Gorenstein cone, define*

$$\tilde{S}(C,t) = \sum_{C_1 \text{ face of } C} S(C_1,t)(-1)^{\dim(C)-\dim(C_1)} G([C_1,C],t)$$

Example 11.27 *If C is the Gorenstein cone over the Veronese polyhedron of degree 4 of $\mathbb{P}^3$, we have for the faces $C_1 \subset C$*

$\dim(C_1)$	*number of faces of C of this dimension*	$S(C_1,t)$	$G([C_1,C],t)$
0	1	1	1
1	4	1	1
2	6	$1+3t$	1
3	4	$1+12t+3t^2$	1
4	1	$1+31t+31t^2+t^3$	1

hence

$$\tilde{S}(C,t) = t + 19t^2 + t^3$$

Example 11.28 *If C is the Gorenstein cone over the degree 5 Veronese polyhedron from Example 11.21, the $S(C_1, t)$ for the faces $C_1 \subset C$ are*

$\dim(C_1)$	*number of faces of C of this dimension*	$S(C_1, t)$	$G([C_1, C], t)$
0	1	1	1
1	5	1	1
2	10	$1 + 4t$	1
3	10	$1 + 18t + 6t^2$	1
4	5	$1 + 52t + 68t^2 + 4t^3$	1
5	1	$1 + 121t + 381t^2 + 121t^3 + t^4$	1

hence

$$\tilde{S}(C, t) = t + 101t^2 + 101t^3 + t^4$$

Theorem 11.29 *[Batyrev, Dais, 1996] Let $C \subset M_{\mathbb{R}}$ be a Gorenstein cone supported on a reflexive polyhedron Δ. If X is an ample non-degenerate Calabi-Yau hypersurface of dimension d in $\mathbb{P}(\Delta) = \operatorname{Proj}\mathbb{C}[C \cap M]$, then*

$$\begin{aligned} E_{st}(X, u, v) &= (uv)^{-1}(-u)^{\dim(C)}\,\tilde{S}\left(C, u^{-1}v\right) + (uv)^{-1}\,\tilde{S}\left(C^{\vee}, uv\right) \\ &\quad + (uv)^{-1} \sum_{0 \subsetneq C_1 \subsetneq C} (-u)^{\dim(C_1)}\,\tilde{S}\left(C_1, u^{-1}v\right)\tilde{S}\left(C_1^{\vee}, uv\right) \end{aligned}$$

where the sum goes over the faces C_1 of C.

We may write this formula as

$$E_{st}(X, u, v) = (uv)^{-1} \sum_{C_1 \subset C} (-u)^{\dim(C_1)}\,\tilde{S}\left(C_1, u^{-1}v\right)\tilde{S}\left(C_1^{\vee}, uv\right)$$

Corollary 11.30 *Consider the setup of Theorem 11.29. If X° is an ample non-degenerate Calabi-Yau hypersurface of dimension d in $\mathbb{P}(\Delta^*) =$ $\operatorname{Proj}\mathbb{C}[C^{\vee} \cap N]$, then the stringy E-functions of X and X° satisfy the mirror duality relation*

$$E_{st}(X; u, v) = (-u)^d\, E_{st}\left(X^{\circ}; u^{-1}, v\right)$$

Example 11.31 *For the quadric $K3$ surface in $\mathbb{P}^3$ given by a general section in the degree 4 Veronese polytope, we obtain*

$\dim(C_1)$	*number of faces of C of this dimension*	$\tilde{S}(C_1, t)$	$\tilde{S}(C_1^*, t)$
0	1	1	$t + t^2 + t^3$
1	4	0	0
2	6	$3t$	0
3	4	$3t + 3t^2$	0
4	1	$t + 19t^2 + t^3$	1

hence

$$
\begin{aligned}
E_{st}(X,u,v) &= 1 + uv + (uv)^2 \\
&\quad + u^2 + 19uv + v^2 \\
&= 1 + \left(u^2 + 20uv + v^2\right) + (uv)^2
\end{aligned}
$$

Example 11.32 *For the quintic Calabi-Yau threefold in $\mathbb{P}^4$ given by a general section in the degree* 5 *Veronese polytope*

$\dim(C_1)$	*number of faces of C of this dimension*	$\tilde{S}(C_1,t)$	$\tilde{S}(C_1^*,t)$
0	1	1	$t+t^2+t^3+t^4$
1	5	0	0
2	10	$4t$	0
3	10	$6t+6t^2$	0
4	5	$4t+44t^2+4t^3$	0
5	1	$t+101t^2+101t^3+t^4$	1

hence

$$
\begin{aligned}
E_{st}(X,u,v) &= 1 + uv + (uv)^2 + (uv)^3 \\
&\quad - \left(u^3 + 101u^2v + 101uv^2 + v^3\right) \\
&= 1 + uv - \left(u^3 + 101u^2v + 101uv^2 + v^3\right) + (uv)^2 + (uv)^3
\end{aligned}
$$

11.4.4 String-theoretic Hodge formula for complete intersections

Let $\Delta \subset M_{\mathbb{R}}$ be a reflexive polytope and $Y = \mathbb{P}(\Delta)$ the corresponding Gorenstein toric Fano variety, denote by $\Sigma \subset N_{\mathbb{R}}$ the normal fan of Δ, and let $\Sigma(1) = I_1 \cup ... \cup I_c$ be a nef partition, i.e., $E_j = \sum_{v \in I_j} D_v$ are Cartier divisors, spanned by global sections and $\sum_{j=1}^{c} E_j = -K_Y$. Denote by $\Delta_j = \Delta_{E_j}$ the polytope of sections of E_j and by X a Calabi-Yau complete intersection given by general sections $s_j \in H^0(Y, \mathcal{O}_Y(E_j))$ for $j = 1, ..., c$.

Define

$$Z = \mathbb{P}\left(\mathcal{O}_Y(E_1) \oplus ... \oplus \mathcal{O}_Y(E_c)\right)$$

with canonical projection

$$\pi : Z \to Y$$

Then $\pi_* \mathcal{O}_Z(1) = \mathcal{O}_Y(D_1) \oplus ... \oplus \mathcal{O}_Y(D_c)$ and

$$H^0(Z, \mathcal{O}_Z(1)) \cong H^0(Y, \mathcal{O}_Y(E_1)) \oplus ... \oplus H^0(Y, \mathcal{O}_Y(E_c))$$

so $(s_1, ..., s_c)$ corresponds to a section $s \in H^0(Z, \mathcal{O}_Z(1))$. Let $\bar{X}$ be the zero set of s.

As X is transversal to the toric strata of Y

$$E_{st}(X,u,v) = E_{st}(Y,u,v) - E_{st}(Y\backslash X,u,v)$$

Proposition 11.33 *[Batyrev, Borisov, 1996-I]* $\pi\mid_{Z\backslash\bar{X}}$: $Z\backslash\bar{X}\to Y\backslash X$ *is in the Zariski topology a locally trivial* $\mathbb{C}^{c-1}$*-bundle, hence*

$$E_{st}(Y\backslash X,u,v) = (uv)^{1-c}\,E_{st}\left(Z\backslash\bar{X},u,v\right)$$

As Z is a $\mathbb{P}_{\mathbb{C}}^{c-1}$-bundle over Y

$$E_{st}(Y,u,v) = \left((uv)^{c}-1\right)^{-1}(uv-1)\,E_{st}(Z,u,v)$$

Proposition 11.34 *[Batyrev, Borisov, 1996-I] The sheaf* $\mathcal{O}_Z(1)$ *is Cartier, spanned by global sections, the morphism*

$$\alpha : Z \to W = \operatorname{Proj}\bigoplus\nolimits_{k\geq 0} H^0\left(Z,\mathcal{O}_Z(k)\right)$$

is crepant, $\mathcal{O}_Z(c)$ *is the anticanonical sheaf on* Z *and* W *is a Gorenstein toric Fano variety.*

$\alpha\left(\bar{X}\right)$ *is an ample hypersurface in* W*.*

Note that

$$W = \operatorname{Proj}\mathbb{C}\left[C\cap(\mathbb{Z}^r\oplus M)\right]$$

with the cone

$$C = \left\{\left(\lambda_1,...,\lambda_c,\sum_{i=1}^{c}\lambda_i m_i\right)\in(\mathbb{Z}^r\oplus M)_{\mathbb{R}}\mid \lambda_i\in\mathbb{R}_{\geq 0},\ m_i\in\Delta_i,\ i=1,...c\right\}$$

which is a Gorenstein cone with respect to $w\in N$ uniquely defined by

$$\begin{aligned}&\langle m,w\rangle = 0 \text{ for all } m\in M_{\mathbb{R}}\subset(\mathbb{Z}^r\oplus M)_{\mathbb{R}}\\&\langle e_i,w\rangle = 1 \text{ for all } i=1,...,c\end{aligned}$$

and has reflexive supporting polyhedron.

Observing that

$$\begin{aligned}E_{st}(Z,u,v) &= E_{st}(W,u,v)\\E_{st}\left(Z\backslash\bar{X},u,v\right) &= E_{st}\left(W\backslash\alpha\left(\bar{X}\right),u,v\right)\end{aligned}$$

we have

$$\begin{aligned}
E_{st}(X,u,v) &= E_{st}(Y,u,v) - E_{st}(Y\backslash X,u,v) \\
&= ((uv)^c-1)^{-1}(uv-1)\,E_{st}(Z,u,v) - (uv)^{1-c}\,E_{st}\left(Z\backslash\bar{X},u,v\right) \\
&= ((uv)^c-1)^{-1}(uv-1)\,E_{st}(W,u,v) - (uv)^{1-c}\,E_{st}\left(W\backslash\alpha\left(\bar{X}\right),u,v\right) \\
&= ((uv)^c-1)^{-1}(uv-1)\,E_{st}(W,u,v) \\
&- (uv)^{1-c}\left(E_{st}(W,u,v) - E_{st}\left(\alpha\left(\bar{X}\right),u,v\right)\right)
\end{aligned}$$

The stringy E-function $E_{st}(W,u,v)$ can be computed by the following Proposition 11.35, which shows equality of the stringy E-function and the original string-theoretic E-function defined by Batyrev and Dais in [Batyrev, Dais, 1996].

Proposition 11.35 *[Borisov, Mavlyutov, 2003] Let $X=\bigcup_{i\in I}X_i$ be a stratified algebraic variety of dimension n with the following properties (satisfied by W):*

- *X has at most Gorenstein toroidal singularities such that for each $i\in I$ the singularities of X along the stratum X_i of codimension c_i are given by some c_i-dimensional finite rational polyhedral cone σ_i. This is equivalent to X being locally isomorphic to $\mathbb{C}^{n-c_i}\times U(\sigma_i)$ at all points $x\in X_i$.*
- *There is a desingularization $\pi:\overline{X}\to X$ such that its restriction to the preimage if X_i is a locally trivial fibration in the Zariski topology.*
- *For all points $x\in X_i$ the preimage of an analytic neighborhood of x under π is analytically isomorphic the product of a complex disc and a preimage of a neighborhood of $\{0\}$ in $U(\sigma_i)$ under a resolution of singularities of $U(\sigma_i)$ such that the isomorphism is compatible with the resolutions.*

Then

$$E_{st}(X,u,v)=\sum_{i\in I}E(X_i,u,v)\cdot S(\sigma_i,uv)$$

Hence if we denote by P the Eulerian poset of the faces of the cone C with rank function

$$\begin{array}{rccc}\rho: & P & \to & \{0,...,\dim(C)\} \\ & F & \mapsto & \dim(F)\end{array}$$

then

$$E_{st}(W,u,v)=\sum_{\substack{x\in P\\ x>\min(P)}}(uv-1)^{\rho(x)-1}\,S(x^*,uv)$$

In order to compute $E_{st}\left(\alpha\left(\bar{X}\right), u, v\right)$ we can apply Section 11.4.3 to the Gorenstein cone C.

Theorem 11.36 *[Batyrev, Borisov, 1996-I] Let $X \subset Y = \mathbb{P}(\Delta)$ and $X^\circ \subset Y^\circ = \mathbb{P}(\nabla)$ be general complete intersections of dimension d defined by nef partitions, which are dual to each other with respect to the construction by Batyrev and Borisov as given in Section 2.2. Then the stringy E-functions of X and X° satisfy the mirror duality relation*

$$E_{st}(X; u, v) = (-u)^d E_{st}\left(X^\circ; u^{-1}, v\right)$$

11.4.5 Remarks on a tropical computation of the stringy E-function

Consider the setup from Section 9. So denote by $Y = X(\Sigma)$ the toric Fano variety given by the fan Σ over the Fano polytope Δ^*, and denote by S the Cox ring of Y. Let $\mathfrak{X} \subset Y \times \operatorname{Spec}\mathbb{C}[[t]]$ be the Calabi-Yau degeneration given by the ideal $I \subset \mathbb{C}[t] \otimes S$ with monomial special fiber given by $I_0 \subset S$.

Recall that

$$\nabla = C_{I_0}(I) \cap \{w_t = 1\}$$

and ∇^* is a Fano polytope.

A first approximation of a tropical expression of the stringy E-function of the general fiber X° of $\mathfrak{X}^\circ$ would be

$$E_{st}(X^\circ, u, v) = \sum_{\substack{x \in BF_{I_0}(I) \\ \dim(x) > 0}} (uv-1)^{\dim(x)-1} S\left(x^\vee, uv\right)$$

where

$$x^\vee \subset C_{I_0}(I)^\vee$$

denotes the face dual to x of the Gorenstein cone $C_{I_0}(I)^\vee$ over the Fano polytope ∇^*, i.e., x^* is the cone over

$$\operatorname{dual}(x \cap \{w_t = 1\}) \subset \operatorname{dual}(B(I)) \subset \operatorname{Poset}(\nabla^*)$$

Of course this will not work due to the nature of the singularities of the reducible X_0°.

One may ask for a formula for $E_{st}(X, u, v)$ in terms of the data given by the Gorenstein cones $C_{I_0}(I)^\vee$ and $C_{I_0^\circ}(I^\circ)^\vee$ and the subfans $BF_{I_0}(I)^\vee \subset \operatorname{Poset}\left(C_{I_0}(I)^\vee\right)$ and $BF_{I_0^\circ}(I^\circ)^\vee \subset \operatorname{Poset}\left(C_{I_0^\circ}(I^\circ)^\vee\right)$. This formula should be mirror symmetric with respect to the tropical mirror construction, i.e., should satisfy

$$E_{st}(X; u, v) = (-u)^d E_{st}\left(X^\circ; u^{-1}, v\right)$$

when exchanging $BF_{I_0}(I)$ and $C_{I_0}(I)$ with $BF_{I_0^\circ}(I^\circ)$ and $C_{I_0^\circ}(I^\circ)$ and should specialize to the formula for hypersurfaces from Section 11.4.3.

12 Implementation of the tropical mirror construction

In order to implement the tropical mirror construction, the following packages for the computer algebra systems Macaulay2 [Grayson, Stillman, 2006] and Maple [Maple, 2000] have been written by the author:

12.1 mora.m2

The Macaulay2 library mora.m2 provides an implementation of the standard basis algorithm.

Polynomials are represented as elements in the Macaulay2 type PolynomialRing and ideals are represented via the type Ideal.

- Monomial orderings:

  Denoting by M the semigroup of monomials in a polynomial ring, they are implemented as functions $f : M \times M \to \{\mathsf{true}, \mathsf{false}\}$ comparing two monomials, where $f(m_1, m_2) = \mathsf{true}$ if and only if $m_1 > m_2$.

  The following monomial orderings as defined in Section 1.4.1 are provided by mora.m2. They are selected by the global variable monord.

  - lexicographical lp
  - reverse lexicographical rp
  - degree reverse lexicographical dp
  - negative lexicographical ls

  The following weight orderings depend on a weight vector specified by the global variable ww of Macaulay2 type List with rational entries, whose length is the number of variables of the polynomial ring.

  - weighted reverse lexicographical wp
  - weighted lexicographical Wp
  - local weighted reverse lexicographical ws
  - local weighted lexicographical Ws

  The matrix ordering Mat depends on a matrix mm of Macaulay2 type Matrix with rational entries. The number of columns of Mat is the number of variables of the polynomial ring.

- $\mathsf{L}(f)$

  Computes the lead monomial of the polynomial f with respect to the semigroup ordering specified by monord.

- $\mathsf{SPolynomial}(f,g)$

  Returns the s-polynomial

  $$\mathrm{SPolynomial}(f,g)=\frac{\mathrm{lcm}\left(L\left(f\right),L\left(g\right)\right)}{L\left(f\right)}f-\frac{LC\left(f\right)}{LC\left(g\right)}\frac{\mathrm{lcm}\left(L\left(f\right),L\left(g\right)\right)}{L\left(g\right)}g$$

  of the polynomials f and g in the given polynomial ring with semigroup ordering monord.

- $\mathsf{NFG}(f,G)$

  Computes the Gröbner normal form of the polynomial f with respect to the finite Macaulay2 type list G of polynomials and semigroup ordering monord via Algorithm 1.174.

- $\mathsf{redNFG}(f,G)$

  Returns the Gröbner reduced normal form of the polynomial f with respect to the list G and semigroup ordering monord, using Algorithm 1.176.

- $\mathsf{NF}(f,G)$

  Computes the Mora normal form of the polynomial f with respect to the list G and semigroup ordering monord by Algorithm 1.179.

- $\mathsf{Std}(G)$

  Implements Algorithm 1.187 to compute a standard basis of the ideal $\langle G\rangle$ for a list G of elements in a polynomial ring and semigroup ordering monord.

- $\mathsf{Minimize}(G)$

  Given a list G of polynomials computes an interreduced subset with respect to the semigroup ordering monord.

- $\mathsf{MStd}(G)$

  Returns a minimal standard basis of the ideal $\langle G\rangle$ for a list G of elements in a polynomial ring and semigroup ordering monord.

- ReduceGb (G)

  Given a minimal Gröbner basis G of the ideal $\langle G\rangle$ with respect to monord, returns a reduced Gröbner basis of $\langle G\rangle$ via Algorithm 1.199.

- ReduceStd (G)

  Applying Algorithm 1.201 takes a minimal standard basis G of the ideal $\langle G\rangle$ with respect to monord and computes a reduced standard basis of $\langle G\rangle$ by the Gröbner normal form. If the reduction does not terminate, the procedure stops after a finite number of reductions of each element of G specified by the global variable iterlimit.

The global variable verbose $\in \{0,1,2\}$ controls the output of intermediate results, e.g., of syzygies in Gröbner computations.

Example 12.1 *Load the package and create a polynomial ring:*

```
load "mora.m2";
R=QQ[x,y,z];
```

Lead monomials with respect to various orderings:

```
f=x^4+y^7+z^5+x^4*y*z+x^3*y^3;
monord=lp;
L(f)
x^4*y*z

monord=dp;
L(f)
y^7

monord=ls;
L(f)
z^5

monord=ds;
L(f)
x^4

monord=Wp;
ww={2,1,-1};
L(f)
x^3*y^3

monord=Wp;
ww={-3,-1,-2}
L(f)
```

y^7

```
monord=Mat;
MM=matrix {{-3,-1,-2},{1,0,0},{0,1,0},{0,0,1}};
L(f)
```

y^7

Computing standard bases, division with remainder:

```
monord=lp;
G={x*y-1,y^2-1};
std(G);
```

*{x*y-1, y^2-1, x-y}*

```
GB=minimalstd(G);
```

{y^2-1, x-y}

```
f=x^2*y+x*y^2+y^2;
NFB(f,G)
```

x+y^2+y

```
NFB(f,GB)
```

2y+1

```
redNFB(f,GB)
```

y+1/2

```
G={x^2+y,x*y+x};
GB=minimalstd(G)
```

*{x^2+y, x*y+x, y^2+y}*

```
f=x^2-y^2;
redNFB(f,G)
```

y^2+y

```
redNFB(f,GB)
```

0

Mora normal form and Gröbner normal form for local orderings:

```
monord=ls;
NF(x,{x-x^2})
```

0

```
iterlimit=50;
NFB(x,{x-x^2})
```

x^51

```
monord=ls;
f=z^2+y*z+y^2+x^2;
```

```
G={x,z};
NF(f,G)
```
y^2+x^2

Minimal standard bases of the ideal $\langle G\rangle$ for the monomial orderings dp, lp, ds and ls:

```
G={x^6+x^5*y^2,y^4-x^2*y^3};
monord=dp;
minimalstd(G)
```
*{x^6+x^5*y^2, -x^2*y^3+y^4, x^6*y+x^3*y^4, -x^7+x*y^6, x^2*y^5+x*y^7, y^8+x^7}*

```
monord=lp
minimalstd(G)
```
*{x^6+x^5*y^2, -x^2*y^3+y^4, x*y^6+y^8, -y^9+y^6}*

```
monord=ds
minimalstd(G)
```
*{x^6+x^5*y^2,y^4-x^2*y^3}*

```
monord=ls
GB=minimalstd(G)
```
*{x^5*y^2+x^6, y^4-x^2*y^3, x^7*y^3-x^7}*

```
iterlimit=10^6;
reducestd(GB)
```
*{x^5*y^2+x^6, y^4-x^2*y^3, x^7}*
Note that y^3-1 is a unit in $R_>$ for the negative lexicographical ordering $>=ls$ and $x^7y^3-x^7=x^7\left(y^3-1\right)$.

The ideal of a line and a plane in the global setting and in the local ring $\mathbb{Q}\left[x,y,z\right]_{\langle x,y,z\rangle}$:

```
G={x*y+y,x*z+z};
monord=dp;
minimalstd(G)
```
*{x*y+y,x*z+z}*

```
NF(x,GB)
```
x

```
NF(y,GB)
```
y

```
NF(z,GB)
```
z

```
monord=ls;
```

```
GB=minimalstd(G);
{x*y + y, x*z + z}
NF(x,GB)
x
NF(y,GB)
0
NF(z,GB)
0
reducestd(GB)
{y, z}
```

Note that $1+x$ *is a unit in* $R_>$ *for the negative lexicographical ordering* >= *ls.*

12.2 homology.m2

The Macaulay2 library homology.m2 provides the following functions:

Let C be a cell complex given as a list, where the d-th element is a list of the faces of dimension d, and each face is given as a list of vertices.

- IsSimplicial (C)

  Checks if C is simplicial.

- AssociatedChainComplex (C, R)

  Associates to the simplicial complex C the associated chain complex with coefficients in the Macaulay2 type ring R. The resulting chain complex is represented as a Macaulay2 type chain complex. The orientation of the cells of C is represented by the ordering of the vertices in the lists representing the faces of C and the boundary maps are given by

$$\partial\left(v_{i_0},...,v_{i_d}\right)=\sum_{j=0}^{d}(-1)^j\left(v_{i_0},...,\widehat{v_{i_j}},...,v_{i_d}\right)$$

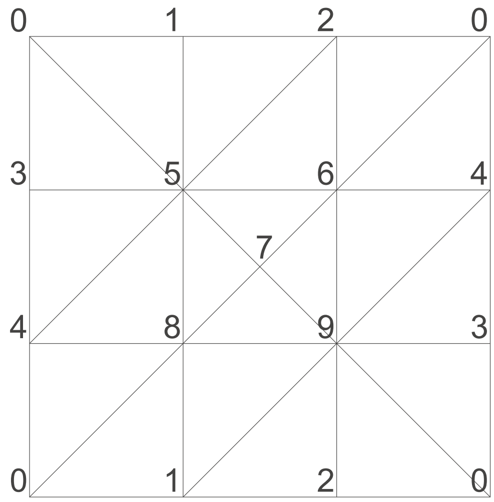
- BoundaryMap (C, R, d)

  Returns above boundary map $\partial : C_d \to C_{d-1}$.

- HomologyChainComplex (C, R)

  Computes a list with the homology groups of C with coefficients in R.

Example 12.2 *Consider the following triangulation of the Klein bottle*




```
load "homology.m2";

C={{{0},{1},{2},{3},{4},{5},{6},{7},{8},{9}},
{{0,1},{1,2},{0,2},{0,8},{1,8},{1,9},{2,9},{0,9},{0,3},{4,8},
{8,9},{3,9},{3,4},{4,5},{5,8},{7,8},{7,9},{6,7},{5,7},{3,5},
{5,6},{6,9},{4,6},{4,9},{0,4},{0,5},{1,5},{2,5},{2,6},{0,6}},
{{0,1,8},{1,2,9},{2,0,9},{0,3,9},{1,9,8},{0,8,4},{3,4,5},{4,8,5},
{5,8,7},{7,8,9},{5,7,6},{6,7,9},{4,6,9},{3,4,9},{0,3,5},{0,5,1},
{1,5,2},{2,5,6},{0,2,6},{0,6,4}}};

cC=SimplicialChainComplex(C,ZZ)
```

$0 \leftarrow \mathbb{Z}^{10} \leftarrow \mathbb{Z}^{30} \leftarrow \mathbb{Z}^{20} \leftarrow 0$

```
HomologyChainComplex(cC)
```

$(\mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}/2, 0)$

```
cC=SimplicialChainComplex(C,QQ);
HomologyChainComplex(cC)
```

$(\mathbb{Q}, \mathbb{Q}, 0)$

```
cC=SimplicialChainComplex(C,ZZ/2);
HomologyChainComplex(cC)
```

$(\mathbb{Z}/2, \mathbb{Z}/2 \oplus \mathbb{Z}/2, \mathbb{Z}/2)$

12.3 stanleyfiltration.m2

The Macaulay2 library stanleyfiltration.m2 provides the following functions:

- StanleyDecomposition (I)

  Implements Algorithm 6.66 to compute a Stanley decomposition

$$S/I \cong \bigoplus_{(D,\sigma)\in\mathcal{S}} S_\sigma\left(-[D]\right)$$
$$\mathcal{S} \subset \{(D,\sigma) \mid D \in \mathrm{WDiv}_T\left(Y''\right), \, D \text{ effective}, \, \sigma \in \Sigma''\}$$

  of a monomial ideal I in the polynomial ring S, where Σ'' is the fan over the simplex on the variables of S and $Y'' = \mathbb{A}^{\Sigma''(1)} = \operatorname{Spec} S$.

  The ring S is represented via the Macaulay2 type Ring and I via the Macaulay2 type Ideal.

  The output is a set of tuples (m, P) representing $(D, \sigma) \in \mathcal{S}$, where m is a monomial in S defining the divisor D and P is a set of variables of S generating the cone $\sigma \in \Sigma''$.

- StanleyFiltration (I)

  Returns a list with a Stanley filtration of the monomial ideal $I \subset S$. The elements of the list are represented in the same way as for the output of StanleyDecomposition.

- MonomialIdealsFixedHilbertPolynomial (S, P, A, B)

  Returns the set of monomial ideals in the multigraded polynomial ring $S = \mathbb{Q}[y_1, ..., y_r]$ with multigraded Hilbert polynomial $P \in \mathbb{Q}[t_1, ..., t_a]$, where $A \in \mathbb{Z}^{d\times r}$ is the presentation matrix of the Chow group of a smooth toric variety Y and B is the irrelevant ideal of Y.

Example 12.3 *Consider the ideal*

$$I = \langle y_1y_2, y_0y_3\rangle \subset S = \mathbb{C}[y_0, ..., y_3]$$

```
load "stanleyfiltration.m2";

S=QQ[y_0..y_3];

I=ideal(y_1*y_2,y_0*y_3);

StanleyFiltration(I)
```

`{{1, {y_0,y_1}}, {y1, {y_0,y_2}}, {y0,{y_1,y_3}}, {y0*y1, {y_2,y_3}}}`

This corresponds to the Stanley decomposition

$$
\begin{aligned}
S/I &= 1 \cdot \mathbb{C}\left[y_2, y_3\right] \oplus y_1 \cdot \mathbb{C}\left[y_1, y_3\right] \oplus y_0 \cdot \mathbb{C}\left[y_0, y_2\right] \oplus y_0 y_1 \cdot \mathbb{C}\left[y_0, y_1\right] \\
I &= \langle y_0, y_1\rangle \cap \langle y_0, y_2\rangle \cap \langle y_1, y_3\rangle \cap \langle y_2, y_3\rangle
\end{aligned}
$$

and to the Stanley filtration given by the Stanley decompositions

$$
\begin{aligned}
S/\left\langle y_1, y_0\right\rangle &= 1 \cdot \mathbb{C}\left[y_2, y_3\right] \\
S/\left\langle y_1 y_2, y_0\right\rangle &= 1 \cdot \mathbb{C}\left[y_2, y_3\right] \oplus y_1 \cdot \mathbb{C}\left[y_1, y_3\right] \\
S/\left\langle y_1 y_2, y_0 y_3, y_0 y_1\right\rangle &= 1 \cdot \mathbb{C}\left[y_2, y_3\right] \oplus y_1 \cdot \mathbb{C}\left[y_1, y_3\right] \oplus y_0 \cdot \mathbb{C}\left[y_0, y_2\right] \\
S/\left\langle y_1 y_2, y_0 y_3\right\rangle &= 1 \cdot \mathbb{C}\left[y_2, y_3\right] \oplus y_1 \cdot \mathbb{C}\left[y_1, y_3\right] \oplus y_0 \cdot \mathbb{C}\left[y_0, y_2\right] \oplus y_0 y_1 \cdot \mathbb{C}\left[y_0, y_1\right]
\end{aligned}
$$

12.4 tropicalmirror

In the Maple package tropicalmirror we provide an implementation of the tropical mirror construction given in the Sections 9 and 8. It also contains an implementation of the algorithms from Section 6 computing the Gröbner and Bergman fan.

In addition to standard Maple packages, tropicalmirror assumes the convex package for convex geometry [Franz, 2006] to be present. For local Gröbner computations tropicalmirror allows to call:

- Macaulay2 with mora.m2.
- Macaulay2 with Lazard ordering.
- Singular with built in monomial orderings.
- Singular with Lazard ordering.

The weight orderings can be represented in Macaulay2 and Singular as Wp, wp or by a matrix ordering. tropicalmirror assumes that the following variables of type string are present:

- `runM2` with the command running Macaulay2 in the shell.
- `runSingular` with the command running Singular in the shell.

- `stdSystem` with value `M2` or `Singular` selecting the computer algebra system for Gröbner calculations.

- `pathConvex` with the path to the convex package.

Let $N = \mathbb{Z}^n$, $P \subset N_{\mathbb{R}}{=}N \otimes \mathbb{R}$ be a Fano polytope, Σ the fan over the faces of P and $Y = X(\Sigma)$ the corresponding toric Fano variety of dimension n as defined in Section 7.2. The polytope P is represented as type polytope and the fan Σ as type fan in the convex package. Choosing a numbering of rays of Σ, let A be a Maple type matrix presenting the Chow group of Y via

$$0 \rightarrow \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^r \rightarrow A_{n-1}(Y) \rightarrow 0$$

as given in Section 1.3.4. Let v be a list of names for the variables corresponding to the rows of A formed by the minimal lattice generators of the rays of Σ. Denote the prime T-Weil divisor of Y corresponding to the j-th row of A by D_j.

The package tropicalmirror provides the following functions. They are organized in a way to avoid multiple computations of the same result.

- FanOverFaces (P)

  Returns the fan over the faces of the polytope P containing 0 as defined in Section 7.2.

- RandomPolynomial(A, v, a, c)

  Let a be an element of $\mathbb{Z}^r$ corresponding the Weil divisor $D = \sum_{r\in\Sigma(1)} a_j D_j$ representing the class $[D] \in A_{n-1}(Y)$. The function RandomPolynomial returns a Maple type polynomial $f \in S_{[D]}$ in the variables given by the Maple type list v with coefficients in $\{1, ..., c-1\}$ such that all monomials in $S_{[D]}$ appear in the polynomial. The Cox polynomial f is obtained as explained in Section 1.3.8 and corresponds to a generic linear combination of the lattice points of

  $$\Delta_D = \{m \in M_{\mathbb{R}} \mid \langle m, \hat{r}\rangle \geq -a_r \forall r \in \Sigma(1)\}$$

  with $M = \mathrm{Hom}(N, \mathbb{Z})$, which form a T-invariant basis of the space of global sections

  $$H^0(Y, \mathcal{O}_Y(D)) \cong \bigoplus_{m\in\Delta_D\cap M} \mathbb{C}x^m$$

  of the reflexive sheaf $\mathcal{O}_Y(D)$ as explained in Section 1.3.4.

- ReduceGenerators (v, t, gI)

  Let $gI = [f_1, ..., f_r]$ be a list of Maple type polynomials representing Cox homogeneous elements in $\mathbb{C}[t] \otimes_{\mathbb{C}} S$ such that for each polynomial f_j the degree 0 part with respect to the t-degree is a monomial m_j in S. The function ReduceGenerators removes all terms of $f_j - m_j$ which are divisible by some m_i. Up to first order this amounts to Gröbner reduction of gI.

- AssociatedFirstOrderDegeneration (v, t, gI)

  Deletes all terms of t-degree bigger than 1 from the polynomials $f_j \in \mathbb{C}[t] \otimes_{\mathbb{C}} S$ in the list $gI = [f_1, ..., f_r]$.

- SpecialFiberGroebnerCone (A, v, t, gI)

  Suppose $\mathfrak{X} \subset \mathbb{A}^1 \times Y$ is a flat family of Calabi-Yau varieties of dimension d with monomial special fiber, given by the ideal $I \subset \mathbb{C}[t] \otimes_{\mathbb{C}} S$ generated by the Cox homogeneous elements f_j of the list $gI = [f_1, ..., f_r]$. Assume that the monomials of f_j of t-degree 0 are minimal generators of the monomial ideal I_0 of the special fiber of $\mathfrak{X}$. The function SpecialFiberGroebnerCone returns the special fiber Gröbner cone $C_{I_0}(I)$ as defined in Section 9.6. It is represented by the convex type cone.

- GroebnerFan (Σ, A, v, t, s, gI)

  Computes the ideal $J \subset \mathbb{C}[t, s] \otimes_{\mathbb{C}} S$ of the projective closure $\overline{\mathfrak{X}} \subset \mathbb{P}^1 \times Y$ of the flat family $\mathfrak{X} \subset \mathbb{A}^1 \times Y$ given by the ideal I generated by the elements $f_j \in \mathbb{C}[t] \otimes_{\mathbb{C}} S$ of the list $gI = [f_1, ..., f_r]$. Returns the Gröbner fan of J as a subfan of $\mathbb{R} \oplus N_{\mathbb{R}}$, computed as explained in Section 6.3. It is represented by the convex package type fan.

- BergmanFan $(\Sigma, A, v, t, s, gI, GF)$

  Computes the Bergman subfan of the Gröbner fan GF as explained in Section 6, where GF is the result returned by GroebnerFan (v, t, s, A, gI). The result is represented by the convex package type fan.

- AssociatedAnticanonicalSectionsPolytope (C)

  Intersects the special fiber Gröbner cone $C \subset \mathbb{R} \oplus N_{\mathbb{R}}$ with the hyperplane $\{w_t = 1\}$ and returns the resulting polytope in $N_{\mathbb{R}}$.

- AssociatedFanoPolytope (C)

  Computes $\nabla = C \cap \{w_t = 1\} \subset N_{\mathbb{R}}$ and returns the polytope $\nabla^* \subset M_{\mathbb{R}}$.

- FacePoset (∇)

  Returns the complex of faces of a polytope $\nabla \subset N_{\mathbb{R}}$, represented as a list of lists $L = [L_{-1}, L_0, ..., L_n]$. The list L_j contains the faces of ∇ of dimension j and each face is represented by the convex package type face of polyhedron.

- VertexRepresentation (B)

  Given a complex B represented as a list of lists, where each face is of typ face of polyhedron, returns a list of lists, where each face is represented as a list of its vertices.

- ChowGroup (A)

  Returns a group isomorphic to the Chow group $A_{n-1}(Y)$ of $Y = X(\Sigma)$, given as the cokernel of a diagonal matrix A' of the same dimensions as A.

- ChowGroupAction (A)

  Computes isomorphisms $W \in \mathrm{GL}(n, \mathbb{Z})$ and $U = (u_{ij}) \in \mathrm{GL}(r, \mathbb{Z})$

$$\begin{array}{cccccccc} 0 \rightarrow & \mathbb{Z}^n & \xrightarrow{A} & \mathbb{Z}^r & \rightarrow & A_{n-1}(Y) & \rightarrow 0 \\ & \downarrow W & & \downarrow U & & \downarrow & \\ 0 \rightarrow & \mathbb{Z}^n & \xrightarrow{A'} & \mathbb{Z}^r & \rightarrow & H & \rightarrow 0 \end{array}$$

  such that A' is a matrix with non-zero entries only on the diagonal. As explained in Section 1.3.9, the group

$$G(\Sigma) = \mathrm{Hom}_{\mathbb{Z}}(A_{n-1}(Y), \mathbb{C}^*)$$

  acts on the affine Cox space $\mathrm{Hom}_{sg}(\mathrm{WDiv}_T(Y), \mathbb{C}) \cong \mathbb{C}^r$ by

$$\begin{array}{l} G(\Sigma) \times \mathrm{Hom}_{sg}(\mathrm{WDiv}_T(Y), \mathbb{C}) \rightarrow \mathrm{Hom}_{sg}(\mathrm{WDiv}_T(Y), \mathbb{C}) \\ \qquad (g, a) \mapsto \begin{array}{cccc} ga: & \mathrm{WDiv}_T(Y) & \rightarrow & \mathbb{C} \\ & D_r & \mapsto & g([D_r])\, a(D_r) \end{array} \end{array}$$

  hence

$$G(\Sigma)' = \mathrm{Hom}_{\mathbb{Z}}(H, \mathbb{C}^*)$$

  acts by

$$\begin{array}{c} G(\Sigma)' \times \mathbb{C}^r \rightarrow \mathbb{C}^r \\ ((t_j), (a_j)) \mapsto \left(\prod_{i=1}^r t_i^{u_{ij}} a_j\right)_{j=1,\ldots,r} \end{array}$$

Defining

$$T = \left(\prod_{i=1}^{r} t_i^{u_{ij}}\right)_{j=1,\dots,r}$$

the function ChowGroupAction returns the list $[A', T]$.

- IrrelevantIdeal (Σ, A, v)

  Returns the irrelevant ideal $B(\Sigma) \subset S$ as defined in Section 1.3.9. The variables in the list v corresponds to the rows of A.

- BergmanSubfanOfGroebnerCone (Σ, A, v, t, gI, C)

  Computes the Bergman subfan of the fan of faces of C as defined in Section 9.8 for the ideal I generated by the elements of the list gI.

- BergmanSubcomplexOfSectionsPolytope $(\Sigma, A, v, t, gI, posetNabla)$

  Returns the Bergman subcomplex $B(I)$ as defined in Section 9.8, i.e., the intersection of the output of BergmanSubfanOfGroebnerCone with the hyperplane $\{w_t = 1\}$. The result is a subcomplex of the complex of faces $posetNabla$ = FacePoset (∇) of $\nabla = C \cap \{w_t = 1\} \subset N_{\mathbb{R}}$ and is represented as a list of faces of ∇ of the form $[..., B_j, ...]$ where B_j is a list of faces of dimension j. For practical reasons it is useful to fix a numbering of the faces in each dimension, so we represent B_j as a list. Each face is represented by the convex package type face of a polyhedron.

- SpecialFiberIdeal $(AMirror, z, posetNabla, B)$

  Suppose z is a list of names for the variables of the Cox ring of $Y^{\circ} = X(\mathrm{NF}(\nabla))$ corresponding the rays of the normal fan of ∇, which are numbered by the rows in the matrix $AMirror$. Suppose $posetNabla$ = FacePoset (∇) and B is the Bergman subcomplex of $posetNabla$ as given by the function BergmanSubcomplexOfSectionsPolytope. Then SpecialFiberIdeal returns the ideal

  $$I_0^{\circ} = \bigcap_{F \in B_d} \langle z_{G^*} \mid G \text{ a facet of } \nabla \text{ with } F \subset G \rangle$$

  which gives the subvariety $X_0^{\circ} \subset Y^{\circ}$ as defined in Section 9.11. The ideal I_0° is represented as a list of minimal generators.

- ToricStrataDecomposition $(posetDelta, I_0)$

  Returns the subcomplex $\mathrm{Strata}_{\Delta}(I_0) \subset \mathrm{Poset}(\Delta)$ as defined in Section 9.3. It is represented as a list of lists and each face is of the convex package type face of polyhedron.

- ComplexOfInitialIdeals (v, t, gI, C)

  Gives the complex of initial ideals $\mathrm{in}_F(I)$ for the faces F of B. It is represented as a list of lists in the same way as the Bergman subcomplex B. Each ideal $\mathrm{in}_F(I)$ is represented by a list containing a standard basis with respect to a monomial ordering in the interior of C.

- DualComplex (A, v, t, inI, B)

  Computes the dual complex $\mathrm{dual}(B(I)) \subset \mathrm{Poset}(\nabla^*) \subset M_{\mathbb{R}}$ as given by the map dual defined in Section 9.7. Here B denotes the Bergman subcomplex $B(I)$ as returned by BergmanSubcomplexOfSectionsPolytope and inI is the complex of initial ideals as returned by ComplexOfInitialIdeals. The complex $\mathrm{dual}(B(I))$ is represented as a list of lists in the same way as the Bergman subcomplex B. The faces of $\mathrm{dual}(B(I))$ are represented by the convex package type face of a polyhedron.

- CombinatorialDualization $(posetNabla, B)$

  If B is a subcomplex of the complex $posetNabla =$ FacePoset (∇), then the poset of dual faces $F^* \subset \nabla^*$ is returned. It is represented as a list of lists in the same way as B and the faces are represented by the convex package type face of a polyhedron.

  Suppose B is the Bergman subcomplex $B(I)$ of FacePoset (∇) as returned by BergmanSubcomplexOfSectionsPolytope, then by Proposition 9.26 we have

  CombinatorialDualization $(posetNabla, B) =$ DualComplex (A, v, t, inI, B)

  where inI is the complex of initial ideals as returned by ComplexOfInitialIdeals.

- EqualityofFaceComplexes (B_1, B_2)

  Given two complexes B_1 and B_2 represented as a list of lists of faces of the same polyhedron, returns $true$ if $B_1 = B_2$, i.e., if the lists in each dimension agree up to permutation, otherwise returns $false$.

- MirrorComplex (P, A, v, t, inI, B)

  Suppose B denotes the Bergman subcomplex $B(I)$ of $posetNabla =$ FacePoset (∇) as returned by BergmanSubcomplexOfSectionsPolytope and inI is the complex of initial ideals as given by ComplexOfInitialIdeals. Then the function MirrorComplex returns the complex $\mu(B(I)) \subset$ FacePoset (Δ) where $\Delta = P^*$. It is represented as a list of lists in the same way as B.

The faces of $\mu(B(I))$ are represented by the convex package type face of a polyhedron.

- LimitComplex $(Iirr, A, v, t, B)$

  If B denotes the Bergman subcomplex as returned by the function BergmanSubcomplexOfSectionsPolytope and $Iirr$ the irrelevant ideal of $X(\Sigma)$ as returned by the function IrrelevantIdeal, then the limit complex $\lim(B)$ is computed as described in Section 9.10. The limit complex is represented as a list of lists in the same way as B. The stratum corresponding to the face F is represented by the ideal I_F as defined in Section 9.10 and I_F is given by a list with minimal generators.

Example 12.4 *Consider the monomial degeneration $\mathfrak{X}$ of the complete intersection of two general quadrics in $\mathbb{P}^3$ defined by the ideal $I \subset \mathbb{C}[t] \otimes \mathbb{C}[x_0, ..., x_3]$ as considered in the examples in Section 8. To give a computation of the ideal I_F of $\lim(F)$ in a non-simplicial setting, we consider a face $F \in \mu(B(I)) \cong \mathrm{Strata}_\Delta(I_0)$ of the Bergman complex of the mirror. We have*

$$\Delta = \mathrm{convexhull}\{(3,-1,-1),(-1,3,-1),(-1,-1,3),(-1,-1,-1)\}$$

and

$$\mu(B(I)) = \left(\begin{array}{c} \{\{(-1,3,-1)\},\{(-1,-1,-1)\},\{(-1,-1,3)\},\{(3,-1,-1)\}\} \\ \left\{\begin{array}{c} \{(-1,3,-1),(-1,-1,-1)\},\{(-1,-1,-1),(3,-1,-1)\}, \\ \{(-1,3,-1),(-1,-1,3)\},\{(3,-1,-1),(-1,-1,3)\} \end{array}\right\} \end{array}\right)$$

omitting empty dimensions. The rays of Σ° are the rows of the presentation matrix

$$A^\circ = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 0 & -1 \\ 2 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

of $A_2(Y^\circ)$ and we denote the corresponding variables of the Cox ring S° by $y_1, ..., y_8$. The irrelevant ideal of Y° is

$$\begin{aligned} B(\Sigma^\circ) = {} & \langle y_1, y_7\rangle \cap \langle y_3, y_5\rangle \cap \langle y_2, y_8\rangle \cap \langle y_4, y_6\rangle \cap \langle y_2, y_4\rangle \cap \langle y_5, y_7\rangle \\ & \cap \langle y_1, y_3, y_8\rangle \cap \langle y_1, y_3, y_6\rangle \cap \langle y_1, y_6, y_8\rangle \cap \langle y_3, y_6, y_8\rangle \end{aligned}$$

Let F be the face

$$F = \text{convexhull}\{(-1,3,-1),(-1,-1,3)\}$$
$$w = (-1,3,-1) + (-1,-1,3) = (-2,2,2) \in \text{int}(F)$$

and $a(t) \in (K^*)^3$ with $\text{val}(a(t)) = w$ and $c(t) = c_J \cdot t^J + \text{hot}$ a Cox arc representing $a(t)$, i.e., $J^t + \ker(A^t)$ is the space of solutions of the linear system of equations $A^t J^t = w^t$. The intersection with this affine space with the positive orthant has the minimal 0-dimensional strata

$$(2,0,0,0,0,0,0,2),(0,0,0,2,2,0,0,0),(0,0,2,0,4,0,0,2),(2,0,0,4,0,2,0,0)$$

corresponding to the ideals

$$\langle y_1, y_8\rangle, \langle y_4, y_5\rangle, \langle y_3, y_5, y_8\rangle, \langle y_1, y_4, y_6\rangle \subset S^\circ$$

As $B(\Sigma^\circ) \subset \langle y_3, y_5, y_8\rangle$ and $B(\Sigma^\circ) \subset \langle y_1, y_4, y_6\rangle$ the limit face $\lim(F)$ is given by any of the ideals

$$\langle y_1, y_8\rangle, \langle y_4, y_5\rangle$$

hence also by

$$\langle y_1, y_4, y_5, y_8\rangle$$

the ideal of all facets of ∇ containing $\lim(F)$. Figure 12.1 shows the limit face $\lim(F) \subset B(I) \subset \nabla$ and the numbering of the facets of ∇ by the variables of S°.

- DualLimitComplex (P, B)

  Given the Bergman subcomplex $B \subset \nabla$, this function computes for all faces F the faces H of P such that $\dim(F \cap H) = \dim(F)$ and among those returns the set of minimal faces with respect to inclusion. The output forms the dual limit complex of B and is represented as a list of lists compatible to B.

- LatticePoints (dB)

  Returns the set of lattice points of the complex $dB \subset P \subset \mathbb{Z}^n \otimes \mathbb{R}$, which is represented as a list of lists and each face F of dB is a face of the polyhedron P.

- DeformationsFromCombinatorialData (A, v, dB)

  Suppose $dB = \text{dual}(B(I))$ as given by the function DualComplex. The function DeformationsFromCombinatorialData returns the complex of first order deformations of X_0, represented as a list of lists in the same way as $\mathrm{dual}\left(B\left(I\right)\right)$. Each face is a set of degree 0 Cox Laurent monomials corresponding to the lattice points of the corresponding face of $\mathrm{dual}\left(B\left(I\right)\right)$.

- FirstOrderDegenerationFromCombinatorialData $(I0, defs, c)$

  Suppose I_0 is the ideal of the special fiber of $\mathfrak{X}$ given as a list $I0$ of monomial generators and $defs$ is the complex of first order deformations as returned by the complex DeformationsFromCombinatorialData then the list

$$\left[m + \sum_{\alpha \text{ in a face of } defs} t \cdot c_\alpha \cdot \alpha\left(m\right) \mid m \in I0\right]$$

  is returned.

- ExtendFirstOrderPfaffian (gI)

  If the ideal $I^1 \subset \mathbb{C}\left[t\right] / \langle t^2 \rangle \times S$ generated by gI is Pfaffian with syzygy matrix A, a list with the Pfaffians of A in $\mathbb{C}\left[t\right] \times S$ is returned to extend the family defined by I^1 as explained in Section 10.2.

- ModuliDimStanleyReisner (dB)

  The function ModuliDimStanleyReisner computes the number of lattice points of $dB = \mathrm{dual}\left(B\left(I\right)\right)$ as given by DualComplex and the number r_0 of roots of Y among them, and returns

$$\left|\mathrm{dual}\left(B\left(I\right)\right) \cap M\right| - \dim Y - r_0$$

  If $Y = \mathbb{P}\left(\Delta\right)$ where Δ is a Veronese polytope of $\mathbb{P}^n$ this number is $h^{1,\dim(X)-1}\left(X\right)$ of the general fiber X of $\mathfrak{X}$, as discussed in Section 11.3.

- StringyEFunctionOfGorensteinCone (C)

  Given a Gorenstein cone C returns

$$(uv)^{-1} \sum_{C_1 \subset C} (-u)^{\dim(C_1)} \tilde{S}\left(C_1, u^{-1}v\right) \tilde{S}\left(C_1^\vee, uv\right)$$

Example 12.5 *Let $\mathfrak{X} \subset \mathbb{P}\left(\Delta\right) \times \mathrm{Spec}\, \mathbb{C}\left[\left[t\right]\right]$ be the monomial degeneration of an elliptic curve given as the complete intersection of two general quadrics*

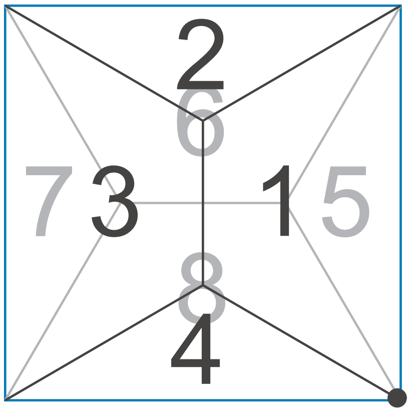


Figure 12.1: $\lim(F) \subset \nabla$ for $F \in \mathrm{Strata}(X_0)$ as in Example 12.4, and the numbering of the facets of ∇ by the variables of the Cox ring for the complete intersection of two general quadrics in $\mathbb{P}^3$

in $\mathbb{P}^3$*, as considered in the examples in Section 8. So let* $\mathfrak{X}$ *be given by the ideal*

$$I = \langle t \cdot g_1 + x_1 x_2,\ t \cdot g_2 + x_0 x_3 \rangle \subset \mathbb{C}[t] \otimes S$$

where $g_1, g_2 \in S = \mathbb{C}[x_0, ..., x_3]$ *are general quadrics reduced with respect to* $I_0 = \langle x_1 x_2, x_0 x_3 \rangle$*.*

```
runM2:="M2":
runSingular:="Singular":
stdSystem:="M2":
pathConvex:="/usr/local/convex":
read("tropicalmirror"):
v:=[x0,x1,x2,x3]:
A:=matrix([[-1,-1,-1],[1,0,0],[0,1,0],[0,0,1]]):
P:=convexhull(op(convert(A,listlist))):
```

$$P := polytope(3, 3, 4, 4)$$

```
Sigma:=FanOverFaces(P):
```

$$Sigma := FAN(3, 3, 0, 4, [0, 0, 4])$$

```
g1:=RandomPolynomial(A,v,matrix([[2],[0],[0],[0]]),13):
g2:=RandomPolynomial(A,v,matrix([[2],[0],[0],[0]]),13):
gI:=[t*g1+x1*x2,t*g2+x0*x3]:
```

```
gI:=ReduceGenerators(v,t,gI);
```

$gI := [2tx0^2 + tx1x0 + 8tx2x0 + tx1^2 + tx3x1 + 9tx3^2 + 12tx2x3 + 8tx2^2 + x1x2,$
$15tx0^2 + 8tx1x0 + 11tx2x0 + 16tx1^2 + 16tx3x1 + 14tx2x3 + 15tx2^2 + 7tx3^2 + x0x3]$

```
gI1:=AssociatedFirstOrderDegeneration(v,t,gI):
C:=SpecialFiberGroebnerCone(A,v,t,gI);
```

$$C := cone(4, 4, 0, 8, 8)$$

```
rays(C);
```

$$[[1,1,0,1],[1,0,1,1],[1,0,-1,-1],[1,-1,0,-1],$$
$$[1,1,0,0],[1,0,1,0],[1,0,0,1],[1,-1,-1,-1]]$$

```
Nabla:=AssociatedAnticanonicalSectionsPolytope(C);
```

$$\nabla := polytope(3, 3, 8, 8)$$

```
vertices(Nabla);
```

$[[1,0,1],[0,1,1],[0,-1,-1],[-1,0,-1],[0,0,1],[-1,-1,-1],[0,1,0],[1,0,0]]$

```
PMirror:=AssociatedFanoPolytope(C):
vertices(PMirror);
```

$[[-1,-1,0],[-1,-1,2],[0,2,-1],[2,0,-1],[0,0,-1],[0,0,1],[1,-1,0],[-1,1,0]]$

```
SigmaMirror:=FanOverFaces(PMirror):
# By fixing the presentation matrix AMirror of the Chow group of
# YMirror=X(SigmaMirror), we choose a numbering of the Cox variables
# of YMirror compatible with Example 12.4.
AMirror:=matrix([[0,2,-1], [0,0,-1], [2,0,-1], [0,0,1], [-1,1,0],
[-1,-1,0], [1,-1,0], [-1,-1,2]]):
ChowGroup(AMirror);
```

$$\left[\left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{array}\right], "ZZ/2+ZZ\hat{\ }5"\right]$$

```
ChowGroupAction(AMirror);
```

$$\left[\left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \end{array}\right], [s4, \frac{s5s8}{s3s6}, \frac{s6s8}{s3s4}, s5, \frac{s6s7s8}{s3^2s4^2}, s6, s7, s8]\right]$$

```
IirrMirror:=IrrelevantIdeal(SigmaMirror,A,v);
```

$IirrMirror := [y3y4y7y8, y1y4y5y8, y2y3y6y7, y1y2y5y6, y4y5y6y7y8, y2y5y6y7y8, y1y2y3y4y7, y1y2y3y4y5]$

```
posetNabla:=FacePoset(Nabla);
```

$posetNabla :=$
$[[face(0,8)],$
$[face(1,4), face(1,4), face(1,4), face(1,4), face(1,3), face(1,3), face(1,3), face(1,3)],$
$[face(2,2), face(2,2), face(2,2), face(2,2), face(2,2), face(2,2), face(2,2), face(2,2), face(2,2), face(2,2), face(2,2), face(2,2), face(2,2), face(2,2)],$
$[face(4,1), face(4,1), face(4,1), face(4,1), face(3,1), face(3,1), face(3,1), face(3,1)],$
$[face(8,0)]]$

```
# Numbers of faces of dimensions -1,0,1,2,3 are 1,8,14,8,1
VertexRepresentation(posetNabla);
```

$[[[]],$
$[[[1,0,1]], [[0,1,1]], [[0,-1,-1]], [[-1,0,-1]], [[0,0,1]], [[-1,-1,-1]], [[0,1,0]], [[1,0,0]]],$
$[[[0,1,0],[1,0,0]], [[0,0,1],[-1,-1,-1]], [[0,1,1],[-1,0,-1]], [[0,-1,-1],[-1,0,-1]], [[0,1,1],[0,0,1]], [[1,0,1],[0,0,1]], [[-1,0,-1],[-1,-1,-1]], [[0,-1,-1],[-1,-1,-1]], [[-1,0,-1],[0,1,0]], [[0,1,1],[0,1,0]], [[0,-1,-1],[1,0,0]], [[1,0,1],[1,0,0]], [[1,0,1],[0,1,1]], [[1,0,1],[0,-1,-1]]],$
$[[[1,0,1],[0,1,1],[0,1,0],[1,0,0]], [[0,-1,-1],[-1,0,-1],[0,1,0],[1,0,0]], [[1,0,1],[0,-1,-1],[0,0,1],[-1,-1,-1]], [[0,1,1],[-1,0,-1],[0,0,1],[-1,-1,-1]], [[1,0,1],[0,1,1],[0,0,1]], [[0,-1,-1],[-1,0,-1],[-1,-1,-1]], [[0,1,1],[-1,0,-1],[0,1,0]], [[1,0,1],[0,-1,-1],[1,0,0]]],$
$[[[1,0,1],[0,1,1],[0,-1,-1],[-1,0,-1],[0,0,1],[-1,-1,-1],[0,1,0],[1,0,0]]]]$

```
BF:=BergmanSubfanOfGroebnerCone(Sigma,A,v,t,gI,C);
```

$BF := [[cone(4,0,0,0,0)],$
$[cone(4,1,0,1,1), cone(4,1,0,1,1), cone(4,1,0,1,1), cone(4,1,0,1,1)],$
$[cone(4,2,0,2,2), cone(4,2,0,2,2), cone(4,2,0,2,2), cone(4,2,0,2,2)],$
$[],$
$[]]$

```
B:=BergmanSubcomplexOfSectionsPolytope(Sigma,A,v,t,gI,posetNabla):
```

VertexRepresentation(B);

$$[[],$$
$$[[[1,0,1]],[[0,1,1]],[[0,-1,-1]],[[-1,0,-1]]],$$
$$[[[1,0,1],[0,-1,-1]],[[1,0,1],[0,1,1]],[[0,-1,-1],[-1,0,-1]],[[0,1,1],[-1,0,-1]]],$$
$$[],$$
$$[]]$$

 # Numbers of faces of dimensions -1,0,1,2,3 are 0,4,4,0,0

y:=[y1,y2,y3,y4,y5,y6,y7,y8];

I0Mirror:=SpecialFiberIdeal(AMirror,y,posetNabla,B);

$$I0mirror := [y5y6y7y8, y1y6y7y8, y2y5y7y8, y1y2y7y8, y3y5y6y8, y1y3y6y8, y2y3y5y8, y1y2y3y8, y4y5y6y7, y1y4y6y7, y2y4y5y7, y1y2y4y7, y3y4y5y6, y1y3y4y6, y2y3y4y5, y1y2y3y4]$$

inC:=ComplexOfInitialIdeals(v,t,gI,C):

dualB:=DualComplex(A,v,t,inI,B):

 # Numbers of faces of dimensions -1,0,1,2,3 are 0,0,4,4,0

VertexRepresentation(dualB);

$$[[],$$
$$[],$$
$$[[[0,2,-1],[-1,1,0]],[[-1,-1,0],[0,0,-1]],[[0,0,1],[-1,-1,2]],[[1,-1,0],[2,0,-1]]],$$
$$[[[-1,0,0],[0,2,-1],[-1,-1,0],[0,0,-1],[-1,1,0],[0,1,-1]],$$
$$[[0,-1,0],[1,-1,0],[-1,-1,0],[2,0,-1],[0,0,-1],[1,0,-1]],$$
$$[[0,0,1],[0,2,-1],[0,1,0],[-1,1,0],[-1,0,1],[-1,-1,2]],$$
$$[[0,0,1],[1,-1,0],[2,0,-1],[0,-1,1],[1,0,0],[-1,-1,2]]],$$
$$[]]$$

Covering(dualB);

$$[[],$$
$$[[[[-1,1,0]],[[0,2,-1]]],\ [[[0,0,-1]],[[-1,-1,0]]],\ [[[-1,-1,2]],[[0,0,1]]],$$
$$[[[2,0,-1]],[[1,-1,0]]]],$$
$$[[[[0,2,-1],[0,0,-1]],[[-1,-1,0],[-1,1,0]]],[[[2,0,-1],[0,0,-1]],[[1,-1,0],[-1,-1,0]]],$$
$$[[[-1,1,0],[-1,-1,2]],[[0,0,1],[0,2,-1]]],[[[1,-1,0],[-1,-1,2]],[[0,0,1],[2,0,-1]]]],$$
$$[],$$
$$[]]$$

```
cdualB:=CombinatorialDualization(posetNabla,B):
EqualityofFaceComplexes(dualB,cdual);
```

$$true$$

```
limB:=LimitComplex(Sigma,A,v,t,B);
```

$$\begin{array}{l} limB := [[], \\ [[x3, x1, x0], [x1, x2, x3], [x0, x2, x1], [x3, x2, x0]], \\ [[x1, x3], [x2, x3], [x0, x1], [x0, x2]], \\ [], \\ []] \end{array}$$

```
BMirror:=MirrorComplex(P,A,v,t,inI,B):
# Numbers of faces of dimensions -1,0,1,2,3 are 0,4,4,0,0
VertexRepresentation(BMirror);
```

$$\begin{array}{l} [[], \\ [[[-1, 3, -1]], [[-1, -1, -1]], [[-1, -1, 3]], [[3, -1, -1]]], \\ [[[-1, -1, -1], [-1, 3, -1]], [[-1, -1, -1], [3, -1, -1]], [[-1, 3, -1], [-1, -1, 3]], \\ [[3, -1, -1], [-1, -1, 3]]], \\ [], \\ []] \end{array}$$

```
cdualBMirror:=CombinatorialDualization(posetDelta,BMirror):
duallimB:=DualLimitComplex(P,B);
# Numbers of faces of dimensions -1,0,1,2,3 are 0,0,4,4,0
EqualityofFaceComplexes(duallimB,cdualBMirror);
```

$$true$$

```
Covering(duallimB);
```

$$\begin{array}{l} [[], \\ [[[[1, 0, 0]], [[0, 0, 1]]], [[[0, 0, 1]], [[0, 1, 0]]], [[[1, 0, 0]], [[-1, -1, -1]]], [[[-1, -1, -1]], [[0, 1, 0]]]], \\ [[[[1, 0, 0]], [[-1, -1, -1], [0, 0, 1]]], [[[0, 0, 1]], [[0, 1, 0], [1, 0, 0]]], \\ [[[-1, -1, -1]], [[0, 1, 0], [1, 0, 0]]], [[[0, 1, 0]], [[-1, -1, -1], [0, 0, 1]]]], \\ [], \\ []] \end{array}$$

```
LatticePoints(duallimB);
```

$$[[],$$
$$[],$$
$$[[[0,0,1],[1,0,0]],[[0,1,0],[0,0,1]],[[-1,-1,-1],[1,0,0]],[[0,1,0],[-1,-1,-1]]],$$
$$[[[-1,-1,-1],[0,0,1],[1,0,0]],[[0,1,0],[0,0,1],[1,0,0]],$$
$$[[0,1,0],[-1,-1,-1],[1,0,0]],[[0,1,0],[-1,-1,-1],[0,0,1]]],$$
$$[]]$$

```
DefsMirror:=DeformationsFromCombinatorialData(AMirror,y,duallimB);
```

$$DefsMirror := [[],$$
$$[],$$
$$[[\tfrac{y4y8^2}{y1y2y3},\tfrac{y3^2y7}{y5y6y8}],[\tfrac{y1^2y5}{y6y7y8},\tfrac{y4y8^2}{y1y2y3}],[\tfrac{y2y6^2}{y1y3y4},\tfrac{y3^2y7}{y5y6y8}],[\tfrac{y1^2y5}{y6y7y8},\tfrac{y2y6^2}{y1y3y4}]],$$
$$[[\tfrac{y2y6^2}{y1y3y4},\tfrac{y4y8^2}{y1y2y3},\tfrac{y3^2y7}{y5y6y8}],[\tfrac{y1^2y5}{y6y7y8},\tfrac{y4y8^2}{y1y2y3},\tfrac{y3^2y7}{y5y6y8}],$$
$$[\tfrac{y1^2y5}{y6y7y8},\tfrac{y2y6^2}{y1y3y4},\tfrac{y3^2y7}{y5y6y8}],[\tfrac{y1^2y5}{y6y7y8},\tfrac{y2y6^2}{y1y3y4},\tfrac{y4y8^2}{y1y2y3}]],$$
$$[]]$$

```
FirstOrderDegenerationFromCombinatorialData(IOMirror,DefsMirror):
IOmirrorCartier:=CartierMonomials(IOMirror,SigmaMirror,AMirror,y):
```

$$[y1y2y3y4, y5y6y7y8]$$

```
ts1:=ToricStrataDecomposition(posetNabla,IOmirrorCartier):
ts2:=ToricStrataDecomposition(posetNabla,IOmirror):
EqualityofFaceComplexes(ts1,ts2);
```

$$true$$

```
FirstOrderDegenerationFromCombinatorialData(IOmirrorCartier,DefsMirror):
```

$$\left[t \cdot y2^2y6^2 + t \cdot y4^2y8^2 + y1y2y3y4,\ t \cdot y1^2y5^2 + t \cdot y3^2y7^2 + y5y6y7y8\right]$$

13 Perspectives

13.1 Tropical computation of string cohomology

Stringy E-functions and tropical geometry share many technical concepts, for example formal arcs, which are used in motivic integration to prove key theorems on stringy E-functions, as indicated in Section 1.2. As explained in Section 11.4.3, Batyrev and Dais give in [Batyrev, Dais, 1996] a formula to compute the stringy E-function for a general anticanonical toric hypersurface. The stringy E-function of a hypersurface in $\mathbb{P}(\Delta)$ is computed in terms of combinatorial data of the reflexive polytope Δ. As explained in Section 11.4.4, Batyrev and Borisov represent in [Batyrev, Borisov, 1996-I] complete intersections in $Y = \mathbb{P}(\Delta)$ given by general sections $s_j \in H^0(Y, \mathcal{O}_Y(E_j))$ with $\sum_{j=1}^{c} E_j = -K_Y$ as the zero set of a section of

$$Z = \mathbb{P}(\mathcal{O}_Y(D_1) \oplus ... \oplus \mathcal{O}_Y(D_c)) \to Y$$

so reducing the computation of the stringy E-function to the case of a hypersurface and the bundle Z.

The tropical mirror construction, as given in Section 9, provides the additional data given by the complexes $B(I) \subset \text{Poset}(\nabla)$ and $\lim(B(I)) \subset \text{Poset}(\Delta)$. So it is natural to ask whether it is possible to give a direct formula for the stringy E-function of the general complete intersection Calabi-Yau in terms of this data, as indicated in Section 11.4.5.

Further evidence to expect that the stringy E-function $E_{st}(X)$ of X should be computable from the tropical data, is given the fact that the special fiber X_0 of $\mathfrak{X}$ is a union of toric varieties and, as noted in Section 11, the stringy E-function respects stratifications. See also Section 11.4.1 for an explicit formula for the stringy E-function of a toric variety.

So, in general one may ask if the stringy E-function of the generic element of the degeneration $\mathfrak{X}$ is computable in terms of the tropical data.

13.2 Hilbert schemes and moduli spaces

The multigraded Hilbert scheme described in Sections 6.6.1-6.6.8 for smooth toric varieties may be generalized to the non-smooth and further to the non-simplicial setup by using the ideas of Section 9.3. For reduced monomial ideals I_0 the saturation $(I_0 : B(\Sigma)^\infty)$ is generalized by the ideal I_0^Σ in the non-simplicial setup: Consider a toric variety $Y = X(\Sigma)$ given by the Fano polytope $P \subset N_\mathbb{R}$, $\Sigma = \Sigma(P)$, let $\Delta = P^* \subset M_\mathbb{R}$ and $S = \mathbb{C}[y_r \mid r \in \Sigma(1)]$ be the Cox ring of Y. Given a monomial ideal $I_0 \subset S$ we associate to I_0 the complex of strata $\mathrm{Strata}_{\Delta}(I_0)$ with faces of dimension s given by

$$\mathrm{Strata}_{\Delta}(I_0)_s = \left\{ F \mid \begin{array}{l} F \text{ a face of } \Delta \text{ of } \dim(F) = s \text{ with} \\ \mathrm{facets}_F(\Delta) \cap \mathrm{facets}_m(\Delta) \neq \emptyset \\ \text{for all monomial } m \in I_0 \end{array} \right\}$$

where

$$\begin{aligned} \mathrm{facets}_F(\Delta) &= \{G \mid G \text{ facet of } \Delta \text{ with } F \subset G\} \\ \mathrm{facets}_m(\Delta) &= \{G \mid G \text{ facet of } \Delta \text{ with } y_{G^*} \mid m\} \end{aligned}$$

To this complex we can associate the ideal

$$I_0^{\Sigma} = \left\langle \prod_{v \in J} y_v \mid J \subset \Sigma(1) \text{ with } \mathrm{supp}(B(I)) \subset \bigcup_{v \in J} F_v \right\rangle \subset S$$

Replacing I_0 by I_0^{Σ} also amounts to passing to a reduced ideal, so without modification this can only work for the local Hilbert scheme around a reduced point, but this is what is relevant for the tropical mirror construction.

Generalizing the multigraded Hilbert scheme, also the state polytope, as defined in Section 6.6.9, can be generalized to the non-simplicial setup.

Local moduli stacks may be computed by taking the quotient of the local Hilbert scheme around a given ideal by a (in general non-reductive) automorphism group, generalizing the ideas of Section 11.3. For geometric invariant theory in the non-reductive setting see [Doran, Kirwan, 2006].

Let $\Delta \subset M_{\mathbb{R}}$ be a reflexive polytope and consider an anticanonical toric hypersurface in $Y = \mathbb{P}(\Delta)$. Generalizing the ideas from Batyrev's Hodge formulas for toric hypersurfaces, as outlined in Section 11.2, the subset

$$\Xi_0 = \Delta \cap M - \{0\} - \bigcup_{\mathrm{codim}\, Q = 1} \mathrm{int}_M(Q)$$

of the set of lattice points of the polytope of sections Δ of $-K_Y$ plays the key role in the construction of the simplified complex moduli space of anticanonical hypersurfaces. By mirror symmetry this set is also used in the construction of the Kähler moduli space of the mirror. See, e.g., [Aspinwall, Greene, Morrison, 1993] and [Cox, Katz, 1999, Section 6] for details. Now consider the setup of the tropical mirror construction with degeneration $\mathfrak{X}$ given by I. Then the set Ξ_0 generalizes to the set of lattice points of the support of $\mathrm{dual}(B(I)) \subset \mathrm{Poset}(\nabla^*)$ which do not correspond to a trivial deformations of the special fiber. Note that for hypersurfaces $\nabla^* = \Delta$. So one may ask if the complex $\mathrm{dual}(B(I))$ may be useful for the construction of moduli spaces.

13.3 Integrally affine structures

Gross and Siebert use in [Gross, Siebert, 2003], [Gross, Siebert, 2006] and [Gross, Siebert, 2007] toric degenerations, integrally affine structures and the discrete Legendre transform to give a mirror construction.

They consider a degeneration $f : \mathfrak{X} \to S$ of Calabi-Yau varieties whose total space is a complex analytic space its base is a complex disc S and which has a special fiber whose normalization is a disjoint union of toric varieties. Outside a set of codimension 2 any point x in the total space is assumed to have a neighborhood U_x such that there is an affine toric variety Y_x, a regular function f_x given by a monomial and a commutative diagram

$$\begin{array}{lcl} U_x & \to & Y_x \\ \downarrow f\mid_{U_x} & & \downarrow f_x \\ S & \to & \mathbb{C} \end{array}$$

with open embeddings $U_x \to Y_x$ and $S \to \mathbb{C}$.

Given this data a manifold with an integral affine structure with singularities is constructed, i.e., a topological manifold such that outside a finite union of locally closed submanifolds of codimension at least 2 there are charts whose transition functions are integral affine transformations. Furthermore a polyhedral decomposition of this manifold is constructed. This process is reversible, i.e., from an integral affine manifold with singularities and a polyhedral decomposition one can construct a degeneration. Using the descrete Legendre transform one can obtain a mirror integral affine manifold with polyhedral decomposition from which one obtains the mirror degeneration.

So one may ask if it is possible to obtain from the polytopes Δ and ∇ and the embedded subcomplexes $B(I) \subset \text{Poset}(\nabla)$ and $\lim(B(I)) \subset \text{Poset}(\Delta)$ integral affine structures and how the tropical mirror construction relates to the construction by Gross and Siebert.

13.4 Torus fibrations

Consider the setup of the tropical mirror construction. So let Y be a toric Fano variety given by a Fano polytope $P \subset N_{\mathbb{R}}$ with Cox ring S and a monomial degeneration $\mathfrak{X}$ of Calabi-Yau varieties of dimension d which is given by the ideal $I \subset S \otimes \mathbb{C}[t]$.

The monomial special fiber X_0 has a degenerate torus fibration over the sphere $\text{Strata}(X_0) \cong \lim(B(I)) \subset \text{Poset}(\Delta)$ with $\Delta = P^*$: The strata of X_0 of dimension $s = 0, ..., d$ are complex tori $(\mathbb{C}^*)^s$ which contain $(S^1)^s$.

In the same way also the mirror special fiber X_0° has a degenerate torus fibration over $B(I)$. The dimensions of the tori and their base faces are related via the map lim by $s \leftrightarrow d - s$. This agrees with the large-small interchange of T-duality.

One may ask if the data provided by the spheres $B(I) \subset \text{Poset}(\nabla)$ and $\lim(B(I)) \subset \text{Poset}(\Delta)$ and the complexes $\text{dual}(B(I))$ and $\lim(B(I))^*$ of deformations in the initial ideals provide sufficient information to obtain a SYZ-fibration of the general fiber of $\mathfrak{X}$ as introduced in [Strominger, Yau, Zaslow, 1996]. Furthermore one may ask how these fibrations relate to mirror symmetry via T-duality.

13.5 Mirrors for further Stanley-Reisner Calabi-Yau degenerations

In Section 9.3 we connected the combinatorial representation of monomial ideals in the Cox ring of a toric variety Y in the special case of $Y = \mathbb{P}^n$ to the Stanley-Reisner setup:

Let $Y = \mathbb{P}(\Delta) \cong \mathbb{P}^n$ where Δ is the degree $n+1$ Veronese polytope and let Σ be the normal fan of Δ. Let $\mathcal{R} = \Sigma(1)$ be the set of rays of Σ and $S = \mathbb{C}[y_r \mid r \in \mathcal{R}]$ the homogeneous coordinate ring of Y. The faces of the simplex Δ^* correspond to the subsets of $\Sigma(1)$, i.e., the set of variables of the homogeneous coordinate ring S.

Let Z be a simplicial subcomplex of $\text{Poset}(\Sigma)$. In the following we represent the faces as subsets of $\mathcal{R}$, so $\text{Poset}(\Sigma)$ is represented as the complex $2^{\mathcal{R}}$ of all subsets of $\mathcal{R}$ and $Z \subset 2^{\mathcal{R}}$. Any face F of $\text{Poset}(\Sigma) \cong 2^{\mathcal{R}}$ can be considered as a square free monomial

$$y_F = \prod\nolimits_{r \in F} y_r$$

Then the monomial ideal generated by the y_F for the non-faces of Z

$$I_Z = \left\langle \prod\nolimits_{r \in F} y_r \mid F \in 2^{\mathcal{R}} \text{ not a face of } Z \right\rangle \subset S$$

is the Stanley-Reisner ideal corresponding to $Z \subset 2^{\mathcal{R}}$ and $A_Z = S/I_Z$ is the Stanley-Reisner ring of Z. Note that for any monomial ideal $I_Z \subset S$ the ring S/I_0 is $\mathbb{Z}^{\mathcal{R}}$-graded. The complex Z defines the affine scheme $\mathbb{A}_Z = \text{Spec}(A_Z)$ and the projective scheme $\mathbb{P}_Z = \text{Proj}(A_Z)$.

The complex Z relates to the complex of strata $\text{Strata}_\Delta(I_Z)$, which we defined in Section 9.3, by the isomorphism of complexes

$$\begin{array}{rccl} & \text{Poset}(\Delta) & \overset{\cong}{\to} & 2^{\mathcal{R}} \\ & \cup & & \cup \\ \text{comp}: & \text{Strata}_\Delta(I_Z) & \overset{\cong}{\to} & Z \\ & F & \mapsto & \{r \in \mathcal{R} \mid r \not\subset \text{hull}(F^*)\} \end{array}$$

In [Altmann, Christophersen, 2004-II] and [Altmann, Christophersen, 2004-I] the deformation theory of Stanley-Reisner rings is addressed, computing the first order deformations and obstructions. We give a short outline of the computation of the first order deformations of $\mathbb{P}_Z \subset \mathbb{P}^n$ by Altmann and Christophersen.

Consider the following notation. Given a subcomplex $Z \subset 2^{\mathcal{R}}$ we denote by

$$\operatorname{vert}(Z) = \{r \in \mathcal{R} \mid \{r\} \in Z\}$$

the set of vertices of Z. If $a \in 2^{\mathcal{R}}$ is a face then we can define the complex of faces of a as

$$\operatorname{Poset}(a) = \left\{b \in 2^{\mathcal{R}} \mid b \subset a\right\}$$

the boundary of a as

$$\partial a = \left\{b \in 2^{\mathcal{R}} \mid b \subsetneqq a\right\}$$

and the link of a in Z

$$\operatorname{lk}(a, Z) = \{b \in Z \mid b \cap a = \emptyset,\, b \cup a \in Z\}$$

An element $c \in \mathbb{Z}^{\mathcal{R}}$ has a support $\operatorname{supp}(c) \in 2^{\mathcal{R}}$ defined as

$$\operatorname{supp}(c) = \{r \in \mathcal{R} \mid c_r \neq 0\}$$

Let S be a polynomial K-algebra mapping onto A with $A \cong S/I$ for some ideal I and

$$0 \to R \to F \to S \to A \to 0$$

with free F a presentation of A as an S-module. If M is an A-module then define

$$T^1(A/K, M) = \operatorname{coker}\left(\operatorname{Der}_K(S, M) \to \operatorname{Hom}_A\left(I/I^2, A\right)\right)$$

In the Stanley-Reisner setup write $T^1_{A_Z} = T^1(A_Z/\mathbb{C}, A_Z)$. The grading of A_Z induces a grading on $T^1_{A_Z}$.

For $c \in \mathbb{Z}^{\mathcal{R}}$ homomorphisms in $\operatorname{Hom}_S(I_Z, S/I_Z)_c$ can be represented by Cox Laurent monomials.

Computation of T^1 reduces to links of faces:

Theorem 13.1 *[Altmann, Christophersen, 2004-II] Let $D \in \mathbb{Z}^{\mathcal{R}}$ and write $D = D_+ - D_-$ where $D_+, D_- \in \mathbb{Z}^{\mathcal{R}}_{\geq 0}$ with disjoint support. Denote by $a =$ $\operatorname{supp}(D_+)$ and $b = \operatorname{supp}(D_-)$.*

Then $T^1_{A_Z,D} = 0$ unless $a \in Z$, $D_- \in \{0,1\}^{\mathcal{R}}$ and $b \neq \emptyset$. Suppose these conditions are satisfied. $T^1_{A_Z,D}$ depends only on a and b, so write $T^1_{A_Z,a-b}$ for $T^1_{A_Z,D}$. Then

$$T^1_{A_Z,a-b} = 0$$

unless $a \in Z$ *and* $b \subset \operatorname{vert}(\operatorname{lk}(a, Z))$ *and if these conditions are satisfied*

$$T^1_{A,a-b}(Z) \cong T^1_{A,\emptyset-b}(\operatorname{lk}(a, T))$$

Suppose that Z is a **combinatorial manifold**, i.e., for all faces $a \in Z$ the link $\operatorname{lk}(a, Z)$ is a sphere of dimension $\dim(Z) - \dim(a) - 1$.

Lemma 13.2 *[Altmann, Christophersen, 2004-II] For* $b \in 2^{\Sigma(1)}$ *with* $|b| \geq 2$ *it is equivalent:*

- $T^1_{A,\emptyset-b}(Z) \neq \emptyset$
- $\dim\left(T^1_{A,\emptyset-b}(Z)\right) = 1$
- *It holds*
$$Z = \left\{ \begin{array}{ll} L * \partial b & \text{if } b \notin Z \\ L * \partial b \cup \partial L * \operatorname{Poset}(b) & \text{if } b \in Z \end{array} \right\}$$
where the geometric realization of L *is a* $\dim(Z) + 1 - |b|$ *sphere. In any case* Z *is a sphere.*

Theorem 13.3 *[Altmann, Christophersen, 2004-II] If* Z *is a manifold, then*

$$T^1_{A_Z} = \sum_{\substack{D \in \mathbb{Z}^{\mathcal{R}} \\ a = \operatorname{supp}(D) \in Z}} T^1_{<0}(\operatorname{lk}(a, Z))$$

with

$$T^1_{<0}(\operatorname{lk}(F, Z)) = \sum T^1_{\emptyset-b}(\operatorname{lk}(F, Z))$$

where the sum goes over all $b \subset \operatorname{lk}(F, Z)$ *with* $|b| \geq 2$ *and*

$$\operatorname{lk}(F, Z) = \left\{ \begin{array}{ll} L * \partial b & \text{if } b \notin \operatorname{lk}(F, Z) \\ L * \partial b \cup \partial L * \operatorname{Poset}(b) & \text{if } b \in \operatorname{lk}(F, Z) \end{array} \right\}$$

where the geometric realization L *is a* $\dim(\operatorname{lk}(F, Z)) + 1 - |b|$ *sphere.*

Note that all $T^1_{\emptyset-b}(\operatorname{lk}(F, Z))$ *are of dimension one.*

From this one can compute for case of manifolds of dimension ≤ 2:

Proposition 13.4 *[Altmann, Christophersen, 2004-II] If Z is a manifold of dimension ≤ 2, then $T^1_<(Z)$ is trivial or*

$\dim(Z)$	Z		$\dim(T^1_<(Z))$
0	$\partial\Delta_1$	*two points*	1
1	E_3	*triangle*	4
1	E_4	*quadrangle*	2
2	$\partial\Delta_3$	*tetrahedron*	11
2	$\Sigma(E_3)$	*suspension of a triangle*	5
2	$\Sigma(E_4)$	*octahedron*	3
2	$\Sigma(E_m)$	*suspension of an m-gon, $m \geq 5$*	1
2	$C(m,2)$, $m \geq 6$	*cyclic polytope*	1

Here Δ_m denotes the m-simplex, E_m the m-gon and $\Sigma(C)$ the suspension of C, i.e., the double pyramid on C.

This can be applied to compute $T^1_<(Z)$ for the links in a threefold.

Proposition 13.5 *[Altmann, Christophersen, 2004-II] Given a simplicial complex $Z \subset 2^{\Sigma(1)}$ and the corresponding Stanley-Reisner ideal I_Z we have*

$$T^1_{\mathbb{P}_Z/\mathbb{P}^n} = H^0\left(\mathbb{P}_Z, N_{\mathbb{P}_Z/\mathbb{P}^n}\right) \cong \operatorname{Hom}_S(I_Z, S/I_Z)_0$$

The kernel of $\operatorname{Hom}_S(I_Z, S/I_Z)_0 \to T^1_{A_Z,0}$ is generated by the homomorphisms $x_{r_1}\frac{\partial}{\partial x_{r_2}}$ and

$$\dim\left(\operatorname{Hom}_S(I_Z, S/I_Z)_0\right) = \dim\left(T^1_{A_Z,0}\right) + (n+1)^2$$

As outlined above for T^1, the methods given by Altmann and Christophersen allow computation of the first order deformations and obstructions, hence should provide the necessary data to apply the tropical mirror construction given in Section 9.

Let $X_0 \subset Y$ be defined by a Stanley-Reisner ideal. We may consider, if existent, a component of the local Hilbert scheme of X_0 such that a degeneration $\mathfrak{X}$ with general tangent vector in this component given by an ideal I satisfies $C_{I_0}(I) \cap \{w_t = 0\} = \{0\}$. The first order deformations in the tangent space of the $\mathfrak{X}$-component form the complex $\operatorname{dual}(B(I))$ as defined in Section 9.7 and span the Fano polytope P°, which gives the embedding toric Fano variety $Y^\circ = X(\Sigma)$ with $\Sigma = \Sigma(P^\circ)$ for the mirror fibers.

As an example, in [Grünbaum, Sreedharan, 1967] an enumeration of all combinatorial types of simplicial 4-polytopes with 7 and 8 vertices is given. These correspond to reduced monomial Calabi-Yau threefolds X_0 in $\mathbb{P}^6$ and $\mathbb{P}^7$ via the Stanley-Reisner construction. For codimension 4, due to the lack of a structure theorem analogous to Theorem 10.2, smoothing of X_0 has to be addressed by the deformation theory of Stanley-Reisner rings.

13.6 Deformations and obstructions of a non-simplicial generalization of Stanley-Reisner rings

Consider the setup of the previous Section 13.5. So let $Y = \mathbb{P}(\Delta) \cong \mathbb{P}^n$ with the degree $n+1$ Veronese polytope Δ, $\Sigma = \mathrm{NF}(\Delta)$ and $S = \mathbb{C}[y_r \mid r \in \Sigma(1)]$ the Cox ring of Y. The faces of the simplex Δ^* correspond to the subsets of $\Sigma(1)$, i.e., the set of variables of the homogeneous coordinate ring S. Let Z be a simplicial subcomplex of $\mathrm{Poset}(\Sigma)$ representing faces as sets of rays and $I_Z \subset S$ the corresponding Stanley-Reisner ideal. As noted in the previous Section 13.5 and Section 9.3 the isomorphism

$$\begin{array}{rccl} & \mathrm{Poset}(\Delta) & \overset{\cong}{\to} & 2^{\Sigma(1)} \\ & \cup & & \cup \\ \mathrm{comp}: & \mathrm{Strata}_{\Delta}(I_Z) & \overset{\cong}{\to} & Z \\ & F & \mapsto & \{r \in \Sigma(1) \mid r \not\subset \mathrm{hull}(F^*)\} \end{array}$$

transfers the combinatorial data to a subcomplex of $\mathrm{Poset}(\Delta)$ and

$$I_Z = \left\langle \prod_{v \in J} y_v \mid J \subset \Sigma(1) \text{ with } \mathrm{supp}(\mathrm{Strata}_{\Delta}(I_Z)) \subset \bigcup_{v \in J} F_v \right\rangle$$

Note that this also works if $Y = X(\Sigma)$ is a toric variety such that Σ is the fan over the faces of a simplex Δ^*. The dual description of the ideal I_Z via the subcomplex $\mathrm{Strata}_{\Delta}(I_Z) \subset \mathrm{Poset}(\Delta)$ should allow for a reformulation of the formulas for T^1 and T^2 by Altmann and Christophersen in terms of the complex $\mathrm{Strata}_{\Delta}(I_Z)$. So one may ask if this allows for a non-simplicial generalization of these formulas.

13.7 Mirrors of Calabi-Yau varieties given by ideals with Pfaffian resolutions in the Cox rings of toric Fano varieties

Let $Y = X(\Sigma)$ be a $\mathbb{Q}$-Gorenstein toric variety of dimension n with Cox ring S. We call a subscheme $X \subset Y$ of codimension 3 Pfaffian, if

1. there is a vector bundle $\mathcal{F}$ on Y of rank $2k+1$ for some $k \in \mathbb{Z}_{\geq 0}$
2. and a skew symmetric map $\varphi : \mathcal{F}(D) \to \mathcal{F}^*$ for some divisor D such that φ degenerates to rank $\leq 2k-2$ in codimension 3
3. X is scheme theoretically the degeneracy locus of φ.

If X is Pfaffian in Y given by the skew symmetric map $\varphi : \mathcal{F}(D) \to \mathcal{F}^*$ and $\det(\mathcal{F}^*) = \mathcal{O}_X(E)$ then the resolution of X is of the form

$$0 \to \mathcal{O}_Y(D-2E) \to \mathcal{F}(D-E) \to \mathcal{F}^*(-E) \to \mathcal{O}_X$$

$\omega_X^\circ \cong \mathcal{O}_X(-D+2E+K_Y)$ and X is locally defined by the Pfaffians of φ.

One may ask if the following generalization of Walters theorem from Section 10.1 holds: If X is an equidimensional, locally Gorenstein subscheme $X \subset Y$ of dimension $n-3$, $\omega_X^\circ \cong \mathcal{O}_X(D)$ for some divisor D and some parity condition similar to that in Theorem 10.4 is satisfied then X is Pfaffian.

With respect to these topics see also [Eisenbud, Popescu, Walter, 2000].

We call X globally defined by Pfaffians, if X is Pfaffian where $\mathcal{F} = \mathcal{O}_Y(E_1) \oplus ... \oplus \mathcal{O}_Y(E_r)$ is a direct sum with divisors E_j. Then X is defined by the Pfaffians of φ in the Cox ring S.

Generalizing the work of Tonoli in [Tonoli, 2000] one may construct Pfaffian Calabi-Yau varieties in toric Fano varieties.

Suppose $\mathfrak{X} \subset Y \times \operatorname{Spec}\mathbb{C}[t]$ is a monomial degeneration of Calabi-Yau varieties with fibers in the toric Fano variety Y with Cox ring S given by the ideal $I \subset \mathbb{C}[t] \otimes S$ as in the setup of the tropical mirror construction. Let $\mathfrak{X}^\circ \subset Y^\circ \times \operatorname{Spec}\mathbb{C}[t]$ be the mirror degeneration with fibers in the toric Fano variety Y° with Cox ring S° given by the ideal $I^\circ \subset \mathbb{C}[t] \otimes S^\circ$. For hypersurfaces and complete intersections we have:

Assume that $\mathfrak{X}$ is the degeneration associated to a general anticanonical toric hypersurface in a Gorenstein toric Fano variety as given in Section 3.1. Then the mirror degeneration is again a degeneration of toric hypersurfaces.

If $\mathfrak{X}$ is the degeneration associated in Section 3.1 to a general Calabi-Yau complete intersection given by a nef partition in the Gorenstein toric Fano variety Y°, then the mirror degeneration $\mathfrak{X}^\circ \subset Y^\circ \times \operatorname{Spec}\mathbb{C}[t]$ can be defined by an ideal $I^\circ \subset \mathbb{C}[t] \otimes S^\circ$ with Koszul resolution.

So if $\mathfrak{X}$ can be defined by an ideal $J \subset I$ with Pfaffian resolution we may ask: Is there always a birational model $\hat{\mathfrak{X}}^\circ \subset \hat{Y}^\circ \times \operatorname{Spec}\mathbb{C}[t]$ of the mirror degeneration $\mathfrak{X}^\circ \subset Y^\circ \times \operatorname{Spec}\mathbb{C}[t]$ which has fibers in a toric Fano variety $\hat{Y}^\circ = X(\Sigma^\circ)$, $\hat{\Sigma}^\circ(1) \subset \Sigma^\circ(1)$ with Cox ring $\hat{S}^\circ$ and is defined by an ideal $\hat{J}^\circ \subset \mathbb{C}[t] \otimes \hat{S}^\circ$ with Pfaffian resolution?

13.8 Tropical geometry and mirror symmetry over finite fields

Consider the Fermat one parameter family of quintics threefold hypersurfaces given by

$$f(x,t) = \sum_{i=1}^{5} x_i^5 + 5t \cdot x_1 x_2 x_3 x_4 x_5$$

in projective space over $\mathbb{F}_q$ with $q = p^r$, $p \neq 5$. Denote by $N_r(t)$ the number of solutions of $f(x,t)$ in $\mathbb{P}^4_{\mathbb{F}_q}$. In [Candelas, de la Ossa, Villegas, 2000] these numbers are computed in terms of the periods and in [Candelas, de la Ossa, Villegas, 2004] the structure of the ζ-function

$$\zeta(s,t) = \exp\left(\sum_{r=1}^{\infty} N_r(t) \frac{s^r}{r}\right)$$

is discussed and related to mirror symmetry.

Non-Archimedian and p-adic geometry share many similarities. Also, as observed by L. Tabera, tropical geometry behaves well with respect to finite fields. One may ask how the ζ-function relates to the tropical data associated to monomial degenerations.

13.9 Tropical curves and the A-model instanton numbers

Mikhalkin gives in [Mikhalkin, 2005] a formula enumerating curves of arbitrary genus in toric surfaces via tropical geometry. He computes the finite number of curves of genus g and degree d passing through $3d - 1 + g$ points in general position, i.e., the Gromov-Witten invariants of $\mathbb{P}^2$, by counting tropical curves via lattice paths of length $3d - 1 + g$ in the degree d Veronese polytope Δ of $\mathbb{P}^2$. This generalizes to other toric surfaces by replacing the polytope Δ.

In the context of Calabi-Yau varieties and mirror symmetry we are interested in the A-model correlation functions defined via Gromov-Witten invariants. The instanton numbers appearing in the Gromov-Witten invariants are related to the number of rational curves of given degree on the Calabi-Yau variety.

Consider the setup of the tropical mirror construction. One may ask if it is possible to compute instanton numbers, Gromov-Witten invariants and the A-model correlation functions of the general fiber of the monomial degeneration $\mathfrak{X}$ in terms of a tropical curve count using the components of the special fiber of $\mathfrak{X}$.

13.10 GKZ-hypergeometric differential equations and quantum cohomology rings of Calabi-Yau varieties

Consider a Calabi-Yau hypersurface in a toric Fano variety $\mathbb{P}(\Delta)$ of dimension n given by a reflexive polytope $\Delta \subset M_{\mathbb{R}}$. For this setup the Gelfand-Kapranov-Zelevinski hypergeometric systems are analysed in [Hosono, 1998] in the context of mirror symmetry. Via the local Torelli theorem one can give a local coordinate on the moduli space in terms of period integrals. For the hypersurface given by

$$f_c = \sum_{m\in\Delta^*\cap M} c_m x^m$$

we have one canonical period integral

$$\Pi\left(c\right) = \frac{1}{\left(2\pi i\right)^n}\int_{C_0}\frac{1}{f_c\left(x\right)}\prod\nolimits_{i=1}^{n}\frac{dx_i}{x_i}$$

for the cycle

$$C_0 = \{|x_i| = 1 \mid i = 1,...,n\} \subset T = \mathrm{Hom}_{\mathbb{Z}}\left(M, \mathbb{C}^*\right)$$

As shown by Batyrev in [Batyrev, 1993] the period integral satisfies the following GKZ-hypergeometric system associated to $\mathcal{A} = \{1\} \times (\Delta^* \cap N)$ and the exponent $\beta = 1 \times 0$. With the lattice

$$L = \left\{(l_\delta) \in \mathbb{Z}^{\mathcal{A}} \mid \sum_{\delta\in\mathcal{A}}^{n} l_\delta \delta = 0\right\}$$

of relations on the elements of $\mathcal{A}$ (see also Section 1.3.14) this system of differential equations is given by

$$\left(\prod\nolimits_{l_\delta>0}\left(\frac{\partial}{\partial c_\delta}\right)^{l_\delta} - \prod\nolimits_{l_\delta<0}\left(\frac{\partial}{\partial c_\delta}\right)^{-l_\delta}\right)\Psi\left(c\right) = 0 \text{ for } l \in L$$

$$\left(\sum_{\delta\in\mathcal{A}}\delta \cdot c_\delta\frac{\partial}{\partial c_\delta} - \beta\right)\Psi\left(c\right) = 0$$

The GKZ-hypergeometric system relates to the period integrals about the maximal degeneration point of the hypersurface degeneration.

We may ask for a generalization of the hypersurface setup using the complexes involved in the tropical mirror construction.

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Zusammenfassung

Mirror Symmetry and Tropical Geometry

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Unter Verwendung von Tropischer Geometrie entwickelt die vorliegende Arbeit eine Mirrorkonstruktion für monomiale Degenerationen von Calabi-Yau-Varietäten in torischen Fanovarietäten. Die Konstruktion reproduziert die Mirrorkonstruktionen von Batyrev für Calabi-Yau Hyperflächen und von Batyrev und Borisov für Calabi-Yau vollständige Durchschnitte. Wir wenden die Konstruktion auf Pfaffsche Beispiele an, insbesondere erhalten wir den von Rødland gefundenen Mirror für die Calabi-Yau 3-Faltigkeit in $\mathbb{P}^6$ von Grad 14 gegeben durch die Pfaffschen einer allgemeinen linearen schiefsymmetrischen $7 \times 7-$Matrix.

Die Arbeit stellt das notwendige Hintergrundwissen zur Verfügung, das in die tropische Mirrorkonstruktion eingeht, wie torische Geometrie, Gröbnerbasen, Tropische Geometrie, Hilbertschemata und Deformationen. Die tropische Mirrorkonstruktion liefert einen Algorithmus, den wir an einer Reihe von Beispielen illustrieren.

Using tropical geometry we propose a mirror construction for monomial degenerations of Calabi-Yau varieties in toric Fano varieties. The construction reproduces the mirror constructions by Batyrev for Calabi-Yau hypersurfaces and by Batyrev and Borisov for Calabi-Yau complete intersections. We apply the construction to Pfaffian examples and recover the mirror given by Rødland for the degree 14 Calabi-Yau threefold in $\mathbb{P}^6$ defined by the Pfaffians of a general linear 7×7 skew-symmetric matrix.

We provide the necessary background knowledge entering into the tropical mirror construction such as toric geometry, Gröbner bases, tropical geometry, Hilbert schemes and deformations. The tropical approach yields an algorithm which we illustrate in a series of explicit examples.

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