**Local Spectral Methods In The Theory Of Banach Function Algebras**

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# Contents

**Abstract**

A variant of Domar's theorem [**17**] on the existence of a largest subharmonic minorant of a given function is introduced. This theorem is used to obtain criteria for decomposability properties of operators with "thin" spectra satisfying certain growth conditions for the resolvent.

Main aim of the thesis is to provide sufficient conditions under which Banach function algebras of Dales-Davie type admit partitions of unity or are even regular.

Let $K \subset \mathbb{C}$ be a perfect, compact set and $(M_p)_{p\in\mathbb{N}_\circ}$ be a sequence of positive reals such that

$$M_\circ = 1 \qquad \text{and} \qquad \frac{M_p}{M_q M_{p-q}} \geq \binom{p}{q}, \qquad (q = 0, \cdots, p).$$

The normed algebras $\mathcal{D}_q(K, \{M_p\})$, $q \in \{1, \infty\}$ of all infinitely complex differentiable functions $f$ on $K$ satisfying $\|f\|_{\{M_p\},1} = \sum_{p=0}^{\infty} \frac{\|f^{(p)}\|_K}{M_p} < \infty$ and $\|f\|_{\{M_p\},\infty} = \sup_{p\in\mathbb{N}_\circ} \frac{\|f^{(p)}\|_K}{M_p} < \infty$ are not complete and their completions are not semisimple in general. By assuming the closability of $d/dz$ in $C(K)$, it is shown that the completions denoted by $\tilde{\mathcal{D}}_q(K, \{M_p\})$ are again Banach function algebras and that $\tilde{\mathcal{D}}_1(K, \{M_p\})$ is natural on $K$ under mild conditions on the sequence $(M_p)_{p\in\mathbb{N}_\circ}$. Further, by means of the variant of Domar's theorem normality criteria for $\tilde{\mathcal{D}}_q(K, \{M_p\})$ are given. In the case $q = 1$, natural regularity conditions are also obtained.

Let $K$ be a perfect, compact set such that $d/dz$ is closable with closure $\tilde{d}$ and $(\mathcal{M}_p)_{p\in\mathbb{N}_\circ}$ be a sequence of bounded, positive functions on $K$ satisfying

$$\mathcal{M}_\circ(z) = 1 \quad \text{and} \quad \frac{\mathcal{M}_p(z)}{\mathcal{M}_q(z)\mathcal{M}_{p-q}(z)} \geq \binom{p}{q} \qquad (z \in K,\ 0 \leq q \leq p).$$

The above results (of completion, naturality, normality and regularity) are carried over to the localised algebras $\mathcal{D}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ with norms $\|f\|_{\{\mathcal{M}_p\},1} := \sum_{p=0}^{\infty} \left\| \frac{f^{(p)}}{\mathcal{M}_p} \right\|_K < \infty$ and $|||f|||_{\{\mathcal{M}_p\},\infty} := \sup_{p\in\mathbb{N}_\circ} \left\| \frac{\tilde{d}^p f}{\mathcal{M}_p} \right\|_K < \infty$, respectively. Furthermore, sufficient conditions are given, which ensure the regularity of Banach function algebras $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ on certain $K$ having positive Lebesgue measure.

**Abstrakt**

Eine Variante von Domars Satz [**17**] über die Existenz einer größten subharmonischen Minorante einer gegebenen Funktion wird bewiesen. Mit Hilfe dieses Satzes erhält man Kriterien für Zerlegbarkeitseigenschaften für Operatoren mit "dünnen" Spektren, deren Resolventen gewisse Wachstumsbedingungen erfüllen.

Hauptziel der Arbeit ist, die Angabe von Kriterien für die Existenz von Partitionen der Eins oder sogar für die Regularität bei Banachfunktionenalgebren vom Dales-Davie Typ.

Sei $K \subset \mathbb{C}$ eine perfekte, kompakte Menge und $(M_p)_{p\in\mathbb{N}_\circ}$ eine positive reelle Folge, so dass

$$M_\circ = 1 \qquad \text{und} \qquad \frac{M_p}{M_q M_{p-q}} \geq \binom{p}{q}, \qquad (q = 0, \cdots, p).$$

Im Allgemeinen sind die normierten Algebren $\mathcal{D}_q(K, \{M_p\})$, $q \in \{1, \infty\}$, aller unendlich oft komplex differenzierbaren Funktionen $f$ auf $K$ mit der Norm

$$\|f\|_{\{M_p\},1} = \sum_{p=0}^{\infty} \frac{\|f^{(p)}\|_K}{M_p} < \infty \quad \text{bzw.} \quad \|f\|_{\{M_p\},\infty} = \sup_{p\in\mathbb{N}_\circ} \frac{\|f^{(p)}\|_K}{M_p} < \infty$$

nicht vollständig und ihre Vervollständigungen nicht halbeinfach. Unter der Voraussetzung der Abschließbarkeit von $d/dz$ in $C(K)$, wird gezeigt, dass die Vervollständigungen $\tilde{\mathcal{D}}_q(K, \{M_p\})$ wieder Banachfunktionenalgebren sind und dass $\tilde{\mathcal{D}}_1(K, \{M_p\})$ unter milden Bedingungen an die Folge $(M_p)_{p\in\mathbb{N}_\circ}$ natürlich ist. Ferner, werden Kriterien für die Normalität von $\tilde{\mathcal{D}}_q(K, \{M_p\})$ mit Hilfe der Variante des Satzes von Domar gezeigt. Im Fall $q = 1$, werden auch Kriterien für die Regularität gezeigt.

Sei nun $K$ eine perfekte, kompakte Menge und sei $d/dz$ abschliessbar mit der Abschließung $\tilde{d}$ sowie $(\mathcal{M}_p)_{p\in\mathbb{N}_\circ}$ eine Folge positiver, beschränkter Funktionen auf $K$ mit den Eigenschaften,

$$\mathcal{M}_\circ(z) = 1 \quad \text{und} \quad \frac{\mathcal{M}_p(z)}{\mathcal{M}_q(z)\mathcal{M}_{p-q}(z)} \geq \binom{p}{q} \qquad (z \in K,\ 0 \leq q \leq p).$$

Die genannten Resultate (bzgl. Vervollständigung, Natürlichkeit, Normalität und Regularität) werden auf die lokalen Algebren $\mathcal{D}_1(K, \{\mathcal{M}_p\})$ und $D_\infty(K, \{\mathcal{M}_p\})$ mit Normen

$$\|f\|_{\{\mathcal{M}_p\},1} := \sum_{p=0}^{\infty} \left\|\frac{f^{(p)}}{\mathcal{M}_p}\right\|_K < \infty \quad \text{bzw.} \quad |||f|||_{\{\mathcal{M}_p\},\infty} := \sup_{p\in\mathbb{N}_\circ} \left\|\frac{\tilde{d}^p f}{\mathcal{M}_p}\right\|_K < \infty$$

übertragen. Ferner, werden hinreichende Bedingungen für die Regularität der Banachfunktionenalgebren $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ und $D_\infty(K, \{\mathcal{M}_p\})$ für gewisse Kompakta $K$ mit positiven Lebesgue-Maß angegeben.

# Introduction

An important class of bounded linear operators is the class of decomposable operators in the sense of C. Foiaş [**26**]. These operators have a rich local spectral theory and a rich invariant subspace lattice (see e.g. in [**13**], [**38**], [**32**]) and play a significant role in obtaining regular Banach algebras (see [**32**]). Many authors have obtained decomposability results for operators satisfying a suitable growth condition of the resolvent provided that $T$ has thin spectrum (e.g. H. G. Tillmann [**36**], L. Waelbroeck, [**39**] E. M. Dyn'kin ([**19**] , [**20**]), Kellay-Zarrabi [**30**]) $\cdots$. Localised versions of such statements have also been considered by various authors (e.g. Ljubic-Macaev [**33**] , H. J. Sussmann [**35**] , Albrecht-Ricker [**4**]).

In [**4**], E. Albrecht and W. J. Ricker have shown that an operator $T$ is residually decomposable and has non-trivial hyperinvariant subspaces, when the resolvent satisfies the growth condition;

$$(\mathbf{a})\|R(\xi,T)\| \leq \exp\Big(\exp\Big(\frac{C}{\operatorname{dist}(\xi,\sigma(T))^{\alpha}}\Big)\Big),$$

for positive constants $C$ and $0 < \alpha < 1$ and $\sigma(T)$ is thin in $\Omega$, i.e. $\lambda(\Omega\cap\sigma(T)) = 0$, for an open set $\Omega \subset \mathbb{C}$ and assuming that for each $z \in \Omega$, there exists some open neighbourhood $U(z)$ of $z$ in $\Omega$ and some $\varepsilon > 0$ such that:

$$(\mathbf{1}) \qquad I := \iint_{U(z)} \Big(\log^{+}\log^{+}\|R(\xi,T)\|\Big)^{1+\varepsilon}\, d\lambda(\xi) < \infty.$$

For a subharmonic function $\mu$ and a given non-negative, upper semi-continuous function $F$ on an open, connected set $E \subset \mathbb{R}^n$, such that

$$(\mathbf{2}) \qquad \mu(x) \leq F(x), \quad \text{for every } x \in E$$

Y. Domar in [**17**] provided the local boundedness of the function $M(x) := \sup_{\mu\in\{F^{+}\}} \mu(x)$, by considering the conditions on $E$ and on $F$, where $\{F^{+}\}$ represents the class of all non-negative subharmonic functions $\mu$ on $E$ satisfying (**2**). Therefore, condition (**1**) was needed in [**4**] in order to be able to apply the result of Y. Domar [**17**] on uniform boundedness of families of subharmonic functions majorized by Lebesgue measurable functions satisfying growth conditions of the above mentioned type. However, the examples of (fractal) spectra considered in [**4**] have many exposed straight lines and upper box dimension $< 2$. Therefore, of particular interest would be to find conditions under which operators with thin spectra, satisfying appropriate growth conditions for the resolvent are decomposable. Criteria of this kind may be helpful in other areas of operator theory. One interesting application can be seen in Banach function algebras of Dales-Davie type.

Let $K \subset \mathbb{C}$ be a perfect, compact set and $(M_p)_{p=0}^{\infty}$ be a sequence of positive reals such that

$$M_\circ = 1 \qquad \text{and} \qquad \frac{M_p}{M_q M_{p-q}} \geq \binom{p}{q}, \qquad (q = 0, \cdots, p).$$

For $q = \{1, \infty\}$, define the normed spaces of complex ultra-differentiable functions as;

$$\mathcal{D}_q(K, \{M_p\}) := \left\{ f \in \mathcal{D}^\infty(K); \|f\|_{\{M_p\},q} < \infty \right\},$$

where

$$\|f\|_{\{M_p\},1} := \sum_{p=0}^{\infty} \frac{1}{M_p} \|f^{(p)}\|_K \quad \text{and} \quad \|f\|_{\{M_p\},\infty} := \sup_{p \in \mathbb{N}_\circ} \frac{1}{M_p} \|f^{(p)}\|_K.$$

Here, $\mathcal{D}^\infty(K)$ is the space of infinitely complex differentiable functions on $K$ and $\|\cdot\|_K$ denotes the supremum norm on $K$. $\mathcal{D}_1(K, \{M_p\})$ is a normed function algebra and has been first studied by Dales and Davie in [**15**]. In this paper, the authors assumed for a set $K$ to be a finite union of uniformly regular sets which gives Banach function algebras $\mathcal{D}^k(K)$ and $\mathcal{D}_1(K, \{M_p\})$, where $\mathcal{D}^k(K)$ is the space of $k$-times complex differentiable functions on $K$ such that $d^k f/dz^k$ is continuous. $\mathcal{D}_\infty(K, \{M_p\})$ normed algebras on a closed unit interval $I$ has only been considered in [**15**] as a special example, where it has been shown that for a particular sequence $(M_p)_{p=0}^{\infty}$ satisfying mild conditions, $\mathcal{D}_\infty(K, \{M_p\})$ is a Banach function algebra. Other than this point, not much has been said about $\mathcal{D}_\infty(K, \{M_p\})$ spaces. For a uniformly regular set $K \subset \mathbb{C}$, [**15**] provided a condition on the sequence $(M_p)_{p \in \mathbb{N}_\circ}$ under which the Banach function algebra $\mathcal{D}_R(K, \{M_p\})$ is natural, where $\mathcal{D}_R(K, \{M_p\})$ is the closed subalgebra in $\mathcal{D}_1(K, \{M_p\})$, generated by the rational functions with poles off $K$. Moreover, for the same set $K$, a condition on $(M_p)_{p \in \mathbb{N}_\circ}$ without proof has been mentioned to assure the naturality of $\mathcal{D}_1(K, \{M_p\})$ algebras. Further, a special example of $(M_p)_{p=0}^{\infty}$ yielded a natural quasianalytic Banach function algebra on such sets $K$. However, in [**25**] it has been pointed out that the normed algebra $\mathcal{D}_1(K, \{M_p\})$ is in general not complete, even for nicer sets $K$.

For a perfect, compact set $K \subset \mathbb{C}$ having infinitely many components, [**10**] showed that the spaces $\mathcal{D}^k(K)$ and $\mathcal{D}_1(K, \{M_p\})$ are incomplete and provided an example of a set $K$ which is the image of a rectifiable Jordan arc resulting incomplete space $\mathcal{D}^1(K)$ and hence $\mathcal{D}_1(K, \{M_p\})$. The authors also mentioned that due to a result of [**29**], the completeness of $\mathcal{D}^1(K)$ in [**15**] is equally valid if, $K$ is a finite union of pointwise regular sets. Further, the completions of the normed algebras have also been investigated by weakening the differentiability requirement on the functions (in terms of $\mathcal{F}$ derivatives and rectifiable paths in $K$). In a later paper of Abtahi and Honary [**1**] following the assumptions on the set $K$ from [**15**], a sufficient condition on the sequence $(M_p)_{p \in \mathbb{N}_\circ}$ to assure the naturality of $\mathcal{D}_1(K, \{M_p\})$ has been provided. In [**16**], the concept of $\mathcal{F}$-derivative has been replaced by a weaker condition and semirectifiability of the set $K$ has been assumed to guarantee the completion of $\mathcal{D}^1(K)$ to be a Banach function algebra. Moreover, they showed that whenever $K$ is $\mathcal{F}$-regular, the completion and the normed algebra $\mathcal{D}^1(K)$ are equal. Furthermore, sufficient conditions on $K$ for the incompleteness of $\mathcal{D}^1(K)$ have also been identified. These conditions and some other results will be used in Chapter 3 to study completions of function algebras $\mathcal{D}_q(K, \{M_p\})$, $q = \{1, \infty\}$.

Chapter 1 deals with a variant of Domar's theorem (Theorem 1.6), which provides a wide range of applications in obtaining the decomposability of an operator $T$ on a complex Banach space $X$. One direct consequence is Corollary 1.11, which can be considered as a variant of Theorem 3.3 [**4**]. Furthermore, when the resolvent satisfies a finite order growth condition, a condition on the Lebesgue measure of the level sets

$$A_n(z) := \left\{ w \in U(z); e^{-e^{n+1}} \leq \operatorname{dist}(w, \sigma(T)) < e^{-e^n} \right\}$$

is identified, which assures the existence of the condition (**1**) and thus residual decomposability of $T$ and non-trivial hyperinvariant subspaces for $T$.

In chapter 2, we have shown that how growth conditions of a non-negative, continuous function $G$ defined on the complement of a given thin, compact set $K \subset \mathbb{C}$ imply integrability results even in situations where the set $K$ has Lebesgue measure zero and upper box dimension $= d, 1 < d \leq 2$. An example of the Sierpinski carpet having Lebesgue measure zero and box dimension $\frac{\log 8}{\log 3}$ is provided for the case considered in [**4**], i.e. when the function $G$ satisfies growth condition of type (**a**). Other growth estimates taken into consideration for the function $G$ are;

$$(\mathbf{b}) \quad |G(z)| \leq \exp\left(\frac{C}{\operatorname{dist}(z, K)^{\alpha}}\right), \qquad \alpha > 0, \quad (z \in \mathbb{C} \setminus K);$$

$$\text{and} \quad (\mathbf{c}) \quad |G(z)| \leq \frac{C}{\operatorname{dist}(z, K)^{\alpha}}, \qquad \alpha \geq 1, \quad (z \in \mathbb{C} \setminus K).$$

for positive constant $C$. In addition, integrability criteria for the function $G$ using Domar's variant are shown and examples of sets $K$ (having Lebesgue measure zero and upper box dimension $d \leq 2$) are provided for each of the growth estimate of type (**a**) $-$ (**c**), where the upper box dimension is calculated either using the grid method or is given by a grid dimension function.

As completeness of Dales-Davie algebras is not always possible so, in Chapter 3, one focus point is the completion of such algebras. The chapter starts with the basic facts about Banach function algebras, in relevance with the decomposability of the operator of multiplication followed by few standard examples of the sequence $(M_p)_{p\in\mathbb{N}_\circ}$ satisfying nice properties. It turned out that the completions denoted by $\tilde{\mathcal{D}}_q(K, \{M_p\})$, $q \in \{1, \infty\}$ are again Banach function algebras on $K$, provided that the operator $d/dz$ is closable in $C(K)$. Further, by using the method of Abtahi and Honary in [**1**], it is shown in Theorem 3.22 that $\tilde{\mathcal{D}}_1(K, \{M_p\})$ is natural on $K$ under a mild condition on the sequence $(M_p)_{p\in\mathbb{N}_\circ}$ which is satisfied in the standard examples.

The second focus point of the chapter is to derive conditions under which the Banach function algebras $\tilde{\mathcal{D}}_q(K, \{M_p\})$, $q \in \{1, \infty\}$ are regular on $K$. For this purpose, by using Domar's variant (from Chapter 1), first the decomposability of $\tilde{M}_z$-the operator of multiplication by the coordinate function $z$-is shown on Banach function algebras $\tilde{\mathcal{D}}_q(K, \{M_p\})$, under an integral condition for the associated entire function $\sum_{p=0}^{\infty} \frac{p!}{M_p} z^p$. Results from introductory facts yield that, in this case $\tilde{\mathcal{D}}_q(K, \{M_p\})$ are normal Banach function algebras on $K$. Thus, from the normality and the naturality, the regularity of the Banach function algebra $\tilde{\mathcal{D}}_1(K, \{M_p\})$ on $K$ is obtained.

Chapter 4 deals with the localisations of such Banach function algebras. We introduce a sequence $(\mathcal{M}_p)_{p=0}^{\infty}$ of bounded, positive functions on a perfect, compact set $K$ with

$$\mathcal{M}_\circ(z) = 1 \qquad \text{and} \qquad \frac{\mathcal{M}_p(z)}{\mathcal{M}_q(z)\mathcal{M}_{p-q}(z)} \geq \binom{p}{q}$$

for all $z \in K,\ p \in \mathbb{N}_\circ,\ 0 \leq q \leq p$. For $q = \{1, \infty\}$, the corresponding normed function algebras are given by:

$$\mathcal{D}_q(K, \{\mathcal{M}_p\}) := \left\{ f \in \mathcal{D}^\infty(K); \|f\|_{\{\mathcal{M}_p\},q} < \infty \right\},$$

where $\|f\|_{\{\mathcal{M}_p\},1} := \sum_{p=0}^{\infty} \left\| \frac{f^{(p)}}{\mathcal{M}_p} \right\|_K$ and $\|f\|_{\{\mathcal{M}_p\},\infty} := \sup_{p \in \mathbb{N}_\circ} \left\| \frac{f^{(p)}}{\mathcal{M}_p} \right\|_K$.

Some properties of these normed algebras are observed and natural examples of $(\mathcal{M}_p)_{p=0}^{\infty}$ are given.

It is observed that, similar to the constant situation, assuming the closability of $d/dz$ the completions of $\mathcal{D}_q(K, \{\mathcal{M}_p\})$ are Banach function algebras, denoted by $\tilde{\mathcal{D}}_q(K, \{\mathcal{M}_p\})$.

As in the constant situation, the naturality of the Banach function algebra $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$, along with the examples of the sequences $(\mathcal{M}_p)_{p=0}^{\infty}$ satisfying the condition for naturality, are also shown. Let $\tilde{d}$ be the closure of the differential operator $d/dz$ in $C(K)$ with domain $\mathcal{D}$. For $q = \infty$, we define the normed spaces as

$$D_\infty(K, \{\mathcal{M}_p\}) := \left\{ f \in \cap_{p \in \mathbb{N}_\circ} \mathcal{D}(\tilde{d}^p); |||f|||_{\{\mathcal{M}_p\},\infty} := \sup_{p \in \mathbb{N}_\circ} \left\| \frac{\tilde{d}^p f}{\mathcal{M}_p} \right\|_K < \infty \right\}$$

Using similar methods as in the constant case for the completion and local inverse-closedness, it is observed that $D_\infty(K, \{\mathcal{M}_p\})$ is a natural Banach function algebra on $K$. Following the methods of the constant case, we have first shown the decomposability of the operator of multiplication $\tilde{\mathcal{M}}_z$ in Theorem 4.15 and hence, the regularity of Banach function algebras $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ on $K$.

More interesting results would be to show the regularity of such algebras on a thin set $K$ having positive Lebesgue measure. For this purpose, we have considered the set $K$ as the union of two compact sets such that one subset $S$ of $K$ has positive area and $K \setminus S$ has zero Lebesgue measure. Following [**31**], a sketch of such sets is given in Section 4.5. Further, it is shown that, when the set $K$ is the spectrum of an operator $T \in \mathcal{L}(X)$ satisfying the local grid dimension conditions (from Chapter 2) at a point $z \in \sigma(T) \setminus S$, the operator $T$ is residually decomposable. Moreover, if the set $S$ is totally disconnected, $T$ is decomposable.

This helped in proving the decomposability of $\tilde{\mathcal{M}}_z$ and hence, the regularity of $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ on such sets $K$.

We conclude the work with an interesting example where

$$\mathcal{M}_p(z) = p!\Big(\prod_{k=1}^{p} \log(e+k)\Big)^{\gamma(z)}, \quad p \in \mathbb{N}_\circ,$$

$\gamma(z) := 1 + \operatorname{dist}(z, S)$, $S$ is a compact, totally disconnected subset of the compact set $K$ and $K$ satisfies some grid dimension condition.

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CHAPTER 1

# Domar's Variant

In 1957 and later in 1988, Domar [**17**], [**18**] gave sufficient conditions on the existence of a largest subharmonic minorant of a given function. To get a familiarity of Domar's result and the construction, it would be appropriate to define the basic terms first.

## 1.1. Preliminaries

In section 1 and 2, we will be dealing with higher dimensional space $\mathbb{R}^k$, for $k > 1$, except at few places mentioned below. Let $E$ be an open, connected subset of $\mathbb{R}^k$.

DEFINITION 1.1. *A real-valued function $u(x) : E \to [-\infty, \infty)$ is* `subharmonic`*, if it satisfies the following conditions:*

a) *$u(x)$ is upper semi-continuous (u.s.c.) on $E$, i.e. for each $x_\circ \in E$, $\lim_{x \to x_\circ} \sup u(x) \leq u(x_\circ)$.*
b) *For every $y = (y_1, \cdots, y_k) \in E$ and $r > 0$, satisfying $\overline{B(y,r)} \subset E$ and for every real-valued function $h(x)$, harmonic on $B(y,r)$ and continuous on $\overline{B(y,r)}$ and satisfies $u(x) \leq h(x)$ on $\partial B(y,r)$, it holds that $u(x) \leq h(x)$ on $B(y,r)$.*

The semicontinuity guarantees that $u$ is (Borel) measurable and bounded above on every compact subset of $E$. Every harmonic function (by the maximum principle) is subharmonic. Also, finite sums of subharmonic functions, and maximum of finite collections of subharmonic functions are subharmonic.

REMARK 1.2. *The function $z \mapsto u(z) = \log \|f(z)\|$, is subharmonic, where $f$ is an analytic function on an open set $\Omega \subset \mathbb{C}$ with values in a Banach space. For the proof, see e.g. (Lemma 3.4.6, [**6**]). Also, $\log^+ \|f(z)\| := \max(\log \|f(z)\|, 0)$ and $\|f(z)\|^p = \exp(p \log \|f(z)\|)$ are subharmonic functions on $\Omega$ for $p > 0$.*

*For a harmonic function $h$ on $\Omega$, $|h|^p$ is subharmonic for $p \geq 1$, see e.g. (Chapter 1, Section 6, [**28**]).*

We recall few notions about subharmonic minorants and follow the notations of Domar [**17**]. Let $E \subseteq \mathbb{R}^k$ be an open, connected set and $F : E \to [0, \infty]$ be an upper semi continuous function. Define the class $\{F\}$ of all subharmonic functions $u(x)$ on $E$ such that

$$u(x) \leq F(x), \ \forall \ x \in E. \tag{1.1}$$

Denote $\{F^+\}$ the class of all non-negative subharmonic functions on $E$ satisfying (1.1) and define $M(x) := \sup\limits_{u \in \{F^+\}} u(x)$.

We give a brief introduction of Domar's Theorem (Theorem 2) [**17**]. It is clear from the property of subharmonic functions that for a finite collection of subharmonic functions in the class $\{F\}$, also have their maximum in the same class. The question arose that under what conditions, the supremum $M(x)$ of any collection of subharmonic minorants for the function $F$, also belong to the class $\{F\}$. The problem was trivial for one dimensional case, i.e. $k = 1$. For higher dimensions, Sjöberg and Brelot independently, gave a necessary and sufficient condition for the function $M(x)$ to be subharmonic.

THEOREM 1.3. *$M(x)$ is subharmonic if and only if it is bounded on every compact subset of $E$.*

Having the above information in hand, Domar provided the following result, restricting his direction in finding the conditions on the set $E$ and on the function $F$, under which $M(x)$ is bounded. The following Theorem was known to Beurling in the case $k = 2$, but no proof is available.

THEOREM 1.4 (Theorem 2, [**17**]). *$M(x)$ is bounded on every compact subset of $E$, if for some $\varepsilon > 0$,*

$$\int\limits_E \left[\log^+ F(x)\right]^{k-1+\varepsilon} dx < \infty.$$

The above theorem is still true if for every compact subset $C$ of $E$, there exists an $\varepsilon > 0$ such that, $\int\limits_C [\log^+ F(x)]^{k-1+\varepsilon}\, dx < \infty$. However, Domar's results are more general and remains true, if the subharmonic function is assumed to be measurable instead of upper semi continuous.

## 1.2. New version of Domar's result

In order to prove the improvement of Domar's result (Theorem 2, [**17**]), we need the Lemma 1 [**17**]. For the convenience, the statement of Lemma 1 [**17**] has been mentioned here. Throughout the discussion, the measure considered will be the Lebesgue measure. As in [**17**], we define the set;

$$E_\nu := \left\{x \in E : e^\nu \le u(x) < e^{\nu+1}\right\} \quad \text{and} \quad l_\nu := \lambda(E_\nu),$$

where $u \in \{F^+\}$ and $\lambda$ is the measure of the set $E_\nu$.

LEMMA 1.5. *(Lemma 1 [**17**]) Let $D$ be a positive constant and $\gamma$ a positive integer, both so large that $\dfrac{e}{D^k S_1} + \dfrac{1}{e^\gamma} \le 1$, where $S_1$ is the volume of the $k$-dimensional unit ball. Then the following is true:*
*If for some integer $\nu$ and some point $x_\nu \in E$, $u(x_\nu) \ge e^\nu$ and $S_R(x_\nu) \subset E$, where $R > D(l_{\nu-\gamma} + \cdots + l_\nu)^{\frac{1}{k}}$, then $S_R(x_\nu)$ contains a point $x_{\nu+1}$, where $u(x_{\nu+1}) \ge e^{\nu+1}$.*

Now, we give a variant of Theorem 2 in [**17**], which will be used throughout the work. Later, we give few examples of the function $f$, defined in the following theorem.

THEOREM 1.6. *Let $f : [0,\infty) \to [0,\infty)$ be a monotone increasing function such that for some $a > 0$, the integral $\int_a^\infty 1/f(t)\, dt$ exists. If*

$$\int_E f\big(\log^+ F(x)\big)^{k-1}\, dx < \infty,$$

*where $E$ is an open, connected set in $\mathbb{R}^k$. Then, $M(x)$ is bounded on every compact subset of $E$.*

PROOF. Let $x_n \in E$ and $u \in \{F^+\}$ such that for all $n \in \mathbb{N}$, $u(x_n) \geq e^n$. We will use $D$ and $\gamma$ same as in Lemma 1.5, satisfying $n > \gamma$ and follow the proof of Theorem 2 [**17**].

By Lemma 1.5 we see that, every ball $S_R(x_n)$ centred at $x_n$ with radius $R > D\sum_{\nu=n}^{m}(l_{\nu-\gamma} + \cdots + l_\nu)^{\frac{1}{k}}$ contains either a boundary point of $E$ or some point $x_m \in E$, such that $u(x_m) \geq e^m$.

Since $u(x)$ is bounded on every compact subset of $E$, it follows that:

$$\begin{aligned}
\operatorname{dist}(x_n, \partial E) &\leq D\sum_{\nu=n}^{\infty}(l_{\nu-\gamma} + \cdots + l_\nu)^{\frac{1}{k}} \\
&\leq D\sum_{\nu=n}^{\infty}(l_{\nu-\gamma}^{\frac{1}{k}} + \cdots + l_\nu^{\frac{1}{k}}) \\
&\leq D(\gamma+1)\sum_{\nu=n-\gamma}^{\infty} l_\nu^{\frac{1}{k}} \\
&= D(\gamma+1)\sum_{\nu=n-\gamma}^{\infty} \frac{1}{f(\nu)^{\frac{k-1}{k}}} \cdot f(\nu)^{\frac{k-1}{k}} l_\nu^{\frac{1}{k}}.
\end{aligned}$$

Using the Hölder's inequality, we get:

$$\operatorname{dist}(x_n, \partial E) \leq D(\gamma+1)\Big(\sum_{\nu=n-\gamma}^{\infty} \frac{1}{f(\nu)}\Big)^{\frac{k-1}{k}} \cdot \Big(\sum_{\nu=n-\gamma}^{\infty} f(\nu)^{k-1} l_\nu\Big)^{\frac{1}{k}}$$

Note that the left term of the above product

$$\delta_n := D(\gamma+1)\Big(\sum_{\nu=n-\gamma}^{\infty} \frac{1}{f(\nu)}\Big)^{\frac{k-1}{k}}$$

is independent of $u$, and hence, by the monotonicity of the function $f$ and definition of the set $E_\nu$, we have:

$$\begin{aligned}
\operatorname{dist}(x_n, \partial E) &\leq \delta_n\Big(\int_E f(\log^+ u(x))^{k-1}\, dx\Big)^{\frac{1}{k}} \\
&\leq \delta_n\Big(\int_E f(\log^+ F(x))^{k-1}\, dx\Big)^{\frac{1}{k}}
\end{aligned}$$

where $\delta_n \to 0$, when $n \to \infty$. Therefore, if for some point $x \in E$, we have $M(x) > e^n$, we conclude that:

$$\operatorname{dist}(x, \partial E) \leq \delta_n\Big(\int_E f(\log^+ F(x))^{k-1}\, dx\Big)^{\frac{1}{k}}$$

which shows that $M(x)$ is bounded on subsets of $E$ having a positive distance to $\partial E$. ☐

Examples 1.7. *Let $\varepsilon > 0$ and for some $a > 0$,*

1) *consider $f(t) = t^{1+\varepsilon}$. Then, we get the statement of Theorem 2* [**17**] *and thus, the integral exists.*
2) *if $f(t) = t(\log^+ t)^{1+\varepsilon}$ then, $\int_a^\infty f(t)\ dt$ exists.*
3) *let $f(t) = t(\log^+ t)(\log^+ \log^+ t)^{1+\varepsilon}$. Then, $\int_a^\infty f(t)\ dt$ exists.*

Using the iterative process, we may find many examples like that of (2) and (3), such that the integral exists.

## 1.3. Applications to local spectral theory

From here onwards, the complex plane $\mathbb{C}$ will be under discussion. Let $\Omega$ be an open, connected subset in $\mathbb{C}$, $\mathcal{L}(X)$ denotes the set of all bounded linear operators on a Banach space $X$ and $\lambda$ represents the planar Lebesgue measure. Further, we denote by $\sigma(T)$ and $\rho(T) = \mathbb{C} \setminus \sigma(T)$, the spectrum and the resolvent of an operator $T \in \mathcal{L}(X)$.

Recall from [**8**] that a bounded linear operator $T$ on a Banach space $X$ is said to have *Bishop's property* $(\beta)$ on an open set $\Omega \subseteq \mathbb{C}$, if for every open set $U \subset \Omega$ and every sequence of analytic functions $f_n : U \to X$ with the property that $(T - z)f_n(z) \to 0$, as $n \to \infty$, uniformly on all compact subsets of $U$, it follows that $f_n(z) \to 0$, as $n \to \infty$, locally uniformly on $U$.

Let $\mathcal{O}(U, X)$ denote the space of all analytic functions from $U$ into $X$. Define the operator $T_U : \mathcal{O}(U, X) \to \mathcal{O}(U, X)$ by $(T_U f)(z) := (T - z)f(z)$, for all $f \in \mathcal{O}(U, X)$ and $z \in U$. In terms of the operator $T_U$, property $(\beta)$ is characterised as; An operator $T \in \mathcal{L}(X)$ has property $(\beta)$ on $\Omega$, if and only if, for every open set $U \in \Omega$, the operator $T_U$ on $\mathcal{O}(U, X)$ is injective and has closed range (for proof see e.g. Proposition 1.2.6, [**32**]). In the next lemma, we state some important results of property $(\beta)$ which will be used later (for proof see Lemma 1.2 [**4**]) .

Lemma 1.8. *Let $T \in \mathcal{L}(X)$ and $\Omega \subset \mathbb{C}$ be open. Then the following hold:*

a) *Let $K$ be a compact, totally disconnected set in $\mathbb{C}$. If $T$ has property $(\beta)$ on $\Omega \setminus K$, then $T$ has property $(\beta)$ on $\Omega$.*
b) *If $\lambda(\Omega \cap \sigma(T)) = 0$, then $T$ has property $(\beta)$ on $\Omega$ if and only if for each sequence $(f_n)_{n=1}^\infty$ in $\mathcal{O}(\Omega, X)$ with $(T - z)f_n(z) \to 0$ uniformly on all compact subsets of $\Omega$, the sequence $(f_n)_{n=1}^\infty$ is locally uniformly bounded on $\Omega$.*
c) *If, for each $z \in \Omega$ the operator $T$ has property $(\beta)$ on some open neighbourhood $U(z) \subset \Omega$ of $z$, then $T$ has property $(\beta)$ on $\Omega$.*

In fact, property $(\beta)$ is closely related to the decomposability of an operator. Decomposable operators were first introduced by C. Foiaş in 1963 [**26**]. To see the things more clear, let us define few terms and basic results concerning them.

Definition 1.9. *A bounded linear operator $T$ on a Banach space $X$ is called* `decomposable`*, if for every open cover $\{U, V\}$ of $\mathbb{C}$, there exist $T$-invariant closed linear subspaces $Y$ and $Z$ of $X$ such that $\sigma(T|Y) \subset U$, $\sigma(T|Z) \subset V$ and $X = Y + Z$.*

It is known that every decomposable operator has property $(\beta)$ [**32**]. However, the converse is not true in general. We further define that, an operator $T \in \mathcal{L}(X)$ is called *residually (or S-)decomposable with residuum S*, if for each open cover of $\mathbb{C} = U \cup V$, and a closed set $S \subset \sigma(T)$ such that $S \subset U$ and $S \cap \overline{V} = \emptyset$, there exist $T$-invariant closed subspaces $Y$ and $Z$ of $X$, such that $X = Y + Z$, $\sigma(T|Y) \subset U$ and $\sigma(T|Z) \subset V$ (see [**37**]). It is easy to note that for an open set $\Omega \subset \mathbb{C}$ with residuum $\emptyset = S := \sigma(T) \setminus \Omega$, $S$-decomposability implies decomposability in the sense of C. Foiaş [**26**]. From the duality results of an operator $T$ in (Section 3) [**3**] it follows that, if $T$ and it's transpose $T^*$ both have property $(\beta)$ on an open set $\Omega$, then $T$ and $T^*$ are residually decomposable with residuum $\sigma(T) \setminus \Omega$. Recall that a linear subspace $Y$ of $X$ is called *$T$-hyperinvariant* if $RY \subset Y$, for every $R \in \mathcal{L}(X)$ commuting with $T$.

For further properties of decomposable operators, we refer to the books of [**13**] and [**32**].

**1.3.1. Relation between subharmonic minorants and property $(\beta)$.** Domar's results and it's variant assures us that a non-negative upper semi-continuous function has a subharmonic minorant. In [**4**], these subharmonic minorants have been used to show the existence of property $(\beta)$ and hence, (residual) decomposability of an operator $T \in \mathcal{L}(X)$.

In fact, from Lemma 1.8(b) to show that an operator $T$ has Bishop's property $(\beta)$ on $\Omega$ with $\lambda(\Omega \cap \sigma(T)) = 0$, it suffices to show that the sequence $(f_n)_{n=1}^{\infty}$ is locally uniformly bounded on $\Omega$. Recall that (e.g. from [**6**], Lemma 3.4.6) for $f \in \mathcal{O}(\Omega, X)$ and $z \in \Omega$, $\log \|f(z)\|$ is subharmonic. Let for any compact subset $K$ of $\Omega$, for each open neighbourhood $U \subset \Omega$ of $K$ and for all $n \in \mathbb{N}$, consider the subharmonic function $u_n(z) := \log \|f_n(z)\|$, with the convention that $\log 0 := -\infty$. Then to show the uniform boundedness of $(f_n)_{n=1}^{\infty}$ on $K$, it would be sufficient to find a subharmonic majorant for the sequence $(u_n)_{n=1}^{\infty}$.

Before presenting an immediate consequence of Domar's variant, we fix the condition of the function $f$ introduced in Theorem 1.6 in the following definition.

DEFINITION 1.10. *A function $f : [0, \infty) \to [0, \infty)$ is said to satisfy* `condition` *$(\eta)$ if, $f$ is monotone increasing and for some $a > 0$, the integral*

$$I(f) = \int_a^{\infty} \frac{1}{f(t)}\, dt < \infty.$$

Now, we give a corollary of Theorem 1.6.

COROLLARY 1.11. *Let $T$ be a bounded linear operator on a Banach space $X$ and $\Omega \subseteq \mathbb{C}$ be open. Assume that for each $z \in \Omega$, and a neighbourhood $U(z) \subset \Omega$ of $z$, there exists a function $f_z$ satisfying condition $(\eta)$ such that the following integral exists;*

$$I := \iint\limits_{U(z)} f_z(\log^+ \log^+ \|R(\zeta, T)\|)\, d\lambda(\zeta) < \infty \tag{1.2}$$

*Then, $T$ is residually decomposable with residuum $S := \sigma(T) \setminus \Omega$.*

PROOF. Let $(g_n)_{n=1}^{\infty}$ be a sequence in $\mathcal{O}(U(z), X)$ such that

$$(\zeta - T)g_n(\zeta) \to 0, \quad \text{for} \;\; n \to \infty,$$

uniformly on all compact subsets of $U(z)$ and let $K$ be a compact subset of $U(z)$. For $n \geq n_\circ$, we have $\|(\zeta - T)g_n(\zeta)\| \leq 1$ on some open neighbourhood $V$ of $K$ with $\overline{V} \subset U(z)$.

Note that $\zeta \mapsto \log \|g_n(\zeta)\|$ is subharmonic on $U(z)$ and, for $n \geq n_\circ$,

$$\log \|g_n(\zeta)\| \leq \log^+ \|R(\zeta, T)\| \text{ on } V.$$

As, $\iint\limits_{U(z)} f_z(\log^+ \log^+ \|R(\zeta, T)\|)\ d\lambda(\zeta)$ is bounded, we conclude from Theorem 1.6 with $E = V$ that $\big(\log \|g_n\|\big)_{n=1}^{\infty}$ is uniformly bounded on $V$. Hence also $\big(\|g_n\|\big)_{n=1}^{\infty}$ is uniformly bounded on $V$. By Lemma 1.8(b) and (c), $T$ has property $(\beta)$ on $\Omega$. As in the proof of Theorem 3.3 in [**4**], we see using similar arguments for $T^*$ that $T^*$ has also property $(\beta)$ on $\Omega$. Thus, from Theorem 20 [**3**] , $T$ is residually decomposable with residuum $S = \sigma(T) \setminus \Omega$. □

From the above Corollary, it is interesting to note that if $\Omega \cap \sigma(T)$ contains at least two points or if $\sigma(T) \setminus \Omega$ and $\Omega \cap \sigma(T)$ are non-empty, then $T$ has a non-trivial hyperinvariant subspace.

The above Corollary is a local version and by considering Examples 1.7$(i)$ for the function $f_z$, i.e. $f_z(t) = t^{1+\varepsilon}$, for some $\varepsilon > 0$ and for all $t \geq 0$ we obtain the statement of Theorem 3.3 in [**4**].

REMARK 1.12. *A global version of Corollary 1.11 can be formulated as:*

*Let $T$ and $\Omega$ be as in Corollary 1.11 and $f$ satisfies the condition $(\eta)$. Then, $T$ is residually decomposable with residuum $\sigma(T) \setminus \Omega$, if the following holds:*

$$I_\Omega := \iint\limits_{\Omega} f(\log^+ \log^+ \|R(\zeta, T)\|)\ d\lambda(\zeta) < \infty. \tag{1.3}$$

REMARK 1.13. *Let $T$ and $\Omega$ be as in Corollary 1.11 with residuum $S = \sigma(T) \setminus \Omega$. If $S$ is totally disconnected, then $T$ is decomposable.*

PROOF. Following the arguments as in the proof of Corollary 1.11, we obtain that $T$ has property $(\beta)$ on $\Omega$. Hence, also on $\mathbb{C}$ as $S$ is totally disconnected by Lemma 1.8(a). From the proof of Theorem 3.3 [**4**], $T^*$ has also property $(\beta)$ on $\Omega$ and by Lemma 1.8(a) and the totally disconnectedness of $S$, also on $\mathbb{C}$. Thus $T$ is decomposable. □

We combine Corollary 1.11 and Remark 1.13 in the following result which will be useful in obtaining decomposability in localised situation.

COROLLARY 1.14. *Let $T \in \mathcal{L}(X)$ and $S$ be a compact, totally disconnected subset of $\sigma(T)$. Assume that for each $z \in \sigma(T) \setminus S$, there exists a closed square $Q_z \subset \sigma(T) \setminus S$ with centre $z$ and a function $f_z$ satisfying condition $(\eta)$, such that the following integral exists;*

$$I_{Q_z} = \iint\limits_{Q_z} f_z(\log^+ \log^+ \|R(\zeta, T)\|)\ d\lambda(\zeta) < \infty.$$

*Then, $T$ is decomposable.*

**1.3.2. Application to decomposability.** Let $\Omega \subset \mathbb{C}$ be an open set. An operator $T$ is said to have *thin spectrum* in $\Omega$, if $\lambda(\Omega \cap \sigma(T)) = 0$. Next theorem gives us an improvement of Theorem 3.3 [**4**], using the Domar's variant.

THEOREM 1.15. *Let $T \in \mathcal{L}(X)$ and $\Omega \subseteq \mathbb{C}$ be open, such that $\lambda(\Omega \cap \sigma(T)) = 0$. Suppose that for each $z \in \Omega$ and a neighbourhood $V(z) \subset \Omega$ of $z$, there exists some constant $C(z) > 0$ and a function $f_z$ satisfying:*

i) *condition $(\eta)$;*
ii) *for all $C > 0$, we have $K(C) := \lim_{t\to\infty} \sup \frac{f_z(C+t)}{f_z(t)} < \infty$,*

*such that for all $n \in \mathbb{N}$, $\lambda(A_n(z)) \leq \frac{C(z)}{f_z(n)^2}$, where,*

$$A_n(z) := \left\{ w \in V(z) | e^{-e^{n+1}} \leq \operatorname{dist}(w, \sigma(T)) < e^{-e^n} \right\}.$$

*Assume that the resolvent satisfies the condition of the form;*

$$\|R(\zeta, T)\| \leq C_\circ \operatorname{dist}(\zeta, \sigma(T))^{-k}, \qquad \forall\, \zeta \in \rho(T),$$

*for positive constants $C_\circ$ and $k \geq 1$. Then, $T$ is residually decomposable with residuum $S := \sigma(T) \setminus \Omega$.*

In particular, if $\Omega \cap \sigma(T)$ contains at least two points or if $\sigma(T) \setminus \Omega$ and $\Omega \cap \sigma(T)$ are non-empty, then $T$ has a non-trivial hyperinvariant subspace.

PROOF. Fix an arbitrary $z \in \Omega$. If $\operatorname{dist}(z, \sigma(T)) \geq e^{-e}$, then we have with $\varepsilon := \frac{1}{2} e^{-e}$ a neighbourhood $V_\varepsilon(z) \subset \Omega$ of $z$ so that:

$$\iint\limits_{V_\varepsilon(z)} f_z\big(\log^+ \log^+ \|(\zeta - T)^{-1}\|\big)\, d\lambda(\zeta) < \infty$$

which shows from Corollary 1.11 that $T$ is residually decomposable.

Suppose now that $\operatorname{dist}(z, \sigma(T)) < e^{-e}$. Let $V(z)$ be as in the theorem. Then,

$$U(z) := V(z) \cap \big\{ w \in \Omega | \operatorname{dist}(w, \sigma(T)) < e^{-e} \big\}$$

is a neighbourhood of $z$ and $U(z) \subset \cup_{n=1}^{\infty} A_n(z)$.

For $\zeta \in A_n(z)$, we have by the growth condition of $R(\zeta, T)$, that $\operatorname{dist}(\zeta, \sigma(T)) \geq e^{-e^{n+1}}$, which implies $\|(\zeta - T)^{-1}\| \leq C_\circ e^{ke^{n+1}}$. Then,

$$\begin{aligned}
\log^+ \log^+ \|(\zeta - T)^{-1}\| &\leq \log^+ \log^+ (C_\circ e^{ke^{n+1}}) \\
&\leq \log(\log C_\circ + ke^{n+1}) \leq \log(C_1(k) e^{n+1}) \\
&\leq \log C_1(k) + n + 1 \leq C_2(k) + n
\end{aligned}$$

where $C_1(k)$, $C_2(k) > 0$ are constants independent of $n$. Let

$$I = \iint\limits_{U(z)} f_z\big(\log^+ \log^+ \|(\zeta - T)^{-1}\|\big)\, d\lambda(\zeta) \tag{1.4}$$

Then, by the disjointness of $A_n(z),\ n \in \mathbb{N}$, one obtains:

$$
\begin{aligned}
I &\leq \sum_{n=1}^{\infty} \iint_{A_n(z)} f_z\big(\log^+ \log^+ \|(\zeta - T)^{-1}\|\big)\ d\lambda(\zeta) \\
&\leq \sum_{n=1}^{\infty} \iint_{A_n(z)} f_z(C_2(k) + n)\ d\lambda(\zeta) \\
&= \sum_{n=1}^{\infty} f_z(C_2(k) + n)\lambda(A_n(z)) \\
&\leq \sum_{n=1}^{\infty} \frac{C(z) f_z(C_2(k) + n)}{f_z(n)^2} \\
&\leq C(z) K(C_2(k)) \sum_{n=1}^{\infty} \frac{1}{f_z(n)} \qquad (1.5)
\end{aligned}
$$

Since, by assumption $i)$ in the theorem, the sum on the right hand side of (1.5) converges, which implies that the integral $I$ in (1.4) exists.

Now, to show that $T$ is residually decomposable with residuum $S = \sigma(T) \setminus \Omega$ and the fact that $T$ has a non-trivial hyperinvariant subspace follows directly from the proof of Corollary 1.11 and the remarks after it. $\square$

REMARK 1.16.

1) *This Theorem also applies to the functions given in Examples 1.7.*
2) *The above theorem also gives the improvement of Theorem 3.3 in* [**4**], *when the original result of Domar (Theorem 2,* [**17**]*) is used. In that case, we assume that for some* $\delta(z) > 0$, *some constant* $C(z) > 0$ *and for all* $n \in \mathbb{N}$,
$$\lambda(A_n(z)) \leq \frac{C(z)}{n^{2+\delta(z)}}$$
*where* $A_n(z)$ *is same as in the Theorem 1.15.*
3) *This Theorem applies to* `hyponormal` *operators* $T$ *on a Hilbert space* $\mathcal{H}$, *i.e. when* $T^*T \geq TT^*$. *Since the resolvent of a hyponormal operator satisfies linear order growth condition, it is clear that for* $k, C_\circ = 1$, *the resolvent estimate in Theorem 1.15 will be* $\|R(\zeta, T)\| \leq \operatorname{dist}(\zeta, \sigma(T))^{-1}$.

**1.3.3. Application to normality.** An application of Domar's variant (Theorem 1.6) to obtain normal (function) algebras will be provided here. We mention here that, more normality results on some interesting Banach function algebras will be presented in Chapters 3 and 4. Recall that a function algebra $\mathcal{A}$ is said to be *normal* on a locally compact set $K$, if given a compact set $K_1$ and a closed set $K_2$ such that $K_1 \cap K_2 = \emptyset$, there exists $f \in \mathcal{A}$ with $f|K_2 = 0$ and $f|K_1 = 1$.

One observes that, the normality of certain algebras of functions as obtained in Theorem 3.10 [**4**] can be improved by using the variant of Domar's Theorem from Section 1.2. For this reason, we first define the algebra of functions as discussed in [**4**].

Let $h : [0, \infty) \to [0, \infty)$ be a monotone increasing function such that $h(t) \geq t$, for all $t \geq 0$. Let $K$ be a compact set having Lebesgue measure zero and define

$\mathcal{D}_K(h)$ be the algebra of all complex-valued bounded $C^1$-functions such that:

$$\|\phi\|_h := \sup_{z\in\mathbb{C}} |f(z)| + \sup_{z\in\mathbb{C}\setminus K} \left| \frac{1}{h(\mathrm{dist}(z,K))} \cdot \frac{\partial f}{\partial \bar{z}}(z) \right| < \infty.$$

Then, $\mathcal{D}_K(h)$ is a commutative Banach algebra. Denote, $\mathcal{D}_{\circ,K}(h)$ be the closure in $\mathcal{D}_K(h)$ of the subalgebra of all those functions $\phi$ in $\mathcal{D}_K(h)$, which are analytic in some (individual) neighbourhood $U_\phi$ of $K$. We further define an ideal $\mathcal{J}(K)$ in $\mathcal{D}_{\circ,K}(h)$ as the collection of all $\phi \in \mathcal{D}_K(h)$ vanishing in some neighbourhood $V_\phi$ of $K$ and $\mathcal{I}(K) := \overline{\mathcal{J}(K)}$. We consider the quotient algebra $\mathcal{Q}_K(h) := \mathcal{D}_{\circ,K}(h)/\mathcal{I}(K)$ and $[\phi]_K$ represents the equivalence class of $\phi \in \mathcal{D}_{\circ,K}(h)$ in the quotient algebra $\mathcal{Q}_K(h)$. In particular, using Remark 1.12, we get the following result:

THEOREM 1.17. *Let $f$ be a function satisfying condition $(\eta)$, $K$ be a compact set with $\lambda(K) = 0$ and $h$ be a function as defined above.*
*If, there exists an open neighbourhood $U$ of $K$ such that for some $t \in (0,1)$, we have:*

$$I := \iint\limits_U f\big(\log^+ \log^+ g(\mathrm{dist}(z,K))\big)\, d\lambda(z) < \infty$$

*where $g(\mathrm{dist}(z,K)) := h(\mathrm{dist}(z,K) - th(\mathrm{dist}(z,K)))$. Then, the algebra $\mathcal{R}_K(h)$ of all restrictions of functions from $\mathcal{D}_{\circ,K}(h)$ to $K$ is normal.*

PROOF. The proof follows the proof of Theorem 3.10 [**4**]. $\square$

CHAPTER 2

# Integrability criteria of functions near "thin" sets

In this chapter we investigate in detail the integrability criteria of a non-negative, continuous function $G$ defined on the complement of a given thin, compact set $K \subset \mathbb{C}$, when it satisfies some growth estimates near a given compact set $K$. We obtain the integrability criteria under two situations, in one using Domar's theorem (Theorem 2, [**17**]) and in other using Domar's variant (Theorem 1.6). In both situations, we will discuss three types of growth conditions of a function $G$, namely, $(i)$ finite order growth, $(ii)$ exponential growth and $(iii)$ double exponential growth . Since, these growth conditions will be used frequently throughout the chapter, we mention them separately.

Growth Conditions (GC)

Let $K \subset \mathbb{C}$ be a thin compact set and $G$ be a non-negative, continuous function on $\mathbb{C}\backslash K$. We will consider the following growth estimates for the function $G$;

$$(GC1) \qquad |G(z)| \leq \frac{C}{\operatorname{dist}(z,K)^{\alpha}}, \qquad \alpha \geq 1, \quad (z \in \mathbb{C} \setminus K);$$

$$(GC2) \qquad |G(z)| \leq \exp\Big(\frac{C}{\operatorname{dist}(z,K)^{\alpha}}\Big), \qquad \alpha > 0, \quad (z \in \mathbb{C} \setminus K);$$

$$(GC3) \qquad |G(z)| \leq \exp\Big(\exp\Big(\frac{C}{\operatorname{dist}(z,K)^{\alpha}}\Big)\Big), \qquad \alpha > 0, \quad (z \in \mathbb{C} \setminus K),$$

where $C$ is a positive constant. Each particular growth condition $(GC1)-(GC3)$ will be provided with individual examples.

## 2.1. Introductory facts and dimensions

A set having a fine, irregular structure and which contains copies of itself at arbitrarily small scales is usually defined as *fractal* . Most of the fractals are self-similar, for example Cantor set , von Koch curve. Much of the details of the fractals are not essential here, so we refer to the books of K. Falconer [**23, 24**] for more examples and detailed structure of such sets.

Fractals are studied and measured using dimension, usually known as *Fractal dimension*, which is greater than the topological dimension of a structure. Roughly speaking, by fractal dimension of a set we mean, the ratio of $m$ copies of the set scaled by a factor $r$. There are different ways to measure this dimension, but we will only discuss two types, i.e. Hausdorff dimension and Box dimension and will use only the variant of the later one. For other types of dimensions we suggest to Chapter 3, [**23**].

We mention here that the following discussion (about the fundamental facts of the above two dimensions) will deal with arbitrary sets in $\mathbb{R}^n$.

**2.1.1. Hausdorff dimension.** Hausdorff dimensions are based on the concept of measures. Let $F \subset \mathbb{R}^n$ be a non-empty set and $\{F_i\}$ be an $\varepsilon$-cover of $F$ for some $\varepsilon > 0$, i.e. $\{F_i\}$ is a countable covering of $F$, where each $F_i,\ i \in \mathbb{N}$ has diameter at most $\varepsilon$.

Let $|F_i|,\ i \in \mathbb{N}$ denotes the diameter of a set and $s > 0$. Then, for any $\varepsilon > 0$ we define the set $\mathcal{H}^s_\varepsilon(F)$ by

$$\mathcal{H}^s_\varepsilon(F) := \inf\Big\{\sum_{i=1}^{\infty} |F_i|^s : \{F_i\} \text{ is an } \varepsilon - \text{cover of } F\Big\}.$$

As $\varepsilon$ decreases, the infimum $\mathcal{H}^s_\varepsilon(F)$ increases and thus, we define the *s-dimensional Hausdorff measure* as:

$$\mathcal{H}^s(F) := \lim_{\varepsilon \to 0} \mathcal{H}^s_\varepsilon(F).$$

It is easy to check that $\mathcal{H}^s(F)$ satisfies the definition of a measure. Hence, we denote and define the *Hausdorff dimension* ; $\dim_H F$ to be the infimum of $s$ for which the Hausdorff measure is zero, i.e.

$$\dim_H F = \inf\{s : \mathcal{H}^s(F) = 0\}.$$

In other words,

$$\mathcal{H}^s(F) = \begin{cases} \infty, & \text{if } s < \dim_H F \\ 0, & \text{if } s > \dim_H F \end{cases} \tag{2.1}$$

Hausdorff dimension has different variants but in most of the examples it is not simple to calculate the Hausdorff measure of a set.

**2.1.2. Box-counting dimension.** Box-counting dimension or usually known as Box dimension is commonly used when dealing with self-similar sets. The basic idea is to cover a (bounded) set $F \subset \mathbb{R}^n$ by the minimum number of sets $N_\varepsilon(F)$ of diameter at most $\varepsilon$. As $\varepsilon$ becomes small, the number $N_\varepsilon(F)$ becomes large. If there exists some $d > 0$ such that

$$N_\varepsilon(F) \sim 1/\varepsilon^d, \qquad \text{as } \varepsilon \to 0$$

then, $d$ is the box dimension of $F$. But this $d$ only exists if there is some positive constant $k$ such that

$$\lim_{\varepsilon \to 0} \frac{N_\varepsilon(F)}{1/\varepsilon^d} = k.$$

Since, both sides of the above equation are positive, taking logarithm of both sides we get:

$$\lim_{\varepsilon \to 0}\big[\log N_\varepsilon(F) - d\log(1/\varepsilon)\big] = \log k$$
$$d = \lim_{\varepsilon \to 0} \frac{\log k - \log N_\varepsilon(F)}{\log \varepsilon} = -\lim_{\varepsilon \to 0} \frac{\log N_\varepsilon(F)}{\log \varepsilon}.$$

We denote the box dimension of a set $F$ by $\dim_B F$ and formally define it as follows:

DEFINITION 2.1. *Let $F \subset \mathbb{R}^n$ be a non-empty, bounded set and $\varepsilon > 0$. Denote by $N_\varepsilon(F)$ the minimum number of sets of diameter at most $\varepsilon$ which cover $F$. Then, the* `lower box dimension` *and* `upper box dimension` *of $F$ are defined as:*

$$\underline{\dim}_B F = \underline{\lim}_{\varepsilon\to 0} \frac{\log N_\varepsilon(F)}{-\log\varepsilon}$$
$$\text{and} \quad \overline{\dim}_B F = \overline{\lim}_{\varepsilon\to 0} \frac{\log N_\varepsilon(F)}{-\log\varepsilon}$$

*respectively. If the above limits exist and are equal, then the common value is defined as the* `Box dimension` *, i.e.*

$$\dim_B F = \lim_{\varepsilon\to 0} \frac{\log N_\varepsilon(F)}{-\log\varepsilon}.$$

When computing the box dimension, structure of a set $F \subset \mathbb{R}^n$ plays an important role and helps in deciding to find an appropriate covering of it. Other than boxes, a set $F$ can be covered in various ways, e.g. by discs or stars of diameter $\varepsilon$. Few commonly used coverings are:

($i$) closed balls of radius $\varepsilon$;
($ii$) cubes of side length $\varepsilon$;
($iii$) $\varepsilon$-mesh cubes intersecting with the set $F$;
($iv$) sets of diameter at most $\varepsilon$;
($v$) disjoint balls of radius $\varepsilon$ with centres in the set $F$.

It has been shown in Section 3.1 [**23**] that, all the coverings $(i)-(v)$ mentioned above are in fact, equivalent definitions of the box dimension of a set $F$. However, in type $(i)-(iv)$, $N_\varepsilon(F)$ is the smallest number covering $F$ and in type $(v)$, $N_\varepsilon(F)$ is the largest number of such disjoint balls.

Note that when computing $\overline{\dim}_B F$ and $\underline{\dim}_B F$, not every $\varepsilon$ need to be considered for $\varepsilon \to 0$. In fact, by choosing a decreasing sequence $(\varepsilon_i)_{i\in\mathbb{N}_\circ}$ with $\varepsilon_i \to 0$ for $i \to \infty$ such that $\varepsilon_{i+1} > c\, \varepsilon_i$, for some constant $0 < c < 1$, then

$$\overline{\dim}_B F = \overline{\lim_{i\to\infty}} \frac{\log N_{\varepsilon_i}}{-\log\varepsilon_i}$$

(see [**23**], p. 41).

It is clear that the lower box dimension is (slightly) less than the upper box dimension. Also, from the arguments before Example 3.3 [**23**], we observe that:

$$\dim_H(F) \le \underline{\dim}_B(F) \le \overline{\dim}_B(F). \tag{2.2}$$

REMARK 2.2. *Let $F \subset \mathbb{R}^n$ be any bounded set having upper box dimension $< n$. Then, $F$ has zero volume.*

PROOF. It is clear from (2.2) that $F$ has Hausdorff dimension $\dim_H F < n$. Hence, $F$ has zero volume by (2.1). □

Moreover, we see from Proposition 2.5 [**23**] that a set $F \subset \mathbb{R}^n$ having $\dim_H F < 1$ is totally disconnected, e.g. Cantor Set having $\dim_H = \dim_B = \log 2/\log 3$ (see Example 3.3, [**23**]).

We will use grid method which is similar to type (*iii*) to calculate the (upper) box dimension and give a sketch of it in $\mathbb{R}^2$.

Let $F \subset \mathbb{R}^2$ be a non-empty, bounded set and $k \in \mathbb{N}_\circ$. In grid method, we cover the complex plane $\mathbb{C}$ by squares of side length $\varepsilon_k$ parallel to the axes, i.e. squares of the form $[n_1\varepsilon_k, (n_1+1)\varepsilon_k] \times [n_2\varepsilon_k, (n_2+1)\varepsilon_k]$, where $n_1, n_2 \in \mathbb{Z}$. Then, $N_{\varepsilon_k}(F)$ denotes the number of squares of side length $\varepsilon_k$ intersecting with $F$. We divide the grids of side length $\varepsilon_k$ into grids of side length $\varepsilon_{k+1}$ and consider the subsquares having non-empty intersection with $F$. Continuing in this way and letting $k \to \infty$, we obtain a finer cover of a set $F$. We will consider upper box dimension when dealing with sets.

## 2.2. Integrability and dimension conditions

In this section we will use the concept of upper box dimension for a compact set $K \subset \mathbb{C}$. Further, we obtain the integrability results of a non-negative, continuous function $G$ using Domar's result when a function $G$ satisfies a certain growth near a given thin compact set $K$ having some upper box dimension $d$.

First, we give some elementary results of monotone increasing functions which will be used frequently in the following discussion and later deal with the growth estimates of a function $G$ and upper box dimension of a compact set $K$.

LEMMA 2.3. *Let $x_1, x_2, \cdots, x_m \in \mathbb{R}$ and $g$ be a real-valued monotone increasing function. Then, for the function $g$ and for the logarithm function, we have the following estimates:*

$$g\big(\max_{1\le j\le m} x_j\big) = \max_{1\le j\le m} g(x_j) \le \sum_{j=1}^{m} g(x_j),$$

*and*

$$\log \sum_{j=1}^{m} x_j \le \log 2^{m-1} + \sum_{j=1}^{m} \log x_j = \log\Big(2^{m-1} \prod_{j=1}^{m} x_j\Big).$$

PROOF. The estimates can be easily proved by using the monotonicity of the functions and induction with respect to $m$. □

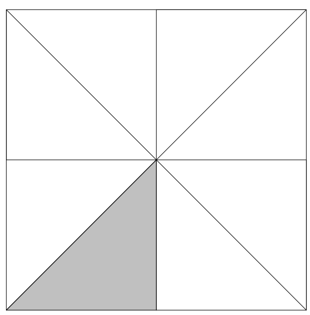

FIGURE 2.1. Square $Q_a$

LEMMA 2.4. *Let $h : (0,\infty) \to (0,\infty)$ be a monotone increasing function such that for some $\delta > 0$,*

$$I(h) := \int_0^1 h\Big(\frac{2}{t^{1+1/\delta}}\Big)\ dt < \infty. \tag{2.3}$$

*Let $Q_a$ be a square of side length $a \le 1$. Then, the following estimate holds;*

$$J_a(h) := \iint\limits_{Q_a} h\Big(\frac{1}{\operatorname{dist}(z,\partial Q_a)}\Big)\ d\lambda(z) \le 2a^2\left[I(h) + h\Big(\frac{2}{a^{1+\delta}}\Big)\right].$$

PROOF. We decompose $Q_a$ into 8 congruent triangles $\triangle_1, \cdots, \triangle_8$, as shown in Figure 2.1, which gives:

$$J_a(h) = \sum_{j=1}^{8} \iint\limits_{\triangle_j} h\Big(\frac{1}{\operatorname{dist}(z,\partial Q_a)}\Big)\ d\lambda(z).$$

Define, $J_j(h) := \iint\limits_{\triangle_j} h(\frac{1}{\operatorname{dist}(z,\partial Q_a)})\ d\lambda(z)$, for $j = 1, \cdots, 8$.

With an appropriate choice of the coordinates for $\triangle_{j's}$, $j = 1, \cdots, 8$, we have:

$$J_j(h) = \int_0^{\frac{a}{2}} \int_0^x h(\frac{1}{y})\ dydx.$$

Integrating by parts and changing the variable by $x = \frac{a}{2}t$, we get:

$$J_j(h) = \int_0^{\frac{a}{2}} (\frac{a}{2} - x)h(\frac{1}{x})\ dx \le \frac{a^2}{4}\int_0^1 h(\frac{2}{at})\ dt.$$

For some $\delta > 0$, the above integral can be written as:

$$\begin{aligned} J_j(h) =& \frac{a^2}{4}\Big[\int_0^{a^\delta} h(\frac{2}{at})\ dt + \int_{a^\delta}^1 h(\frac{2}{at})\ dt\Big] \\ \le& \frac{a^2}{4}\Big[\int_0^{a^\delta} h(\frac{2}{t^{1+1/\delta}})\ dt + \int_{a^\delta}^1 h(\frac{2}{a^{1+\delta}})\ dt\Big] \\ \le& \frac{a^2}{4}\Big[h(\frac{2}{a^{1+\delta}}) + I(h)\Big] \end{aligned}$$

Thus, by symmetry, we get that:

$$J_a(h) \le 2a^2\left[I(h) + h\big(\frac{2}{a^{1+\delta}}\big)\right].$$

□

To acquire the integrability results of a function $G$ for each growth estimate $(GC1) - (GC3)$, respective grid dimension conditions of a set $K$ will be defined. Before discussing the types of the growth estimates and the respective dimension conditions, we give some details of a general scheme of covering a set $K$ using the grid method.

Covering of a set $K$

Let $Q \subset \mathbb{C}$ be a closed square of side length $a$ and $K \subset \mathbb{C}$ be a compact subset of $Q$ with $\lambda(K) = 0$ and upper box dimension $d \leq 2$. Then, for each $k \in \mathbb{N}$, using the notion of grid dimension from [**23**], we consider the coverings $Q_k$ of $Q$ by squares of side length $a_k$ parallel to the axes and $\operatorname{int} R_1 \cap \operatorname{int} R_2 \neq \emptyset$, if and only if, $R_1 = R_2$ for all $R_1, R_2 \in Q_k$.

Let $N_{a_k}$ denote the minimal number of squares $R \in Q_k$ of side length $a_k$ having non-empty intersection with $K$. Thus, the area $F_k$ of the minimal union $A_k$ of squares from $Q_k$ having non-empty intersection with $K$ will be:

$$F_k := N_{a_k} a_k^2. \tag{2.4}$$

Let $k_\circ \in \mathbb{N}$ and $N_Q$ denote the minimal number of squares from $Q_{k_\circ}$ covering $Q$. Then, for $A_Q$ the minimal union of squares from $Q_{k_\circ}$ covering $Q$, we have for all $k \geq k_\circ$, that:

$$Q = A_Q \supseteq A_{k_\circ} \supseteq A_{k_\circ+1} \supseteq \cdots \supseteq A_k \supseteq A_{k+1}. \tag{2.5}$$

Since, the gridlines$_{k+1}$ of side length $a_{k+1}$ contains the gridlines$_k$ of side length $a_k$, which gives $a_k \to 0$, for $k \to \infty$. Thus, $\lambda\big(\cap_{k=k_\circ}^{\infty} A_k\big) = \lim_{k\to\infty} F_k = 0$.

We need to show that for some $\varepsilon > 0$, the integral

$$I_Q := \iint\limits_{Q} \Big(\log^+ \log^+ |G(z)|\Big)^{1+\varepsilon} \, d\lambda(z) < \infty. \tag{2.6}$$

Using (2.5), we see that $I_Q$ can be further written as:

$$\begin{aligned} I_Q \leq & \iint\limits_{A_Q \setminus A_{k_\circ}} \Big(\log^+ \log^+ |G(z)|\Big)^{1+\varepsilon} \, d\lambda(z) + \\ & + \sum_{k=k_\circ}^{\infty} \iint\limits_{A_k \setminus A_{k+1}} \Big(\log^+ \log^+ |G(z)|\Big)^{1+\varepsilon} \, d\lambda(z). \end{aligned} \tag{2.7}$$

In particular, we will use (2.7), when the growth conditions of a function $G$ near $K$ and the upper box dimension of $K$ are known and that $K$ has Lebesgue measure zero.

We fix the notations for the proceeding results in the following situation:

$$\underline{\textit{Situation } S1} \begin{cases} Q \subset \mathbb{C} \text{ is a closed square of side length } a; \\ K \subset Q \text{ is compact having upper box dimension } d; \\ Q_k,\ k \in \mathbb{N} \text{ is the covering of } Q \text{ having grids of side length} \\ a_k \text{ such that } a_k \to 0, \text{ as } k \to \infty; \\ N_k := N_{a_k} \text{ is the number of subsquares from } Q_k \\ \text{having non-empty intersection with } K; \\ G \text{ is a non-negative, continuous function on } Q \setminus K. \end{cases}$$

**2.2.1. Double exponential growth.** Now, we consider the case $(GC3)$, i.e. when a function $G$ satisfies the double exponential growth condition and the compact set $K$ has upper box dimension $d < 2$.

THEOREM 2.5. *Consider the situation $S1$ such that $K$ has upper box dimension $d < 2$. Assume that $G$ satisfies the following condition;*

$$|G(z)| \leq \exp\Big(\exp\Big(\frac{C}{\operatorname{dist}(z,K)^\alpha}\Big)\Big) \qquad (z \in Q \setminus K)$$

*where $C$ and $\alpha$ are positive constants and $\alpha < \min\{1, 2-d\}$. Then, for some $\varepsilon > 0$, (2.6) holds, i.e.*

$$I_Q = \iint\limits_Q \Big(\log^+ \log^+ |G(z)|\Big)^{1+\varepsilon} \, d\lambda(z) < \infty.$$

PROOF. Let $a_k := 2^{-k}a$ be the side length of the squares $Q_k$. As, $K$ has upper box dimension $d < 2$, we have from Remark 2.2 that $\lambda(K) = 0$. Thus, by the definition of upper box dimension, we have that:

$$\overline{\lim_{k\to\infty}} \frac{\log N_k}{\log \frac{1}{a_k}} = \overline{\lim_{k\to\infty}} \frac{\log N_k}{\log(\frac{2^k}{a})} \leq d < 2 - \alpha.$$

In particular, there is some $k_\circ \in \mathbb{N}$ such that,

$$\forall\ k \geq k_\circ : \quad \frac{\log N_k}{\log(\frac{2^k}{a})} \leq d + \delta,$$

where $\delta > 0$ is chosen such that $d + \delta < 2 - \alpha$. Hence, from (2.4) and the above inequality, we get that:

$$F_k = N_k a^2 2^{-2k} \leq a^{2-(d+\delta)} 2^{-k(2-(d+\delta))}, \qquad k \geq k_\circ. \tag{2.8}$$

For $k \geq k_\circ$ and a square $R$ from $Q_k$ with $\operatorname{int} R \cap K = \emptyset$, we have; $\operatorname{dist}(z,K) \geq \operatorname{dist}(z, \partial R)$. Thus, for $\varepsilon > 0$ with $d + \delta + \alpha(1+\varepsilon) < 2$ and $\alpha + \varepsilon < 1$, and from the growth condition we get:

$$\begin{aligned} I_R :=& \iint\limits_R \Big(\log^+ \log^+ |G(z)|\Big)^{1+\varepsilon} \, d\lambda(z) \\ \leq& \iint\limits_R \Big(\frac{C}{\operatorname{dist}(z,\partial R)^\alpha}\Big)^{1+\varepsilon} \, d\lambda(z) = C_1 \iint\limits_R \frac{1}{\operatorname{dist}(z,\partial R)^{\alpha(1+\varepsilon)}} \, d\lambda(z) \end{aligned}$$

where $C_1 = C^{1+\varepsilon}$. Using Lemma 3.7 [**4**], we get that:

$$\begin{aligned} I_R \leq& C_1 \frac{2^{1+\alpha(1+\varepsilon)} 2^{-k(2-\alpha(1+\varepsilon))} a^{2-\alpha(1+\varepsilon)}}{(2-\alpha(1+\varepsilon))(1-\alpha(1+\varepsilon))} \\ \leq& C_2 a^{2-\alpha(1+\varepsilon)} 2^{-k(2-\alpha(1+\varepsilon))} \end{aligned}$$

where $C_2$ is a positive constant independent of $k$. Thus, from (2.7) we obtain for $I_Q$ that:

$$\begin{aligned} I_Q \leq& C_2 a^{2-\alpha(1+\varepsilon)} \Big( (N_Q - N_{k_\circ}) 2^{-k_\circ(2-\alpha(1+\varepsilon))} + \sum_{k=k_\circ}^{\infty} (4N_k - N_{k+1}) 2^{-(k+1)(2-\alpha(1+\varepsilon))} \Big) \\ \leq& C_2 a^{-\alpha(1+\varepsilon)} 2^{k_\circ \alpha(1+\varepsilon)} (\lambda(A_Q) - F_{k_\circ}) + C_2 a^{-\alpha(1+\varepsilon)} \sum_{k=k_\circ}^{\infty} (F_k - F_{k+1}) 2^{(k+1)\alpha(1+\varepsilon)} \end{aligned}$$

Note that, for $\mathtt{K} > k_\circ$, we have:

$$
\begin{aligned}
S_{\mathtt{K}} :=& - F_{k_\circ} 2^{k_\circ \alpha(1+\varepsilon)} + \sum_{k=k_\circ}^{\mathtt{K}} (F_k - F_{k+1}) 2^{(k+1)\alpha(1+\varepsilon)} \\
=& \sum_{k=k_\circ}^{\mathtt{K}} F_k (2^{(k+1)\alpha(1+\varepsilon)} - 2^{k\alpha(1+\varepsilon)}) - F_{\mathtt{K}+1} 2^{(\mathtt{K}+1)\alpha(1+\varepsilon)} \\
=& \sum_{k=k_\circ}^{\mathtt{K}} F_k 2^{k\alpha(1+\varepsilon)} (2^{\alpha(1+\varepsilon)} - 1) - F_{\mathtt{K}+1} 2^{(\mathtt{K}+1)\alpha(1+\varepsilon)}
\end{aligned}
$$

By (2.8), we have for $k \geq k_\circ$,

$$
\begin{aligned}
F_k 2^{k\alpha(1+\varepsilon)} &\leq a^{2-(d+\delta)} 2^{-k(2-(d+\delta))} \cdot 2^{k\alpha(1+\varepsilon)} \\
&= c(a, d, \delta) 2^{-k[2-(d+\delta)-\alpha(1+\varepsilon)]}
\end{aligned}
$$

where $c$ is a positive constant depending on $a, d$ and $\delta$ only. Note that $d + \delta + \alpha(1+\varepsilon) < 2$. Hence, $\lim_{\mathtt{K}\to\infty} S_{\mathtt{K}}$ exists and

$$
I_Q \leq C_2 a^{-\alpha(1+\varepsilon)} \Big( \lambda(A_Q) 2^{k_\circ \alpha(1+\varepsilon)} + \lim_{\mathtt{K}\to\infty} S_{\mathtt{K}} \Big) < \infty.
$$

□

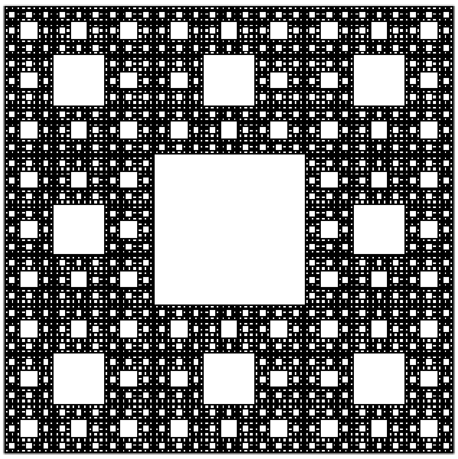

FIGURE 2.2. Sierpinski carpet $Q_s$ at 4th step

EXAMPLE 2.6. *Let $Q$ be a unit square and at some $p$-th step, $p \in \mathbb{N}$;*

$$
\begin{aligned}
\varepsilon_p =& \frac{1}{3^p} = \textit{side length of the squares} \;; \\
N_p =& 8^p = \textit{number of the remaining squares} \;.
\end{aligned}
$$

*Then, the resulting compact* `set` *$Q_s$ known as* `Sierpinski carpet`*, is a fractal set of Lebesgue measure zero and box dimension ≈ 1.89.*

PROOF. Subdivide $Q$ into $3^2$ equal subsquares and delete 1 square of side length $\frac{1}{3}$ (here we delete the central square). Subdivide the remaining $N_1 = 8$ subsquares into equal subsquares of side length $\frac{1}{3^2}$. Delete from each of them, the central subsquare, resulting $N_2 = 8^2$ remaining subsquares of side length $\frac{1}{3^2}$.

Continuing iteratively in the same way with the remaining subsquares, we get a decreasing sequence $\varepsilon_p = \frac{1}{3^p}$, which converges to zero as $p$ tends to $\infty$. Hence,

$$\lim_{p\to\infty} \frac{\log N_p}{\log 1/\varepsilon_p} = \lim_{p\to\infty} \frac{\log 8^p}{\log 3^p} = \frac{\log 8}{\log 3}.$$

It is clear from (2.2) that $\dim_H(Q_s) \leq \frac{\log 8}{\log 3} < 2$ and thus from (2.1), $Q_s$ has Lebesgue measure zero. □

In the following subsections, we will consider that a compact set $K$ satisfies some grid dimension condition. Since, these conditions will be used frequently throughout the discussion, we first set the relevant notations and define the conditions.

We will use the notations from the situation $S1$ and in addition, denote and define the *grid dimension function* of a set $K$ as:

$$D(a_k) := \frac{\log N_k}{\log \frac{1}{a_k}}, \quad k \in \mathbb{N}.$$

Using the above notation along with the notations in situation $S1$, we define the types of grid dimension condition as follows:

Grid dimension Condition of type $(2, \delta)$

Let $a_k := 2^{-k(k+1)/2}a,\ k \in \mathbb{N}$. Then, the *grid dimension condition* will be of *type* $(2, \delta)$ if, there exists $\delta > 0$ such that:

$$\forall k \geq k_1, \quad D(a_k) = \frac{\log N_k}{\log \frac{1}{a_k}} \leq 2 - (1+\delta)\frac{\log\log \frac{1}{a_k}}{\log \frac{1}{a_k}}. \tag{2.9}$$

Grid dimension Condition of type $(3, \delta)$

Consider the side length $a_k := 2^{-2^k}a,\ k \in \mathbb{N}$. Then, we call the *grid dimension condition* of *type* $(3, \delta)$ if, there exists $\delta > 0$ such that:

$$\forall k \geq k_1, \quad D(a_k) = \frac{\log N_k}{\log \frac{1}{a_k}} \leq 2 - (1+\delta)\frac{\log(\log\log \frac{1}{a_k})}{\log \frac{1}{a_k}}. \tag{2.10}$$

Grid dimension Condition of type $(1, \alpha)$

For the side length $a_k := 2^{-k}a,\ k \in \mathbb{N}$, the *grid dimension condition* will be called of *type* $(1, \alpha)$ if, there exists $C > 0$ and $\alpha \geq 0$ such that:

$$\forall k \geq k_1, \quad D(a_k) = \frac{\log N_k}{\log \frac{1}{a_k}} \leq 1 + \frac{\log(Ck^\alpha)}{\log \frac{1}{a_k}}. \tag{2.11}$$

**2.2.2. Exponential growth.** This section deals with the case, when a function $G$ satisfies the exponential growth condition $(GC2)$ and the compact set $K$ has upper box dimension 2.

THEOREM 2.7. *Consider the situation* $S1$. *Assume that there exists* $\delta > 0$ *such that* $K$ *satisfies grid dimension condition of type* $(2, \delta)$ *and that* $G$ *satisfies the condition of the form:*

$$|G(z)| \leq \exp\Big(\frac{C}{\operatorname{dist}(z,K)^{\alpha}}\Big), \qquad (z \in Q \setminus K)$$

*where* $C$ *and* $\alpha$ *are positive constants. Then, for some* $\varepsilon > 0$, (2.6) *holds, i.e.*

$$I_Q = \iint\limits_{Q} \Big(\log^+ \log^+ |G(z)|\Big)^{1+\varepsilon} \, d\lambda(z) < \infty.$$

PROOF. Since for each $k \in \mathbb{N}$, the subsquares have side length $a_k = 2^{-\frac{k(k+1)}{2}} a$, we have from (2.4) that; $F_k = N_k 2^{-k(k+1)} a^2$.

By (2.9), there exists $k_\circ \in \mathbb{N}$, such that:

$$\forall\ k \geq k_\circ : \quad F_k \leq \tilde{\tilde{C}} \cdot \frac{1}{k^{2(1+\delta)}} \text{ and } \ a_k = 2^{-\frac{k(k+1)}{2}} a \leq 1. \tag{2.12}$$

Same as in the proof of Theorem 2.5, we observe that for $k \geq k_\circ$ we have, $\operatorname{dist}(z,K) \geq \operatorname{dist}(z, \partial R)$, where $R$ is a square from $Q_k$ such that $\operatorname{int} R \cap K = \emptyset$. Then, for some $\varepsilon > 0$, such that $\varepsilon < \delta$ and from the growth condition of the function $G$, we have:

$$\begin{aligned} I_R = & \iint\limits_{R} \Big(\log^+ \log^+ |G(z)|\Big)^{1+\varepsilon} \, d\lambda(z) \\ \leq & \iint\limits_{R} \Big(\log(e+C) + \log \frac{1}{\operatorname{dist}(z,\partial R)^{\alpha}}\Big)^{1+\varepsilon} \, d\lambda(z) \\ \leq & C_1 \iint\limits_{R} \Big(\log \frac{1}{\operatorname{dist}(z,\partial R)^{\alpha}}\Big)^{1+\varepsilon} \, d\lambda(z) \end{aligned}$$

where $C_1$ is a positive constant depending on $\varepsilon$ only. Applying Lemma 2.4 with $h(x) := \big(\log x^{\alpha}\big)^{1+\varepsilon}$ and for some $\delta_\circ > 0$, we get that:

$$I_R \leq C_1 \Big[2a_k^2 \big\{I(h) + h\big(\frac{2}{a_k^{1+\delta_\circ}}\big)\big\}\Big] \tag{2.13}$$

where $I(h) = \int_0^1 h\Big(\frac{2}{t^{1+1/\delta_\circ}}\Big) \, dt = \int_0^1 \Big(\log\big(\frac{2}{t^{1+1/\delta_\circ}}\big)^{\alpha}\Big)^{1+\varepsilon} \, dt = M < \infty$ by (2.3) and

$$\begin{aligned} h\big(\frac{2}{a_k^{1+\delta_\circ}}\big) = & \Big(\log \frac{2^{\alpha}}{a_k^{\alpha(1+\delta_\circ)}}\Big)^{1+\varepsilon} = \Big(\log \frac{2^{\alpha} 2^{\alpha(1+\delta_\circ)(k^2+k)/2}}{a^{\alpha(1+\delta_\circ)}}\Big)^{1+\varepsilon} \\ = & \Big[\alpha \log 2 + \alpha(1+\delta_\circ)\big\{\big(\frac{k^2+k}{2}\big)\log 2 + \log\frac{1}{a}\big\}\Big]^{1+\varepsilon} \\ = & \big[c + c_\circ(k^2+k)\big]^{1+\varepsilon} \leq \tilde{C}(k(k+1))^{1+\varepsilon} \end{aligned}$$

where $\tilde{C}, c$ and $c_\circ$ are positive constants independent of $k$. Putting the above values in (2.13), we get:

$$\begin{aligned} I_R \leq & C_1\big[2 \cdot 2^{-k(k+1)} a^2 \{M + \tilde{C}(k(k+1))^{1+\varepsilon}\}\big] \\ \leq & C_2 a^2 2^{-k(k+1)} (k(k+1))^{1+\varepsilon} \end{aligned}$$

where $C_2$ is a positive constant independent of $k$. Thus, from (2.7), we obtain for $I_Q$ that:

$$\begin{aligned}
I_Q \leq & (N_Q - N_{k_\circ})C_2 a^2 2^{-k_\circ(k_\circ+1)}(k_\circ(k_\circ+1))^{1+\varepsilon} + \\
& + \sum_{k=k_\circ}^{\infty} (2^{2(k+1)}N_k - N_{k+1})C_2 a^2 2^{-(k+1)(k+2)}((k+1)(k+2))^{1+\varepsilon} \\
\leq & C_2\lambda(A_\Omega)(k_\circ(k_\circ+1))^{1+\varepsilon} - C_2 F_{k_\circ}(k_\circ(k_\circ+1))^{1+\varepsilon} + \\
& + \sum_{k=k_\circ}^{\infty} C_2(F_k - F_{k+1})((k+1)(k+2))^{1+\varepsilon}
\end{aligned}$$

Note that, for $\mathtt{K} > k_\circ$, we have:

$$\begin{aligned}
S_{\mathtt{K}} := & - F_{k_\circ}(k_\circ(k_\circ+1))^{1+\varepsilon} + \sum_{k=k_\circ}^{\mathtt{K}} (F_k - F_{k+1})((k+1)(k+2))^{1+\varepsilon} \\
= & \sum_{k=k_\circ}^{\mathtt{K}} F_k((k+1)(k+2))^{1+\varepsilon} - F_k(k(k+1))^{1+\varepsilon} - F_{\mathtt{K}+1}((\mathtt{K}+1)(\mathtt{K}+2))^{1+\varepsilon} \\
= & \sum_{k=k_\circ}^{\mathtt{K}} F_k(k+1)^{1+\varepsilon}((k+2)^{1+\varepsilon} - k^{1+\varepsilon}) - F_{\mathtt{K}+1}((\mathtt{K}+1)(\mathtt{K}+2))^{1+\varepsilon}
\end{aligned}$$

Applying Mean Value Theorem to above, we have that for some $\xi \in (k, k+2)$,

$$\begin{aligned}
(k+2)^{1+\varepsilon} - k^{1+\varepsilon} = & 2(1+\varepsilon)\xi^{\varepsilon} \leq 2(1+\varepsilon)(k+2)^{\varepsilon} \\
\leq & C_\circ(1+\varepsilon)(k+1)^{\varepsilon} = \tilde{C}_\circ(k+1)^{\varepsilon}
\end{aligned}$$

Thus,

$$\begin{aligned}
S_{\mathtt{K}} \leq & \sum_{k=k_\circ}^{\mathtt{K}} \tilde{C}_\circ F_k(k+1)^{1+2\varepsilon} - F_{\mathtt{K}+1}((\mathtt{K}+1)(\mathtt{K}+2))^{1+\varepsilon} \\
\leq & \sum_{k=k_\circ}^{\mathtt{K}} C' F_k k^{1+2\varepsilon} - F_{\mathtt{K}+1}((\mathtt{K}+1)(\mathtt{K}+2))^{1+\varepsilon}
\end{aligned}$$

From (2.12), we have that for $k \geq k_\circ$ and $\delta_\circ := \delta - \varepsilon$ that:

$$C' F_k k^{1+2\varepsilon} \leq C'' \frac{k^{1+2\varepsilon}}{k^{2+2\delta}} = C'' \frac{1}{k^{1+2\delta-2\varepsilon}} \leq \frac{C''}{k^{1+\delta_\circ}}.$$

Hence, $\lim_{\mathtt{K}\to\infty} S_{\mathtt{K}}$ exists and

$$I_Q \leq \lambda(A_Q)C_2(k_\circ(k_\circ+1))^{1+\varepsilon}) + C_2 \lim_{\mathtt{K}\to\infty} S_{\mathtt{K}} < \infty.$$

□

Remark 2.8. *It is clear from the grid dimension condition of type* $(2,\delta)$ *that* $\lambda(K) = 0$.

PROOF. In deed, from (2.9), we see that:

$$\log N_k \le \log \frac{1}{a_k^2(\log \frac{1}{a_k})^{1+\delta}}$$

$$N_k a_k^2 \le \frac{1}{(\log \frac{1}{a_k})^{1+\delta}}.$$

The right hand side of the above inequality tends to 0 as $k \to \infty$ and thus,

$$\lambda(K) \le N_k a_k^2 \to 0, \text{ for } k \to \infty$$

implying that $\lambda(K) = 0$ under condition (2.9). □

Now, we provide an example of a compact set $K$ where the above result can be applied.

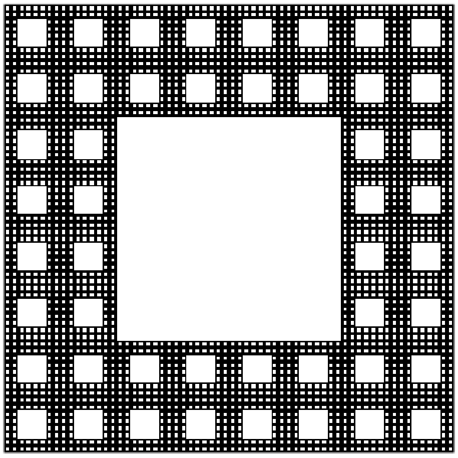

FIGURE 2.3. Fractal set $Q(m)$ at 3rd step for $m = 8$

EXAMPLE 2.9. *Let $Q$ be a closed unit square and the side lengths of grids at some $p$-th step are given by;*

$$\varepsilon_p = \frac{1}{m^{\sum_{j=1}^{p} j}} = \frac{1}{m^{\frac{p(p+1)}{2}}}, \qquad p \in \mathbb{N},\ m \in \mathbb{Z}^+;$$

$$N_p = m^{p^2} n^p = \text{number of squares left of side length } \varepsilon_p \text{ in } Q,\ n < m.$$

*Then, the resulting compact set $Q(m)$ is a fractal set of box dimension 2 and Lebesgue measure* 0.

PROOF. For $m = 8$, subdivide $Q$ into $8^2$ equal sub-squares of side length $\varepsilon_1 = \frac{1}{8}$. Delete a square of side length $= \frac{1}{2}$ (of $Q$). Divide each of the remaining subsquares $N_1 = 6 \cdot 8$ of side length $\varepsilon_1$, into equal subsquares of side length $\varepsilon_2 = \frac{1}{8^2} \cdot \varepsilon_1 = \frac{1}{8^3}$. Delete from each of the remaining subsquares $N_1$, a square of side length $\frac{1}{2} \cdot \varepsilon_1 = \frac{1}{2} \cdot \frac{1}{8}$. Now, from the remaining subsquares $N_2 = 8^{2^2} 6^2$ of side length $\varepsilon_2$, divide each of them into equal subsquares of side length $\varepsilon_3 = \frac{1}{8^3} \cdot \varepsilon_2 = \frac{1}{8^6}$. Delete from each of them a square of side length $\frac{1}{2} \cdot \varepsilon_2 = \frac{1}{2} \cdot \frac{1}{8^3}$.

Proceeding inductively, in the same way with the remaining subsquares, we get that for $p \in \mathbb{N}$ at some $p-th$ step, a decreasing sequence $\varepsilon_p$ converging to zero, for $p$ tending to $\infty$. Hence,

$$\begin{aligned}
\lim_{p\to\infty} \frac{\log N_p}{\log \frac{1}{\varepsilon_p}} &= \lim_{p\to\infty} \frac{\log 8^{p^2} 6^p}{\log 8^{\frac{p(p+1)}{2}}} \\
&= \lim_{p\to\infty} \frac{p^2 \log 8 + p \log 6}{\frac{p(p+1)}{2} \log 8} \\
&= \lim_{p\to\infty} \left[ \frac{2p^2}{p^2+p} + \frac{2\log 6}{(p+1)\log 8} \right] = 2.
\end{aligned}$$

To show that $Q(m)$ has Lebesgue measure zero, we first denote;

$$n_p = \text{ total number of squares removed of side length } a_p.$$

From the construction above, we notice that for $Q(m)$,

$$n_p = 8^{(p-1)^2} 6^{p-1} \quad \text{and} \quad a_p = \frac{1}{2} \cdot \varepsilon_{p-1} = \frac{1}{2} \cdot \frac{1}{8^{\frac{p(p-1)}{2}}}.$$

Hence, the total area removed

$$\begin{aligned}
&= \sum_{p=1}^{\infty} n_p a_p^2 = \sum_{p=1}^{\infty} 8^{(p-1)^2} 6^{p-1} \times \left( \frac{1}{2} \cdot \varepsilon_{p-1} \right)^2 \\
&= \left( \frac{1}{2} \cdot \frac{1}{8^{\frac{p(p-1)}{2}}} \right)^2 = 1
\end{aligned}$$

which shows that $Q(m)$ has Lebesgue measure zero.

□

**2.2.3. Finite order growth.** This section deals with the case when a function $G$ satisfies the finite order growth condition $(GC1)$ and the compact set $K$ has upper box dimension 2.

THEOREM 2.10. *Consider the situation $S1$ and assume that there exists $\delta > 0$, such that the set $K$ satisfies the grid dimension condition of type $(3, \delta)$. Moreover, suppose that $G$ satisfies the growth condition of the form:*

$$|G(z)| \le \frac{C}{\operatorname{dist}(z, K)^{\alpha}} \qquad (z \in Q \setminus K)$$

*where $C$ and $\alpha$ are positive constants with $\alpha \ge 1$. Then, for some $\varepsilon > 0$, (2.6) holds, i.e.*

$$I_Q = \iint_Q \left( \log^+ \log^+ |G(z)| \right)^{1+\varepsilon} d\lambda(z) < \infty.$$

PROOF. Since for each $k \in \mathbb{N}$, the subsquares have side length $a_k = 2^{-2^k} a$, which gives from (2.4) that: $F_k = N_k 2^{-2^{k+1}} a^2$.

By (2.10), there exists $k_\circ \in \mathbb{N}$, such that:

$$\forall\, k \ge k_\circ : \quad F_k \le \tilde{\tilde{C}} \cdot \frac{1}{k^{1+\delta}} \text{ and } a_k = 2^{-2^k} a \le 1. \tag{2.14}$$

Notice that for $k \geq k_\circ$ and a square $R$ from $Q_k$ with $\operatorname{int} R \cap K = \emptyset$; $\operatorname{dist}(z, K) \geq \operatorname{dist}(z, \partial R)$. Then, from the growth condition and for some $\varepsilon > 0$, such that $\varepsilon < \delta$, we have:

$$\begin{aligned} I_R =& \iint\limits_R \Big( \log^+ \log^+ |G(z)| \Big)^{1+\varepsilon} \, d\lambda(z) \\ \leq& \iint\limits_R \Big( \tilde{C_1} + \log\log \frac{1}{\operatorname{dist}(z, \partial R)^\alpha} \Big)^{1+\varepsilon} \, d\lambda(z) \end{aligned}$$

where $\tilde{C_1} := \log(2\log(e + C))$. Thus,

$$I_R \leq C_1 \iint\limits_R \Big( \log\log \frac{1}{\operatorname{dist}(z, \partial R)^\alpha} \Big)^{1+\varepsilon} \, d\lambda(z)$$

where $C_1$ is a positive constant depending on $\varepsilon$ only. Applying Lemma 2.4 with $h(x) := \big( \log\log x^\alpha \big)^{1+\varepsilon}$, we get for some $\delta_\circ > 0$ that:

$$I_R \leq C_1 \Big[ 2a_k^2 \big\{ I(h) + h\big( \frac{2}{a_k^{1+\delta_\circ}} \big) \big\} \Big] \tag{2.15}$$

where $I(h) = \int_0^1 h\big(\frac{2}{t^{1+1/\delta_\circ}}\big) \, dt = \int_0^1 \big( \log\log \frac{2^\alpha}{t^{(1+1/\delta_\circ)\alpha}} \big)^{1+\varepsilon} \, dt = M < \infty$ by (2.3) and

$$\begin{aligned} h\Big( \frac{2}{a_k^{1+\delta_\circ}} \Big) =& \Big( \log\log \frac{2^\alpha}{a_k^{\alpha(1+\delta_\circ)}} \Big)^{1+\varepsilon} = \Big( \log\log \big( \frac{2^\alpha 2^{\alpha(1+\delta_\circ)2^k)}}{a^{\alpha(1+\delta_\circ)}} \big) \Big)^{1+\varepsilon} \\ =& \Big( \log \big( \alpha \log 2 + \alpha(1+\delta_\circ) \big\{ 2^k \log 2 + \log(1/a) \big\} \big) \Big)^{1+\varepsilon} \\ \leq& (\log(c + c_\circ 2^k))^{1+\varepsilon} \leq \big( k \log 2 + \tilde{C_2} \big)^{1+\varepsilon} \leq C_2 k^{1+\varepsilon} \end{aligned}$$

Putting the above values in (2.15), we get:

$$\begin{aligned} I_R \leq& C_1 \big( 2 \cdot 2^{-2^{k+1}} a^2 \big[ M + C_2 k^{1+\varepsilon} \big] \big) \\ \leq& \tilde{c_1} 2^{-2^{k+1}} a^2 + \tilde{C_2} 2^{-2^{k+1}} a^2 k^{1+\varepsilon} \\ \leq& C_3 a^2 2^{-2k+1} k^{1+\varepsilon} \end{aligned}$$

where $C_3$ is a positive constant independent of $k$. Thus, from (2.7), we obtain for $I_Q$ that:

$$\begin{aligned} I_Q \leq& (N_Q - N_{k_\circ}) 2^{-2^{k_\circ+1}} C_3 a^2 k_\circ^{1+\varepsilon} + \\ &+ \sum_{k=k_\circ}^{\infty} (2^{2^{k+1}} N_k - N_{k+1}) 2^{-2^{k+2}} C_3 a^2 (k+1)^{1+\varepsilon} \\ \leq& C_3 \lambda(A_\Omega) k_\circ^{1+\varepsilon} - C_3 F_{k_\circ} k_\circ^{1+\varepsilon} + \\ &+ \sum_{k=k_\circ}^{\infty} C_3 F_k (k+1)^{1+\varepsilon} - C_3 F_{k+1} (k+1)^{1+\varepsilon} \end{aligned}$$

Note that, for $\mathtt{K} > k_\circ$, we have:

$$
\begin{aligned}
S_{\mathtt{K}} :=& - F_{k_\circ} k_\circ^{1+\varepsilon} + \sum_{k=k_\circ}^{\mathtt{K}} (F_k - F_{k+1})(k+1)^{1+\varepsilon} \\
=& \sum_{k=k_\circ}^{\mathtt{K}} F_k((k+1)^{1+\varepsilon} - k^{1+\varepsilon}) - F_{\mathtt{K}+1}(\mathtt{K}+1)^{1+\varepsilon}
\end{aligned}
$$

Applying Mean Value Theorem to above, we have that for some $\xi \in (k, k+1)$:

$$
(k+1)^{1+\varepsilon} - k^{1+\varepsilon} =(1+\varepsilon)\xi^{\varepsilon} \leq C_\circ (k+1)^{\varepsilon} \leq \tilde{C} k^{\varepsilon}
$$

Thus,

$$
S_{\mathtt{K}} \leq \sum_{k=k_\circ}^{\mathtt{K}} F_k \tilde{C} k^{\varepsilon} - F_{\mathtt{K}+1}(\mathtt{K}+1)^{1+\varepsilon}
$$

From (2.14), we have that for $k \geq k_\circ$, $\delta > 0$,

$$
F_k \tilde{C} k^{\varepsilon} \leq \frac{C_4}{k^{1+\delta-\varepsilon}} \leq \frac{C_4}{k^{1+\delta_\circ}}
$$

Hence, $\lim_{\mathtt{K}\to\infty} S_{\mathtt{K}}$ exists and

$$
I_Q \leq \lambda(A_Q) C_3 k_\circ^{1+\varepsilon} + C_3 \lim_{\mathtt{K}\to\infty} S_{\mathtt{K}} < \infty.
$$

□

REMARK 2.11. *Same as in Remark 2.8, we observe that grid dimension condition of type* $(3,\delta)$ *implies* $\lambda(K) = 0$.

PROOF. In fact, from (2.10), we see that:

$$
\begin{aligned}
\log N_k &\leq \log\Big(\frac{1}{a_k^2(\log\log\frac{1}{a_k})^{1+\delta}}\Big) \\
N_k a_k^2 &\leq \frac{1}{(\log\log\frac{1}{a_k})^{1+\delta}}.
\end{aligned}
$$

Clearly, right hand side of the above inequality tends to 0, as $k \to \infty$ and hence,

$$
\lambda(K) \leq N_k a_k^2 \to 0, \text{ for } \ k \to \infty.
$$

□

EXAMPLE 2.12. *Consider a unit square* $Q$*. Let* $m \in \mathbb{Z}^+$ *and* $\varepsilon_p$ *and* $N_p,\ p \in \mathbb{N}$ *are given by;*

$$
\begin{aligned}
\varepsilon_p =& \frac{1}{m^{2^{p-1}}} = \textit{side length of the squares} \ ; \\
N_p =& (\frac{3}{4})^p \cdot m^{2^p} = \textit{number of subsquares left in } Q \textit{ of side length } \ \varepsilon_p.
\end{aligned}
$$

*Then, we obtain a compact fractal set* $Q_\infty$ *of Lebesgue measure zero and box dimension 2.*

Proof. Let $m = 8$. Subdivide $Q$ into $8^2$ equal subsquares and delete a square of side length $a_1 = \frac{1}{2}$ (of $Q$). Divide each of the remaining subsquares $N_1 = 6 \cdot 8 = \frac{3}{4}8^2$ into equal subsquares of side length $\frac{1}{8^2}$. Remove from each of them, a square of side length $a_2 = \frac{1}{2}\varepsilon_1 = \frac{1}{2}\frac{1}{8}$. Thus, at the second step, we are left with $N_2 = (\frac{3}{4})^2 8^{2^2}$ subsquares of side length $\varepsilon_2 = \frac{1}{8^2}$. Divide each of the remaining $N_2$ subsquares into equal subsquares of side length $\varepsilon_3 = \frac{1}{8^{2^2}}$. Delete from each of them a subsquare of side length $a_3 = \frac{1}{2} \cdot \varepsilon_2 = \frac{1}{2} \cdot \frac{1}{8^2}$.

Continuing in this manner, at some $p - th$ step, for $p \in \mathbb{N}$, we obtain a decreasing sequence of side lengths $\varepsilon_p$ converging to zero. Hence,

$$\begin{aligned}\lim_{p\to\infty} \frac{\log N_p}{\log \frac{1}{\varepsilon_p}} &= \lim_{p\to\infty} \frac{\log(\frac{3}{4})^p \cdot 8^{2^p}}{\log 8^{2^{p-1}}} \\ &= \lim_{p\to\infty} \frac{p \log \frac{3}{4} + 2^p \log 8}{2^{p-1} \log 8} = 2.\end{aligned}$$

Observe that at some $p$-th step, $p \in \mathbb{N}$; the number of squares removed =

$$n_p = (\frac{3}{4})^{p-1} \cdot 8^{2^{p-1}}, \text{ of side length } a_p = \frac{1}{2} \cdot \varepsilon_{p-1} = \frac{1}{2} \cdot \frac{1}{8^{2^{p-2}}}.$$

Therefore, the total area removed will be $\sum_{p=1}^{\infty} n_p a_p^2 = 1$. Thus, showing that $Q_\infty$ has Lebesgue measure zero.

□

**2.2.4. Box dimension near to 1.** Here we deal with the sets having upper box dimension $d \geq 1$ and the function satisfies a special growth condition which is weaker than the double exponential one.

Theorem 2.13. *Let the situation $S1$ hold and assume that there exists $C > 0$ and $\alpha \geq 0$ such that the set $K$ satisfies the grid dimension condition of type $(1, \alpha)$. Let $\varepsilon > 0$ and define $f(t) := t(\log^+ t)^{1+\varepsilon}$, for $t \geq 0$ satisfying condition $(\eta)$. Moreover, suppose that for $C_1 > 0$ and some $\beta > 3 + \alpha$, a function $G$ satisfies the following growth condition;*

$$|G(z)| \leq \exp\Big(\exp\Big(\frac{C_1}{\operatorname{dist}(z,K)\big(\log \frac{1}{\operatorname{dist}(z,K)}\big)^\beta}\Big)\Big) \quad (z \in Q \setminus K).$$

*Then,*

$$I_Q = \iint\limits_Q f\big(\log^+ \log^+ |G(z)|\big)\, d\lambda(z) < \infty.$$

Proof. It is evident from the grid dimension condition of type $(1, \alpha)$ that $\overline{\dim}_B(K) < 2$ and thus from Remark 2.2, we get that $\lambda(K) = 0$. By the growth condition of the function $G$ and the monotonicity of the function $f$, we have that:

$$f\big(\log^+ \log^+ |G(z)|\big) \leq f\Big(\frac{C_1}{\operatorname{dist}(z,K)\big(\log \frac{1}{\operatorname{dist}(z,K)}\big)^\beta}\Big).$$

Observe that for $k \geq k_\circ$ we have $\operatorname{dist}(z, K) \geq \operatorname{dist}(z, \partial R)$, where $R$ is a square from $Q_k$ with $\operatorname{int} R \cap K = \emptyset$. Thus, for some $\varepsilon > 0$ with $\beta > 3 + \alpha + \varepsilon$ we obtain:

$$\begin{aligned} I_R = & \iint_R f\big(\log^+ \log^+ |G(z)|\big)\ d\lambda(z) \\ \leq & \iint_R \Big( \frac{C_1}{\operatorname{dist}(z, \partial R)\big(\log \frac{1}{\operatorname{dist}(z,\partial R)}\big)^\beta} \Big) \cdot \\ & \cdot \Big( \log^+ \Big( \frac{C_1}{\operatorname{dist}(z, \partial R)\big(\log \frac{1}{\operatorname{dist}(z,\partial R)}\big)^\beta} \Big)\Big)^{1+\varepsilon}\ d\lambda(z) \end{aligned}$$

By an appropriate choice of coordinates from the proof of Lemma 2.4, we have:

$$\begin{aligned} I_R \leq & 8 \int_0^{a_k/2} (\frac{a_k}{2} - x) \frac{C_1}{x\big(\log \frac{1}{x}\big)^\beta} \Big( \log \frac{C_1}{x\big(\log \frac{1}{x}\big)^\beta} \Big)^{1+\varepsilon}\ dx \\ \leq & 2a_k^2 \int_0^1 (\frac{2C_1}{a_k t (\log(2/a_k t))^\beta}) \Big( \log (\frac{2C_1}{a_k t (\log(2/a_k t))^\beta}) \Big)^{1+\varepsilon}\ dt \\ = & 4a_k C_1 \int_0^1 (\frac{1}{t(\log(2/a_k t))^\beta}) \Big( \log \frac{2}{a_k t} + \log C_1 + \log \frac{1}{(\log 2/a_k t)^\beta} \Big)^{1+\varepsilon}\ dt \\ = & 4a_k C_1 \int_{\log 2/a_k}^\infty \frac{1}{u^\beta} \Big( u + \log C_1 + \log \frac{1}{u^\beta} \Big)^{1+\varepsilon}\ du \end{aligned}$$

Since, $a_k < a_\circ : \log C_1 < \log(2/a_k)$, which gives:

$$\begin{aligned} I_R \leq & 8 a_k 2^{1+\varepsilon} C_1 \beta^{1+\varepsilon} \int_{\log 2/a_k}^\infty \frac{u^{1+\varepsilon}}{u^\beta}\ du \\ \leq & C_2 a_k (\log \frac{2}{a_k})^{2+\varepsilon-\beta} \\ = & C_2 2^{-k} a (\log(2^{1+k}/a))^{2+\varepsilon-\beta} \leq C_3 2^{-k} a k^{2+\varepsilon-\beta} \end{aligned}$$

where $C_2$ and $C_3$ are independent of $k$. Thus, from (2.7), we obtain for $I_Q$ that:

$$I_Q \leq (N_Q - N_{k_\circ}) C_3 2^{-k_\circ} a k_\circ^{2+\varepsilon-\beta} + \sum_{k=k_\circ}^\infty (4N_k - N_{k+1}) C_3 2^{-(k+1)} a (k+1)^{2+\varepsilon-\beta}$$

Note that, for $\mathtt{K} > k_\circ$, we have:

$$\begin{aligned} S_{\mathtt{K}} := & - N_{k_\circ} 2^{-k_\circ} a k_\circ^{2+\varepsilon-\beta} + \sum_{k=k_\circ}^{\mathtt{K}} (4N_k - N_{k+1}) 2^{-(k+1)} a (k+1)^{2+\varepsilon-\beta} \\ = & - N_{k_\circ} 2^{-k_\circ} a k_\circ^{2+\varepsilon-\beta} + \sum_{k=k_\circ}^{\mathtt{K}} (2 \cdot 2^{-k} N_k - N_{k+1} 2^{-(k+1)}) a (k+1)^{2+\varepsilon-\beta} \\ = & \sum_{k=k_\circ}^{\mathtt{K}} N_k 2^{-k} a [2(k+1)^{2+\varepsilon-\beta} - k^{2+\varepsilon-\beta}] - N_{\mathtt{K}+1} 2^{-(\mathtt{K}+1)} a (\mathtt{K}+1)^{2+\varepsilon-\beta} \\ = & \sum_{k=k_\circ}^{\mathtt{K}} N_k 2^{-k} a k^{2+\varepsilon-\beta} \Big[ 2 (\frac{k+1}{k})^{2+\varepsilon-\beta} - 1 \Big] - N_{\mathtt{K}+1} 2^{-(\mathtt{K}+1)} a (\mathtt{K}+1)^{2+\varepsilon-\beta} \end{aligned}$$

For $k \geq k_\circ$, $\left(2\left(\frac{k+1}{k}\right)^{2+\varepsilon-\beta} - 1\right) < \frac{1}{2}$, we have from (2.11) that:

$$N_k 2^{-k} a k^{2+\varepsilon-\beta} \leq C k^{2+\alpha+\varepsilon-\beta}.$$

Note that $\beta > 3 + \alpha + \varepsilon$, which gives that $\lim_{\mathtt{K}\to\infty} S_{\mathtt{K}}$ exists and

$$I_Q \leq N_Q C_3 2^{-k_\circ} a k_\circ^{2+\varepsilon-\beta} + C_3 \lim_{\mathtt{K}\to\infty} S_{\mathtt{K}} < \infty.$$

$\square$

Observe that, if $K$ is a rectifiable arc of length $l$, then it satisfies grid dimension condition of type $(1, \alpha)$ for $\alpha = 0$.
Indeed, for $\alpha = 0$, (2.11) becomes:

$$\begin{aligned} \log N_k \leq & \log \frac{1}{a_k} + \log C \\ \Rightarrow & N_k a_k \leq C \end{aligned}$$

where $C$ can be the treated as the length of $K$ by [**30**] (see (5.1) and proof of Proposition 5.4).

## 2.3. Further criteria of integrability

In this section, we consider the situation when a function $G$ satisfies some growth condition near a given thin, compact set $K$ and obtain the integrability criteria of a function $G$ using variant of Domar's Theorem (Theorem 1.6). In particular, we consider a closed square $Q \subset \mathbb{C}$ of side length $a$ and a compact subset $K$ of $Q$ such that $\lambda(K) = 0$. We give a general construction of the covering of $K$ by following a method as given in [**4**] (after Lemma 3.7).

Let $Q \subset \mathbb{C}$ be a closed square of side length $a$ and $K \subset Q$ be a compact set. Let $\{m_i : i, \in \mathbb{N}\}$ be a sequence of positive integers. Subdivide the square $Q$ into a family $Q_1$ of $m_1^2$ closed congruent subsquares $q$. Let $Q_1^*$ be the set of all closed subsquares $q \in Q_1$ having non-empty intersection with $K$, i.e. $Q_1^* = \{q \in Q_1 | q \cap K \neq \emptyset\}$. Define,

$$K_1 := \cup_{q \in Q_1^*} q$$

be the compact subset obtained by the union of all $q \in Q_1^*$. Denote by $n_1$, the number of removed subsquares $q \in Q_1$.

Divide each of the remaining $m_1^2 - n_1$ subsquares of side length $\frac{a}{m_1}$ into a family $Q_2$ of $m_2^2$ closed congruent subsquares. Similar as in the first step, we define $Q_2^*$ the set of all closed subsquares $q \in Q_2$ of side length $\frac{a}{m_1 m_2}$ having non-empty intersection with $K$, and obtain the compact set $K_2$ as:

$$K_2 := \cup_{q \in Q_2^*} q.$$

Let $n_2$ denote the number of removed subsquares of side length $\frac{a}{m_1 m_2}$. Continuing iteratively, we obtain a compact set $K_\infty := \cap_{j=1}^\infty K_j = K$. In fact, for some $z \notin K$, $\exists$ $j$ such that $a_j := \frac{a}{\prod_{k=1}^j m_k} < \frac{1}{2}\operatorname{dist}(z, K)$. Let $\tilde{Q}_j := \left\{ q \text{ of side length } \frac{a}{\prod_{k=1}^j m_k} \right\}$ such that $\exists$ $q_* \in \tilde{Q}_j$ with $z \in q_*$. Thus, $q_* \cap K = \emptyset$, as $\operatorname{diam} q_* = \frac{\sqrt{2}a}{\prod_{k=1}^{j} m_k}$. Also, $\frac{1}{m_j} \to 0$, as $j \to \infty$. Thus, the total area removed is $\sum_{j=1}^{\infty} n_j a_j^2 = \sum_{j=1}^{\infty} n_j \frac{a^2}{\prod_{k=1}^{j} m_k^2} = a^2$.

Throughout the section, we fix the following notations from the above construction:

Notations

$$\begin{aligned}
&Q = \text{closed square of side length } \ a, ; \\
&K = \text{compact subset of } \ Q \text{ having } \lambda(K) = 0; \\
&n_l = \text{number of subsquares removed from } Q \text{ at some } l - th \text{ step }, \ l \in \mathbb{N}; \\
&a_l = \frac{a}{\prod_{k=1}^{l} m_k} = \text{side length of subsquares, where} \\
&\qquad \frac{1}{m_k}, 1 \leq k \leq l \text{ is the side length at some } k - th \text{ step, respectively }.
\end{aligned}$$

Next, we use this construction and Domar's variant to obtain a general scheme under which the integral (2.6) holds. For this, we recall some conditions in the following situation.

$$\underline{\textit{Situation } S2}\begin{cases}
Q, \ K, \ n_l \text{ and } a_l \text{ as in Notations };\\
G \text{ is a non-negative, continuous function on } Q \setminus K;\\
f \text{ is a function satisfying condition } (\eta);\\
g : (0,\infty) \to (0,\infty) \text{ is a monotone increasing function };\\
h \text{ is a non-negative, monotone, increasing function defined by,}\\
h(x) := f(g(x)) \text{ such that for some } \delta > 0, \text{ the following integral exists;}\\
I(h) = \int_0^1 h\Big(\frac{2}{t^{1+1/\delta}}\Big)\ dt = \int_0^1 f\Big(g\Big(\frac{2}{t^{1+1/\delta}}\Big)\Big)\ dt < \infty.
\end{cases}$$

THEOREM 2.14. *Consider the situation $S2$ and assume that:*

$$\sum_{l=1}^{\infty} n_l a_l^2 h\Big(\frac{2}{a_l^{1+\delta}}\Big) < \infty. \tag{2.16}$$

*Then,*

$$\iint\limits_{Q} f\big(\log^+\log^+|G(z)|\big)\ d\lambda(z) < \infty. \tag{2.17}$$

PROOF. For a deleted subsquare $Q_{a_l}$ of side length $a_l$, we have from Lemma 2.4 that:

$$J_{a_l}(h) = \iint\limits_{Q_{a_l}} h\Big(\frac{1}{\operatorname{dist}(z,\partial Q_{a_l})}\Big)\ d\lambda(z) \leq 2a_l^2\left[I(h) + h\Big(\frac{2}{a_l^{1+\delta}}\Big)\right]$$

where, $I(h) = M < \infty$ from the situation $S2$ and $h(\frac{2}{a_l^{1+\delta}}) = f(g(\frac{2}{a_l^{1+\delta}}))$. Thus, summing over all deleted subsquares we get from (2.16) that:

$$S := \sum_{l=1}^{\infty} n_l J_{a_l}(h) \leq 2\sum_{l=1}^{\infty} n_l a_l^2 \left[M + h\Big(\frac{2}{a_l^{1+\delta}}\Big)\right] < \infty. \tag{2.18}$$

This further gives that:

$$\iint\limits_{Q} f\big(\log^+\log^+|G(z)|\big)\ d\lambda(z) \le \iint\limits_{Q\setminus K} h\Big(\frac{1}{\operatorname{dist}(z,K)}\Big)\ d\lambda(z)$$
$$= \iint\limits_{Q\setminus K} f(g(\operatorname{dist}(z,K)^{-1}))\ d\lambda(z)$$
$$\le S < \infty.$$

by (2.18). This completes the proof. □

Next, we consider few cases of the monotone increasing function $g$, in order to get a better picture of Theorem 2.14. It is important to know that all the functions $f$ given in Examples 1.7, can be easily used for Theorem 2.14 and for the particular cases discussed below.

COROLLARY 2.15. *Let the situation $S2$ holds and assume that for constants $C, \delta$ and $\alpha > 0$ such that $\alpha(1+1/\delta) < 1$, the function $G$ satisfies the growth condition;*

$$|G(z)| \le \exp\big(\exp C \operatorname{dist}(z,K)^{-\alpha}\big), \quad \forall z \in Q \setminus K.$$

*Further, assume that*

$$\sum_{l=1}^{\infty} n_l a_l^2 f\Big(\frac{C2^\alpha}{a_l^{\alpha(1+\delta)}}\Big) < \infty. \tag{2.19}$$

*Then,* (2.17) *holds, i.e.*

$$\iint\limits_{Q} f\big(\log^+\log^+|G(z)|\big)\ d\lambda(z) < \infty.$$

PROOF. Define a monotone increasing function $g$ as $g(x) := Cx^\alpha$. By our assumption on $\alpha$, we observe from the situation $S2$ that $I(h) = \int_0^1 f\big(\frac{C}{t^{\alpha(1+1/\delta)}}\big)\ dt = M < \infty$. Therefore, applying Lemma 2.4 to a deleted subsquare $Q_{a_l}$ of side length $a_l$ and summing over all the deleted subsquares, we get from (2.18), from the definition of the function $g$, the monotonicity of the functions $f$ and $g$ and from the assumption (2.19) that:

$$S = \sum_{l=1}^{\infty} n_l J_{a_l}(h) \le 2\sum_{l=1}^{\infty} n_l a_l^2\Big(M + h\Big(\frac{2}{a_l^{1+\delta}}\Big)\Big)$$
$$= 2\sum_{l=1}^{\infty} n_l a_l^2\Big(M + f\Big(g\Big(\frac{2}{a_l^{1+\delta}}\Big)\Big)\Big) < \infty$$

Hence, using similar arguments as in the proof of Theorem 2.14, we obtain that (2.17) holds. □

REMARK 2.16. *Note that considering the function $f$ from the Examples 1.7, it is clear that $I(h) = \int_0^1 h\big(\frac{2}{t^{1+1/\delta}}\big)\ dt < \infty$.*

EXAMPLE 2.17. *Let $K$ be the Sierpinski carpet as in Example 2.6. If $\alpha < \big(\frac{1}{1+\delta}\big)\big(2 - \frac{\log 8}{\log 3}\big)$ then,* (2.17) *holds.*

PROOF. We see from Example 2.6 that, $n_l = 8^{l-1}$ and $a_l = 1/3^l$, $l \in \mathbb{N}$. Putting the above values in (2.19) and considering Examples 1.7(i) for the sake of simplicity, i.e. $f(t) = t^{1+\varepsilon}$, $t \geq 0$ and for some $\varepsilon > 0$ such that $\alpha + \varepsilon < 1$, we get:

$$S \leq \sum_{l=1}^{\infty} \Big(\frac{8}{9}\Big)^l \tilde{M}(C, \alpha, \varepsilon) 3^{\alpha l(1+\varepsilon)(1+\delta)} < \infty$$

when $\alpha < (\frac{1}{1+\delta})\big(2 - \frac{\log 8}{\log 3}\big)$. □

COROLLARY 2.18. *Consider the situation S2 and assume that the following condition holds;*

$$|G(z)| \leq \exp\big(C \operatorname{dist}(z, K)^{-k}\big), \quad \forall z \in Q \setminus K.$$

*for positive constants C and $k \geq 1$. Further, assume that for some $\delta > 0$,*

$$\sum_{l=1}^{\infty} n_l a_l^2 f\Big(\log \frac{C2^k}{a_l^{k(1+\delta)}}\Big) < \infty. \tag{2.20}$$

*Then,* (2.17) *holds, i.e.*

$$\iint_Q f\big(\log^+ \log^+ |G(z)|\big)\, d\lambda(z) < \infty.$$

PROOF. Define a monotone increasing function $g$ as, $g(x) := \log(Cx^k)$. For a deleted subsquare $Q_{a_l}$ of side length $a_l$, we see from the monotonicity of the functions $f$ and $g$ that:

$$h(\frac{2}{a_l^{1+\delta}}) = f\Big(\log \frac{C2^k}{a_l^{k(1+\delta)}}\Big).$$

In particular, $I(h) = \int_0^1 f\big(g(\frac{2}{t^{1+1/\delta}})\big)\, dt < \infty$. Therefore, by (2.20) and using same arguments as in the previous Corollary 2.15, we get that (2.18) and hence, (2.17) holds. □

EXAMPLE 2.19. *Let K be a fractal set Q(m), as explained in Example 2.9. If, the following sum*

$$\sum_{l=1}^{\infty} n_l 2^{-2\tilde{C}l^2} f(\log C2^k\ 2^{(1+\delta)k\tilde{C}l^2}) < \infty,$$

*where $C, \tilde{C}$ are positive constants, then* (2.20) *holds.*

PROOF. In deed, at some $l-th$ step, recall from Example 2.9 for $m = 8$ that:

$$n_l = 6^{l-1} 8^{(l-1)^2}, a_l = \frac{1}{2} \cdot \frac{1}{8^{\frac{l(l-1)}{2}}} = \frac{1}{2^{1+3l(l-1)/2}} = \frac{1}{2^{\tilde{C}l^2}}.$$

For the sake of simplicity and for some $\varepsilon > 0$, consider $f(t) = t^{1+\varepsilon},\ t \geq 0$, from the Examples 1.7, we get that:

$$\begin{aligned} S \leq & \sum_{l=1}^{\infty} (\frac{6}{8})^l \left[ (\log C 2^k 2^{\tilde{C} l^2 (1+\delta) k})^{1+\varepsilon} \right] \\ \leq & 2^{1+\varepsilon} \sum_{l=1}^{\infty} (\frac{6}{8})^l \left[ (\log C 2^k)^{1+\varepsilon} + (k(1+\delta)\tilde{C} \log 2)^{1+\varepsilon} l^{2(1+\varepsilon)} \right] \\ \leq & C_\circ \sum_{l=1}^{\infty} (\frac{6}{8})^l \left[ C_1(k,\varepsilon) + C_2(k,\delta,\varepsilon) l^{2(1+\varepsilon)} \right] \\ \leq & C_3(k,\delta,\varepsilon) \sum_{l=1}^{\infty} (\frac{6}{8})^l l^{2(1+\varepsilon)} < \infty, \end{aligned}$$

where $C_\circ, C_1, C_2, C_3$ are positive constants. $\square$

Corollary 2.20. *Let situation S2 hold and assume that for $k \geq 1$,*

$$|G(z)| \leq C \operatorname{dist}(z, K)^{-k}, \qquad \forall z \in Q \setminus K.$$

*Further, assume that for some $\delta > 0$ and positive constant $C \geq e$*

$$\sum_{l=1}^{\infty} n_l a_l^2 f\Big( \log\log \frac{C 2^k}{a_l^{k(1+\delta)}} \Big) < \infty. \tag{2.21}$$

*Then,* (2.17) *holds, i.e.*

$$\iint\limits_{Q} f\big( \log^+ \log^+ |G(z)| \big)\ d\lambda(z) < \infty.$$

Proof. Define a function $g$ as, $g(x) := \log\log(Cx^k)$. Similar as in the proof of Corollary 2.18, one observes that $I(h) < \infty$ for $h = f \circ g$. Applying Lemma 2.4 to a deleted subsquare $Q_{a_l}$ and summing over all the deleted subsquares $n_l$ of side length $a_l$, we get from (2.18) that:

$$\begin{aligned} S = & \sum_{l=1}^{\infty} n_l J_{a_l}(h) \leq 2 \sum_{l=1}^{\infty} n_l a_l^2 \Big[ M + h(\frac{2}{a_l^{1+\delta}}) \Big] \\ < & 2 \sum_{l=1}^{\infty} n_l a_l^2 \Big[ M + f(\log\log \frac{C 2^k}{a_l^{k(1+\delta)}}) \Big] < \infty \end{aligned}$$

which is clear from the choice of the function $g$. Thus, from the proof of the Theorem 2.14, (2.17) holds. $\square$

Example 2.21. *Consider the compact set $K = Q_\infty$, as explained in Example 2.12. If*

$$\sum_{l=2}^{\infty} n_l 2^{-2^{l+1}} f(\log\log C 2^k\ 2^{k\tilde{C}(\delta) 2^l}) < \infty,$$

*for constants $C > 0, \tilde{C}(\delta) > 0$, then,* (2.17) *holds.*

PROOF. In fact, for some $\varepsilon > 0$ and $f(t) = t^{1+\varepsilon},\ t \geq 0$ from the Examples 1.7 and $n_l = (\frac{3}{4})^{l-1} \cdot 8^{2^{l-1}}$ and $a_l = \frac{1}{2} \cdot \frac{1}{8^{2^{l-2}}} = \frac{1}{2^{1+3\cdot 2^{l-2}}} = \frac{1}{2^{C_1 2^l}}$, the sum $S$ in (2.18) for $M(\delta, k, \varepsilon) > 0$ becomes:

$$\begin{aligned} S &\leq \sum_{l=2}^{\infty} (\frac{3}{4})^l (\log\log C 2^k 2^{\tilde{C}(\delta) k 2^l})^{1+\varepsilon} \\ &\leq \sum_{l=2}^{\infty} (\frac{3}{4})^l M(\delta, k, \varepsilon) l^{1+\varepsilon} < \infty \end{aligned}$$

and hence, (2.17) holds. □

CHAPTER 3

# Banach function algebras of complex ultra-differentiable functions

## 3.1. Preliminary results

Let $K$ be a compact Hausdorff space. The space $C(K)$ of all complex-valued continuous functions on $K$ is a commutative Banach algebra with pointwise operations and the supremum norm $\|\cdot\|_K$ on $K$.

Definition 3.1. *A subalgebra $\mathcal{A}$ of the algebra $C(K)$ of all complex-valued continuous functions on $K$ is called a* `Banach function algebra` *(see e.g. in* [**15, 14**]*) if,*
*(i) it separates the points of $K$,*
*(ii) contains the constant functions and*
*(iii) is endowed with an algebra norm $\|\cdot\|_{\mathcal{A}}$ such that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a Banach algebra.*

In the following lemma, we collect some obvious facts concerning Banach function algebras.

Lemma 3.2. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach function algebra on a compact Hausdorff space $K$. Then, the following hold:*

a) *For all $f \in \mathcal{A}$ we have $f(K) \subset \sigma_{\mathcal{A}}(f)$, where $\sigma_{\mathcal{A}}(f)$ denotes the spectrum of $f$ in $\mathcal{A}$.*
b) *For all $f \in \mathcal{A}$ we have*

$$\|f\|_K = \sup_{z\in K} |f(z)| \leq r(f) \leq \|f\|_{\mathcal{A}}$$

*where $r(f)$ is the spectral radius of $f$ in $\mathcal{A}$. In particular, the inclusion map $J : \mathcal{A} \to C(K)$ is continuous.*
c) *$\mathcal{A}$ is semisimple.*
d) *The set $\Delta(\mathcal{A})$ of all multiplicative linear functionals contains the set $\{\delta_t\ ; t \in K\}$ of all point evaluations $\delta_t : f \mapsto f(t)$, $f \in \mathcal{A}$, $t \in K$.*

An element $f$ of a Banach function algebra $\mathcal{A}$ on a compact Hausdorff space $K$ is said to have *natural spectrum* if, $f(K) = \sigma_{\mathcal{A}}(f)$.

We say that $f \in \mathcal{A}$ has *locally natural spectrum* if, for all $g \in \mathcal{A}$ and all $\lambda \in \mathbb{C} \setminus f(\operatorname{supp} g)$ the function

$$z \mapsto \Big(\frac{g}{\lambda - f}\Big)(z) := \begin{cases} \frac{g(z)}{\lambda - f(z)} & z \in \operatorname{supp} g \\ 0 & z \in K \setminus \operatorname{supp} g \end{cases}$$

is in $\mathcal{A}$. In particular, every function $f \in \mathcal{A}$ having locally natural spectrum have natural spectrum.

Recall from [**15**] that a Banach function algebra $\mathcal{A}$ on a compact Hausdorff space $K$ satisfying the property that $\Delta(\mathcal{A}) = \{\delta_t\ ; t \in K\}$ will be called *natural*. Notice that, in a natural Banach function algebra, every element has natural spectrum.

A subalgebra $\mathcal{A}$ of $C(K)$ is said to be *inverse closed* , if it is a full subalgebra of $C(K)$, i.e. if $1/f \in \mathcal{A}$ for all $f \in \mathcal{A}$ with $f(z) \neq 0$ for all $z \in K$. This is the case if and only if, all $f \in \mathcal{A}$ have natural spectrum.

An algebra $\mathcal{A}$ of continuous functions on a compact Hausdorff space $K$ is said to be *normal* on $K$ if for all closed $F_1$, $F_2 \subset K$ with $\overline{F_1 \cap F_2} = \emptyset$, there exists a function $f \in \mathcal{A}$ such that $f \equiv 1$ on $F_1$ and $f \equiv 0$ on $F_2$.

$\mathcal{A}$ is said to be *regular* on $K$, if for every compact subset $H$ of $K$ and every $z_\circ \in K \setminus H$ there exists some $f \in \mathcal{A}$ satisfying $f|_H \equiv 0$ and $f(z_\circ) = 1$.

A commutative unital Banach algebra $\mathcal{A}$ is by definition *normal* (resp. *regular*), if the set $\hat{\mathcal{A}}$ of its Gelfand transforms is normal (resp. regular) on the structure space $\Delta(\mathcal{A})$.

If $K$ is a compact subset of $\mathbb{C}$, we denote by $\mathrm{Rat}(K)$ the algebra of all rational functions on $K$ with poles off $K$ and by $R(K)$ the uniform closure of $\mathrm{Rat}(K)$ in $C(K)$

Proposition 3.3. *Let $\mathcal{A}$ be a Banach function algebra on a compact set $K \subset \mathbb{C}$ with $\mathrm{Rat}(K) \subset \mathcal{A}$. Then, $\mathcal{A}$ will be natural, if one of the following hold:*

a) *$\mathrm{Rat}(K)$ is dense in $\mathcal{A}$.*
b) *$R(K) = C(K)$ and $\mathcal{A}$ is inverse closed.*

Proof. a) As the set $\{\delta_z; z \in K\}$ of all point evaluations is contained in the space $\Delta(\mathcal{A})$ of all multiplicative linear functionals on $\mathcal{A}$, we have to prove only that $\Delta(\mathcal{A}) \subseteq \{\delta_z; z \in K\}$. Since, $\mathrm{Rat}(K) \subset \mathcal{A}$, the spectrum $\sigma_{\mathcal{A}}(id_K)$ of $id_K : z \mapsto z$ coincides with $K$ and we obtain $\phi(id_K) \in K$ for all $\phi \in \Delta(\mathcal{A})$. Therefore, $\phi(f) = f(\phi(id_K))$ for all $f \in \mathrm{Rat}(K)$. Hence, (by the continuity of $\phi$ and $\delta_{\phi(id_K)}$) also for all $f$ in the closure of $\mathrm{Rat}(K)$ in $\mathcal{A}$.

b) Assume that there exists some $\phi \in \Delta(\mathcal{A}) \setminus \{\delta_z; z \in K\}$. Thus, for each $z \in K$, there exists some $f_z \in \ker \phi$ such that $f_z(z) \neq 0$ and hence $f_z(w) \neq 0$ for all $w \in K \cap U_z$, for some neighbourhood $U_z$ of $z$. As, $K$ is compact we find finitely many open sets $U_1, \cdots, U_n$ with $K \subset \cup_{j=1}^n U_j$ and functions $f_1, \cdots, f_n \in \mathcal{A}$ such that $f_j(z) \neq 0$ for all $z \in U_j \cap K$, $j = 1, \cdots, n$.

Thus,

$$\gamma := \min_{z \in K} \sum_{j=1}^{n} |f_j|^2 > 0.$$

Because of $R(K) = C(K)$, there exist functions $g_1, \cdots, g_n \in \mathrm{Rat}(K)$ such that, for $j = 1, \cdots, n$,

$$\|\overline{f_j} - g_j\|_K < \frac{\gamma}{\sum_{j=1}^{n} \|f_j\|_K}$$

It follows that

$$\Big\| \sum_{j=1}^{n} |f_j|^2 - \sum_{j=1}^{n} f_j g_j \Big\|_K < \gamma$$

In particular the function $h := \sum_{j=1}^n f_j g_j$ is in $\ker \phi$ and has no zero in $K$. As $\mathcal{A}$ is inverse closed, we have the contradiction that $1 = \sum_{j=1}^n f_j g_j h^{-1} \in \ker \phi$.

□

If $\mathcal{R}$ is a commutative Banach algebra with a unit element, then we write $Dec(\mathcal{R})$ for the set of all $a \in \mathcal{R}$ such that the operator $M_a$ of multiplication by $a$ is decomposable on $\mathcal{R}$. This is a closed subalgebra of $\mathcal{R}$ by a result of Apostol [**5**] (see also [**32**], Proposition 4.4.9).

Moreover, $\mathcal{R}$ contains a unique maximal closed subalgebra which is regular (see [**2**], Theorem 2.4 in the semisimple case and [**32**], Theorem 4.3.6 for the general case). Further, if $\mathcal{R}$ is semisimple, then $\mathcal{R}$ is regular if and only if $\mathcal{R} = Dec(\mathcal{R})$ by a result of Frunză [**27**].

Proposition 3.4. *If $\mathcal{A}$ is a Banach function algebra on a compact Hausdorff space $K$ which is normal on $K$ then $Dec(\mathcal{A})$ coincides with the set of all $f \in \mathcal{A}$ having locally natural spectrum.*

Proof. "$\Longrightarrow$" Suppose that $f \in \mathcal{A}$ is given such that $M_f$ is decomposable on $\mathcal{A}$. As the inclusion mapping $J : \mathcal{A} \to C(K)$ intertwines the operators of multiplication on $\mathcal{A}$ and $C(K)$ by $f$, we obtain from Theorem 1.2.23 [**32**] for all closed $F \subset \mathbb{C}$,

$$\mathcal{E}_{\mathcal{A}}(F) \subseteq \mathcal{E}_{C(K)}(F) = \big\{ h \in C(K) \,; \operatorname{supp} h \subset f^{-1}(F) \big\}$$

where $\mathcal{E}_{\mathcal{A}}(F)$ respectively $\mathcal{E}_{C(K)}(F)$ represent the spectral capacities for $M_f$ on $\mathcal{A}$ respectively on $C(K)$. Thus,

$$\mathcal{E}_{\mathcal{A}}(F) \subseteq \big\{ h \in \mathcal{A} \,; \operatorname{supp} h \subset f^{-1}(F) \big\} =: \mathcal{E}_\circ(F).$$

Notice that $\mathcal{E}_\circ(F)$ is a closed $M_f$ invariant subspace of $\mathcal{A}$. Fix $h \in \mathcal{E}_\circ(F)$. By the decomposability of $M_f$, for each $n \in \mathbb{N}$, there exist $\varphi_1$, $\varphi_2 \in \mathcal{A}$ with

$$\varphi_1 \in \mathcal{E}_{\mathcal{A}}(V_n) \quad \text{where } V_n = \left\{ \lambda \in \mathbb{C} \,; \operatorname{dist}(\lambda, F) = \frac{1}{n} \right\}$$

$$\varphi_2 \in \mathcal{E}_{\mathcal{A}}(W_n) \quad \text{where } W_n = \left\{ \lambda \in \mathbb{C} \,; \operatorname{dist}(\lambda, F) \geq \frac{1}{2n} \right\}$$

and $\varphi_1 + \varphi_2 \equiv 1$.

In particular,

$$\operatorname{supp} \varphi_2 \subset \left\{ z \in K \,; \operatorname{dist}(f(z), F) \geq \frac{1}{2n} \right\}$$

It follows that $\varphi_2 h \equiv 0$ and hence $h = \varphi_1 h = M_h \varphi_1 \in \mathcal{E}_{\mathcal{A}}(V_n)$ as $M_h$ commutes with $M_f$. Hence, $h \in \cap_{n=1}^{\infty} \mathcal{E}_{\mathcal{A}}(V_n) = \mathcal{E}_{\mathcal{A}}(F)$ and we conclude that $\mathcal{E}_{\mathcal{A}}(F) = \mathcal{E}_\circ(F)$.

Fix now $h \in \mathcal{A}$ and let $\lambda$ be in $\mathbb{C} \setminus f(\operatorname{supp} h)$. Thus, $h \in \mathcal{E}_{\mathcal{A}}(f(\operatorname{supp} h))$ and $\lambda \notin \sigma(M_f | \mathcal{E}_{\mathcal{A}}(f(\operatorname{supp} h)))$.

Hence, $T := (\lambda - M_f|\mathcal{E}_{\mathcal{A}}(f(\operatorname{supp} h)))^{-1}$ exists in $\mathcal{L}(\mathcal{E}_{\mathcal{A}}(f(\operatorname{supp} h)))$. Because of $(\lambda - f)Th \equiv h$, we see that $Th \equiv \frac{h}{\lambda - f} \in \mathcal{A}$. Thus, has locally natural spectrum.

"$\Longleftarrow$" Suppose now that $f \in \mathcal{A}$ has locally natural spectrum. In particular,

$$\sigma(M_f, \mathcal{A}) = \sigma_{\mathcal{A}}(f) = f(K).$$

Let now $U_1, U_2 \subseteq \mathbb{C}$ be open with $\sigma(M_f, \mathcal{A}) = f(K) \subset U_1 \cup U_2$. Then $f^{-1}(U_j)$ is open in $K$, $(j = 1, 2)$ and $f^{-1}(U_1) \cup f^{-1}(U_2) = K$.

Moreover, $K_j := K \setminus f^{-1}(U_j)$ are disjoint compact subsets of $K$. As $K$ is normal, there exist disjoint closed subsets $F_1, F_2$ of $K$ such that $K_j \subset \operatorname{int} F_j$ $(j = 1, 2)$. Now, $\mathcal{A}$ is normal on $K$. Hence, there exists $\varphi \in \mathcal{A}$ with $\varphi \equiv 1$ on $F_1$ and $\varphi \equiv 0$ on $F_2$.

For all $g \in \mathcal{A}$ we then have $g = g_1 + g_2$ with $g_1, g_2 \in \mathcal{A}$ given by

$$g_1 = \varphi g, \qquad g_2 = (1 - \varphi)g.$$

Hence,

$$\operatorname{supp} g_1 \subseteq K \setminus \operatorname{int} F_2 \subset f^{-1}(U_1)$$

and

$$\operatorname{supp} g_2 \subseteq K \setminus \operatorname{int} F_1 \subset f^{-1}(U_2)$$

and $\mathcal{A} = X_1 + X_2$, where

$$X_1 = \{h \in \mathcal{A}\ ; \operatorname{supp} h \subseteq K \setminus \operatorname{int} F_2\}$$

$$X_2 = \{h \in \mathcal{A}\ ; \operatorname{supp} h \subseteq K \setminus \operatorname{int} F_1\}$$

are closed $M_f-$invariant subspaces of $\mathcal{A}$.

Fix now $j \in \{1, 2\}$ and let $\lambda \in \mathbb{C} \setminus f(K \setminus \operatorname{int} F_{3-j})$. Then $M_f|X_j$ is injective as $\lambda - f$ has no zeros on $K \setminus \operatorname{int} F_{3-j}$. If $h \in X_j$, then $\frac{h}{\lambda - f} \in \mathcal{A}$ as $f$ has locally natural spectrum and $(\lambda - M_f)\frac{h}{\lambda - f} = h$. Thus, $\sigma(M_f|X_j) \subseteq f(K \setminus \operatorname{int} F_{3-j}) \subset U_j$ for $j = 1, 2$ and $M_f$ is decomposable. □

Notice that for the direction "$\Rightarrow$", we did not need the fact that $\mathcal{A}$ is normal on $K$. In the case $K \subset \mathbb{C}$ and $f(z) \equiv z$, this proof actually shows:

COROLLARY 3.5. *Let $\mathcal{A}$ be a Banach function algebra on a compact set $K \subseteq \mathbb{C}$. If the operator $M_z$ of multiplication by the coordinate function is decomposable on $\mathcal{A}$, then $\mathcal{A}$ is normal on $K$.*

COROLLARY 3.6. *Let $\mathcal{A}$ be a Banach function algebra on $K$ such that* $\operatorname{Rat}(K)$ *is dense in $\mathcal{A}$. If $M_z$ is decomposable on $\mathcal{A}$, then $\mathcal{A}$ is a regular Banach function algebra and $\Delta(\mathcal{A}) = \{\delta_w\ ; w \in K\}$.*

PROOF. By [**13**] Theorem 1.10, Chapter 2, we have $\operatorname{Rat}(K) \subset Dec(\mathcal{A})$. Hence, $Dec(\mathcal{A}) = \mathcal{A}$ (as $Dec(\mathcal{A})$ is closed) and by the theorem of Frunză [**27**], $\mathcal{A}$ is a regular Banach function algebra. The fact that $\Delta(\mathcal{A}) = \{\delta_w; w \in K\}$ holds has already been shown in Proposition 3.3(a).

□

## 3.2. Algebras of differentiable functions

From here onwards, throughout the thesis, $K$ is a non-empty perfect, compact subset of $\mathbb{C}$, unless stated otherwise. For $k \in \mathbb{N}$, let $\mathcal{D}^k(K)$ be the space of all $k-times$ complex differentiable functions on $K$ such that $f^{(k)} := \frac{d^k f}{dz^k}$ is continuous. For $f \in \mathcal{D}^k(K)$, the norm on $\mathcal{D}^k(K)$ will be defined as:

$$\|f\|_k := \sum_{j=0}^{k} \frac{1}{j!}\|f^{(j)}\|_K, \qquad f \in \mathcal{D}^k(K). \tag{3.1}$$

It is clear that $\|\cdot\|_k$ is submultiplicative and $\mathcal{D}^k(K)$ is a normed (function) algebra. We further define $\mathcal{D}^\infty(K)$, the algebra of all infinitely complex differentiable functions on $K$, i.e.

$$\mathcal{D}^\infty(K) = \bigcap_{k\in\mathbb{N}} \mathcal{D}^k(K)$$

endowed with the supremum norm on $K$.

Following [**15**], we define certain normed algebras of infinitely complex differentiable functions on $K$. Let $(M_p)_{p\in\mathbb{N}_\circ}$ be a sequence of positive reals satisfying:

$$i)\ M_\circ = 1, \qquad ii)\ \frac{M_p}{M_q M_{p-q}} \geq \binom{p}{q}, \quad (q = 0, \cdots, p). \tag{3.2}$$

Define a subalgebra $\mathcal{D}_1(K, \{M_p\})$ of $\mathcal{D}^\infty(K)$ as:

$$\mathcal{D}_1(K, \{M_p\}) := \left\{ f \in \mathcal{D}^\infty(K); \|f\|_{\{M_p\},1} := \sum_{p=0}^{\infty} \frac{1}{M_p}\|f^{(p)}\|_K < \infty \right\}$$

where $\|\cdot\|_K$ is the supremum norm on $K$. We can relate the algebras $\mathcal{D}^k(K)$ and $\mathcal{D}_1(K, \{M_p\})$ by setting $M_p = p!, \quad (p = 0, \cdots, k)$ and $\frac{1}{M_p} = 0, \quad (p = k+1, \cdots)$.

As in [**34**] and in [**15**], given a sequence $(M_p)_{p\in\mathbb{N}_\circ}$ as above, let $m_p := \left(\frac{M_p}{p!}\right)^{1/p}$, $p \in \mathbb{N}$ be a sequence of positive reals with $m_\circ = 1$. Then, $\mathcal{D}_1(K, \{M_p\})$ can be rewritten as:

$$\mathcal{D}_1(K, \{M_p\}) = \left\{ f \in \mathcal{D}^\infty(K); \|f\|_{\{M_p\},1} = \sum_{p=0}^{\infty} \frac{1}{p! m_p^p}\|f^{(p)}\|_K < \infty \right\}$$

The submultiplicativity of the norm can be seen by using (3.2)($ii$). In terms of $(m_p)_{p=0}^\infty$, (3.2)($ii$) will be:

$$m_q^q m_{p-q}^{p-q} \leq m_p^p, \quad \text{for } 0 \leq q \leq p, \quad p,\ q \in \mathbb{N}_\circ \tag{3.3}$$

Similarly, we define:

$$\mathcal{D}_\infty(K, \{M_p\}) := \left\{ f \in \mathcal{D}^\infty(K); \|f\|_{\{M_p\},\infty} := \sup_{p\in\mathbb{N}_\circ} \frac{1}{M_p}\|f^{(p)}\|_K < \infty \right\}$$

where the multiplication is continuous w.r.t. the above norm provided:

$$\sup_{p\in\mathbb{N}_\circ} \sum_{q=0}^{p} \frac{m_q^q m_{p-q}^{p-q}}{m_p^p} < \infty. \tag{3.4}$$

Equivalently saying that:

$$\sup_{p\in\mathbb{N}_\circ}\sum_{q=1}^{p-1}\frac{m_q^q m_{p-q}^{p-q}}{m_p^p}<\infty.$$

Indeed, let $f,g\in\mathcal{D}_\infty(K,\{M_p\})$ and $h:=fg$. Then,

$$\|h\|_{\{M_p\},\infty}=\sup_{p\in\mathbb{N}_\circ}\frac{1}{M_p}\|h^{(p)}\|_K=\sup_{p\in\mathbb{N}_\circ}\frac{1}{M_p}\|(fg)^{(p)}\|_K$$

Using the Leibniz rule and $M_p=p!m_p^p$, $p\in\mathbb{N}_\circ$, we get:

$$\begin{aligned}\|h\|_{\{M_p\},\infty}&\le\sup_{p\in\mathbb{N}_\circ}\frac{1}{p!m_p^p}\sum_{q=0}^{p}\binom{p}{q}\|f^{(q)}\|_K\|g^{(p-q)}\|_K\\&=\sup_{p\in\mathbb{N}_\circ}\sum_{q=0}^{p}\frac{1}{q!(p-q)!m_q^q m_{p-q}^{p-q}}\|f^{(q)}\|_K\|g^{(p-q)}\|_K\cdot\frac{m_q^q m_{p-q}^{p-q}}{m_p^p}\\&=\sup_{p\in\mathbb{N}_\circ}\sum_{q=0}^{p}\frac{\|f^{(q)}\|_K}{q!m_q^q}\frac{\|g^{(p-q)}\|_K}{(p-q)!m_{p-q}^{p-q}}\cdot\frac{m_q^q m_{p-q}^{p-q}}{m_p^p}\\&\le\|f\|_{\{M_p\},\infty}\|g\|_{\{M_p\},\infty}\cdot\sup_{p\in\mathbb{N}_\circ}\sum_{q=0}^{p}\frac{m_q^q m_{p-q}^{p-q}}{m_p^p}<\infty\end{aligned}$$

by (3.4). Thus, the multiplication is continuous and $\|\cdot\|_{\{M_p\},\infty}$ is equivalent to some submultiplicative norm. Throughout this chapter we shall assume that condition (3.4) is fulfilled whenever we consider situations with $q=\infty$.

It has been mentioned in [**25**] that the algebra $\mathcal{D}_1(K,\{M_p\})$ is in general not complete. However, in [**15**] and in later papers of Honary [**29**] following Dales [**15**], the completeness of the algebras $\mathcal{D}^k(K)$ and $\mathcal{D}_1(K,\{M_p\})$ has been shown by assuming some conditions on the perfect, compact set $K$. We point out that not much attention has been given to $\mathcal{D}_\infty(K,\{M_p\})$ algebras except in [**15**], where they appear as a special example on a closed unit interval.

Observe that for a perfect, compact set $K\subset\mathbb{C}$, the normed function algebras $\mathcal{D}_q(K,\{M_p\})$, $q\in\{1,\infty\}$ contains the rational functions on $K$ with poles off $K$ under some conditions on the sequence $(M_p)_{p\in\mathbb{N}_\circ}$. For $\mathcal{D}_1(K,\{M_p\})$, a condition has been given in [**15**] .

REMARK 3.7. *Let $K$ be a perfect, compact set in $\mathbb{C}$. Then,* $\mathrm{Rat}(K)\subset\mathcal{D}_q(K,\{M_p\})$, $q\in\{1,\infty\}$, *if and only if $m_p\to\infty$, for $p\to\infty$.*

PROOF. $\Rightarrow$ Define the rational function $h(z)=\frac{1}{\xi-z}$ for some $\xi\in\mathbb{C}\backslash K$. Then,

$$\|h\|_{\{M_p\},q}\le\left\|\frac{1}{m_p^p d(\xi)^{p+1}}\right\|_{\{M_p\},q}\tag{3.5}$$

will be finite for all $\xi$ only if $\lim_{p\to\infty}m_p=\infty$, where $d(\xi):=\mathrm{dist}(\xi,K)$. Thus, showing that $\mathrm{Rat}(K)\subset\mathcal{D}_q(K,\{M_p\})$, $q\in\{1,\infty\}$.

$\Leftarrow$ Conversely, we know from the norm that

$$\|h\|_{\{M_p\},\infty}\le\frac{1}{d(\xi)}\sup_{p\in\mathbb{N}_\circ}\frac{1}{m_p^p d(\xi)^p}$$

Assume that $\lim_{p\to\infty} m_p \neq \infty$. Then there exists a subsequence $(p_k)_{k=0}^{\infty}$ and some $c > 0$ such that

$$m_{p_k} \to c, \quad \text{for } k \to \infty.$$

Then, for some $k_\circ > 0$, we have

$$\frac{c}{2} \leq m_{p_k} \leq 2c \quad \text{and} \quad d(\xi) < \frac{1}{4c}.$$

Thus, for $k \geq k_\circ$

$$\frac{1}{(m_{p_k} d(\xi))^{p_k}} \geq \frac{1}{(2cd(\xi))^{p_k}} \geq 2^{p_k} \to \infty, \quad \text{for } k \to \infty$$

which shows that $h \notin \mathcal{D}_\infty(K, \{M_p\})$ and hence not in $\mathcal{D}_1(K, \{M_p\})$.

□

EXAMPLES 3.8. *Few standard examples of the sequence* $(m_p)_{p=0}^{\infty}$ *are:*

*i)* $m_p = p!^{s/p}$, *ii)* $m_p = p^s$, *for* $s > 0$.

*iii)* $m_p = (\log(e + p))^s$, *iv)* $m_p = (\prod_{k=1}^{p} \log(e + k))^{s/p}$, *for* $s \geq 1$,

*where* $m_\circ = 1$ *in each case.*

The fact that these sequences satisfy condition (3.4) is a consequence of the following lemma.

LEMMA 3.9. *Let* $(m_p)_{p\in\mathbb{N}}$ *be any of the sequences of positive reals as given in the above Example with* $m_\circ = 1$. *Define*

$$S_{n,k}(s) := \frac{m_k^k m_{n-k}^{n-k}}{m_n^n}.$$

*Then,*

(a) $S_n(s, k)$ *is symmetric in* $k$ *and* $n - k$ *and monotone decreasing in* $k$ *for* $1 \leq k \leq n/2$.

(b) $S_n(s) := \sum_{k=1}^{n-1} S_n(s, k) \to 0$, *as* $n \to \infty$.

PROOF. For example $(iv)$, we observe that by using similar arguments as in the proof of Lemma 3.3 [**15**], both $(a)$ and $(b)$ hold.

$(a)$ It is obvious that $S_n(s, k)$ is symmetric in $k$ and $n - k$ for the above examples. For examples $(i)$ and $(ii)$, there is nothing to be shown for $n = 2, 3$. Let now $n \geq 4$ and $2 \leq k \leq n/2$. In the first case we have:

$$\begin{aligned} k!^s (n-k)!^s =& (k-1)!^s (n-k+1)!^s \cdot \frac{k^s}{(n-k+1)^s} \\ \leq& (k-1)!^s (n-k+1)!^s \end{aligned}$$

because $k \leq n/2 \leq n - k + 1$.

In the second case observe that:

$$\begin{aligned} k^{ks}(n-k)^{(n-k)s} =& (k-1)^{(k-1)s}(n-k+1)^{(n-k+1)s} \cdot \frac{k^s}{(n-k+1)^s} \cdot \\ & \cdot \frac{\left(1 + \frac{1}{k-1}\right)^{(k-1)s}}{\left(1 + \frac{1}{n-k}\right)^{(n-k)s}} \\ \leq& (k-1)^{(k-1)s}(n-k+1)^{(n-k+1)s} \end{aligned}$$

as $k < n-k+1$ and $k-1 < n-k$.

For third example, notice that for $x \in \mathbb{R}^+$, the function $f(x) = \left(\frac{\log(e+x)}{\log(e+x-1)}\right)^{x-1}$ is monotone increasing. Hence, we obtain that:

$$\left(\frac{\log(e+k)}{\log(e+k-1)}\right)^{s(k-1)} (\log(e+k))^s \leq \left(\frac{\log(e+n-k+1)}{\log(e+n-k)}\right)^{s(n-k)} (\log(e+n-k+1))^s$$

as $k < n-k+1$, for $1 \leq k \leq n/2$. Thus, from above we get that:

$$(\log(e+k))^{sk} (\log(e+n-k))^{s(n-k)} \leq (\log(e+k-1))^{s(k-1)} (\log(e+n-k+1))^{s(n-k+1)}$$

which shows that $S_{n,k}(s)$ is monotone decreasing in $k$ for $1 \leq k \leq n/2$.

$(b)$ We first show it for example $(ii)$. Let $\varepsilon > 0$.

$(ii)$ For the second example we have:

$$\begin{aligned} S_n(s) =& \sum_{k=1}^{n-1} \frac{k^{ks}(n-k)^{(n-k)s}}{n^{ns}} \\ \leq& 2 \sum_{1\leq k\leq n/2} \frac{k^{ks}(n-k)^{(n-k)s}}{n^{ns}} \end{aligned}$$

Fix $k_\circ$ so that $1 < k_\circ < n/2$, we have:

$$\begin{aligned} S_n(s) \leq& 2 \sum_{1\leq k<k_\circ} \left(\frac{k}{n}\right)^{sk} + 2 \sum_{k_\circ\leq k\leq n/2} \left(\frac{k}{n}\right)^{sk} \\ <& 2 \sum_{1\leq k<k_\circ} \left(\frac{k}{n}\right)^{sk} + 2 \sum_{k=k_\circ}^{\infty} \frac{1}{2^{sk}} \\ <& 2k_\circ \left(\frac{k_\circ}{n}\right)^{s} + 2\frac{2^{-sk_\circ}}{1-2^{-s}} \end{aligned}$$

Thus, if $k_\circ$ is chosen such that $\frac{2^{1-sk_\circ}}{1-2^{-s}} < \varepsilon/2$ and if then $n_\circ$ is such that $n_\circ > 2k_\circ$ and $2\frac{k_\circ^{1+s}}{n_\circ^s} < \varepsilon/2$ then, for all $n \geq n_\circ$ we have, $S_n(s) < \varepsilon$.

Hence, $S_n(s) \to 0$ for $n \to \infty$.

$(i)$ For example $(i)$, we have that:

$$S_{n,k}(s) = \frac{m_k^k m_{n-k}^{n-k}}{m_n^n} = \frac{k!^s(n-k)!^s}{n!^s}.$$

Using Stirling's formula for $x \in \mathbb{R}^+$, i.e.

$$x! = \sqrt{2\pi} x^{x+1/2} \exp^{-x+\mu(x)},$$

where $0 < \mu(x) < 1/12x$, we obtain:

$$\begin{aligned} S_{n,k}(s) \leq& (2\pi)^{s/2} \frac{k^{ks}(n-k)^{(n-k)s}}{n^{ns}} \cdot \left(\frac{k(n-k)}{n}\right)^{s/2} \cdot (e^{1/k}e^{1/(n-k)})^{s/12} \\ \leq& (2\pi)^{s/2} \frac{k^{ks}(n-k)^{(n-k)s}}{n^{ns}} \cdot \left(\frac{k(n-k)}{n}\right)^{s/2} e^{s/6} \end{aligned}$$

Thus, for $1 \le k \le n/2$ and for some positive constant $C$ only depending on $s$, we get:

$$\begin{aligned} S_{n,k}(s) \le & C_s \, \frac{k^{ks}(n-k)^{(n-k)s}}{n^{ns}} \cdot \Big(\frac{k(n-k)}{n}\Big)^{s/2} \\ \le & C_s \Big(\frac{k}{n}\Big)^{ks} \Big(\frac{k}{n}\Big)^{s/2} \end{aligned}$$

Now,

$$\begin{aligned} S_n(s) = \sum_{k=1}^{n-1} S_{n,k}(s) \le & 2C_s \sum_{1\le k\le n/2} \Big(\frac{k}{n}\Big)^{ks} \max_{1\le k\le n/2} \Big(\frac{k}{n}\Big)^{s/2} \\ \le & 2C_s \sum_{1\le k\le n/2} \Big(\frac{k}{n}\Big)^{ks} \Big(\frac{1}{2}\Big)^{s/2} \\ \le & 2\tilde{C}_s \sum_{1\le k\le n/2} \Big(\frac{k}{n}\Big)^{ks} \end{aligned}$$

where $\tilde{C}_s$ is a positive constant independent of $k$ and $n$. Using the above result from example $(ii)$, we obtain for $\varepsilon > 0$, for fixed $k_\circ$ so that $1 < k_\circ < n/2$ and $n_\circ > 2k_\circ$ that:

$$S_n(s) \le \tilde{C}_s \varepsilon,$$

which shows that $\lim_{n\to\infty} S_n(s) = 0$.

$(iii)$ For example $(iii)$, we proceed similar as in the proof of Lemma 3.3 in [**15**].

Let $0 < \varepsilon < 1$ be given and choose $N \in \mathbb{N}$ such that $N > 2/\varepsilon$ and

$$\forall n \ge N : \quad \frac{\log(e+1+\log n)}{\log(e+n)} < \min\big\{\frac{\varepsilon}{5}, e^{-2}\big\}.$$

Fix an arbitrary $n \ge N$ and let $k$ be the smallest integer greater than $\log n$. For $j = 1, \cdots, n$ we have

$$\begin{aligned} S_{n,j}(s) = & \frac{(\log(e+j))^{js}(\log(e+n-j))^{(n-j)s}}{(\log(e+n))^{ns}} \\ = & S_{n,n-j}(s) \end{aligned}$$

Hence, because of $s \ge 1$,

$$S_n(s) = \sum_{j=1}^{n-1} S_{n,j}(s) \le 2 \sum_{1\le j\le n/2} S_{n,j}(s)$$

Hence, by $(a)$ for $(iii)$,

$$\begin{aligned} S_n(s) = & 2 \sum_{1\le j\le k} \Big(\frac{\log(e+j)}{\log(e+n)}\Big)^{sj} + 2 \sum_{k<j\le n/2} S_{n,k}(s) \\ \le & 2 \sum_{1\le j\le k} \Big(\frac{\log(e+k)}{\log(e+n)}\Big)^{j} + n \cdot e^{-2k} \end{aligned}$$

where we used the fact that $s \ge 1$ and

$$S_{n,k}(s) \le \Big(\frac{\log(e+k)}{\log(e+n)}\Big)^{ks} \le e^{-2ks} \le e^{-2k}.$$

We obtain

$$\begin{aligned} S_n(s) \leq& 2\sum_{j=1}^{\infty}\left(\varepsilon/5\right)^j + n\cdot e^{-2\log n} \\ \leq& 2\cdot\frac{\varepsilon}{5-\varepsilon}+\frac{1}{n}<\varepsilon. \end{aligned}$$

Hence, $(b)$ holds.

□

## 3.3. Completions of normed function algebras of differentiable functions

Let $C(K)^{k+1}$ be the space of $(k+1)-tuples$ of all continuous complex functions on $K$. An element $f \in C(K)^{k+1}$ will be of the form $f = (f_j)_{j=0}^k$. $C(K)^{k+1}$ endowed with the norm given by

$$\|f\|_k^{\tilde{}} := \sum_{j=0}^{k}\frac{1}{j!}\|f_j\|_K, \qquad (f_j)_{j=0}^k \in C(K)^{k+1}$$

is a Banach algebra, where $\|\cdot\|_K$ denotes the supremum norm on $K$. Submultiplicativity of norm can be seen by defining the multiplication as follows:

For $f = (f_j)_{j=0}^k$ and $g = (g_j)_{j=0}^k$ in $C(K)^{k+1}$, define $fg =: h = (h_j)_{j=0}^k$, where,

$$h_j = \sum_{\nu=0}^{j}\binom{j}{\nu} f_\nu g_{j-\nu}, \qquad j = 0, \cdots, k.$$

Moreover, $(C(K)^{k+1}, \|\cdot\|_k^{\tilde{}})$ is a unital Banach algebra with the unit element $e := (e_j)_{j=0}^k$, where,

$$e_j := \begin{cases} 1, & \text{j=0} \\ 0, & \text{j=1, } \cdots\text{, k.} \end{cases}$$

Defining a unital isometric homomorphism $J : \mathcal{D}^k(K) \to C(K)^{k+1}$ by $J(f) := (f^{(j)})_{j=0}^k$, for all $f \in \mathcal{D}^k(K)$, we observe that $J(\mathcal{D}^k(K))$ is a subalgebra of $C(K)^{k+1}$ and its closure $\overline{J(\mathcal{D}^k(K))}$ may be identified with $\tilde{\mathcal{D}}^k(K)$, the completion of $\mathcal{D}^k(K)$. We give the details for $k = 1$ only and generalises for $k > 1$.

Proposition 3.10. *For a perfect, compact set $K \subset \mathbb{C}$ the following are equivalent:*

(a) *The completion $\tilde{\mathcal{D}}^1(K)$ of $\mathcal{D}^1(K)$ is a Banach function algebra on $K$.*
(b) *$\tilde{\mathcal{D}}^1(K)$ is semisimple.*
(c) *The operator $d/dz$ is closable in $C(K)$.*

Proof. $(a) \implies (b)$ is obvious.
$(b) \implies (c)$ Notice that $J : \mathcal{D}^1(K) \to C(K)^2$ as defined above by $J(f) = (f, f')$ is an isometric monomorphism and $\|\cdot\|_1^{\tilde{}}$ is just the graph norm on the domain $\mathcal{D}^1(K)$ of $d/dz$ and $J(\mathcal{D}^1(K))$ coincides with the graph of $d/dz$. If $\overline{J(\mathcal{D}^1(K))}$ is not a graph then there exists some $(0, g) \in \overline{J(\mathcal{D}^1(K))}$, with $g \neq 0$. Because of $(0, g)^2 = (0, 0)$, $g \in \mathrm{rad}(\overline{J(\mathcal{D}^1(K))})$. As $\overline{J(\mathcal{D}^1(K))}$ may be identified with $\tilde{\mathcal{D}}^1(K)$, this proves the implication $(b) \implies (c)$.

$(c) \Longrightarrow (a)$ Suppose now that $d/dz$ is closable and let $\tilde{d}$ be the closure of $d/dz$. Then $\tilde{\mathcal{D}}^1(K)$ coincides with the domain $\mathcal{D}(\tilde{d})$ of $\tilde{d}$ (endowed with the graph norm $\|\cdot\|_{\tilde{d}}$ w.r.t. $\tilde{d}$).

In particular, we have:

$$\|f\|_K \leq \|f\|_{\tilde{d}} = \|f\|_K + \|\tilde{d}f\|_K, \quad \text{for all } f \in \tilde{\mathcal{D}}^1(K).$$

Thus, $\tilde{\mathcal{D}}^1(K)$ is a Banach function algebra on $K$. □

Recall that a compact set $K \subset \mathbb{C}$ is *semirectifiable*, if the union of all rectifiable Jordan arcs in $K$ is dense in $K$.

REMARK 3.11. *If $K$ is a compact, semirectifiable set in $\mathbb{C}$, then $d/dz$ is closable in $C(K)$.*
*Indeed, by Theorem 6.3 in* [**16**] *the algebra $\tilde{\mathcal{D}}^1(K)$ is semisimple and $d/dz$ hence closable by Proposition 3.10(b).*

More general, the completion $\tilde{\mathcal{D}}^k(K)$ will be a Banach function algebra if $d/dz$ is closable in $C(K)$ endowed with the norm $\|\cdot\|_k^{\tilde{}}$ inherited from $C(K)^{k+1}$.

As seen from the above proposition that in general, $J(\mathcal{D}^k(K))$ is not closed in $C(K)^{k+1}$. An easy example for this situation is the following mentioned by Bishop in [**7**].

EXAMPLE 3.12. *Let $K$ be a perfect, compact, totally disconnected set in $\mathbb{C}$. For every $n \in \mathbb{N}$, there exists a finite open covering of $K$ by pairwise disjoint sets $U_{\alpha,n}$, $\alpha \in A_n$, such that $\operatorname{diam} U_{\alpha,n} < \frac{1}{n}$ for all $\alpha \in A_n$. Fix $t_{\alpha,n} \in K \cap U_{\alpha,n}$. Let $(f_n)_{n=1}^{\infty}$ be the sequence of functions given by*

$$f_n(z) := z - t_{\alpha,n}, \quad \text{for all } z \in U_{\alpha,n}, \qquad n \in \mathbb{N}.$$

*Thus, $f_n^{(j)}$ exists for all $j \in \mathbb{N}_\circ$ and*

$$f_n^{(1)} \equiv 1, \qquad f_n^{(j)} \equiv 0, \quad \text{for all } j > 1.$$

*It follows that, for $j \neq 1$,*

$$f_n^{(j)} \to 0, \quad \text{uniformly on } K \text{ for } n \to \infty$$

*and*

$$f_n^{(1)} \to 1, \quad \text{uniformly on } K \text{ for } n \to \infty.$$

*Thus,*

$$\overline{J(\mathcal{D}^k(K))} \neq J(\mathcal{D}^k(K)).$$

*and $d/dz$ is not even closable in $C(K)$.*

Bishop provided an example in [**7**], where $K$ is a compact, Jordan arc in $\mathbb{C}$, having no rectifiable subarcs. In particular, it has been shown for this example that given a tuple $(f_n)_{n=0}^{k}$ of continuous functions on $K$, there exists a sequence $(p_n)_{n=0}^{\infty}$ of polynomials such that each $f_i, 0 \leq i \leq k$, is approximated uniformly on $K$ by $p_n^{(i)}$.

**3.3.1. Completions of normed function algebras of complex ultra-differentiable functions.** Define $l^1(\mathbb{N}_\circ, C(K), \{M_p\})$ to be the sequence space of functions on $K$ with a weight sequence $(M_p)_{p\in\mathbb{N}_\circ}$ by:

$$l^1(\mathbb{N}_\circ, C(K), \{M_p\}) := \left\{ f = (f_p)_{p=0}^{\infty} \in C(K)^{\mathbb{N}_\circ}; |f|_{\{M_p\},1} := \sum_{p=0}^{\infty} \frac{1}{M_p} \|f_p\|_K < \infty \right\}$$

where $\|\cdot\|_K$ is the supremum norm on the set $K$. We endow $l^1(\mathbb{N}_\circ, C(K), \{M_p\})$ with the norm $|\cdot|_{\{M_p\},1}$.

Let $f = (f_p)_{p=0}^{\infty}$ and $g = (g_p)_{p=0}^{\infty} \in l^1(\mathbb{N}_\circ, C(K), \{M_p\})$. Then the multiplication $fg =: h = (h_p)_{p=0}^{\infty}$ is defined as:

$$h_p := \sum_{\nu=0}^{p} \binom{p}{\nu} f_\nu g_{p-\nu}.$$

It is easy to check that the above multiplication is continuous w.r.t. the norm $|\cdot|_{\{M_p\},1}$ , and hence $l^1(\mathbb{N}_\circ, C(K), \{M_p\})$ is a Banach algebra.

In a similar way, the space $l^\infty(\mathbb{N}_\circ, C(K), \{M_p\})$ is defined as:

$$l^\infty(\mathbb{N}_\circ, C(K), \{M_p\}) := \left\{ f = (f_p)_{p=0}^{\infty} \in C(K)^{\mathbb{N}_\circ}; \sup_{p\in\mathbb{N}_\circ} \frac{1}{M_p} \|f_p\|_K < \infty \right\}$$

We endow $l^\infty(\mathbb{N}_\circ, C(K), \{M_p\})$ with the norm $|\cdot|_{\{M_p\},\infty}$. The multiplication in $l^\infty(\mathbb{N}_\circ, C(K), \{M_p\})$ is defined as in $l^1(\mathbb{N}_\circ, C(K), \{M_p\})$. We observe that with the additional condition (3.4), the multiplication is continuous w.r.t. the norm $|\cdot|_{\{M_p\},\infty}$. Thus, $l^\infty(\mathbb{N}_\circ, C(K), \{M_p\})$ is a Banach algebra and may be endowed with an equivalent submultiplicative norm.

For $q = \{1, \infty\}$, we denote by $\mathcal{D}_q(K, \{M_p\})$ any of the algebras $\mathcal{D}_1(K, \{M_p\})$ or $\mathcal{D}_\infty(K, \{M_p\})$ and by $l^q(\mathbb{N}_\circ, C(K), \{M_p\})$ any of the algebras $l^1(\mathbb{N}_\circ, C(K), \{M_p\})$ or $l^\infty(\mathbb{N}_\circ, C(K), \{M_p\})$, respectively.

The mapping

$$J : \mathcal{D}_q(K, \{M_p\}) \to l^q(\mathbb{N}_\circ, C(K), \{M_p\}),$$

$$\text{defined by,} \quad J(f) := (f^{(p)})_{p=0}^{\infty}, \quad \text{for all } f \in \mathcal{D}_q(K, \{M_p\})$$

is an isometric algebra monomorphism from $\mathcal{D}_q(K, \{M_p\})$ into $l^q(\mathbb{N}_\circ, C(K), \{M_p\})$.

The completion $\tilde{\mathcal{D}}_q(K, \{M_p\})$ of the normed algebra $\mathcal{D}_q(K, \{M_p\})$ may then be identified with the closure of $J(\mathcal{D}_q(K, \{M_p\}))$ in $l^q(\mathbb{N}_\circ, C(K), \{M_p\})$, denoted by $J_q(K, \{M_p\})$

Next, we give a criterion for the completion $\tilde{\mathcal{D}}_q(K, \{M_p\})$ of the normed algebra $\mathcal{D}_q(K, \{M_p\})$ to be a Banach function algebra.

PROPOSITION 3.13. *Let $K \subseteq \mathbb{C}$ be a perfect, compact set such that $d/dz$ is closable in $C(K)$. Let $J$ be the isometric embedding of $\mathcal{D}_q(K, \{M_p\}), q = \{1, \infty\}$ in $l^q(\mathbb{N}_\circ, C(K), \{M_p\})$ as defined above with closure $J_q(K, \{M_p\})$, where as usual $(M_p)_{p\in\mathbb{N}_\circ}$ satisfying* (3.2) *and in addition, $(m_p)_{p=0}^{\infty}$ satisfying* (3.4) *for $q = \infty$. Let $P_\circ$ be the projection from $l^q(\mathbb{N}_\circ, C(K), \{M_p\})$ to $C(K)$ defined as:*

$$P_\circ f := f_\circ, \qquad f \in l^q(\mathbb{N}_\circ, C(K), \{M_p\}),$$

*then $P_\circ|J_q(K,\{M_p\})$ is an algebra monomorphism. Hence, if we endow its range $\tilde{\mathcal{D}}_q(K,\{M_p\}) := P_\circ(J_q(K,\{M_p\}))$ with the norm given by*

$$\|h\|^{\tilde{}}_{\{M_p\},q} := |g|_{\{M_p\},q}$$

*for all $h \in \tilde{\mathcal{D}}_q(K,\{M_p\})$, where $g = (g_k)_{k=0}^\infty \in J_q(K,\{M_p\})$ with $g_\circ = h$, then, $\tilde{\mathcal{D}}_q(K,\{M_p\})$ may be considered as the completion of $\mathcal{D}_q(K,\{M_p\})$ and $\tilde{\mathcal{D}}_q(K,\{M_p\})$ is a Banach function algebra on $K$.*

Proof. To show that $P_\circ|_{J_q(K,\{M_p\})}$ is an isometry, it suffices to prove the injectivity of $P_\circ|_{J_q(K,\{M_p\})}$.

Assume that this is not the case. Then, there exists some $0 \neq g = (g_p)_{p\in\mathbb{N}_\circ} \in J_q(K,\{M_p\})$ with $g_\circ \equiv 0$ and $g_p \not\equiv 0$ for some $p \in \mathbb{N}$. Let $p_\circ \in \mathbb{N}$ be minimal with that property, so that $g_{p_\circ-1} \equiv 0$ and $g_{p_\circ} \not\equiv 0$.

By the definition of $J_q(K,\{M_p\})$, there exists a sequence $(f_n)_{n=1}^\infty$ in $\mathcal{D}_q(K,\{M_p\})$ such that

$$(f_n^{(p)})_{p\in\mathbb{N}_\circ} \to g \quad \text{for } n \to \infty.$$

In particular,

$$f_n^{(p_\circ-1)} \to g_{p_\circ-1} \equiv 0 \quad \text{and} \quad \frac{df_n^{(p_\circ-1)}}{dz} = f_n^{(p_\circ)} \to g_{p_\circ} \not\equiv 0$$

uniformly on $K$, for $n \to \infty$ in contradiction to the fact that $d/dz$ is closable in $(C(K), \|\cdot\|_K)$. Hence, $P_\circ|_{J_q(K,\{M_p\})}$ is an isometric algebra isomorphism from $J_q(K,\{M_p\})$ to $\tilde{\mathcal{D}}_q(K,\{M_p\})$ and $\tilde{\mathcal{D}}_q(K,\{M_p\})$ is a Banach function algebra on $K$.

□

**3.3.2. Some regularity results on a set $K$.** Some criteria for completeness of $\mathcal{D}^1(K)$ algebras have been presented in [**16**], by assuming different conditions on the compact set $K$. Next, we recall some definitions concerning the regularity of a set $K$.

Definition 3.14. *Let $K$ be a compact subset of the complex plane $\mathbb{C}$. $K$ is said to be* `regular at a point` *$z \in K$, if there exists a positive constant $k_z$ and for each $w \in K$ there is a rectifiable path $\gamma$ from $z$ to $w$ in $K$, such that $|\gamma| \leq k_z|z-w|$, where $|\cdot|$ denotes the length of a rectifiable path.*

*A set $K$ is* `point-wise regular`*, if it is regular at every point $z$ of $K$. Further, $K$ is called* `uniformly regular`*, if there exists a positive constant $k$, such that for all $z$ and $w$ in $K$ and for a rectifiable path $\gamma$ from $z$ to $w$ in $K$, $|\gamma| \leq k|z-w|$.*

From Theorem 1.6 [**15**], a sufficient condition for function algebras $\mathcal{D}^k(K)$ and $\mathcal{D}_1(K,\{M_p\})$ to be complete is that $K$ should be a finite union of uniformly regular sets. However, if $K \subset \mathbb{C}$ is a perfect, compact set with infinitely many components then $\mathcal{D}_1(K,\{M_p\})$ is never complete (by Theorem 2.3 in [**10**]).

Definition 3.15. *Let $K \subset \mathbb{C}$ be a perfect, compact set. A point $a \in K$ will be called a* `UC-point`*, if there exists a sequence $(z_n)_{n=1}^\infty$ of points $z_n \in K \setminus \{a\}$*

*converging to $a$ for $n \to \infty$ with the property that the family $(\Delta_n)_{n=1}^{\infty}$ of linear functionals $\Delta_n : \mathcal{D}^1(K) \to \mathbb{C}$ with*

$$\Delta_n(f) := \frac{f(z_n) - f(a)}{z_n - a}, \qquad (f \in \mathcal{D}^1(K),\ n \in \mathbb{N})$$

*is uniformly continuous on $(\mathcal{D}^1(K), \|\cdot\|_1)$.*

Note that for all $f \in \mathcal{D}^1(K)$, we have in this situation $\Delta_n(f) \to f'(a)$ for $n \to \infty$. The set of all *$UC$ – points* in $K$ will be denoted by $UC(K)$.

EXAMPLE 3.16. *Let $b$ be a point in a perfect, compact set $K$ such that there exists a sequence $(z_n)_{n=1}^{\infty}$ of points in $K \setminus \{b\}$ converging to $b$ and a sequence of rectifiable paths $(\gamma_n)_{n=1}^{\infty}$ with $(\gamma_n)(0) = z_n$ and $(\gamma_n)(1) = b$ such that the length $|\gamma_n|$ of $\gamma_n$ satisfies*

$$|\gamma_n| \leq C_b |z_n - b|$$

*for all $n \in \mathbb{N}$, with a constant $C_b$ independent of $n$. Then, $b \in UC(K)$.*

PROOF. By the Fundamental Theorem of Calculus for rectifiable paths (in the form of Theorem 3.3 of [**10**], see [**9**] for the proof), we have

$$\begin{aligned}\Big|\frac{f(z_n) - f(b)}{z_n - b}\Big| &= \Big|\frac{1}{z_n - b}\int_{\gamma_n} f'\ dz\Big| \\ &\leq \|f'\|_K \cdot \frac{|\gamma_n|}{|z_n - b|} \leq C_b \|f'\|_K.\end{aligned}$$

□

This example shows in particular that if, $K \subset \mathbb{C}$ is a locally pointwise regular, compact set in the sense of [**10**], then $UC(K) = K$.

EXAMPLE 3.17. *Let $K := \{it\ ; -1 \leq t \leq 1\} \cup \big\{x + i \sin\frac{1}{x}\ ; x \in (0,1]\big\}$. Then, $UC(K) = K$ but $K$ is not locally pointwise regular, as $K$ is not path connected.*

REMARK 3.18. *If $K$ is a perfect, compact set and $I$ is a (not necessarily closed) interval in $\mathbb{R}$. If $\gamma : I \to \mathbb{C}$ is a non-constant piecewise smooth path with $\gamma(I) \subseteq K$, then $\gamma(I) \subset UC(K)$.*

REMARK 3.19. *If $K$ is a swiss cheese or one of the fractal sets considered in the previous Chapter, then $UC(K)$ is dense in $K$.*

PROPOSITION 3.20. *Let $K \subseteq \mathbb{C}$ be a perfect, compact set with $\overline{UC(K)} = K$. Then $d/dz$ is closable and the completion $\tilde{\mathcal{D}}^1(K)$ of $\mathcal{D}^1(K)$ is a Banach function algebra.*

PROOF. We may identify $\tilde{\mathcal{D}}^1(K)$ with the closure $J_1$ of $J := \{(f, f')|f \in \mathcal{D}^1(K)\}$ in $C(K)^2$.

Fix an arbitrary $a \in UC(K)$. Thus, there exists a sequence $(z_n)_{n=1}^{\infty}$ in $K \setminus \{a\}$ and a constant $C > 0$ such that

$$\Big|\frac{f(z_n) - f(a)}{z_n - a}\Big| \leq C\big(\|f\|_K + \|f'\|_K\big)$$

for all $n \in \mathbb{N}$ and all $f \in \mathcal{D}^1(K)$. As $J$ is dense in $J_1$, we conclude that:

$$\Big|\frac{f(z_n)-f(a)}{z_n-a}\Big| \leq C\big(\|f\|_K + \|g\|_K\big)$$

holds for all $n \in \mathbb{N}$ and all $(f,g) \in J_1$. In particular,

$$\Big|\frac{f(z_n)-f(a)}{z_n-a} - g(a)\Big| \leq (C+1)\big(\|f\|_K + \|g\|_K\big)$$

for all $n \in \mathbb{N}$ and all $(f,g) \in J_1$. As,

$$\Big|\frac{f(z_n)-f(a)}{z_n-a} - f^{'}(a)\Big| \to 0, \quad \text{for } n \to \infty$$

and all $(f, f^{'})$ in the dense subset $J$ of $J_1$, we conclude that:

$$\lim_{n\to\infty} \frac{f(z_n)-f(a)}{z_n-a} = g(a)$$

for all $(f,g) \in J_1$. Therefore, if $(f_k)$ is a sequence in $\mathcal{D}^1(K)$ such that

$$f_k \to 0 \quad \text{and} \quad f_k^{'} \to g \in C(K) \quad \text{for } k \to \infty,$$

uniformly on $K$, then in particular $(0,g) \in J_1$ and hence, we must have $g(a) = 0$ for all $a \in UC(K)$. As, $UC(K)$ is dense in $K$, we conclude that $g \equiv 0$ on $K$. Thus showing that, $d/dz$ is a closable linear operator in $C(K)$. □

## 3.4. Naturality

Proposition 3.21. *Let $K$ be a perfect, compact set in $\mathbb{C}$ with $d/dz$ closable in $C(K)$. Assume that* $\mathrm{Rat}(K) \subset \mathcal{D}_q(K, \{M_p\})$, $q = \{1, \infty\}$ *and let* $R_q(K, \{M_p\}) := \overline{\mathrm{Rat}(K)}^{\|\cdot\|_{\{M_p\},q}^{\tilde{}}}$ *be its closure in* $\tilde{\mathcal{D}}_q(K, \{M_p\})$. *Then,* $R_q(K, \{M_p\})$ *will be a natural Banach function algebra on $K$.*

Proof. The proof follows directly from Proposition 3.13 and Proposition 3.3$(a)$. □

It is still unknown that whether $\mathrm{Rat}(K)$ is (always) dense in $\mathcal{D}_q(K, \{M_p\})$, $q = \{1, \infty\}$ or not. This open question has also been mentioned in [**16**] for $\mathcal{D}^1(K)$ algebras and in [**25**] for $\mathcal{D}_1(K, \{M_p\})$ algebras on a perfect, compact set $K$.

Recall from Theorem 1.6 [**15**] that, for a compact set $K \subset \mathbb{C}$, which is a finite union of uniformly regular sets, the algebras $\mathcal{D}^k(K)$ and $\mathcal{D}_1(K, \{M_p\})$ are Banach function algebras. In [**15**], for such sets $K$ a condition on the sequence $(m_p)_{p=0}^{\infty}$, i.e.

$$\sum_{q=1}^{p-1} \frac{m_q^q m_{p-q}^{p-q}}{m_p^p} \to 0, \quad \text{as} \quad p \to \infty$$

has been given, which leads to a natural Banach function algebra $\mathcal{D}_1(K, \{M_p\})$, where as usual $m_p = \left(\frac{M_p}{p!}\right)^{1/p}$, $p \in \mathbb{N}_\circ$.

For a Banach function algebra $\mathcal{D}_1(K, \{M_p\})$, [**1**] provided general conditions on $(M_p)_{p=0}^{\infty}$ to obtain their naturality, when $K \subset \mathbb{C}$ is a uniformly regular compact set. They showed that whenever $P_p := \frac{M_p}{p!}$, $p \in \mathbb{N}_\circ$ (where, in our situation $P_p = m_p^p$ and where $(M_p)_{p\in\mathbb{N}_\circ}$ satisfying (3.2)) satisfies any of the following condition, the resulting Banach function algebra $\mathcal{D}_1(K, \{M_p\})$ will be natural:

$(i)$ $\sup\left\{\frac{m_k^k m_{n-k}^{n-k}}{m_{n-1}^{n-1}} : k, n \in \mathbb{N}\right\} < \infty$,

$(ii)$ $m_n^{2n} \leq m_{n-1}^{n-1} m_{n+1}^{n+1}, \quad \forall\, n \in \mathbb{N}$,

$(iii)$ $\max_{1\leq k\leq n-1} \frac{m_k^k m_{n-k}^{n-k}}{m_n^n} \to 0, \quad$ as $n \to \infty$

Following [**1**], we define $d(M) = \lim_{n\to\infty}\left(\frac{n!}{M_n}\right)^{1/n}$ and the sequence $(A_t)_{t=1}^{\infty}$ by:

$$A_t := \sup\left\{\frac{1}{m_n^n}\prod_{k=1}^{n}(m_k^k)^{a_k} : n \geq t, (a_1, \cdots, a_n) \in S(t,n)\right\},$$

where $m_n^n = P_n = \frac{M_n}{n!}$ and $S(t,n)$ is a set of all $a = (a_1, \cdots, a_n) \in \mathbb{Z}^{+n}$ such that $\sum_{k=1}^n a_k = t$ and $\sum_{k=1}^n k a_k = n$, for $t, n \in \mathbb{N}$ with $n \geq t$.

From Theorem 2.7 [**1**], a necessary condition for a complete function algebra $\mathcal{D}_1(K, \{M_p\})$ to be natural is that $d(M) = 0$, where as mentioned before that, in general $\mathcal{D}_1(K, \{M_p\})$ and hence, $\mathcal{D}_\infty(K, \{M_p\})$ are not always complete. In [**1**] Corollary 3.5, Corollary 3.6 and Corollary 3.8, the authors showed that whenever $K$ is a uniformly regular compact set in $\mathbb{C}$ and $m_n^n = P_n = \frac{M_n}{n!}, \quad (n \in \mathbb{N}_\circ)$ satisfies any one condition $(i) - (iii)$ of above, it gives $(A_t)^{1/t} \to 0$, for $t \to \infty$, which further implies that $d(M) = 0$ and thus yielding a natural Banach function algebra $\mathcal{D}_1(K, \{M_p\})$.

Theorem 3.22. *Let $K$ be a perfect, compact set in $\mathbb{C}$ and that $d/dz$ is closable in $C(K)$. Let $(M_p)_{p=0}^{\infty}$ be a sequence of positive reals satisfying* (3.2)*. Define the sequences $(m_p)_{p=0}^{\infty}$ and $(A_t)_{t=1}^{\infty}$ as above such that $\lim_{t\to\infty}(A_t)^{1/t} = 0$. Then, $\tilde{\mathcal{D}}_1(K, \{M_p\})$, the completion of $\mathcal{D}_1(K, \{M_p\})$ is a natural Banach function algebra on $K$.*

Proof. From Proposition 3.13, it is clear that $\tilde{\mathcal{D}}_1(K, \{M_p\})$ is a Banach function algebra. To show that $\tilde{\mathcal{D}}_1(K, \{M_p\})$ is natural, we see from the proof of Theorem 3.3 [**1**] that:

$$\|f^p\|_{\{M_p\},1} \leq \sum_{t=1}^{n}\binom{n}{t}\|f\|_K^{n-t}\Big[\underbrace{(A_t)^{1/t}(\|f\|_{\{M_p\},1} - \|f\|_K)}_{=:\varepsilon_t}\Big]^t,$$

for all $f \in \mathcal{D}_1(K, \{M_p\})$ and hence by the continuity of the norm $\|\cdot\|_{\{M_p\},1}^{\tilde{}}$ of $\tilde{\mathcal{D}}_1(K, \{M_p\})$, also for all $f \in \tilde{\mathcal{D}}_1(K, \{M_p\})$.

As, $\varepsilon_t = A_t^{1/t}(\|f\|_{\{M_p\},1} - \|f\|_K) \to 0$, as $t \to \infty$, we obtain by using Lemma 3.1 [**1**] that,

$$\limsup_{p\to\infty}(\|f^p\|_{\{M_p\},1}^{\tilde{}})^{1/p} \leq \|f\|_K, \quad \forall f \in \tilde{\mathcal{D}}_1(K, \{M_p\}). \tag{3.6}$$

Using Theorem 1.3 [**1**], we obtain that $\tilde{\mathcal{D}}_1(K, \{M_p\})$ is a natural Banach function algebra on $K$.

□

In particular, by Remark 3.11, this Theorem also applies to compact, semirectifiable sets in $\mathbb{C}$.

REMARK 3.23. *From the proof of Corollary* 3.5*, Corollary* 3.6 *and Corollary* 3.8 *in* [**1**]*, we observe that anyone of the condition* (*i*) − (*iii*) *above implies* $\lim_{m\to\infty} A_m^{1/m} = 0$*, even for the normed algebras* $\mathcal{D}_q(K, \{M_p\})$*,* $q = \{1, \infty\}$*.*

EXAMPLES 3.24. *Consider the sequences* $(m_p)_{p=0}^{\infty}$ *given in Examples 3.8. Then, they satisfy condition* (*iii*)*, i.e.*

$$\max_{1\le k\le n-1} \frac{m_k^k m_{n-k}^{n-k}}{m_n^n} \to 0, \quad \textit{as } n \to \infty, \tag{3.7}$$

*and* $\lim_{m\to\infty} A_m^{1/m} = 0$*.*

PROOF. By Lemma 3.9, it is clear that (3.7) holds for the Examples 3.8. Further, from the proof of Corollary 3.8 [**1**], $A_m^{1/m} \to 0$, as $m \to \infty$, thus yielding a natural Banach function algebra $\tilde{\mathcal{D}}_1(K, \{M_p\})$ under the assumptions of the above Theorem 3.22.

□

## 3.5. Locally inverse closed algebras

Recall from the introductory section that an algebra $\mathcal{A}$ of functions on a compact Hausdorff set $K$ is inverse-closed if, $\frac{1}{f} \in \mathcal{A}$ for all $f \in \mathcal{A}$ with $f(z) \neq 0$, for all $z$ in $K$. Equivalently saying that, each $f \in \mathcal{A}$ has natural spectrum. In [**34**], Rudin has considered the inverse-closedness of the algebras $C\{M_n\}$ of all complex functions $f$ on the real line, for which there exist constants $\beta_f$ and $B_f$ such that:

$$\|D^{(n)} f\|_K \le \beta_f B_f^n M_n, \quad n \in \mathbb{N}_\circ.$$

REMARK 3.25. *Let $K$ be a perfect, compact set and $H \neq 0$ be a compact subset of $K$. Let $f \in \mathcal{D}^k(K)$ for $0 \le k \le \infty$, such that $f(z) \neq 0$ for all $z \in H$. Then $\frac{1}{f}$ is $k -$ times continuously differentiable in all points of $H$.*

Next two results will deal with the local inverse-closedness of $\mathcal{D}_1(K, \{M_p\})$ algebra. For the proof, we follow the ideas of [**34**].

LEMMA 3.26. *Let* $(m_p)_{p=0}^{\infty}$ *and* $(\tilde{m}_p)_{p=0}^{\infty}$ *be monotone increasing sequences such that*

$$\lim_{p\to\infty} \frac{\tilde{m}_p}{m_p} = 0, \quad \textit{for all } p \in \mathbb{N}_\circ. \tag{3.8}$$

*Let $\mathcal{D}_1(K, \{M_p\})$ be the algebra defined before, where $K$ is a perfect, compact set in $\mathbb{C}$ and $\mathcal{D}_1(K, \{\tilde{M}_p\})$ be the corresponding algebra for $\tilde{M}_p := p!\tilde{m}_p^p$, $p \in \mathbb{N}_\circ$.*

a) *Let $H$ be a non-empty compact subset of $K$ and $f \in \mathcal{D}_1(K, \{\tilde{M}_p\})$ with the property that $f$ has no zero in $H$. Then, $1/f$ is infinitely often continuously complex differentiable in all points of $H$ and*

$$\|1/f\|_{\{M_p\},1,H} := \sum_{p=0}^{\infty} \frac{\|D^p(1/f)\|_H}{M_p} < \infty.$$

b) *Let $g \in \mathcal{D}_1(K, \{M_p\})$ and $f \in \mathcal{D}_1(K, \{\tilde{M}_p\})$ such that $|f(z)| > 0$ for all $z \in \operatorname{supp} g$, then $h = g/f \in \mathcal{D}_1(K, \{M_p\})$ with the norm $\|\cdot\|_{\{M_p\},1}$.*

PROOF. *a*) Choose a positive constant $\sigma$ on a neighbourhood of $H$ and define:

$$r_n := \frac{\sigma}{2\|f\|_{\{\tilde{M}_n\},1}\tilde{m}_n}, \quad n \in \mathbb{N}_\circ,$$

where,

$$\|f\|_{\{\tilde{M}_n\},1} = \sum_{n=0}^{\infty} \frac{1}{\tilde{M}_n}\|f^{(n)}\|_K = \sum_{n=0}^{\infty} \frac{1}{n!\tilde{m}_n{}^n}\|f^{(n)}\|_K.$$

Now, fix $n$ and $x_\circ \in H$, and define:

$$Q(z) = f(x_\circ) + Df(x_\circ)z + \cdots + \big(D^n f\big)(x_\circ)\frac{z^n}{n!}$$

Then, for $|z| \le r_n$, we have:

$$\begin{aligned}
|Q(z)| \ge & |f(x_\circ)| - \Big|\sum_{j=1}^{n} \frac{f^{(j)}(x_\circ)z^j}{j!}\Big| \\
\ge & \sigma - \sum_{j=1}^{n} \frac{|f^{(j)}(x_\circ)|}{j!} \frac{\sigma^j}{2^j\|f\|_{\{\tilde{M}_n\},1}\tilde{m}_n^j} \\
\ge & \sigma - \frac{\sigma}{2\|f\|_{\{\tilde{M}_n\},1}} \sum_{j=1}^{n} \frac{|f^{(j)}(x_\circ)|}{j!\tilde{m}_n^j}\Big(\frac{\sigma}{2\|f\|_{\{\tilde{M}_n\},1}}\Big)^{j-1} \\
\ge & \sigma - \frac{\sigma}{2\|f\|_{\{\tilde{M}_n\},1}}\|f\|_{\{\tilde{M}_n\},1} = \frac{\sigma}{2}.
\end{aligned}$$

Now, the first $n-$derivatives of $Q$ at $z = 0$ are equal to the first $n-$derivatives of $f$ at $x = x_\circ$, i.e.

$$\big(D^n\frac{1}{f}\big)(x_\circ) = \big(D^n\frac{1}{Q}\big)(0).$$

Using the Cauchy formula, we get:

$$\big(D^n\frac{1}{f}\big)(x_\circ) = \frac{n!}{2\pi i}\int_{|z|=r_n} \frac{dz}{z^{n+1}Q(z)}.$$

Then,

$$\Big|\big(D^n\frac{1}{f}\big)(x_\circ)\Big| \le \frac{n!}{r_n^n}\frac{2}{\sigma} = \frac{n!2}{\sigma}\Big(\frac{2\|f\|_{\{\tilde{M}_n\},1}\tilde{m}_n}{\sigma}\Big)^n$$

and

$$\|D^n\big(\frac{1}{f}\big)\|_H \le \frac{n!2}{\sigma}\Big(\frac{2\|f\|_{\{\tilde{M}_n\},1}\tilde{m}_n}{\sigma}\Big)^n \tag{3.9}$$

Further,

$$\begin{aligned}
\Big\|\frac{1}{f}\Big\|_{\{M_p\},1,H} = \sum_{p=0}^{\infty} \frac{\|D^p(\frac{1}{f})\|_H}{p!m_p^p} & \le \frac{2}{\sigma}\sum_{p=0}^{\infty}\Big(\frac{2\|f\|_{\{\tilde{M}_n\},1}\tilde{m}_p}{\sigma m_p}\Big)^p \\
& \le \frac{2}{\sigma}\sum_{p=0}^{\infty}\Big(\frac{\tilde{m}_p}{m_p}\Big)^p\Big(\frac{2\|f\|_{\{\tilde{M}_n\},1}}{\sigma}\Big)^p < \infty
\end{aligned} \tag{3.10}$$

from (3.9), the assumption (3.8) and that $f \in \mathcal{D}_1(K, \{\tilde{M}_p\})$.

*b*) We need to show that $g/f \in \mathcal{D}_1(K,\{M_p\})$, where

$$(\frac{g}{f})(z) := \begin{cases} 0 & , z \notin \operatorname{supp} g \\ (\frac{g}{f})(z) & , z \in \operatorname{supp} g \end{cases}$$

Then, by the Leibniz rule, $D^n(\frac{g}{f}) \equiv 0$ outside $\operatorname{supp} g$ and for $x_\circ \in \operatorname{supp} g$, we have:

$$\Big|(D^n(\frac{g}{f}))(x_\circ)\Big| \le \sum_{j=0}^{n} \binom{n}{j} \Big|(D^j \frac{1}{f})(x_\circ)\Big| \Big|(D^{n-j} g)(x_\circ)\Big|$$

and by using (3.9) with $H := \operatorname{supp} g$, we get:

$$\begin{aligned} \|D^n(\frac{g}{f})\|_{\operatorname{supp} g} &\le \sum_{j=0}^{n} \binom{n}{j} \Big(\frac{2\|f\|_{\{\tilde{M}_n\},1} \tilde{m}_j}{\sigma}\Big)^j \|g^{(n-j)}\|_K \frac{j!2}{\sigma} \\ &= \sum_{j=0}^{n} \frac{n!}{(n-j)!} \frac{2}{\sigma} \Big(\frac{2\|f\|_{\{\tilde{M}_n\},1} \tilde{m}_j}{\sigma}\Big)^j \|g^{(n-j)}\|_K \end{aligned}$$

Further,

$$\sum_{p=0}^{\infty} \frac{\|D^p(\frac{g}{f})\|_{\operatorname{supp} g}}{p! m_p^p} \le \sum_{p=0}^{\infty} \sum_{j=0}^{p} \frac{1}{(p-j)!} \frac{2}{\sigma} \Big(\frac{2\|f\|_{\{\tilde{M}_n\},1} \tilde{m}_j}{\sigma}\Big)^j \frac{\|g^{(p-j)}\|_K}{m_p^p}$$

By using (3.3) for $(m_p)_{p\in\mathbb{N}_\circ}$ and (3.8), we get that:

$$\Big\|\frac{g}{f}\Big\|_{\{M_p\},1} \le \|g\|_{\{M_p\},1} \Big\|\frac{1}{f}\Big\|_{\{M_p\},1,H} < \infty.$$

□

In the next two results, for the proof we follow the ideas in the proof of Lemma 3.4 in [**15**].

LEMMA 3.27. *Let $K$ be a perfect, compact subset of $\mathbb{C}$ and let $(m_n)_{n=0}^{\infty}$ be a monotone increasing sequence with $m_\circ = 1$ as defined before, such that*

$$\sum_{n=2}^{\infty} \max_{1\le k\le n-1} \frac{m_k^k m_{n-k}^{n-k}}{m_n^n} < \infty. \tag{3.11}$$

*Then, $\mathcal{D}_1(K,\{M_p\})$ is locally inverse closed in the following sense:*

a) *For all compact subsets $\emptyset \ne H$ of $K$ and all $f \in \mathcal{D}_1(K,\{M_p\})$ with $f(z) \ne 0$ for all $z \in H$, the function $1/f$ is infinitely often continuously complex differentiable in a neighbourhood of $H$ and*

$$\|1/f\|_{\{M_p\},1,H} = \sum_{p=0}^{\infty} \frac{\|D^p(1/f)\|_H}{M_p} < \infty.$$

b) *Let $g \in \mathcal{D}_1(K,\{M_p\})$ such that $f(z) \ne 0$ for all $z \in \operatorname{supp} g$, then $g/f \in \mathcal{D}_1(K,\{M_p\})$.*

PROOF. *a*) If $f \in \mathcal{D}_1(K,\{M_p\})$ has no zero in $H$, then there exists some $\delta > 0$ and an open neighbourhood $U$ of $H$ such that:

$$\inf_{z\in U\cap K} |f(z)| > \delta.$$

Without loss of generality we may assume that $\delta = 1$. In particular, $1/f$ is infinitely often continuously complex differentiable on $U \cap K$. As in [**15**], proof of Lemma 3.4, we now use the fact that $f \cdot \frac{1}{f} \equiv 1$ on $U \cap K$ and obtain for all $n \geq 1$

$$\beta_n \leq \sum_{k=1}^{n} \alpha_k \beta_{n-k} \frac{m_k^k m_{n-k}^{n-k}}{m_n^n}$$

where

$$\alpha_p := \frac{\|f^{(p)}\|_H}{M_p} \leq \|f\|_{\{M_p\},1} \text{ and } \beta_p := \frac{\|D^p(1/f)\|_H}{M_p} \text{ for all } p \in \mathbb{N}_\circ.$$

In particular, $\beta_\circ = \|1/f\|_H < 1$. Hence, for $n \geq 1$,

$$\begin{aligned}\beta_n \leq & \sum_{k=1}^{n-1} \alpha_k \beta_{n-k} \frac{m_k^k m_{n-k}^{n-k}}{m_n^n} + \alpha_n \\ \leq & \|f\|_{\{M_p\},1} \max_{1\leq k \leq n-1} \frac{m_k^k m_{n-k}^{n-k}}{m_n^n} \cdot \sum_{j=1}^{n-1} \beta_j + \alpha_n \\ \leq & \|f\|_{\{M_p\},1} C_n S_{n-1} + \alpha_n\end{aligned}$$

where $S_\circ = 0$ and

$$S_{n-1} := \sum_{j=1}^{n-1} \beta_j \quad \text{and} \quad C_n := \max_{1 \leq k \leq n-1} \frac{m_k^k m_{n-k}^{n-k}}{m_n^n}$$

Thus, for all $n \geq 1$, we have

$$S_n \leq S_{n-1}\big(1 + \|f\|_{\{M_p\},1} C_n\big) + \alpha_n.$$

Inductively, we obtain, $S_1 \leq \alpha_1$ and for $n > 1$:

$$\begin{aligned}S_n \leq & \sum_{j=1}^{n} \alpha_j \prod_{k=j+1}^{n} \big(1 + \|f\|_{\{M_p\},1} C_k\big) \\ \leq & \sum_{j=1}^{n} \alpha_j \prod_{k=1}^{\infty} \big(1 + \|f\|_{\{M_p\},1} C_k\big) \\ \leq & \|f\|_{\{M_p\},1} \cdot C(f)\end{aligned}$$

where $C(f) := \prod_{k=1}^{\infty} \big(1 + \|f\|_{\{M_p\},1} C_k\big)$ exists, as $\sum_{k=1}^{\infty} \|f\|_{\{M_p\},1} C_k < \infty$ by (3.11) and product over empty index set is defined to be 1. Hence,

$$\begin{aligned}\|1/f\|_{\{M_p\},1,H} = & \|1/f\|_H + \lim_{n\to\infty} S_n \\ \leq & \|1/f\|_H + \|f\|_{\{M_p\},1} C(f) < \infty.\end{aligned}$$

*b*) Consider a non-empty compact subset $H$ of $K$ such that $H := \operatorname{supp} g$. Since, $f(z) \neq 0$, for all $z \in \operatorname{supp} g$, we define a function:

$$h(z) := \big(\frac{g}{f}\big)(z) = \begin{cases} 0 & , z \notin \operatorname{supp} g \\ (\frac{g}{f})(z) & , z \in \operatorname{supp} g \end{cases}$$

Since we know from $a$) that $\|1/f\|_{\{M_p\},1,H} < \infty$, therefore, for $z \in \operatorname{supp} g$, using the Leibniz rule, the fact that $M_p = p! m_p^p$ and (3.3), for $0 \le q \le p \le \infty$, we have:

$$\frac{\|D^p(g/f)\|_H}{M_p} \le \sum_{q=0}^{p} \frac{1}{q!(p-q)! m_q^q m_{p-q}^{p-q}} \|D^q(1/f)\|_H \|D^{(p-q)}(g)\|_K$$

Thus,

$$\begin{aligned}\|g/f\|_{\{M_p\},1,H} \le & \sum_{p=0}^{\infty}\sum_{q=0}^{p} \frac{\|D^q(1/f)\|_H}{q! m_q^q} \frac{\|D^{(p-q)}(g)\|_K}{(p-q)! m_{p-q}^{p-q}} \\ \le & \|1/f\|_{\{M_p\},1,H} \|g\|_{\{M_p\},1} < \infty.\end{aligned}$$

□

EXAMPLES 3.28. *For $s > 1$ the condition (3.11) is satisfied for the sequences given by*

$$m_p = p!^{s/p} \qquad \text{and} \qquad m_p = p^s, \quad p \in \mathbb{N},$$

*where $m_\circ = 1$ in both cases.*

PROOF. We have seen in Lemma 3.9 that $S_{n,k}(s)$ is monotone decreasing in $k$ for $1 \le k \le n/2$ and symmetric in $k$ and $n-k$, its maximum must be attained for some $k \in \mathbb{N}$ with $k \le n/2$.
In the first case:

$$\max_{1 \le k \le n-1} \frac{m_k^k m_{n-k}^{n-k}}{m_n^n} \le \frac{m_1^1 m_{n-1}^{n-1}}{m_n^n} = \frac{1}{n^s}$$

and in the second case:

$$\max_{1 \le k \le n-1} \frac{m_k^k m_{n-k}^{n-k}}{m_n^n} \le \frac{m_1^1 m_{n-1}^{n-1}}{m_n^n} = \frac{(n-1)^{(n-1)s}}{n^{sn}} \le \frac{1}{n^s}$$

Thus, condition (3.11) is satisfied in both cases for $s > 1$. □

Next, we give a lemma which will be useful later in showing the regularity of a Banach function algebra.

LEMMA 3.29. *Let $K$ be a perfect, compact subset of $\mathbb{C}$ and $(m_n)_{n=0}^{\infty}$ be a monotone increasing sequence of positive reals with $m_\circ = 1$ and*

$$\sum_{k=1}^{n-1} \frac{m_k^k m_{n-k}^{n-k}}{m_n^n} \to 0, \ \text{as} \ \ n \to \infty. \tag{3.12}$$

*Then, $\mathcal{D}_\infty(K, \{M_p\})$ is locally inverse closed as follows:*

a) *For all compact subsets $\emptyset \ne H$ of $K$ and all $f \in \mathcal{D}_\infty(K, \{M_p\})$ with $f(z) \ne 0$ for all $z \in H$, the function $1/f$ is infinitely often continuously complex differentiable in a neighbourhood of $H$ and*

$$\|1/f\|_{\{M_p\},\infty,H} := \sup_{p \in \mathbb{N}_\circ} \frac{\|D^p(1/f)\|_H}{M_p} < \infty.$$

b) *Let $g \in \mathcal{D}_\infty(K, \{M_p\})$ such that $f(z) \ne 0$ for all $z \in \operatorname{supp} g$, then $g/f \in \mathcal{D}_\infty(K, \{M_p\})$.*

Proof. *a)* We follow the proof of Lemma 3.4 [**15**]. As, $f \in \mathcal{D}_\infty(K,\{M_p\})$ has no zero in $H$, there exists some $\delta > 0$ and an open neighbourhood $U$ of $H$ such that:

$$\inf_{z\in U\cap K} |f(z)| > \delta.$$

Without loss of generality we may assume that $\delta = 1$. Then, $1/f$ is infinitely often continuously complex differentiable on $U \cap K$.

Choose some constant $C > 0$ such that $\|f\|_{\{M_p\},\infty} < C < \infty$ and define

$$\alpha_p := \frac{\|f^{(p)}\|_H}{M_p}, \quad \beta_p := \frac{\|D^p(1/f)\|_H}{M_p} \quad \text{for all } p \in \mathbb{N}_\circ.$$

Since $f \cdot \frac{1}{f} \equiv 1$ on $U \cap K$, we have for all $n \geq 1$

$$\beta_n \leq \sum_{k=1}^{n} \alpha_k \beta_{n-k} \frac{m_k^k m_{n-k}^{n-k}}{m_n^n}.$$

In particular, $\beta_\circ = \|1/f\|_H < 1$. Hence, for $n \geq 1$, we get:

$$\beta_n \leq \sum_{k=1}^{n-1} \alpha_k \beta_{n-k} \frac{m_k^k m_{n-k}^{n-k}}{m_n^n} + \alpha_n.$$

Choose a positive constant $N = N(C)$ such that

$$\sum_{k=1}^{n-1} \frac{m_k^k m_{n-k}^{n-k}}{m_n^n} < \frac{1}{2C}, \quad \text{for } n \geq N.$$

Then, for $n \leq N$, we have, $\beta_n \leq C_\circ$, where $C_\circ = C_\circ(C) \geq 2C$. For $n \geq N$, we have

$$\beta_{n+1} \leq \frac{1}{2} \max\{\beta_k \ ; k \leq n\} + C$$

which gives $\beta_{n+1} \leq C_\circ$.

Thus, by induction on $n$, we obtain that:

$$\beta_n \leq C_\circ, \quad \text{for all } n \in \mathbb{N}_\circ.$$

Hence,

$$\|1/f\|_{\{M_p\},\infty,H} = \sup_{p\in\mathbb{N}_\circ} \frac{\|D^p(1/f)\|_H}{M_p} = \sup_{p\in\mathbb{N}_\circ} \beta_p \leq C_\circ < \infty.$$

*b)* If $f, g \in \mathcal{D}_\infty(K,\{M_p\})$ such that $f$ has no zero on $H := \operatorname{supp} g$, then from *a)*, $\|1/f\|_{\{M_p\},\infty,H} < \infty$. Moreover, by Leibniz rule, using $M_p = p! m_p^p$, for all $p \in \mathbb{N}_\circ$ and (3.12), we obtain:

$$\begin{aligned}\|g/f\|_{\{M_p\},\infty} &= \sup_{p\in\mathbb{N}_\circ} \frac{\|D^p(g/f)\|_H}{M_p} \leq \sup_{p\in\mathbb{N}_\circ} \sum_{q=0}^{p} \frac{\|D^q(1/f)\|_H \|D^{(p-q)}(g)\|_K}{p! m_p^p} \\ &\leq \|1/f\|_{\{M_p\},\infty,H} \|g\|_{\{M_p\},\infty} \cdot \sup_{p\in\mathbb{N}_\circ} \sum_{q=0}^{p} \frac{m_q^q m_{p-q}^{p-q}}{m_p^p} < \infty.\end{aligned}$$

□

Examples 3.30. *Examples for this situation have already been given and proved in Lemma 3.9.*

## 3.6. Order of growth and coefficients of an entire function

This section shows the relation between order of growth and coefficients of entire functions and applications will be given from $(m_p)_{p\in\mathbb{N}_\circ}$ sequences.

PROPOSITION 3.31. *Let* $F(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an entire function and for positive constants* $c, d, \alpha$ *with* $0 < \alpha \leq 1$, $h(r) = c\exp(dr^\alpha)$. *Then, the following hold:*

a) *If* $M(r) := \max_{|z|=r} |F(z)| \leq \exp(h(r))$, *then*

$$|a_n| \leq \Big(\frac{2d\exp}{\log\frac{n}{c\alpha}}\Big)^{\frac{n}{\alpha}}.$$

b) *If* $|a_n| \leq A(n) := \Big(\frac{C}{\log\frac{n}{c_1}}\Big)^{\frac{n}{\alpha}}$, *with positive constants* $C, c_1$ *and* $\alpha \in (0,1]$ *then*

$$M(r) = \max_{|z|=r} |F(z)| \leq \exp(c_2\exp(d_2r^\alpha))$$

*for some positive constants* $c_2$ *and* $d_2$.

PROOF. $(a)$ Define a function $u(r) := rh'(r) = cd\alpha r^\alpha \exp(dr^\alpha)$. For $r \geq d^{-1/\alpha}$, define monotone increasing functions $g_1(r)$ and $g_2(r)$ such that

$$g_1(r) := c\alpha\exp(dr^\alpha) \leq u(r) \leq c\alpha\exp(2dr^a) =: g_2(r) \tag{3.13}$$

Since, the inverse functions of $g_1(r), g_2(r)$ and $u(r)$ exist, we get:

$$g_1^{-1}(t) = \Big(\frac{1}{d}\log\frac{t}{c\alpha}\Big)^{1/\alpha}$$

and

$$g_2^{-1}(t) = \Big(\frac{1}{2d}\log\frac{t}{c\alpha}\Big)^{1/\alpha}$$

From (3.13) we obtain, for $t \geq g_2((\frac{1}{d})^{1/\alpha})$ that

$$g_2^{-1}(t) \leq u^{-1}(t) \leq g_1^{-1}(t) \tag{3.14}$$

From [**22**] Section 2.2, Theorem 2, we have that:

$$\begin{aligned}|a_n| \leq& \frac{\exp(h(u^{-1}(n)))}{(u^{-1}(n))^n} \leq \frac{\exp(h(g_1^{-1}(n)))}{(g_2^{-1}(n))^n}\\ =&\Big(\frac{2d\exp}{\log\frac{n}{c\alpha}}\Big)^{n/\alpha}\end{aligned}$$

$(b)$ From the assumption

$$|a_n| \leq A(n) = \Big(\frac{C}{\log\frac{n}{c_1}}\Big)^{n/\alpha}$$

we obtain:

$$\begin{aligned}|a_n|r^n =& \exp\Big(\frac{n}{\alpha}\log r^\alpha + \log a_n\Big)\\ \leq& \exp\Big(\frac{n}{\alpha}\log\frac{Cr^\alpha}{\log\frac{n}{c_1}}\Big)\\ =& \exp\Big(\frac{n}{\alpha}\Big\{\log r^\alpha + \log C - \log\log\frac{n}{c_1}\Big\}\Big)\end{aligned}$$

Let $r \geq \exp(1/\alpha) =: r_\circ$ and $\varepsilon > 0$. Then,

$$\log \frac{Cr^\alpha}{\log \frac{n}{c_1}} < -\varepsilon$$
$$\text{if and only if} \quad n \geq c_1 \exp(Cr^\alpha \exp(\varepsilon)) =: t_2 = t_2(\varepsilon)$$

Thus, for $r \geq r_\circ$ and $n \geq t_2(\varepsilon)$, we have $|a_n| r^n \leq \exp(-\varepsilon n/\alpha)$. Also,

$$\log \frac{C}{\log \frac{n}{c_1}} \geq 0, \quad \text{if and only if} \quad n \leq c_1 \exp(C) =: t_1$$

Because of $t_2 = t_2(\varepsilon) > t_1$, we have:

$$\begin{aligned}\sum_{n=0}^{\infty} a_n r^n &= S_1 + S_2 + S_3 \\ &= \sum_{n \leq t_1} a_n r^n + \sum_{t_1 < n \leq t_2} a_n r^n + \sum_{n > t_2} a_n r^n\end{aligned}$$

Now, $S_1 := \sum_{n \leq t_1} a_n r^n$ is a polynomial of order $\leq t_1$.

$$\begin{aligned}S_2 \leq \sum_{t_1 < n \leq t_2} |a_n| r^n &\leq \sum_{t_1 < n \leq t_2} \exp\left(\frac{n}{\alpha} \log cr^\alpha\right) \\ &\leq t_2 \exp\left(\frac{t_2}{\alpha} \log cr^\alpha\right) \\ &= c_1 \exp(Cr^\alpha \exp(\varepsilon)) \exp(\tilde{c_1} \exp(Cr^\alpha \exp(\varepsilon)) \log r)\end{aligned}$$

where $\log r < \exp(\varepsilon r^\alpha)$. Hence, for $r \geq r_1(\varepsilon)$, we get:

$$\begin{aligned}S_2 &\leq c_1 \exp(Cr^\alpha \exp(\varepsilon)) \exp(\tilde{c_1} \exp(C \exp(\varepsilon) + \varepsilon) r^\alpha) \\ &\leq \exp(2\tilde{c_1} \exp(C \exp(\varepsilon) + \varepsilon) r^\alpha)\end{aligned}$$

$$\begin{aligned}S_3 \leq \sum_{n > t_2} |a_n| r^n &\leq \sum_{n > t_2} \exp(-n\varepsilon/\alpha) \\ &\leq \frac{\exp(-t_2\varepsilon/\alpha)}{1 - 1/\exp(\varepsilon/\alpha)} = C(\varepsilon)\end{aligned}$$

Thus, there exists $\tilde{c_1} > 0$ such that

$$M(r) \leq \sum_{n=0}^{\infty} |a_n| r^n \leq \exp(2\tilde{c_1} \exp(C \exp(\varepsilon) + \varepsilon) r^\alpha)$$

for all $r \geq r_1(\varepsilon) \geq r_\circ$. Hence,

$$M(r) \leq \exp(c_2 \exp(d_1 r^\alpha))$$

for some constant $d_1 \geq C$ and $c_2 > \tilde{c_1}$.

$\square$

Corollary 3.32. *Let $0 < \alpha \leq 1$, $m_\circ = 1$ and $m_n^n = \left(\prod_{k=1}^n \log(k+e)\right)^{1/\alpha}$ for all $n \in \mathbb{N}$. If $(a_n)_{n=0}^\infty$ is a sequence satisfying $|a_n| \leq m_n^{-n}$, then for all $\varepsilon \in (0, 1-\alpha)$ the entire function $z \mapsto F(z) = \sum_{n=0}^\infty a_n z^n$ satisfies*

$$M(r) = \max_{|z|=r} |F(z)| \leq \exp(c \exp(dr^{\alpha+\varepsilon}))$$

*with some positive constants $c = c(\varepsilon)$ and $d = d(\varepsilon)$.*

PROOF. Fix $\delta \in (0,1)$ such that $\frac{\alpha}{1-\delta} < \alpha + \varepsilon$. By the choice of $\delta$, we can write $m_n^n$ as:

$$m_n^n = \Big(\prod_{1\le k\le\tilde{\delta}n} \log(k+e)\Big)^{1/\alpha} \cdot \Big(\prod_{\tilde{\delta}n\le k\le n} \log(k+e)\Big)^{1/\alpha},$$

where $\tilde{\delta}n$ is the smallest integer $\geq \delta n$. Since the left term of the product is always $\geq 1$, we have:

$$\begin{aligned} m_n^n \geq & \prod_{\tilde{\delta}n\le k\le n} \big(\log(k+e)\big)^{1/\alpha} \\ \geq & \big(\log(\delta n + e)\big)^{\frac{n(1-\delta)}{\alpha}} \\ \geq & \big(\log(\delta n + e)\big)^{\frac{n}{\alpha+\varepsilon}} \\ = & \big(\log(n(\delta + e/n)\big)^{\frac{n}{\alpha+\varepsilon}} \geq \big(\log(\delta n)\big)^{\frac{n}{\alpha+\varepsilon}} \end{aligned}$$

for $n$ larger than some $n_1$. Hence, $|a_n| \leq m_n^{-n} \leq \Big(\frac{1}{\log(\delta n)}\Big)^{\frac{n}{\alpha+\varepsilon}}$. From Proposition 3.31($b$), we obtain that:

$$M(r) \leq \exp(c\exp(dr^{\alpha+\varepsilon})).$$

□

PROPOSITION 3.33. *Let $(m_n)_{n=0}^{\infty}$ be a monotone increasing sequence with $m_\circ = 1$ and $m_n \to \infty$, for $n \to \infty$. Let $(a_n)_{n=0}^{\infty}$ be a sequence such that*

$$|a_n| \leq \frac{1}{m_n^n}, \qquad \textit{for all} \quad n \in \mathbb{N}_\circ.$$

*Consider the entire function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ and put $M(r) := \sup_{|z|=r} |F(z)| \leq \sum_{n=0}^{\infty} \big(\frac{r}{m_n}\big)^n$. Define*

$$n_\circ(r) := \min\{n \in \mathbb{N} : r + 1 \leq m_n\}$$

*then, for $r \geq 1$,*

$$M(r) \leq n_\circ(r)\exp(n_\circ(r)\cdot\log r) + (r+1).$$

PROOF. For $r \geq 1$, $M(r)$ can be estimated as:

$$\begin{aligned} M(r) &\leq \sum_{n=0}^{n_\circ(r)-1} \big(\frac{r}{m_n}\big)^n + \sum_{n=n_\circ(r)}^{\infty} \big(\frac{r}{m_n}\big)^n \\ &\leq n_\circ(r)\cdot \max_{0\le n\le n_\circ(r)} \big(\frac{r}{m_n}\big)^n + \sum_{n=n_\circ(r)}^{\infty} \big(\frac{r}{r+1}\big)^n \\ &\leq n_\circ(r) r^{n_\circ(r)} + \big(\frac{r}{r+1}\big)^{n_\circ(r)} \frac{1}{1-\frac{r}{r+1}} \end{aligned}$$

Thus, $M(r) \leq n_\circ(r)\exp(n_\circ(r)\cdot\log r) + (r+1)$ □

EXAMPLES 3.34.

(1) *Let* $m_\circ = 1$ *and* $m_n = n^s$, *for* $s > 0$, $n \in \mathbb{N}$. *Then, by the definition of* $n_\circ(r)$ *and* $m_n$,

$$r + 1 \le n^s \Leftrightarrow n \ge (r+1)^{1/s}.$$

*Thus,* $n_\circ(r) \le (r+1)^{1/s} + 1$ *and*

$$\begin{aligned} M(r) &\le ((r+1)^{1/s}+1)\exp\big(((r+1)^{1/s}+1)\log r\big) + r + 1 \\ &\le c\exp(dr^{1/s}\log r) \end{aligned}$$

*for positive constants* $c, d$ *only depending on* $s$ *and not on* $r$.

(2) *Let* $m_\circ = 1$ *and* $m_n = (\log(e+n))^{1/\alpha}$, *for* $\alpha \in (0,1]$.

*Then, by definition of* $n_\circ(r)$, *we have:*

$$\begin{aligned} r + 1 &\le (\log(e+n))^{1/\alpha} \\ &\Leftrightarrow \exp(r+1)^\alpha - e \le n \end{aligned}$$

*Hence,* $n_\circ(r) \le \exp(r+1)^\alpha$, *and we obtain:*

$$\begin{aligned} M(r) &\le \exp(r+1)^\alpha \exp\big(\exp(r+1)^\alpha \log r\big) + r + 1 \\ &\le \exp(c\exp(dr^\alpha)\log r) \end{aligned}$$

*for some positive constants* $c$ *and* $d$ *only depending on* $\alpha$.

## 3.7. Regular Banach function algebras

We need a criterion to show that the operator of multiplication by the coordinate function is decomposable on $\tilde{\mathcal{D}}_q(K, \{M_p\})$, for $q = \{1, \infty\}$.

THEOREM 3.35. *Let* $K \subset \mathbb{C}$ *be a perfect, compact set with* $\lambda(K) = 0$. *Let* $\tilde{M}_z$ *be the operator of multiplication on* $l^q(\mathbb{N}_\circ, C(K), \{M_p\})$, $q = \{1, \infty\}$ *defined by:*

$$\tilde{M}_z(g) := J(\mathrm{id}_K)\cdot g = \big(zg_p + pg_{(p-1)}\big)_{p=0}^\infty, \quad g = (g_p)_{p=0}^\infty \in l^q(\mathbb{N}_\circ, C(K), \{M_p\}),$$

*where same as in Section 3.3.1,* $J$ *is the isometric algebra monomorphism from* $\mathcal{D}_q(K, \{M_p\})$ *to* $l^q(\mathbb{N}_\circ, C(K), \{M_p\})$.

*Assume that* $m_p \to \infty$, *for* $p \to \infty$ *and define an entire function* $\omega : \mathbb{C} \to \mathbb{C}$ *by,*

$$\omega(\xi) := \sum_{p=0}^\infty \frac{\xi^{p+1}}{m_p^p}, \qquad \xi \in \mathbb{C}.$$

*Moreover, suppose that* $V(K)$ *be an open, bounded neighbourhood of* $K$ *and* $f$ *satisfy the condition* $(\eta)$. *If, the integral*

$$I_V := \iint\limits_{V(K)} f\Big(\log^+\log^+\omega\big(\frac{1}{\mathrm{dist}(\xi, K)}\big)\Big)\, d\lambda(\xi) < \infty,$$

*then,* $\tilde{M}_z$ *is decomposable.*

PROOF. For $\xi \in \mathbb{C}\setminus K$, define the function $h \in \mathrm{Rat}(K)$ by;

$$h(z) = \frac{1}{\xi - z}, \quad \text{for all } z \in K.$$

Then,

$$\frac{h^{(p)}}{M_p} = \frac{p!}{(\xi - z)^{p+1}M_p} \quad \text{and} \quad \frac{\|h^{(p)}\|_K}{M_p} \le \frac{d(\xi)^{p+1}}{m_p^p}$$

where $d(\xi) := \frac{1}{\mathrm{dist}(\xi,K)}$. Hence,

$$\|h\|_{\{M_p\},1} = \sum_{p=0}^{\infty} \frac{(d(\xi))^{p+1}}{m_p^p} = \omega(d(\xi))$$

and

$$\|h\|_{\{M_p\},\infty} = \sup_{p\in\mathbb{N}_\circ} \frac{(d(\xi))^{p+1}}{m_p^p} \leq \omega(d(\xi)).$$

Note that, $(\xi - \tilde{M}_z)^{-1}$ is the operator of multiplication by $J(h)$. Therefore, for some positive constant $C$ and $q = \{1,\infty\}$, we obtain:

$$\|(\xi - \tilde{M}_z)^{-1}\| \leq C\|h\|_{\{M_p\},q} \leq C\cdot\omega(d(\xi)).$$

The decomposability of $\tilde{M}_z$ is now immediate from Remark 1.12 for $T = \tilde{M}_z$, $\Omega = V(K)$ and $K = \sigma(\tilde{M}_z)$ and from Remark 1.13 with $K \setminus \Omega = \emptyset$. $\square$

Remark

From the above Theorem, it is interesting to note that $\tilde{M}_z$ is decomposable on $l^q(\mathbb{N}_\circ, C(K), \{M_p\})$, $q \in \{1,\infty\}$ and on all closed subalgebras of $l^q(\mathbb{N}_\circ, C(K), \{M_p\})$ such that $J_q(K, \{M_p\}) \subset l^q(\mathbb{N}_\circ, C(K), \{M_p\})$, $(q = \{1,\infty\})$ or which contain $f_{\mathrm{id}} := (\mathrm{id}, 1, 0, \cdots)$ and $J(h)$ for all $h \in \mathrm{Rat}(K)$, where $J_q(K,\{M_p\})$ is the closure of $J(\mathcal{D}_q(K,\{M_p\}))$ in $l^q(\mathbb{N}_\circ, C(K), \{M_p\})$ as defined in Section 3.3.1.

Corollary 3.36. *Assume that all conditions of Theorem 3.35 hold and $d/dz$ is closable in $C(K)$. Then, $\tilde{\mathcal{D}}_q(K,\{M_p\})$, for $q = \{1,\infty\}$ will be normal on $K$.*

Proof. Since all conditions of Theorem 3.35 hold, we get that $\tilde{M}_z$ is decomposable and thus by Corollary 3.5 $\tilde{\mathcal{D}}_q(K,\{M_p\})$, for $q = \{1,\infty\}$ is normal on $K$. $\square$

Combining Theorem 3.22 and Corollary 3.36, we obtain a sufficient condition for $\tilde{\mathcal{D}}_1(K,\{M_p\})$ to be a natural, normal Banach function algebra on $K$ (and hence a regular Banach function algebra).

Corollary 3.37. *Let $d/dz$ be closable and assume that all conditions of Theorem 3.35 hold. Let $(A_m)_{m=0}^{\infty}$ be a sequence as defined in Section 3.4 such that $\lim_{m\to\infty}(A_m)^{1/m} = 0$. Then, $\tilde{\mathcal{D}}_1(K,\{M_p\})$ is a regular Banach function algebra on $K$.*

Proof. From Theorem 3.22 we obtain that $\tilde{\mathcal{D}}_1(K,\{M_p\})$ is natural on $K$. Further, Corollary 3.36 gives that $\tilde{\mathcal{D}}_1(K,\{M_p\})$ is normal on $K$ and hence (by the definitions of normality and naturality) regular Banach function algebra on $K$. $\square$

Proposition 3.38. *Let $K$ be a perfect, compact set in $\mathbb{C}$ with upper box dimension $d < 2$. Assume that $d/dz$ is closable and in addition that for some positive constants $C, c_1$ and $0 < \alpha < \min\{1, 2-d\}$,*

$$\frac{1}{m_p} \leq \Big(\frac{C}{\log\frac{p}{c_1}}\Big)^{1/\alpha}. \tag{3.15}$$

*Then, $\tilde{M}_z$ will be decomposable and $\tilde{\mathcal{D}}_q(K,\{M_p\})$ for $q = \{1,\infty\}$ will be normal on $K$.*

*In particular, if* $\lim_{t\to\infty}(A_t)^{1/t} = 0$*, where* $(A_t)_{t=0}^{\infty}$ *is a sequence as defined in Section 3.4, then* $\tilde{\mathcal{D}}_1(K, \{M_p\})$ *is a regular Banach function algebra.*

PROOF. It is clear from (3.15) that $m_p \to \infty$, for $p \to \infty$. Since, $\overline{\dim}_B(K) < 2$, which clearly shows from Remark 2.2 that, $\lambda(K) = 0$. Thus, following the proof of Theorem 3.35, we observe that the norm of the rational function $h \in \mathrm{Rat}(K)$ defined by;

$$h(z) = \frac{1}{\xi - z}, \quad \xi \in \mathbb{C} \setminus K, \quad \text{for all } z \in K$$

will be:

$$\begin{aligned}\|h\|_{\{M_p\},q} \leq& \omega\Big(\frac{1}{d(\xi)}\Big) = \sum_{p=0}^{\infty} \frac{1}{m_p^p (d(\xi))^{p+1}} \\ \leq& \sum_{p=0}^{\infty} \Big(\frac{C}{\log \frac{p}{c_1}}\Big)^{p/\alpha} \frac{1}{(d(\xi))^{p+1}}\end{aligned}$$

where $d(\xi) := \mathrm{dist}(\xi, K)$.

From Proposition 3.31($b$), we get for positive constants $c_2$ and $d_2$ that:

$$\|h\|_{\{M_p\},q} \leq \exp(c_2 \exp(d_2 (d(\xi))^{\alpha})).$$

This shows from Theorem 2.5 that, for a closed square $Q$ containing $K$ with $|G(\xi)| = \|(\xi - z)^{-1}\|$ and for some $\varepsilon > 0$, the integral

$$I_Q = \iint\limits_{Q} \Big( \log^+ \log^+ \|(\xi - z)^{-1}\|\Big)^{1+\varepsilon} \, d\lambda(\xi)$$

is finite. Thus, by Theorem 3.35 for $V(K) = \partial Q$ and $f(t) = t^{1+\varepsilon}$, $t \geq 0$, $\tilde{M}_z$ is decomposable. Further, from Corollary 3.5, we obtain the normality of $\tilde{\mathcal{D}}_q(K, \{M_p\})$, for $q = \{1, \infty\}$ on $K$.

It is clear from Theorem 3.22 that $\tilde{\mathcal{D}}_1(K, \{M_p\})$ is natural on $K$ and hence, a regular Banach function algebra. □

The following Propositions hold for $(m_p)_{p=0}^{\infty}$, $p \in \mathbb{N}_\circ$, as in Examples 3.8.

PROPOSITION 3.39. *Let* $K \subset \mathbb{C}$ *be a perfect, compact set. Assume that* $d/dz$ *is closable in* $C(K)$ *and that there exists* $\delta > 0$ *such that* $K$ *satisfies the grid dimension condition of type* $(2, \delta)$.
*Moreover, assume that for some* $\alpha > 0$*:*

$$\limsup_{p\to\infty} \frac{\log p}{\log m_p} =: \alpha < \infty. \tag{3.16}$$

*Then,* $\tilde{M}_z$ *is decomposable and* $\tilde{\mathcal{D}}_q(K, \{M_p\})$, $q \in \{1, \infty\}$ *are normal on* $K$*.*
*In particular, if* $\lim_{t\to\infty}(A_t)^{1/t} = 0$*, where* $(A_t)_{t=0}^{\infty}$ *is a sequence as defined in Section 3.4, then* $\tilde{\mathcal{D}}_1(K, \{M_p\})$ *is a regular Banach function algebra.*

PROOF. As in the proof of the previous Proposition, we see that for the function $z \mapsto \frac{1}{\xi - z}$, $\xi \in \mathbb{C} \setminus K$, for all $z \in K$, the norm can be estimated as:

$$\Big\|\frac{1}{\xi - z}\Big\|_{\{M_p\},q} \leq \sum_{p=0}^{\infty} \frac{1}{m_p^p (d(\xi))^{p+1}} = \omega\Big(\frac{1}{d(\xi)}\Big)$$

where as usual, $d(\xi) = \mathrm{dist}(\xi, K)$.

We see from Theorem 2.2.2 [**11**] that, (3.16) denotes the order of the entire function $\omega$, which is finite and thus, from formula 2.1.5 [**11**] we obtain that, for some $\varepsilon > 0$,

$$\Big\|\frac{1}{\xi - z}\Big\|_{\{M_p\},q} \leq c_1 \exp\Big(\frac{1}{(d(\xi))^{\alpha+\varepsilon}}\Big),$$

where $c_1$ is a positive constant. Therefore, from Theorem 2.7, we see for a closed square $Q \supset K$ and $|G(\xi)| = \|(\xi - z)^{-1}\|$ that for some $\varepsilon > 0$, the integral $I_Q$ (mentioned in the proof of the previous Proposition) is finite. Hence, from Theorem 3.35, for $f(t) = t^{1+\varepsilon}$, $t \geq 0$ and $V(K) = \partial Q$, the operator $\tilde{M}_z$ is decomposable and from Corollary 3.5, $\tilde{\mathcal{D}}_q(K, \{M_p\})$, for $q = \{1, \infty\}$ are normal on $K$.

It is clear from Theorem 3.22 that $\tilde{\mathcal{D}}_1(K, \{M_p\})$ is natural on $K$ and hence, a regular Banach function algebra. $\square$

Proposition 3.40. *Let $K$ be a perfect, compact set in $\mathbb{C}$. Assume that $d/dz$ is closable and that there exists $\delta > 0$ such that $K$ satisfies the grid dimension condition of type $(3, \delta)$. Then, $\tilde{M}_z$ is decomposable and $\tilde{\mathcal{D}}^k(K)$, $k \in \mathbb{N}$, is normal on $K$.*

Proof. Clearly, from the proof of previous Propositions we see that, the norm of the function $z \mapsto \frac{1}{\xi - z}$, $\quad \xi \in \mathbb{C} \setminus K$, $\quad$ for all $z \in K$ can be estimated as:

$$\Big\|\frac{1}{\xi - z}\Big\|_k = \sum_{j=0}^{k} \frac{1}{j!}\Big\|\frac{1}{(\xi - z)^{(j+1)}}\Big\|_K \leq \frac{C}{(d(\xi))^{k+1}},$$

for some constant $C > 0$ and $d(\xi) = \mathrm{dist}(\xi, K)$. Further, from Theorem 2.10 for a closed square $Q \supset K$ and $|G(\xi)| = \|(\xi - z)^{-1}\|$ and for some $\varepsilon > 0$, we obtain that the integral $I$ (as mentioned in the proof of previous Propositions) is finite which gives the decomposability of $\tilde{M}_z$ and from Corollary 3.5 we obtain the normality of $\tilde{\mathcal{D}}^k(K)$ on $K$.

$\square$

CHAPTER 4

# Localisations of complex ultra-differentiable Banach function algebras

## 4.1. Introductory localised algebras

Let $K$ be a perfect, compact set in $\mathbb{C}$. A sequence $(\mathcal{M}_p)_{p=0}^{\infty}$ of bounded, positive functions on $K$ will be called a $l^1$-*algebra sequence* if it satisfies the following conditions:

$$\mathcal{M}_\circ(z) = 1 \quad \text{and} \quad \frac{\mathcal{M}_p(z)}{\mathcal{M}_q(z)\mathcal{M}_{p-q}(z)} \geq \binom{p}{q} \tag{4.1}$$

for all $z \in K, \quad p \in \mathbb{N}_\circ, \quad 0 \leq q \leq p$.

As in the constant situation, we associate to a sequence $(\mathcal{M}_p)_{p=0}^{\infty}$ of bounded, positive functions the sequence $(\mathsf{m}_p)_{p=0}^{\infty}$ given by:

$$\mathsf{m}_p(z) := \Big(\frac{\mathcal{M}_p(z)}{p!}\Big)^{1/p}, \qquad z \in K, \quad p \in \mathbb{N}_\circ.$$

Then, (4.1) is equivalent to

$$\mathsf{m}_\circ(z) = 1 \quad \text{and} \quad \mathsf{m}_q^q(z)\mathsf{m}_{p-q}^{p-q}(z) \leq \mathsf{m}_p^p(z), \tag{4.2}$$

for all $z \in K, \quad p \in \mathbb{N}_\circ, \quad 0 \leq q \leq p$.

EXAMPLES 4.1. *Let $\alpha$ be a bounded, positive real function on $K$. Examples of non-constant $\mathsf{m}_p(z)$ will be:*

$$(i)\ p^{\alpha(z)}, \quad (ii)\ p!^{\alpha(z)/p}, \quad \text{where } \alpha(z) > 0,\ \forall z \in K,\ p \in \mathbb{N}_\circ.$$

*If $\alpha(z) \geq 1$ on $K$, then*

$$(iii)\ (\prod_{k=1}^{p} \log(e+k))^{\alpha(z)/p}, \quad (iv)\ (\log(e+p))^{\alpha(z)}, \quad z \in K, \quad p \in \mathbb{N}_\circ,$$

*where $(i) - (iv)$ also satisfy* (4.2).

As in the constant situation, the space of functions given by

$$\mathcal{D}_1(K, \{\mathcal{M}_p\}) := \Big\{f \in \mathcal{D}^\infty(K); \|f\|_{\{\mathcal{M}_p\},1} := \sum_{p=0}^{\infty} \Big\|\frac{f^{(p)}}{\mathcal{M}_p}\Big\|_K < \infty\Big\}$$

and endowed with the norm $\|\cdot\|_{\{\mathcal{M}_p\},1}$ will then be a normed algebra with respect to the pointwise defined operations of addition, multiplication and multiplication by scalars.

Indeed, we have by the Leibniz rule and (4.2) for all $f, g \in \mathcal{D}^\infty(K),\ p \in \mathbb{N}_\circ$, that:

$$\Big\|\frac{(fg)^{(p)}}{\mathcal{M}_p}\Big\|_K = \Big\|\frac{(fg)^{(p)}}{p!\mathsf{m}_p^p}\Big\|_K = \Big\|\sum_{q=0}^{p}\frac{f^{(q)}g^{(p-q)}}{q!(p-q)!\mathsf{m}_p^p}\Big\|_K$$
$$\leq \sum_{q=0}^{p}\Big\|\frac{f^{(q)}}{q!\mathsf{m}_q^q}\Big\|_K\Big\|\frac{g^{(p-q)}}{(p-q)!\mathsf{m}_{p-q}^{p-q}}\Big\|_K.$$

Hence, for all $f, g \in \mathcal{D}_1(K, \{\mathcal{M}_p\})$, we obtain $fg \in \mathcal{D}_1(K, \{\mathcal{M}_p\})$ and

$$\|fg\|_{\{\mathcal{M}_p\},1} \leq \|f\|_{\{\mathcal{M}_p\},1}\|g\|_{\{\mathcal{M}_p\},1}.$$

If a sequence $(\mathcal{M}_p)_{p=0}^\infty$ of bounded, positive functions on $K$ with associated sequence $(\mathsf{m}_p)_{p=0}^\infty$ satisfies (4.1) (equivalently (4.2)) and

$$\sup_{p\in\mathbb{N}_\circ}\Big\|\sum_{q=0}^{p}\frac{\mathsf{m}_q^q\mathsf{m}_{p-q}^{p-q}}{\mathsf{m}_p^p}\Big\|_K < \infty, \tag{4.3}$$

or equivalently:

$$\sup_{p\in\mathbb{N}}\Big\|\sum_{q=1}^{p-1}\frac{\mathsf{m}_q^q\mathsf{m}_{p-q}^{p-q}}{\mathsf{m}_p^p}\Big\|_K < \infty,$$

then, it will be called a $l^\infty$*-algebra sequence.* In this case the normed space given by

$$\mathcal{D}_\infty(K, \{\mathcal{M}_p\}) := \Big\{f \in \mathcal{D}^\infty(K); \|f\|_{\{\mathcal{M}_p\},\infty} := \sup_{p\in\mathbb{N}_\circ}\Big\|\frac{f^{(p)}}{\mathcal{M}_p}\Big\|_K < \infty\Big\}$$

with the norm $\|\cdot\|_{\{\mathcal{M}_p\},\infty}$ will be an algebra. As in the case of constant sequences, the continuity of the multiplication is seen as follows:

$$\Big\|\frac{(fg)^{(p)}}{\mathcal{M}_p}\Big\|_K = \Big\|\sum_{q=0}^{p}\frac{f^{(q)}g^{(p-q)}}{q!\mathsf{m}_q^q(p-q)!\mathsf{m}_{p-q}^{p-q}}\cdot\frac{\mathsf{m}_q^q\mathsf{m}_{p-q}^{p-q}}{\mathsf{m}_p^p}\Big\|_K$$
$$\leq \|f\|_{\{\mathcal{M}_p\},\infty}\|g\|_{\{\mathcal{M}_p\},\infty}\cdot\Big\|\sum_{q=0}^{p}\frac{\mathsf{m}_q^q\mathsf{m}_{p-q}^{p-q}}{\mathsf{m}_p^p}\Big\|_K$$

for all $f, g \in \mathcal{D}_\infty(K, \{\mathcal{M}_p\})$ and $\mathcal{D}_\infty(K, \{\mathcal{M}_p\})$ may be endowed with an equivalent submultiplicative norm. Obviously, $\mathcal{D}_1(K, \{\mathcal{M}_p\}) \subset \mathcal{D}_\infty(K, \{\mathcal{M}_p\})$ with continuous inclusion mapping.

LEMMA 4.2. *Let $\alpha : K \to (0, \infty)$ be continuous and consider the sequences $(\mathsf{m}_p)_{p\in\mathbb{N}_\circ}$ given in Examples 4.1 (i) and (ii). If $\alpha \geq 1$ on $K$, then we also consider the Examples 4.1 (iii) and (iv). In all these examples we have*

$$\Big\|\sum_{k=1}^{n-1}\frac{\mathsf{m}_k^k\mathsf{m}_{n-k}^{n-k}}{\mathsf{m}_n^n}\Big\|_K \to 0,\ \textit{for}\ \ n \to \infty.$$

*In particular these sequences satisfy (4.3) and hence are $l^\infty-$algebra sequences.*

PROOF. The proof can be given by direct computations adopting the proof of Lemma 3.9 and the proof of Lemma 3.3 in [**15**]. □

$\mathcal{D}_1(K, \{\mathcal{M}_p\})$ contains the set of all (restrictions to $K$ of holomorphic) polynomials in $z$.

To ensure that $\mathrm{Rat}(K) \subset \mathcal{D}_1(K, \{\mathcal{M}_p\})$ we introduce an additional condition in the following lemma.

LEMMA 4.3. *Let $K$ be a perfect, compact set in $\mathbb{C}$ and $(\mathcal{M}_p)_{p=0}^{\infty}$ be a $l^1$-algebra sequence of continuous, positive functions on $K$. Then, the following hold:*

(a) *If the associated sequence $(\mathsf{m}_p)_{p=0}^{\infty}$ satisfies the condition;*

$$\|\mathsf{m}_p^{-1}\|_K \to 0, \quad \textit{for} \quad p \to \infty. \tag{4.4}$$

*Then, $Rat(K) \subset \mathcal{D}_1(K, \{\mathcal{M}_p\}) \subset \mathcal{D}_\infty(K, \{\mathcal{M}_p\})$.*

(b) *If $K$ has an empty interior and* $\mathrm{Rat}(K) \subset \mathcal{D}_\infty(K, \{\mathcal{M}_p\})$ *then* (4.4) *is satisfied.*

PROOF. $(a)$ As $\mathcal{D}_1(K, \{\mathcal{M}_p\})$ is an algebra containing all polynomials in $z$, using partial fraction decomposition for rational functions, it suffices to show that all functions

$$h_\xi : z \mapsto \frac{1}{\xi - z}, \quad z \in K, \quad \xi \in \mathbb{C} \setminus K,$$

are in $\mathcal{D}_1(K, \{\mathcal{M}_p\})$. Indeed, for $\xi \in \mathbb{C} \setminus K$, we have for all $p \in \mathbb{N}_\circ$

$$\begin{aligned} \Big\| \frac{h_\xi^{(p)}}{\mathcal{M}_p} \Big\|_K &\le \frac{1}{\mathrm{dist}(\xi, K)^{p+1}} \cdot \|\mathsf{m}_p^{-p}\|_K \\ &= \frac{\|\mathsf{m}_p^{-1}\|_K^p}{\mathrm{dist}(\xi, K)^{p+1}}. \end{aligned}$$

Hence, by (4.4) the sequence

$$\sum_{p=0}^{\infty} \Big\| \frac{h_\xi^{(p)}}{\mathcal{M}_p} \Big\|_K \le \frac{1}{\mathrm{dist}(\xi, K)} \sum_{p=0}^{\infty} \frac{\|\mathsf{m}_p^{-1}\|_K^p}{\mathrm{dist}(\xi, K)^p}$$

converges and we have $h_\xi \in \mathcal{D}_1(K, \{\mathcal{M}_p\})$.

$(b)$ Assume that (4.4) is not fulfilled. We show that there exists some $\xi \in \mathbb{C} \setminus K$ such that $h_\xi \notin \mathcal{D}_\infty(K, \{\mathcal{M}_p\})$.

If (4.4) is not satisfied then, there exists some $C > 0$ with

$$\limsup_{p \to \infty} \|\mathsf{m}_p^{-1}\|_K > C$$

and there exists a subsequence $(p_k)_{k=1}^{\infty}$ of $(p)_{p=0}^{\infty}$ such that

$$\|\mathsf{m}_{p_k}^{-1}\|_K > C \quad \text{for all } k \in \mathbb{N}.$$

As $K$ is compact and $\mathsf{m}_k$ is continuous, there exists $z_k \in K$ with

$$\frac{1}{\mathsf{m}_{p_k}(z_k)} = \|\mathsf{m}_{p_k}^{-1}\|_K.$$

Using Bolzano-Weierstrass and passing to some subsequence, we may assume that the sequence $(z_k)_{k=1}^{\infty}$ converges to some $z_\circ \in K$ for $k \to \infty$.

Then there exists some $k_\circ$ such that, for all $k \ge k_\circ$, we have

$$\frac{1}{\mathsf{m}_{p_k}(z_k)} = \|\mathsf{m}_{p_k}^{-1}\|_K > C \quad \text{and} \quad |z_k - z_\circ| < C/3.$$

As $K$ has empty interior, there exists some $\xi_\circ \in \mathbb{C} \setminus K$ with $|\xi_\circ - z_\circ| < C/3$. Hence, $|\xi_\circ - z_k| < 2C/3$ for $k \geq k_\circ$. We obtain for all $k > k_\circ$ that:

$$\begin{aligned} \|h_{\xi_\circ}\|_{\{\mathcal{M}_p\},\infty} \geq \left\| \frac{h_{\xi_\circ}^{(p_k)}}{\mathcal{M}_{p_k}} \right\|_K &\geq |h_{\xi_\circ}^{(p_k)}(z_k)| \cdot \mathcal{M}_{p_k}^{-1}(z_k) \\ &= \frac{\|\mathsf{m}_{p_k}^{-1}\|_K^{p_k}}{|\xi_\circ - z_k|^{p_k+1}} \geq \frac{3}{2C} \cdot (\frac{3}{2})^{p_k} \to \infty, \end{aligned}$$

for $k \to \infty$. Hence, $h_{\xi_\circ} \notin \mathcal{D}_\infty(K, \{\mathcal{M}_p\})$. □

Examples 4.4. *The examples in Lemma 4.2 satisfy* (4.4).

Proof. It can be easily seen that for examples $(i) - (iii)$ in Lemma 4.2, there is nothing to prove. For $(iv)$, let $k_\circ \in (0, 1)$ and $\tilde{k}p$ be the smallest integer $\geq k_\circ p,\ p \in \mathbb{N}_\circ$. Then, using similar arguments from the proof of Corollary 3.32, we get:

$$\begin{aligned} \mathsf{m}_p(z) &\geq \prod_{k=\tilde{k}p}^{p} (\log(e+k))^{\alpha(z)/p} \\ &\geq (\log(e + k_\circ p))^{p(1-k_\circ)\alpha(z)/p} \\ &= (\log(e + k_\circ p))^{(1-k_\circ)\alpha(z)} \geq (\log(e + k_\circ p))^{1-k_\circ} \\ \frac{1}{\mathsf{m}_p(z)} &\leq \frac{1}{(\log(e + k_\circ p))^{1-k_\circ}} \\ \text{Hence,}\quad \sup_{z \in K} \frac{1}{\mathsf{m}_p(z)} &\leq \sup_{z \in K} \frac{1}{(\log(e + k_\circ p))^{1-k_\circ}} \to 0, \quad \text{as } p \to \infty. \end{aligned}$$

□

## 4.2. Completions of complex-ultra differentiable functions in localised case

Define $l^1(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ to be the sequence space of functions on $K$ with a weight sequence $(\mathcal{M}_p)_{p=0}^\infty$ by:

$$l^1(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\}) := \left\{ f = (f_p)_{p=0}^\infty \in C(K)^{\mathbb{N}_\circ};\ |f|_{\{\mathcal{M}_p\},1} := \sum_{p=0}^\infty \left\| \frac{f_p}{\mathcal{M}_p} \right\|_K < \infty \right\}$$

$l^1(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ endowed with the norm $|\cdot|_{\{\mathcal{M}_p\},1}$ and the multiplication as defined in Section 3.3.1 is a Banach algebra.

In a similar way, the space $l^\infty(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ is defined as:

$$l^\infty(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\}) := \left\{ f = (f_p)_{p=0}^\infty \in C(K)^{\mathbb{N}_\circ};\ \sup_{p \in \mathbb{N}_\circ} \left\| \frac{f_p}{\mathcal{M}_p} \right\|_K < \infty \right\}$$

We endow $l^\infty(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ with the norm $|\cdot|_{\{\mathcal{M}_p\},\infty}$ and define the multiplication same as in $l^1(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$. One observes that by using the condition (4.3), the multiplication in $l^\infty(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ is continuous with respect to the norm $|\cdot|_{\{\mathcal{M}_p\},\infty}$. Thus, $l^\infty(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ is a Banach algebra.

For $q = \{1, \infty\}$, let $\mathcal{D}_q(K, \{\mathcal{M}_p\})$ denote any of the algebras $\mathcal{D}_1(K, \{\mathcal{M}_p\})$ or $\mathcal{D}_\infty(K, \{\mathcal{M}_p\})$, respectively and $l^q(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ denote any of the algebras $l^1(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ or $l^\infty(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$, respectively.

The mapping

$$J : \mathcal{D}_q(K, \{\mathcal{M}_p\}) \to l^q(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\}),$$

$$\text{defined by} \quad J(f) := (f^{(p)})_{p=0}^\infty, \quad \text{for all} \quad f \in \mathcal{D}_q(K, \{\mathcal{M}_p\})$$

is an isometric algebra monomorphism from $\mathcal{D}_q(K, \{\mathcal{M}_p\})$ into $l^q(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$.

The completion of $\mathcal{D}_q(K, \{\mathcal{M}_p\})$ denoted by $\tilde{\mathcal{D}}_q(K, \{\mathcal{M}_p\})$ may then be identified with $J_q(K, \{\mathcal{M}_p\}) :=$ the closure of $J(\mathcal{D}_q(K, \{\mathcal{M}_p\}))$ in $l^q(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$. Throughout the discussion, by $l^q$-algebra sequence ($q = \{1, \infty\}$), we mean a sequence $(\mathcal{M}_p)_{p \in \mathbb{N}_\circ}$ (or the associated sequence $(\mathsf{m}_p)_{p \in \mathbb{N}_\circ}$) of bounded, positive functions on a set $K$.

Next result gives us a criterion for the completion $\tilde{\mathcal{D}}_q(K, \{\mathcal{M}_p\})$ of the normed algebra $\mathcal{D}_q(K, \{\mathcal{M}_p\})$ to be a Banach function algebra on the set $K$.

Proposition 4.5. *Let $K \subseteq \mathbb{C}$ be a perfect, compact set and $(\mathcal{M}_p)_{p=0}^\infty$ be a $l^q$-algebra sequence. Let $J$ and $J_q(K, \{\mathcal{M}_p\})$ for $q = \{1, \infty\}$ be as defined above such that $d/dz$ is closable in $C(K)$. If $P_\circ$ is the projection from $l^q(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ to $C(K)$ by $(g_p)_{p=0}^\infty \mapsto g_\circ$ onto the zero-component then $P_\circ | J_q(K, \{\mathcal{M}_p\})$ is an algebra monomorphism. Hence, if we endow its range $\tilde{\mathcal{D}}_q(K, \{\mathcal{M}_p\}) := P_\circ(J_q(K, \{\mathcal{M}_p\}))$ with the norm given by:*

$$\|h\|^{\sim}_{\{\mathcal{M}_p\}, q} := |g|_{\{\mathcal{M}_p\}, q}$$

*for all $h \in \tilde{\mathcal{D}}_q(K, \{\mathcal{M}_p\})$, where $g = (g_k)_{k=0}^\infty \in J_q(K, \{\mathcal{M}_p\})$ with $g_\circ = h$ then, $\tilde{\mathcal{D}}_q(K, \{\mathcal{M}_p\})$ may be considered as the completion of $\mathcal{D}_q(K, \{\mathcal{M}_p\})$ and $\tilde{\mathcal{D}}_q(K, \{\mathcal{M}_p\})$ is a Banach function algebra.*

Proof. As in the constant situation, to show that $P_\circ|_{J_q(K, \{\mathcal{M}_p\})}$ is an isometry, it is sufficient to show the injectivity of $P_\circ|_{J_q(K, \{\mathcal{M}_p\})}$. We follow the proof as in the constant case in Proposition 3.13.

If $P_\circ|_{J_q(K, \{\mathcal{M}_p\})}$ is not injective then, there exists some $0 \neq g = (g_p)_{p \in \mathbb{N}_\circ} \in J_q(K, \{\mathcal{M}_p\})$ with $g_\circ \equiv 0$ and $g_p \not\equiv 0$ for some $p \in \mathbb{N}$. Thus, for some minimal $p_\circ \in \mathbb{N}$ with this property,

$$g_{p_\circ - 1} \equiv 0 \quad \text{and } g_{p_\circ} \not\equiv 0.$$

By the definition of $J_q(K, \{\mathcal{M}_p\})$, there exists a sequence $(f_n)_{n=1}^\infty$ in $\mathcal{D}_q(K, \{\mathcal{M}_p\})$ such that for all $p \in \mathbb{N}_\circ$,

$$\left\| \frac{f_n^{(p)} - g_p}{\mathcal{M}_p} \right\|_K \to 0, \quad \text{for } n \to \infty \tag{4.5}$$

Thus, for all $p \in \mathbb{N}_\circ$, we obtain from the boundedness of $\mathcal{M}_p$ and from (4.5) that:

$$\|f_n^{(p)} - g_p\|_K \leq \|\mathcal{M}_p\|_K \cdot \left\| \frac{f_n^{(p)} - g_p}{\mathcal{M}_p} \right\|_K \to 0, \quad \text{for } n \to \infty.$$

In particular,

$$f_n^{(p_\circ - 1)} \to g_{p_\circ - 1} \equiv 0 \quad \text{and } f_n^{(p_\circ)} \to g_{p_\circ} \not\equiv 0,$$

uniformly on $K$, for $n \to \infty$ contradicting the closability of $d/dz$ in $C(K)$. Hence, $P_\circ|_{J_q(K,\{\mathcal{M}_p\})}$ is injective and the result follows from the constant case.

$\square$

## 4.3. Naturality in localised case

We define $d(M)$ as introduced in [**1**] for $l^q$-algebra sequences, $q = \{1, \infty\}$ as:

$$d(\mathcal{M}) := \lim_{n\to\infty} \left(\frac{n!}{\|\mathcal{M}_n\|_K}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{\|\mathsf{m}_n\|_K}, \quad n \in \mathbb{N}.$$

We notice that Proposition 3.21 holds for $l^q$-algebra sequences $q = \{1, \infty\}$ as well. For convenience, we give the statement as follows:

PROPOSITION 4.6. *Let $K$ be a perfect, compact set in $\mathbb{C}$ with $d/dz$ closable in $C(K)$ and $(\mathcal{M}_p)_{p=0}^\infty$ be a $l^q$-algebra sequence. Assume that* $\mathrm{Rat}(K) \subset \mathcal{D}_q(K, \{\mathcal{M}_p\})$*, $q = \{1, \infty\}$. Define $R_q(K, \{\mathcal{M}_p\}) := \overline{\mathrm{Rat}(K)}^{\|\cdot\|_{\{\mathcal{M}_p\},q}^{\tilde{}}}$, i.e. the closure of* $\mathrm{Rat}(K)$ *in $\tilde{\mathcal{D}}_q(K, \{\mathcal{M}_p\})$. Then, $R_q(K, \{\mathcal{M}_p\})$ will be a natural Banach function algebra on $K$.*

PROOF. The proof follows directly from Proposition 4.5 and Proposition 3.3$(a)$. $\square$

Following [**1**], we use some notations and definitions from combinatorial analysis in this setting.

Let $t, n \in \mathbb{N}$ such that $n \geq t$ and define $S(t, n)$ be the set of all $a = (a_1, \cdots, a_n) \in \mathbb{Z}^+$ such that $\sum_{k=1}^n a_k = t$ and $\sum_{k=1}^n k a_k = n$. Since (4.2) holds for all $p, q \in \mathbb{N}_\circ$, $0 \leq q \leq p$, $z \in K$, we have that $(\mathsf{m}_k^k(z))^{a_k} \leq \mathsf{m}_{ka_k}^k(z)$, for all $z \in K$, $k \in \mathbb{N}$ and thus:

$$\prod_{k=1}^n (\mathsf{m}_k^k(z))^{a_k} \leq \mathsf{m}_{\sum_{k=1}^n ka_k}^{\sum_{k=1}^n ka_k}(z) = \mathsf{m}_n^n, \quad \forall z \in K.$$

DEFINITION 4.7. *For $\mathsf{m}_k^k(z) = \frac{\mathcal{M}_k(z)}{k!}$, and for all $k \in \mathbb{N}_\circ$, $z \in K$, we define the sequence $(\mathsf{A}_t)_{t=1}^\infty$ by:*

$$\mathsf{A}_t(z) := \sup\left\{\frac{1}{\mathsf{m}_n^n(z)} \prod_{k=1}^n (\mathsf{m}_k^k(z))^{a_k} : n \geq t, (a_1, \cdots, a_n) \in S(t, n),\ z \in K\right\},$$

*where $S(t, n)$ is defined above.*

Next result shows that under some assumption on the sequence $(\mathsf{A}_t)_{t=1}^\infty$, the completion of $\mathcal{D}_1(K, \{\mathcal{M}_p\})$ is a natural Banach function algebra on a perfect, compact set $K$.

THEOREM 4.8. *Let $K$ be a perfect, compact set in $\mathbb{C}$ and that $d/dz$ is closable in $C(K)$. Let $(\mathcal{M}_p)_{p=0}^\infty$ be a $l^1$-algebra sequence and $\lim_{t\to\infty} \|\mathsf{A}_t\|_K^{1/t} = 0$. Then, $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ is a natural Banach function algebra on $K$.*

PROOF. From Proposition 4.5 we see that $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ is a Banach function algebra. To obtain the naturality, we follow the proof of Theorem 3.3 [**1**].

For every $f \in \mathcal{D}_1(K, \{\mathcal{M}_p\})$ and $F(y) := y^p,\ p \in \mathbb{N}$, we obtain using the Faà di Bruno's formula that:

$$\Big|\frac{(F\circ f)^{(n)}(z)}{\mathcal{M}_n(z)}\Big| \le \sum_{t=0}^{n} \frac{|F^{(t)}(f(z))|}{\mathcal{M}_n(z)} \sum \frac{n!}{\prod a_k!} \prod_{k=1}^{n} \Big(\frac{|f^{(k)}(z)|}{k!}\Big)^{a_k}$$
$$\le \sum_{t=0}^{\min\{p,n\}} t! \binom{p}{t} \frac{|f(z)|^{p-t}}{\mathcal{M}_n(z)} \sum \frac{n!}{\prod a_k!} \prod_{k=1}^{n} \Big(\frac{|f^{(k)}(z)|}{k!}\Big)^{a_k}$$

where the inner summation is taken over $a = (a_1, \cdots, a_n) \in S(t, n)$. Interchanging, the order of summation, we get:

$$\sum_{n=0}^{\infty}\Big|\frac{(F\circ f)^{(n)}(z)}{\mathcal{M}_n(z)}\Big| \le \sum_{t=0}^{p} \binom{p}{t} |f(z)|^{p-t} t! \sum_{n=t}^{\infty} \sum \frac{n!}{\prod a_k!} \frac{1}{\mathcal{M}_n(z)} \prod_{k=1}^{n} \Big(\frac{|f^{(k)}(z)|}{k!}\Big)^{a_k}$$
$$= \sum_{t=0}^{p} \binom{p}{t} |f(z)|^{p-t} t! \sum_{n=t}^{\infty} \sum \frac{\prod_{k=1}^{n} \mathsf{m}_k^k(z)^{a_k}}{\prod a_k! \mathsf{m}_n^n(z)} \prod_{k=1}^{n} \Big(\frac{|f^{(k)}(z)|}{\mathcal{M}_k(z)}\Big)^{a_k}$$
$$\le \sum_{t=0}^{p} \binom{p}{t} |f(z)|^{p-t} \mathsf{A}_t(z) t! \sum_{n=t}^{\infty} \sum \frac{1}{\prod a_k!} \prod_{k=1}^{n} \Big(\frac{|f^{(k)}(z)|}{\mathcal{M}_k(z)}\Big)^{a_k}$$

Taking the supremum norm on $K$ and using formula (2) [**1**], we obtain:

$$\|f^p\|_{\{\mathcal{M}_p\},1} \le \sum_{t=1}^{p} \binom{p}{t} \|f\|_K^{p-t} \Big[\underbrace{\|\mathsf{A}_t\|_K^{1/t}(\|f\|_{\{\mathcal{M}_p\},1} - \|f\|_K)}_{=:\varepsilon_t}\Big]^t,$$

for all $f \in \mathcal{D}_1(K, \{\mathcal{M}_p\})$. Hence, from the proof of Theorem 3.22, using [**1**] Lemma 3.1 and Theorem 1.3 and the fact that $\lim_{t\to\infty} \|\mathsf{A}_t\|_K^{1/t} = 0$, we obtain that $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ is a natural Banach function algebra on $K$.

$\square$

REMARK 4.9. *As in Remark 3.23, we observe in view of the proof of Corollary* 3.8 [**1**] *that, if*

$$\max_{1\le k\le n-1} \Big\|\frac{\mathsf{m}_k^k \mathsf{m}_{n-k}^{n-k}}{\mathsf{m}_n^n}\Big\|_K \to 0, \text{ as } n \to \infty \tag{4.6}$$

*then,* $\lim_{t\to\infty} \|\mathsf{A}_t\|_K^{1/t} = 0$, *even for the normed algebras* $\mathcal{D}_q(K, \{\mathcal{M}_p\})$, $q = \{1, \infty\}$.

EXAMPLES 4.10. *It follows from the Lemma 4.2 that the examples given there satisfy condition* (4.6). *By the proof of Corollary* 3.8 [**1**], *we obtain that* (4.6) *implies* $\lim_{t\to\infty} \|\mathsf{A}_t\|_K^{1/t} = 0$.

**4.3.1.** $D_\infty(K, \{\mathcal{M}_p\})$ **natural algebras.** Throughout the section, $K \subset \mathbb{C}$ is a perfect, compact set and $\tilde{d}$ is the closure of the differential operator $d/dz$ in $C(K)$ with domain $\mathcal{D}$.

LEMMA 4.11. *Let* $K$, $\tilde{d}$, $\mathcal{D}$ *be as above and* $f$, $g \in \mathcal{D}$. *Then,* $f+g$, $\alpha f$, $fg$, $1/f \in \mathcal{D}$, *for all* $\alpha \in \mathbb{C}$.

PROOF. It is easy to show that $f+g$, $\alpha f$ and $fg$ are in $\mathcal{D}$, for all $\alpha \in \mathbb{C}$. Since the proofs are elementary, so we will only show that $1/f \in \mathcal{D}$.

To show that $1/f \in \mathcal{D}$, let there exists $(f_n)_{n\in\mathbb{N}_\circ} \in \mathrm{dom}(d/dz)$ such that

$$f_n \to f, \quad \text{uniformly on } \ K, \quad f \in \mathcal{D}.$$

and

$$f_n' \to \tilde{d}(f), \qquad \frac{1}{f_n} \to \frac{1}{f} \quad \text{uniformly on } \ K.$$

Then,

$$\Big(\frac{1}{f_n}\Big)' = \frac{-f_n'}{f_n^2} \to \frac{-\tilde{d}(f)}{f^2} = \tilde{d}(1/f).$$

Thus, showing that $1/f \in \mathcal{D}$ and $\tilde{d}(\frac{1}{f}) = \frac{-\tilde{f}}{f^2}$. □

Let $(\mathcal{M}_p)_{p\in\mathbb{N}_\circ}$ be a $l^\infty$-algebra sequence and define a corresponding normed space $D_\infty(K,\{\mathcal{M}_p\})$ by

$$D_\infty(K,\{\mathcal{M}_p\}) := \Big\{ f \in \cap_{p\in\mathbb{N}_\circ}\mathcal{D}(\tilde{d}^p); |||f|||_{\{\mathcal{M}_p\},\infty} := \sup_{p\in\mathbb{N}_\circ} \Big\| \frac{\tilde{d}^p f}{\mathcal{M}_p} \Big\|_K < \infty \Big\}$$

We observe from Lemma 4.11 and in the arguments before Lemma 4.2 that, $|||\cdot|||_{\{\mathcal{M}_p\},\infty}$ is equivalent to some submultiplicative norm and hence, $D_\infty(K,\{\mathcal{M}_p\})$ is a normed algebra.

THEOREM 4.12. *Let $K, \tilde{d}$ and $\mathcal{D}$ be as mentioned before and $(\mathcal{M}_p)_{p\in\mathbb{N}_\circ}$ be a $l^\infty$- algebra sequence. Define a mapping*

$$\tilde{J} : D_\infty(K,\{\mathcal{M}_p\}) \to l^\infty(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\}) \quad by$$
$$\tilde{J}(f) := (\tilde{d}^p f)_{p\in\mathbb{N}_\circ}, \qquad for\ all \ \ f \in D_\infty(K,\{\mathcal{M}_p\})$$

*Then, $\tilde{J}$ is an isometric algebra monomorphism from $D_\infty(K,\{\mathcal{M}_p\})$ to the sequence algebra $l^\infty(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$.*
*Let $P_\circ$ be the projection from $l^\infty(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ to $C(K)$ by $(g_p)_{p=0}^\infty \mapsto g_\circ$, then $P_\circ|\tilde{J}(D_\infty(K,\{\mathcal{M}_p\}))$ is an algebra monomorphism. Hence, $D_\infty(K,\{\mathcal{M}_p\}) := P_\circ(\tilde{J}(D_\infty(K,\{\mathcal{M}_p\}))$ is a Banach function algebra on $K$.*

PROOF. We only need to show that the range of $\tilde{J}$ is closed and the rest follows by the isometry of the projection $P_\circ$. Let $g = (g_n)_{n\in\mathbb{N}_\circ}$ be a Cauchy sequence in $\tilde{J}(D_\infty(K,\{\mathcal{M}_p\}))$, then there exists $f = (f_p)_{p=0}^\infty$ in $l^\infty(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ such that

$$g_n \to f, \quad \text{in } \ l^\infty(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\}).$$

Thus,

$$g_{n,p} \to f_p, \quad \text{uniformly on } \ K, \ \forall\, p \in \mathbb{N}_\circ$$
$$\text{and} \quad \tilde{d}g_{n,p} = g_{n,p+1} \to f_{p+1}, \quad \text{uniformly on } \ K.$$

Since, $\tilde{d}$ is closed, we get $\tilde{d}f_p = f_{p+1}$, which shows that $f = (\tilde{d}^p f_\circ)_{p=0}^\infty$ is in $\tilde{J}(D_\infty(K,\{\mathcal{M}_p\}))$.

Hence, $\tilde{J}(D_\infty(K,\{\mathcal{M}_p\}))$ is closed in $l^\infty(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ and thus complete. The result follows from the proof of Proposition 4.5, by showing the injectivity of the projection $P_\circ$ with respect to the closed operator $\tilde{d}$. □

THEOREM 4.13. *Let $K, \tilde{d}$ and $\mathcal{D}$ be as before and $(\mathcal{M}_p)_{p=0}^{\infty}$ be a $l^{\infty}$-algebra sequence such that*

$$\Big\| \sum_{k=1}^{n-1} \frac{\mathsf{m}_k^k \mathsf{m}_{n-k}^{n-k}}{\mathsf{m}_n^n} \Big\|_K \to 0, \quad \text{as } n \to \infty.$$

*Then, $D_\infty(K, \{\mathcal{M}_p\})$ is inverse-closed, i.e. for all $f \in D_\infty(K, \{\mathcal{M}_p\})$ with $f(z) \neq 0$, for all $z \in K$, the function $1/f \in D_\infty(K, \{\mathcal{M}_p\})$ and*

$$|||1/f|||_{\{\mathcal{M}_p\},\infty} = \sup_{p \in \mathbb{N}_\circ} \Big\| \frac{\tilde{d}^p(1/f)}{\mathcal{M}_p} \Big\|_K < \infty. \tag{4.7}$$

PROOF. Using similar arguments as in the proof of Lemma 4.11 we see that $1/f \in \mathcal{D}(\tilde{d}^p)$, for all $p \in \mathbb{N}_\circ$. To show that $1/f \in D_\infty(K, \{\mathcal{M}_p\})$ we proceed similar as in the proof of [**15**] Lemma 3.4.

Without loss of generality we may assume that $\inf_{z\in K} |f(z)| \geq 1$. Put $C := |||f|||_{\{\mathcal{M}_p\},\infty}$.

We define for all $p \in \mathbb{N}_\circ$, $z \in K$

$$\alpha_p(z) := \frac{|(\tilde{d}^p f)(z)|}{\mathcal{M}_p(z)}, \qquad \beta_p(z) := \frac{|\tilde{d}^p(1/f)(z)|}{\mathcal{M}_p(z)}$$

Using $f \cdot (1/f) \equiv 1$ on $K$ and the Leibniz rule for $\tilde{d}$ we obtain for $n \geq 1$,

$$\beta_n(z) \leq \sum_{k=1}^{n} \alpha_k(z) \beta_{n-k}(z) \frac{\mathsf{m}_k^k(z) \mathsf{m}_{n-k}^{n-k}(z)}{\mathsf{m}_n^n(z)}.$$

Choose $N > 0$ so that

$$\Big\| \sum_{k=1}^{n-1} \frac{\mathsf{m}_k^k \mathsf{m}_{n-k}^{n-k}}{\mathsf{m}_n^n} \Big\|_K < \frac{1}{2C}, \quad \text{for } n \geq N.$$

Then, $\max_{0\leq n\leq N} \|\beta_n\|_K \leq M$, where we may assume $M \geq 2C$. Hence, for $n > N$, we have

$$\beta_{n+1}(z) \leq \sum_{k=1}^{n} \alpha_k(z) \beta_{n+1-k}(z) \frac{\mathsf{m}_k^k(z) \mathsf{m}_{n+1-k}^{n+1-k}(z)}{\mathsf{m}_{n+1}^{n+1}(z)} + \frac{\alpha_n(z)}{|f(z)|}$$

As $1/|f(z)| \leq 1$, we get

$$\begin{aligned} \beta_{n+1}(z) \leq & C\big(\max_{1\leq k\leq n} \beta_k(z)\big) \sum_{k=1}^{n} \frac{\mathsf{m}_k^k(z) \mathsf{m}_{n+1-k}^{n+1-k}(z)}{\mathsf{m}_{n+1}^{n+1}(z)} + C \\ \leq & \frac{1}{2} \max_{1\leq k\leq n} \beta_k(z) + C \end{aligned}$$

By induction we obtain $\|\beta_n\|_K \leq M$, for all $n \in \mathbb{N}_\circ$. Hence

$$|||1/f|||_{\{\mathcal{M}_p\},\infty} = \sup_{n\in\mathbb{N}_\circ} \|\beta_n\|_K \leq M < \infty$$

which shows that $1/f \in D_\infty(K, \{\mathcal{M}_p\})$.

∎

REMARK 4.14. *Recall from the introductory section in Chapter 3 that a Banach algebra $\mathcal{A}$ is inverse-closed if and only if every element $f \in \mathcal{A}$ has a natural spectrum. Thus, if a Banach function algebra $\mathcal{A}$ is inverse-closed, then it is natural. In other words, Theorem 4.12 and Theorem 4.13 show that $D_\infty(K, \{\mathcal{M}_p\})$ is a natural Banach function algebra on $K$.*

## 4.4. Regularity in localised algebras

As in the constant case, we require a (set of) condition(s), which yields decomposability of the operator of multiplication by the coordinate function on the Banach function algebras $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$. Considering similar assumptions as in Theorem 3.35 for the localised case, we obtain the following:

THEOREM 4.15. *Let $K$ be a perfect, compact set with $\lambda(K) = 0$ and $\tilde{\mathcal{M}}_z$ be the operator of multiplication on $l^q(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$, $q = \{1, \infty\}$, as defined in Theorem 3.35. Assume that* (4.4) *holds and define an entire function $\omega : \mathbb{C} \to \mathbb{C}$ by,*

$$\omega(\xi) := \sum_{p=0}^{\infty} \xi^{p+1} \|\mathsf{m}_p^{-1}\|_K^p, \qquad \xi \in \mathbb{C}.$$

*Moreover, suppose that $V(K)$ be an open, bounded neighbourhood of $K$ and $f$ satisfies condition $(\eta)$. If the integral*

$$I_V = \iint\limits_{V(K)} f\Big(\log^+ \log^+ \omega\big(\frac{1}{\mathrm{dist}(\xi, K)}\big)\Big)\, d\lambda(\xi) < \infty,$$

*then, $\tilde{\mathcal{M}}_z$ is decomposable.*

PROOF. Following the proof of Theorem 3.35, we observe that for $\xi \in \mathbb{C} \setminus K$ and the rational function $h$ in $\mathrm{Rat}(K)$ given by; $h(z) = \frac{1}{\xi - z}$, $\forall z \in K$, we have:

$$\frac{h^{(p)}}{\mathcal{M}_p(z)} = \frac{p!}{(\xi - z)^{p+1}\mathcal{M}_p(z)} \leq d(\xi)^{p+1} \mathsf{m}_p^{-p}(z)$$

where $d(\xi) := \frac{1}{\mathrm{dist}(\xi, K)}$. Thus,

$$\Big\|\frac{h^{(p)}}{\mathcal{M}_p}\Big\|_K \leq d(\xi)^{p+1} \|\mathsf{m}_p^{-1}\|_K^p$$

and for $q = \{1, \infty\}$,

$$\|h\|_{\{\mathcal{M}_p\}, q} \leq \omega(d(\xi)).$$

Let $J$ and $\tilde{J}$ be the isometric algebra monomorphisms from $\mathcal{D}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ to $l^1(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ and $l^\infty(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$, respectively. Then, following the proof of Theorem 3.35, we obtain that $(\xi - \tilde{\mathcal{M}}_z)^{-1}$ is the operator of multiplication by $J(h)$ and also by $\tilde{J}(h)$.

Thus, for some $C > 0$ and $q = \{1, \infty\}$, we have:

$$\|(\xi - \tilde{\mathcal{M}}_z)^{-1}\| \leq C\|h\|_{\{\mathcal{M}_p\}, q} \leq C\omega(d(\xi)).$$

The decomposability of $\tilde{\mathcal{M}}_z$ follows directly, from Remark 1.12 and Remark 1.13 for $T = \tilde{\mathcal{M}}_z$, $\Omega = V(K)$, $\sigma(\tilde{\mathcal{M}}_z) = K$ and with $\sigma(\tilde{\mathcal{M}}_z) \setminus V(K) = \emptyset$. $\square$

Similar as in the remarks after Theorem 3.35, we notice that $\tilde{\mathcal{M}}_z$ is decomposable on all closed subalgebras of $l^q(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$, $q = \{1, \infty\}$ such that $J_1(K, \{\mathcal{M}_p\}) \subset l^1(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ and $\tilde{J}(D_\infty(K, \{\mathcal{M}_p\})) \subset l^\infty(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ or which contain $g_{\mathrm{id}} := (\mathrm{id}, 1, 0, \cdots)$ and $\tilde{J}(h)$ and $J(h)$ for all $h \in \mathrm{Rat}(K)$, where $J_1(K, \{\mathcal{M}_p\})$ is the closure of $J(\mathcal{D}_1(K, \{\mathcal{M}_p\}))$ in $l^1(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$.

Corollary 4.16. *Let $d/dz$, $\tilde{d}$ be as in Section 4.3.1 such that $d/dz$ is closable and $(\mathcal{M}_p)_{p=0}^\infty$ be a $l^q$-algebra sequence for $q = \{1, \infty\}$. Assume that all conditions of Theorem 4.15 hold. Then, $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ will be normal on $K$.*

Proof. From Theorem 4.15 we obtain the decomposability of $\tilde{\mathcal{M}}_z$ and thus by Corollary 3.5, the proof follows immediately. □

Next results deals with the regularity of Banach function algebras $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$, respectively.

Corollary 4.17. *Let $d/dz$, $\tilde{d}$ be as in Corollary 4.16 and $(\mathcal{M}_p)_{p=0}^\infty$ be a $l^1$-algebra sequence such that $\lim_{t\to\infty} \|\mathsf{A}_t\|_K^{1/t} = 0$. Assume that all conditions of Theorem 4.15 hold. Then, $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ is a regular Banach function algebra.*

Proof. From the assumptions and Theorem 4.8 we obtain that $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ is natural on $K$. Further, Corollary 4.16 gives that $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ is normal on $K$ and hence, (by the definitions of normality and naturality) regular Banach function algebra on $K$. □

Corollary 4.18. *Let $d/dz$ and $\tilde{d}$ be as in Corollary 4.16 and assume that all conditions of Theorem 4.15 hold. Let $(\mathcal{M}_p)_{p\in\mathbb{N}_\circ}$ be a $l^\infty$-algebra sequence such that*

$$\Big\| \sum_{k=1}^{n-1} \frac{\mathsf{m}_k^k \mathsf{m}_{n-k}^{n-k}}{\mathsf{m}_n^n} \Big\|_K \to 0, \quad \text{as} \quad n \to \infty.$$

*Then, $D_\infty(K, \{\mathcal{M}_p\})$ is a regular Banach function algebra on $K$.*

Proof. The naturality of $D_\infty(K, \{\mathcal{M}_p\})$ is clear from Theorem 4.13 and the normality from Corollary 4.16. Hence, the result follows. □

## 4.5. Sets of positive Lebesgue measure and (residual)decomposability

This section deals with the (residual)decomposability results of an operator $T$ on a Banach space $X$, when its resolvent satisfies some growth conditions near the spectrum $\sigma(T)$ having positive area. We consider $\sigma(T)$ a perfect set which is the union of two compact sets $K_1$ and $K_2$ such that $K_1$ is a connected set of Lebesgue measure zero and $K_2$ is a set having positive Lebesgue measure. For $K_1$, we use the same construction of covering as in Section 2.2, where now from the covering of $K_1$, $Q$ is a closed square of side length $a$ having empty intersection with $K_2$.

Example 4.19. *For $K_2$, sets like (Cantor)dust or arcs having positive area as explained in* [**31**] *can be considered.*

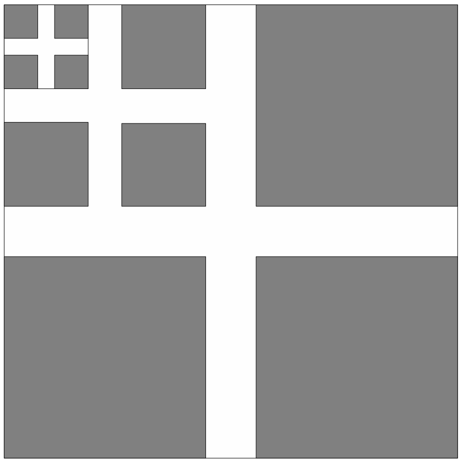

FIGURE 4.1. Set of positive area

*An easy example of a set $K$ can be the union of the Sierpinski carpet with Cantor dust or the union of one of the fractal sets discussed in Example 2.9 or in Example 2.12 with an arc having positive area.*

We give a rough sketch of sets having positive area here. Using the method as explained in [**31**], we construct a compact set $K_2 \subset K$ as follows:

<u>Construction</u>

We start with a square $F_\circ$ of side length $a$ and for $0 < \varepsilon < a^2$ with a sequence of positive numbers $(\varepsilon_j)_{j=1}^{\infty}$ such that $\sum_{j=1}^{\infty} \varepsilon_j = \varepsilon$. We take out a 'plus shaped' region of area $\varepsilon_1 > 0$ centred in the square $F_\circ$. Call the remaining set $F_1$. From each of the remaining four congruent squares, we take out a 'plus shaped' region of total area $\varepsilon_2 > 0$.

Proceeding in this way, taking out from the set $F_{k-1}$ in step $k$, 'plus shaped' regions of total area $\varepsilon_k > 0$, we obtain a sequence $(F_j)_{j=0}^{\infty}$ of compact sets such that

$$F_{j+1} \subset F_j \quad \text{for all } j.$$

The totally disconnected compact set $F := \cap_{j=0}^{\infty} F_j$ has area $a^2 - \sum_{j=1}^{\infty} \varepsilon_j = a^2 - \varepsilon > 0$.

We set the situation as follows and use the definition of upper box dimension at a point in this situation.

$$\underline{\textit{Situation } S3}\begin{cases} T \text{ is a bounded linear operator on a Banach space } X; \\ S \text{ is a compact subset of } \sigma(T). \end{cases}$$

DEFINITION 4.20. *Let $F \subset \mathbb{R}^2$ be a bounded set. Then, $F$ has* `upper box dimension` $\leq d$ `at` $z$*, if there exists a closed neighbourhood $F_1$ of $z$, such that the upper box dimension of $F_1 \cap F \leq d$.*

Equivalently saying that,

$$d_F(z) := \inf_{F_1}\{\delta \mid \overline{\dim}_B(F_1 \cap F) \leq \delta\}.$$

We discuss the (residual)decomposability of an operator $T$ by finding a relation between the growth conditions of the resolvent and upper box dimension at a point of the spectrum of $T$.

THEOREM 4.21. *Consider the Situation $S3$. Assume that for each $z \in \sigma(T)\backslash S$ we have $d_{\sigma(T)}(z) < 2$ and there exists some closed square $Q_z$ with centre $z$ having empty intersection with $S$ and $\overline{\dim}_B(Q_z \cap \sigma(T)) < 2$ such that the resolvent satisfies the condition;*

$$\|R(\xi, T)\| \leq \exp\Big(\exp\Big(\frac{C(z)}{\operatorname{dist}(\xi, \partial Q_z \cup (Q_z \cap \sigma(T)))^{\alpha(z)}}\Big)\Big),$$

*for some constants $C(z) > 0$ and $0 < \alpha(z) < \min\{1, 2 - d_{\sigma(T)}(z)\}$ and for all $\xi \in Q_z \setminus \partial Q_z \cup (Q_z \cap \sigma(T))$. Then, $T$ is residually decomposable with residuum $S$. If $S$ is totally disconnected then, $T$ is decomposable.*

PROOF. Fix an arbitrary point $z \in \sigma(T) \setminus S$ and a closed square $Q_z$ as in the statement of the theorem. We observe from Theorem 2.5 that by setting the function $|G(\xi)| = \|R(\xi, T)\|$, $Q_z = Q$ and $K = \partial Q_z \cup (Q_z \cap \sigma(T))$ and for some $\varepsilon(z) > 0$, the integral

$$I_{Q_z} = \iint\limits_{Q_z} \Big(\log^+ \log^+ \|R(\xi, T)\|\Big)^{1+\varepsilon(z)} \, d\lambda(\xi)$$

is finite. Thus, from the proof of Corollary 1.11 with $f_z(t) = t^{1+\varepsilon(z)}$, $t \geq 0$, $T$ is residually decomposable.

If $S$ is totally disconnected, then $T$ is decomposable from Corollary 1.14. ☐

THEOREM 4.22. *Consider the situation $S3$ and assume that for each $z \in \sigma(T)\backslash S$, there exists a closed square $Q(z)$ with centre $z$ having empty intersection with $S$ such that for some constant $\delta(z) > 0$, $Q(z) \cap \sigma(T)$ satisfies the grid dimension condition of type $(2, \delta(z))$ and that the resolvent satisfies the condition;*

$$\|R(\xi, T)\| \leq \exp\Big(\frac{C(z)}{\operatorname{dist}(\xi, \partial Q(z) \cup (Q(z) \cap \sigma(T)))^{\alpha(z)}}\Big),$$

*for positive constants $C(z)$ and $\alpha(z)$ and for all $\xi \in Q(z) \setminus \partial Q(z) \cup (Q(z) \cap \sigma(T))$. Then, $T$ is residually decomposable with residuum $S$. If $S$ is totally disconnected then, $T$ is decomposable.*

PROOF. Using Theorem 2.7 along the lines of the proof of Theorem 4.21, the results hold. ☐

THEOREM 4.23. *Consider the situation $S3$ and assume that for each $z \in \sigma(T)\backslash S$, there exists a closed square $Q(z)$ with centre $z$ having empty intersection* *with $S$ such that for some constant $\delta(z) > 0$, $Q(z) \cap \sigma(T)$ satisfies the grid dimension condition of type $(3, \delta(z))$ and that the resolvent satisfies the condition;*

$$\|R(\xi, T)\| \leq \frac{C(z)}{\operatorname{dist}(\xi, \partial Q(z) \cup (Q(z) \cap \sigma(T)))^{\alpha(z)}},$$

*for positive constants $C(z)$ and $\alpha(z) \geq 1$ and for all $\xi \in Q(z) \setminus \partial Q(z) \cup (Q(z) \cap \sigma(T))$. Then, $T$ is residually decomposable with residuum $S$. If $S$ is totally disconnected then, $T$ is decomposable.*

Proof. Using Theorem 2.10 along the lines of the proof of Theorem 4.21, the results hold. □

This theorem gives an immediate application to hyponormal operators on a Hilbert space $\mathcal{H}$, which we state in the following remark.

Remark 4.24. *Let $T$ be a hyponormal operator on a Hilbert space $\mathcal{H}$ such that $\sigma(T, \mathcal{H}) = S \cup (\sigma(T, \mathcal{H}) \setminus S)$ satisfies the grid dimension condition of type $(3, \delta(z))$, where $S$ is totally disconnected and $\varpi := \sigma(T, \mathcal{H}) \setminus S$ has Lebesgue measure zero. Then, $T$ is decomposable. Moreover, $R(\sigma(T, \mathcal{H})) = C(\sigma(T, \mathcal{H}))$ by* [**12**].

Theorem 4.25. *Consider the situation $S3$ and assume that for each $z \in \sigma(T) \setminus S$, there exists a closed square $Q(z)$ with centre $z$ having empty intersection with $S$ such that for positive constants $C(z)$ and $\alpha(z)$, $Q(z) \cap \sigma(T)$ satisfies the grid dimension condition of type $(1, \alpha(z))$ and that for some $\varepsilon(z) > 0$, $C_1(z) > 0$, $\beta(z) > 3 + \alpha(z)$ and a monotone increasing function $f_z(t) = t(\log^+ t)^{1+\varepsilon(z)}, t \geq 0$ satisfying condition $(\eta)$, the resolvent satisfies the growth;*

$$\|R(\xi, T)\| \leq \exp\Big(\exp\Big(\frac{C_1(z)}{\operatorname{dist}(\xi, K_\circ(z))\big(\log \frac{1}{\operatorname{dist}(\xi, K_\circ(z))}\big)^{\beta(z)}}\Big)\Big),$$

*for all $\xi \in Q(z) \setminus K_\circ(z)$, where $K_\circ(z) := \partial Q(z) \cup (Q(z) \cap \sigma(T))$ Then, $T$ is residually decomposable with residuum $S$. If $S$ is totally disconnected then, $T$ is decomposable.*

Proof. Following the arguments as in the proof of Theorem 4.21 and using Theorem 2.13, the results hold. □

## 4.6. Regular Banach function algebras on sets of positive area

In this section, we deal with the regularity of Banach function algebras $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ on a given perfect, compact set $K$ having positive area and satisfying some local grid dimension conditions.

Theorem 4.26. *Let $K \subset \mathbb{C}$ be a perfect, compact set and $S$ be a compact subset of $K$. Assume that for each $\zeta \in K \setminus S$, there exists a closed square $Q'_\zeta$ centred at $\zeta$ having empty intersection with $S$ such that $\lambda(Q'_\zeta \cap K) = 0$. Let $\tilde{\mathcal{M}}_z$ be the operator of multiplication on $l^q(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$, $q = \{1, \infty\}$ as defined in Theorem 3.35. Assume that* (4.4) *holds and define an entire function $\omega : \mathbb{C} \to \mathbb{C}$ by,*

$$\omega(\xi) := \sum_{p=0}^{\infty} \xi^{p+1} \|\mathsf{m}_p^{-1}\|^p_{Q'_\zeta \cap K}, \qquad \xi \in \mathbb{C}. \tag{4.8}$$

*Moreover, suppose that for all $\zeta \in K \setminus S$ a function $f_\zeta$ is given that satisfies condition $(\eta)$. If for all $\zeta \in K \setminus S$, the integral*

$$I = \iint\limits_{Q'_\zeta} f_\zeta\Big(\log^+\log^+\omega\big(\frac{1}{\mathrm{dist}(\xi, \partial Q'_\zeta \cup (Q'_\zeta \cap K))}\big)\Big)\, d\lambda(\xi)$$

*is finite, then $\tilde{\mathcal{M}}_z$ is residually decomposable with residuum $S$. If $S$ is totally disconnected then, $\tilde{\mathcal{M}}_z$ is decomposable.*

Proof. Define the function $h \in \mathrm{Rat}(K)$ by;

$$h(u) = \frac{1}{\xi - u}, \quad \xi \in \mathbb{C} \setminus K, \quad \text{for all} \quad u \in K.$$

$$\text{Then,} \quad \frac{h^{(p)}}{\mathcal{M}_p(u)} = \frac{1}{(\xi - u)^{p+1}\mathsf{m}_p^p(u)} \quad \text{and}$$

$$\Big\|\frac{h^{(p)}}{\mathcal{M}_p}\Big\|_K = \sup_{u\in K}\frac{1}{|\xi - u|^{p+1}\mathsf{m}_p^p(u)}$$

Fix an arbitrary point $\zeta \in K \setminus S$ and a closed square $Q_\zeta$ centred at $\zeta$. Let $Q'_\zeta$ be a closed square with centre $\zeta$ contained in $\mathrm{int}\, Q_\zeta$. Then, if now $\xi$ is in $Q'_\zeta \setminus K$,

$$\Big\|\frac{h^{(p)}}{\mathcal{M}_p}\Big\|_K \le \sup_{u\in(Q'_\zeta\cap K)}\frac{\mathsf{m}_p^{-p}(u)}{|\xi - u|^{p+1}} + \sup_{u\in K\setminus Q_\zeta}\frac{\mathsf{m}_p^{-p}(u)}{|\xi - u|^{p+1}} \tag{4.9}$$

Since $|\xi - u| \ge \mathrm{dist}(\xi, \partial Q'_\zeta \cup (Q'_\zeta \cap K))$ and also notice that $|\xi - u|$ is bounded below on $K \setminus Q_\zeta$, i.e. there exists some $\delta > 0$, such that $\delta < |\xi - u|$. Then, (4.9) becomes,

$$\Big\|\frac{h^{(p)}}{\mathcal{M}_p}\Big\|_K \le \sup_{u\in(Q'_\zeta\cap K)}\frac{\mathsf{m}_p^{-p}(u)}{\mathrm{dist}(\xi, \partial Q'_\zeta \cup (Q'_\zeta \cap K))^{p+1}} + \sup_{u\in K\setminus Q_\zeta}\frac{\mathsf{m}_p^{-p}(u)}{\delta^{p+1}}$$

Hence,

$$\begin{aligned}\|h\|_{\{\mathcal{M}_p\},1} &\le \sum_{p=0}^{\infty}\sup_{u\in(Q'_\zeta\cap K)}\frac{\mathsf{m}_p^{-p}(u)}{\mathrm{dist}(\xi, \partial Q'_\zeta \cup (Q'_\zeta \cap K))^{p+1}} + \sum_{p=0}^{\infty}\sup_{u\in K\setminus Q_\zeta}\frac{\mathsf{m}_p^{-p}(u)}{\delta^{p+1}}\\ &= \omega\Big(\frac{1}{\mathrm{dist}(\xi, \partial Q'_\zeta \cup (Q'_\zeta \cap K))}\Big) + C(\delta),\end{aligned}$$

where $C(\delta)$ is a positive constant. Similarly,

$$\|h\|_{\{\mathcal{M}_p\},\infty} \le \omega\Big(\frac{1}{\mathrm{dist}(\xi, \partial Q'_\zeta \cup (Q'_\zeta \cap K))}\Big) + C(\delta).$$

Thus, from the proof of Theorem 4.15 we see that, $(\xi - \tilde{\mathcal{M}}_z)^{-1}$ is the operator of multiplication by $J(h)$ and $\tilde{J}(h)$, as in the proof of Theorem 4.15. Hence,

$$\|(\xi - \tilde{\mathcal{M}}_z)^{-1}\| \le \tilde{C}\|h\|_{\{\mathcal{M}_p\},q} \le C_\circ\, \omega(\frac{1}{d(\xi)})$$

for some constants $\tilde{C}, C_\circ > 0$, where $d(\xi) := \operatorname{dist}(\xi, \partial Q'_\zeta \cup (Q'_\zeta \cap K))$. Therefore,

$$\iint\limits_{Q'_\zeta} f_\zeta\Big(\log^+\log^+\|(\xi - \tilde{\mathcal{M}}_z)^{-1}\|\Big)\, d\lambda(\xi) < \infty.$$

Hence, from Corollary 1.11 and its proof for $T = \tilde{\mathcal{M}}_z$ and $K = \sigma(\tilde{\mathcal{M}}_z)$, we obtain that $\tilde{\mathcal{M}}_z$ is residually decomposable. Further, if $S$ is totally disconnected, then the result follows from Corollary 1.14. □

Corollary 4.27. *Assume that all conditions of Theorem 4.26 hold and let $d/dz$, $\tilde{d}$ be as in Section 4.3.1 such that $d/dz$ is closable in $C(K)$ and $(\mathcal{M}_p)_{p\in\mathbb{N}_\circ}$ be a $l^q$-algebra sequence, $q = \{1, \infty\}$. Then, the Banach function algebras $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ will be normal on $K$.*

Proof. From Theorem 4.26 we obtain the decomposability of $\tilde{\mathcal{M}}_z$ and thus, the proof follows immediately from Corollary 3.5. □

Next result deals with the regularity of Banach function algebras $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ on $K$.

Corollary 4.28. *Let $d/dz$ and $\tilde{d}$ be as in Corollary 4.27 and assume that all conditions of Theorem 4.26 hold. Let $(\mathcal{M}_p)_{p=0}^\infty$ be a $l^q$-algebra sequence ($q = \{1, \infty\}$) such that:*

(*a*) $\lim_{t\to\infty} \|\mathsf{A}_t\|_K^{1/t} = 0$;

(*b*) $\left\| \sum_{k=1}^{n-1} \frac{\mathsf{m}_k^k \mathsf{m}_{n-k}^{n-k}}{\mathsf{m}_n^n} \right\|_K \to 0, \quad$ *as* $n \to \infty$.

*Then, $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ are regular Banach function algebras on $K$.*

Proof. Considering $(\mathcal{M}_p)_{p=0}^\infty$ be a $l^1$-algebra sequence and from the assumption ($a$), we obtain from Theorem 4.8 that $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ is natural on $K$.

Similarly, for $(\mathcal{M}_p)_{p=0}^\infty$ be a $l^\infty$-algebra sequence and from the assumption ($b$), we obtain from Theorem 4.13 that $D_\infty(K, \{\mathcal{M}_p\})$ is natural on $K$.

Further, Corollary 4.27 gives that $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ are normal on $K$ and hence, (by the definitions of normality and naturality) regular Banach function algebras on $K$. □

<u>*Situation S*4</u>
- $K \subset \mathbb{C}$ is a perfect, compact set ;
- $S$ is a compact, totally disconnected subset of $K$;
- $\tilde{d}$ is the closure of $d/dz$, such that $d/dz$ is closable in $C(K)$;
- $(\mathcal{M}_p)_{p\in\mathbb{N}_\circ}$ is a $l^q$ – algebra sequence .

Proposition 4.29. *Consider the situation S4. Assume that for each $\zeta \in K \setminus S$, there exists a closed square $Q_\zeta$ with centre $\zeta$ having empty intersection with $S$ such that $Q_\zeta \cap K$ has upper box dimension $\le d(\zeta) < 2$ and that for some positive constants $C(\zeta)$, $c_1(\zeta)$ and $\alpha(\zeta) < \min\{1, 2 - d(\zeta)\}$,*

$$\|\mathsf{m}_p^{-1}\|_{Q_\zeta \cap K} \le \left(\frac{C(\zeta)}{\log \frac{p}{c_1(\zeta)}}\right)^{1/\alpha(\zeta)}. \tag{4.10}$$

*Then, $\tilde{\mathcal{M}}_z$ will be decomposable and $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ will be normal on $K$.*

PROOF. It is evident from (4.10) that $\|\mathsf{m}_p^{-1}\|_{Q_\zeta \cap K} \to 0$, as $p \to \infty$. For $\xi \in \mathbb{C} \setminus K$, define the rational function

$$h(u) = \frac{1}{\xi - u}, \ \forall\, u \in K.$$

Thus, following the proof of Theorem 4.26, we fix an arbitrary point $\zeta \in K \setminus S$ and a closed square $R_\zeta$ centred at $\zeta$. Let $Q_\zeta$ be a closed square centred at $\zeta$ with $Q_\zeta \subset \operatorname{int} R_\zeta$. Then, for $\xi \in Q_\zeta \setminus K$ and for some $\delta(\zeta) > 0$, we have for $q = \{1, \infty\}$ that:

$$\|h\|_{\{\mathcal{M}_p\},q} \le \omega\Big(\frac{1}{d(\xi)}\Big) + C(\delta).$$

where $\omega$ is an entire function as in (4.8), $d(\xi) := \operatorname{dist}(\xi, \partial Q_\zeta \cup (Q_\zeta \cap K)$ and $C(\delta)$ is a positive constant. Hence, from (4.10), we obtain:

$$\|h\|_{\{\mathcal{M}_p\},q} \le \tilde{C}(\delta) \sum_{p=0}^{\infty} \frac{1}{d(\xi)^{p+1}} \Big(\frac{C(\zeta)}{\log \frac{p}{c_1(\zeta)}}\Big)^{p/\alpha(\zeta)}.$$

From Proposition 3.31(b), we get that:

$$\|h\|_{\{\mathcal{M}_p\},q} \le \exp(c_2(\zeta) \exp(d_2(\zeta) d(\xi)^{\alpha(\zeta)})),$$

for some positive constants $c_2(\zeta)$ and $d_2(\zeta)$. From the proof of Theorem 4.26, we observe that for some positive constant $C_\circ$;

$$\|(\xi - \tilde{\mathcal{M}}_z)^{-1}\| \le C_\circ \|h\|_{\{\mathcal{M}_p\},q}.$$

Thus from Theorem 4.21 with $T = \tilde{\mathcal{M}}_z$ and $K = \sigma(\tilde{\mathcal{M}}_z)$, one observes that $\tilde{\mathcal{M}}_z$ is decomposable. Further, from Corollary 4.27, the Banach function algebras $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ are normal on $K$. □

PROPOSITION 4.30. *Consider the situation S4. Assume that for each $z \in K \setminus S$, there exists a closed square $Q(z)$ with centre $z$ having empty intersection with $S$ such that for some constant $\delta(z) > 0$, $Q(z) \cap K$ satisfies the grid dimension condition of type $(2, \delta(z))$ and that for some $\alpha(z) > 0$:*

$$\limsup_{p \to \infty} \frac{\log p}{\log \|\mathsf{m}_p\|_{Q(z) \cap K}} =: \|\alpha\|_{Q(z) \cap K} < \infty. \tag{4.11}$$

*Then, $\tilde{\mathcal{M}}_\zeta$ will be decomposable and $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ will be normal on $K$.*

PROOF. Following the proof of the previous Proposition, we fix an arbitrary point $z \in K \setminus S$ and a closed square $R(z)$ centred at $z$. Let $Q(z)$ be a closed square centred at $z$ with $Q(z) \subset \operatorname{int} R(z)$. Further, for $\xi \in \mathbb{C} \setminus K$, define the function $h \in \operatorname{Rat}(K)$ by;

$$h(u) = \frac{1}{\xi - u}, \ \forall\, u \in K$$

If, now $\xi \in Q(z) \setminus K$ and for $q = \{1, \infty\}$ we have

$$\|h\|_{\{\mathcal{M}_p\},q} \le \omega\Big(\frac{1}{d(\xi)}\Big) + C(\delta)$$

where as above, $\omega$ is an entire function as in (4.8), $d(\xi) := \mathrm{dist}(\xi, \partial Q(z) \cup (Q(z) \cap K)$ and $C(\delta)$ is a positive constant. Further, we observe from Theorem 2.2.2. [**11**] that, (4.11) denotes the order of the entire function $\omega$, which is finite and thus from formula 2.1.5 [**11**], one obtains that, for some $\varepsilon(z) > 0$,

$$\|h\|_{\{\mathcal{M}_p\},q} \leq c_1(z) \exp\Big(\frac{1}{d(\xi)^{\alpha(z)+\varepsilon(z)}}\Big),$$

where $c_1(z)$ is a positive constant. Thus, from the proof of Theorem 4.26, we have,

$$\|(\xi - \tilde{\mathcal{M}}_\zeta)^{-1}\| \leq C_\circ \|h\|_{\{\mathcal{M}_p\},q}$$

for some positive constant $C_\circ$ and the operator $\tilde{\mathcal{M}}_\zeta$ is decomposable from Theorem 4.22 with $T = \tilde{\mathcal{M}}_\zeta$ and $K = \sigma(\tilde{\mathcal{M}}_\zeta)$. Further, from Corollary 4.27, the Banach function algebras $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ are normal on $K$. $\square$

Combining the above Propositions 4.29 and 4.30 with Corollary 4.28, we observe that $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ are regular Banach function algebras on $K$.

PROPOSITION 4.31. *Consider the situation S4. Assume that for each $z \in K \setminus S$, there exists a closed square $Q(z)$ with centre $z$ having empty intersection with $S$ such that for some constant $\delta(z) > 0$, $Q(z) \cap K$ satisfies the grid dimension condition of type $(3, \delta(z))$ . Then, $\tilde{\mathcal{M}}_\zeta$ will be decomposable and $\tilde{\mathcal{D}}^k(K)$ will be normal on $K$.*

PROOF. Following the proof of Proposition 4.29, we fix a point $z \in K \setminus S$ and a closed square $R(z)$ centred at $z$. Let $Q(z)$ be a closed square centred at $z$ with $Q(z) \subset \mathrm{int}\, R(z)$. Define the function $h \in \mathrm{Rat}(K)$ by;

$$h(u) = \frac{1}{\xi - u}, \quad \xi \in \mathbb{C} \setminus K, \ \forall u \in K,$$

It is clear from the proof of the Proposition 4.29 that if, now for $\xi \in Q(z) \setminus K$ we have:

$$\|h\|_k = \sum_{p=0}^{k} \frac{1}{p!} \Big\| \frac{1}{(\xi - u)^{p+1}} \Big\|_{Q(z)\cap K} \leq \frac{C(z)}{d(\xi)^{p+1}},$$

for some positive constant $C(z)$ and $d(\xi) := \mathrm{dist}(\xi, \partial Q(z) \cup (Q(z) \cap K)$. Moreover, again from the proof of Theorem 4.26

$$\|(\xi - \tilde{\mathcal{M}}_\zeta)^{-1}\| \leq C_\circ \|h\|_k$$

for some positive constant $C_\circ$. Hence, $\tilde{\mathcal{M}}_\zeta$ is decomposable from Theorem 4.23 with $T = \tilde{\mathcal{M}}_\zeta$ and $\sigma(\tilde{\mathcal{M}}_\zeta) = K$. The normality of $\tilde{\mathcal{D}}^k(K)$ on $K$ is clear from Corollary 3.5. $\square$

Recall that the local upper box dimension of a compact set $K$ at a point $z \in K$ is given by

$$d_K(z) = \inf_Q \{\delta \ ; \ \overline{\dim}_B(Q \cap K) \leq \delta\}$$

where the infimum is taken over all closed neighbourhoods of $z$.

PROPOSITION 4.32. *Let $K$ be a perfect, compact set and let $S \subset K$ be totally disconnected and closed. Consider the sequence $(\mathcal{M}_p)_{p\in\mathbb{N}_\circ}$ of functions on $K$ given by*

(i) $\mathcal{M}_p(z) = p!\prod_{k=1}^{p}(\log(e+k))^{1+\operatorname{dist}(z,S)}$,
(ii) $\mathcal{M}_p(z) = p!(\log(e+p))^{p(1+\operatorname{dist}(z,S))}$

*for all $z \in K$, $p \in \mathbb{N}_\circ$. If*

$$d_K(z) + \frac{1}{1+\operatorname{dist}(z,S)} < 2 \tag{4.12}$$

*for all $z \in K \setminus S$ then the operator $\tilde{\mathcal{M}}_\xi$ is decomposable on $l^q(\mathbb{N}_\circ, C(K), \{\mathcal{M}_p\})$ and on the completions $\tilde{\mathcal{D}}_q(K,\{\mathcal{M}_p\})$ of $\mathcal{D}_q(K,\{\mathcal{M}_p\})$, $q = \{1,\infty\}$.*

PROOF. Of course, the sequence $(\mathcal{M}_p)_{p\in\mathbb{N}_\circ}$ is an $l^q$-algebra sequence satisfying condition (4.4). Let $z$ be an arbitrary point in $K \setminus S$. By the definition of $d_K(z)$ and by the continuity of $z \mapsto d_K(z)$ there exists a closed square $Q$ with centre $z$ such that

$$\overline{\dim}_B(Q\cap K) + \underbrace{\sup_{\xi\in Q\cap K}\frac{1}{1+\operatorname{dist}(\xi,S)}}_{=:\alpha} < 2.$$

Fix a smaller closed square $Q' \subset \operatorname{int} Q$ and let $w$ be an arbitrary point in $\operatorname{int} Q' \setminus K$. For the function

$$\xi \mapsto h(\xi) := \frac{1}{w-\xi}$$

we have by the proof of Theorem 4.26

$$\|h\|_{\{\mathcal{M}_p\},q} \le \omega\Big(\frac{1}{\operatorname{dist}(w,\partial Q' \cup (Q'\cap K))}\Big) + C$$

where

$$C := \sum_{p=0}^{\infty}\sup_{\xi\in K\setminus Q}\mathsf{m}_p^{-p}(\xi)\cdot\delta^{-(p+1)}$$

with $\delta := \operatorname{dist}(Q',\partial Q)$ and

$$\omega(\eta) := \sum_{p=0}^{\infty}\sup_{\xi\in Q'\cap K}\mathsf{m}_p^{-p}(\xi)\eta^{p+1}, \quad \eta\in\mathbb{C}.$$

Because

$$\gamma := \sup_{\xi\in K}\mathsf{m}_p^{-1}(\xi) \le (\log(e+k_\circ p)^{-(1-k_\circ)} \to 0, \quad \text{as } p\to\infty$$

in both cases, where $k_\circ \in (0,1)$ (see proof of Example 4.4), we see that the constant $C$ is finite and that $\omega$ is an entire function. More precisely, choosing $\varepsilon > 0$ such that

$$\overline{\dim}_B(Q\cap K) + \sup_{\xi\in Q\cap K}\frac{1}{1+\operatorname{dist}(\xi,S)} + \varepsilon < 2$$

$$\text{and} \quad \varepsilon + \sup_{\xi\in Q\cap K}\frac{1}{1+\operatorname{dist}(\xi,S)} < 1$$

we see from Corollary 3.32 and Examples 3.34(2) that there are constants $c, d > 0$ such that

$$\omega\Big(\frac{1}{\operatorname{dist}(w, \partial Q' \cup (Q' \cap K))}\Big) \leq \exp\Big(c \exp\Big(\frac{d}{\operatorname{dist}(w, \partial Q' \cup (Q' \cap K))^{\alpha+\varepsilon}}\Big)\Big)$$

Hence, we have

$$\|h\|_{\{\mathcal{M}_p\},q} \leq \exp\Big(c_1 \exp\Big(\frac{d}{\operatorname{dist}(w, \partial Q' \cup (Q' \cap K))^{\alpha+\varepsilon}}\Big)\Big)$$

with $c_1, d > 0$ not depending on $w$.

As in both cases the sequences $(\mathcal{M}_p)_{p\in\mathbb{N}_\circ}$ are $l^q-$algebra sequences and $h \in \mathcal{D}_q(K, \{\mathcal{M}_p\})$, the inverse of $w - \tilde{\mathcal{M}}_\xi$ is given by the multiplication by $h$ and we obtain for its operator norm

$$\|(w - \tilde{\mathcal{M}}_\xi)^{-1}\| \leq \exp\Big(\exp\Big(\frac{d_1}{\operatorname{dist}(w, \partial Q' \cup (Q' \cap K))^{\alpha+\varepsilon}}\Big)\Big)$$

for positive constant $d_1$. By Theorem 4.21, $\tilde{\mathcal{M}}_\xi$ is decomposable.

□

Corollary 4.33. *Let $K$ be a perfect, compact set in $\mathbb{C}$ such that $d/dz$ is closable with closure $\tilde{d}$ and let $S$ be a totally disconnected, closed subset of $K$. If $(\mathcal{M}_p)_{p\in\mathbb{N}_\circ}$ is one of the sequences considered in the previous Proposition, then $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ and $D_\infty(K, \{\mathcal{M}_p\})$ are regular, natural Banach function algebras.*

Proof. By Theorem 4.8 in connection with Examples 4.10, the algebra $\tilde{\mathcal{D}}_1(K, \{\mathcal{M}_p\})$ is a natural Banach function algebra. For $D_\infty(K, \{\mathcal{M}_p\})$ this follows from Theorem 4.13 and Lemma 4.2.

As the operator of the multiplication with the variable is decomposable on $\tilde{\mathcal{D}}_q(K, \{\mathcal{M}_p\})$, $q = \{1, \infty\}$ by Proposition 4.32 and as $\tilde{\mathcal{D}}_\infty(K, \{\mathcal{M}_p\}) \subseteq D_\infty(K, \{\mathcal{M}_p\})$ both algebras are normal. From the definition of naturality and regularity we see that they are regular Banach algebras. □

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# List of Symbols

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