Resolution of Inverse Scattering Problems for the full three-dimensional Maxwell-Equations in Inhomogeneous Media using the Approximate Inverse

Dissertation
zur Erlangung des akademischen Grad
des Doktors der Naturwissenschaften
der Mathematischen-Naturwissenschaftlichen Fakultäten
der Universität des Saarlandes

von

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Saarbrücken
2006

Tag des Kolloquiums:	13. Juli 2006
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”Mein Herr, erbarme Dich ihrer,
so wie sie mich aufgezogen haben,
als ich klein war.”

with gratitude to my parents and hope for my children.

”Wasser, das durch ein Leck in das Boot dringt,
führt zum Untergang des Bootes.
Wasser unter dem Boot jedoch trägt es.”

with recognition to my ”Doktorvater”.

”Von Anfang an, begehren wir nach altem Wein,
dürsten nach ihm ans Ende der Welt.
Dieser Wein aus dem Becher des Seins
ist nicht sauer.
Schon sein Duft betört und berauscht uns.
Indes die Durstigen nach Wasser suchen,
sucht auch das Wasser die Durstigen.
... Wenn du in deinen Intellekt verliebt bist
und meinst, du seiest den Anbetern der Form
überlegen, dann bedenke:
Dieser Intellekt ist nur ein Strahl des
universalen Intellekt, der auf deine Sinne fällt.
Betrachte ihn als leichten Goldüberzug
über deinem Kupfer.
Weise wünschen sich Selbstbeherrschung;
Kinder Süßigkeiten.”

Jalaluddin Rumi.

Kurze Zusammenfassung

Ziel dieser Arbeit ist die Entwicklung einer neuen Methode zur Lösung des inversen elektromagnetischen Streuproblems in inhomogenen Medien unter Verwendung von elektromagnetischen Nahfelddaten. Die vollständigen drei-dimensionalen zeit-harmonischen MAXWELL-Gleichungen lassen sich äquivalenterweise als ein System von Kontrastquellen-Integralgleichung formulieren, das hier zur mathematischen Modellierung des Problems benutzt wird. Das inverse Problem ist schlechtgestellt und nichtlinear. Durch Ausnutzung des bekannten Konzeptes der äquivalenten Quellen lässt sich das inverse Streuproblem in zwei Aufgaben unterteilen: zum einen das inverse Quellenproblem, das linear aber schlechtgestellt ist, zum anderen das inverse Mediumproblem, das zwar stabiler als das ursprüngliche Problem ist, allerdings nichtlinear bleibt. Weiter sind die Komponenten der elektromagnetischen Felder so gekoppelt, dass die komponentenweise Behandlung des Systems nicht direkt ausführbar ist. Wir führen das neue Konzept der verallgemeinerten induzierten Quellen ein, um die Vektorintegralgleichungen in skalaren Lippmann-Schwinger Gleichungen zu entkoppeln. Abdullah und Louis [AL99] analysierten den für die 2-D Lippmann-Schwinger Gleichung eingeführten skalaren Streuoperator bei sphärischer Messgeometrie mit Hilfe der Singulärwertzerlegung. Zur Anwendung der Methode der approximativen Inversen analysieren wir den skalaren Streuoperator in der 3-D sphärischen Anordnung, wobei wir die Singulärwertzerlegung herleiten und eine Basis für den Nullraum bestimmen. Weiter wenden wir die von Louis [Lou99] eingeführte Fehleranalyse der Methode der approximativen Inverse an, um Fehlerabschätzungen der regularisierten Lösungen des skalaren inversen Streuproblems mit sphärischer Messgeometrie zu erhalten. Die lineare Inversion wird mit der effizienten und stabilen Methode der approximativen Inversen durchgeführt. Damit wird zuerst die verallgemeinerte induzierte Quelle aus den Daten jedes Experimentes bestimmt. Die äquivalenten Quelle lässt sich dann daraus herleiten. Mit Hilfe des neuen Konzeptes der verallgemeinerten induzierten Quellen adaptieren wir die nichtlineare Version des KACZMARZ-Verfahrens zur Entwicklung eines iterativen Verfahrens. Damit werden die Materialeigenschaften aus den berechneten äquivalenten Quellen rekonstruiert. Einige numerische Simulationen belegen die analytischen Ergebnisse für die entwickelte Methode und zeigen die praktische Brauchbarkeit.

Abstract

A new method is developed in this work to solve the inverse electromagnetic scattering problem in inhomogeneous media using near-field measurements. The modeling is based on the formulation as contrast source integral equations of the full three-dimensional time-harmonic Maxwell-model. This inverse problem is ill-posed and nonlinear. The known idea of using equivalent sources splits inverse scattering into two subproblems: the inverse source problem, which is linear and ill-posed, and the inverse medium problem, which is more stable but nonlinear. We introduce the concept of generalized induced source to recast the system of intertwined vector equations, describing the electromagnetic inverse source problem, into decoupled scalar scattering problems. We utilize the method of the approximate inverse to recover the induced source for each experiment. We consider in three-dimensional setting the spherical scattering operator introduced by Abbdullah and Louis [Abd98] for 2-D acoustic waves. We derive its singular-value decomposition and determine a basis for its null space. We further apply some results about error estimate from [Lou99] to the scalar problem in three-dimensions with spherical set-up. The nonlinear version of the algorithm of Kaczmarz is then adapted, using the generalized induced source, to derive an iterative scheme for the resolution of the inverse medium problem. Numerical simulations illustrate the efficiency and practical usefulness of the developed method.

Zusammenfassung

Ziel dieser Arbeit ist die Entwicklung einer neuen Methode zur Lösung des inversen Streuproblems für die vollständigen drei-dimensionalen zeit-harmonischen MAXWELL-Gleichungen in inhomogenen Medien unter Verwendung von elektromagnetischen Nahfelddaten.
Dazu stelle man sich vor, dass ein in einem homogenen Medium eingebettetes Objekt von einfallenden elektromagnetischen Wellen bestrahlt wird.
Das inverse elektromagnetische Streuproblem besteht nun darin, aus äußeren Messungen des gestreuten elektromagnetischen Feldes die elektromagnetischen und geometrischen Eigenschaften des Objektes zu rekonstruieren.

Nun beschreiben wir die mathematische Modellierung des Problems. Die Frequenz $w > 0$ des einfallenden elektromamgnetischen Feldes $(\mathbf{E}^{inc}, \mathbf{H}^{inc})$ sei fest. Weiter sei das Medium im Hintergrund homogen mit der dielektrischen Konstante $\varepsilon_0 > 0$ und der magnetischen Permeabilität $\mu_0 > 0$.
Die zeit-harmonische MAXWELL-Gleichungen im isotropen Medien lassen sich äquivalenterweise durch das System der Integralgleichungen

$$\begin{aligned}\mathbf{E}(x) &= \mathbf{E}^{inc}(x) - k^2 \int_{\mathbb{R}^3} g_k(x,y)\, f_e(y)\, \mathbf{E}(y)\, dy \\ &+ iw\mu_0 \int_{\mathbb{R}^3} \nabla_y g_k(x,y) \times f_m(y)\, \mathbf{H}(y)\, dy \\ &+ \int_{\mathbb{R}^3} \nabla_y g_k(x,y)\, \nabla \cdot (f_e \mathbf{E})(y)\, dy, \qquad (1)\end{aligned}$$

$$\begin{aligned}\mathbf{H}(x) &= \mathbf{H}^{inc}(x) - k^2 \int_{\mathbb{R}^3} g_k(x,y) f_m(y)\, \mathbf{H}(y)\, dy \\ &- iw\varepsilon_0 \int_{\mathbb{R}^3} \nabla_y g_k(x,y) \times f_e(y)\, \mathbf{E}(y)\, dy \\ &+ \int_{\mathbb{R}^3} \nabla_y g_k(x,y)\, \nabla \cdot (f_m \mathbf{H})(y)\, dy, \qquad (2)\end{aligned}$$

formulieren, wobei $g_k(x,y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}$, $x \neq y$, $x, y \in \mathbb{R}^3$, die Green'sche Funktion in $\mathbb{R}^3$ für den HELMHOLTZ-Operator $(\Delta + k^2)$ mit der Wellenzahl $k := w\sqrt{\varepsilon_0 \mu_0}$ ist.
Das komplexwertige drei-dimensionale Vektorfeld $\mathbf{E}$ bzw. $\mathbf{H}$ bezeichnet hier das gesamte elektrische bzw. magnetische Feld. Das gestreute elektrische bzw. magnetische Feld ist durch $\mathbf{E}^s := \mathbf{E} - \mathbf{E}^{inc}$ bzw. $\mathbf{H}^s := \mathbf{H} - \mathbf{H}^{inc}$ gegeben.
Die Materialeingenschaften des Streuers sind durch die Kontrastfunktionen

$$f_e := (1 - \varepsilon_0^{-1}\tilde{\varepsilon}) \quad \text{and} \quad f_m := (1 - \mu_0^{-1}\tilde{\mu})$$

mit

$$\tilde{\varepsilon}(x) := \varepsilon(x) + i\frac{\sigma_e(x)}{w} \quad \text{und} \quad \tilde{\mu}(x) := \mu(x) + i\frac{\sigma_m(x)}{w},\ x \in \mathbb{R}^3,$$

beschrieben, wobei $\varepsilon > 0$, $\mu > 0$, $\sigma_e \geq 0$ bzw. $\sigma_m \geq 0$, die dielektrische Permittivität, die magnetische Permeabilität, die elektrische Konduktivität bzw. die magnetische Konduktivität bezeichnet. Weiter setzen wir vorraus, dass die Kontrastfunktionen kompakten Träger haben, die im beschränkten Gebiet $\Omega \subset \mathbb{R}^3$ mit glatten Rand Γ enthalten sind.
Die Gleichungen (1)-(2) lassen sich in der Operatorform

$$\mathcal{T}(f\mathbf{u}) = \mathbf{u}^s$$

mit

$$\mathbf{u} := (\mathbf{E}, \mathbf{H}), \mathbf{u}^s := (\mathbf{E}^s, \mathbf{H}^s),\ f := (f_e, f_m),$$

schreiben.
Wenn $\mathbf{M}_\Gamma$ der Messoperator auf dem Rand Γ bezeichnet, dann lautet das inverse Streuproblem:
"Aus gemessenen Daten $\mathbf{d}_j = \mathbf{M}_\Gamma \mathbf{u}_j^s,\ \mathbf{d}_j(x) = (d_j^i(x))_i,\ x \in \Gamma$, bestimme die Kontrastfunktion f, so dass

$$\mathbf{M}_\Gamma\, \mathcal{T}(f\mathbf{u}_j) = \mathbf{d}_j$$

für alle Experimente j gilt".
Bei der Lösung dieses Problems begegnet man drei Hauptschwierigkeiten:

1. Die Schlechtgestelltheit, was bei inversen Problemen typisch ist.
2. Die Nichtlinearität. Obwohl der Operator $\mathcal{T}$ linear ist, kann die Abhängigkeit des Feldes $\mathbf{u}_j$ von der Kontrastfunktion f stark nichtlinear sein.
3. Die Kopplung der Komponenten der elektromagnetischen Felder, was die komponentenweise Behandlung der Vektorgleichungen des Systems (1)-(2) behindert.

Durch Ausnutzung des bekannten Konzeptes der äquivalenten Quellen $\mathbf{q} = f\mathbf{u}$, lässt sich das inverse Streuproblem in zwei Aufgaben unterteilen:

1. Das inverse Quellenproblem, das linear aber schlechtgestellt ist. Für jedes Experiment j besteht die Aufgabe hier in der Bestimmung der äquivalenten Quelle $\mathbf{q}_j$ aus den gemessenen Daten $\mathbf{d}_j$, so dass

   $$\mathbf{M}_\Gamma\, \mathcal{T}\mathbf{q}_j = \mathbf{d}_j$$

   gilt.
2. Das inverse Mediumproblem, das zwar stabiler als das ursprüngliche Problem ist, allerdings nichtlinear bleibt. Hier wird die Rekonstruktion der Kontrastfunktion f aus den erhaltenen äquivalenten Quellen $\mathbf{q}_j$ erzielt .

Wir führen das neue Konzept der verallgemeinerten induzierten Quellen (auf English: *general induced sources*) $\mathbf{w} = (w^i)_i$ ein, so dass

$$A\, w^i = (\mathcal{T}\, \mathbf{q})_i$$

in kartesischen Koordinaten gilt. Der Operator A bezeichnet den skalaren Streuoperator in 3-D. Das inverse Quellenproblem lässt sich dann in die skalare Probleme

$$M_\Gamma \, A \, w^i = d^i$$

entkoppeln.
Bei sphärischer Messgeometrie analysierten Abdullah und Louis [AL99] den für 2-D akustischen Wellen eingeführten skalaren Streuoperator mit Hilfe der Singulärwertzerlegung.
Zur Anwendung der Methode der approximativen Inversen analysieren wir den Streuoperator $A_\Gamma = M_\Gamma A$ in 3-D sphärischen Anordnung, wobei wir die Singulärwertzerlegung herleiten und eine Basis für den Nullraum bestimmen.
Weiter wenden wir die vom Louis [Lou99] eingeführte Fehleranalyse der Methode der approximativen Inverse an, um Fehlerabschätzungen der regularisierten Lösungen des skalaren inversen Streuproblems mit sphärischen Messgeometrie zu erhalten.

Die lineare Inversion wird mit der effizienten und stabilen Methode der approximativen Inversen durchgeführt. Damit wird zuerst die verallgemeinerte induzierte Quelle $\mathbf{w}_j$ aus den Daten jedes Experimentes j bestimmt. Die äquivalenten Quelle $\mathbf{q}_j$ lässt sich dann aus $\mathbf{w}_j$ herleiten.

Mit Hilfe des neuen Konzeptes der verallgemeinerten induzierten Quellen adaptieren wir die nichtlineare Version des KACZMARZ-Verfahrens zur Entwicklung eines iterativen Verfahrens. Damit werden die Materialeigenschaften aus den berechneten äquivalenten Quellen rekonstruiert.

Einige numerische Simulationen belegen die analytische Ergebnisse für die entwickelte Methode und zeigen die praktische Brauchbarkeit.

Introduction

... When I considered what people generally want in calculating,
I found: it is always a number.
Alkhawarizmi (Algorithmus), a bagdadi mathematician died in 850.

Inverse scattering problems are not strange to everyday life. For example, when we want to guess the content of a closed box without opening it, we usually knock on it and from the sound coming back we might be able to make an idea about the content, or at least know whether the box is empty or not. In other words, by knocking we send an acoustic wave, which we call incident wave, and we expect to guess the content of the box by comparing the outcoming sound with the one that would exist if the box were empty. This simple example set an inverse scattering problem for acoustic waves.

In the acoustic or electromagnetic inverse scattering problems, the goal is to acquire knowledge about acoustic or electromagnetic properties of a buried object from measurements, taken outside the target region, of incident-wave deformations due to the contrast between the object and its background.

The applications of inverse scattering problems for electromagnetic waves arise in numerous areas. Typically, in medical imaging to detect anomalies like mycrocardial infraction, leukaemia or cancerous tumours [SBS$^+$00], in geophysics for the exploration of minerals [Zhd02], in industry to make nondestructive testung [LJ02], in military for target identification to mention only a few. The endeavour to broaden the range of applications to further areas, such as the recent investigation for humanitarian mine detection, has been also motivating multi-disciplinary research efforts to adapt old techniques and develop new methods to make advances in the efficiency, robustness and stability of inversion algorithms.

The available inverse scattering algorithms are mainly divided into two classes:

- Linear methods. They are based on approximations like the Born/Rytov approximation, which are valid for media with low contrasts, see for example [Lan87] and the references therein. About linear methods, we further refer to [MMH⁺02] for the synthetic aperture focusing techniques, [Zhd02] for the quasi-analytical method. For the inversion from far-field data see [CK98], [Kre01], [CHP03].

- Nonlinear methods. The material properties are here recovered iteratively from an intial guess. These methods generally make use of forward solvers for the direct Maxwell-model obtained from the discretization of the system of partial differential equations [DBABC99], [Voe03], [NB04], [BSSP04], [Hab04], or the integral equations [AvdB02].

We present in this work a new method to solve the inverse electromagnetic scattering problem in inhomogeneous media.
For a fixed angular frequency $w > 0$ and the constants $\varepsilon_0 > 0$, $\mu_0 > 0$, let $k := w\sqrt{\varepsilon_0\mu_0}$ be the wave number and $g_k(x,y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}$, $x \neq y$, $x, y \in I\!R^3$, be the Green's function in $I\!R^3$ for the Helmholtz operator $(\Delta + k^2)$.
From the time-harmonic Maxwell-equations in 3-D in an isotropic medium, we may derive the system of integral equations

$$\begin{aligned}\mathbf{E}(x) &= \mathbf{E}^{inc}(x) - k^2\int_{I\!R^3} g_k(x,y)\, f_e(y)\,\mathbf{E}(y)\,dy \\ &+ iw\mu_0\int_{I\!R^3}\nabla_y g_k(x,y)\times f_m(y)\,\mathbf{H}(y)\,dy \\ &+ \int_{I\!R^3}\nabla_y g_k(x,y)\,\nabla\cdot(f_e\mathbf{E})(y)\,dy, \qquad (3)\end{aligned}$$

$$\begin{aligned}\mathbf{H}(x) &= \mathbf{H}^{inc}(x) - k^2\int_{I\!R^3} g_k(x,y) f_m(y)\,\mathbf{H}(y)\,dy \\ &- iw\varepsilon_0\int_{I\!R^3}\nabla_y g_k(x,y)\times f_e(y)\,\mathbf{E}(y)\,dy \\ &+ \int_{I\!R^3}\nabla_y g_k(x,y)\,\nabla\cdot(f_m\,\mathbf{H})(y)\,dy, \qquad (4)\end{aligned}$$

where the constitutive properties of the medium are described by the electric and magnetic contrast functions

$$f_e := (1-\varepsilon_0^{-1}\varepsilon) \quad \text{and} \quad f_m := (1-\mu_0^{-1}\mu)$$

assumed to be sufficiently smooth with compact support embedded in a bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary Γ.
The complexe-valued three-dimensional vector fields $\mathbf{E}$ and $\mathbf{H}$ denote the time-harmonic electromagnetic total fields. The scattered fields are defined by $\mathbf{E}^s := \mathbf{E}-\mathbf{E}^{inc}$ and $\mathbf{H}^s := \mathbf{H}-\mathbf{H}^{inc}$. The scalar functions ε and μ are defined by $\varepsilon(x) := \tilde{\varepsilon}(x) + i\,\frac{\sigma_e(x)}{w}$ and $\mu(x) := \tilde{\mu}(x) + i\frac{\sigma_m(x)}{w}$, $x \in I\!R^3$, where $\tilde{\varepsilon} > 0$, $\tilde{\mu} > 0$, $\sigma_e \geq 0$, $\sigma_m \geq 0$, are the electric permittivity, the magnetic permeability, the electric conductivity and

the magnetic conductivity, respectively.
We write the system of integral equations (1)-(2) as an operator equation

$$\mathcal{T}(f\mathbf{u}) = \mathbf{u}^s$$

with $\mathbf{u} = (\mathbf{E}, \mathbf{H}), \mathbf{u}^s = (\mathbf{E}^s, \mathbf{H}^s)$ and $f = (f_e, f_m)$.
If $\mathbf{M}_\Gamma$ denotes the measurement operator on the boundary, the inverse problem reads:
"To the measured data $\mathbf{d}_j = \mathbf{M}_\Gamma \mathbf{u}_j^s$, determine the contrast function f such that

$$\mathbf{M}_\Gamma \, \mathcal{T}(f\mathbf{u}_j) = \mathbf{d}_j$$

for all experiments j."
In this inverse problem, we have to tackle with three major aspects of difficulty:

1. Ill-posedness, due to non-uniqueness and ill-conditioning which are inherent of almost all inverse scattering problems.
2. Nonlinearity. Although the operator $\mathcal{T}$ is linear, the dependence of the field $\mathbf{u}_j$ on the contrast function f may be highly nonlinear.
3. The coupling of the cartesian components of the electromagnetic fields.

Using the equivalent or induced sources $\mathbf{q}_j = f\mathbf{u}_j$, inverse scattering is splitted into two subproblems:

1. The inverse source problem, which is linear and ill-posed. It is concerned with determining the equivalent source $\mathbf{q}_j$ from the data $\mathbf{d}_j$ subject to

   $$\mathbf{M}_\Gamma \, \mathcal{T}\mathbf{q}_j = \mathbf{d}_j$$

   for each experiment j.
2. The inverse medium problem, which is nonlinear. It consists of recovering the contrast function f from the equivalent sources $\mathbf{q}_j$ obtained from multiple experiments.

We introduce the concept of generalized induced source $\mathbf{w} = (w^i)_i$ satisfying

$$A \, w^i = (\mathcal{T} \, \mathbf{q})_i$$

in cartesian cordinates, where A is the scalar scattering operator in 3-D. This reduces the vector inverse source problem into decoupled scalar problems

$$M_\Gamma \, A \, w_j^i = d_j^i.$$

For a spherical configuration, Abdullah and Louis [AL99] analysed the scattering operator for 2-D acoustic waves using the singular value decomposition. We analyse in this work the spherical scattering operator $A_\Gamma = M_\Gamma A$ in 3-D, where we derive the singular value decomposition and determine a basis for the null space.

We utilize the method of the approximate inverse to recover, for each experiment j, the generalized induced source $\mathbf{w}_j$, from which we can determine the equivalent source $\mathbf{q}_j$. The method of the approximate inverse is an efficient and stable regularization method for linear ill-posed problems. To briefly describe this method, let $T : X \longrightarrow Y$ denote a linear non-degenarate compact operator between the Hilbert spaces $X \subset L_2(\Omega)$ and $Y \subset L_2(\Gamma)$, on some measurable sets Ω and Γ, endowed with scalar products denoted $< \cdot, \cdot >$.
To solve the equation

$$T f = g,$$

we consider for $e_\gamma^x \in X,\, x \in \Omega,$ the adjoint problems

$$T^* \psi_\gamma^x = e_\gamma^x,\, x \in X,$$

to determine the function $\psi_\gamma(x, y) = \psi_\gamma^x(y),\, x \in \Omega, y \in \Gamma,$ called reconstruction kernel. If e_γ^x is a mollifier, which satisfies $< f, e_\gamma^x > \to f(x)$ as $\gamma \to 0$ for $f \in X, x \in \Omega,$ then by computing $< g, \psi_\gamma^x >$ we can obtain an approximation f_γ to the solution f at the reconstruction point $x \in \Omega,$ since it holds

$$f_\gamma(x) := < f, e_\gamma^x > = < f, T^* \psi_\gamma^x > = < T f, \psi_\gamma^x > = < g, \psi_\gamma^x > .$$

The usage of invariance properties of the operator T reduces the resolution of the adjoint problems to few reconstruction points.
Further, we use results due to Louis [Lou99] to derive an error estimate for the regularized solution obtained by the method of the approximate inverse applied to the spherical scattering operator .

The nonlinear version of the algorithm of Kaczmarz had been applied to inverse scattering problems by Natterer and Wübbeling [NW95] for ultrasound tomography and by Dorn *et al* [DMR00] and Vögeler [Voe03], for electromagnetic problems. Using the generalized induced source, we adapt this algorithm to derive an iterative scheme for the resolution of our nonlinear problem. The contrast function f is then determined from the recovered equivalent sources $\mathbf{q}_j$.
We finally make few numerical simulations for objects with simple geometry to check the validity of the method.

The developed method presents many advantages:

- The efficiency and stability of the linear inversion due to the utilization of the approximate inverse for regularization.
- An estimate of location, shape, size of the sought-for object and an approximation of the contrast function, can be rapidly obtained.
- The iterative scheme deals with high contrasts, beyond the Born/Rytov approximation.
- As the method is derived in a quite general setting, it can be used for a wide range of applications.

In the first chapter of this work, we outline the modeling of the direct scattering. We first recall the physical background in the framework of the fundamental equations of Maxwell to model the electromagnetic wave propagation in inhomogeneous media. For a fixed frequency, the electromagnetic scattering in time-harmonic regime is splitted into the direct source problem treated in section 2 and the direct medium problem considered in section 3. The first problem deals with the radiation in some homogeneous domain from a given electromagnetic source. The next one is concerned with scattering of monochromatic incident wave by an inhomogeneous medium with prescribed electromagnetic properties. The integration of the Maxwell-equations leads to the contrast source integral equations. This formulation provides the basic model for the inverse problem. We close this chapter with stating the well-posedness of the direct scattering problem in inhomogeneous media for acoustic and electromagnetic waves in the unified setting of equations of Lippmann-Schwinger type.

Inverse scattering is considered in chapter 2. To comprehend inverse scattering problems in acoustics and electromagnetics, we start with an abstract formulation. We next discuss the practical relevance of the electromagnetic inverse scattering and give a brief overview of the main available methods for the inversion of near-field electromagnetic data. In the next section, we recall the uniqueness result due to Ola, Päivärinta and Somersalo. The procedure to make the inversion of the electromagnetic scattering operators using the new concept of generalized induced source is presented in section 4. For a spherical geometrical configuration, we treat in the last section the scalar scattering operator, where we derive the singular value decomposition and a basis for the null space, further we study its smoothing properties in Sobolev spaces.

The third chapter treats the application of the method of the approximate inverse to solve the inverse source and the inverse medium scattering problems. We first recall fundamental results about regularization with the approximate inverse, which we apply to the spherical scattering operator. We derive the corresponding reconstruction kernel using the singular value decomposition. We describe the resolution of the inverse source problem for nonmagnetic media, then for objects with both electric and magnetic properties. The nonlinear scheme to solve the inverse medium problem is discussed in the last section.

The last chapter is devoted to the presentation and discussion of the numerical results.

In the conclusion, we summurize the features characterizing this method.
Finally, we give an outlook about some interesting issues for further investigations.

It is a pleasure to express my cordially thanks to Univ.-Prof. Dr. Alfred K. Louis for the chance to belong to his team and for his steady support. He has been providing congential environement and facilities for doing research and supervised the present work with expertise, encouragement and patience.

I would like also to thank my present and former colleagues in the team-Louis for their cooperativeness and friendship.
I am particularly thankful to Dr. Uwe Schmitt for his helpfullness and willingness to review the first drafts of this work.
I am also gratefully indebted to Dr. Peter Jonas who kindly helped me to make a good professional start in the group.

Many thanks are due to Dr. Werner Karpen for his enlightments about some physical issues as we were collaborating in the BMBF-project ”Humanitarian Mine Detection.”

I gratefully acknowledge the academical, financial or personal help I recieved from many people in Germany, particularly from many lecturers and friends from Kaiserslautern and Saarbrücken, as well as from the DAAD-organization.

Last but not least, I would like to express my deepest gratitude to my parents Amor and Menana, my brothers Anen and Mohamed Majdi, my uncle Abderrahmen and finally to my spiritual father, for their love, support and assistance.

Saarbrücken, im April 2006 Aref Lakhal

Contents

Chapter 1

Direct electromagnetic scattering

A well-founded knowledge of the theory of the direct problem is always required for investigating the inversion. We devote, therefore, this first chapter to outline the forward scattering models, which are relevant for this work. We start with the fundamental equations of Maxwell to model the electromagnetic wave propagation in inhomogeneous media and give a brief discussion of the physical background. In time-harmonic regime, we consider two problems related to electromagnetic scattering. The first one is the direct source problem, which is treated in section 2. It is concerned with the radiation in some homogeneous domain from a given electromagnetic source. The next one is the direct medium problem considered in section 3. It deals with scattering by an inhomogeneous medium with prescribed electromagnetic properties, which is illuminated by a single-frequency electromagnetic wave. The integration of the Maxwell-equations leads to the contrast source integral equations. This formulation serves as the basic model for the inverse problem. In the closing section, we state, in the unified framework of Lippmann-Schwinger-type equations, the well-posedness of the direct scattering problem in inhomogeneous media for acoustic and electromagnetic waves.

The theory on the direct electromagnetic scattering is well-established and the literature on the subject is quite abundant. For the content of this chapter we mainly refer to [Mue69], [CK98] and [DL90b].

1.1 Maxwell equations

The propagation of electromagnetic waves is governed [1] by the *Maxwell equations*

$$\nabla \times \mathcal{H} = \mathcal{J}_c + \mathcal{J}_e + \frac{\partial \mathcal{D}}{\partial t}, \tag{1.1}$$

$$\nabla \times \mathcal{E} = \mathcal{M} - \frac{\partial \mathcal{B}}{\partial t}. \tag{1.2}$$

The complex-valued three dimensional vector fields $\mathcal{E} = \mathcal{E}(x,t) \in \mathbb{C}^3$ and $\mathcal{H} = \mathcal{H}(x,t) \in \mathbb{C}^3$ denote the electric and magnetic fields in a space position $x \in \mathbb{R}^3$, at a time $t \in \mathbb{R}$.
The *electric displacement* $\mathcal{D}$ and the *magnetic flux density* $\mathcal{B}$ are related to the electromagnetic properties of the underlying medium by supplementary relations called *constitutive equations*. For a *linear isotropic* medium we have

$$\mathcal{D} = \varepsilon\mathcal{E} \quad \text{and} \quad \mathcal{B} = \mu\mathcal{H},$$

where ε is the *electric permittivity* and μ is the *magnetic permeability*.
In the more general case when the medium is anisotropic, ε and μ are symmetric two-dimensional tensors. Each tensor is identified to a vector formed with the corresponding eigenvalues. The medium is isotropic when all the eigenvalues of each tensor are equal, see [Kon90] and [BW99] for more details.
We assume that the medium is isotropic and the physical properties do not change in the course of time, $i.e.$ we have only the spatial dependence of the real-valued functions $\varepsilon = \varepsilon(x)$ and $\mu = \mu(x)$. These functions take the constant values $\varepsilon = \varepsilon_0$ and $\mu = \mu_0$ in *free space* void of any matter.
The *conduction current density* $\mathcal{J}_c$ is expressed for the underlying medium in terms of the electric field by means of *Ohm's law*

$$\mathcal{J}_c = \sigma_e\mathcal{E},$$

where $\sigma_e = \sigma_e(x)$ is the *electric conductivity* of the medium ranging between $\sigma_e = 0$ for a nonconducting medium and $\sigma_e = \infty$ for a *perfect conductor*.
The generation of the electromagnetic waves is due to the *extraneous electric current source density* $\mathcal{J}_e$ and to the *magnetic current source density* $\mathcal{M}$. To take advantage of the symmetry of Maxwell equations we may use the real-valued function $\sigma_m = \sigma_m(x)$ as the *magnetic conductivity* and a complex-valued vector field $\mathcal{J}_m = \mathcal{J}_m(x,t)$ as the *extraneous magnetic current density* such that

$$\mathcal{M} = -\mathcal{J}_m + \sigma_m\mathcal{H}.$$

For a fixed angular frequency $w > 0$, we assume the current sources to be *time-harmonic*

$$\mathcal{J}_e(x,t) = \mathbf{J}_e(x)e^{-iwt} \quad \text{and} \quad \mathcal{J}_m(x,t) = \mathbf{J}_m(x)e^{-iwt},$$

[1] To denote differential operators in a classical sense, we use the nabla operator ∇ $i.e.$ ∇, $\nabla\cdot$ and $\nabla\times$ stand for gradient, divergence and curl, respectively.

as well as the electromagnetic fields

$$\mathcal{E}(x,t) = \mathbf{E}(x)e^{-iwt} \quad \text{and} \quad \mathcal{H}(x,t) = \mathbf{H}(x)e^{-iwt}.$$

A field varying arbitrarily in time may be, using Fourier analysis, represented as a sum of harmonic fields. We therefore lose no generality by considering only harmonic time-dependence.
The Maxwell equations are reduced in the time-harmonic case to

$$\nabla \times \mathbf{H} + iw\tilde{\varepsilon}\,\mathbf{E} = \mathbf{J}_e\ , \tag{1.3}$$
$$\nabla \times \mathbf{E} - iw\tilde{\mu}\,\mathbf{H} = -\mathbf{J}_m, \tag{1.4}$$

where the *electric index* $\tilde{\varepsilon}$ and the *magnetic index* $\tilde{\mu}$ are defined by

$$\tilde{\varepsilon}(x) := \varepsilon(x) + i\frac{\sigma_e(x)}{w} \quad \text{and} \quad \tilde{\mu}(x) := \mu(x) + i\frac{\sigma_m(x)}{w}, \quad x \in \mathbb{R}^3,$$

with

$$\varepsilon > 0, \quad \mu > 0, \quad \sigma_e \geq 0, \quad \sigma_m \geq 0.$$

This nomenclature is motivated by the connection with a parameter used in acoustics, which is *the refractive index* [2] defined by

$$n(x) = \varepsilon_0^{-1}\tilde{\varepsilon}(x).$$

Alternatively to the angular frequency w we may also use *the wave number* k defined by

$$k := w\sqrt{\varepsilon_0\mu_0}.$$

In most practical situations, we may suppose that the extraneous current sources are only generated in a bounded region and the medium is only inhomogeneous in a bounded volume.
Let Ω and Ω' be two bounded domains in $\mathbb{R}^3$ with smooth boundaries $\partial\Omega$ and $\partial\Omega'$, respectively. We assume that the closure sets $\overline{\Omega} := \partial\Omega \cup \Omega$ and $\overline{\Omega'} := \partial\Omega' \cup \Omega'$ do not intersect $i.e.$ $\overline{\Omega} \cap \overline{\Omega'} = \emptyset$.
Furthermore, we suppose

1. the complex-valued functions $\tilde{\varepsilon}$ and $\tilde{\mu}$ are continuously differentiable on Ω and constant outside with $\varepsilon(x) = \varepsilon_0$ and $\mu(x) = \mu_0$ for $x \notin \Omega$.

2. the complex-valued vector fields $\mathbf{J}_e$ and $\mathbf{J}_m$ are continuous on Ω', vanish outside $i.e.$ $\mathbf{J}_e(x) = \mathbf{J}_m(x) = 0$ for $x \notin \Omega'$, and admit divergence fields $\nabla \cdot \mathbf{J}_e$ and $\nabla \cdot \mathbf{J}_m$, which are continuous on Ω'.

We call *direct (electromagnetic) scattering problem* the determination of electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ solving the time-harmonic Maxwell equations in $\mathbb{R}^3$, for given $\tilde{\varepsilon}$, $\tilde{\mu}$, $\mathbf{J}_e$, $\mathbf{J}_m$.

[2] In [BW99], the refractive index is alternatively defined as $n = (\varepsilon_0^{-1}(\varepsilon + i\frac{4\pi\sigma}{\omega}))^{\frac{1}{2}}$.

From physical point of view, the electromagnetic waves generated from the *source region* Ω' are impinging *the scatterer* occupying the domain Ω. Due to inhomogeneity of the electromagnetic properties in the medium, the *incident field*, which is the field that would exist if no scatterer present, induces secondary waves referred to as *the scattered* electromagnetic field.
The direct scattering problem may be spitted, due to the linearity of the Maxwell equations, into two subproblems

- *The direct source problem* consisting of the determination, in some homogeneous domain, of the radiating fields from given electric and magnetic sources.
- *The direct medium problem* dealing with the determination of the scattered electromagnetic field for a given incident field impinging a scatterer with prescribed electromagnetic properties.

Notations. In this work we use the following notations. For an open set $\Omega \subset \mathbb{R}^d, d \in \mathbb{N}$, $D \in \{\mathbb{R}, \mathbb{C}\}$, we denote $C(\Omega, D)$ or $C(\Omega)$ the space of continuous function on Ω, $C^p(\Omega, D)$ or simply $C^p(\Omega)$, $p \in \mathbb{N} \cup \{0, \infty\}$, the space of p times continuously differentiable functions on Ω, $C^{\alpha,p}(\Omega), \alpha \in (0,1)$, $p \in \mathbb{N} \cup \{0, \infty\}$, the Hölder spaces and $C_0^p(\Omega)$ the space of p continuously differentiable function compactly supported[3] in Ω.
If Ω is a measurable set, then $L_2(\Omega)$ denote the set of square integrable functions endowed with the usual scalar product denoted by $< \cdot, \cdot >$.
If X is a function space, then X^n, $n \in \mathbb{N}$, $n \geq 2$, denotes the space of vector-valued functions with each of the n components in X.
For $x, y \in \mathbb{R}^3$, the scalar product is denoted by $x \cdot y$ or $< x, y >$ and the vector product by $x \times y$ or $x \wedge y$.
If A is a linear operator, then $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote its null space and its range, respectively.

[3] $i.e.$ the support $supp(f) := \overline{\{x, f(x) \neq 0\}}$ of a function f is contained in a compact set embedded in Ω.

1.2 The direct source problem

In this section we are concerned with the radiation in a homogeneous medium from given electromagnetic sources. After a brief discussion of the connection between the Maxwell and the Helmholtz equations and defining the radiation conditions for them, we recall the Stratton-Chu representation formulae for the Maxwell equation in a homogeneous medium. These are the basic tool to derive the solution of the direct source problem. The electromagnetic field is then given by an integral representation, which can be considered as the mathematical formulation of the Hyugens principle, see for instance [Kon90] p. 382.

Definition 1.2.1. *Let $\Omega \subset \mathbb{R}^3$ be an open set and F be a complex-valued vector field on $\mathbb{R}^3$.*
We say that F is a regular current field on Ω if F is continuous on Ω and admits a divergence field $\nabla \cdot F$, which is continuous on Ω.
We say that F is a regular electromagnetic field on Ω if F is continuous on Ω and admits a curl field $\nabla \times F$, which is continuous on Ω.

Let Ω be a bounded domain in $\mathbb{R}^3$ with a regular boundary $\partial\Omega$. For w, ε_0, μ_0, positive constants, we consider the direct source problem of determining the regular electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$, on $\mathbb{R}^3 \backslash \partial\Omega$, solving

$$\nabla \times \mathbf{H} + iw\,\varepsilon_0\,\mathbf{E} = \mathbf{J}_e\,, \tag{1.5}$$
$$\nabla \times \mathbf{E} - iw\,\mu_0\,\mathbf{H} = -\mathbf{J}_m, \tag{1.6}$$

where $\mathbf{J}_e$ and $\mathbf{J}_m$ are given regular current fields on Ω with support in $\overline{\Omega}$ *i.e.*

$$\mathbf{J}_e(x) = \mathbf{J}_m(x) = 0 \text{ for } x \notin \overline{\Omega}.$$

Outside Ω the equations (1.5) and (1.6) are homogeneous. If $\mathbf{E}$, $\mathbf{H} \in C^1(\mathbb{R}^3 \backslash \overline{\Omega})$ are solutions to these equations, then they admit on $\mathbb{R}^3 \backslash \overline{\Omega}$ vanishing divergence fields

$$\nabla \cdot \mathbf{E} = 0, \tag{1.7}$$
$$\nabla \cdot \mathbf{H} = 0, \tag{1.8}$$

since $\nabla \cdot \nabla \times \mathbf{E} = 0$ and $\nabla \cdot \nabla \times \mathbf{H} = 0.$
If we further suppose that $\mathbf{E}, \mathbf{H} \in C^2(\mathbb{R}^3 \backslash \overline{\Omega}),$ we can use the identity

$$\nabla \times \nabla \times \mathbf{E} = -\Delta\,\mathbf{E} + \nabla(\nabla \cdot \mathbf{E}), \tag{1.9}$$

to decouple the equations (1.5) and (1.6), then obtain on $\mathbb{R}^3 \backslash \overline{\Omega}$ the pair of *vector Helmholtz equations*

$$(\boldsymbol{\Delta} + k^2)\;\mathbf{E} = 0, \tag{1.10}$$
$$(\boldsymbol{\Delta} + k^2)\;\mathbf{H} = 0, \tag{1.11}$$

where $k := w\sqrt{\varepsilon_0 \mu_0}$.
For divergence-free fields, the Maxwell equations are reduced to the Helmholtz equation. Each cartesian component u of the electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ must be

solution to the *(scalar) Helmholtz equation* $(\Delta + k^2)u = 0$ in $\mathbb{R}^3 \backslash \overline{\Omega}$.
Physically relevant solutions to the Helmholtz equation are obtained by considering the so called *Sommerfeld radiation condition*, which specifies the appropriate geometric attenuation of a solution to the Helmholtz equation and imposes its outgoing character.

Definition 1.2.2. *Let $k > 0$ and Ω be a bounded domain in $\mathbb{R}^3$. A solution $u \in C^2(\mathbb{R}^3 \backslash \overline{\Omega}, \mathbb{C})$ to the scalar Helmholtz equation*

$$(\Delta + k^2)u = 0 \quad \textit{in} \quad \mathbb{R}^3 \backslash \overline{\Omega}, \tag{1.12}$$

satisfies the Sommerfeld radiation condition if

$$\lim_{r \to \infty} r \left(\frac{\partial u}{\partial r} - iku \right) = 0, \tag{1.13}$$

uniformly in all directions $\frac{x}{|x|}$, where $r = |x|$, $x \in \mathbb{R}^3 \backslash \{0\}$.

The fundamental solution $g_k \in C^\infty(\mathbb{R}^3 \backslash \{0\})$ to the scalar Helmholtz equation in $\mathbb{R}^3$, which satisfies the Sommerfeld radiation condition, is given by

$$g_k(x) = \frac{1}{4\pi} \frac{e^{ik|x|}}{|x|} \quad \text{for } x \in \mathbb{R}^3 \backslash \{0\}.$$

Indeed, a straightforward computation yields

$$(\Delta + k^2) g_k = 0 \quad \text{in} \quad \mathbb{R}^3 \backslash \{0\}, \tag{1.14}$$

and

$$\frac{\partial g_k}{\partial r}(x) - ikg_k(x) = -\frac{1}{4\pi} \frac{e^{ikr}}{r^2} \quad \text{with} \quad r := |x|, x \in \mathbb{R}^3 \backslash \{0\}. \tag{1.15}$$

We may, in the sequel, abuse the notation and denote the Green's function in $\mathbb{R}^3$ for the Helmholtz Operator $(\Delta + k^2)$, $k > 0$, also

$$g_k(x, y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|} \quad \text{for} \quad x \neq y, x, y \in \mathbb{R}^3.$$

Lemma 1.2.3. *For $\varphi \in C_0(\mathbb{R}^3)$ the volume potential given by*

$$u(x) = \int_{\mathbb{R}^3} g_k(x, y) \varphi(y) dy \tag{1.16}$$

is well defined for $x \in \mathbb{R}^3$.
If $\varphi \in C_0(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$, then $u \in C^2(\mathbb{R}^3)$ and

$$(\Delta + k^2)u = -\varphi. \tag{1.17}$$

Furthermore u satisfies the Sommerfeld radiation condition.

For the proof of this lemma as well as further results concerned with the Helmholtz equation we refer to [CK98] and [DL90a].
Analogously to the Helmholtz equation, we need to supplement the Maxwell equations also by some kind of boundary conditions at infinity called *the Silver-M üller radiation conditions.*

Definition 1.2.4. *Let the regular electromagnetic fields* $\mathbf{E}, \mathbf{H},$ *be solutions to the homogeneous Maxwell equations*

$$\begin{aligned} \nabla \times \mathbf{H} + iw\ \varepsilon_0\, \mathbf{E} &= 0, \\ \nabla \times \mathbf{E} - iw\ \mu_0\, \mathbf{H} &= 0, \end{aligned}$$

in $\mathbb{R}^3 \backslash \overline{\Omega},$ *where* Ω *is a bounded domain in* $\mathbb{R}^3$.
The solutions $\mathbf{E}, \mathbf{H},$ *are called radiating if they satisfy one of the Silver-M ülller radiation conditions*

$$\lim_{|x| \to \infty} (\mathbf{H} \times x - |x| \mathbf{E}) = 0 \tag{1.18}$$

or

$$\lim_{|x| \to \infty} (\mathbf{E} \times x + |x| \mathbf{H}) = 0, \tag{1.19}$$

uniformly in all directions $\frac{x}{|x|},\ x \in \mathbb{R}^3 \backslash \{0\}$.

The Silver-Müller and the Sommerfeld conditions are closely related to each other. As one would expect from equations (1.10) and (1.11), the Silver-Müller radiation condition for smooth solutions to the homogeneous Maxwell equations is equivalent to the Sommerfeld radiation condition for each of the cartesian components.
To construct a solution of the direct source problem, which satisfies the Silver-Müller radiation condition and investigate its uniqueness, the representation formulae due to Stratton-Chu play a key role. These formulae are for the Maxwell equations similar to the Green's integral formula (D.1) for the scalar Helmholtz equation.

Lemma 1.2.5 (Stratton-Chu)**.** *Let* $w,\ \varepsilon_0,\ \mu_0$ *be positive constants,*

$$g_k(x,y) := \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|},\ x \neq y, x \in \mathbb{R}^3,$$

with $k := w\sqrt{\varepsilon_0 \mu_0}$.
Let Ω *be a regular bounded domain in* $\mathbb{R}^3$ *and* $\mathbf{n}$ *denote the normal unit vector to the boundary* $\partial\Omega$ *directed into the exterior of* Ω.

For the regular current fields $\mathbf{J}_e,\ \mathbf{J}_m$ *on* $\Omega,$ *we suppose that* $\mathbf{E}$ *and* $\mathbf{H}$ *are regular electromagnetic fields, which solve in* Ω *the Maxwell equations*

$$\begin{aligned} \nabla \times \mathbf{H} + iw\ \varepsilon_0\, \mathbf{E} &= \mathbf{J}_e, \\ \nabla \times \mathbf{E} - iw\ \mu_0\, \mathbf{H} &= -\mathbf{J}_m. \end{aligned}$$

Then for $x \in \Omega$ *we have*

$$\begin{aligned}
\mathbf{E}(x) &= iw\mu_0 \int_\Omega g_k(x,y)\,\mathbf{J}_e(y)\,dy \\
&+ \int_\Omega \nabla_y g_k(x,y) \times \mathbf{J}_m(y)\,dy \\
&- i(w\varepsilon_0)^{-1} \int_\Omega \nabla_y g_k(x,y) \nabla \cdot \mathbf{J}_e(y)\,dy \\
&- iw\mu_0 \int_{\partial\Omega} g_k(x,y)\,(\mathbf{n}(y) \times \mathbf{H}(y))\,d\sigma(y) \\
&+ \int_{\partial\Omega} \nabla_y g_k(x,y) \times (\mathbf{n}(y) \times \mathbf{E}(y))\,d\sigma(y) \\
&- \int_{\partial\Omega} \nabla_y g_k(x,y)(\mathbf{n}(y) \cdot \mathbf{E}(y))\,d\sigma(y),
\end{aligned} \tag{1.20}$$

$$\begin{aligned}
\mathbf{H}(x) &= iw\varepsilon_0 \int_\Omega g_k(x,y)\,\mathbf{J}_m(y)\,dy \\
&- \int_\Omega \nabla_y g_k(x,y) \times \mathbf{J}_e(y)\,dy \\
&- i(w\mu_0)^{-1} \int_\Omega \nabla_y g_k(x,y) \nabla \cdot \mathbf{J}_m(y)\,dy \\
&+ iw\varepsilon_0 \int_{\partial\Omega} g_k(x,y)(\mathbf{n}(y) \times \mathbf{E}(y))\,d\sigma(y) \\
&+ \int_{\partial\Omega} \nabla_y g_k(x,y) \times (\mathbf{n}(y) \times \mathbf{H}(y))\,d\sigma(y) \\
&- \int_{\partial\Omega} \nabla_y g_k(x,y)(\mathbf{n}(y) \cdot \mathbf{H}(y))\,d\sigma(y).
\end{aligned} \tag{1.21}$$

For $x \in \mathbb{R}^3 \backslash \overline{\Omega}$ *the right hand side in (1.20) and (1.21) vanish identically.*

We see from the Stratton-Chu representation formulae that the electromagnetic fields, which are solutions to the Maxwell equations inside a regular bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$, are completely determined from the volume densities of electric and magnetic current sources $\mathbf{J}_e$ and $\mathbf{J}_m$ in Ω and the value of the electromagnetic fields on the boundary $\partial\Omega$. This suggests to consider two subproblems called interior and exterior Maxwell problems, where the equations (1.5) and (1.6) are to be solved in Ω and in $\mathbb{R}^3 \backslash \overline{\Omega}$, respectively.
The uniqueness of the solution to the Maxwell equations for prescribed boundary conditions follows immediately from the Stratton-Chu formulae.

Theorem 1.2.6. *Let* w, ε_0 *and* μ_0 *be positive constants,* Ω *be a regular bounded domain in* $\mathbb{R}^3$ *and* $\mathbf{n}$ *denote the normal unit vector to the boundary* $\partial\Omega$ *directed into the exterior of* Ω.

We consider the homogeneous Maxwell equations

$$\nabla \times \mathbf{H}(x) + iw\ \varepsilon_0\, \mathbf{E}(x) \;=\; 0, \tag{1.22}$$
$$\nabla \times \mathbf{E}(x) - iw\ \mu_0\, \mathbf{H}(x) \;=\; 0. \tag{1.23}$$

If $\mathbf{E},\, \mathbf{H} \in C(\overline{\Omega}) \cap C^1(\Omega)$ *are solutions to the homogeneous Maxwell equations in* Ω *such that*

$$(\mathbf{n} \times \mathbf{E})(x) = (\mathbf{n} \times \mathbf{H})(x) = 0 \quad \textit{for} \quad x \in \partial\Omega, \tag{1.24}$$

then

$$\mathbf{E} = \mathbf{H} = 0 \quad in \quad \overline{\Omega}.$$

If $\mathbf{E},\, \mathbf{H} \in C(\mathbb{R}^3 \backslash \Omega) \cap C^1(\mathbb{R}^3 \backslash \overline{\Omega})$ *are radiating solutions to the homogeneous Maxwell equations in* $\mathbb{R}^3 \backslash \overline{\Omega}$ *such that*

$$(\mathbf{n} \times \mathbf{E})(x) = 0 \quad \textit{for} \quad x \in \partial\Omega, \tag{1.25}$$

then

$$\mathbf{E} = \mathbf{H} = 0 \quad in \quad \mathbb{R}^3 \backslash \Omega.$$

The proof of this theorem can be found in [Mue69]. It is to be noticed that for the exterior problem, an additional condition is assumed there, namely

$$\mathbf{E}(x) = O\left(\frac{1}{|x|}\right), |x| \to \infty,$$

to hold uniformly in all directions $\frac{x}{|x|}, x \in \mathbb{R}^3 \backslash \{0\}$.
However, this condition is systematically fulfilled by radiating solutions of the Maxwell equations, see [CK98].
In the Stratton-Chu integral representations the electromagnetic fields $\mathbf{E}$ and $\mathbf{H}$ are expressed in terms of their values on the boundary on the one hand, and in terms of the electric and magnetic source densities $\mathbf{J}_e,\, \mathbf{J}_m$ and their divergence fields $\nabla \cdot \mathbf{J}_e,\, \nabla \cdot \mathbf{J}_m$ on the other hand.
If $\rho_e(x,t) = \varrho_e(x)e^{-iwt}$ and $\rho_m(x,t) = \varrho_m(x)e^{-iwt}$ are the volume densities of electric and "magnetic" charges [4] in a position $x \in \mathbb{R}^3$ at a time $t \in \mathbb{R}$, then from the charge-conservation law we have the equations

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathcal{J}_e = 0 \quad \text{and} \quad \frac{\partial \rho_m}{\partial t} + \nabla \cdot \mathcal{J}_m = 0,$$

which imply

$$\varrho_e(x) = -iw^{-1} \nabla \cdot \mathbf{J}_e \quad \text{and} \quad \varrho_m(x) = -iw^{-1} \nabla \cdot \mathbf{J}_m.$$

Consequently, the volume integrals in the identities (1.20) and (1.21) relate the electromagnetic fields to their electric and magnetic sources given as volume densities of currents and charges.
The question now is what the boundary integrals are due to.

[4] Magnetic charges are only virtual charges without actual physical existence.

Theorem 1.2.7. *Let* w, ε_0, μ_0 *be positive constants and*

$$g_k(x,y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|},\ x \neq y, x, y \in \mathbb{R}^3,$$

with $k := w\sqrt{\varepsilon_0\mu_0}$.
Let Ω *be a regular bounded domain in* $\mathbb{R}^3$ *and* $\mathbf{n}$ *denote the normal unit vector to the boundary* $\partial\Omega$ *directed into the exterior of* Ω.
If the complex-valued fields $\mathbf{J}_e$ *and* $\mathbf{J}_m$ *are regular current fields on* Ω, *vanishing outside* $i.e.$ $\mathbf{J}_e(x) = \mathbf{J}_m(x) = 0$ *for* $x \notin \overline{\Omega}$.
Then the fields $\mathbf{E}$ *and* $\mathbf{H}$ *defined by*

$$\begin{aligned}
\mathbf{E}(x) &= iw\mu_0 \int_\Omega g_k(x,y)\mathbf{J}_e(y)\,dy \\
&+ \int_\Omega \nabla_y g_k(x,y) \times \mathbf{J}_m(y)\,dy \\
&- i(w\varepsilon_0)^{-1} \int_\Omega \nabla_y g_k(x,y)\,(\nabla_y \cdot \mathbf{J}_e(y))\,dy \\
&+ i(w\varepsilon_0)^{-1} \int_{\partial\Omega} \nabla_y g_k(x,y)(\mathbf{n}(y)\cdot\mathbf{J}_e(y))\,d\sigma(y) \qquad (1.26)
\end{aligned}$$

$$\begin{aligned}
\mathbf{H}(x) &= iw\varepsilon_0 \int_\Omega g_k(x,y)\mathbf{J}_m(y)\,dy \\
&- \int_\Omega \nabla_y g_k(x,y) \times \mathbf{J}_e(y)\,dy \\
&- i(w\mu_0)^{-1} \int_\Omega \nabla_y g_k(x,y)\,(\nabla_y \cdot \mathbf{J}_m(y))\,dy \\
&+ i(w\mu_0)^{-1} \int_{\partial\Omega} \nabla_y g_k(x,y)(\mathbf{n}(y)\cdot\mathbf{J}_m(y))\,d\sigma(y), \qquad (1.27)
\end{aligned}$$

are regular electromagnetic fields on Ω *and* $\mathbb{R}^3\backslash\overline{\Omega}$, *respectively, which satisfy*

$$\begin{aligned}
\nabla \times \mathbf{H}(x) + iw\ \varepsilon_0\,\mathbf{E}(x) &= \mathbf{J}_e(x)\ , \\
\nabla \times \mathbf{E}(x) - iw\ \mu_0\,\mathbf{H}(x) &= -\mathbf{J}_m(x),
\end{aligned}$$

for $x \in \Omega$ *and*

$$\begin{aligned}
\nabla \times \mathbf{H}(x) + iw\ \varepsilon_0\,\mathbf{E}(x) &= 0, \\
\nabla \times \mathbf{E}(x) - iw\ \mu_0\,\mathbf{H}(x) &= 0.
\end{aligned}$$

for $x \in \mathbb{R}^3\backslash\overline{\Omega}$.
Moreover the fields $\mathbf{E}$ *and* $\mathbf{H}$ *fulfill the Silver-Müller conditions.*

We see from this theorem that discontinuities of the current fields $\mathbf{J}_e$ and $\mathbf{J}_m$, namely of their normal components,which occur through the boundary, are also sources of electromagnetic fields. These sources, called *surface currents*, are responsible for the discontinuity of the fields $\mathbf{E}$ and $\mathbf{H}$ on the boundary.

Corollary 1.2.8. *Let* w, ε_0, μ_0 *be positive constants and*

$$g_k(x,y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|},\; x \neq y, x, y \in \mathbb{R}^3,$$

with $k := w\sqrt{\varepsilon_0\mu_0}$.
Let Ω *be a regular bounded domain in* $\mathbb{R}^3$ *and* $\mathbf{n}$ *denote the normal unit vector to the boundary* $\partial\Omega$ *directed into the exterior of* Ω.
Let the complex-valued fields $\mathbf{J}_e$ *and* $\mathbf{J}_m$ *be regular current fields on* Ω, *vanishing outside* $i.e.\, \mathbf{J}_e(x) = \mathbf{J}_m(x) = 0$ *for* $x \notin \overline{\Omega}$.
If we further suppose

$$\mathbf{n}(y)\cdot\mathbf{J}_e(y) = \mathbf{n}(y)\cdot\mathbf{J}_m(y) = 0 \quad \textit{for} \quad y \in \partial\Omega. \tag{1.28}$$

then the fields given by

$$\begin{aligned}\mathbf{E}(x) &= iw\mu_0 \int_\Omega g_k(x,y)\mathbf{J}_e(y)\,dy \\ &+ \int_\Omega \nabla_y g_k(x,y)\times\mathbf{J}_m(y)\,dy \\ &- i(w\varepsilon_0)^{-1}\int_\Omega \nabla_y g_k(x,y)\,(\nabla_y\cdot\mathbf{J}_e(y))\,dy\end{aligned} \tag{1.29}$$

and

$$\begin{aligned}\mathbf{H}(x) &= iw\varepsilon_0 \int_\Omega g_k(x,y)\mathbf{J}_m(y)\,dy \\ &- \int_\Omega \nabla_y g_k(x,y)\times\mathbf{J}_e(y)\,dy \\ &- i(w\mu_0)^{-1}\int_\Omega \nabla_y g_k(x,y)\,(\nabla_y\cdot\mathbf{J}_m(y))\,dy\end{aligned} \tag{1.30}$$

are in $C^1(\mathbb{R}^3,\mathbb{C}^3)$ *and are radiating solutions to the Maxwell equations*

$$\nabla\times\mathbf{H} + iw\,\varepsilon_0\,\mathbf{E} = \mathbf{J}_e\,, \tag{1.31}$$

$$\nabla\times\mathbf{E} - iw\,\mu_0\,\mathbf{H} = -\mathbf{J}_m. \tag{1.32}$$

For the proof of theorem 1.2.7 and Corollary 1.2.8 we refer to [Mue69]. The last corollary gives the solution of the direct source problem as a sum of three volume integrals. The electric field can be seen as generated by three types of volume sources. The first integral shows the radiation of the volume density of electrical currents, the second integral reflects the contribution due to the electromagnetic coupling and the last one is the contribution due to the volume density of electrical charges. A similar interpretation is also valid for the magnetic field.
To guarantees the regularity of the electromagnetic fields we made an assumption excluding surface currents in our modeling of the direct source problem. For more details about the effects of the surface currents we refer to [VB91].

1.3 The direct medium problem

In this section we deal with the determination of electromagnetic fields in an inhomogeneous medium excited by an extraneous incident field when the electromagnetic material properties are prescribed. In the first subsection, we use the concept of induced or equivalent sources to reduce the direct medium problem into a direct source problem. We then derive the formulation as a system of integro-differential equations. In the second section we give the formulation of the direct medium problem as a boundary value problem and state existence and uniqueness result in a weaker space setting. This formulation is relevant as far as inversion from boundary measurement is concerned. In the last section we define the operator formalism used throughout this work. The equations of Lippmann-Schwinger type give a unified setting for modeling scattering in inhomogeneous media and highlights the similarities between scattering problems in acoustics and in electromagnetics.

1.3.1 Integral equation formulation

Let Ω be a regular bounded domain in $\mathbb{R}^3$ with boundary $\partial\Omega,\ w>0$ be a fixed frequency and $\varepsilon_0, \mu_0,$ be positive constants .
For an inhomogeneous medium with prescribed electromagnetic properties ε and μ satisfying

$$\varepsilon-\varepsilon_0,\ \mu-\mu_0\ \in C_0^1(\Omega,\mathbb{C}) \tag{1.33}$$

with

$$\mathrm{Re}(\varepsilon)>0,\ \mathrm{Re}(\mu)>0, \tag{1.34}$$

we consider the following direct medium problem.
Let $\mathbf{E}^i, \mathbf{H}^i \in C^1(\mathbb{R}^3,\mathbb{C}^3)$ be solutions in $\mathbb{R}^3$ to the homogeneous Maxwell equations

$$\nabla\times\mathbf{H}^i + iw\ \varepsilon_0\,\mathbf{E}^i \;=\; 0, \tag{1.35}$$
$$\nabla\times\mathbf{E}^i - iw\ \mu_0\,\mathbf{H}^i \;=\; 0. \tag{1.36}$$

We want to find $\mathbf{E}, \mathbf{H} \in C^1(\mathbb{R}^3,\mathbb{C}^3)$ which are solutions to the Maxwell equations

$$\nabla\times\mathbf{H} + iw\ \varepsilon\,\mathbf{E} \;=\; 0, \tag{1.37}$$
$$\nabla\times\mathbf{E} - iw\ \mu\,\mathbf{H} \;=\; 0, \tag{1.38}$$

in $\mathbb{R}^3$, such that the scattered fields $\mathbf{E}^s$ and $\mathbf{H}^s$ defined by

$$\mathbf{E}^s : \;=\; \mathbf{E}-\mathbf{E}^i, \tag{1.39}$$
$$\mathbf{H}^s : \;=\; \mathbf{H}-\mathbf{H}^i, \tag{1.40}$$

satisfy the Silver-Müller radiation condition

$$\lim_{|x|\to\infty}(\mathbf{H}^s\times x-|x|\mathbf{E}^s)=0. \tag{1.41}$$

If we introduce the *induced* electric and magnetic current sources $\tilde{\mathbf{J}}_e$ and $\tilde{\mathbf{J}}_m$ defined by

$$\begin{aligned}\tilde{\mathbf{J}}_e &:= iw(\varepsilon_0 - \varepsilon)\mathbf{E}, & (1.42)\\ \tilde{\mathbf{J}}_m &:= iw(\mu_0 - \mu)\mathbf{H}, & (1.43)\end{aligned}$$

the direct medium problem is then reduced to a direct source problem in homogeneous medium for the induced source densities $\tilde{\mathbf{J}}_e$ and $\tilde{\mathbf{J}}_m$. Indeed, From the equations (1.35)-(1.43) we get

$$\begin{aligned}\nabla \times \mathbf{H}^s + iw\ \varepsilon_0\, \mathbf{E}^s &= \tilde{\mathbf{J}}_e\ , & (1.44)\\ \nabla \times \mathbf{E}^s - iw\ \mu_0\, \mathbf{H}^s &= -\tilde{\mathbf{J}}_m, & (1.45)\end{aligned}$$

where the scattered fields $\mathbf{E}^s$, $\mathbf{H}^s$ have to satisfy the radiating condition (1.41).
If we denote

$$g_k(x,y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|},\ x \neq y, x \in \mathbb{R}^3.$$

with $k := w\sqrt{\varepsilon_0\mu_0}$, we get from Corollary 1.2.8 that the electromagnetic fields given by

$$\begin{aligned}\mathbf{E}(x) = \mathbf{E}^i(x) \quad &+ \quad iw\mu_0 \int_\Omega g_k(x,y)\tilde{\mathbf{J}}_e(y)\,dy\\ &+ \quad \int_\Omega \nabla_y g_k(x,y) \times \tilde{\mathbf{J}}_m(y)\,dy\\ &- \quad i(w\varepsilon_0)^{-1}\int_\Omega \nabla_y g_k(x,y)\,(\nabla_y \cdot \tilde{\mathbf{J}}_e(y))\,dy \qquad (1.46)\end{aligned}$$

and

$$\begin{aligned}\mathbf{H}(x) = \mathbf{H}^i(x) \quad &+ \quad iw\varepsilon_0 \int_\Omega g_k(x,y)\tilde{\mathbf{J}}_m(y)\,dy\\ &- \quad \int_\Omega \nabla_y g_k(x,y) \times \tilde{\mathbf{J}}_e(y)\,dy\\ &- \quad i(w\mu_0)^{-1}\int_\Omega \nabla_y g_k(x,y)\,(\nabla_y \cdot \tilde{\mathbf{J}}_m(y))\,dy, \qquad (1.47)\end{aligned}$$

are solutions to the direct medium problem.
Inversely, the solutions to the integral equation (1.46) and (1.47) are also solutions to the Maxwell equations as we can see from the next theorem.

Theorem 1.3.1. *Let ε, μ satisfy the conditions (1.33),(1.34) and $\mathbf{E}^i, \mathbf{H}^i \in C^1(\mathbb{R}^3, \mathbb{C}^3)$ be solution in $\mathbb{R}^3$ to the homogeneous Maxwell equations (1.35)-(1.36).*

If $\mathbf{E}, \mathbf{H} \in C(\mathbb{R}^3, \mathbb{C}^3)$ *are solutions to the integral equations*

$$\begin{aligned} \mathbf{E}(x) &= \mathbf{E}^i(x) - w^2\mu_0 \int_\Omega g_k(x,y)\,(\varepsilon_0 - \varepsilon(y))\,\mathbf{E}(y)\,dy \\ &+ iw \int_\Omega \nabla_y g_k(x,y) \times (\mu_0 - \mu(y))\,\mathbf{H}(y)\,dy \\ &- \int_\Omega \nabla_y g_k(x,y)\,(\frac{\nabla\varepsilon}{\varepsilon}(y) \cdot \mathbf{E}(y))\,dy \end{aligned} \tag{1.48}$$

and

$$\begin{aligned} \mathbf{H}(x) &= \mathbf{H}^i(x) - w^2\varepsilon_0 \int_\Omega g_k(x,y)(\mu_0 - \mu(y))\mathbf{H}(y)\,dy \\ &- iw \int_\Omega \nabla_y g_k(x,y) \times (\varepsilon_0 - \varepsilon(y))\mathbf{E}(y))\,dy \\ &- \int_\Omega \nabla_y g_k(x,y)\,(\frac{\nabla\mu}{\mu}(y) \cdot \mathbf{H}(y))\,dy, \end{aligned} \tag{1.49}$$

for $x \in \mathbb{R}^3$, *then we have*

$$\nabla \cdot (\varepsilon_0 - \varepsilon)\mathbf{E} = -\varepsilon_0 \frac{\nabla\varepsilon}{\varepsilon} \cdot \mathbf{E}, \tag{1.50}$$

$$\nabla \cdot (\mu_0 - \mu)\mathbf{H} = -\mu_0 \frac{\nabla\mu}{\mu} \cdot \mathbf{H}. \tag{1.51}$$

Furthermore, $\mathbf{E}$ *and* $\mathbf{H}$ *are in* $C^1(\mathbb{R}^3, \mathbb{C}^3)$ *and the scattered fields*

$$\mathbf{E}^s : = \mathbf{E} - \mathbf{E}^i, \tag{1.52}$$

$$\mathbf{H}^s : = \mathbf{H} - \mathbf{H}^i, \tag{1.53}$$

are radiating solution to the Maxwell equations

$$\nabla \times \mathbf{H}^s + iw\,\varepsilon_0\,\mathbf{E}^s = \mathbf{J}_e\ , \tag{1.54}$$

$$\nabla \times \mathbf{E}^s - iw\,\mu_0\,\mathbf{H}^s = -\mathbf{J}_m, \tag{1.55}$$

in $\mathbb{R}^3$ *with*

$$\mathbf{J}_e := iw(\varepsilon_0 - \varepsilon)\mathbf{E}, \tag{1.56}$$

$$\mathbf{J}_m := iw(\mu_0 - \mu)\mathbf{H}. \tag{1.57}$$

Remark. The relations (1.50) and (1.51) following from the integral equations (1.48) and (1.49) are necessary to derive the Maxwell-equations from the equations (1.46) and (1.47).

The scattering of electromagnetic fields from an inhomogeneous medium with prescribed electromagnetic properties and given incident fields is governed by a system of integro-differential equations. The contrast of the electromagnetic properties between an object and its background creates equivalent sources of currents and charges. The scattered electromagnetic field is interpreted as being radiated from these sources.

Definition 1.3.2. *Let* $\varepsilon,\ \mu$ *satisfy the conditions (1.33),(1.34) and* $\mathbf{E}^{\,i}, \mathbf{H}^i \in C^1(\mathbb{R}^3, \mathbb{C}^3)$ *be solution in* $\mathbb{R}^3$ *to the homogeneous Maxwell equations (1.35)-(1.36).*
We call

$$\begin{aligned} \mathbf{E}(x) &= \mathbf{E}^i(x) - k^2 \int_\Omega g_k(x,y)\, f_e(y)\, \mathbf{E}(y)\, dy \\ &+ iw\mu_0 \int_\Omega \nabla_y g_k(x,y) \times f_m(y)\, \mathbf{H}(y)\, dy \\ &+ \int_\Omega \nabla_y g_k(x,y)\, \nabla \cdot (f_e\, \mathbf{E})(y))\, dy, \end{aligned} \tag{1.58}$$

$$\begin{aligned} \mathbf{H}(x) &= \mathbf{H}^i(x) - k^2 \int_\Omega g_k(x,y) f_m(y)\, \mathbf{H}(y)\, dy \\ &- iw\varepsilon_0 \int_\Omega \nabla_y g_k(x,y) \times f_e(y)\, \mathbf{E}(y))\, dy \\ &+ \int_\Omega \nabla_y g_k(x,y)\, \nabla \cdot (f_m\, \mathbf{H})(y)\, dy, \end{aligned} \tag{1.59}$$

the electromagnetic field integral equations with the electric contrast f_e *and the magnetic contrast* f_m *defined by*

$$f_e := (1 - \varepsilon_0^{-1}\varepsilon), \tag{1.60}$$
$$f_m := (1 - \mu_0^{-1}\mu). \tag{1.61}$$

If $\mu = \mu_0$, *the equation*

$$\begin{aligned} \mathbf{E}(x) &= \mathbf{E}^i(x) - k^2 \int_\Omega g_k(x,y)\, f_e(y)\, \mathbf{E}(y)\, dy \\ &+ \int_\Omega \nabla_y g_k(x,y)\, \nabla \cdot (f_e\, \mathbf{E})(y)\, dy \end{aligned} \tag{1.62}$$

is called the electrical field integral equation for a nonmagnetic medium with contrast f_e.

To emphasize the dependence of the problem on the frequency, we inserted the wave number $k = w\sqrt{\varepsilon_0\mu_0}$ in the equations (1.48), (1.49) to obtain the equations (1.58) and (1.59).
We distinguish between two behaviors, propagation and diffusion. Propagative fields are evinced by the first two integrals on the right-hand side of equations (1.58) and (1.59), however the diffusive aspect is due to the divergence operator in the last integrals. In the quasi-static regime, when k is small, the diffusive behavior is prevailing.

1.3.2 Boundary value problem

Electromagnetic inversion from near-field data involve boundary measurments of the electromagnetic field. It is therefore worth to consider a local formulation of the direct

medium problem as a boundary value problem.
Furthermore, the uniqueness result for the inverse medium problem relies on the weak formulation of the direct problem. For the functional space setting, we use the Sobolev spaces, $H^s, s \in \mathbb{R},$ as they are defined in [DL88]. For the sake of clarity, we use the notations $\mathbf{grad}$, $\mathbf{div}$ and $\mathbf{curl}$ for the gradient, divergence and curl operators in the weak sense and we denote $\mathbf{Div}$ the surface divergence [5] operator.
Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary Γ. For $x \in \Gamma,$ let $\mathbf{n}(x),$ denote the outward normal unit vector to the boundary.

Definition 1.3.3. *For $s \in \mathbb{R},$ we define*

$$TH^s(\Gamma) := \{\mathbf{f} \in H^s(\Gamma)^3,\, \mathbf{n}\cdot\mathbf{f} = 0\}$$

as the Sobolev space of tangential fields.
For $s \geq 1,$ we define the space

$$TH^s_{Div}(\Gamma) := \{\mathbf{f} \in TH^s(\Gamma)^3,\, \mathbf{Div}\,\mathbf{f} \in H^s(\Gamma)\},$$

endowed with the norm

$$||\mathbf{f}||_{TH^s_{Div}(\Gamma)} := ||\mathbf{f}||_{H^s(\Gamma)^3} + ||\mathbf{Div}\,\mathbf{f}||_{H^s(\Gamma)},$$

and the space

$$H^s_{Div}(\Omega) := \{\mathbf{f} \in H^s(\Omega)^3,\, \mathbf{Div}(\mathbf{n} \times \mathbf{f}_{|\Gamma}) \in H^{s-\frac{1}{2}}(\Gamma)\},$$

endowed with the norm

$$||\mathbf{f}||_{H^s_{Div}(\Omega)} := ||\mathbf{f}||_{H^s(\Gamma)^3} + ||\mathbf{Div}\,\mathbf{f}||_{H^s(\Gamma)}.$$

The spaces $H^s_{Div}(\Omega)$ arise naturally through the tangential trace mapping on the boundary, see D.0.11.

Theorem 1.3.4. *Let w, ε_0 and μ_0 be positive contants, ε and μ satisfy*

$$\varepsilon - \varepsilon_0 \in C^1_0(\Omega, \mathbb{C}),\; \mu - \mu_0 \in C^1_0(\Omega, \mathbb{R}) \tag{1.63}$$

with

$$Re(\varepsilon) > 0,\; Im(\varepsilon) > 0,\; \mu > 0, \tag{1.64}$$

Then the boundary value problem

$$\begin{aligned} \mathbf{curl}\,\mathbf{H} + iw\,\varepsilon\,\mathbf{E} &= 0, \\ \mathbf{curl}\,\mathbf{E} - iw\,\mu\,\mathbf{H} &= 0, \end{aligned}$$

with the boundary condition

$$\mathbf{n} \times \mathbf{E} = \mathbf{f} \quad on \quad \Gamma \tag{1.65}$$

for given $\mathbf{f} \in TH^{\frac{1}{2}}_{Div}(\Gamma)$,
admits a unique (weak) solution $(\mathbf{E}, \mathbf{H}) \in H^1_{Div}(\Omega) \times H^1_{Div}(\Omega)$.

[5] for more details about the surface divergence we refer to [CK98] p.167.

This result can be found in [Isa98] p. 137. If $\mathrm{Im}(\varepsilon) = 0$, theorem 1.3.4 holds except for a discrete set of resonance frequencies $\{w_n\}_n$.
We see from theorem 1.3.4 that the direct medium problem formulated as a boundary value problem is uniquely solvable if the tangential component of the electrical field is prescribed on the boundary. A similar result can be derived if the tangential component of the magnetic field is prescribed on the boundary.

1.4 Equation of Lippmann-Schwinger type

For w, ε, μ and k positive constants with $k := w\sqrt{\varepsilon\mu}$, let

$$g_k(x,y) := \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|},\ x \neq y, x \in \mathbb{R}^3,$$

and Ω, Ω', $\Omega \subset \Omega'$, be two bounded domains in $\mathbb{R}^3$ with smooth boundaries.

Definition 1.4.1. *The (first) scalar scattering operator A is defined for $f \in C_0(\Omega)$ by*

$$Af(x) := \int_\Omega g_k\,(x,y)\,f(y)\,dy, \quad x \in \mathbb{R}^3.$$

The second scalar scattering operator $\mathbf{A}'$ is defined [6] for $f \in C_0(\Omega)$ by

$$\mathbf{A}' f(x) := \int_\Omega \nabla_y\, g_k\,(x,y)\,f(y)\,dy, \quad x \in \mathbb{R}^3.$$

For $N \in \mathbb{N}, N > 1$, the vector scattering operator $\mathbf{A}$ is defined for $\mathbf{f} = (f_i)_{i=1,..,N} \in C_0(\Omega)^N$ by

$$\mathbf{Af} := (A\,f_i)_{i=1,..,N}.$$

The first coupling operator $\mathbf{K_d}$ is defined for $\mathbf{f} \in C_0^1(\Omega)^3$ by

$$\mathbf{K_d}\,\mathbf{f}(x) := \int_\Omega \nabla_y\, g_k\,(x,y)\,\mathbf{div}\,\mathbf{f}\,(y)\,dy, \quad x \in \mathbb{R}^3.$$

The second coupling operator $\mathbf{K_r}$ is defined for $\mathbf{f} \in C_0(\Omega)^3$ by

$$\mathbf{K_r}\,\mathbf{f}(x) := \int_\Omega \nabla_y\, g_k\,(x,y) \,\times\, \mathbf{f}(y)\,dy, \quad x \in \mathbb{R}^3.$$

Hilbert-spaces provide a better setting for the inverse problem. Before extending the operators defined above onto spaces of square integrable functions, we recall the definition of some spaces, for more details see [DL88].

[6] The notation should not be confused with the usual notation for the derivative.

Definition 1.4.2. *We define the space*

$$H(\mathbf{div},\Omega) := \{\mathbf{f} \in L_2(\Omega)^3,\ \mathbf{div}\,\mathbf{f} \in L_2(\Omega)\},$$

endowed with the norm

$$||\mathbf{f}||_{H(\mathbf{div},\Omega)} := \left(||\mathbf{f}||^2_{L_2(\Omega)^3} + ||\mathbf{div}\,\mathbf{f}||^2_{L_2(\Omega)}\right)^{\frac{1}{2}},$$

and the space

$$H(\mathbf{curl},\Omega) := \{\mathbf{f} \in L_2(\Omega)^3,\ \mathbf{curl}\,\mathbf{f} \in L_2(\Omega)\},$$

endowed with the norm

$$||\mathbf{f}||_{H(\mathbf{curl},\Omega)} := \left(||\mathbf{f}||^2_{L_2(\Omega)^3} + ||\mathbf{curl}\,\mathbf{f}||^2_{L_2(\Omega)^3}\right)^{\frac{1}{2}}.$$

We denote

$$H(\mathbf{div}_0,\Omega) := \{\mathbf{f} \in L_2(\Omega)^3,\ \mathbf{div}\,\mathbf{f} = 0\}$$

and

$$H(\mathbf{curl}_0,\Omega) := \{\mathbf{f} \in L_2(\Omega)^3,\ \mathbf{curl}\,\mathbf{f} = 0\}.$$

The closure of $C_0^\infty(\Omega)$ in $H^s(\Omega), s \in \mathbb{R},$ is denoted $H_0^s(\Omega)$. The closure of $C_0^\infty(\Omega)^3$ in $H(\mathbf{div},\Omega)$ is denoted $H_0(\mathbf{div},\Omega)$ and in $H(\mathbf{curl},\Omega)$ is denoted $H_0(\mathbf{curl},\Omega)$.

Lemma 1.4.3. *We can extend the operators defined above to*

1. $A \in \mathcal{L}\,(L_2\,(\Omega), H^2\,(\Omega^{'}))$.
2. $\mathbf{A}^{'} \in \mathcal{L}\,(L_2\,(\Omega), H^1\,(\Omega^{'})^3)$.
3. $\mathbf{A} \in \mathcal{L}\,(L_2\,(\Omega)^N, H^2\,(\Omega^{'})^N)$.
4. $\mathbf{K_d} \in \mathcal{L}\,(H\,(\mathbf{div},\Omega), H\,(\mathbf{curl}_0,\Omega^{'}) \cap H^1(\Omega^{'})^3)$.
5. $\mathbf{K_r} \in \mathcal{L}\,(L_2\,(\Omega), H^1\,(\mathbf{div}_0,\Omega^{'}))$.

Proof. The first assertion is proved in [CK98] p.215 and it immediately implies the third one.
The second assertion may be proved similarly to the first one, where the kernel $g_{\,k}\,(x,y)$ is replaced by $\nabla_y\, g_k\,(x,y),$ which is also weakly singular.
In the fourth assertion, the operator $\mathbf{K_d}$ can be extended boundedly, since A' is bounded and we have

$$||\mathbf{K_d f}||_{H^1(\Omega)^3} \leq ||A'||\ ||\mathbf{div}\,\mathbf{f}||_{L_2(\Omega)} \leq ||A'||\ ||\mathbf{f}||_{H\,(\mathbf{div},\Omega)} \quad \text{for} \quad \mathbf{f} \in H\,(\mathbf{div},\Omega).$$

It remains to show that

$$\mathbf{curl}\,\mathbf{K_d}\,\mathbf{f} = 0 \quad \text{for} \quad \mathbf{f} \in H\,(\mathbf{div},\Omega).$$

Let $f \in L_2(\Omega)$, $\mathbf{grad}\, Af$ makes sense since $Af \in H^2(\Omega^{'})$.
Using the identity

$$\nabla_x g_k(x,y) = \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|} \left(\frac{1}{|x-y|} - ik \right) \frac{y-x}{|x-y|} = -\nabla_y g_k(x,y), \quad x \neq y,$$

we get

$$\mathbf{grad}\, Af(x) = \nabla_x \int_\Omega g_k(x,y) f(y) dy = -\int_\Omega \nabla_y\, g_k(x,y) f(y) dy = -\mathbf{A}^{'} f(x)$$

for $x \in \Omega^{'}$. It yields

$$\mathbf{grad} A = -\mathbf{A}^{'} \quad on \quad L_2(\Omega).$$

For $f \in H(\mathbf{div}, \Omega)$, we have

$$\mathbf{K_d}\, \mathbf{f}(x) = -\mathbf{grad}\, A\, \mathbf{div}\, \mathbf{f}(x) \quad \text{for} \quad x \in \Omega^{'},$$

and conclude

$$\mathcal{R}(\mathbf{K_d}) \subset \mathcal{N}(\mathbf{curl}).$$

We have $\mathbf{K_r} \in \mathcal{L}(L_2(\Omega), H^1(\Omega^{'})^3)$ as $\mathbf{A}^{'} \in \mathcal{L}(L_2(\Omega), H^1(\Omega^{'}))$.
Let $\mathbf{f} \in C_0^1(\Omega)$ and $x \in \Omega'$. From the identity

$$\mathbf{curl}\,(g_k(x,y)\mathbf{f}(y)) = \nabla_y\, g_k(x,y) \times \mathbf{f}(y) + g_k(x,y)\, \mathbf{curl}\, \mathbf{f}(y), \quad y \in \Omega,\, y \neq x,$$

and as the integral of the left-hand side vanishes by virtue of the curl theorem, see for example [DL90b] p.311, we get

$$\begin{aligned}
\mathbf{K_r f}(x) &= \int_\Omega \nabla_y\, g_k(x,y) \times \mathbf{f}(y)\, dy \\
&= -\int_\Omega \nabla_x\, g_k(x,y) \times \mathbf{f}(y)\, dy \\
&= -\int_\Omega \nabla_x \times (g_k(x,y)\, \mathbf{f}(y))\, dy \\
&= -\nabla_x \times \int_\Omega g_k(x,y)\, \mathbf{f}(y)\, dy \\
&= -\mathbf{curl}\, \mathbf{A}\, \mathbf{f}(x), \quad x \in \Omega.
\end{aligned}$$

Since $C_0^1(\Omega)$ is dense in $L_2(\Omega)$, we get

$$\mathbf{K_r} = -\mathbf{curl}\, \mathbf{A} \quad on \quad L_2(\Omega),$$

and consequently

$$\mathcal{R}(\mathbf{K_r}) \subset \mathcal{N}(\mathbf{div}).$$

□

We collect some intermediate results from the proof above in the following corollary.

Corollary 1.4.4. *We have*

$$\begin{aligned} \mathbf{grad}\, A &= -\mathbf{A}^{'} && \text{on} \quad L_2(\Omega), \\ \mathbf{K_d} &= -\mathbf{grad}\, A\, \mathbf{div} && \text{on} \quad H(\text{div}, \Omega), \\ \mathbf{K_r} &= -\mathbf{curl}\, \mathbf{A} && \text{on} \quad L_2(\Omega)^3. \end{aligned}$$

Definition 1.4.5. *The electromagnetic scattering operator $\mathcal{T}_{em}$ is defined for $\mathbf{f} = (\mathbf{f}_e, \mathbf{f}_m) \in C_0^1(\Omega)^3 \times C_0^1(\Omega)^3$ by*

$$\mathcal{T}_{em}\, \mathbf{f} := (\mathcal{T}_e\, \mathbf{f}, \mathcal{T}_m\, \mathbf{f})$$

with

$$\mathcal{T}_e\, \mathbf{f} = k^2\, \mathbf{A}\, \mathbf{f}_e - \mathbf{K_d}\, \mathbf{f}_e - i\, \omega\, \mu_0\, \mathbf{K_r}\, \mathbf{f}_m, \tag{1.66}$$

$$\mathcal{T}_m\, \mathbf{f} = k^2\, \mathbf{A}\, \mathbf{f}_m - \mathbf{K_d}\, \mathbf{f}_m + i\, \omega\, \varepsilon_0\, \mathbf{K_r}\, \mathbf{f}_e. \tag{1.67}$$

The nonmagnetic scattering operator $\mathbf{T}_e$ is defined for $\mathbf{f} \in C_0^1(\Omega)^3$ by

$$\mathbf{T}_e\, \mathbf{f} = k^2\, \mathbf{A}\, \mathbf{f} - \mathbf{K_d}\, \mathbf{f}. \tag{1.68}$$

Lemma 1.4.6. *The nonmagnetic scattering operator $\mathbf{T}_e$ can be extended to a bounded linear operator from $H\,(\mathbf{div}, \Omega)$ to $L_2\,(\Omega^{'})^3$.*
The electromagnetic scattering operator $\mathcal{T}_{em}$ can be extended to a bounded linear operator from $H\,(\mathbf{div}, \Omega) \times H\,(\mathbf{div}, \Omega)$ to $L_2\,(\Omega^{'})^3 \times L_2\,(\Omega^{'})^3$.

Definition 1.4.7. *Let $n \in \mathbb{N}$, X be a Banach-algebra and $T \in \mathcal{L}(X^N)$.*
For fixed $f \in X$ and $u^0 \in X^N$, the equation

$$u = u^0 - T f u, \quad u \in X^N, \tag{1.69}$$

is said to be of Lippmann-Schwinger type for T.
Let $V \subset X$. If for every $f \in V$ and $u^0 \in X^N$, the equation (1.69) admits a unique solution u, the nonlinear operator $\mathcal{A}$ defined on $V \times X^N$ by

$$\mathcal{A}\,(f, u^0) = u$$

is called a Lippmann-Schwinger operator for T.

Examples

1. The scattering of time-harmonic acoustic waves in an inhomogeneous medium is modeled by an integral equation of Lippmann-Schwinger type with $N = 1, X = C\,(\Omega)$ for the operator $T = k^2 A$, $k > 0$, where A is the scalar scattering operator, see [CK98] p. 216. This equation is refered to as the scalar Lippmann-Schwinger equation.

2. Let $N = 3$, $X = C^1\,(\Omega) \times C^1\,(\Omega)$. The system of electromagnetic equations is of Lippmann-Schwinger type for the operator $T = \mathcal{T}_{em}$ with

   $$u^0 := (E^i, H^i), \quad u := (E, H), \quad f := (f_e, f_m).$$

   We call this equation the electromagnetic Lippmann-Schwinger equation.

3. Let $N = 3, X \in C^1(\Omega)$. The electrical field integral equation for a nonmagnetic medium with contrast f_e is of Lippmann-Schwinger type for the operator $T = T_e$ with
$$u^0 := E^i, \quad u := E, \quad f := f_e.$$
We call this equation the nonmagnetic Lippmann-Schwinger equation.

Theorem 1.4.8. *(Well-posedness of the direct medium problem)*

1. *Let $X = C(\Omega)$. For $f \in C^{1,\alpha}(\Omega)$ with compact support in $\Omega, \alpha \in (0,1)$, there exists a unique solution $u \in X$ to the scalar Lippmann-Schwinger equation and this solution depends continuously with respect to the maximum norm on the incident field $u^0 \in X$.*

2. *Let $X = C^1(\Omega)^3 \times C^1(\Omega)^3$. For $f \in C^{1,\alpha}(\Omega) \times C^{1,\alpha}(\Omega)$ with compact support in $\Omega, \alpha \in (0,1)$, there exists a unique solution $u \in X$ to the electromagnetic Lippmann-Schwinger equation and this solution depends continuously with respect to the maximum norm on the incident field $u^0 \in X$.*

3. *Let $X = C^1(\Omega)^3$. For $f \in C^{1,\alpha}(\Omega)$ with compact support in $\Omega, \alpha \in (0,1)$, there exists a unique solution $u \in X$ to the nonmagnetic Lippmann-Schwinger equation and this solution depends continuously with respect to the maximum norm on the incident field $u^0 \in X$.*

Proof. The proof of the first assertion can be found in [CK98]. For the proof of the third one see [Mue69], the second assertion is a particular case of it. □

Remarks.

1. We lose no generality when we restrict the Lippmann-Schwinger equations to the bounded domain Ω, instead of the whole space $\mathbb{R}^3$, since the values of the fields outside Ω are completely determined from the values of the fields on Ω by using the integral operator.

2. For the analysis of the nonmagnetic Lippmann-Schwinger operator in the setting of Sobolev spaces we refer to [AB01].

3. The Helmholtz operator $\Delta + k^2$ acting componentwise on vector wave fields can be seen as the scalar approximation to the operator $-\nabla \times \nabla + k^2$.
This assumption is particularly valid for fields with weak divergence. Nevertheless, there is an essential difference between the two operators. The Helmholz operator is elliptic, however the operator $-\nabla \times \nabla + k^2$ is not.

4. Although the discretized form of the integral equations does not lead to sparse matrices like the partial differential equations do, it is advantageous to use the Lippmann-Schwinger equation to model the forward problem since the boundary conditions are already embedded in the integral equation formulation. Furthermore many properties of elliptic partial differential equations, such as the Lax-Milgram lemma, are not valid for the non-elliptic Maxwell-Model.

Chapter 2

Inverse electromagnetic scattering

Inverse scattering is a central problem in diverse areas of application. The goal is the determination of the characteristics, like the shape, location, acoustic or electromagnetic constitutive properties, of a buried object embedded in some known background. We are intereseted in the inversion carried from near-filed measurements of the scattered waves using sensors placed outside the target-object illuminated by a set of monochromatic incident fields. For the scattering of electromagnetic waves, we base our modeling on the contrast source integral equations corresponding to full three-dimensional Maxwell equations. The formulation as equations of Lippmann-Schwinger type on hand for the Helmholtz and the Maxwell models provides a comprehensive framework for inverse scattering problems in inhomogeneous media for acoustic and electromagnetic waves.
We start this chapter with an abstract formulation of the inverse scattering problems. We next highlight the practical relevance of the electromagnetic inverse scattering and discuss some methods to accomplish the inversion. We further present for the inverse medium problem a uniqueness result due to Ola, Päivärinta and Somersalo. In the third section, we introduce a procedure to make the inversion of the electromagnetic and nonmagnetic scattering operators. The last section is devoted to treat the scalar scattering operator when the geometrical configuration is spherical.

2.1 Formulation of the inverse problems

The bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega,$ denotes the target region where the material parameters are to be determined. The contrast functions are supposed to be compactly supported in Ω and are elements of the *parameter space* $V(\Omega)$, which can be injected into a subspace of $L_2(\Omega)^N$. The N-dimensional wave fields are elements of the *state space* $U(\Omega)^N \subset L_2(\Omega)^N$. Let the *source space* $X(\Omega)^N$ be a subspace of $L_2(\Omega)^N$ such that $f\,u \in X(\Omega)^N$ for every $f \in V(\Omega)$ and $u \in U(\Omega)^N$. The data are measured on the subset Γ of $\mathbb{R}^3,$ which is in general a surface exterior

to the target region. The *data space* $Y(\Gamma)^P$ is a subspace of $L_2(\Gamma)^P$. We notice that in many experimental setting, the dimension P of the measured data may differ from the dimension of the wave fields N.
The scattering operator denoted by $T : X(\Omega)^N \to U(\Omega)^N$ associates to a source ϕ the scattered field $u^s := T\phi$.
Let $I \subset \mathbb{N}$ denote the experiment-set with p elements. For each $j \in I$, we are given an operator $M^j : U(\Omega)^N \to Y(\Gamma)^P$ called the *measurement operator* for the j^{th}- experiment, which to an element u_j from the state-space $U(\Omega)^N$ associates an element d_j from the data-space $Y(\Gamma)^P$.
The operator $T_\Gamma^j = M^j T$ is called the restricted scattering operator.

Definition 2.1.1. *For $j \in I$, let the incident fields $v_j \in U(\Omega)^N$ be given. The inverse medium problem for the operator T and the measurement operators $(M^j)_{j \in I}$ reads : To the data $d_j \in Y(\Gamma)^P$, determine the contrast function $f \in V(\Omega)$, such that there exists a wave field $u_j \in U(\Omega)^N$ satisfying the data fitting equation and the equation of Lippmann-Schwinger type given by the system:*

$$\begin{cases} M^j u_j^s = d_j, \\ u_j^s = T(f\, u_j), \end{cases}$$

with $u_j^s := u_j - v_j$, for every $j \in I$.

We can already see two major difficulties featuring the inverse medium problem in general:

- The ill-posedness, which is typical for inverse problems. It is here twofold since the inverse problem is under-determined and ill-conditioned. In general, the measurements are taken on a subset Γ, which can be immersed in a manifold of lower order than the target region Ω. The operator $M_\Gamma T$ admits, therefore, a big null space. Furthermore, data are endowed with an unavoidable error, which are related to measurement conditions. This causes instability when making the inversion.

- The nonlinearity. The dependence of the field u on the contrast function f is in many applications highly nonlinear.

In comparison with the acoustic case, the electromagnetic case presents mainly two further aspects of difficulty:

- Mathematically, the system is much more complicated since the underlying operators are integro-differential operators acting on three-dimensional vector fields. Furthermore, all components of the vector fields are intertwined, which requires the simultaneous treatment of all components.

- Physically, besides the propagation, a diffusive behavior and an electromagnetic coupling are involved.

Definition 2.1.2. *If the contrast function* f *and the total field* $(u_j)_j$ *are solution to the inverse medium problem, then* $\phi_j := f\,u_j, j \in I$*, is called equivalent or induced (current) source for the* j^{th}*-experiment.*
The determination of the equivalent source $\phi_j, j \in I$*, satisfying*

$$T_\Gamma^j \phi_j = d_j$$

is called inverse source problem for the j^{th}*-experiment.*

We see that the solution of the inverse medium problem provides the contrast function f as an identification parameter for the material, which should be the same for all experiments $j \in I$. The total wave field u_j and the induced source $\phi_j = f\,u_j$ are, however, depending on the experiment j. For each $j \in I$ we have a new inverse source problem.

2.2 Applications and methods of inversion

Inversion algorithms are closely related to the field of application and to technological considerations. Technical limitations usually compromise the expectations of the inversion method and its relevance for practice. The acting frequency, the size and strength of the scatterer, the measurement conditions are crucial criteria for the modeling and inversion of electromagnetic data, see [dH00], [NH00].

Microwave tomography is a competitive noninvasive imaging technique in several practical areas. For medical application, microwave radiation can be used, without as much caution as for x-ray tomography, to identify the properties and to detect the anomalies of biological tissues by determining the electromagnetic characteristics like the electric permittivity and conductivity, see [SBS$^+$00]. The operating frequencies are typically in GHz range, [Kon90] p.13. The resolution is, however, not as accurate as the x-ray tomography since it does never provide the millimeter precision reached when using high-energetic radiations.

For geophysical prospecting, popular devices are the ground penetrating radars (GPR). They operate at frequencies ranging from radio to microwave domain, typically larger than 100 MHz, see [DBABC99]. Due to the attenuation of the wave at these frequencies, the underground penetration remains limited.
Alternatively, one may use metal detectors operating at the lower frequency-band of radio waves ($\leq$ 300 KHz). This involves electromagnetic induction tomography, which is a promising tool for imaging electromagnetic parameters like the electrical conductivity with relevance for many practical problems, such as the recent application to humanitarian mine detection. Nevertheless, a typical feature for this tomographical method in comparison with the microwave imaging is the diffusive character of the waves. Therefore, the treatment of the full three-dimensional Maxwell Model is required.

Many linear inversion methods are based on the Born/Rytov approximation, see [Lan87], where the scattered field is neglected with respect to the incident field. This leads to a linearized model for the scattering, whose validity is, however, confined to scatterers with low-contrast. Electromagnetic imaging involving techniques from the diffraction tomography, see [Nat01], [NW01], [KS88], are based on linear models using Born approximation or further extensions of it like the iterated Born approximation, or relying on empirical approximation such as the quasi-analytical method due to Zdhanov [Zhd02] or the synthetic aperture focusing techniques used in [MMH+02]. These methods remain with limited scope of application. The treatment of the nonlinear model is therefore necessary.

The approach in [NH00], [NB04], [Hab04], is to use nonlinear optimization techniques to reduce the mismatch between measured data and the predicted fields as a functional of the sought-for material parameters. In their iterative method, the high cost of the full Newton iteration is circumvented by using some approximation of the Jacobian. This leads to a scheme of Quasi-Newton type. Discretized Maxwell's equations are used for the forward model.

Dorn *et al.* in [DBABC99], [DBABP02], derive a nonlinear inversion of electromagnetic data for geophysical applications using the adjoint operator of the Fréchet derivative of the residual operator. Their iterative scheme can be considered as a generalization to nonlinear problems of the algebraic reconstruction technique for linear systems (ART) in x-ray tomography. The nonlinear iterations rely on a forward solver of the boundary value problem for the Maxwell equations. The method of Vögeler [Voe03] for microwave tomography utilizes also similar techniques. The modeling of the forward problem is alternatively formulated there as an initial value problem.

The source-type methods consist in the splitting of the nonlinear inverse medium scattering problem into two subproblems. The first one is the inverse source problem. It is linear but ill-posed. One attempts here, to recover the equivalent sources realizing the best fitting to the data. The second problem is in general well-posed, however nonlinear. It consists in the derivation of the material properties from the recovered equivalent source by solving the constitutive relations. Although, the equivalent source may already provide a good guess about the scatterer, see [AL99], [DMR00], the full reconstruction of the material properties from the equivalent source recovered from a single experiment is in general unfeasible. This is due to the non uniqueness of the solution of the inverse source problem related to the existence of non-radiating sources. These are sources ϕ_j that generate scattered fields u_j^s with $M^j u^s = 0$. Non-radiating sources and the fields they generate have been investigated for many years as a typical feature of the inverse source problem for the Helmholtz and the Maxwell equations, see [DW73], [BC77], [CWO+94], [DM98]. Hence, to recover the contrast function one should combine the outcome of many experiments.

Many algorithms to solve the inverse scattering problem for the Helmholtz or the Maxwell equations can be considered as source-type inversion. One approach is to use the resolution of the inverse source problem as a built-in component of an iterative algorithm. Typical examples, in the framework of the contrast source integral equation in two dimensional case, are the method of Habashy *et al.* [HGS93], [HODH95], and the contrast source inversion method due to van den Berg and Kleinmann [vdBK97],[vdB01]. Some of these Methods have been already extended to the electromagnetic case, see [AvdB02] and [BSSP04]. A common feature of all these schemes is the simultaneous treatment of all data and the resolution of the forward problem in each iteration, which obviously enhance the computation and storage effort dramatically. Abdullah and Louis [AL99] and later Wallacher [Wal02] proceeded differently. They first made a linear inversion to compute an approximate value of the equivalent source, which corresponds to the component with minimal energy called the minimal-norm solution. On a subspace orthogonal to the recovered component, a nonlinear optimization is executed to compute the contrast function, which realizes the best fitting for all experiments, see also [HODH95]. The optimization is accomplished with respect to an analytically precomputed basis. This means that no forward computation is required within the iterations of the nonlinear part of the scheme. The price for that is to determine the singular value decomposition of the restricted scattering operator T_Γ^j and a basis for its null space. The linear inversion is regularized using the approximate inverse. It yields, therefore, a stable and efficient scheme for the two dimensional case, which also works within a setting of incomplete data [Wal02].

2.3 Uniqueness

We present in this section a uniqueness theorem for the inverse medium problem formulated as a boundary value problem for the Maxwell's equations. This result is due to Ola, Päivärinta and Somersalo, see [OPS93], [OS96]. The proof for the electromagnetic case can be seen as an extension of the uniqueness theorem developed by Sylvester Uhlman [SU87] and Nachmann [Nac88] for the inverse conductivity problem in two-dimension. The global uniqueness theorem, derived there, states that the material parameters are uniquely determined from the Dirichlet-to-Neumann map. In the electromagnetic case the impedance map plays the role played by the Dirichlet-to-Neumann map in scalar inverse problems.

Definition 2.3.1. *Let w, ε_0 and μ_0 be positive constants, ε and μ satisfy*

$$\varepsilon - \varepsilon_0 \in C_0^1(\Omega, \mathbb{C}), \; \mu - \mu_0 \in C_0^1(\Omega, \mathbb{R})$$

with

$$\textit{Re}(\varepsilon) > 0, \; \textit{Im}(\varepsilon) > 0, \; \mu > 0.$$

The impedance map is defined by

$$\begin{array}{rcl} Z : TH_{Div}^{\frac{1}{2}}(\Gamma) & \longrightarrow & TH_{Div}^{\frac{1}{2}}(\Gamma) \\ \mathbf{f} & \longmapsto & \mathbf{n} \times \mathbf{E}, \end{array}$$

such that $(\mathbf{E}, \mathbf{H}) \in H^1_{Div}(\Omega) \times H^1_{Div}(\Omega)$ *are solutions to the Maxwell equations*

$$\mathbf{curl}\,\mathbf{H} + iw\,\varepsilon\,\mathbf{E} = 0, \tag{2.1}$$

$$\mathbf{curl}\,\mathbf{E} - iw\,\mu\,\mathbf{H} = 0, \tag{2.2}$$

with the boundary condition

$$\mathbf{n} \times \mathbf{H} = \mathbf{f} \quad on \quad \Gamma. \tag{2.3}$$

Theorem 2.3.2. *The impedence map* Z *determines uniquely the coefficients* ε, μ *with the above properties.*

Besides the theoretical importance of theorem 2.3.2, it shows the amount of data required to uniquely reconstruct the electromagnetic properties of a medium from boundary measurements.
Further results on uniqueness of inverse scattering can be found in [Isa01]. For electromagnetic inversion from local surface measurements see [Liu99].

2.4 Inversion of the em-scattering operators

In this section, we introduce a new procedure based on the concept of generalized induced sources. It plays a key role in the method developed in this work.
As before mentioned, the system modeling electromagnetic scattering involves integro-differential operators acting on vector fields with intertwined components, which requires a simultaneous treatment of all equations of the system. To decouple this system, we define the electromagnetic and nonmagnetic scattering potentials. This generalizes the concept of equivalent current sources, where the diffusion and the electromagnetic coupling are also considered. Using the introduced potentials, we derive a factorization of the electromagnetic and nonmagnetic scattering operators. As a consequence, we can recast the inverse electromagnetic scattering into a scalar inverse source problem for each component of the field.
Besides the decoupling of the underlying equations in the electromagnetic case, we use the described procedure to develop, in section 4 of the third chapter, an iterative scheme to solve the nonlinear inverse medium problem.

We begin first with the next lemma. The identities given there can be seen as a generalization of the Green's formulea to vector fields.

Lemma 2.4.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^3$ *with a smooth boundary* $\partial\Omega$ *and* $\mathbf{n}(x)$ *denote the unit vector normal to the boundary at* $x \in \partial\Omega$*. Then we have*

$$\int_\Omega \mathbf{v}(x) \cdot \mathbf{grad}\,\varphi(x)\,dx = -\int_\Omega \mathbf{div}\,\mathbf{v}(x)\,\varphi(x)\,dx + \int_{\partial\Omega} \mathbf{v}(x) \cdot \mathbf{n}(x)\,\varphi(x)\,d\sigma(x)$$

for $\mathbf{v} \in H\,(\mathbf{div}, \Omega)$, $\varphi \in H^1(\Omega)$ *and*

$$\int_\Omega \mathbf{u}(x) \cdot \mathbf{curl}\,\mathbf{v}(x)\,dx = \int_\Omega \mathbf{curl}\,\mathbf{u}(x) \cdot \mathbf{v}(x)\,dx + \int_{\partial\Omega} [\mathbf{u}(x) \wedge \mathbf{n}(x)] \cdot \mathbf{v}(x)\,d\sigma(x)$$

for $\mathbf{u} \in H(\mathbf{curl}, \Omega)$, $\mathbf{v} \in H^1(\Omega)^3$.

Proof. For the proof of this lemma we refer to [DL90b] p. 206 and p. 207. □

Let w, ε, μ be positive constants and

$$g_k(x,y) := g_k(|x-y|) = \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \; x \neq y, x \in \mathbb{R}^3,$$

with $k := w\sqrt{\varepsilon\mu}$.

Lemma 2.4.2. *If* Ω *is a bounded domain in* $\mathbb{R}^3$ *with a smooth boundary, then*

$$\int_\Omega \mathbf{curl}\,(g_k(x,y)\,\mathbf{f}(y))\,dy = 0 \qquad \textit{for} \qquad \mathbf{f} \in H_0^1(\Omega)^3, \tag{2.6}$$

$$\int_\Omega \mathbf{grad}\,(g_k(x,y)\,f(y))\,dy = 0 \qquad \textit{for} \qquad f \in H_0^1(\Omega). \tag{2.7}$$

Proof. Let $\mathbf{f} \in C_0^\infty(\Omega)^3$ whose support is contained in a compact set $K \subset \Omega$ with a smooth boundary ∂K.
Let $x \in \mathbb{R}^3, x \notin \partial K$, B_ε denote a ball, with center x and radius $\varepsilon > 0$, contained in K if $x \in K$ and $B_\varepsilon = \emptyset$ if $x \notin K$. Further, let $K_\varepsilon := K \setminus B_\varepsilon$ and $(\mathbf{e}_i)_{i=1,2,3}$ be the canonical basis in $\mathbb{R}^3$.
Let $\mathbf{n}(y)$ denote the outward unit vector normal to the boundary at $y \in \partial B_\varepsilon \cup \partial K$.
If we apply the second identity from lemma 2.4.1 to $\mathbf{u} = \mathbf{e}_i$, $i \in \{1,2,3\}$, and $\mathbf{v}(y) := g_k(x,y)\,\mathbf{f}(y) \in H^1(K_\varepsilon)$, we get

$$\int_{K_\varepsilon} \nabla_y \times (g_k(x,y)\,\mathbf{f}(y)) \cdot \mathbf{e}_i\,dy = \int_{\partial B_\varepsilon} g_k(x,y)\,\mathbf{f}(y) \cdot\, (\mathbf{e}_i \wedge \mathbf{n}(y))\,d\sigma(y),$$

since $\mathbf{e}_i$ is constant and

$$\int_{\partial K} g_k(x,y)\,\mathbf{f}(y) \cdot\, (\mathbf{e}_i \wedge \mathbf{n}(y))\,d\sigma(y) = 0$$

for $\mathbf{f} \in C_0^\infty(K)^3$.
From the inequalities

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|, \; |\mathbf{a} \wedge \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}| \quad \text{for} \quad \mathbf{a}, \mathbf{b} \in \mathbb{R}^3,$$

and

$$|g_k(x,y)| = (4\pi\varepsilon)^{-1}, \; y \in \partial B_\varepsilon,$$

we obtain the estimate

$$\left| \int_{\partial B_\varepsilon} g_k(x,y)\,[\,\mathbf{f}(y) \cdot\, (\mathbf{e}_i \wedge \mathbf{n}(y))\,]\,d\sigma(y) \right| \leq \|\mathbf{f}\|_\infty \int_{\partial B_\varepsilon} |(g_k(x,y)|\;d\sigma(y) = \varepsilon \|\mathbf{f}\|_\infty.$$

When ε tends to zero, we get

$$\int_K \mathbf{curl}\big(g_k(x,y)\,\mathbf{f}(y)\big)\cdot\mathbf{e}_i\,dy = 0, \quad \text{for} \quad i \in \{1,2,3\}. \tag{2.8}$$

The first formula has been proved for $\mathbf{f} \in C_0^\infty(\Omega)^3$. By a density argument it also holds for $\mathbf{f} \in H_0^1(\Omega)^3$.
To prove the second identity, we proceed similarly to above by applying the first formula from lemma 2.4.1 to $\mathbf{v} = \mathbf{e}_i$ and $\varphi(y) = g_k(x,y)\,f(y)$ for $f \in H_0^1(\Omega)$. □

We introduce now the electromagnetic and the nonmagnetic scattering potentials, which play a key role to perform the inversion of the electromagnetic scattering operators.

Definition 2.4.3. *The first electromagnetic scattering potential* $\mathcal{V}$ *is defined for* $\mathbf{f} = (\mathbf{f}_e, \mathbf{f}_m) \in H^2(\mathbb{R}^3)^3 \times H^2(\mathbb{R}^3)^3$ *by*

$$\mathcal{V}\,\mathbf{f} := (\mathcal{V}_e\,\mathbf{f}, \mathcal{V}_m\,\mathbf{f})$$

with

$$\mathcal{V}_e\,\mathbf{f} \;=\; k^2\,\mathbf{f}_e + \mathbf{grad}\,(\mathbf{div}\,\mathbf{f}_e) + i\,\omega\,\mu\,\mathbf{curl}\,\mathbf{f}_m, \tag{2.9}$$
$$\mathcal{V}_m\,\mathbf{f} \;=\; k^2\;\mathbf{f}_m + \mathbf{grad}\,(\mathbf{div}\,\mathbf{f}_m) - i\,\omega\;\varepsilon\,\mathbf{curl}\,\mathbf{f}_e. \tag{2.10}$$

The nonmagnetic scattering potential $\mathbf{P}$ *is defined for* $\mathbf{f} \in H^2(\mathbb{R}^3)^3$ *by*

$$\mathbf{P}\,\mathbf{f} = k^2\,\mathbf{f} + \mathbf{grad}\,(\mathbf{div}\,\mathbf{f}). \tag{2.11}$$

Remarks.

1. If $\mathbf{f}$ is an induced source we also call $\mathcal{V}\,\mathbf{f}$ and $\mathbf{P}\,\mathbf{f}$ (electromagnetic) generalized induced source and (nonmagnetic) generalized induced source, respectively.

2. The nonmagnetic potential involves two terms, the induced current source $k^2\,\mathbf{f}$, also present in the scalar Lippmann-Schwinger equation, and the diffusive term $\mathbf{grad}\,(\mathbf{div}\,\mathbf{f})$. If the material is magnetic we have to consider in the electromagnetic scattering potential a further term related to the coupling between the electric and magnetic fields.

Theorem 2.4.4. *Let* Ω *be a bounded domain in* $\mathbb{R}^3$ *with a smooth boundary. We have*
$T_e = \mathbf{A}\,\mathbf{P}$ *on* $H_0^2(\Omega)^3$,
$\mathcal{T}_{em} = \mathbf{A}\,\mathcal{V}$ *on* $H_0^2(\Omega)^3 \times H_0^2(\Omega)^3$.

Proof. Let $\mathbf{f} \in C_0^\infty(\Omega)^3$ and $x \in \mathbb{R}^3$.
If we use the the identity

$$\nabla_y\;(g_k(x,y)\,\varphi(y)) = (\nabla_y\,g_k(x,y))\;\varphi(y) + g_k(x,y)\,\nabla\varphi(y),\; y \neq x,$$

with $\varphi := \nabla \cdot \mathbf{f}$, we may write

$$\begin{aligned} \mathbf{K_d}\,\mathbf{f}(x) &:= \int_\Omega \nabla_y\, g_k(x,y)\,\nabla\cdot\mathbf{f}(y)\,dy \\ &= \int_\Omega \nabla_y\,\big(g_k(x,y)\,\nabla\cdot\mathbf{f}(y)\big)\,dy - \int_\Omega g_k(x,y)\,\nabla(\nabla\cdot\mathbf{f})(y)\,dy. \end{aligned}$$

By lemma 2.4.2, we see that the first term vanishes and we obtain

$$\mathbf{K_d}\,\mathbf{f}(x) = -\int_\Omega g_k(x,y)\,\mathbf{grad}\,\mathbf{div}\,\mathbf{f}(y)\,dy \tag{2.12}$$

for $x \in \mathbb{R}^3$.
Since $C_0^\infty(\Omega)^3$ is dense in $H_0^2(\Omega)^3$, the identity (2.12) remains valid for $\mathbf{f} \in H_0^2(\Omega)^3$.
On the other hand, from the identity

$$\nabla_y \times \big(g_k(x,y)\,\mathbf{f}(y)\big) = \nabla_y\, g_k(x,y) \times \mathbf{f}(y) + g_k(x,y)\,\nabla_y \times \mathbf{f}(y),\ x \neq y,$$

we get

$$\begin{aligned} \mathbf{K_r}\,\mathbf{f}(x) &= \int_\Omega \nabla_y\, g_k(x,y) \times \mathbf{f}(y)\,dy \\ &= \int_\Omega \nabla_y \times \big(g_k(x,y)\,\mathbf{f}(y)\big)\,dy - \int_\Omega g_k(x,y)\,\nabla_y \times \mathbf{f}(y)\,dy. \end{aligned}$$

The first term vanishes, as we can see from lemma 2.4.2, and by the density of $C_0^\infty(\Omega)^3$ in $H_0^2(\Omega)^3$ we obtain

$$\mathbf{K_r}\,\mathbf{f}\,(x) = -\int_\Omega g_k(x,y)\,\mathbf{curl}\,\mathbf{f}(y)\,dy,\ x \in \mathbb{R}^3, \tag{2.13}$$

for $\mathbf{f} \in H_0^2(\Omega)$.
If we now insert (2.12) and (2.13) into (1.66), (1.67) and (1.68), it yields

$$\mathbf{T_e}\,\mathbf{f}\,(x) = \int_\Omega g_k(x,y)\,\big[k^2\,\mathbf{f} + \mathbf{grad}\,(\mathbf{div}\,\mathbf{f})\big](y)\,dy = \mathbf{A}\,\mathbf{P}\,\mathbf{f}(x),\ \ x \in \mathbb{R}^3, \mathbf{f} \in H_0^2(\Omega)^3,$$

and

$$\begin{aligned} \mathcal{T}_e\,\mathbf{f}\,(x) &= \int_\Omega g_k(x,y)\,\big[k^2\,\mathbf{f_e} + \mathbf{grad}\,(\mathbf{div}\,\mathbf{f_e}) + i\,\omega\,\mu\,\mathbf{curl}\,\mathbf{f_m}\big](y)\,dy = \mathbf{A}\,\mathcal{V}_e\,\mathbf{f}(x), \\ \mathcal{T}_m\,\mathbf{f}\,(x) &= \int_\Omega g_k(x,y)\,\big[k^2\,\mathbf{f_m} + \mathbf{grad}\,(\mathbf{div}\,\mathbf{f_m}) - i\,\omega\,\varepsilon\,\mathbf{curl}\,\mathbf{f_e}\big](y)\,dy = \mathbf{A}\,\mathcal{V}_m\,\mathbf{f}(x), \end{aligned}$$

for $x \in \mathbb{R}^3$, $\mathbf{f} = (\mathbf{f_e}, \mathbf{f_m}) \in H_0^2(\Omega)^3 \times H_0^2(\Omega)^3$. □

The factorization given in the theorem above shows that the electromagnetic and nonmagnetic scattering operators can be seen as a scalar scattering operator acting on

each component of the electromagnetic and nonmagnetic scattering potentials. For the inversion, we have to determine the induced current source from the electromagnetic scattering potential. In other words to invert the operators $\mathcal{V}$ and $\mathbf{P}$.
In the following theorem, we show how to compute the divergence of the induced current source from the scattering potentials.

Theorem 2.4.5. *Let* $\mathbf{g} \in H_0^\alpha(\Omega)^3,\ \alpha \geq 1.$
If we have

$$\mathbf{P}\,\mathbf{f} = \mathbf{g} \quad \textit{with} \quad \mathbf{f} \in H_0^{\alpha+2}(\Omega)^3,$$

then

$$\mathbf{div}\,\mathbf{f}(x) = -\int_\Omega g_k(x,y)\,\mathbf{div}\,\mathbf{g}(y)\,dy,\ \ x \in \Omega. \tag{2.14}$$

Let $\mathbf{g} = (\mathbf{g}_\text{e}, \mathbf{g}_\text{m}) \in H_0^\alpha(\Omega)^3 \times H_0^\alpha(\Omega)^3,\ \alpha \geq 1.$
If we have

$$\mathcal{V}\,\mathbf{f} = \mathbf{g} \quad \textit{with} \quad \mathbf{f} = (\mathbf{f}_\text{e}, \mathbf{f}_\text{m}) \in H_0^{\alpha+2}(\Omega)^3 \times H_0^{\alpha+2}(\Omega)^3,$$

then

$$\mathbf{div}\,\mathbf{f}_\text{e}(x) \;=\; -\int_\Omega g_k(x,y)\,\mathbf{div}\,\mathbf{g}_\text{e}(y)\,dy, \tag{2.15}$$

$$\mathbf{div}\,\mathbf{f}_\text{m}(x) \;=\; -\int_\Omega g_k(x,y)\,\mathbf{div}\,\mathbf{g}_\text{m}(y)\,dy, \tag{2.16}$$

for $x \in \Omega$.

Proof. It is enough to prove (2.15) and (2.16), since (2.14) ensues by setting $\mathbf{f}_\mathbf{e} = \mathbf{f}$, $\mathbf{f_m} = 0, \mathbf{g_e} = \mathbf{g}$ and $\mathbf{g_m} = 0$.
Let $\alpha \geq 1,\ \mathbf{g} = (\mathbf{g}_e, \mathbf{g}_m) \in H^\alpha(\Omega)^3 \times H^\alpha(\Omega)^3$ and $\mathbf{f} = (\mathbf{f}_e, \mathbf{f}_m) \in H_0^{\alpha+2}(\Omega)^3 \times H_0^{\alpha+2}(\Omega)^3$ such that

$$\begin{cases} k^2\,\mathbf{f_e} + \mathbf{grad}\,(\mathbf{div}\,\mathbf{f_e}) + i\,\omega\,\mu\,\mathbf{curl}\,\mathbf{f_m} = \mathbf{g_e}, \\ k^2\,\mathbf{f_m} + \mathbf{grad}\,(\mathbf{div}\,\mathbf{f_m}) - i\,\omega\,\varepsilon\,\mathbf{curl}\,\mathbf{f_e} = \mathbf{g_m}. \end{cases}$$

If we apply the divergence operator to the equations above, we get

$$\begin{cases} (\Delta + k^2)\,\mathbf{div}\,\mathbf{f_e} = \mathbf{div}\,\mathbf{g_e}, \\ (\Delta + k^2)\,\mathbf{div}\,\mathbf{f_m} = \mathbf{div}\,\mathbf{g_m}. \end{cases}$$

since $\mathbf{div}\,\mathbf{curl} = 0$.
Let $x \in \Omega,\ B_\varepsilon$ denote a ball contained in Ω with center x and radius $\varepsilon > 0$.

Let $(f, g) \in \{(\mathbf{div}\,\mathbf{f_e}, \mathbf{div}\,\mathbf{g_e}), (\mathbf{div}\,\mathbf{f_m}, \mathbf{div}\,\mathbf{g_m})\}$. From the second Green's formula (D.1) we have

$$\int_{\Omega\setminus B_\varepsilon} g_k(x,y)\,(\Delta_y + k^2)\,f(y)\,dy = \int_{\Omega\setminus B_\varepsilon} (\Delta_y + k^2)\,g_k(x,y)\,f(y)\,dy$$
$$+ \int_{\partial\Omega} (g_k(x,y)(\nabla f\cdot\mathbf{n})(y) - f(y)(\nabla_y g_k\cdot\mathbf{n})(y))\,d\sigma(y)$$
$$- \int_{\partial B_\varepsilon} (g_k(x,y)(\nabla f\cdot\mathbf{n})(y) - f(y)(\nabla_y g_k\cdot\mathbf{n})(y))\,d\sigma(y),$$

where $\mathbf{n}(y)$ denotes the outward unit vector normal at y to the underlying boundary . The volume integral on the right hand side vanishes since g_k is a fundamental solution to the Helmholtz equation. The boundary integral on $\partial\Omega$ also vanishes since $f \in H_0^2(\Omega)$.
For $x \neq y$ we have

$$g_k(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|} \quad \text{and} \quad \nabla_y g_k(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}\left(\frac{1}{|x-y|} - ik\right)\frac{x-y}{|x-y|}.$$

As $\mathbf{n}(y) = \frac{x-y}{|x-y|}$ and $|x-y| = \varepsilon$ for $y \in B_\varepsilon$, we obtain after a straightforward computation

$$|\int_{\partial B_\varepsilon} (g_k(x,y)(\nabla f\cdot\mathbf{n})(y) - f(y)(\nabla_y g_k(x,y)\cdot\mathbf{n}(y)))\,d\sigma(y) - f(x)|$$
$$\leq \varepsilon(1+k)\,(||f||_{B_0,\infty} + ||\nabla f||_{B_0,\infty}) + |\frac{1}{4\pi\varepsilon^2}\int_{\partial B_\varepsilon} f(y)d\sigma(y) - f(x)|,$$

where B_0 is a ball in Ω containing $\overline{B_\varepsilon}$. It follows from the Sobolev embedding theorem, see [Ada75], that we can choose continuous representatives for f and ∇f. The maximum norms $||f||_{B_0,\infty}$ and $||\nabla f||_{B_0,\infty}$ are therefore finite. Passing to the limit when ε tends to zero, using the mean value theorem, we get

$$\lim_{\varepsilon\to 0}\int_{\partial B_\varepsilon} (g_k(x,y)(\nabla f\cdot\mathbf{n})(y) - f(y)(\nabla_y g_k(x,y)\cdot\mathbf{n}(y)))\,d\sigma(y) = f(x).$$

Hence, for $x \in \Omega$ we have

$$f(x) = -\int_\Omega g_k(x,y)\,g(y)\,dy.$$

□

For the inversion of the electromagnetic scattering potential we will make use of a second scattering potential.

Definition 2.4.6. *We define the second electromagnetic scattering potential* $\mathcal{W}$ *for* $\mathbf{f} = (\mathbf{f}_{\mathrm{e}}, \mathbf{f}_{\mathrm{m}}) \in H(\mathbf{curl}, \mathbb{R}^3) \times H(\mathbf{curl}, \mathbb{R}^3)$ *by*

$$\mathcal{W}\,\mathbf{f} = \mathbf{g},$$

where $\mathbf{g} := (\mathbf{g}_{\mathrm{e}}, \mathbf{g}_{\mathrm{m}})$ *is given by*

$$\begin{cases} \mathbf{g}_{\mathrm{e}} = \mathbf{f}_{\mathrm{e}} \,-\, \frac{i\,\omega\,\mu}{k^2}\,\mathrm{curl}\,\mathbf{f}_{\mathrm{m}}, \\ \mathbf{g}_{\mathrm{m}} = \mathbf{f}_{\mathrm{e}} \,+\, \frac{i\,\omega\,\varepsilon}{k^2}\,\mathrm{curl}\,\mathbf{f}_{\mathrm{e}}. \end{cases} \tag{2.17}$$

Lemma 2.4.7. *On* $H^{\alpha}(\mathbb{R}^3), \alpha \geq 3,$ *we have*

$$\mathcal{V}\,\mathcal{W} \,= \mathcal{W}\,\mathcal{V} = \boldsymbol{\Delta} + k^2.$$

Proof. Let $\mathbf{f} = (\mathbf{f_e}, \mathbf{f_m})$ and $\mathbf{g} = (\mathbf{g_e}, \mathbf{g_m})$ such that

$$\mathcal{V}\,\mathbf{f} = \mathbf{g}, \tag{2.18}$$

that is

$$\begin{cases} k^2\,\mathbf{f_e} + \mathbf{grad}\,(\mathbf{div}\,\mathbf{f_e}) + i\,\omega\,\mu\,\mathbf{curl}\,\mathbf{f_m} = \mathbf{g_e}, \\ k^2\,\mathbf{f_m} + \mathbf{grad}\,(\mathbf{div}\,\mathbf{f_m}) - i\,\omega\,\varepsilon\,\mathbf{curl}\,\mathbf{f_e} = \mathbf{g_m}. \end{cases} \tag{2.19}$$

If we apply the $\mathbf{curl}$ operator to the system (2.19), we get

$$\begin{cases} \mathbf{curl}\,\mathbf{f_e} = k^{-2}\left[\,\mathbf{curl}\,\mathbf{g_e} - i\,\omega\,\mu\,\mathbf{curl}\,(\mathbf{curl}\,\mathbf{f_m})\,\right], \\ \mathbf{curl}\,\mathbf{f_m} = k^{-2}\left[\,\mathbf{curl}\,\mathbf{g_m} + i\,\omega\,\varepsilon\,\mathbf{curl}\,(\mathbf{curl}\,\mathbf{f_e})\,\right], \end{cases} \tag{2.20}$$

since $\mathbf{curl}\,\mathbf{grad} = \mathbf{0}$.
From the identity

$$\mathbf{curl}\,(\mathbf{curl}) = -\boldsymbol{\Delta} + \mathbf{grad}\,(\mathbf{div}) \tag{2.21}$$

and as $k^2 = \omega^2\mu\,\varepsilon$, it follows when we insert $\mathbf{curl}\,\mathbf{f_e}$ and $\mathbf{curl}\,\mathbf{f_m}$ from (2.20) into (2.19) that

$$\begin{cases} (\boldsymbol{\Delta} + k^2\,)\,\mathbf{f_e} = \mathbf{g_e} - \frac{i\,\omega\,\mu}{k^2}\mathbf{curl}\,\mathbf{g_m}, \\ (\boldsymbol{\Delta} + k^2)\,\mathbf{f_m} = \mathbf{g_m} + \frac{i\,\omega\,\varepsilon}{k^2}\mathbf{curl}\,\mathbf{g_e}. \end{cases} \tag{2.22}$$

This shows that

$$\mathcal{W}\,\mathcal{V} = (\boldsymbol{\Delta} + k^2). \tag{2.23}$$

Inversely, let $\mathbf{f} = (\mathbf{f_e}, \mathbf{f_m})$ and $\mathbf{g} = (\mathbf{g_e}, \mathbf{g_m})$ such that

$$\mathcal{W}\,\mathbf{g} = \mathbf{f}, \tag{2.24}$$

that is

$$\begin{cases} \mathbf{f_e} = \mathbf{g_e} - \frac{i\,\omega\,\mu}{k^2}\,\mathbf{curl}\,\mathbf{g_m}, \\ \mathbf{f_m} = \mathbf{g_m} + \frac{i\,\omega\,\varepsilon}{k^2}\,\mathbf{curl}\,\mathbf{g_e}. \end{cases} \tag{2.25}$$

To the system (2.25) we apply on the one hand, the $\mathbf{div}$ operator

$$\begin{cases} \mathbf{div}\,\mathbf{f_e} = \mathbf{div}\,\mathbf{g_e}, \\ \mathbf{div}\,\mathbf{f_m} = \mathbf{div}\,\mathbf{g_m}, \end{cases} \tag{2.26}$$

on the other hand, the **curl** operator

$$\begin{cases}\mathbf{curl}\,\mathbf{f_e} = \mathbf{curl}\,\mathbf{g_e} - \frac{i\,\omega\,\mu}{k^2}\,\mathbf{curl}\,(\mathbf{curl}\,\mathbf{g_m}),\\ \mathbf{curl}\,\mathbf{f_m} = \mathbf{curl}\,\mathbf{g_m} + \frac{i\,\omega\,\varepsilon}{k^2}\,\mathbf{curl}\,(\mathbf{curl}\,\mathbf{g_e}).\end{cases} \tag{2.27}$$

We further make use of the identity (2.21) and insert $\mathbf{div}\,\mathbf{g_e}$, $\mathbf{div}\,\mathbf{g_m}$ from (2.26) and $\mathbf{curl}\,\mathbf{g_e}$, $\mathbf{curl}\,\mathbf{g_m}$ from (2.27) into (2.25) to get

$$\begin{cases}\mathbf{f_e} = \mathbf{g_e} - \frac{i\,\omega\,\mu}{k^2}\left[\,\mathbf{curl}\,\mathbf{f_m} - \frac{i\,\omega\,\varepsilon}{k^2}\left(-\boldsymbol{\Delta}\,\mathbf{g_e} + \mathbf{grad}\,(\mathbf{div}\,\mathbf{f_e})\right)\right],\\ \mathbf{f_m} = \mathbf{g_m} + \frac{i\,\omega\,\mu}{k^2}\left[\,\mathbf{curl}\,\mathbf{f_e} + \frac{i\,\omega\,\varepsilon}{k^2}\left(-\boldsymbol{\Delta}\,\mathbf{g_m} + \mathbf{grad}\,(\mathbf{div}\,\mathbf{f_m})\right)\right].\end{cases} \tag{2.28}$$

It yields

$$\begin{cases}k^2\,\mathbf{f_e} + \mathbf{grad}\,(\mathbf{div}\,\mathbf{f})_\mathbf{e} + i\,\omega\,\mu\,\mathbf{curl}\,\mathbf{f_m} = (\boldsymbol{\Delta} + k^2)\,\mathbf{g_e},\\ k^2\,\mathbf{f_m} + \mathbf{grad}\,(\mathbf{div})\,\mathbf{f_m} - i\,\omega\,\varepsilon\,\mathbf{curl}\,\mathbf{f_e} = (\boldsymbol{\Delta} + k^2)\,\mathbf{g_m}.\end{cases} \tag{2.29}$$

Hence,

$$\mathcal{V}\,\mathcal{W} = \boldsymbol{\Delta} + k^2.$$

□

Theorem 2.4.8. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded domain with a smooth boundary,* $\mathbf{M}_\Gamma \in \mathcal{L}(H^1(\Omega)^6, L_2(\Gamma)^6)$ *be a measurement operator with* $\Gamma \subset \Omega \setminus \mathbb{R}^3$.
For $\mathbf{d} \in L_2(\Gamma)^6$, *let* $\mathbf{q} \in H_0^2(\Omega)^6$ *be a solution to*

$$\mathbf{A}_\Gamma \mathbf{q} = \mathbf{d} \tag{2.30}$$

with $\mathbf{A}_\Gamma := \mathbf{M}_\Gamma \mathbf{A}$.
Let $\mathbf{f} \in H_0^2(\Omega)^6$ *such that* $\mathbf{f} = -\mathcal{W}\mathbf{A}\mathbf{q}$ *on* Ω. *Then* $\mathbf{f}$ *is a solution to*

$$\mathbf{M}_\Gamma \mathcal{T}_{em} \mathbf{f} = \mathbf{d}.$$

Proof. For $\mathbf{d} \in L_2(\Gamma)^6$, let $\mathbf{q} \in H_0^2(\Omega)^6$ be solution to (2.30) and $\mathbf{f} := -\mathcal{W}\mathbf{A}\mathbf{q}$. Using theorem 2.4.4 we get

$$\mathbf{M}_\Gamma \mathcal{T}_{em} \mathbf{f} = \mathbf{M}_\Gamma \mathbf{A} \mathcal{V} \mathbf{f} = \mathbf{A}_\Gamma \mathcal{V} \mathbf{f} = \mathbf{A}_\Gamma \mathbf{q} = \mathbf{d},$$

since it holds

$$\mathcal{V}\mathcal{W}\mathbf{A}\mathbf{q} = -\mathbf{q} \tag{2.31}$$

by virtue of lemma 2.4.7 and lemma 1.2.3 applied componentwise.
The proof of the theorem is then finished □

The previous theorem sketches already the steps we are going to follow to solve the inverse source and inverse medium problems in the electromagnetic case:

1. For each experiment, solve the equation (2.30) componentwise as a scalar inverse source problem to recover the first scattering potential of the induced source.

2. For each experiment, determine the equivalent source from the generalized induced source.

3. Determine the contrast functions from the recovered equivalent sources for multiple experiments.

In the first step, the problem is scalar, linear and ill-posed, in the second step it is vector, linear and well-posed, in the last step it is nonlinear.
In the procedure described above, we made no specification of the measurement operator, which means that it can cover a wide range of applications. In the next section, we specify the geometrical configuration for the measurements as being spherical.

2.5 Spherical scattering operator

As a target region we take now the ball $\Omega = B_\rho$ centered at the origin, with radius $\rho > 0$. We suppose that the measurements are taken on the sphere $\Gamma = \partial B_R$ centered at the origin, with radius $R \geq \rho$. If the measurement operator M_Γ is the trace mapping γ_0 on the boundary Γ, see appendix D, and $A = A_\Omega$ is the scalar scattering operator, then for $T = k^2 A_\Omega$, the restricted scattering operator $T_\Gamma = M_\Gamma T$ is reduced, up to the factor k^2, to the spherical scattering operator defined in the following.

Definition 2.5.1. *Let $k > 0$ and $0 < \rho \leq R$. The spherical scattering operator $A_{\rho,R} \in \mathcal{L}(L_2(B_\rho), L_2(\partial B_R))$, also denoted A, is defined for $\phi \in L_2(B_\rho)$ by*

$$A_{\rho,R}\phi(x) = \int_{B_\rho} g_k(x,y)\phi(y)dy,\ x \in \partial B_R,$$

where

$$g_k(x,y) := g_k(|x-y|) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|},\ x \neq y,\ x,y \in \mathbb{R}^3.$$

The adjoint operator $A^* \in \mathcal{L}(L_2(\partial B_R), L_2(B_\rho))$ of A is then given by

$$A^*_{\rho,R}\psi(x) = \int_{\partial B_R} \overline{g_k(|x-y|)}\psi(y)dy,\ x \in B_\rho,\ \psi \in L_2(\partial B_R).$$

Remark. To avoid confusion we denote the scalar scattering operator given in definition 1.4.3 by A_Ω and the spherical scattering operator by A.
For the notations used in the sequel we refer to the appendices A and B.

Abdullah and Louis already investigated in [AL99] the spherical scattering operator for acoustic scattering in 2-D, where the kernel g_k is given by the Hankel function of the first kind

$$g_k(x,y) = \frac{i}{4}H_0(k|x-y|),\ x \neq y,\ x,y \in \mathbb{R}^2.$$

The singular value decomposition is derived there from the addition theorem for the Hankel function. Further, based on the Lommel theorem for Bessel functions, Wallacher[Wal02]

determined an orthonormal basis for the null space, This provides a complete setting in 2-D to determine the radiating and the nonradiating components of the equivalent sources.
In this section we analyze the spherical scattering operator in 3-D.

2.5.1 Singular value decomposition

The singular value decomposition is a powerful tool to analyse compact operators. It gives much insight into the ill-posedness character of the inverse problem and can be used to derive regularization algorithms, see [Lou89]. In this section, we determine a singular value decomposition for the spherical scattering operator and investigate the behavior of the singular values.

Theorem 2.5.2. *The spherical scattering operator* $A = A_{\rho,R}$ *is compact and admits the singular value system* $\{v_n^m,\, u_n^m;\, \sigma_n\}_{n\in\mathbb{N},\, m=-n,\dots,n.}$ *given by*

$$\begin{aligned} \sigma_n &= kR\,|h_n^{(1)}(kR)|\,\alpha_n, \\ v_n^m(r\omega) &= \gamma_n j_n(kr)\,Y_n^m(\omega) \quad \textit{for} \quad r>0, w\in S^2, \\ u_n^m(R\theta) &= R^{-1}\,Y_n^m(\theta) \quad \textit{for} \quad \theta\in S^2, \end{aligned}$$

where

$$\begin{aligned} \alpha_n &= \alpha_n(\rho) := \|\, j(k.)\,\|_{L_2((0,\rho),\,r^2dr)} = (\int_0^\rho j_n(kr)^2\,r^2 dr\,)^{\frac{1}{2}}, \\ \beta_n &= \beta_n(R) = \begin{cases} \frac{\overline{h_n^{(1)}(kR)}}{|h_n^{(1)}(kR)|} & \textit{if} \quad h_n^{(1)}(kR)\neq 0 \\ 1 & \textit{if} \quad h_n^{(1)}(kR) = 0 \end{cases}, \\ \gamma_n &= -i\,\frac{\beta_n}{\alpha_n}. \end{aligned}$$

Before proving theorem 2.5.2, we give the following lemma about the asymptotic behavior of the singular values for large order.

Lemma 2.5.3.

$$\sigma_n = \mathcal{O}\left(n^{-\frac{3}{2}}\,(\frac{\rho}{R})^n\right),\ n\to\infty.$$

Proof of lemma 2.5.3. From the series representation of the spherical Bessel and Neumann functions (B.7) and (B.8), it holds

$$j_n(r) = \frac{2^n\,n!}{(2n+1)!}\,r^n\left(1+\mathcal{O}(\frac{1}{n})\right),\ n\to\infty, \tag{2.32}$$

uniformly on compact subsets of $[\,0,\infty)$ and

$$h_n(r) = -i\frac{(2n-1)!}{2^n\,n!}\,r^{-(n+1)}\left(1+\mathcal{O}(\frac{1}{n})\right),\ n\to\infty, \tag{2.33}$$

uniformly on compact subsets of $(0, \infty)$.
It yields after a straightforward computation

$$\sigma_n := kR \left| h_n^{(1)}(kR) \right| \Big(\int_0^\rho (j_n(kr))^2 r^2 d \Big)^{\frac{1}{2}} = \mathcal{O}\left(n^{-\frac{3}{2}} (\frac{\rho}{R})^n \right), \; n \to \infty.$$

□

Proof of theorem 2.5.2. Let $I := \{(n,m), n \in \mathbb{N}, m = -n, \dots, n\}$. Since the spherical harmonics $\{Y_n^m\}_{(n,m)\in I}$ form a complete orthonormal system in $L_2(S^2)$, the system $\{u_n^m\}_{(n,m)\in I}$ is also complete and orthonormal in $L_2(\partial B_R)$ and the system $\{v_n^m\}_{(n,m)\in I}$ is orthonormal in $L_2(B_\rho)$.
We begin first with proving

$$A^* u_n^m = \sigma_n v_n^m \quad for \quad (n,m) \in I. \tag{2.34}$$

Let $(n,m) \in I$ and $x \in B_\rho$ with polar coordinates $r := |x| \neq 0, \; \omega := \frac{x}{|x|} \in S^2$. We have

$$\begin{aligned} [\,A^* u_n^m\,](x) &= \int_{RS^2} \overline{g_k(|x-y|)}\, u_n^m(y)\, d\sigma_R(y) \\ &= R \int_{S2} \overline{g_k\left((r^2 + R^2 - 2rR < \omega, \theta >)^{\frac{1}{2}}\right)}\, Y_n^m(\theta)\, d\theta, \end{aligned}$$

where we made the substitution $y = R\theta$.
Applying the Funk-Hecke theorem A.0.1 we get,

$$[\,A^* u_n^m\,](r\omega) = R I_n^R(r)\, Y_n^m(\omega)$$

with

$$I_n^R(r) := 2\pi \int_{-1}^{+1} \overline{g_k\left((r^2 + R^2 - 2rRt)^{\frac{1}{2}}\right)}\, P_n(t)\, dt,$$

where P_n is the Legendre polynomial.
Noticing that

$$g_k(t) = \frac{1}{4\pi} \frac{e^{ikt}}{t} = \frac{1}{4\pi} (\frac{\cos kt}{t} + i \frac{\sin kt}{t}) = \frac{ik}{4\pi} h_0^{(1)}(kt),$$

we may write

$$\overline{I_n^R(r)} = \frac{ik}{2} (\frac{\pi}{2})^{\frac{1}{2}} \int_0^\pi \frac{H_{\frac{1}{2}}^{(1)}\left(\left((kr)^2 + (kR)^2 - 2k^2 rR\cos\theta\right)^{\frac{1}{2}}\right)}{\left((kr)^2 + (kR)^2 - 2k^2 rR\cos\theta\right)^{\frac{1}{2}}} P_n(\cos\theta) \sin\theta \, d\theta,$$

which is the Gegenbauer integral [1].
It yields

$$\begin{aligned}\overline{I_n^R(r)} &= ik\,(\frac{\pi}{2}) \quad \frac{H^{(1)}_{n+\frac{1}{2}}(kR)}{(kR)^{\frac{1}{2}}} \quad \frac{J_{n+\frac{1}{2}}(kr)}{(kr)^{\frac{1}{2}}} \\ &= ik \quad h_n^{(1)}(kR) \quad j_n(kr).\end{aligned}$$

Finally, we get

$$[\,A^* u_n^m\,]\,(r\omega) = -ik\,R\,\overline{h_n^{(1)}(kR)}\,j_n(kr)\,Y_n^m(\omega) = \,\sigma_n v_n^m(r\omega).$$

Analogously, if we substitute Y_n^m by $\overline{Y_n^m}$ we get

$$\overline{A^*\,\overline{u_n^m}(r\omega)} = ikR\,h_n^{(1)}(kR)\,j_n(kr)\,Y_n^m(\omega). \tag{2.35}$$

We show now

$$Av_n^m = \,\sigma_n u_n^m.$$

Let $\theta \in S^2$. We have

$$\begin{aligned}Av_n^m(R\theta) &= \int_0^\rho \int_{S^2} g_k\,((r^2 + R^2 - 2rR < \omega, \theta >)^{\frac{1}{2}})\,v_n^m(r\omega)\,r^2\,drd\omega \\ &= \gamma_n \int_0^\rho j_n(kr)\,(\int_{S^2} g_k\,((r^2 + R^2 - 2rR < \omega, \theta >)^{\frac{1}{2}})\,Y_n^m(\omega)\,d\omega\,)\,r^2 dr \\ &= R^{-1}\gamma_n \int_0^\rho j_n(kr)\,\overline{A^*\,\overline{u_n^m}\,(r\theta)}\,r^2 dr.\end{aligned}$$

From (2.35), we obtain

$$Av_n^m = \sigma_n u_n^m.$$

Hence, the selfadjoint operator AA^* has the eigenvalues $\sigma_n^2,\ n \in \mathbb{N},$ and the corresponding eigenfunctions $u_n^m,\ (n, m) \in I,$ form a complete orthonormal system in $L_2(\partial B_R)$ and *a fortiori* in $\overline{\mathcal{R}(A)}$. The functions $v_n^m = A^* u_n^m,\ (n, m) \in I,$ build up a complete orthonormal system of $\overline{\mathcal{R}(A^*)}$. It remains to show that A is compact.
If $\rho < R,$ the kernel g_k is free from singularity and therefore is square integrable on $\partial B_R \times B_\rho$. The operator A is in this case a Hilbert-Schmidt operator [2]. Thus, it is compact.
If $\rho = R,$ the kernel g_k is not square integrable on $\partial B_R \times B_R$ and we have to argument differently. One proves first that A^* is compact, where a sequence $(A_N^*)_{N\in\mathbb{N}}$ of degenarte operators, *i.e.* $\dim \mathcal{R}(A_N^*) < \infty,$ is constructed to converge to A^* in the operator norm. Although this procedure is quite classical, we give it in details for the

[1] See Watson (17) p. 367 for $\nu = \frac{1}{2}$ with the notation used there.
[2] See [DL90b] p. 30

sake of clarity.
For every $f \in L_2(RS^2)$ we have the uniquely determined expansion

$$f(R\omega) = \sum_{(n,m)\in I} f_n^m \, Y_n^m(\omega), \quad \omega \in S^2,$$

where the sequence $\{f_n^m\}_I$ is square-summable.
For $N \in \mathbb{N}$, we denote by f_N the truncated expansion of f given by

$$f_N(R\omega) = \sum_{(n,m)\in I, n\leq N} f_n^m \, Y_n^m(\omega), \quad \omega \in S^2,$$

and define the operator A_N^* by

$$A_N^* f := A^* f_N, \quad f \in L_2(\partial B_R).$$

Let $\varepsilon > 0,\ N \in \mathbb{N}$. Using the Cauchy-Schwartz inequality we have

$$\begin{aligned} \|(A^* - A_N^*)f\|^2_{L_2(B_\rho)} &= \|\sum_{n=N+1}^{\infty} \sum_{m=-n}^{+n} f_n^m \, A^* \, Y_n^m\|^2_{L_2(B_\rho)} \\ &\leq (\sum_{n=N+1}^{\infty} \sum_{m=-n}^{+n} |f_n^m|^2) \sum_{n=N+1}^{\infty} \sum_{m=-n}^{+n} \|A^* \, Y_n^m\|^2_{L_2(B_\rho)}. \end{aligned}$$

From (2.34) it follows

$$\|(A^* - A_N^*)f\|^2_{L_2(\Omega)} \quad \leq \quad R^2 \, \|f\|^2_{L_2(\partial B_R)} (\sum_{n=N+1}^{\infty} (2n+1)\sigma_n^2)$$

for any $f \in L_2(RS^2)$.
We suppose that N is large enough, from lemma 2.5.3 it holds

$$\sigma_n \leq C \, (\frac{\rho}{R})^n \, n^{-\frac{3}{2}}, \quad n \geq N, \tag{2.36}$$

with some positive constant C. The convergence of the series

$$\sum_{n\in\mathbb{N}} (2n+1)\sigma_n^2$$

yields

$$\|(A^* - A_N^*)f\|_{L_2(B_\rho)} \quad \leq \quad \varepsilon \|f\|_{L_2(RS^2)} \tag{2.37}$$

for any $f \in L_2(RS^2)$.
Hence,

$$\lim_{N\to\infty} \|(A^* - A_N^*)\|_{\mathcal{L}(L_2(\partial B_R),\, L_2(B_\rho))} = 0\,. \tag{2.38}$$

The adjoint operator A^* is then compact. It follows from the theorem of Schauder that the operator A is compact as well. □

Remark. Since the range $\mathcal{R}(A)$ of the compact operator A is not closed in $L_2(\partial B_R)$, the Moore-Penrose or pseudo-inverse operator $A^\dagger$ defined for $g \in \mathcal{D}^\dagger(A) := \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp$ by

$$A^\dagger g = \sum_{(n,m)\in I} \sigma_n^{-1} < g, u_n^m > v_n^m$$

is not bounded. Hence, the problem

$$Af = g$$

is ill-posed and regularization techniques are necessary for the inversion, see [Lou89],[Nat01]. This issue will be further discussed in chapter 3.

In theorem 2.5.2, the coefficients α_n are given as integrals of Bessel functions. To avoid numerical integration of highly oscillating integrands we rather use the first formula in the following lemma.

Lemma 2.5.4. *We have*

$$\alpha_n^2 = \frac{\rho^3}{2}\left(j_n(k\rho)^2 - j_{n+1}(k\rho)j_{n-1}(k\rho)\right). \tag{2.39}$$

Furthermore, if $k\rho = \vartheta\nu$ *with* $\nu = (n+\frac{1}{2})$ *and* $0 < \vartheta < 1$, *then it holds*

$$\alpha_n^2 \le \frac{\rho^2}{4k}\vartheta^{-2}(1+\vartheta^2)^{\frac{1}{2}}\nu^{-1}e^{-(\frac{2\nu}{3})(1-\vartheta^2)^{\frac{3}{2}}}. \tag{2.40}$$

Proof. From [Luk62] p. 255, we have the identities

$$\begin{aligned}\int_0^\rho C_\nu(r)^2 r dr &= \frac{\rho^2}{2}\left(C_\nu(k\rho)^2 - C_{\nu+1}(k\rho)C_{\nu-1}(k\rho)\right), &(2.41)\\ &= \frac{\rho^2}{2}\left((1-(\frac{\nu}{k\rho})^2)C_\nu(k\rho)^2 - C_\nu^{'}(k\rho)\right), &(2.42)\end{aligned}$$

where C_ν is a cylindrical function of order ν and $C_\nu^{'}$ is its derivative.
The identity (2.39) follows immediately from (2.41) and (B.2).
Let $\nu = (n+\frac{1}{2}), n$ integer, such that $k\rho \in (\nu-1,\nu)$ and let $\vartheta = k\rho\nu^{-1}$. From lemma (B.0.2), we have

$$\begin{aligned}0 &\le J_\nu(\nu\vartheta) \le (2\pi\nu)^{-\frac{1}{2}}(1-\vartheta^2)^{-\frac{1}{4}}e^{-(\frac{\nu}{3})(1-\vartheta^2)^{\frac{3}{2}}},\\ 0 &\le \vartheta J_\nu'(\nu\vartheta) \le (2\pi\nu)^{-\frac{1}{2}}(1+\vartheta^2)^{\frac{1}{4}}e^{-(\frac{\nu}{3})(1-\vartheta^2)^{\frac{3}{2}}}.\end{aligned}$$

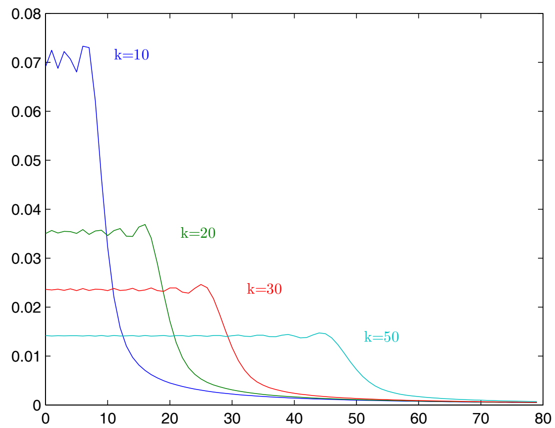

Figure 2.1: Singular values of of $A_{R,\rho}$ for $R = \rho = 1,\ k \in \{10, 20, 30, 50\}$.

Thus, using 2.42 we get

$$\begin{aligned}
\alpha_n^2 &= \frac{\pi}{2k}\int_0^\rho (J_\nu(kr))^2\, rdr \\
&\leq \frac{\rho^2}{8k}\left(\vartheta^{-2}(1-\vartheta^2)^{\frac{1}{2}} + \vartheta^{-2}(1+\vartheta^2)^{\frac{1}{2}}\right)\nu^{-1}e^{-(\frac{2\nu}{3})(1-\vartheta^2)^{\frac{3}{2}}} \\
&\leq \frac{\rho^2}{8k}\vartheta^{-2}(1+\vartheta^2)^{\frac{1}{2}}\left(1+(\frac{1-\vartheta^2}{1+\vartheta^2})^{\frac{1}{2}}\right)\nu^{-1}e^{-(\frac{2\nu}{3})(1-\vartheta^2)^{\frac{3}{2}}} \\
&\leq \frac{\rho^2}{4k}\vartheta^{-2}(1+\vartheta^2)^{\frac{1}{2}}\nu^{-1}e^{-(\frac{2\nu}{3})(1-\vartheta^2)^{\frac{3}{2}}}.
\end{aligned}$$

□

Remark. Asymptotically, the singular values of the spherical scattering operator A behave like $\left(n^{-\frac{3}{2}}\left(\frac{\rho}{R}\right)^n\right)$, $n \to \infty$, as we can see from lemma 2.5.3. Nevertheless, from the estimate in lemma 2.40 we see that there exists a transition region about $k\rho$, where the singular values decay exponentially, see figures (2.1), and (2.2). In other words, the singular values of order larger than $k\rho$ can be neglected in comparaison with those of order smaller than $k\rho$. The spherical scattering operator acts radially as a low-pass filter damping all oscillations related to Bessel functions of order larger than $k\rho$.

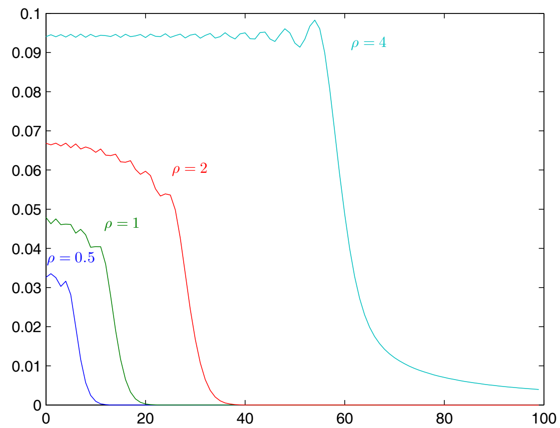

Figure 2.2: Singular values of $A_{R,\rho}$ for $k = 15, R = 4, \rho \in \{0.5, 1, 2, 4\}$.

2.5.2 Null space

The characterization of non-radiating sources is important for solving the inverse source problem. We aim in this subsection to construct a complete orthonormal basis for the null space of the spherical scattering operator in the three-dimensional case.
Within the framework of a boundary value problem for the Helmholtz equation, Devaney *et al* [DM98] constructed in 3-D a basis for non-radiating solutions. It is, however, nonorthogonal. In 2-D, Wallacher derived in [Wal02] an orthonormal basis and showed its completeness using the Lommel theorem for Bessel functions.
A powerful tool to derive orthonormal bases is to formulate the problem as a spectral decomposition of a selfadjoint compact operator.
We start with the next lemma, which states that an eigenmode of the Laplace operator satisfying some orthogonality relation on the boundary, is either an element of the null space $\mathcal{N}(A)$ or an element of $\overline{\mathcal{R}(A^*)}$.

Lemma 2.5.5. *For $\lambda > 0,\ \lambda \neq k,$ let $\phi \in C^2(\overline{B_\rho})$ such that*

$$\Delta\phi = -\lambda^2\,\phi \quad \textit{in} \quad B_\rho.$$

Then

$$\phi \in \mathcal{N}(A) \quad \textit{iff} \quad \left[\int_{\partial B_\rho} \partial_n\phi\,\overline{v} - \phi\,\overline{\partial_n\, v}\,d\sigma = 0 \quad \textit{for every} \quad v \in \mathcal{R}\,(A^*)\right], \quad (2.43)$$

where ∂_n denotes the outward normal derivative to the boundary ∂B_ρ.

Proof. Since $\mathcal{N}(A) = \mathcal{R}(A^*)^\perp,$ we have

$$\phi \in \mathcal{N}(A) \quad \text{iff} \quad \Big[\int_{B_\rho} \phi(x)\,\overline{v(x)}\,dx \,=0 \quad \text{for every} \quad v \in \mathcal{R}\,(A^*) \Big].$$

From theorem B.0.3 it follows for $v \in \mathcal{R}\,(A^*)$ that

$$v \in C^2(\,\overline{B_\rho}\,) \quad \text{and} \quad (\Delta + k^2)v = 0.$$

Hence, the required result follows immediately from the second Green's formula

$$\begin{aligned} -(\lambda^2 - k^2) \int_{B_\rho} \phi(x)\,\overline{v(x)}\,dx &= \int_{B_\rho} \Delta\,\phi(x)\,\overline{v(x)}\, - \phi(x)\,\Delta\overline{v(x)}\,dx \\ &= \int_{\partial B_\rho} \partial_n \phi(x)\,\overline{v(x)}\, - \,\phi(x)\,\partial_n \overline{v(x)}\,d\sigma(x). \end{aligned}$$

□

We take now advantage of the spherical symmetry of the problem to make use of the method of separation of variables, which leads to the spherical Bessel differential operator, see appendix B.

Definition 2.5.6. *For $n \in \mathbb{N},$ let $\mathcal{L}_n$ denote the spherical Bessel differential operator defined by*

$$\mathcal{L}_n f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} - \frac{n(n+1)}{r^2} f \quad for \quad f \in C^{\,2}(0, \infty).$$

Lemma 2.5.7. *Let $n \in \mathbb{N},\ \alpha_n := j_n(k\rho),\ \beta_n := -kj_n^{'}(k\rho),$ with j_n the spherical Bessel function of order n.*
Let $\lambda > 0,\ \lambda \neq k$. If $f = f_{n,\,\lambda} \in C^{\,2}(\,\overline{B_\rho}\,)$ is a solution to

$$\begin{cases} \mathcal{L}_n f = -\lambda^2 f, \\ \alpha_n\, f^{'}(\rho) \,+\, \beta_n\, f(\rho) = 0 \end{cases}, \tag{2.44}$$

then for $m \in \{-n, \ldots, n,\}$, the function

$$\phi^m_{n,\,\lambda}(x) := f_{n,\,\lambda}(\,|x|\,)\,Y^m_n(\,\frac{x}{|x|}\,),\ x \in \overline{B}_\rho \setminus \{0\}, \tag{2.45}$$

satisfies

$$\phi^m_{n,\,\lambda} \in \mathcal{N}(A).$$

Proof. If $\phi^m_{n,\,\lambda}, m \in \{-n, \cdots n,\}, n \in \mathbb{N},$ is given by (2.45), then we have $\phi^m_{n,\,\lambda} \in C^{\,2}(\,\overline{B_\rho}\,)$ and the Laplace operator in polar coordinates reads

$$\Delta\phi^m_{n,\,\lambda}\,(x) = \Big[\frac{\partial^2}{\partial r^2} \,+\, \frac{2}{r}\frac{\partial}{\partial r} \,+\, \frac{1}{r^2}\Delta_B\Big]\,f(r)\,Y^m_n(\xi)$$

for $x \in B_\rho \setminus \{0\}$, $r = |x|$ and $\xi = \frac{x}{|x|}$, where Δ_B is the Laplace-Beltrami operator. From (A.7) we obtain

$$\Delta\, \phi^m_{n,\,\lambda} = -\lambda^2\, \phi^m_{n,\,\lambda}.$$

Let $l \in \mathbb{N}$, $p \in \{-l, \ldots, l\}$, with the notations of theorem 2.5.2 we check for

$$v^p_l(x) = \gamma_n j_n(\, k\, |x|\,)\, Y_n(\frac{x}{|x|}), \quad x \in \overline{B}_\rho \setminus \{0\},$$

the orthogonality relation (2.43)

$$\int\limits_{\partial B_\rho} \left[\, \partial_n\, \phi^m_{n,\,\lambda}\,(x)\, \overline{v^p_l(x)} \,-\, \phi^m_{n,\,\lambda}\,(x)\, \partial_n\, \overline{v^p_l(x)} \right] d\sigma(x)$$

$$= \gamma_n\, [\, \alpha_n\, f^{'}_{n,\,\lambda}(\rho) \,+\, \beta_n\, f_{n,\,\lambda}(\rho)\,] \int\limits_{S^2} Y^m_n(\xi)\, \overline{Y^p_l(\xi)}\, d\xi$$

$$= \gamma_n\, \delta_{nl}\, \delta_{mp}\, [\, \alpha_n\, f^{'}_{n,\,\lambda}(\rho) \,+\, \beta_n\, f_{n,\,\lambda}(\rho)\,] = 0,$$

where δ denotes the Kronecker symbol.
Since v^p_l, $l \in \mathbb{N}$, $p = -l, \ldots, l$, form a complete orthonormal system in $\overline{\mathcal{R}\,(A^*)}$, it holds

$$\int\limits_{\partial B_\rho} \left[\, \partial_n\, \phi^m_{n,\,\lambda}\,(x)\, \overline{v(x)} \,-\, \phi^m_{n,\,\lambda}\,(x)\, \overline{\partial_n\, v(x)} \right] d\sigma(x) = 0$$

for every $v \in \overline{\mathcal{R}(A^*)}$.
From the previous lemma we get

$$\phi^m_{n,\,\lambda} \in \mathcal{N}(A) \quad for \quad n \in \mathbb{N},\, m = -n, \ldots, n.$$

□

Remark. Lemma 2.5.7 reduces the determination of a basis for the null space $\mathcal{N}(A)$ to a Sturm-Liouville problem for the differential operator $\mathcal{L}_n$, which is unfortunately singular at zero. The integration of the differential equation for a degenerate Sturm-Liouville problems does not yield a Hilbert-Schmidt integral operator since, due to the singularity, the corresponding Green's function is not square integrable on the interval $(0, \rho)$, see remarks 16 and 17 p. 51 in [DL90b]. We use the procedure described therein to circumvent this difficulty, where we consider the variational formulation, then apply theorem C.0.4. Nevertheless, if we take the whole interval $[0, \rho]$, we have to investigate the trace properties of the involved Sobolev-spaces, , see [DL90b] p.50. To avoid this task, we proceed indirectly. We first isolate the singularity by considering the intervals $[\varepsilon, \rho], \varepsilon > 0$, then make ε tend to zero. For the easiness of the reader, we collect some useful results from [DL90b] in the appendix C.

Without lost of generality we set $\rho = 1$. Furthermore, an appropriate substitution, see [GR00] 8.49 p. 921, reduces the differential operator $\mathcal{L}_n$ to the operator L_n defined by

$$L_n f = f^{''}(r) + \left(n + \frac{1}{2}\right)^2 r^{-2}\, f(r) \quad \text{for} \quad f \in C^2(0, \infty).$$

Lemma 2.5.8. *Let $n \in \mathbb{N} \cup \{0\}, (\alpha, \beta) \in \mathbb{R}^2, \beta \neq 0$. For every $\varepsilon > 0$, there exists a sequence of positive numbers $(\lambda^{\varepsilon}_{n,k})_{k \in \mathbb{N}}$ with $\lambda^{\varepsilon}_{n,k} \to \infty$, as $k \to \infty$, and a complete orthonormal system in $L_2(\varepsilon, 1)$ of smooth functions $(e^{\varepsilon}_{n,k})_{k \in \mathbb{N}}$ satisfying*

$$\begin{cases} L_n \, e^{\varepsilon}_{n,k} = -\lambda^{\varepsilon}_{n,k} \, e^{\varepsilon}_{n,k} \\ \alpha \, \frac{\partial e^{\varepsilon}_{n,k}}{\partial r}(1) - \beta \, e^{\varepsilon}_{n,k}(1) = 0 \\ e^{\varepsilon}_{n,k}(\varepsilon) = 0 \end{cases} \tag{2.46}$$

for every $k \in \mathbb{N}$.

Proof. Let $H = L_2(\varepsilon, 1)$ and $\overline{V}$ be the closure in $H^1(\varepsilon, 1)$ of the space

$$V := \{ f \in C^1([\varepsilon, 1]), f(\varepsilon) = 0, \alpha \, f'(1) - \beta \, f(1) = 0 \}.$$

In the following chain of inclusions

$$C^{\infty}_0([\varepsilon, 1]) \subset \overline{V} \subset H^1(\varepsilon, 1) \subset H,$$

the space $C^{\infty}_0([\varepsilon, 1])$ is dense in H and the injection of $H^1(\varepsilon, 1)$ into H is compact, see [Ada75] p. 144.
The Hilbert space $\overline{V}$ is then dense and compactly injected in H.
We define on $V \times V$ the symmetric bilinear form a_{ε} by

$$a_{\varepsilon}(f, g) = \int_{\varepsilon}^{1} f'(r) \, g'(r) \, dr - \gamma \, f(1) \, g(1) + \nu \int_{\varepsilon}^{1} f(r) \, g(r) \, r^{-2} \, dr, \quad f, g \in V,$$

with

$$\gamma = \begin{cases} \frac{\beta}{\alpha} & \text{if } \alpha \neq 0 \\ 0 & \text{if } \alpha = 0 \end{cases} \tag{2.47}$$

and

$$\nu = (n + \frac{1}{2})^2. \tag{2.48}$$

Lemma C.0.5 implies that a_{ε} is continuous and coercive on $V \times V$. We extend a_{ε} by density on $\overline{V} \times \overline{V}$, into a continuous and coercive bilinear form.
Furthermore, an integration by parts yields

$$\begin{aligned} < -L_n f, g > &= -\int_{\varepsilon}^{1} [\, f''(r) + \nu \, r^{-2} \, f(r) \,] \, g(r) \, dr \\ &= \int_{\varepsilon}^{1} f'(r) \, g'(r) \, dr - \gamma \, f(1) \, g(1) + \nu \int_{\varepsilon}^{1} f(r) \, g(r) \, r^{-2} \, dr \\ &= a_{\varepsilon}(f, g). \end{aligned}$$

for every $f \in C^2([\varepsilon, \rho])$ and $g \in C^1([\varepsilon, \rho])$ satisfying the boundary conditions in (2.46).
We extend the operator L_n onto the space

$$D(-L_n) := \{ f \in H^2(\varepsilon, \rho), f(\varepsilon) = 0, \alpha f^{'}(1) - \beta f(1) = 0 \}$$

such that

$$< -L_n f, g >= a_\varepsilon(f, g) \quad \text{for every } f \in D(-L_n), g \in V.$$

From theorem C.0.4, it follows that the spectrum of the self adjoint operator $(-L_n)$ defined on $D(-L_n)$ is discrete and admits the positive eigenvalues $\{\lambda^{\varepsilon}_{n,k}\}_k$ satisfying

$$0 < \alpha < \lambda^{\varepsilon}_{n,k}, \ \alpha \text{ constant}, \quad \text{and} \quad \lim_{k \to \infty} \lambda^{\varepsilon}_{n,k} = \infty.$$

More over, the eigenfunctions $e^{\varepsilon}_{n,k}$ of the operator $(-L_n)$ normalized in H and associated with $\lambda^{\varepsilon}_{n,k}$, $k \in \mathbb{N}$, built up a complete orthonormal system for H. We further have $e^{\varepsilon}_{n,k} \in C^{\infty}([\varepsilon, 1])$ for $k \in \mathbb{N}$ and the system (2.46) is fulfilled in a classical sense, see the remark following lemma C.0.5.

□

We give now an estimate of the eigenvalues $\lambda^{\varepsilon}_{n,k}$ using the Min-Max formulae given by theorems C.0.6 and the corollary C.0.7.

Lemma 2.5.9. *For $\varepsilon > 0, k \in \mathbb{N}$, we have*

$$\left(\frac{-\frac{\pi}{2} + k\pi}{1-\varepsilon}\right)^2 \le \lambda^{\varepsilon}_{n,k} \le \left[1 + |\gamma| + \left(n + \frac{1}{2}\right)^2\right] \left(\frac{k\pi}{1-\varepsilon}\right)^2.$$

Proof. For $f \in C^1([\varepsilon, 1])$ with $f(\varepsilon) = 0$, from the Cauchy-Schwartz inequality we get

$$|f(x)|^2 = \left|\int_{\varepsilon}^{x} f^{'}(t)\, dt\right|^2 \le x^2 \int_{\varepsilon}^{x} |f'(t)|^2\, dt, \quad x \in [\varepsilon, 1].$$

Hence, for the bilinear form a_ε we have the estimate

$$\int_{\varepsilon}^{1} |f^{'}(x)|^2\, dx - |\gamma|\, f(1)^2 \ \le\ a_\varepsilon(f, f) \ \le\ \left[1 + |\gamma| + \left(n + \frac{1}{2}\right)^2\right] \int_{\varepsilon}^{1} |f^{'}(x)|^2\, dx$$

for $f \in C^1_0([\varepsilon, 1])$, and by density, also for $f \in H^1_0(\varepsilon, 1)$.
The symmetric bilinear forms b_0 and b_1 defined by

$$b_0(f, g) = \int_{\varepsilon}^{1} f^{'}(x)\, g^{'}(x)\, dx, \quad f, g \in H^1_0(\varepsilon, 1),$$

and

$$b_1(f, g) = \int_{\varepsilon}^{1} f^{'}(x)\, g^{'}(x)\, dx - |\gamma|\, f(1)^2, \quad f, g \in H^1(\varepsilon, 1),$$

are associated, in the sense of theorem C.0.4, to the Laplace operator $(-\frac{d^2}{dx^2})$ on $[\varepsilon, 1]$, where the Dirichlet boundary conditions $(f(1) = f(\varepsilon) = 0)$ lead to b_0, and the boundary conditions $\Big(f(\varepsilon) = 0, f^{'}(1) = |\gamma|\, f(1)\Big)$ to b_1.
If we apply theorem C.0.6 and corollary C.0.7, we get when we take the right ordering

$$\mu_k^\varepsilon \leq \lambda_{n\,k}^\varepsilon \leq \big[\, 1 + |\gamma| + \big(\, n + \frac{1}{2}\,\big)^2 \,\big]\, \nu_{n\,k}^\varepsilon, \quad k \in \mathbb{N},$$

where μ_k^ε and $\nu_{n\,k}^\varepsilon$ are the eigenvalues associated to b_1 and b_0, respectively. These eigenvalues are computed in the remark following lemma C.0.7. Using

$$\nu_{n\,k}^\varepsilon = \big(\, \frac{k\,\pi}{1-\varepsilon}\,\big)^2$$

and the estimate

$$\big(\, \frac{-\frac{\pi}{2} + k\,\pi}{1-\varepsilon}\,\big)^2 \leq \mu_k^\varepsilon,$$

we finish the proof of the lemma.

□

In the next lemma we compute the eigenvalues $\lambda_{n,k}^\varepsilon$ and eigenfunctions $e_{n,k}^\varepsilon$ given in lemma 2.5.8.

Lemma 2.5.10. *There exist sequences of real numbers* $(A_{n,\,k}^\varepsilon)_{\,k\in\mathbb{N}}$ *and* $(B_{n,\,k}^\varepsilon)_{\,k\in\mathbb{N}}$ *uniquely determined, such that*

$$e_{n,\,k}^\varepsilon(x) = x^{\frac{1}{2}}\, \big[\, A_{n,\,k}^\varepsilon\, J_{n+\frac{1}{2}}\,(\sqrt{\lambda_{n,\,k}^\varepsilon}\, x) + B_{n,\,k}^\varepsilon\, Y_{n+\frac{1}{2}}\,(\sqrt{\lambda_{n,\,k}^\varepsilon}\, x)\,\big],$$

where $J_{n+\frac{1}{2}}$ *and* $Y_{n+\frac{1}{2}}$ *are Bessel functions of the first and second kind, respectively. Furthermore, for* $k \in \mathbb{N}$, *we have*

$$\lim_{\varepsilon\to 0} \lambda_{n,\,k}^\varepsilon = \lambda_{n,\,k},$$

where $\{\lambda_{n,\,k}^{\frac{1}{2}}\}_k$ *are positive zeros of the function*

$$\varphi_n(r) = \alpha\, r\, J_{n+\frac{1}{2}}^{'}(r) - (\beta - \frac{\alpha}{2})\, J_{n+\frac{1}{2}}(r).$$

When ε *tends to zero, we have*

$$\begin{aligned} \lim_{\varepsilon\to 0} B_{n,\,k}^\varepsilon &= 0, \\ \lim_{\varepsilon\to 0} A_{n,\,k}^\varepsilon &= A_{n,\,k}, \\ \lim_{\varepsilon\to 0} e_{n,\,k}^\varepsilon(x) &= e_{n,\,k}(x),\ x \in [0,1], \end{aligned}$$

where

$$A_{n,\,k} := \big\{\, \int_0^1 [J_{n+\frac{1}{2}}(\sqrt{\lambda_{n,\,k}}\, r)]^2\, r\, dr\,\big\}^{-\frac{1}{2}} \quad \textit{and} \quad e_{n,\,k}(x) := A_{n,\,k}\, x^{\frac{1}{2}}\, J_{n+\frac{1}{2}}\,(\sqrt{\lambda_{n,\,k}}\, x).$$

Proof. For $\varepsilon > 0,\, k \in \mathbb{N} \cup \{0\}, \lambda = \lambda^{\varepsilon}_{n,\,k},$ the functions $e^{\varepsilon}_{n,\,k}$ are solutions to the boundary value problem

$$\begin{cases} y^{''}(r) + (n+\frac{1}{2})^2\, r^{-2}\, y(r) = -\lambda\, y(r) \\ \alpha\, y^{'}(1) - \beta\, y(1) = 0 \\ y(\varepsilon) = 0 \end{cases},$$

If we make the substitution $y(r) = r^{\frac{1}{2}}\, z\,(\sqrt{\lambda}\, r)$, see [GR00] 8.491.5 p. 921, the differential equation is transformed into the Bessel equation

$$z^{''}(t) + \frac{1}{t}\, z^{'}(t) + \big(\, 1 - \frac{(n+\frac{1}{2})^{\,2}}{t^2}\,\big)\, z(t) = 0,$$

whose fundamental system may be given by the pair $\{J_{n+\frac{1}{2}}, Y_{n+\frac{1}{2}}\}$ of Bessel and Neumann functions.
Hence, there exist uniquely determined constants $A^{\,\varepsilon}_{n,\,k}$ and $B^{\,\varepsilon}_{n,\,k}$ such that

$$e^{\varepsilon}_{n,\,k}(r) = r^{\frac{1}{2}}\,\big[\,A^{\varepsilon}_{n,\,k}\, J_{n+\frac{1}{2}}\,(\sqrt{\lambda^{\varepsilon}_{n,\,k}}\, r) + B^{\varepsilon}_{n,\,k}\, Y_{n+\frac{1}{2}}\,(\sqrt{\lambda^{\varepsilon}_{n,\,k}}\, r)\,\big]$$

with

$$\alpha\, {e^{\varepsilon}_{n,\,k}}^{'}(1) - \beta\, e^{\varepsilon}_{n,\,k}(1) = 0$$

and

$$e^{\varepsilon}_{n,\,k}(\varepsilon) = 0.$$

From lemma 2.5.9 we have

$$\lim_{\varepsilon\to 0} \varepsilon\, \sqrt{\lambda^{\varepsilon}_{n,\,k}} = 0$$

Since the Neumann function $Y_{n+\frac{1}{2}}$ is singular at 0, we must have

$$\lim_{\varepsilon\to 0} B^{\varepsilon}_{n,\,k} = 0$$

and

$$\lim_{\varepsilon\to 0} A^{\varepsilon}_{n,\,k} = \Big\{\, \int_0^1 [J_{n+\frac{1}{2}}(\sqrt{\lambda_{n,\,k}}\, r)]^2\, r\, dr\,\Big\}^{-\frac{1}{2}},$$

to fulfill the boundary condition at ε and the normalizing condition $\|e^{\,\varepsilon}_{\,n,\,k}\| = 1$.
Finally, the boundary condition at 1 implies, when ε tends to zero, that $\lambda_{\,n,\,k}$ have to satisfy

$$\varphi_n\,(\sqrt{\lambda_{n,\,k}}) = 0.$$

□

Remarks.

1. The positive zeros of the function φ_n are simple and may be ascendingly arranged in a sequence $(\lambda_{n,\,k})_{k\in\mathbb{N}}$, see also [Wat58] p. 480.

2. We have for $k \in \mathbb{N} \cup \{0\}$ the identity

$$A_{n,\,k}^{-2} = \begin{cases} \frac{1}{2}\left[\, J'_{n+\frac{1}{2}}\,(\sqrt{\lambda_{n,\,k}})\,\right]^2 & \text{if } \alpha = 0 \\ \frac{1}{2\,\lambda_{n,k}}\left[\,\frac{(\beta-\frac{\alpha}{2})^2}{\alpha^2} + \lambda_{n,\,k} - (n+\frac{1}{2})^2\,\right] J_{n+\frac{1}{2}}\,(\sqrt{\lambda_{n,\,k}})^2 & \text{otherwise,} \end{cases}$$

see [AS72] 11.4.5 p. 485.

3. In the sequel we use also the notations $\lambda_{n,\,k}^{\,0} := \lambda_{n,\,k}$, $e_{n,\,k}^{\,0} := e_{n,\,k}$.

Bearing in mind that we seek the limiting case as ε tends to zero, we need to look into the convergence of $e_{n,k}^{\varepsilon}$ in $L_2(0,1)$.

Lemma 2.5.11. *If we extend $e_{n,\,k}^{\,\varepsilon}$ by zero outside $(\varepsilon, 1)$, $\varepsilon > 0$, then the functions $e_{n,\,k}^{\varepsilon}$ are uniformly bounded, in ε, on the interval $(0,1)$ and they converge to $e_{\,n,\,k}$ in $L_2(0,1)$.*

Proof. Let $n \in \mathbb{N} \cup \{0\}$, $k \in \mathbb{N}$ and A be a constant with $|A_{n,\,k}^{\,\varepsilon}| \leq A$ for $\varepsilon \geq 0$.
The Bessel function of the first kind $J_{n+\frac{1}{2}}$ is bounded by 1 on $[0,\infty)$ and the Bessel function of the second kind $Y_{n+\frac{1}{2}}$ is singular at zero and bounded away from zero, see [AS72] p. 362.
We have for ε small enough

$$|\,Y_{n+\frac{1}{2}}\,(x)\,| \leq |\,Y_{n+\frac{1}{2}}\,(\varepsilon)\,|, \quad x \in (\varepsilon, \infty).$$

We recall that

$$\lim_{\varepsilon \to 0} \varepsilon\,\sqrt{\lambda_{n,\,k}^{\,\varepsilon}} = 0.$$

Let $\varepsilon > 0$ be small enough, from the boundary condition at ε in (2.46) it holds

$$B_{n,\,k}^{\varepsilon}\,Y_{n+\frac{1}{2}}\,(\varepsilon\,\sqrt{\lambda_{n,\,k}^{\,\varepsilon}}) = -A_{n,\,k}^{\varepsilon}\,J_{n+\frac{1}{2}}\,(\varepsilon\,\sqrt{\lambda_{n,\,k}^{\,\varepsilon}}).$$

We obtain for $x \in (\varepsilon, 1)$ the estimate

$$\begin{aligned} e_{n,\,k}^{\varepsilon}(x) &:= x^{\frac{1}{2}}\left[\,A_{n,\,k}^{\varepsilon}\,J_{n+\frac{1}{2}}\,(\sqrt{\lambda_{n,\,k}^{\,\varepsilon}}\,x) + B_{n,\,k}^{\varepsilon}\,Y_{n+\frac{1}{2}}\,(\sqrt{\lambda_{n,\,k}^{\,\varepsilon}}\,x)\,\right] \\ &\leq x^{\frac{1}{2}}\left[\,|A_{n,\,k}^{\varepsilon}\,J_{n+\frac{1}{2}}\,(\sqrt{\lambda_{n,\,k}^{\,\varepsilon}}\,x)\,| + |\,A_{n,\,k}^{\varepsilon}\,J_{n+\frac{1}{2}}\,(\sqrt{\lambda_{n,\,k}^{\,\varepsilon}}\,\varepsilon)\,|\,\right] \\ &\leq 2A \end{aligned}$$

which proves that the extended $e_{\,n,\,k}^{\,\varepsilon}$ are uniformly bounded on $(0,1)$ for $\varepsilon \geq 0$.
We can therefore, interchange the limit and the integration to get

$$\lim_{\varepsilon \to 0} \int_0^1 |\,(e_{n,\,k}^{\varepsilon} - e_{n,\,k})(x)\,|^2\,dx = 0$$

The convergence in $L_2(0,1)$ is proved.

□

Theorem 2.5.12. *The system* $\{e_{n,k}\}_{k\in\mathbb{N}}$ *is orthonormal and complete in* $L_2(0,1)$.

Proof. The system $\{e_{n,k}\}_{k\in\mathbb{N}}$ is orthogonal since for $k,l\in\mathbb{N}$ we have

$$-(\lambda_{n,k}-\lambda_{n,l}) < e_{n,k}, e_{n,l} > = < L_n e_{n,k}, e_{n,l} > - < e_{n,k}, L_n e_{n,l} > = 0.$$

To prove completeness, we construct first the "inverse" operator K of $(-L_n)$. We define the operator $K_\varepsilon \in \mathcal{L}\left(L_2(0,1)\right), \varepsilon>0$, by

$$K_\varepsilon f(x) = \begin{cases} \sum_{k=0}^{\infty} \frac{1}{\lambda^\varepsilon_{n,k}} < f, e^\varepsilon_{n,k} > e^\varepsilon_{n,k}(x), & x\in(\varepsilon,1),\\ 0 & x\in[0,\varepsilon]. \end{cases}$$

For $\varepsilon>0$, the operator K_ε is compact, as a limit of finite rank operators, and we have

$$\|K_\varepsilon\| = \frac{1}{\lambda^\varepsilon_{n,1}}.$$

From lemma 2.5.9 , we see that K_ε are uniformly bounded for $\varepsilon>0$, it follows from the Banach-Steinhaus theorem of uniform boundedness that K_ε converge in $\mathcal{L}\left(L_2(0,1)\right)$, as ε tends to zero, to an operator $K\in\mathcal{L}\left(L_2(0,1)\right)$.
The operator K is compact and it is given by

$$Kf = \lim_{\varepsilon\to 0} K_\varepsilon f, \quad f\in L_2(0,1). \tag{2.49}$$

From lemma 2.5.11 it follows

$$K f(x) = \sum_{k=0}^{\infty} \frac{1}{\lambda_{n,k}} < f, e_{n,k} > e_{n,k}(x), \quad x\in(0,1).$$

Let $f\in C_0^2(0,1)$ and ε_0 such that the support of f is contained in $[\varepsilon_0, 1-\varepsilon_0]$ and $0<\varepsilon<\varepsilon_0$. Integrating by parts we get

$$< L_n f, e^\varepsilon_{n,k} > = < f, L_n e^\varepsilon_{n,k} > = -\lambda^\varepsilon_{n,k} < f, e^\varepsilon_{n,k} > \quad \text{for every } k\in\mathbb{N}.$$

It yields

$$-K_\varepsilon L_n f = \sum_{k=0}^{\infty} < f, e^\varepsilon_{n,k} > e^\varepsilon_{n,k} = f.$$

The last equality holds, since $(e^\varepsilon_{n,k})_k$ form a complete orthonormal system in $L_2(\varepsilon,1)$ and vanishes outside $(\varepsilon,1)$. Hence, when ε tends to zero, we get

$$-K L_n f = f \quad \text{for every } f\in C_0^2(0,1).$$

The expansion

$$f(x) = \sum_{k=0}^{\infty} < f, e_{n,k} > e_{n,k}(x), \quad x\in(0,1),$$

holds also for $f\in L_2(0,1)$ since $C_0^2(0,1)$ is dense in $L_2(0,1)$.
The completeness of $\{e_{n,k}\}_{k\in\mathbb{N}}$ in $L_2(0,1)$ has been also proved.

□

Remark. When $\alpha = 0$, theorem 2.5.12 is known as the Fourier-Bessel expansion, which can also be proved using the Lommel theorem for Bessel functions, see [Hoc86]. The advantage of using the techniques from [DL90b] is that the proof only relies on theorem C.0.4.
In general, boundary conditions are part of the features of Sturm-Liouville problems, which explains the complications related to the case when $\alpha \neq 0$.

We come back now to the system $\{f_{n,k}\}_k$ constructed in lemma 2.5.7 and show its completeness in $L_2(0, \rho, r^2 dr)$.

Corollary 2.5.13. *Let $n \in \mathbb{N}$, $\alpha_n := j_n(k\rho)$, $\beta_n := -kj_n^{'}(k\rho) \neq 0$. Let $\lambda_{n,k}$ be the positive zeros of*

$$\alpha_n \, \lambda j_n^{'}(\lambda\rho) \,+\, \beta_n \, j_n(\lambda\rho) = 0. \tag{2.50}$$

Then the system $\{f_{n,k}\}_k$, $f_{n,k}(r) := a_{n,k} j_n(\lambda_{n,k} r), r \in (0, \rho)$, built up a complete orthonormal system of $L_2(0, \rho, r^2 dr)$, where $a_{n,k}$ are normalizing coefficients.

Proof. Let $\rho = 1$. As $e_{n,k}(r) = \left(\frac{\pi}{2}\right)^{-\frac{1}{2}} A_{n,k} \lambda_{n,k}^{\frac{1}{2}} \, r j_n(\lambda_{n,k} r), k \in \mathbb{N}$, from theorem 2.5.12 build up a complete orthonormal system of $L_2(0,1)$ for each fixed $n \in \mathbb{N}$, the functions $\{r^{-1} e_{n,k}(r)\}_k$ form also a complete orthonormal system of $(L_2(0,1), r^2 dr)$. Hence, the corollary follows from the previous theorem with the right choice of (α, β).
□

We can use the bases $\{f_{n,k}\}_k$ from the corollary above and spherical harmonics to construct a complete orthonormal basis of $\mathcal{N}(A)$ building up with $\{v_n^m\}_{(n,m)}$ a complete orthonormal system of $L_2(B_\rho)$.

Theorem 2.5.14. *Let $n \in \mathbb{N}$. For $m \in \{-n, \ldots, n\}$, let*

$$\phi_{n,\,k}^m(x) := f_{n,\,k}(\,|x|\,)\, Y_n^m(\,\frac{x}{|x|}\,), x \in \overline{B}_\rho \setminus \{0\}. \tag{2.51}$$

Then $\{\phi_{n,k}\}_k^m$ built up a complete orthonormal system of $\mathcal{N}(A)$.

Remark. We see from (2.50), that the null space $\mathcal{N}(A)$ obviously depends on the radius of the target region ρ, however, it does not depend on the radius R of the measurement sphere. Therefore, if we have an *apriori* information about the location of the scatterer, we can use ρ as a parameter to control the zeros related to the radial part of the elements of the basis of $\mathcal{N}(A)$.

2.5.3 Sobolev estimate

We close this section with a result about the smoothing properties of the spherical scattering operator in the setting of Sobolev-spaces. As we will see in the next chapter the inverse scattering problem is solved approximately using a regularization method. The smoothing properties of the underlying operator account for the accuracy of the computed solution , see [Lou89] , [Nat01].

In this subsection we use on the Sobolev-space $H^s(\mathbb{R}^n), s \in \mathbb{R},$ the equivalent norm [3]

$$\|f\|_{H^s(\mathbb{R}^n)} = \left(\int\limits_{\mathbb{R}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}, \ f \in H^s(\mathbb{R}^n),$$

where the Fourier transform $\hat{f}$ of f is defined by

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} f(t) e^{-i<\xi,t>} dt.$$

Let Ω be a bounded domain in $\mathbb{R}^n$. For $s \in \mathbb{R},$ we put

$$H_0^s(\Omega) := \{f \in H^s(\mathbb{R}^n), f \text{ is compactly supported in } \Omega\}.$$

Let χ be the characteristic function of $\Omega,$ we define the norm

$$\|f\|_{H^s(\Omega)} = \|\chi f\|_{H^s(\mathbb{R}^n)}.$$

Lemma 2.5.15. *Let $k > 0$ and $R > 0$. The Fourier transform of the function $g_{k,R}$ defined on $\mathbb{R}^3$ by*

$$g_{k,R}(x) := \begin{cases} \frac{1}{4\pi} \frac{e^{ik|x|}}{|x|} & \text{if } |x| < R \\ 0 & \text{if } |x| \geq R \end{cases}$$

is given by

$$\hat{g}_{k,R}(\xi) = -(|\xi|^2 - k^2)^{-1} \left(1 - e^{ikR} (\cos(R|\xi|) - ik \frac{\sin(R|\xi|)}{|\xi|} \right) \quad \text{for} \quad \xi \in \mathbb{R}^3 \backslash \{0\}.$$

Proof. For $\xi \in \mathbb{R}^3,$ we have

$$\begin{aligned} \hat{g}_{k,R}(\xi) &= (2\pi)^{-\frac{3}{2}} \int\limits_{\mathbb{R}^3} g_{k,R}(x) e^{-i<\xi,x>} dx \\ &= (2\pi)^{-\frac{3}{2}} \int\limits_{B_R} g_k(x) e^{-i<\xi,x>} dx \quad \text{for} \quad \xi \in \mathbb{R}^3. \end{aligned}$$

Let $\phi_\xi(x) := e^{-i<\xi,x>}, x \in \mathbb{R}^3$. As $(\Delta + k^2)\phi_\xi = (k^2 - |\xi|^2)\phi_\xi,$ we get

$$\hat{g}_{k,R}(\xi) = (k^2 - |\xi|^2)^{-1} \int\limits_{B_R} g_k(x)(\Delta + k^2)\phi_\xi(x) dx \quad \text{for} \quad |\xi| \neq k.$$

[3] see [Nat01] p. 201 or [DL88] p.96.

Let $\varepsilon > 0,\ B_r$ be the ball centered at the origin with radius $r \in \{\varepsilon, R\}$.
From the second Green's formula, we have

$$\begin{aligned}
\int\limits_{B_R \setminus B_\varepsilon} g_k(x)(\Delta + k^2)\phi_\xi(x)dx &= \int\limits_{B_R \setminus B_\varepsilon} (\Delta + k^2)g_k(x)\phi_\xi(x)dx \\
&+ \int\limits_{\partial B_R} \left(g_k(x)\partial_n\phi_\xi(x) - \partial_n g_k(x)\phi_\xi(x)\right) d\sigma(x) \\
&- \int\limits_{\partial B_\varepsilon} \left(g_k(x)\partial_n\phi_\xi(x) - \partial_n g_k(x)\phi_\xi(x)\right) d\sigma(x),
\end{aligned}$$

where ∂_n denotes the outward normal derivative.
The first integral on the right-hand side vanishes since g_k is the Green's function for $\Delta + k^2$. It remains only to compute the boundary integrals.
From the Funk-Hecke theorem it follows, on the one hand,

$$\begin{aligned}
\int\limits_{\partial B_r} \phi_\xi(x)d\sigma(x) &= r^2 \int\limits_{S^2} e^{-ir<\xi,\omega>}d\omega \\
&= 2\pi r^2 \int\limits_{-1}^{1} e^{-ir|\xi|t}dt \\
&= 4\pi r \frac{\sin(r|\xi|)}{|\xi|},
\end{aligned}$$

on the other hand,

$$\begin{aligned}
\int\limits_{\partial B_r} \partial_n\phi_\xi(x)d\sigma(x) &= -ir^2 \int\limits_{S^2} <\xi,\omega> e^{-ir<\xi,\omega>}d\omega \\
&= -2i\pi|\xi|r^2 \int\limits_{-1}^{1} te^{-ir|\xi|t}dt \\
&= 4\pi r \left(\cos(r|\xi|) - \frac{\sin(r|\xi|)}{r|\xi|}\right).
\end{aligned}$$

As $g_k'(r) = \left(ik - \frac{1}{r}\right) g_k(r)$ with $r \in \{\varepsilon, R\}$, we obtain after a straightforward computation

$$\lim_{\varepsilon \to 0} \int\limits_{\partial B_\varepsilon} \partial_n g_k(x)\phi_\xi(x)d\sigma(x) = -1 \quad \text{and} \quad \lim_{\varepsilon \to 0} \int\limits_{\partial B_\varepsilon} g_k(x)\partial_n\phi_\xi(x) = 0,$$

and

$$\int\limits_{\partial B_R} g_k(x)\partial_n\phi_\xi(x) - \partial_n g_k(x)d\sigma(x) = e^{ikR}\left(\cos(R|\xi|) - ik\frac{\sin(R|\xi|)}{|\xi|}\right).$$

□

Lemma 2.5.16. *There exists a constant* $m > 0$ *such that*

$$|\hat{g}_{k,R}(\xi)| \geq m \, \max(|\xi|, k)^{-1} \, (|\xi| + k)^{-1}.$$

Proof. We write

$$|\hat{g}_{n,k}(\xi)| = (|\xi| + k)^{-1} \left| e^{ikR} \frac{q(|\xi|) - q(k)}{|\xi| - k} \right|$$

with

$$q(t) := \cos(Rt) - ik \frac{\sin(Rt)}{t}, t > 0.$$

Since the continuous function $q(t) - q(k)$ only vanishes at the simple zero $t = k,$ we get

$$\left| \frac{q(|\xi|) - q(k)}{|\xi| - k} \right| \geq m \max(|\xi|, k)^{-1},$$

for some constant $m > 0.$

□

Remark. Contrary to $g_k,$ we see from the proof above that $\hat{g}_{R,k}$ has no singularity at $|\xi| = k.$

Lemma 2.5.17. *Let* $s \geq 0, a > 0.$ *There is a constant* $C > 0$ *such that*

$$\sup_{|\xi| \leq a} |\hat{f}(\xi)| \leq C \, \|f\|_{H_0^s(\Omega)}$$

for $f \in C_0^\infty(\Omega).$

The proof of the lemma can be found in [Nat01] p.44. It is also to mention that some techniques used in the proof of the next theorem are inspired from the same reference.

Theorem 2.5.18. *Let* $s \geq 0$ *and* $A_\Omega : H_0^s(\Omega) \longrightarrow L_2(\Omega)$ *be the scalar scattering operator. Then there exist two positive constants* $c(s)$ *and* $C(s)$ *such that we have*

$$c(s) \, ||f||_{H_0^s(\Omega)} \leq ||A_\Omega f||_{H^{s+2}(\Omega)} \leq C(s) \, ||f||_{H_0^s(\Omega)} \tag{2.52}$$

for every $f \in C_0^\infty(\Omega).$

Proof. First of all, we notice that it holds

$$A_\Omega f(x) = \chi(x) \int_{B_\rho} g_k(x-y) f(y) dy = \int_{B_\rho} g_{k,R}(x-y) f(y) dy \quad \text{for} \quad x \in B_\rho,$$

with $R = 2\rho,$ since $|x-y| < 2\rho$ for $x, y \in B_\rho$.
Let $s \geq 0$ and $f \in C_0^\infty(\Omega)$. We have

$$\begin{aligned}
\|A_\Omega f\|^2_{H^{s+2}(\Omega)} &= \int_{\mathbb{R}^3} (1+|\xi|^2)^{s+2} |\widehat{\chi A_\Omega f}(\xi)|^2 d\xi \qquad (2.53)\\
&= (2\pi)^3 \int_{\mathbb{R}^3} (1+|\xi|^2)^{s+2} |\hat{f}(\xi)|^2 |\hat{g}_{k,R}(\xi)|^2 d\xi \qquad (2.54)\\
&\geq (2\pi)^3\, m^2 \int_{\mathbb{R}^3} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \left(\frac{1+|\xi|^2}{\max(|\xi|,k)(|\xi|+k)}\right)^2 d\xi\\
&\geq (2\pi)^3 \min(\frac{1}{2}, \frac{1}{2k^2})^2\, m^2\, \|f\|^2_{H_0^s(\Omega)},
\end{aligned}$$

by virtue of the convolution theorem for the Fourier transform and lemma 2.5.16.
To prove the inequality on the right-hand side, we split the integration domain in (2.54) into $\{\xi, |\xi| \leq a\}$ and $\{\xi, |\xi| \geq a\}$, where we take $a > \max(k, 1)$ such that

$$\frac{|\xi|^2}{|\xi|^2 - k^2} \leq \sqrt{2} \quad \text{for} \quad |\xi| \geq a. \qquad (2.55)$$

On the one hand, from lemma 2.5.17 we have

$$\begin{aligned}
\int_{|\xi|\leq a} (1+|\xi|^2)^{s+2} |\hat{f}(\xi)|^2 |\hat{g}_{k,R}(\xi)|^2 d\xi &\leq \sup_{|\xi|\leq a} |\hat{f}(\xi)|^2 \int_{|\xi|\leq a} |\hat{g}_{k,R}(\xi)|^2 (1+|\xi|^2)^{s+2} d\xi\\
&\leq C_2(s) \|f\|^2_{H_0^s(\Omega)}
\end{aligned}$$

with some constant $C_2(s) > 0$, since the last integral is finite.
On the other hand, it holds

$$|\hat{g}_{k,R}(\xi)| \leq (2+kR)||\xi|^2 - k^2|^{-1},$$

where we dominated $\cos(R|\xi|)$ and $\frac{\sin(R|\xi|)}{R|\xi|}$ by 1. Using (2.55) we get

$$\begin{aligned}
\int_{|\xi|\geq a} (1+|\xi|^2)^{s+2} |\hat{f}(\xi)|^2 |\hat{g}_{k,R}(\xi)|^2 d\xi &\leq 2(2+kR)^2 \int_{|\xi|\geq a} \varphi(\xi)^2 (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi\\
&\leq 2^3 (2+kR)^2 \int_{|\xi|\geq a} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi\\
&\leq 2^3 (2+kR)^2 \|f\|^2_{H_0^s(\Omega)},
\end{aligned}$$

where $\varphi(\xi) := |\xi|^{-2}(1+|\xi|^2) \leq 2$ for $|\xi| \geq 1$.
The proof is then finished. □

Corollary 2.5.19. *Let* $\Omega = B_\rho \subset \mathbb{R}^3$ *be the ball centered at the origin with radius* $\rho > 0,\ s \geq 0$ *and* $A_\Gamma : H_0^s(\Omega) \longrightarrow L_2(\Gamma)$ *be the restricted scattering operator on the boundary* $\Gamma := \partial B_\rho$. *Then there exist two positive constants* $c(s)$ *and* $C(s)$ *such that we have*

$$c(s)\,||f||_{H_0^s(\Omega)} \leq ||A_\Gamma f||_{H^{s+\frac{3}{2}}(\Omega)} \leq C(s)\,||f||_{H_0^s(\Omega)} \tag{2.56}$$

for every $f \in C_0^\infty(\Omega)$.

Proof. If we write $A_\Gamma = \gamma_0 A_\Omega$ with the trace mapping γ_0 on the boundary Γ. The corollary follows from theorem 2.5.18 and theorem D.0.9. □

The singular value decomposition, the determination of the null space and the study of the smoothing properties for the spherical scattering operator in three-dimension provide us good tools to deal with the regularization of the inverse problem in the next chapter.

Chapter 3

Application of the approximate inverse

The problem (A, X, Y) of solving the equation $Af = g$ for a bounded linear operator A between the Hilbert spaces X and Y is well-posed according to Hadamard if for any data $g \in Y$ there exists a solution f, this solution is unique, and it is stable through perturbation of the right-hand side, in a sense that the solution f^{ε} for the data g^{ε} is closer to f, the closer g^{ε} is to g. It is well known that if A is compact and its range is not closed, the problem (A, X, Y) is in this case rather ill-posed. Since for such an operator the Moore-Penrose or pseudo-inverse $A^{\dagger}$ is unbounded, the condition of stability is violated. Typically, data are endowed with noise, which is an unavoidable error generally related to measurement conditions.
The application of a regularization method is then required to stabilize the solution. It consists in approximating the ill-posed problem by a class of neighboring well-posed problems. In [Lou99], it has been shown that regularization methods can be considered whether as smoothing of the pseudo-inverse or as prewhitening of the data. Regularization algorithms of broad usage such as Thikhonov-Phillips, the truncated singular value decomposition, iterative Landweber or conjugate gradient can be considered as filtered version of the pseudo-inverse. Alternatively, mollifier methods consist in computing smoothed versions of the solutions, see [Lou99] and the references therein.
The approximate inverse, introduced for linear problems presents a unified approach to regularization methods for linear ill-posed problems. It is also extendible to some nonlinear problems, see [Lou96]. Efficient algorithms based on the approximate inverse has been generated for many applications. Recent applications of the approximate inverse can be found in [Sch00] and [AS04] for 3-D tomography, in [DRT00] for local tomography, in [HSS05] for thermo-acoustic computed tomography.
The first application of the approximate inverse to backscattering problems is due to Abbdullah and Louis [Abd98]. Their scheme for 2-D acoustic waves is derived from the Lippmann-Schwinger integral equation, where a linear inversion with the approximate inverse provides the equivalent source. Wallacher in [Wal02] improved the resolution of the nonlinear part in that scheme to recover the medium properties from the

equivalent source. He also extended the scope of application to limited-angel problems.

3.1 The approximate inverse

In this section we recall some fundamental properties of the approximate inverse as a regularization method and describe how to realize an efficient implementation using the invariances of the problem. Three features characterize this method

1. Stability and robustness, due to the precomputation of the reconstruction kernel independently from data.
2. Efficiency, by using the invariance properties of the problem to compute the reconstruction kernel. Once the kernel is is on hand, any new data can be rapidly inverted.
3. Flexibility in the choice of the filter appropriately to the problem.

3.1.1 Regularization with the approximate inverse

Definition 3.1.1. *Let A be a compact operator between the Hilbert spaces X and Y, $A^\dagger$ be its pseudo-inverse. A regularization $\{T_\gamma\}_{\gamma>0}$ of $A^\dagger$ is a family of continuous operators $T_\gamma : Y \longrightarrow X$ such that there exists a mapping*
$\gamma : \mathbb{R}_+ \times Y \longrightarrow \mathbb{R}_+,$ called a regularizing parameter, satisfying

$$\lim_{\varepsilon\to 0} T_{\gamma(\varepsilon,g^\varepsilon)} g^\varepsilon = A^\dagger g$$

for every $g \in \mathcal{D}(A^\dagger) := \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp$ and $g^\varepsilon \in Y$ with $\|g - g^\varepsilon\| \le \varepsilon$.
If T_γ are all linear, then $\{T_\gamma\}_{\gamma>0}$ is called linear regularization.

Let $n, m \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ and $\Gamma \subset \mathbb{R}^m$ be measurable sets, the spaces $X \subset L_2(\Omega)^N$, $Y \subset L_2(\Gamma)^M$ and $Z \subset L_2(\Gamma)^P$, $M, N, P \in \mathbb{N}$, be Hilbert spaces.

Definition 3.1.2. *The operators $\{M_\gamma\}_{\gamma>0} \subset \mathcal{L}(X)$ form an approximation of the identity if*

$$\lim_{\gamma\to 0} M_\gamma f = f \quad \textit{for } f \in X.$$

We say that it has the rate $\alpha > 0$ if there exists a positive constant $C > 0$ such that

$$\|M_\gamma f - f\| \le C\gamma^\alpha \|f\| \quad \textit{for} \quad f \in X. \tag{3.1}$$

If furthermore for $x \in \Omega$, there exists $e_\gamma^x \in X$ such that for $f \in X$

$$M_\gamma f(x) = < f, e_\gamma^x >, \quad x \in \Omega,$$

then the function e_γ defined by

$$e_\gamma(x, y) := e_\gamma^x(y) \quad \textit{for } x, y \in \Omega,$$

is called the mollifier associated to M_γ.

Definition 3.1.3. *Let* $A \in \mathcal{L}(X, Y)$, e_γ *be a mollifier and* $D : \tilde{X} \to Z$ *be a linear operator on the subspace* $\tilde{X} \subset X$.
If we assume that for every $x \in \Omega$, *there exists a minimum-norm solution* $\psi_\gamma^x \in \overline{\mathcal{R}(A)}$ *to the equation*

$$A^* \psi_\gamma^x = D^* e_\gamma^x, \tag{3.2}$$

then we call the operator $S_\gamma : Y \to Z$ *defined for* $g \in Y$ *by*

$$S_\gamma\, g(x) =< g, \psi_\gamma^x >, \quad x \in \Omega, \tag{3.3}$$

the approximate inverse and the function ψ_γ *defined by*

$$\psi_\gamma(x, y) = \psi_\gamma^x(y), \quad x \in \Omega, y \in \Gamma,$$

the reconstruction kernel, to compute the moment D *of the solution.*

Lemma 3.1.4. *We suppose that* Γ *is bounded. For the operator* $A : X \to Y$ *and the mollifier* e_γ, *let* S_γ *be the approximate inverse to compute the moment* $D : \tilde{X} \to Z$.
If, for every $\gamma > 0$, *the mollifier* e_γ^x *satisfies* $\sup_{x \in \Omega} \|D^* e_\gamma^x\| < \infty$ *and* $D^* e_\gamma^x \in \mathcal{R}(A^*), x \in \Omega$, *then*

$$\lim_{\gamma \to 0} S_\gamma\, Af = Df \quad \textit{for } f \in \mathcal{N}(A)^\perp.$$

Proof. Let $\gamma > 0$. For $x \in \Omega$, the equation

$$A^* \psi_\gamma^x = D^* e_\gamma^x$$

admits a unique minimum-norm solution $\psi_\gamma^x \in \overline{\mathcal{R}(A)}$, see [Lou89].
Let $g \in \mathcal{R}(A)$ and $f \in \mathcal{N}(A)^\perp$ be a solution to $Af = g$. Then for $x \in \Omega$, we have

$$S_\gamma\, g(x) =< Af, \psi_\gamma^x >=< f, A^* \psi_\gamma^x >=< f, D^* e_\gamma^x >=< D\, f, e_\gamma^x > .$$

From the Cauchy-Schwartz inequality we get

$$|S_\gamma\, g(x)| = |< f, D^* e_\gamma^x >| \le \|D^* e_\gamma^x\|\, \|f\|.$$

As Γ is assumed to be bounded with measure $|\Gamma|$, it yields

$$\|S_\gamma g\| \le |\Gamma| \sup_{x \in \Omega} \|D^* e_\gamma^x\|\, \|f\|.$$

Hence $S_\gamma g \in L_2(\Omega)^P$.
Since e_γ^x is associated to an approximation of the identity, we have

$$\lim_{\gamma \to 0} S_\gamma\, Af = Df.$$

□

Remarks

1. The adjoint equation (3.2) is independent of the right hand side g. The reconstruction kernel ψ_γ is precomputed independently from errors related to measured data. This yields a stable reconstruction of Df.

2. The operator D gives the moment of the solution f we ought to compute. For instance, an averaged version of f is computed by taking D the identity on $X = Z$. To compute other moments Df of the solution, we simply "transpose" the operator D on the mollifier. This generalizes the procedure already used by Louis in [Lou96] to compute the derivative of the solution. Schuster in [Sch00] applied also this technique to compute the curl of the solution for the Doppler-transform in 3-D tomography. We may similarly extend this method to solve ill-posed equations, where the gradient, divergence or curl of the solution is required. The inversion is then carried in a stable way, without any differentiation of the measured data.

Examples. For the sake of simplicity, let D be the identity on X and $M = N = P = 1$. A mollifier e_γ is said to be of convolution type if

$$e_\gamma(x,y) = \gamma^{-n}\, e^0(\frac{x-y}{\gamma}), \quad \gamma > 0,\ x,y \in \mathbb{R}^n,$$

for some function $e^0 \in L_2(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} e^0(x)\, dx = 1.$$

The last condition is set to ensure that constant functions are invariant under mollification.
For example, to compute local averages of the solution, we may take

$$e^0(x) = \frac{n}{|S^{n-1}|}\, \chi(x),$$

where χ is the characteristic function of the unit ball of $\mathbb{R}^n$ and $|S^{n-1}|$ is the measure of the surface in the unit ball in $\mathbb{R}^n$.
The band-limited filter

$$e^0(x) = \pi^{-n}\, \text{sinc}\,(x),$$

is used to eliminate the high-frequency component of the solution.
The Gauss-function is a very smoothing mollifier. It is given by

$$e^0(x) = (2\pi)^{-\frac{n}{2}}\, \exp\,(-\frac{|x|^2}{2}).$$

In lemma 3.1.4, the required conditions on the mollifier are overdoing. They may be improved if the smoothing properties of the operator A are involved. For this

purpose we consider an appropriate space setting, we refer to [Lou89], [Lou99] for more details.
Let X and Y be two Hilbert spaces, $A : X \longrightarrow Y$ be a non degenerate compact operator with singular system $\{v_n, u_n; \sigma_n\}_{n\in\mathbb{N}}$. For $\nu \in \mathbb{R}$, the spaces

$$X_\nu := \left\{ f \in \mathcal{N}(A)^\perp, \sum_{n\in\mathbb{N}} \sigma_n^{-2\nu} | < f, v_n > |^2 < \infty \right\}$$

endowed with the scalar product

$$< f, g >:= \sum_{n\in\mathbb{N}} \sigma_n^{-2\nu} < f, v_n >< g, v_n >$$

and the norm

$$\|f\|_{X_\nu} := \left(\sum_{n\in\mathbb{N}} \sigma_n^{-2\nu} | < f, v_n > |^2 \right)^{\frac{1}{2}}$$

built up a pre-Hilbert scale $\{X_\nu\}_{\nu\in\mathbb{R}}$.

Definition 3.1.5. *Let X and Y be two Hilbert spaces, $A : X \longrightarrow Y$ be a non degenerate compact operator with singular system $\{v_n, u_n; \sigma_n\}_{n\in\mathbb{N}}$.*

1. *The problem (A, X, Y) is said to be ill-posed with the order $\alpha > 0$ if $\sigma_n = \mathcal{O}(n^{-\alpha})$.*
   If furthermore, $f \in X_\nu$ with $\nu = \frac{\beta}{\alpha}, \beta > 0,$ is solution to the equation $Af = g, g \in \mathcal{R}(A),$ we say the problem is ill-posed of order (α, β).
2. *The problem (A, X, Y) is said to be exponentially ill-posed if there exist positive constants ϱ and C such that $|\log(\sigma_n)| \geq Cn^\varrho$.*

The ill-posedness is measured by the number $\alpha,$ the *a priori information* about the solution is given by the number ν. The spaces $\{X_\nu\}_\nu$ build up a Hilbert-scale to measure the smoothness of the solution, see [Lou99]. The next lemma, see [Lou89] p.50, shows the connection of this concept of smoothness with the classical Sobolev-spaces.

Lemma 3.1.6. *Let $\Omega \subset \mathbb{R}^n$ be open, $A : L_2(\Omega) \longrightarrow L_2(\Omega)$ be a compact operator with singular system $\{v_n, u_n; \sigma_n\}_{n\in\mathbb{N}}$. If*

$$\|f\|_{L_2(\Omega)} \leq \|Af\|_{H^\alpha(\Omega)} \leq \|f\|_{L_2(\Omega)}, \tag{3.4}$$

then there exist positive constants c_1 and c_2 such that

$$c_1\sigma_n \leq \|v_n\|_{H^\alpha(\Omega)} \leq c_2\sigma_n \quad \textit{and} \quad c_1\sigma_n \leq \|u_n\|_{H^\alpha(\Omega)} \leq c_2\sigma_n, \quad n \in \mathbb{N}. \tag{3.5}$$

Lemma 3.1.7. *Let $M_\gamma : X_{-1} \longrightarrow X$ be a family of linear bounded operators such that*

1. $||M_\gamma f|| \leq c_\alpha \gamma^{-\alpha} ||Af||$ *for all* $f \in \mathcal{N}(A)^\perp$,

2. $||M_\gamma f - f|| \leq c_\beta \gamma^\beta \|f\|$ *for all* $f \in \mathcal{N}(A)^\perp$,

with $\alpha > 0, \beta > 0$. *If*

$$\lim_{\gamma,\varepsilon\to 0} \varepsilon\gamma^{-\alpha} = 0,$$

then $T_\gamma := M_\gamma \overline{A^\dagger}$ *is a linear regularization of* $A^\dagger$ *for finding* f, *where the operator* $\overline{A^\dagger} : Y \longrightarrow X_{-1}$ *denotes the extension by density of* $A^\dagger$ *on* Y.
Furthermore, if $Af = g$, $f \in \mathcal{N}(A)^\perp, g, g^\varepsilon \in Y$, *with*

$$\|g - g^\varepsilon\| \leq \varepsilon \quad \textit{and} \quad \|f\|_{X_\nu} \leq \varrho,$$

where $\nu := \frac{\beta}{\alpha}$, *then there exists a regularizing parameter* $\gamma = \gamma(\varepsilon, \varrho, \alpha, \nu)$ *and a constant* $C > 0$ *such that*

$$\|T_\gamma g^\varepsilon - f\| \leq C\varepsilon^{\frac{\nu}{\nu+1}} \varrho^{\frac{1}{\nu+1}}.$$

Proof. See [Lou99]. □

In other words, the regularization is conditioned by the quality of the mollifier for smoothing and approximating. The approximation M_γ of the identity must be at least of the same smoothing order as the operator A.

3.1.2 Invariance properties

To apply the method of the approximate inverse for solving the equation

$$Af = g,$$

we use the solutions to the adjoint equations

$$A^*\psi_\gamma^x = e_\gamma^x \quad \text{for every } x \in \Omega.$$

As the adjoint equations are independent of the measured data, the inversion algorithm is stable. Nevertheless, the gain in stability has apparently its price, since we have to deal with an operator equation of higher order. Indeed, if A is an integral operator with kernel k, then the corresponding adjoint equations can be reduced to determine a kernel ψ_γ, solution to

$$\int_\Gamma \psi_\gamma(x,z)\, \overline{k(z,y)}\, dz = e_\gamma(x,y),\ x, y \in \Omega.$$

However, if the operator A has invariance properties, which is the case in most applications, we do not need to directly solve the adjoint equations for every $x \in \Omega$.

Lemma 3.1.8. *Let* $A \in \mathcal{L}(X,Y)$. *For* $x \in \Omega$, *let the operators* $T_1^x \in \mathcal{L}(X)$ *and* $T_2^x \in \mathcal{L}(Y)$ *satisfy*

$$T_1^x A^* = A^* T_2^x \quad \textit{and} \quad T_3^x A = A T_1^x.$$

Further, let $e_\gamma^x = e_\gamma(x,\cdot), \gamma > 0,$ *be a mollifier satisfying*

$$e_\gamma^x = T_1^x E_\gamma \quad \textit{with} \quad E_\gamma \in \mathcal{D}((A^*)^\dagger).$$

If ϕ_γ *is a solution of*

$$AA^*\phi_\gamma = AE_\gamma, \tag{3.6}$$

then the solution ψ_γ^x *of*

$$AA^*\psi_\gamma^x = Ae_\gamma^x$$

is given by

$$\psi_\gamma^x = T_2^x\phi_\gamma. \tag{3.7}$$

If we further have $E_\gamma \in \mathcal{R}(A^*),$ *then the solution* ψ_γ^x *of*

$$A^*\psi_\gamma^x = e_\gamma^x$$

is given by

$$\psi_\gamma^x = T_2^x\phi_\gamma, \tag{3.8}$$

where ϕ_γ *is the solution of*

$$A^*\phi_\gamma = E_\gamma. \tag{3.9}$$

The usage of invariance properties reduces dramatically the effort of computation and storage. We have only to solve the adjoint equation (3.6) or (3.9) for few reconstruction points, then deduce the reconstruction kernel ψ_γ^x for every $x \in \Omega$ from (3.8) or (3.7). The evaluation of the solution with the precomputed reconstruction kernel requires only a matrix vector multiplication. This makes inversion schemes based on the approximate inverse very efficient.

3.2 Regularization of the scattering operator

We want now to apply the results of the previous section to regularize the spherical scattering operator using the method of the approximate inverse. As we mentioned before, the smoothing properties of the operator and its order of ill-posedness are closely related to each other.

Theorem 3.2.1. *Let* $0 < \rho \leq R$ *and* $A : X \to Y$ *be the spherical scattering operator with* $X = L_2(B_\rho)$ *and* $Y = L_2(\partial B_R)$.

1. *If* $\rho = R,$ *the problem* (A, X, Y) *is ill-posed of order* $\frac{3}{2}$.
2. *If* $\rho < R,$ *the problem* (A, X, Y) *is exponentially ill-posed.*

Proof. This result follows immediately from definition 3.1.5 and lemma 2.5.3. □

Remarks.

1. We can see from corollary 2.5.19 that the order of ill-posedness of the problem (A, X, Y) for the spherical scattering operator A with $\rho = R$, agrees with its order of smoothing on the Sobolev-scale, which is also equal to $\frac{3}{2}$.
   Despite the apparent mildness of the problem (A, X, Y) when $\rho = R$, it creates, when $k\rho$ is small, as much difficulty as severely ill-posed problems. Definition 3.1.5 relates the ill-posedness to the asymptotic behavior of the singular values for large order, however the fast decay of the singular values of the spherical scattering operator begins earlier at the orders close to $k\rho$, as we can see from the remark following lemma 2.5.4. This behavior is comparable to the situation met in computer-tomography for the limited-angle problem for the Radon transform, see [Lou86] and [Nat01] p. 160.

2. The localization of the target domain, by taking ρ smaller, enhances the ill-posedness of the inverse problem.

Corollary 3.2.2. *The norm $\|\cdot\|_1$ on the space X_1 for the restricted scattering operator A with $\rho = R$, is equivalent to the Sobolev-norm $\|\cdot\|_{H^{\frac{3}{2}}(\Omega)}$.*

Proof. This directly follows from the theorem above, lemma 3.1.6 and corollary 2.5.19. □

We look now into the invariance properties of the spherical scattering operator.

Lemma 3.2.3. *Let $A \in \mathcal{L}(L_2(B_\rho), L_2(\partial B_R))$, $0 < \rho \leq R$, be the spherical scattering operator and U be a rotation about the origin of $\mathbb{R}^3$.*
On the spaces $X_1 = L^2(B_\rho), X_2 = L^2(B_R)$, we define the operators $T_i^U \in \mathcal{L}(X_i), i \in \{1, 2\}$, by

$$T_i^U f = f \circ U \quad \textit{for} \quad f \in X_i,\ i = 1, 2.$$

Then we have

$$T_1^U A^* = A^* T_2^U, \tag{3.10}$$
$$A\ T_1^U = T_2^U A. \tag{3.11}$$

Proof. We prove only (3.10), the identity (3.11) can be proved similarly.
Let $\varphi \in X_2, x \in B_\rho$. We have

$$\begin{aligned}
T_1^U A^* \varphi(x) &= A^* \varphi(Ux) \\
&= \int_{\partial B_R} \overline{g_k(|Ux - y|)} \varphi(y)\, d\sigma(y) \\
&= \int_{\partial B_R} \overline{g_k(|x - U^* y|)} \varphi(y)\, d\sigma(y) \\
&= \int_{\partial B_R} \overline{g_k(|x - t|)} \varphi(Ut)\, d\sigma(t) \\
&= A^* T_2^U \varphi(x),
\end{aligned}$$

where we used the substitution $y = Ut$ and the fact that U is orthogonal and preserves ∂B_R invariant. $\square$

Although, the spherical scattering operator is not invariant under translation, the effort to compute its reconstruction kernel may be significantly reduced. The previous lemma incites us to consider mollifiers of the form

$$e_\gamma(x,y) = e_\gamma^0(x-y), x, y \in \mathbb{R}^3,$$

with e_γ^0 is invariant under rotation.

Theorem 3.2.4. *For $\gamma > 0$, let $e_\gamma(x,y) = e_\gamma(|x-y|), x, y \in \mathbb{R}^3$, be a mollifier satisfying $e_\gamma^x := e_\gamma(x,\cdot) \in \mathcal{R}(A^*) \oplus \mathcal{R}(A)^\perp$ for every $x \in B_\rho$. With the notation of theorem 2.5.2, the reconstruction kernel for the spherical scattering operator A is given by*

$$\psi_\gamma(x,y) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \sigma_n^{-1} w_{n,m}^\gamma(x) u_{n,m}(y),\ x \in B_\rho \setminus \{0\}, y \in \partial B_R, \tag{3.12}$$

where the functions $w_{n,m}^\gamma$ for $m = -n, \cdots, n, n \in \mathbb{N}$, are defined by

$$w_{n,m}^\gamma = \phi_n^\gamma(|x|)\overline{Y_n^m(\frac{x}{|x|})} \quad for \quad x \in B_\rho \setminus \{0\}, \tag{3.13}$$

with

$$\phi_n^\gamma(u) := 2\pi\gamma_n \int_0^\rho \int_{-1}^1 r^2 j_n(kr) e_\gamma((u^2 + r^2 - 2rut)^{\frac{1}{2}}) P_n(t) dt dr,\ u > 0, \tag{3.14}$$

where P_n is the Legendre Polynomial of order n.

Proof. Let $\gamma > 0, x \in B_\rho$. Using the singular value system of the operator A, the minimum-norm solution of the equation

$$A^* \psi_\gamma^x = e_\gamma^x$$

gives the reconstruction kernel ψ_γ^x at the point x as an expansion

$$\psi_\gamma^x = \sum_{n\in\mathbb{N}}\sum_{m=-n}^{n} \sigma_n^{-1} < e_\gamma^x, v_{n,m} >_{L_2(B_\rho)} u_{n,m}, \tag{3.15}$$

where the convergence holds in $L^2(\partial B_R)$ for $e_\gamma^x \in \mathcal{R}(A^*) \oplus \mathcal{R}(A^*)^\perp$.
Let $x = u\theta$ with $u > 0, \theta \in S^2$. For $m = -n, \cdots, n, n \in \mathbb{N}$, we have

$$\begin{aligned} < e_\gamma^x, v_{n,m} > &= \int_{B_\rho} e_\gamma(|x-y|)\overline{v_{n,m}(y)} dy \\ &= \gamma_n \int_0^\rho r^2 j_n(kr) \left(\int_{S^2} e_\gamma \left((u^2 + r^2 - 2ur < \theta, \omega >)^{\frac{1}{2}} \right) \overline{Y_n^m(\omega)}\, d\omega \right) dr \end{aligned}$$

From the Funk-Hecke theorem we get

$$\int_{S^2} e_\gamma \left((u^2 + r^2 - 2ur < \theta, \omega >)^{\frac{1}{2}} \right) \overline{Y_n^m(\omega)} d\omega = I_n^\gamma(u, r) \overline{Y_n^m(\theta)}$$

with

$$I_n^\gamma(u, r) = 2\pi \int_{-1}^{1} e_\gamma \left((u^2 + r^2 - 2urt)^{\frac{1}{2}} \right) P_n(t) dt,$$

where P_n is the Legendre polynomial. □

The invariance under rotation of the spherical scattering operator reduces the computation of the convolution $< e_\gamma^x, v_{n,m} >$ to the evaluation of the integral I_n^γ, which is cheaper in terms of computational effort. We can further simplify the computation of the kernel as we can see from the following result.

Corollary 3.2.5. *We maintain the notations of the theorem above. The reconstruction kernel for the spherical scattering operator A satisfies*

$$\psi_\gamma(x, y) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi R} \sigma_n^{-1} \phi_n^\gamma(|x|) P_n\left(\frac{x \cdot y}{|x||y|}\right),\ x \in B_\rho \setminus \{0\}, y \in \partial B_R. \qquad (3.16)$$

Proof. The result follows immediately from theorem 3.2.4 and the identity A.13. □

Remark. If $x_0 \in S^2$ is fixed $e.g.\ x_0 := (0, 0, 1)$, the reconstruction kernel for $x \in B_\rho \setminus \{0\}, y \in \partial B_R$, satisfies

$$\psi_\gamma(x, y) = \psi_\gamma(rx_0, tx_0), \quad \text{with} \quad r = |x|, \frac{t}{|t|} = r^{-1} R^{-1} x \cdot y. \qquad (3.17)$$

This means that we have only to compute the reconstruction kernel at the reconstruction points $\{(rx_0, tx_0)$ then use (3.17).

Theorem 3.2.6. *If $e_\gamma(x) = e_\gamma^0(x - y), x, y \in \mathbb{R}^3, \gamma > 0$, is a mollifier satisfying*

1. $e_\gamma^0 \in H^{\frac{3}{2}}(\mathbb{R}^3)$ *with* $\|e_\gamma^0\|_{H^{\frac{3}{2}}(\mathbb{R}^3)} \le c_\alpha \gamma^{-\alpha}$,

2. $||(2\pi)^{\frac{3}{2}} \widehat{e}_\gamma^0 - 1||_\infty \le c_\beta \gamma^\beta$,

where $c_\alpha, c_\beta > 0, \alpha, \beta > 0$, are constants, then the approximate inverse S_γ defined by (3.12) is a regularization for the pseudo-inverse $A^\dagger$ of the restricted scattering operator A with $\rho = R$.
Furthermore, let $g, g^\varepsilon \in Y$ and $Af = g$ with $f \in \mathcal{N}(A)^\perp$. If

$$\|g - g^\varepsilon\| \le \varepsilon \quad \text{and} \quad \|f\|_{X_\nu} \le \varrho$$

where ε, ϱ are positive constants and $\nu := \frac{\beta}{\alpha}$, then there exists a regularizing parameter $\gamma = \gamma(\varepsilon, \varrho, \alpha, \nu)$ such that

$$\|S_\gamma g^\varepsilon - f\| \le C \varepsilon^{\frac{\nu}{\nu+1}} \varrho^{\frac{1}{\nu+1}} \qquad (3.18)$$

with some constant $C > 0$.

Proof. To prove theorem 3.2.6 we apply lemma 3.1.7.
Let M_γ be the smoothing operator defined for $f \in L_2(\mathbb{R}^3)$ by $M_\gamma f(x) :=< f, e_\gamma^x >$. Let $f \in \mathcal{N}(A)^\perp$ given by the expansion

$$f = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_n^m v_n^m,$$

with respect to the singular functions $\{v_n^m\}_{n,m}$ of A. We have

$$\begin{aligned} ||M_\gamma f||^2 &= \int_{B_R} | < f, e_\gamma^x > |^2 dx \\ &= \int_{B_R} |\sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_n^m < v_n^m, e_\gamma^x > |^2 dx \\ &\le \left(\sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sigma_n^2 |f_n^m|^2 \right) \left(\int_{B_R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} | < \sigma_n^{-1} v_n^m, e_\gamma^x > |^2 dx \right). \\ &\le \|f\|_{X_{-1}}^2 \left(\int_{B_R} \|e_\gamma^x\|_{X_1}^2 dx \right) \end{aligned}$$

From corollary 3.2.2 there exists a constant $c > 0$ such that

$$\|e_\gamma^x\|_{X_1} \le c \|e_\gamma^x\|_{H^{\frac{3}{2}}}.$$

Since $\widehat{e_\gamma^x}(\xi) = e^{i<\xi,x>} \widehat{e_\gamma^0}(\xi)$, we have

$$\|e_\gamma^x\|_{H^{\frac{3}{2}}} = \|e_\gamma^0\|_{H^{\frac{3}{2}}}.$$

For $f \in \mathcal{N}(A)^\perp$ we get

$$\begin{aligned} \|M_\gamma f\| &\le c \text{ vol}(B_R)^{\frac{1}{2}} \|f\|_{X_{-1}} \|e_\gamma^0\|_{H^{\frac{3}{2}}} \\ &\le C \gamma^{-\alpha} \|f\|_{X_{-1}}, \end{aligned}$$

where $\text{vol}(B_R)$ denotes the volume of B_R and $C := c\, c_\alpha \text{ vol}(B_R)^{\frac{1}{2}}$. The first condition in lemma 3.1.7 is then proved.
By virtue of the Parseval identity and the convolution theorem for the Fourier transform, we obtain

$$\begin{aligned} ||M_\gamma f - f||^2 &\le \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} e_\gamma^0(x-y) f(y) dy - f(x) \right|^2 dx \\ &\le \int_{\mathbb{R}^3} |((2\pi)^{\frac{3}{2}} \hat{e}_\gamma^0(\xi) - 1) \hat{f}(\xi)|^2 \, d\xi \\ &\le ||(2\pi)^{\frac{3}{2}} \widehat{e_\gamma^0}(\xi) - 1)||_\infty^2 \int_{\mathbb{R}^3} |\hat{f}(\xi)|^2 \, d\xi \\ &\le (c_\beta \gamma^\beta)^2 ||f||^2, \end{aligned}$$

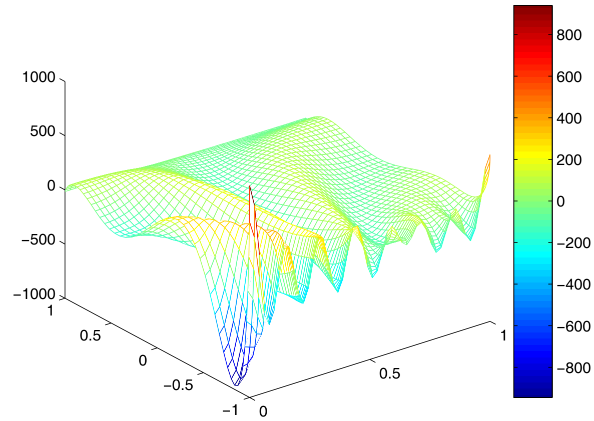

Figure 3.1: The real part of the reconstruction kernel for $k = 15, \gamma = 0.05$, $R = \rho = 1$.

which proves the second condition in lemma 3.1.7 and finishes the proof of the theorem. □

Remark. In the figures 3.1-3.4 we give for the spherical scattering operator $A_{\rho,R}$, plots of the reconstruction kernel computed for the Gauss-mollifier, where ϕ_n^γ is evaluated numerically.

3.3 Resolution of the inverse source problem

We consider a spherical geometry with the ball $\Omega = B_\rho$ as a target region and the sphere $\Gamma = \partial B_R$ as a measurement surface, $0 < \rho \leq R$.
Let $A_\Gamma = A_{\rho,R}$ be the spherical scattering operator and A_Ω be the scalar scattering operator.
For $N \in \{3, 6\}$, the measurement operator $\mathbf{M}_\Gamma$ is defined by

$$\mathbf{M}_\Gamma \mathbf{g} := (\gamma_0 g_j)_j, \mathbf{g} = (g_j)_{j\in\{1,\cdots,N\}} \in H^1(\Omega)^N,$$

where γ_0 is the trace mapping on the boundary Γ.
We denote by $\mathbf{A}_\Omega$ and $\mathbf{A}_\Gamma$ the operators defined by

$$\mathbf{A}_\Omega \mathbf{f} := (A_\Omega f_j)_j, \mathbf{f} = (f_j)_{j\in\{1\cdots N\}} \in L_2(\Omega)^N,$$

and

$$\mathbf{A}_\Gamma \mathbf{f} := (A_\Gamma f_j)_j, \mathbf{f} = (f_j)_{j\in\{1\cdots N\}} \in L_2(\Omega)^N,$$

respectively. To simplify the notations, we may denote A_Ω and A_Γ both by A, and denote $\mathbf{A}_\Omega$ and $\mathbf{A}_\Gamma$ by $\mathbf{A}$.

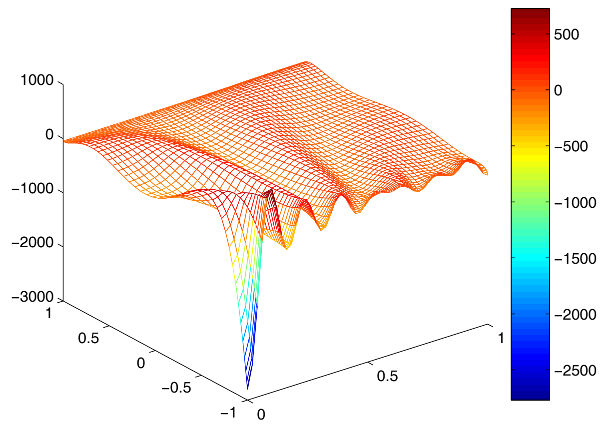

Figure 3.2: The imaginary part of the reconstruction kernel for $k = 15, \gamma = 0.05$, $R = \rho = 1$.

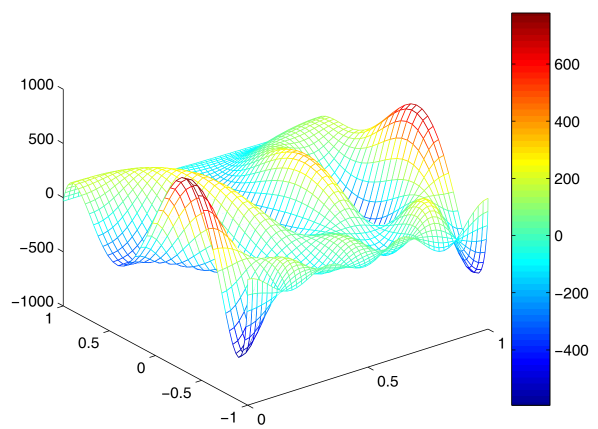

Figure 3.3: The real part of the reconstruction kernel for $k = 15, \gamma = 0.05$, $R = 1, \rho = 0.5$.

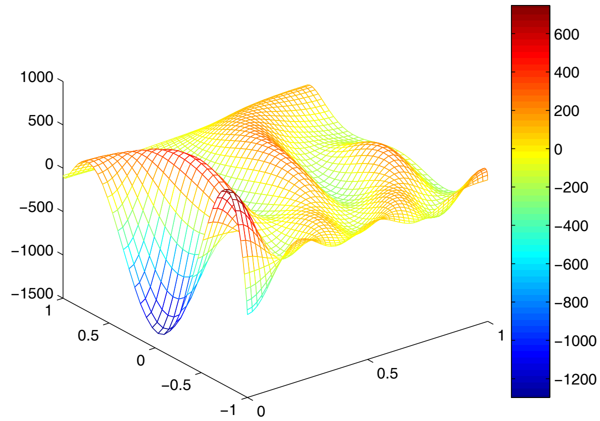

Figure 3.4: The imaginary part of the reconstruction kernel for $k = 15$, $\gamma = 0.05$, $R = 1, \rho = 0.5$.

3.3.1 The nonmagnetic case

We consider the inverse source problem for a nonmagnetic medium.
Given $\mathbf{d} = (d_j)_{j=1,2,3.} \in L_2(\Omega)^3$, determine $\mathbf{q} \in H_0^2(\Omega)^3$ such that

$$\mathbf{M}_\Gamma \mathbf{T}_e \mathbf{q} = \mathbf{d}, \tag{3.19}$$

where

$$\mathbf{T}_e = k^2 \mathbf{A} - \mathbf{K}_d$$

is the nonmagnetic scattering operator.
We may write

$$\mathbf{T}_e = k^2 \mathbf{A} - \mathbf{A}' \,\mathbf{div},$$

where $\mathbf{A}'$ is the second scattering operator.
The equation (3.19) reads then

$$k^2 \mathbf{A}\mathbf{q}(x) - (\mathbf{A}' \,\mathbf{div}\, \mathbf{q})(x) = \mathbf{d}(x), \;\; x \in \Gamma. \tag{3.20}$$

By virtue of theorem 2.4.4 we have

$$\mathbf{T}_e = \mathbf{A}\mathbf{P}, \tag{3.21}$$

where the nonmagnetic scattering potential $\mathbf{P}$ is given by

$$\mathbf{P}\mathbf{q} = k^2 \mathbf{q} + \mathbf{grad}(\mathbf{div}\, \mathbf{q}).$$

Our inverse source problem can thus be reformulated as the following system

$$\begin{cases} \mathbf{A}_\Gamma \mathbf{w} = \mathbf{d} \\ \mathbf{P} \;\; \mathbf{q} = \mathbf{w} \end{cases}. \tag{3.22}$$

We utilize the approximate inverse S_γ of $A_\Gamma^\dagger$ for some smooth mollifier e_γ to compute an approximation to the generalized induced source $\mathbf{w}_\gamma = (w_\gamma^j)_j, \gamma > 0$, obtained by

$$w_\gamma^j = S_\gamma d_j,\ j \in \{1,2,3\}, \tag{3.23}$$

which is the regularized solution of the first equation in (3.22).
When we replace $\mathbf{w}$ by the smoother approximation $\mathbf{w}_\gamma$ in the second equation of (3.22), we get using theorem 2.4.5

$$\mathbf{div}\,\mathbf{q}_\gamma = -A\,\mathbf{div}\,\mathbf{w}_\gamma. \tag{3.24}$$

The evaluation of $v_\gamma := -A\,\mathbf{div}\,\mathbf{w}_\gamma$ determines an approximation to $\mathbf{div}\,\mathbf{q}$, which we insert in the second term of the left-hand side of (3.20). Hence, we can approximate $\mathbf{A}'\,\mathbf{div}\,\mathbf{q}$ in (3.20) by computing

$$\mathbf{c}_\gamma = \mathbf{A}' v_\gamma. \tag{3.25}$$

The problem is reduced to solve the equation

$$\mathbf{A}_\Gamma \mathbf{q} = \mathbf{g}_\gamma \tag{3.26}$$

with the new right-hand side $\mathbf{g}_\gamma = (g_\gamma^j)_{j=1,2,3.}$ given by

$$\mathbf{g}_\gamma = k^{-2}(\mathbf{d} + \mathbf{c}_\gamma). \tag{3.27}$$

To solve the equation (3.26), we apply again the approximate inverse. We finally recover the source $\mathbf{q}_\gamma = (q_\gamma^j)_{j=1,2,3.}$ from

$$q_\gamma^j = S_\gamma g_\gamma^j,\ \ j \in \{1,2,3\}. \tag{3.28}$$

Remarks.

1. Since the operator A is of convolution type we can use FFT techniques, see [Nat01] p. 207, to compute v_γ. The evaluation of $\mathbf{c}_\gamma = \mathbf{A}' v_\gamma$ is only needed on Γ. Besides, the resolution of the inverse source problem involves the application of the approximate inverse twice and the computation of the divergence in (3.24). All these computations can be carried efficiently with a low cost in terms of storage and computational effort.

2. The differentiation in (3.24) to compute the $\mathbf{div}\,\mathbf{w}_\gamma$ is not problematic provided we choose a smooth mollifier e_γ. An alternative way to proceed is to carry the differentiation on the mollifier itself. This approach will be treated for the general case of electromagnetic media.

3.3.2 The electromagnetic case

We consider now the inverse source problem for an electromagnetic medium.
Given $\mathbf{d} = (d_j)_{j=1,\cdots,6} \in L_2(\Omega)^6$, determine $\mathbf{f} \in H_0^2(\Omega)^6$ such that

$$\mathbf{M}_\Gamma \mathcal{T}_{em} \mathbf{f} = \mathbf{d}, \tag{3.29}$$

where $\mathcal{T}_{em}$ is the electromagnetic scattering operator.
We denote $\mathcal{V}$ the first electromagnetic scattering potential and $\mathcal{W}$ the second electromagnetic scattering potential.

Let $\mathcal{X} = (L_2(\Omega))^6$ and $\mathcal{Y} = (L_2(\Gamma))^6$. We define a bilinear form on $\mathcal{X} \times \mathcal{X}$ by

$$((\mathbf{f}, \mathbf{g}))_{\mathcal{X}} = (< f_i, g_i >)_{i=1,\cdots,6}, \quad \text{for} \quad \mathbf{f} = (f_j)_j, \mathbf{g} = (g_j)_j \in \mathcal{X}$$

and a bilinear form on $\mathcal{Y} \times \mathcal{Y}$ by

$$((\mathbf{f}, \mathbf{g}))_{\mathcal{Y}} = (< f_i, g_i >)_{i=1,\cdots,6}, \quad \text{for} \quad \mathbf{f} = (f_j)_j, \mathbf{g} = (g_j)_j \in \mathcal{Y}.$$

We may in the sequel abusively denote $((\cdot,\cdot))_{\mathcal{X}}$ and $((\cdot,\cdot))_{\mathcal{Y}}$ both by $((\cdot,\cdot))$.

Definition 3.3.1. *Let* $\mathbf{E}^x_\gamma = (E^x_{\gamma,j})_j \in L_2(\Omega)^6, \gamma > 0,$ *be a mollifier and*

$$\tilde{\mathbf{E}}^x_\gamma(y) = -\mathcal{W}\mathbf{A}_\Omega\, \mathbf{E}^x_\gamma(y) \tag{3.30}$$

with respect to x, *for* y *fixed. The function* $\mathbf{k}_\gamma : \Omega \times \Gamma \longrightarrow \mathbb{C}^6, \mathbf{k}_\gamma = (k_{\gamma,j})_j$, *defined by*

$$k_{\gamma,j}(x,y) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sigma_n^{-1} u_n^m(y) < \tilde{\mathbf{E}}^x_{\gamma,j}, v_n^m >, j \in \{1 \cdots 6\}, \tag{3.31}$$

is called the reconstruction kernel, and the operator $\mathcal{S}_\gamma$ *defined for* $\mathbf{g} \in (L_2(\Gamma))^6$ *by*

$$\mathcal{S}_\gamma \mathbf{g}(x) = ((\mathbf{g}, \mathbf{k}_\gamma(x,\cdot))), \ x \in \Omega,$$

is called the approximate inverse, for the electromagnetic inverse source problem.

Remark. The idea of using a modified version of the approximate inverse based on the bilinear forms $((\cdot,\cdot))_{\mathcal{X}}$ and $((\cdot,\cdot))_{\mathcal{X}}$, instead of the scalar products of the underlying vector field spaces, had been already applied by Schuster to vector tomography, see [Sch00] for more details.

In the sequel, the mollifier is assumed to be smooth enough to justify the underlying interchanges of integration and differentiation and to permit the passages to the limit.

Lemma 3.3.2. *For a smooth mollifier* $\mathbf{E}^x_\gamma = (E^x_{\gamma,j})_{j=1,\cdots,6}, \gamma > 0,$ *let* $\mathcal{S}_\gamma$ *be the corresponding approximate inverse for the electromagnetic inverse source problem. Then*

$$\mathcal{S}_\gamma \mathbf{g}(x) = ((\tilde{\mathbf{E}}^x_\gamma, \mathbf{A}_\Gamma^\dagger \mathbf{g})), \ \mathbf{g} = (g_j)_j \in L_2(\Gamma)^6, \ x \in \Omega, \tag{3.32}$$

where $\tilde{\mathbf{E}}^x_\gamma(y) = -\mathcal{W}\mathbf{A}_\Omega\, \mathbf{E}^x_\gamma(y)$ *with respect to* x, *for* y *fixed.*

Proof. For $j \in \{1, \cdots, 6\}$, $\mathbf{g} = (g_j)_j \in L_2(\Gamma)^6$, we have

$$\begin{aligned}
(\mathcal{S}_\gamma \mathbf{g})_j &:= < g_j, k_{\gamma,j} > \\
&= \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sigma_n^{-1} < g_j, u_n^m >< \tilde{\mathbf{E}}^x_{\gamma,j}, v_n^m > \\
&= < \tilde{\mathbf{E}}^x_{\gamma,j}, \mathbf{A}_\Gamma^\dagger \mathbf{g} >,
\end{aligned}$$

Thus (3.32) holds since

$$\mathbf{A}_\Gamma^\dagger \mathbf{g} = (A_\Gamma^\dagger g_j)_j,\ \mathbf{g} = (g_j)_j \in L_2(\Gamma)^6.$$

□

Theorem 3.3.3. *Let* $\mathbf{E}_\gamma^x = (E_{\gamma,j}^x)_{j=1,\cdots,6}, \gamma > 0,$ *be a smooth mollifier and* $\mathcal{S}_\gamma$ *be the corresponding approximate inverse for the electromagnetic inverse source problem. For* $\mathbf{d} = (d_j)_j \in L_2(\Gamma)^6$ *let* $\mathbf{f}_\gamma := \mathcal{S}_\gamma \mathbf{d}$. *We suppose that*

$$\lim_{\gamma \to 0} \mathbf{f}_\gamma = \mathbf{f} \quad \textit{with} \quad \mathbf{f} = (f_j)_j \in H_0^2(\Omega)^6.$$

Then it holds

$$\mathbf{M}_\Gamma \mathcal{T}_{em} \mathbf{f} = \mathbf{d}. \tag{3.33}$$

Proof. From lemma 3.3.2 we have

$$\mathbf{f}_\gamma(x) = \mathcal{S}_\gamma \mathbf{d}(x) = ((\tilde{\mathbf{E}}_\gamma^x, \mathbf{A}_\Gamma^\dagger \mathbf{d})) = ((-\mathcal{W}^x \mathbf{A}_\Omega^x\, \mathbf{E}_\gamma^x, \mathbf{A}_\Gamma^\dagger \mathbf{d})),$$

where the index x on the top of the operators indicates that they act on functions depending on x.
It yields

$$\mathbf{f}_\gamma(x) = -\mathcal{W}^x \mathbf{A}_\Omega^x ((\mathbf{E}_\gamma^x,\ \mathbf{A}_\Gamma^\dagger \mathbf{d})) = -\mathcal{W} \mathbf{A}_\Omega \mathbf{q}_\gamma(x)$$

with

$$\mathbf{q}_\gamma(x) = ((\mathbf{A}_\Gamma^\dagger \mathbf{d},\ \mathbf{E}_\gamma^x)) = (S_\gamma^j d_j(x))_j,$$

where S_γ^j is the approximate inverse for the scalar operator A_Γ and the mollifier $E_{\gamma,j}(x,y) = E_{\gamma,j}^x(y), x \in \Omega, y \in \Gamma$.
Passing to the limit, as γ tends to 0, we get

$$\mathbf{f} = -\mathcal{W} \mathbf{A}_\Omega \mathbf{q} \quad \text{with} \quad \mathbf{q} = \lim_{\gamma \to 0} \mathbf{q}_\gamma,$$

The regularizing property of the approximate inverse implies

$$\mathbf{A}_\Gamma \mathbf{q} = \mathbf{d}.$$

The application of theorem 2.4.8 finishes the proof. □

Remarks. We see clearly from our problem the flexibility one can gain by using the approximate inverse for regularization. Thanks to the mollifier, many operations are transposed to the adjoint problem, which is independent of the data. We can therefore choose the mollifier appropriately to withdraw difficulties from the main problem to the adjoint problems. Invariance properties are to be conveyed to the inverse scattering problems for electromagnetic media.

3.4 Resolution of the inverse medium problem

We describe now an iterative procedure to reconstruct the contrast function from data of multiple experiments using the generalized induced source formulation for the inverse scattering problem.

Let X and Y be two Hilbert spaces, $\tilde{V}$ be a subspace of X, the bilinear operator $Q : \tilde{V} \times X \longrightarrow X$ define the product $Q(f,u) = fu$ of $f \in \tilde{V}$ and $u \in X$. Let V be an open subset of $\tilde{V}$. For $f \in V$, we denote $Q_f \in \mathcal{L}(X)$ the operator defined by $Q_f u := Q(f,u)$, $u \in X$.
We consider $T \in \mathcal{L}(X)$ such that for $f \in V$ the operators $(I+TQ_f)$ and $(I+Q_fT)$ are bounded and invertible with $(I+TQ_f)^{-1}$ and $(I+Q_fT)^{-1}$ as inverse operators, respectively.
The nonlinear Lippmann-Schwinger operator $\mathcal{A} : V \times X \longrightarrow X$ for the scattering operator T is, therefore, well-defined and satisfies

$$\mathcal{A}(f,u^0) = (I+TQ_f)^{-1}u^0, \quad f \in V, u^0 \in X. \tag{3.34}$$

We further suppose that there exist $A \in \mathcal{L}(X)$ and $U \in \mathcal{L}(X)$ such that

$$T = AU, \tag{3.35}$$

where U is invertible with inverse U^{-1}.
For the $j-$th experiment, $j \in I := \{1, \cdots, p\}, p \in \mathbb{N}$, let $u_j^0 \in X$ denote the incident field.
We assume, for the sake of simplicity, that we have the same measurement operator $M_\Gamma \in \mathcal{L}(X,Y)$ for all experiments.
We denote $A_\Gamma := M_\Gamma A$ and suppose that $\{S_\gamma\}_{\gamma>0}$ is a regularization for the ill-posed problem (A_Γ, X, Y).

Definition 3.4.1. *We define the (nonlinear) source operator $\phi : V \times X \longrightarrow X$ for $f \in V, u_j^0 \in X$ by*

$$\phi(f,u_j^0) = fu \quad \text{with} \quad u = \mathcal{A}(f,u_j^0). \tag{3.36}$$

We denote $\phi_j(f) := \phi(f,u_j^0)$ and $\phi_j'(f)$ its Fréchet-derivative.
Further, we define the nonlinear operator $F_j : V \longrightarrow Y$ for $f \in V$ by

$$F_j(f) = M_\Gamma T \phi_j(f). \tag{3.37}$$

We denote F_j' the Fréchet-derivative of F_j and F the opeartor

$$F(f) = (F_1(f), \cdots, F_p(f))^T. \tag{3.38}$$

For the data

$$d = (d_1, \cdots, d_p) \in Y^p,$$

the inverse medium problem can be formulated as a least-square problem:

$$F(f) = d. \tag{3.39}$$

The application of the Gauss-Newton method to solve (3.39) requires the inversion of the operator $(F_j'(f)^* F_k'(f))^{-1}_{j,k=1,\cdots,p}$ at each Newton-iteration. For most applications, this challenging computational effort makes the implementation unfeasible with standard computers.
To overcome this difficulty, one approach is to modify the Jacobian operator using some approximation. However, the costs for storage and computation remains a real defiance. In the framework of electromagnetic inversion we refer, about this approach, to the works of Newmann *et al* [NH00],[NB04] and Haber [Hab04].
Alternatively, we use the method applied by Natterer and Wübelling for ultrasound tomography [NW95],[NW01], by Dorn *et al.* [DBABC99],[DBABP02] and also by Vögeler [Voe03] for electromagnetic imaging. The quasi-Newton method of Gutmann and Klibanov [GK93],[GK94]. uses also a similar approach.
The idea is to solve in each step only the linearized equation

$$F_j(f) + F_j'(f)h_j = d_j. \tag{3.40}$$

The minimal norm solution of (3.40) is given by

$$h_j = -F_j'(f)^* C_j^{-1}(F_j(f) - d_j) \quad \text{with} \quad C_j = F_j(f)F_j'(f)^*. \tag{3.41}$$

In each step an approximation f^0 is incremented, using some relaxation factor $w \in (0, 2)$, into a new approximation f^1 as follows

1. $f_0 = f^0$
2. For $j = 1, \cdots, p$:

$$f_j = f_{j-1} + wh_j \quad \text{with} \quad h_j = -F_j'(f_{j-1})C_j^{-1}(F_j(f_{j-1}) - d_j) \tag{3.42}$$

3. $f^1 = f_p$.

This is the extension to nonlinear problems of the Kaczmarz method or algebraic reconstruction technique (ART) of computerized tomography for solving under-determined linear systems, see [Nat01] and [NW01].
The computational effort to evaluate C_j^{-1} is, however, still too demanding. In [NW95], [DBABC99],[DBABP02],[Voe03], the operator C_j, is replaced by an easy to compute and invert matrix, namely a constant factor times the identity.
To avoid such an empirical consideration, we proceed differently to solve the equation (3.42), where we make use of the equivalent source formulation.

Theorem 3.4.2. *Let $j \in I$, $d_j \in \mathcal{D}(A_\Gamma^\dagger)$, $f \in V$. We suppose that for $\gamma > 0$ there exists $h_j^\gamma \in \mathcal{R}(F_j'(f)^*)$ such that*

$$\phi_j'(f)h_j^\gamma = U^{-1}S_\gamma d_j - \phi_j(f). \tag{3.43}$$

Then h_j^γ tends to h_j, as γ tends to 0, with

$$h_j = -F_j'(f)^* \left(F_j'(f)F_j'(f)^*\right)^{-1}(F_j(f) - d_j).$$

Proof. For $h_j^\gamma \in \mathcal{R}(F_j'(f)^*)$ we have

$$h_j^\gamma = F_j'(f)^*(F_j'(f)^*)^\dagger h_j^\gamma = F_j'(f)^* \left(F_j'(f)F_j'(f)^*\right)^{-1} F_j'(f)h_j^\gamma.$$

As $F_j'(f) = M_\Gamma AU\phi_j'(f) = A_\Gamma U\phi_j'(f)$, we get using (3.35) and (3.37)

$$\begin{aligned} h_j^\gamma &= F_j'(f)^* \left(F_j'(f)F_j'(f)^*\right)^{-1} A_\Gamma U\phi_j'(f)h_j^\gamma \\ &= F_j'(f)^* \left(F_j'(f)F_j'(f)^*\right)^{-1} A_\Gamma U \left(U^{-1}S_\gamma d_j - \phi_j(f)\right) \\ &= F_j'(f)^* \left(F_j'(f)F_j'(f)^*\right)^{-1} \left(A_\Gamma S_\gamma d_j - F_j(f)\right) \end{aligned}$$

Since S_γ is a regularization for A_Γ, $A_\Gamma S_\gamma d_j$ tends to d_j, as γ tends to 0, and consequently

$$\lim_{\gamma\to 0} h_j^\gamma = h_j. \tag{3.44}$$

□

Lemma 3.4.3. *Let* $f \in V$. *The Fréchet-derivative* $\phi_j'(f) \in \mathcal{L}(X)$ *of the source operator* $\phi_j(f)$ *for some incident field* $u_j^0 \in X$ *is given by*

$$\phi_j'(f)h = (I + Q_fT)^{-1}h(I + TQ_f)^{-1}u_j^0,\ h \in \tilde{V}. \tag{3.45}$$

Proof. Let $f \in V$. The source operator $\phi_j(f)$ for some incident field $u_j^0 \in X$ is defined by

$$\phi_j(f) = fu \quad \text{with} \quad u = u_j^0 - Tfu.$$

If we multiply both sides of the last equation by f we obtain

$$\phi_j(f) = fu_j^0 - fT\phi_j(f). \tag{3.46}$$

It yields

$$\phi_j'(f)h = hu_j^0 - fT\phi_j'(f)h - hT\phi_j(f),\ h \in \tilde{V}. \tag{3.47}$$

That is

$$(I + Q_fT)(\phi_j'(f)h) = hu \quad \text{with} \quad u = \mathcal{A}(f, u_j^0) = (I + TQ_f)^{-1}u_j^0. \tag{3.48}$$

□

Remarks.

1. At each iteration, we need to solve for given f the forward problem

$$u_j = (I + TQ_f)^{-1}u_j^0$$

just once, from which we deduce $\phi_j(f) = fu_j$.
Let denote

$$q_j^\gamma := U^{-1}S_\gamma d_j - \phi_j(f). \tag{3.49}$$

No further application of the forward solver is required for the determination of h_j^γ satisfying

$$h_j^\gamma (I + TQ_f)^{-1} u_j^0 = (I + Q_f T) q_j^\gamma. \tag{3.50}$$

When the scattering operator T is a convolution operator, which is the case in the underlying applications, we can use FFT techniques to evaluate the right-hand side in (3.50). If we further take the approximate inverse as the regularization S_γ for A_Γ, the computation of q_j^γ can be conducted efficiently and in a stable way with a low computational cost.

2. We do not discuss the issue of convergence here. For linear systems the Kaczmarz method converges for $w \in (0,2)$. The conditions of convergence of this method for linear and nonlinear systems are discussed in [Nat01] and [NW01].

Examples.

1. For scalar inverse scattering problems, we take Q simply the product of scalar-valued functions, U the identity and A the scalar scattering operator.

2. For vector inverse scattering problems, we take the product Q componentwise, that is $Q(f,u) = (f^i u^i)_{i=1,\cdots,3}$ for the nonmagnetic case and $Q(f,u) = (f^i u^i)_{i=1,\cdots,6}$ for the general electromagnetic case. This choice of Q comprehends the general case of anisotropic media. The factorization (3.35) is given by theorem 2.4.4.

3.5 AIGIS-N scheme

For nonmagnetic media, we obtain the inversion scheme **AIGIS** standing for (The **A**pproximate **I**nverse with the **G**eneralized **I**nduced **S**ource) for the resolution of the inverse source problem.
For the resolution of the inverse medium we incorporate the nonlinear-component to obtain **AIGIS-N** with **N** for nonlinear. The scheme reads then:

AIGIS-N

Precomputation
Given a mollifier e_γ,
compute the reconstruction kernel ψ_γ
for the approximate inverse S_γ.

Input:
For each experiment $j \in I = \{1, \cdots, p\}$ we have:

- Incident electrical fields $\mathbf{E}_j^{inc}(x), x \in \Omega$.
- Scattered electrical fields $d^j(x) \in \mathbb{C}^3, x \in \Gamma$.

Linear inversion
For each experiment $j \in I = \{1, \cdots, p\}$:

1. Compute the generalized induced source $\mathbf{w}_\gamma^j = (S_\gamma d_i^j)_{i=1,2,3.}$
2. Compute $v_\gamma^j \simeq \mathbf{div}\, \mathbf{q}^j$: $v_\gamma^j = -A\, \mathbf{div}\, \mathbf{w}_\gamma^j$
3. Compute $\mathbf{c}_\gamma^j = \mathbf{A}'\, v_\gamma^j$
4. Compute $\mathbf{g}_\gamma^j = k^{-2}(\mathbf{d}^j + \mathbf{c}_\gamma^j)$
5. Compute the equivalent source $\mathbf{q}^j = (S_\gamma\, \mathbf{g}_{\gamma,i}^j)_i$

Nonlinear iterations
Given:
Relaxation parameter: w
Initial guess: f^0
Maximal number of iterations: N

For $n = 1, \cdots, N$:

1. $f_0 = f^{n-1}$
2. For $j = 1, \cdots, p$:
   (a) $\mathbf{E}_j = (\mathbf{I} + \mathbf{T}_e\, f_{j-1})^{-1}\mathbf{E}_j^{inc}$
   (b) $h_j^\gamma :\ h_j^\gamma \mathbf{E}_j^0 = (\mathbf{I} + f_{j-1}\, \mathbf{T}_e)(\mathbf{q}_j^\gamma - f_{j-1}\, \mathbf{E}_j)$
   (c) $f_j = f_{j-1} + w\, h_j^\gamma$
3. $f^n = f_p$.

Chapter 4

Numerical results

To numerically test the AIGIS-N scheme for the resolution of the inverse source and inverse medium problems, we consider two objects with simple geometry, namely a sphere and a cylinder. This choice is motivated by the availability of an analytical solution for the forward scattering problem. The dimension and the electromagnetic properties of the buried objects are taken close to realistic values for some medical applications of microwave-tomography, see [SBS$^+$00]. The scatterers are nonmagnetic with electrical contrasts beyond the Born/Rytov approximation, according to the condition (15) in [KS88] p. 214 .

Reconstruction for the MIE-Model

For our first numerical experiment, we use the analytical solution of the Mie scattering problem. It concerns the electromagnetic scattering of a monochromatic plane wave, linearly polarized, by a homogeneous ball with radius $r,$ constant electrical permettivity ε and electrical conductivity $\sigma,$ which is immersed in a homogeneous nonconducting medium with constant electric permittivity ε_0 and magnetic permeability μ_0.
We take a cartesian system of coordinates $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ with the origin at the center of the sphere. The incident electrical field is polarized in the $x-$direction and the wave is propagating in the $z-$direction, that is

$$\mathbf{E}^{inc}(x) = \exp(ikz)\,\mathbf{e}_x$$

with the angular frequency [1] $\omega,$ and the wavenumber $k = w\sqrt{\varepsilon_0\mu_0}$.
We assume that the scatterer is non-magnetic and has the refractive index [2]

$$n = \varepsilon_0^{-1}(\varepsilon + i\frac{\sigma}{\omega}) \tag{4.1}$$

with the dielectric permittivity ε and electric conductivity σ. The contrast function is

$$f = (1-n)\chi, \tag{4.2}$$

[1] The frequency is given by $\nu = \frac{w}{2\pi}$.

[2] In [BW99], the refractive index is alternatively defined by $n = (\varepsilon_0^{-1}(\varepsilon + i\frac{4\pi\sigma}{\omega}))^{\frac{1}{2}}$.

where χ is the characteristic function of the ball centered at the origine with radius r. The Mie or also called Debye formulae provide the analytical solution to time-harmonic Maxwell equations. They are derived using spherical coordinates by matching boundary conditions to determine transmission coefficients, see (57)-(62) in [BW99] p.772 and also [Kon90] p.493. We use these formulae to simulate the scattered field outside the scatterer.

For the numerical experiment we make the following considerations:

1. Incident electrical field:
   (a) Plane wave propagating in the z-direction.
   (b) Monochromatic wave with wavenumber $k = 30\,m^{-1}$ corresponding to the frequency $\nu \simeq 1.43\,GHz$.
   (c) Linearly polarized in $x-$direction.
2. Scatterer:
   (a) Homogeneous spherical sphere centered at the origin with radius $r = 20\,cm$.
   (b) Non-magnetic medium with the electrical contrast function $f = 48 + 14i$.
3. Experimental setting:
   (a) The measurement set Γ is a sphere centered at the origine with radius $R = 1$.
   (b) The target domain is a ball centered at the origine with radius $\rho = 1$.
   (c) Data are the simulated complex-valued scattered electrical field $E^s(x), x \in \Gamma$.
   (d) Noisy data are endowed with a normally distributed random error of 10%.
4. Discretization
   (a) The computational domain is a cube $[-1, 1]^3$.
   (b) The number of grid points in each direction is 64.
5. The mollifier is the Gauss function.

In figures 4.1-22, we give 1-D, 2-D and 3-D representations of the reconstructed complex-valued equivalent source $f\mathbf{E}$.
Reconstructions from noisy data with a random error of 10% are presented in figures 4.19-4.22.
From the obtained results, we can make the following observations:

1. The location is coarsely determined. Dominant oscilliations announce the crossing of the boundary. This can be clearly seen for the component parallel to the direction of polarisation, as well as the transversal component, of the reconstruction.

2. An estimate of the contrast is given by the amplitude of the dominant oscillia- tions for the component parallel to the polarization.

3. The shape is clear, however, this may be also due to the spherical symmetry of the problem.

4. Almost no information is contained in the component of the equivalent source parallel to the direction of propagation.

5. With noisy data we see that the linear inversion is stable.

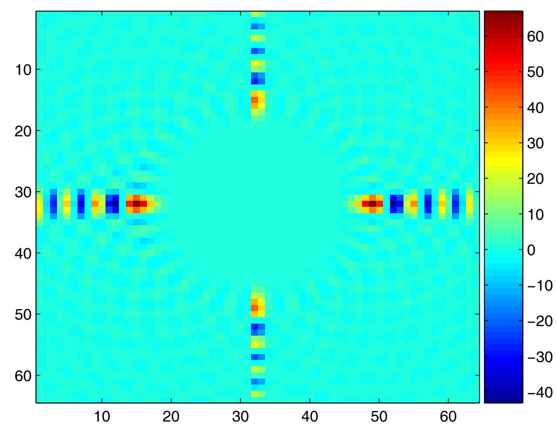

Figure 4.1: The real part of the $x-$component of the recovered equivalent source in the $zy-$plane through the origin.

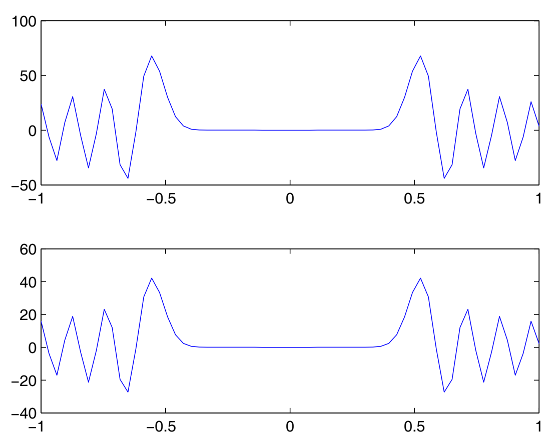

Figure 4.2: The crossing-plot along the axes of the figure above

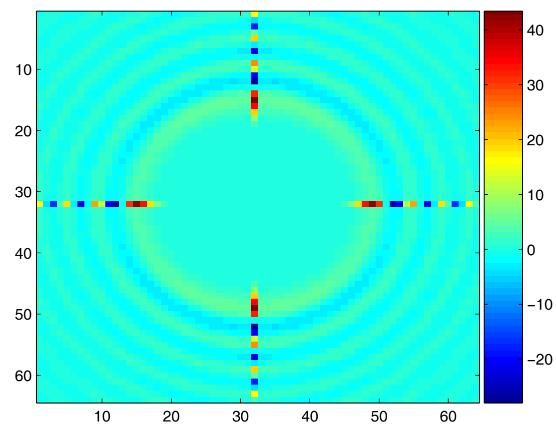

Figure 4.3: The real part of the $x-$component of the recovered equivalent source in the $yx-$plane through the origin.

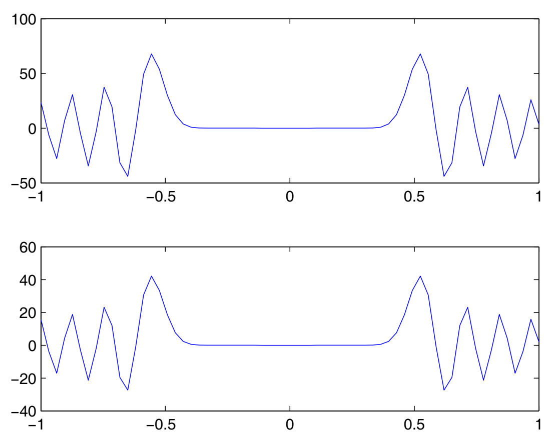

Figure 4.4: The crossing-plot along the axes of the figure above

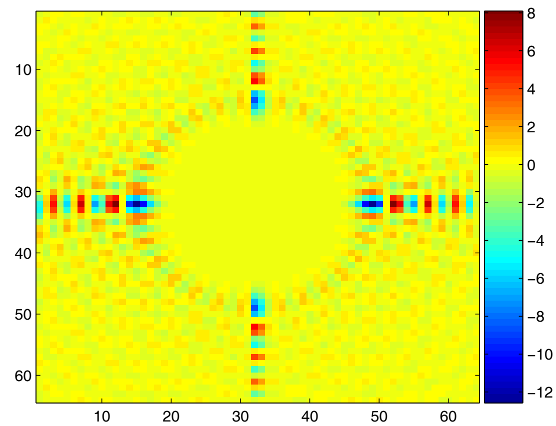

Figure 4.5: The imaginary part of the $x-$component of the recovered equivalent source in the $zy-$plane through the origin.

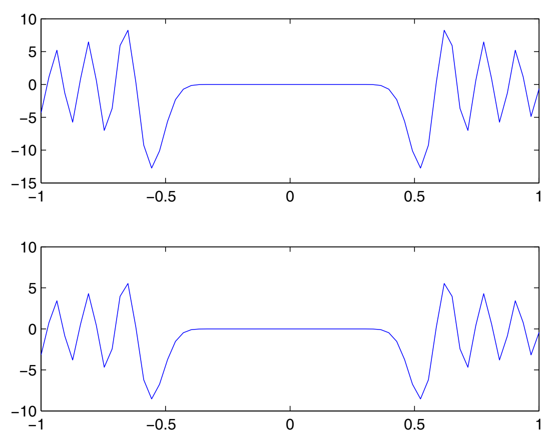

Figure 4.6: The crossing-plots along the axes of the figure above

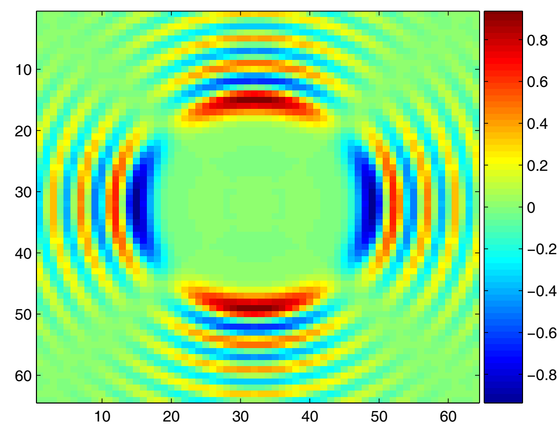

Figure 4.7: The imaginary part of the $y-$component of the recovered equivalent source in the $yx-$plane through the origin.

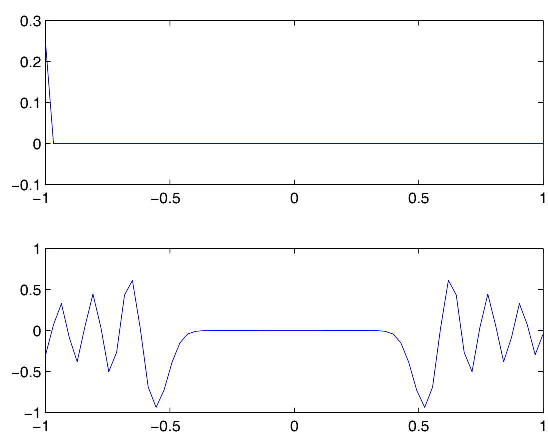

Figure 4.8: The crossing-plots along the axes of the figure above

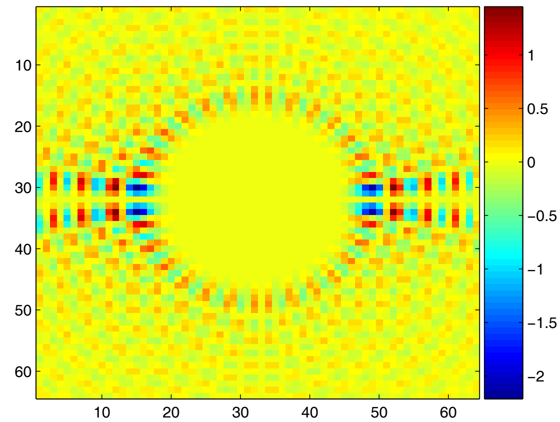

Figure 4.9: The real part of the $y-$component of the recovered equivalent source in the $zy-$plane through the origin.

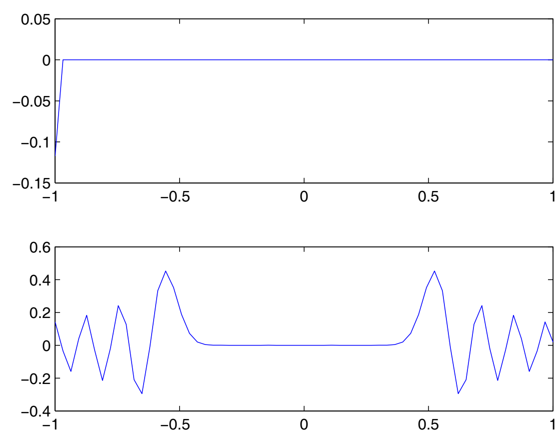

Figure 4.10: The crossing-plot along the axes of the figure above

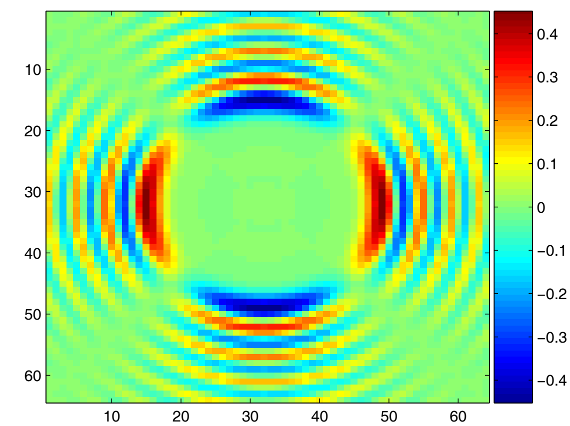

Figure 4.11: The real part of the $y-$component of the recovered equivalent source in the $yx-$plane through the origin.

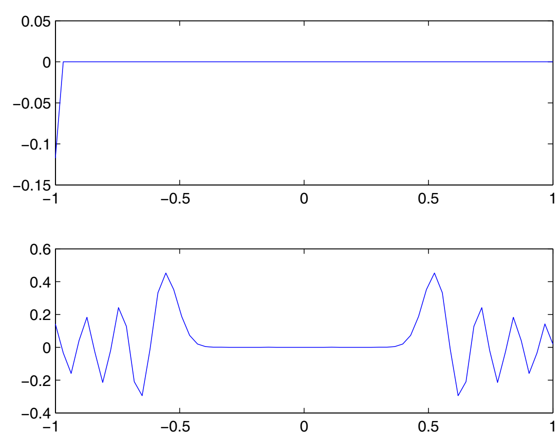

Figure 4.12: The crossing-plot along the axes of the figure above

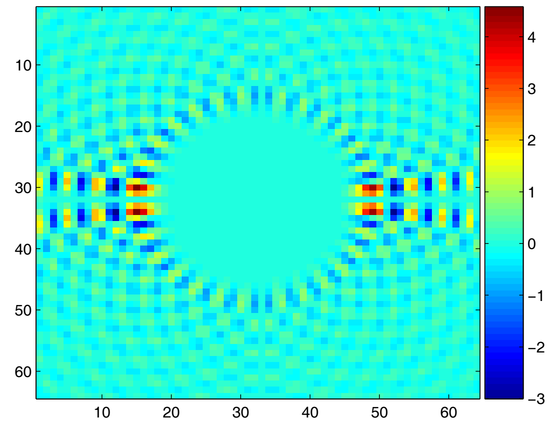

Figure 4.13: The imaginary part of the $y-$component of the recovered equivalent source in the $zy-$plane through the origin.

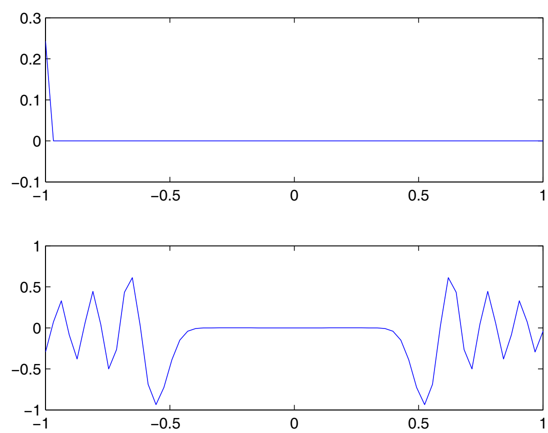

Figure 4.14: The crossing-plots along the axes of the figure above

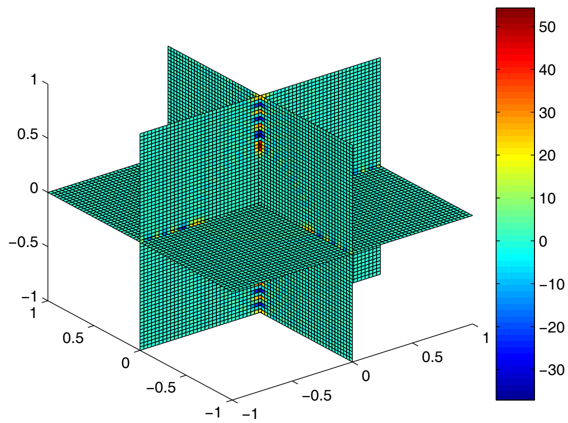

Figure 4.15: Real part of the x-component of the computed equivalent current source.

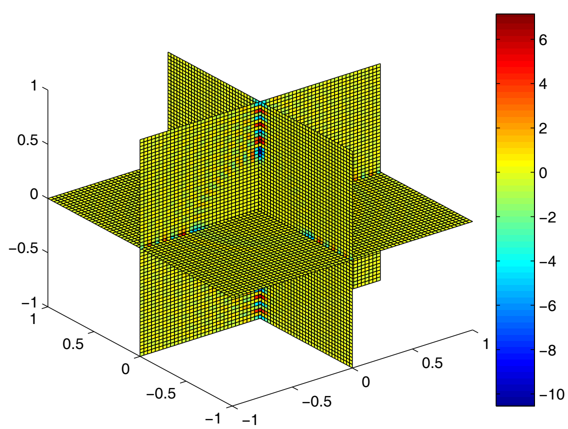

Figure 4.16: Imaginary part of the x-component of the computed equivalent current source .

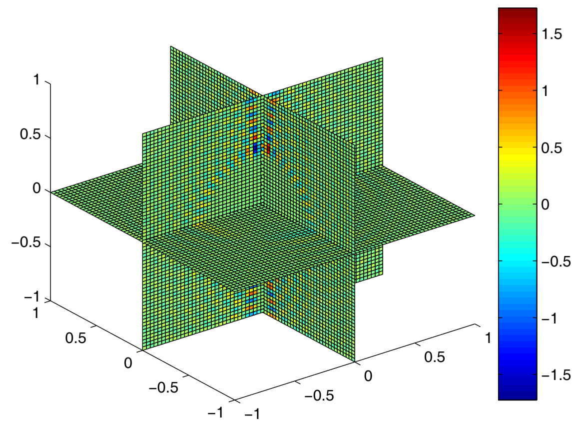

Figure 4.17: Real part of the y-component of the computed equivalent current source.

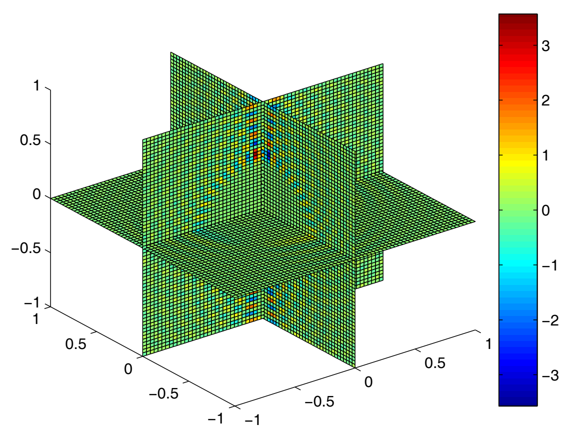

Figure 4.18: Imaginary part of the y-component of the computed equivalent current source .

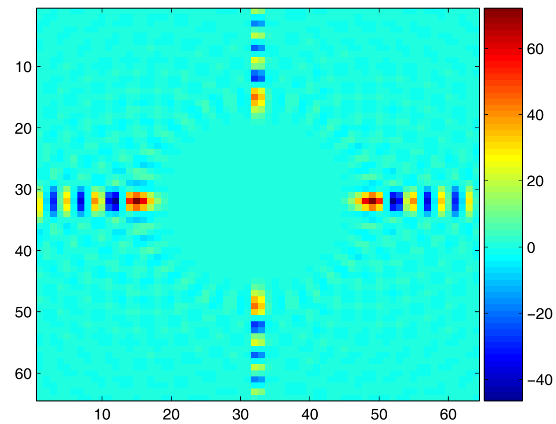

Figure 4.19: The real part of the $x-$ component of the equivalent source in the $zy-$ plane through the origin recovered from noisy data at a level of 10%.

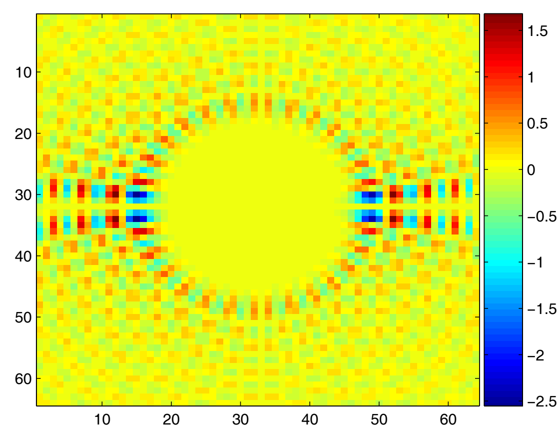

Figure 4.20: The real part of the $y-$ component of the equivalent source in the $zy-$ plane through the origin recovered from noisy data at a level of 10%.

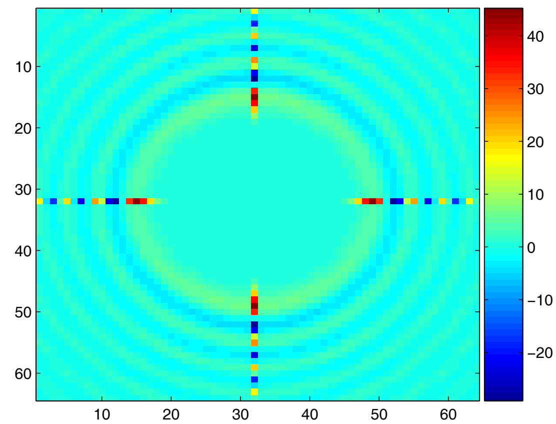

Figure 4.21: The real part of the $x-$ component of the equivalent source in the $yx-$ plane through the origin recovered from noisy data at a level of 10%.

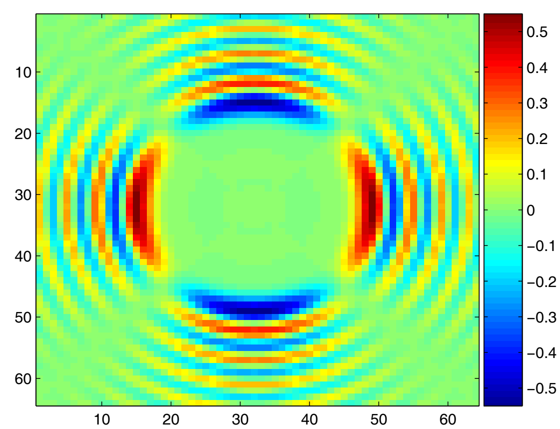

Figure 4.22: The real part of the $y-$ component of the equivalent source in the $yx-$ plane through the origin recovered from noisy data at a level of 10%.

Nonlinear reconstruction for a homogeneous cylinder

To test the nonlinear reconstruction, we take as scatterer a homogeneous cylinder. The analytical solution is also obtained by matching the boundary conditions, see for example [Kon90] p. 497.

1. Incident electrical field properties:
   (a) Plane wave propagating in each of the directions $\mathbf{d}_\alpha \in S^2$, such that $\mathbf{d}_\alpha = \mathbf{d}_\alpha(\alpha, 0)$ in cylindrical coordinates.
   (b) Monochromatic wave at the frequency $\nu \simeq 4\, GHz$.
   (c) Linearly polarized in $z-$direction.
2. Scatterer properties:
   (a) Homogeneous cylinder parallel to the polarization direction z and passing through the point $(0, -0.25, 0)$. It has the axial-length $L = 1\, m$ and the cross-section with the radius $\rho = 25\, cm$.
   (b) Dielectric medium with real-valued contrast function $f = 3$.
3. Experimental setting:
   (a) The measurement set Γ is a sphere centered at the origin with radius $R = 1m$.
   (b) The target domain is a ball centered at origin with radius $\rho = R$.
   (c) Data are the the simulated complexe-valued scattered electrical field $E^s_\alpha(x)$, $x \in \Gamma$ for each direction $\alpha = \frac{2\pi}{m}, m = 0, \cdots, M-1, M = 36$.
4. The mollifier is the Gauss function.

For each direction d_α, we make the linear inversion to recover the equivalent source $\mathbf{q}^\alpha = (q_j^\alpha)_j$.
The main difficulty we face to implement the nonlinear scheme is to incorporate a reliable forward solver for the vector integral equations modeling the direct problem. One would think to formulate these equations in convolution form, and use FFT techniques to solve them. Unfortunately, this approach involves hypersingular kernels, which cause the run-away of the resolution. Hence, the development of a forward solver is not immediate and would timely escape out of the objectives of this work. Commercial programs generally use the formulation as partial differential equations and involve boundary conditions which are unrealistic for our problem.
To convince the reader about the potential of our nonlinear program, we make some simplifications to enable an easy implementation. Since the electrical field is polarized parallely to the axis of the cylinder, the problem has symmetry. We consider, therefore, the reconstruction of the contrast function only in the plane orthogonal to the axis of the cylinder and passing through the origine. Since this problem is predominently 2-D, we can take for the forward solver in the nonlinear iterarions the scattering operator

$$T\, q_j(X) = k^2 \int_{\mathbb{R}} h_k(|X - Y|)\, q_j(Y)\, dY,\; X = X(x, y) \in \mathbb{R}^2,$$

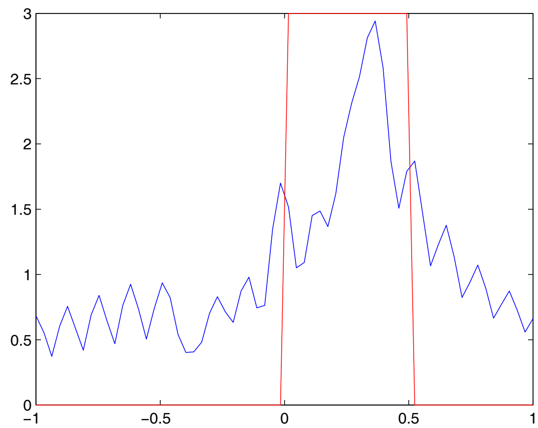

Figure 4.23: The plot of $f(0, y, 0), y \in [-1, 1]$ of the reconstructed contrast function for a dielectric homogeneous cylinder parallel to $z-$direction, centered at (0,-0.25,0) and with cross-section of the Radius $R = 0.25$.

where $h_k(t) = \frac{i}{4} H_0^{(1)}(kt), t > 0,$ is the first kind Hankel function of order zero, see [Kon90]. The dependence of $\mathbf{q}^\alpha$ on z has been neglected. As an intial guess for the contrast function, we take

$$f_0 = \frac{1}{M} \sum_{\alpha=0}^{M-1} \frac{|q_z^\alpha|}{|E_z^\alpha|}$$

The reconstruction results with the AIGIS-N scheme are presented in figures (4.23)-(4.25). We can make the following remarks:

1. The inhomogeneity is detected with satisfacotry approximation for the dielectric constant and the location. The shape reconstruction is however disappointing.
2. Artifacts are infecting the reconstruction. This may be due to the over-simplification of the forward solver. The artifacts depend on how we sweep the incident directions $\mathbf{d}_\alpha$ in the each iteration of the nonlinear scheme.
3. Some further regularization at the nonlinear iterations seems to be requiered.

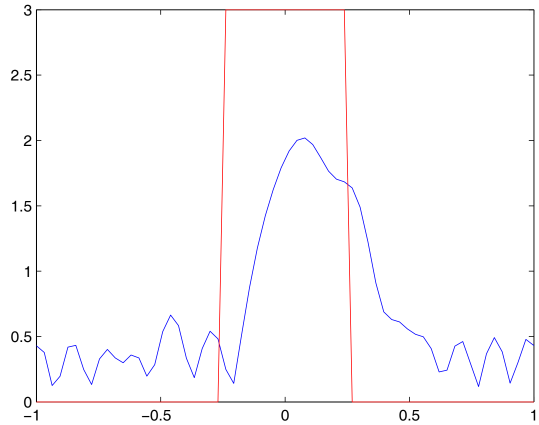

Figure 4.24: The plot of $f(x, -0.25, 0), x \in [-1, 1]$ of the reconstructed contrast function for a dielectric homogeneous cylinder parallel to $z-$direction, centered at (0,-0.25,0) and with cross-section of the Radius $R = 0.25$.

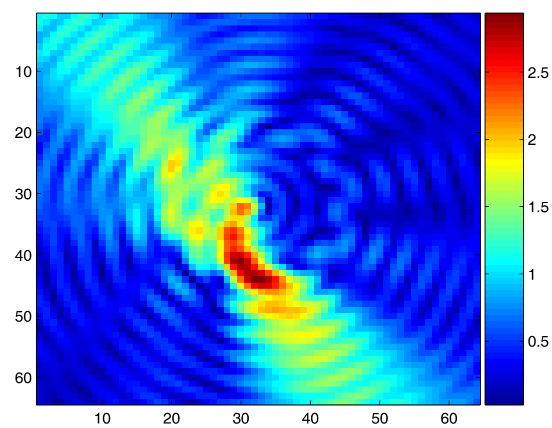

Figure 4.25: Reconstructed contrast function in the xy-plane through the origin of a dielectric homogeneous cylinder parallel to $z-$direction, centered at (0,-0.25,0) and with cross-section of the Radius $R = 0.25$.

Conclusion

We introduced a new method to solve the inverse electromagnetic scattering problem in inhomogeneous media for the full three-dimensional time-harmonic Maxwell-model formulated as contrast source integral equations. This method can be characterized by the following features:

- The utilization of the equivalent source to split inverse scattering into two subproblems: the inverse source problem, which is linear and ill-posed, and the inverse medium problem, which is more stable but nonlinear.
- The introduction of the concept of generalized induced source to recast the vector problem into decoupled scalar problems.
- The utilization of the approximate inverse to recover, for each experiment, the non-radiating induced source.
- The adaptation of the nonlinear version of the algorithm of Kaczmarz, using the generalized induced source, to derive an iterative scheme for the resolution of the inverse medium problem.
- Theoretical results for the spherical configuration permitting the analysis of the method. The application is extendible to other experimental setting.
- The general framework of the method, which is applicable to inhomogeneous electromagnetic media with compact anomalies, which are magnetic or nonmagnetic, isotropic or anisotropic.

From the computational experiments carried at frequencies in the microwave domain for highly-contrasting nonmagnetic isotropic spherical and cylindrical objects, we can make the following observations:

- The linear inversion is efficient, robust and stable even with noisy data.
- From a single experiment, we can rapidly obtain an estimate of the location, the shape, the size of the buried object and an approximation of the sought-for contrast function. This information can be used to restrain the target domain, which consequently reduces the computational effort. Further, to choose the appropriate initial value when executing the iterative scheme.

- For multiple experiments, the resolution of the inverse source problem can be solved in a parallel way.

- Further improvements are required to ameliorate the quality of the reconstruction. The underlying artifacts seem to be related to the over-simplification of the forward solver. They also depend on the way the incident directions are swept in each iteration. However, the iterative scheme is capable to detect inhomogeinities with high contrasts, beyond the Born/Rytov approximation.

Outlook

As interesting issues for further investigations we can mention:

- Since the method is developed in a quite general setting to deal with the problem of inverse electromagnetic scattering in inhomogeneous media with compact anomalies, further experiments are to be executed using simulated and experimental data, for more complicated geometries, for isotropic or anisotropic materials, for media with magnetic properties.

- The adaptation of the method to the limited angle problem for the spherical measurement setting is feasible, similarly to [Lou86] for the limited-angel problem in computer tomography.

- The consideration in this work of the spherical configuration for the experimental setting has been motivated by the available analytical tools for the theoretical investigation of the problem. The method is extendible to other measurement settings.

- The quality of reconstruction should be improved if a reliable forward solver for the three-dimensional volume integral equation is incorporated in the nonlinear iterations.

- A way out to avoid the resolution of the forward problem at each iteration step is to make a precomputation, where the forward solver is applied to the elements of the bases of the radiating and non-radiating sources. The least-square problem is then solved with respect to the underlying bases, see [HODH95] in the framework of 2-D Helmholtz equation.

- We can look for a strategy to choose how to run the looping of the directions of incidence at each iteration of the nonlinear scheme since the quality of the reconstruction is sensible to the way we do it.

- The analysis of the convergence of the nonlinear algorithm is still to be done.

- A sensitivity analysis, see [DBABP02] and [SH95], is required for a better understanding of the nonlinear scheme. It also helps to discuss the dilemma between the nonlinearity and the ill-posedness of the inverse problem and how they are influenced by the strength of the contrast on the one hand, and by the scaling related to the size of the object compared to the wavelength of the incident field, on the other hand. This question is of particular importance at low-frequency regime as we can see from the analysis of the spherical scattering operator.

- The treatment of multi-frequency and dynamical problems with this method is still open.
- We may also investigate the extension of the method to inverse scattering problems of elastic waves, see [Sni01], where more complicated tensorial constitutive relations are involved.
- A key point of the method used here is the factorization of the operator underlying the main problem into a product of two operators. One is linear and ill-posed, the other is more stable but nonlinear. This approach can be used as a technique to treat further nonlinear inverse problems. The advantage of applying the approximate inverse for regularization is that we can withdraw many difficulties emerging within the inversion to adjoint problems, which we set for mollifiers that can be appropriately chosen.

Appendix A

Spherical harmonics

Let S^2 be the unit sphere in $\mathbb{R}^3$ i.e.

$$S^2 = \{ (x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 = 1 \}. \tag{A.1}$$

We notice that in spherical coordinates (r, θ, φ) we have

$$L_2 (\mathbb{R}^3) = L_2 ([0, \infty), r^2 dr) \otimes L_2 (S^2, d\xi) \tag{A.2}$$

with

$$\xi = (\theta, \varphi), \; d\xi = \sin\theta \, d\theta \, d\varphi \, . \tag{A.3}$$

We further have the decomposition of $L_2 (S^2, d\xi)$ into

$$L_2 (S^2, d\xi) = \bigoplus_{n=0}^{+\infty} V_n, \tag{A.4}$$

where V_n is the eigenspace corresponding to the eigenvalue $-n(n+1)$ of the Laplace-Beltrami operator given by

$$\Delta_B = \frac{1}{\sin\theta} [\frac{\partial}{\partial\theta} (\sin\theta) \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\varphi} (\frac{1}{\sin\theta} \frac{\partial}{\partial\varphi})]. \tag{A.5}$$

The subspaces $V_n, \; n \in \mathbb{N}$, have a finite dimension

$$\dim V_n = 2n + 1 \tag{A.6}$$

and are generated by the spherical harmonics $Y_n^m, \; n \in \mathbb{N}, \; m = -n, \ldots, n$, which are eigenfunctions of the operator $(-\Delta_B)$ *i.e.*

$$-\Delta_B \, Y_n^m = n(n+1) Y_n^m. \tag{A.7}$$

The system $\{ Y_n^m \}_{n \in \mathbb{N}, \, m = -n, \ldots, n}$ is complete and orthonormal in $L_2 (S^2)$.
Hence, the space $L_2 (\mathbb{R}^3)$ can be decomposed into

$$L_2 (\mathbb{R}^3) = \bigoplus_{n=0}^{+\infty} W_n \tag{A.8}$$

with

$$W_n = L_2\,(\,[0,\infty), r^2 dr\,) \otimes V_n, \tag{A.9}$$

which means

$$\begin{aligned} W_n = \{\, f \in L_2\,(\mathbb{R}^3\,),\ f(r,\theta,\varphi) = \varphi(r)\, g_n(\theta,\varphi), \\ g_n = \sum_{m=-n}^{+n} a_{nm}\, Y_n^m\ , a_{nm} \in \mathbb{C},\ \varphi \in L_2\,(\,[0,\infty), r^2 dr\,)\,\} \end{aligned} \tag{A.10}$$

In spherical coordinates, the spherical harmonics of order n are given for $m = -n, \ldots, n$ by

$$Y_n^m(\theta,\varphi) := \left[\frac{2n+1}{4\pi}\,\frac{(n-|m|)!}{(n+|m|)!}\right]^{\frac{1}{2}} P_n^{|m|}\,(\cos\theta)\, e^{im\varphi}. \tag{A.11}$$

The functions $P_n^m,\ n \in \mathbb{N},\ m = 0, 1, \ldots, n$ are solutions to the associated Legendre differential equation

$$(1-t^2)\, y^{''}(t) - 2t\, y^{'}(t) + \left[n(n+1) - \frac{m^2}{1-t^2}\right] y(t) = 0, \quad t \in (-1,+1). \tag{A.12}$$

For fixed m the associated Legendre functions P_n^m for $n = m, m+1, \ldots$ form a complete orthogonal system of $L_2\,(\,[-1,+1]\,)$.
The spherical harmonics $Y_n^m,\ n \in \mathbb{N},\ m = -n, \ldots, n$ of order $n \in \mathbb{N}$ satisfy

$$\sum_{m=-n}^{+n} Y_n^m(\xi)\, \overline{Y_n^m(\eta)} = \frac{2n+1}{4\pi}\, P_n\,(<\xi,\eta>) \tag{A.13}$$

for every $\xi, \eta \in S^2$, where $P_n = P_n^0$ is the Legendre polynomial of order n.

Theorem A.0.1 (Funk-Hecke). *For a function h on $[-1,+1]$ and $\theta \in S^2$, we have*

$$\int_{S^2} h\,(\theta.\,\omega)\, Y_n\,(\omega)\, d\omega = \left(2\pi \int_{-1}^{+1} h\,(t)\, P_n\,(t)\, dt\right) Y_n(\theta).$$

Appendix B

Spherical Bessel functions

We consider the spherical Bessel differential equation

$$r^2 y^{''}(r) \;+ 2ry^{'}(r) \;+\; [\, r^2 - n(n+1)\,]\, y(r) = 0, \quad r > 0. \tag{B.1}$$

Particular solutions to (B.1) are the spherical Bessel function of the first kind

$$j_n\,(r) = (\,\frac{\pi}{2r}\,)^{\frac{1}{2}}\; J_{n+\frac{1}{2}}\,(r), \tag{B.2}$$

the spherical bessel function of the second kind

$$y_n\,(r) = (\,\frac{\pi}{2r}\,)^{\frac{1}{2}}\; Y_{n+\frac{1}{2}}\,(r), \tag{B.3}$$

and the spherical Bessel function of the third kind

$$h_n^{(1)}\,(r) \;=\; j_n\,(r) \;+\; iy_n\,(r) = (\,\frac{\pi}{2r}\,)^{\frac{1}{2}}\; H^{(1)}_{n+\frac{1}{2}}\,(r), \tag{B.4}$$

$$h_n^{(2)}\,(r) \;=\; j_n\,(r) \;-\; iy_n\,(r) = (\,\frac{\pi}{2r}\,)^{\frac{1}{2}}\; H^{(2)}_{n+\frac{1}{2}}\,(r), \tag{B.5}$$

where for $\nu \in \mathbb{C} \setminus (-\infty, 0)$, J_ν denote the Bessel function of the first kind, Y_ν denote the Bessel function of the second kind also called Neumann function , $H_\nu^{(1)}$ and $H_\nu^{(2)}$ denote the Bessel functions of the third kind also called first and second Hankel functions. They are all called cylindrical functions and are solutions to the Bessel differential equation

$$r^2\, g^{''}(r) \;+\; r\, g^{'}(r) \;+\; (r^2 - \nu^2)\, g(r) = 0 \tag{B.6}$$

with $\nu = n + \frac{1}{2}$.

Since J_ν and Y_ν are linearly independent for $\nu \notin \mathbb{Z}$, the pairs $\{j_n,\, y_n\}$ and $\{h_n^{(1)},\, h_n^{(2)}\}$ are linearly independent for every $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, we have the series representation

$$j_n(r) = \sum_{p=0}^{\infty} \frac{(-1)^p \, r^{n+2p}}{2^p \, p! \, (1.3 \ldots (2n+2p+1))}. \tag{B.7}$$

$$y_n(r) = -\frac{(2n)!}{2^n n!} \sum_{p=0}^{\infty} \frac{(-1)^p \, r^{2p-n-1}}{2^p \, p! \, (-2n+1)(-2n+3) \ldots (-2n+2p-1)}. \tag{B.8}$$

We need a further result about the asymptotic behaviour of the spherical Bessel functions. Debey's formula, see [AS72] 9.3.7, describes the behavior of $J_\nu(x)$ as both ν and x tend to infinity. The estimate in [Nat01] p.198 gives a more precise description of $J_\nu(\vartheta\nu),\ 0 < \vartheta < 1$. A similar result holds for the derivative $J'_\nu(\nu\vartheta),\ 0 < \vartheta < 1$, see [Wat58] (10) p.255.

Lemma B.0.2. *For $\nu > 0$ and $0 < \vartheta < 1$, we have*

$$0 \le J_\nu(\nu\vartheta) \le (2\pi\nu)^{-\frac{1}{2}} (1-\vartheta^2)^{-\frac{1}{4}} \varphi(\nu,\vartheta), \tag{B.9}$$

$$0 \le \vartheta J'_\nu(\nu\vartheta) \le (2\pi\nu)^{-\frac{1}{2}} (1+\vartheta^2)^{\frac{1}{4}} \varphi(\nu,\vartheta), \tag{B.10}$$

with

$$\varphi(\nu,\vartheta) = e^{-(\frac{\nu}{3})(1-\vartheta^2)^{\frac{3}{2}}}.$$

We finally state the following theorem on solutions to the Helmholtz equation in polar coordinates, called spherical wave functions see [CK98] p. 30.

Theorem B.0.3 (Spherical wave functions)**.** *Let Y_n be a spherical harmonic of order n. Then*

$$v_n(x) = j_n(k\,|x|)\, Y_n(\frac{x}{|x|})$$

is an analytic solution to the Helmholtz equation in $\mathbb{R}^3$ and

$$u_n(x) = h_n^{(1)}(k\,|x|)\, Y_n(\frac{x}{|x|})$$

is a radiating solution, singular at the origin, to the Helmholtz equation in $\mathbb{R}^3 \setminus \{0\}$.

Appendix C

Spectral decomposition

Theorem C.0.4. *Let $V \hookrightarrow H$ be two Hilbert spaces with compact injection, V being dense in H,* $\mathrm{a}(u,v)$ *a continuous symmetric bilinear form on $V \times V$,* **A** *the self-adjoint unbounded operator with domain defined by*

1. $\mathrm{a}(u,v) = (\boldsymbol{A}\, u, v)$ *for all $u \in D(\boldsymbol{A})$ and $v \in V$.*
2. $D(\boldsymbol{A}) := \{\, u \in V, v \mapsto \mathrm{a}(u,v)$ *is continuous on V for the topology on* $H\,\}$.

If a *is coercive i.e. there exists a constant $\alpha > 0$ with*

$$\mathrm{a}(u,u) \geq \alpha \, \|u\|^2 \quad \textit{for all} \quad u \in V,$$

then

1. *the spectrum $\sigma(\boldsymbol{A})$ is discrete and*

$$\sigma(\boldsymbol{A}) = \{\lambda_k\}_{k \in \mathbb{N}} \quad \textit{with } 0 < \alpha \leq \lambda_k \to \infty \textit{ as } k \to \infty.$$

2. *the eigenvectors ω_k of the operator* **A** *normalised in H and associated with the λ_k satisfy the variational equation*

$$\begin{cases} \mathrm{a}(\omega_k, v) = \lambda_k\,(\omega_k, v) & \textit{for all } v \in V, \\ \|\omega_k\|_H = 1. \end{cases}$$

3. *the vector subspace generated by ω_k is dense in V and in H, the functions $\{\omega_k\}_k$ form an orthonormal basis of H.*

Lemma C.0.5. *Let $a, b \in \mathbb{R}$,* $\mathrm{q} \in C^0([a,b])$. *For given $\gamma_1, \gamma_2 \in \mathbb{R}$, let* p *be the bilinear form defined by*

$$\mathrm{p}(u,v) = \gamma_2\, u(b)v(b) - \gamma_1\, u(a)v(a), \quad u, v \in C([a,b]).$$

The symmetric bilinear form a *defined on* $H^1([a,b]) \times H^1([a,b])$ *by*

$$\mathrm{a}\,(u,v) = \int_a^b \left[\, u^{'}(x)\, v^{'}(x) + q(x)\, u(x)\, v(x) \,\right] dx + \mathrm{p}\,(u,v), \quad u,v \in H^1([a,b]).$$

is continuous,
If there exists a constant $M > 0$ *such that if*

$$\mathrm{q}\,(x) \geq M \quad \textit{for all } x \in [a,b], \tag{C.1}$$

then the bilinear form a *is coercive on* $H^1([a,b]) \times H^1([a,b])$*, i.e. there exists* $\alpha > 0$ *such that*

$$\mathrm{a}\,(u,u) \geq \alpha\, \|u\|^{\,2} \quad \textit{for all } u \in V,$$

where

$$\|u\|^{\,2} = \int_a^b \left(|u^{'}|^{\,2} + |u|^{\,2} \right) dx.$$

Remark. Let q satisfy the conditions of the previous lemma. Using theorem C.0.4, we can construct an orthonormal basis in $L_2\,(a,b)$ of eigenfunctions $\{e_{n,\,k}\}_{k\in\mathbb{N}}$ with corresponding eigenvalues $\{\lambda_{n,\,k}\}_{k\in\mathbb{N}}$ satisfying

$$-e_k^{''} + q\, e_k = \lambda_k\, e_k, \quad \lambda_k \neq 0,$$

in a weak sense.
If we further have q of class $C^m, m \in \mathbb{N}$, then e_k is of class C^{m+2} and the differential equation is satisfied in the classical sense, see [DL90b] p. 44.

Theorem C.0.6. *Let* a *and* b *be two symmetric bilinear forms continuous on* $V \times V$ *and coercive on* V*. We assume*

$$\mathrm{a}\,(f,f) \leq \mathrm{b}\,(f,f) \quad \textit{for all } f \in V.$$

If we denote by $\lambda_k\,(\boldsymbol{A})$ *and* $\lambda_k\,(\boldsymbol{B})$ *the n-th eigenvalue of the operator* ***A*** *and* ***B*** *associated with* a *and* b*, respectively, counted with their multiplicities. Then*

$$\lambda_k\,(\boldsymbol{A}) \leq \lambda_k\,(\boldsymbol{B}) \quad \textit{for all } k.$$

Corollary C.0.7. *Let* W *be a closed subspace of* V *dense in* H*. If we denote by* $\{\lambda_k\}$ *the sequence of eigenvalues of the operator defined by* (V,H,b)*, and* $\{\mu_k\}$ *the sequence of the eigenvalues of the operator defined by* (W,H,b)*, then we have*

$$\lambda_k \leq \mu_k \quad \textit{for all } k.$$

Remark. We apply now the above results to estimate the eigenfrequencies of the Laplace operator $(-\frac{d^2}{dx^2})$ on $[\varepsilon, 1]$ for some specified boundary conditions.

Let $\alpha \in \mathbb{R}, \beta \in \mathbb{R} \setminus \{0\}$ be two constants. Let $\lambda > 0$ be an eigenvalue and $f_\lambda \neq 0$ the associated eigenfunction such that

$$\begin{cases} f_\lambda^{''} = -\lambda\, f_\lambda & (0) \\ f_\lambda(\varepsilon) = 0 & (1) \\ \alpha\, f_\lambda^{'}(1) - \beta\, f_\lambda(1) = 0 & (2) \end{cases}$$

The differential equation (0) implies that f_λ has the form

$$f_\lambda(x) = A \cos x\,\sqrt{\lambda} + B \sin x\,\sqrt{\lambda}.$$

To fulfill the boundary conditions (1) and (2), we see after a straight forward computation that $(1-\varepsilon)\lambda^{\frac{1}{2}}$ must be zero of the function

$$\varphi_\varepsilon(z) = \alpha\, z \cos(z) - \beta\,(1-\varepsilon)\sin(z).$$

If $z_k \in [-\frac{\pi}{2} + k\,\pi, \frac{\pi}{2} + k\,\pi], k \in \mathbb{N}$, are the zeros of the function $\varphi_\varepsilon,$ then

$$\lambda \in \{\big(\frac{z_k}{1-\varepsilon}\big)^2,\ k \in \mathbb{N}\}$$

Hence,

$$\big(\frac{-\frac{\pi}{2} + k\,\pi}{1-\varepsilon}\big)^2 \leq\ \lambda\ \leq\ \big(\frac{\frac{\pi}{2} + k\,\pi}{1-\varepsilon}\big)^2 \quad \text{for some } k \in \mathbb{N}.$$

Appendix D

Trace mappings

Let Ω be a bounded domain in $\mathbb{R}^n, n \in \mathbb{N},$ with smooth boundry Γ and $\mathbf{n}(x)$ denote the unit vector normal to the boundary at $x \in \Gamma$ directed into the exterior of Ω.

Theorem D.0.8. *For $u, v \in H^2(\Omega)$ we have the second Green's formula*

$$\int_\Omega u\Delta v - \Delta uv \, dx = -\int_\Gamma u(\nabla v \cdot \mathbf{n}) - (\nabla u \cdot \mathbf{n})v \, d\sigma \tag{D.1}$$

We denote the space

$$\mathcal{D}(\overline{\Omega})^n := \{\varphi_{|\Omega}, \text{the restriction on } \Omega \text{of } \varphi \in C_0^\infty(\mathbb{R}^n)\}.$$

Theorem D.0.9. *Let $m \in \mathbb{N}$. For $u \in \mathcal{D}(\overline{\Omega})^n$, let $\gamma_0 u = u_{|\Gamma}, \gamma_j u = \frac{\partial u}{\partial n}_{|\Gamma}, j \in \mathbb{N}$ where $\frac{\partial}{\partial n}$ is the derivative along the exterior normal to Γ. Then the trace mapping $\gamma_m : u \mapsto \gamma_m u = (\gamma_0, \cdots, \gamma_{m-1} u)$ defined on $\mathcal{D}(\overline{\Omega})^n$ has an extension by density to an isomorphism and homeomorphism from the quotient space $H^m(\Omega)_{|\mathcal{N}(\gamma_n)}$ onto $H^{m-\frac{1}{2}}(\Gamma) \times \cdots \times H^{\frac{1}{2}}(\Gamma)$.*

Theorem D.0.10. *(Trace theorem for $H(\mathbf{div}, \Omega)$).*

1. *The space $\mathcal{D}(\overline{\Omega})^n$ is dense in $H(\mathbf{div}, \Omega)$.*
2. *The trace mapping $\gamma_n : \mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{n}$ defined on $\mathcal{D}(\overline{\Omega})^n$ extends by continuity to a continuous linear mapping ,also denoted γ_n, from $H(\mathbf{div}, \Omega)$ onto $H^{-\frac{1}{2}}(\Gamma)$.*
3. *The kernel of this mapping γ_n is the space $H_0(\mathbf{div}, \Omega)$.*

Theorem D.0.11. *(Trace theorem for $H(\mathbf{curl}, \Omega)$).*

1. *The space $\mathcal{D}(\overline{\Omega})^n$ is dense in $H(\mathbf{div}, \Omega)$.*
2. *the trace map $\gamma_t : \mathbf{v} \mapsto \mathbf{v} \wedge \mathbf{n}$ defined on $\mathcal{D}(\overline{\Omega})^n$ extends by continuity to a continuous linaer mapping ,also denoted γ_t, from $H(\mathbf{curl}, \Omega)$ into $H^{-\frac{1}{2}}(\Gamma)^3$.*
3. *The kernel of this mapping γ_n is the space $H_0(\mathbf{curl}, \Omega)$.*

Remark. The results above are taken from [DL90b]. We refer also to [Ada75].

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