Functional Computation as Concurrent Computation

Joachim Niehren

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Director
Functional Computation as Concurrent Computation

Joachim Niehren

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Joachim Niehren
Programming Systems Lab
German Research Center for Artificial Intelligence (DFKI)
Stuhlsatzenhausweg 3, 66123 Saarbrücken, Germany
niehren@dfki.uni-sb.de

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Abstract

We investigate functional computation as a special form of concurrent computation. As formal basis, we use a uniformly confluent core of the $\pi$-calculus, which is also contained in models of higher-order concurrent constraint programming. We embed the call-by-need and the call-by-value $\lambda$-calculus into the $\pi$-calculus. We prove that call-by-need complexity is dominated by call-by-value complexity. In contrast to the recently proposed call-by-need $\lambda$-calculus, our concurrent call-by-need model incorporates mutual recursion and can be extended to cyclic data structures by means of constraints.
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1 Introduction

We investigate concurrency as unifying computational paradigm in the spirit of Milner [Mil92] and Smolka [Smo94, Smo95a]. Whereas the motivations for both approaches are quite distinct, the resulting formalisms are closely related: The $\pi$-calculus [MPW92] models communication and synchronisation via channels, whereas the $\rho$-calculus [NS94, Smo94, NM95]\footnote{Originally, Smolka’s $\gamma$-calculus [Smo94] and the $\rho$-calculus [NS94] have been technically distinct. In [NM95], they have been combined in a refined version of the $\rho$-calculus. We note that Smolka’s $\gamma$-calculus and Boudol’s $\gamma$-calculus [Bou89] are completely unrelated.} uses logic variables or more generally constraints as inspired by [Mah87, SRP91].

Our motivation in concurrent calculi lies in the design of programming languages. Concurrency enables us to integrate multiple programming paradigms such as functional [Mil92, Smo94, Nie94, Iba95, PT95b], object-oriented [Vas94, PT95a, HSW95, Wal95], and constraint programming [JH91, SSW94]. All these paradigms are supported by the programming language Oz [Smo95a, Smo95b].

In this paper, we model the time complexity of eager and lazy functional computation in a concurrent calculus. The importance of complexity is three-fold:

1. Every implementation-oriented model has to reflect complexity. In the case of lazy functional programming, the consideration of complexity leads to a call-by-need model in contrast to a call-by-name model.

2. A programmer has to reason about the complexity of his programs. In particular for functional programs, denotational semantics are too coarse [San95].

3. Based on the notion of uniform confluence, complexity arguments provide for powerful proof techniques.

Our main technical result is that call-by-need complexity is dominated by call-by-value and call-by-name complexity, i.e. for all closed $\lambda$-expressions $M$:

$$C_{\text{need}}(M) \leq \min \{C_{\text{value}}(M), C_{\text{name}}(M)\}$$

These two estimations can be interpreted as follows: Call-by-need reduction shares the evaluation of functional arguments and evaluates only needed arguments.

As a formal basis, we use a uniformly confluent applicative core of a concurrent calculus that we call $\delta_0$-calculus. This is a proper subset of the polyadic asynchronous $\pi$-calculus [Mil91, HT91, Bou92] and of the $\rho$-calculus [NM95, Smo94], the latter being a foundation of higher-order concurrent constraint programming. The choice of $\delta_0$ has the following advantages:

1. Delay and triggering mechanisms as needed for programming laziness are expressible within $\delta_0$. 
2. Mutually recursive definitions are expressible in a call-by-value and a call-by-need manner.

3. Cyclic data structures and the corresponding equality relations are expressible in an extension of $\delta_0$ with constraints, the $\rho$-calculus.

The $\delta_0$-calculus is defined via expressions, structural congruence, and reduction. Expressions are formed by abstraction, application, composition, and declaration:

$$ E, F ::= x:\overline{y}/E \mid x\overline{y} \mid E[F] \mid (\nu x)E $$

In the terminology of the $\pi$-calculus, abstractions are replicated input-agents and applications are output-agents. Once-only input-agents as in the $\pi$-calculus are not provided, nor constraints or cells as in the $\rho$-calculus.

We identify expressions up to the structural congruence of the $\pi$-calculus. Reduction in $\delta_0$ is defined by the following application axiom:

$$ (x:\overline{y}/E) \mid x\overline{z} \rightarrow (x:\overline{y}/E) \mid E[z/\overline{y}] $$

We do not allow for reduction below abstraction. In terms of the $\lambda$-calculus, this means that we consider standard reduction only.

We embed the call-by-value and the call-by-name $\lambda$-calculus into $\delta_0$, the latter with call-by-need complexity. This is done in two steps: We first extend $\delta_0$ by adding mechanisms for single assignment, delay, and triggering. We obtain a new calculus that we call $\delta$-calculus. Surprisingly $\delta$ can be embedded into $\delta_0$ itself. The idea is to express single assignment by forwarders. In the second step, we encode the above mentioned $\lambda$-calculi into $\delta$. Formulating these embeddings into $\delta$ rather than into $\delta_0$ is motivated by our belief that the abstraction level of $\delta$ is relevant for programming, theory, and implementation.

The notion of single assignment we use in $\delta$ is known from a directed usage of logic variables [Pin87], as for instance in the data-flow language Id [ANP89, BNA91]. Alternatively, we could express single assignment via equational constraints, but these are not available in the $\pi$-calculus. In fact, the directed single assignment mechanism in this paper is motivated by a data-flow discussion for polymorphic typing a concurrent constraint language [Mül96].

The approach of this paper is based on the idea of uniform confluence [Nie94, NS94]. This is a simple criterion that ensures complexity is independent of the execution order. Unfortunately, we can not even expect confluence for $\delta_0$. This is due to expressions such as $x:\overline{y}/E \mid x:\overline{y}/F$ that we consider inconsistent. Inconsistencies may arise dynamically. We can however exclude them statically by a linear type system. In fact, the restriction of $\delta_0$ to well-typed expressions is uniformly confluent and sufficiently rich for embedding $\lambda$-calculi. We note that a well-typed first-order restriction of $\delta_0$ has been proved confluent in [SRP91].

We base all our adequacy proofs for embeddings on a novel technique that combines uniform confluence and shortening simulations [Nie94, NS94]. Shortening simulations are more
powerful than bisimulations, once uniform confluence is available. Nevertheless, the definitions of concrete shortening simulations in this paper are strongly inspired by Milner’s bisimulations in [Mil92].

We are able to compare the complexities of call-by-need and call-by-value in \( \delta \), since up to our embeddings, every call-by-need step is also a call-by-value step. In particular, we do not require in \( \delta \) that a call-by-value function evaluates its arguments before application. This additional freedom compared to the call-by-value \( \lambda \)-calculus does not affect complexity. This is a consequence of the uniform confluence of the well-typed restriction of \( \delta \). We note that the call-by-let \( \lambda \)-calculus introduced in [MOTW95] provides the same kind of freedom.

**Related Work.** Many call-by-need models have been proposed over the last years but none of them has been fully satisfactory.

Our call-by-need model is closely related to the call-by-need \( \lambda \)-calculus of Ariola et al. [AFMOW95]. We show how to embed the call-by-need \( \lambda \)-calculus into \( \delta \) such that complexity is preserved (but not vice versa). The main difference between both approaches is the level on which lazy control is defined. In the case of the call-by-need \( \lambda \)-calculus, laziness is defined on meta level, by evaluation contexts. In the case of the \( \delta \)-calculus, laziness is expressible within the language itself. In other words, the call-by-need \( \lambda \)-calculus is more abstract, or, the \( \delta \)-calculus is more general. The disadvantage of the abstraction level of the call-by-need \( \lambda \)-calculus is that mutual recursion and cyclic data structures are difficult to define. On the other hand side, \( \delta \) is abstract enough for hiding most implementation details. We illustrated this fact by simple complexity reasoning based on shortening simulations and uniform confluence. This technique is again more general than the specialised \( \lambda \)-calculus technique in [AFMOW95].

The setting of the call-by-need \( \lambda \)-calculus is quite similar to Yoshida’s \( \lambda f \)-calculus [Yos93]. She proves that a call-by-need reduction strategy is optimal for weak reduction, but she does not compare call-by-need to call-by-name.

Embeddings of the call-by-value and the call-by-name \( \lambda \)-calculus into the \( \pi \)-calculus have been proposed and proved correct by Milner [Mil92]. An embedding of the call-by-need \( \lambda \)-calculus into the \( \pi \)-calculus is proved correct in [BO95]. The advantage of the embeddings presented here is that they do not need to make use of once-only input channels, which are incompatible with uniform confluence.

Embeddings of the call-by-value and the call-by-name \( \lambda \)-calculus into the \( \rho \)-calculus are presented in [Smo94], the latter with call-by-need complexity. These embeddings motivated those presented here. The difference lies in the usage of constraints for single assignment and triggering. In [Smo94] no proofs are given, but the call-by-value embedding is proved correct in [Nie94]. There, most of the proof techniques presented in this paper have been introduced.

An abstract big-step semantics for call-by-need has been presented by Launchbury [Lau93]. It is complexity sensitive, since computation steps are reflected in proof trees. Launchbury’s
correctness result however does not cover complexity. This is a consequence of using a proof
technique based on denotational semantics.

Many other attempts for call-by-need have been presented. To our knowledge, all of them
are quite implementation oriented such that they suffer from low-level details. We note
the approaches based on explicit substitutions [PS92, ACCL91] and on graph reduction
[Jef94].

Structure of the Report. As a first example we discuss the square function in a concurrent
setting. We define \( \delta_0 \) in Section 3. We then introduce the notion of uniform confluence
and discuss its relationship to complexity and confluence. In Section 5, we prove uniform
confluence for a subset of \( \delta_0 \). In Section 6, we define the \( \delta \)-calculus. Following, we discuss
uniform confluence for \( \delta \). In Sections 8 and 9, we embed the call-by-value, the call-by-name,
and the call-by-need \( \lambda \)-calculus into \( \delta \). We introduce a linear type system in Section 10
and prove that our embeddings fall into the uniformly confluent subset of \( \delta \). In Section
11, we show how to encode single assignment and triggering in \( \delta_0 \). We introduce the simu-
lation proof technique in Section 12 and apply it for proving the adequacy our calculus
embeddings in Sections 13 - 18.

2 The Square Function: An Example

We informally introduce the \( \delta \)-calculus by representing the square function in call-by-value
and call-by-need manner. This motivates our embeddings of \( \lambda \)-calculi into \( \delta \) and indicates
the adequacy results we can expect.

We assume a infinite set of variables ranged over by \( x, y, z, s, \) and \( t \). Sequences of variables
are written as \( \overline{x}, \overline{y}, \ldots \) and integers are denoted with \( n, m, \) and \( k \).
In a concurrent setting, we consider functions as relations with an explicit output argument, for example:

\[ S = \lambda x.x \times x \quad \text{versus} \quad S' = \frac{ztz}{z = x \times x} \]

The expression on the right-hand side is a call-by-value definition of the square function in the \( \delta \)-calculus. The formal parameter \( z \) is the explicit output argument. The expression \( z = x \times x \) is syntactic sugar for an application of a predefined ternary relation \( * \). We assume the following application axiom for all integers \( n, m, k \) and variables \( x \):

\[ x = n \times m \rightarrow x = k \quad \text{if} \quad k = n \times m \]

For forwarding values in equations \( x = n \), we copy them into those positions where they are needed. This kind of administration is definable in many different manners, for instance:

\[ (\nu y)(y = n \mid E) \rightarrow E[n/y] \]

Figure 1 (commented by footnote marks \(^2\) and \(^3\)) illustrates the call-by-value evaluation of the square of \( 2 \times 3 \) in the \( \lambda \)-calculus and the \( \delta \)-calculus. If we ignore forwarding steps, then all possible computations in Figure 1 have length 3. In other words, our call-by-value embedding of the square function preserves time complexity measured in terms of application steps. Ignoring forwarding is correct in the sense that the number of forwarding steps in computations of functional expressions is linearly bounded by the number of application steps. We do no prove this claim formally.

It is interesting that call-by-value evaluation in \( \delta \) is more flexible than in the \( \lambda \)-calculus, as shown by an additional call-by-value computation in our example. This is in the rightmost computation in Figure 1, where the square function is applied before its argument has been evaluated.

For defining a call-by-need square function in a concurrent setting, we need a delay and a triggering mechanism. For this purpose, we introduce two new expressions \( t.E \) and \( tr(t) \). We say that \( E \) is delayed in \( t.E \) until \( t \) is triggered. This behaviour can be provided by the following triggering axiom:

\[ t.E | tr(t) \rightarrow E | tr(t) \]

Note that multiple triggering is possible. A call-by-need version of the square function can be defined as follows:

\[ s' = \frac{xtz}{(z = x \times x | tr(t))} \]

This function can be applied with a delayed argument \( x \) waiting on \( t \) to be triggered. Figure 2 (commented by the footnotemarks \(^2\) and \(^3\)) presents call-by-name and call-by-need computations of the square of \( 2 \times 3 \). Both call-by-name computations have length 4, since the functional argument \( 2 \times 3 \) is evaluated twice. If we ignore triggering and forwarding steps, then our call-by-need computation has length 3. This illustrates that call-by-need

---

\(^2\)Here, \( \rightarrow^* \) stands a forwarding step followed by an application step: \( (\nu y)(z = y \times y \mid y = 6) \rightarrow z = 6 \times 6 \rightarrow z = 36 \).
complexity is dominated by call-by-name and by call-by-value complexity. In this example, the first estimation is proper (raised by sharing), whereas the second is not (since the argument of the square function is needed).

We note that our call-by-need computation in Figure 2 has a direct relative in the call-by-value case, the rightmost computation in Figure 1. This statement holds in general and enables us to compare call-by-need and call-by-value complexity in the δ-calculus.

3 The Applicative Core of the π-Calculus

We define δ₀ as the applicative core of the polyadic asynchronous π-calculus [Mil91, HT91, Bou92] and the ρ-calculus [NM95, Smo94]. Interestingly, δ₀ as formulated here is part of the Oz computation model [Smo94] and the Pict computation model [PT95b], which have been developed independently.

We define the calculus δ₀ via expressions, structural congruence, and reduction. The definition is given in Figures 3 and 4. Expressions are abstractions, applications, compositions, or declarations. An abstraction \( x: \overline{y} \, E \) is named by \( x \), has formal arguments \( \overline{y} \) and body \( E \). An application \( x \, \overline{y} \) of \( x \) has actual arguments \( \overline{y} \). In the standard π-notation, abstractions are replicated input-agents and applications asynchronous output-agents.

Bound variables are introduced as formal arguments of abstractions and by declaration. The set of free variables of an expression \( E \) is denoted by \( \mathcal{V}(E) \). We write \( E =_{\alpha} F \) if \( E \)

\footnote{Here, \( \rightarrow^* \) consists of an application and a triggering step: \( (\nu t)(s'ytz \mid t,y=2\times3) \rightarrow (\nu t)(z=y+y \mid t',t,y=2\times3) \rightarrow z=y+y \mid y=2\times3 \mid (\nu t)(t',t) \). The garbage expression \( (\nu t)(t',t) \) in is omitted in Figure 2:}
and $F$ are equal up to consistent renaming of bound variables. As usual for λ-calculi, we assume all expressions to be α-standardised and omit freeness conditions throughout the paper.

The structural congruence $\equiv$ of $\delta_0$ coincides with that of the $\pi$-calculus. It is the least congruence on expressions satisfying the axioms in Figure 4. With respect to the structural congruence, bound variables can be renamed consistently, composition is associative and commutative, and declaration is equipped with the usual scoping rules.

The reduction $\rightarrow$, synonymously denoted by $\rightarrow_A$, is defined by a single axiom for application. The application axiom uses the simultaneous substitution operator $[\bar{z}/\bar{y}]$, which replaces the components of $\bar{y}$ elementwise by $\bar{z}$. In case of application of $[\bar{z}/\bar{y}]$, we implicitly assume that the sequence $\bar{y}$ is linear and of the same length as $\bar{z}$. Note that reduction is invariant under structural congruence and closed under weak contexts. This means that reduction is applicable below declaration and composition, but not inside of abstraction. In terms of λ-calculi, this means that we consider standard reductions only.

**Example 3.1 (Continuation Passing Style)** The identity function $I = \lambda x.x$ can be defined in $\delta_0$ in continuation passing style: $i: x y/y x$. An application let $i=I$ in $ii$ referred to by $z$ is definable as follows:

\[(vi) (i: x y/y x \mid (\nu y')(ii y' \mid y':e/ze))\]
In composition with $i:xy/yx$ we obtain the following computation:

$$(vy')(iy)(y':e/ze) \rightarrow_A (vy')(y':e/ze) \rightarrow_A zi \rhd (vy')(y':e/ze)$$

**Example 3.2 (Explicit Recursion)** The computation of the following recursive expression does not terminate:

$$xy \rhd x:y/xy \rightarrow_A xy \rhd x:y/xy \rightarrow_A \ldots$$

Compared to the asynchronous $\pi$-calculus [Mil91, Bou92, HT91], $\delta_0$ does not provide for non-replicated input-agents. These are not needed for functional computation and are incompatible with uniform confluence if not restricted linearly [KPT96]. In absence of once-only inputs, it is not clear if the unary restriction of $\delta_0$ is Turing complete.

## 4 Uniform Confluence

We formalise the notions of a calculus, complexity, and uniform confluence as in [Nie94, NS94] and discuss their relationships. These simple concepts will prove extremely useful in the sequel.

The notion of a calculus that we will define extends Klop’s abstract rewrite systems [Klo87] by the concept of a congruence: A **calculus** is a triple $(E, \equiv, \to)$, where $E$ is a set, $\equiv$ an equivalence relation, and $\to$ a binary relation on $E$. Elements of $E$ are called **expressions**, $\equiv$ **congruence**, and $\to$ **reduction** of the calculus. We require that reduction is **invariant under congruence**, i.e., $(\equiv \circ \to \equiv) \subseteq \to$, where $\circ$ stands for relational composition\(^4\). Typical calculi are: $\delta_0, \pi, \rho, \lambda$-calculi, abstract rewrite systems, Turing machines, etc.

A **derivation** in a calculus is a finite or infinite sequence of expressions such that $E_i \rightarrow E_{i+1}$ holds for all subsequent elements. A **derivation of an expression** $E$ is a derivation, whose first element is congruent to $E$. A **computation** of $E$ is a maximal derivation of $E$, i.e. an infinite derivation or a finite one, whose last element is irreducible. The least transitive relation containing $\rightarrow$ and $\equiv$ is denoted with $\rightarrow^*$.

The **length** of a finite derivation $(E_i)_{i=0}^n$ is $n$ and the length of infinite derivation is $\infty$. We call an expression $E$ **uniform** with respect to complexity (and termination), if all its computations have the same length. We define the **complexity** $\mathcal{C}(E)$ of a uniform expression $E$ by the length of its computations. We call a calculus **uniform** if all its expressions are uniform.

We call a calculus **uniformly confluent**, if its reduction and congruence satisfy the following condition (visualised in Figure 5):

\[
(\leftarrow \circ \rightarrow) \subseteq (\rightarrow \circ \leftarrow) \cup \equiv
\]

\(^4\)If $\rightarrow_1$ and $\rightarrow_2$ are two binary relations on some set $E$ and $E, E'' \in E$, then $E \rightarrow_1 o \rightarrow_2 E''$ if and only if there exists $E' \in E$ such that $E \rightarrow_1 E'$ and $E' \rightarrow_2 E''$.\]
Figure 5: Uniform Confluence

Typically, λ-calculi equipped with standard reductions are uniformly confluent, subject to weak reduction.

**Proposition 4.1** A uniformly confluent calculus is confluent and uniform with respect to complexity.

**Proof.** By a standard inductive argument [Nie94] as for the notion of strong confluence [Hue80] (which is weaker than uniform confluence). □

## 5 Uniform Confluence for δ₀

In this Section, we distinguish a uniformly confluent subcalculus of δ₀ that is sufficient for functional computation. We call a δ₀-expression **inconsistent**, if it is of the form:

\[ x:γ/E \mid x:σ/F \]

where \( x:γ/E \not\equiv x:σ/F \). A typical example for non-confluence in the case of inconsistencies is to reduce the expression \( xz \) in composition with \( x:y/\text{sy} \mid x:y/\text{ty} \):

\[ sz \xrightarrow{A} xz \rightarrow_A t \]

These results are irreducible but not congruent under the assumption \( s \not\equiv t \).

We call \( E \) **admissible**, if there exists no expression \( F \) containing an inconsistency and satisfying \( E \rightarrow^* F \). The advantage of this condition is that it is very simple. Unfortunately, it is undecidable if a given expression \( E \) is admissible, since admissibility may depend on the termination of a Turing complete system. This failure is harmless, since we can prove admissibility for all functional expression of \( δ \) with the help of the linear type system in Section 10.

---

5The flexibility provided by the side condition \( x:γ/E \not\equiv x:σ/F \) is needed for encoding multiple triggering in \( δ₀ \). Consider for instance \( [t \rightarrow t] [t \rightarrow t] \equiv t:y/\text{ty} \) as introduced in Section 11.
Theorem 5.1 The restriction of $\delta_0$ to admissible expressions is uniformly confluent.

Together with Proposition 4.1 this implies that all admissible expressions $E$ of $\delta_0$ are uniform with respect to complexity such that $C(E)$ is well-defined.

Proof of Theorem 5.1.
Let $E$ be an admissible $\delta_0$-expression. Every application step on $E$ can be performed on an arbitrary prenex normal form of $E$ (compare [Nie94] for details). Since declarations are not involved during application, we can assume that $E$ is a prenex normal form with an empty declaration prefix. On such $E$, reduction amounts to rewriting on multisets of abstractions and applications.

Let $F_1$ and $F_2$ be expressions such that $F_1 \rightarrow_A F_2$. There exists an application $x_1 z_1$ reduced during the application step $E \rightarrow_A F_1$ and an application $x_2 z_2$ reduced during $E \rightarrow_A F_2$. If these applications are distinct, then we can join $F_1$ and $F_2$ by reducing the respective other one. If both applications coincide then $x_1 = x_2$. Hence, the applied abstractions have to be congruent by admissibility such that $F_1 \equiv F_2$. \qed

6 Single Assignment and Triggering

We extend $\delta_0$ with directed single assignment and triggering. The resulting calculus is called $\delta$. We do not exclude multiple assignment syntactically. This is a matter of the linear type system in Section 10.

For our extension, we need three new types of expressions and two additional reduction axioms. A directed equation\(^5\) $x=y$ is used for single assignment directed from the right to the left. A synchroniser $x.E$ delays the computation of $E$ until $t$ is triggered. A trigger expression $\text{tr}(t)$ triggers a delayed computation waiting on $t$.

\(^5\)The original version of the $\delta$-calculus [Nie94] uses symmetric equations instead of directed ones. This choice does not matter for well-typed expressions.
The structural congruence of \( \delta \) coincides with that of \( \delta_0 \). Its reduction \( \rightarrow \) is a union of three relations, application \( \rightarrow_A \), forwarding \( \rightarrow_F \), and triggering \( \rightarrow_T \):

\[
\rightarrow = \rightarrow_A \cup \rightarrow_F \cup \rightarrow_T
\]

Each of these relations is defined by the corresponding axiom in Figure 6 and the contextual rules in Figure 4.

**Example 6.1 (Single Assignment Style)** The identity function \( I = \lambda x.x \) can be expressed in \( \delta \) by \( i:xy/y=x \). Compared to Example 3.1 we use single assignment instead of continuation passing. An application \( \text{let } i=I \) in (ii) is referred to by \( z \) is represented in \( \delta \) as follows:

\[
(i)(i:xy/y=x | (\nu y')(iiy' | y'iz))
\]

In composition with \( i:xy/y=x \) we obtain the following computation:

\[
(\nu y')(iiy' | y'iz) \rightarrow_A (\nu y')(y'\vdash i | y'iz)
\]

\[
\rightarrow_F (\nu y')(y':xy/y=x | y'iz)
\]

\[
\rightarrow_A (z\vdash i) | (\nu y')(y':xy/y=x)
\]

\[
\rightarrow_F z:xy/y=x | (\nu y')(\ldots)
\]

**Example 6.2 (Call-by-Need Selector Function)** The call-by-need selector function \( F = \lambda xy.x \) can be represented in \( \delta \) by the abstraction \( f:xt_\times yt_\times z/(z=x \mid \text{tr}(t_\times)) \). The symbols \( t_\times \) and \( t_\times \) stand for ordinary variables. Their usage is for triggering the computations of \( x \) and \( y \) respectively. A call-by-need application \( f(ii)(ii) \) has the form:

\[
(\nu x)(\nu t_\times)(\nu y)(\nu t_\times)(f:xt_\times yt_\times z | t_\times, ii x | t_\times, ii y)
\]

In composition with the abstractions named \( i \) and \( f \), we obtain the following computation:

\[
(\nu x)(\nu t_\times)(\nu y)(\nu t_\times)(f:xt_\times yt_\times z | t_\times, ii x | t_\times, ii y)
\]

\[
\rightarrow_A (\nu x)(\nu t_\times)(z=x | \text{tr}(t_\times) | t_\times, ii x | (\nu y)(\nu t_\times)(t_\times, ii y)
\]

\[
\rightarrow_T (\nu y)(\nu t_\times)(z=x | \text{tr}(t_\times) | ii x | (\nu y)(\nu t_\times)(\ldots)
\]

\[
\rightarrow^* z:xy/y=x | (\nu y)(\nu t_\times)(\nu x)(\nu t_\times)(\ldots)
\]

The resulting expression is irreducible. We note that only the needed first argument has been evaluated. The synchroniser \( t_\times, ii y \) for the second argument suspends forever.

### 7 Uniform Confluence for \( \delta \)

For proving a uniform confluence result for \( \delta \), we have to consider how uniform confluence behaves with respect to a union of calculi. We first present a variation of the Hindley-Rosen Lemma [Bar84] for uniform confluence and then apply it to the \( \delta \)-calculus. But the general
results of this Section are also applicable to other unions of calculi such as the call-by-need λ-calculus [AFMOW95] and the ρ-calculus [NM95].

The union of two calculi \((\mathcal{E}, \Xi, \rightarrow_1)\) and \((\mathcal{E}, \Xi, \rightarrow_2)\) is defined by \((\mathcal{E}, \Xi, \rightarrow_1 \cup \rightarrow_2)\). We say that the relations \(\rightarrow_1\) and \(\rightarrow_2\) commute, if

\[
(\rightarrow_1 \circ \rightarrow_2) \subseteq (\rightarrow_2 \circ \rightarrow_1).
\]

**Lemma 7.1 (Reformulation of the Hindley-Rosen Lemma)** The union of two uniformly confluent calculi with commuting reductions is uniformly confluent.

**Proof.** The proof is straightforward.

Note that Lemma 7.1 implies the classical Hindley-Rosen Lemma, since a relation is confluent, if and only if its reflexive transitive closure is uniformly confluent. The next lemma allows us to ignore administrative steps such as forwarding and triggering in the case of \(\delta\):

**Lemma 7.2 (Administrative Steps)** Let \((\mathcal{E}, \Xi, \rightarrow_1)\) be a uniformly confluent calculus and \((\mathcal{E}, \Xi, \rightarrow_2)\) a confluent and terminating calculus such that \(\rightarrow_1\) and \(\rightarrow_2\) commute. If \(E \in \mathcal{E}\), then every computation of \(E\) in the union \((\mathcal{E}, \Xi, \rightarrow_1 \cup \rightarrow_2)\) contains the same number of \(\rightarrow_1\) steps.

**Proof.** The idea is to apply Proposition 4.1 to \((\mathcal{E}, \Xi, \rightarrow_2^* \circ \rightarrow_1 \circ \rightarrow_2^*)\). This calculus is uniform but not uniformly confluent. This deficiency can be remedied by replacing \(\Xi\) with \((\rightarrow \cup \rightarrow_2^*)^*\). The details can be found in [Nie94].

Next, we apply the above results to the \(\delta\)-calculus. We first note that the notion of admissibility carries over from \(\delta_0\) to \(\delta\) without change.

**Proposition 7.3** The relations \(\rightarrow_F\) and \(\rightarrow_T\) terminate. The relation \(\rightarrow_T\) is uniformly confluent and \(\rightarrow_F\) is uniformly confluent when restricted to admissible expressions. The relations \(\rightarrow_{A}, \rightarrow_F,\) and \(\rightarrow_T\) commute pairwise.

**Proof.** Termination is trivial, since \(\rightarrow_F\) decreases the number of directed equations and \(\rightarrow_T\) the number of synchronisers. All other properties can be established by the normal form technique used in the proof of Theorem 5.1.

**Theorem 7.4** The restriction of the \(\delta\)-calculus to admissible expressions is uniformly confluent.

**Proof.** Follows from Theorem 5.1, Proposition 7.3, and Lemma 7.1.
Expressions

\[
\begin{align*}
M, N &::= x \mid V \mid MN \\
V &::= \lambda x. M
\end{align*}
\]

Reduction

\[
\begin{align*}
(\lambda x. M)V &\rightarrow_{\text{value}} M[V/x] \\
(\lambda x. M)N &\rightarrow_{\text{name}} M[N/x]
\end{align*}
\]

Contextual Rules

\[
\begin{array}{c}
M \rightarrow_{\text{value}} M' \\
MN \rightarrow_{\text{value}} M'N \\
MN \rightarrow_{\text{value}} M'N \\
N \rightarrow_{\text{value}} N' \\
MN \rightarrow_{\text{name}} M'N \\
MN \rightarrow_{\text{name}} M'N
\end{array}
\]

Figure 7: The Call-by-Value and the Call-by-Name \(\lambda\)-Calculus

**Theorem 7.5** If \(E\) is admissible, then all computations of \(E\) contain the same number of application steps.

**Proof.** Follows from Theorem 5.1, Proposition 7.3, and Lemma 7.2. \(\square\)

**Definition 7.6** We define the A-complexity \(C^A(E)\) of an admissible \(\delta\)-expression \(E\) as the number of \(\rightarrow_A\) steps in computations of \(E\).

Theorem 7.5 ensures that A-complexity is well defined. We consider forwarding and triggering steps as administrative steps and ignore them in favour of simpler complexity statements and adequacy proofs. However, we could prove for all functional expressions (but not in general) that the number of administrative steps is linearly bound by the number of \(\rightarrow_A\) steps. This would require showing stronger invariants in adequacy proofs.

### 8 Functional Computation in \(\delta\)

We embed the call-by-value and the call-by-name \(\lambda\)-calculus into the \(\delta\)-calculus, the latter with call-by-need complexity.

The call-by-value and the call-by-name \(\lambda\)-calculus are revisited in Figure 7. Note that we consider standard reduction only. A congruence allowing for consistent renaming of bound variables is left implicit as usual.

**Proposition 8.1** The call-by-value and the call-by-name \(\lambda\)-calculus are uniformly confluent.
\[ z = _v MN \overset{\text{def}}{=} (\nu x)(x = _v M \mid (\nu y)(xyz \mid y = _v N)) \]
\[ z = _v \lambda x.M \overset{\text{def}}{=} z:xy/y = _v M \]
\[ z = _v x \overset{\text{def}}{=} z = x \]

Figure 8: Call-by-Value in the \( \delta \)-Calculus

\[ z = _n MN \overset{\text{def}}{=} (\nu x)(x = _n M \mid (\nu y)(xyt_y z \mid y = _n N)) \]
\[ z = _n \lambda x.M \overset{\text{def}}{=} z:xt_x y/y = _n M[x \od t_x/x] \]
\[ z = _n x \od t_x \overset{\text{def}}{=} z = x \mid \text{tr}(t_x) \]

Figure 9: Embedding the Call-by-Name \( \lambda \)-Calculus with Call-by-Need Complexity

**Proof.** The statement for call-by-name is trivial, since call-by-name reduction is deterministic. The proof for call-by-value can be done by a simple induction on the structure of \( \lambda \)-expressions. \( \square \)

Proposition 8.1 allows us to define the call-by-value complexity \( C_{\text{value}}(M) \) and the call-by-name complexity \( C_{\text{name}}(M) \) of a \( \lambda \)-expression \( M \) by the length of its computations in the respective \( \lambda \)-calculus.

Given an arbitrary variable \( z \), Figure 8 presents an embedding \( M \mapsto z = _v M \) of the call-by-value \( \lambda \)-calculus into \( \delta \). The definition of \( z = _v M \) is given up to structural congruence. All variables introduced during this definition are supposed to be fresh.

**Theorem 8.2** For all closed \( \lambda \)-expressions \( M \) and variables \( z \) the call-by-value complexity of \( M \) and the \( \lambda \)-complexity of \( z = _v M \) coincide: \( C_{\text{value}}(M) = C_{\lambda}(z = _v M) \).

**Proof.** A proof simplifies it’s predecessor in [Nie94] is presented Section 14. It is based on a complexity simulation introduced in Section 12 and makes heavy use of uniform confluence for covering the additional freedom provided by call-by-value reduction in \( \delta \). We define our complexity simulation in the style of [Mil92] using explicit substitutions. \( \square \)

An embedding \( z \mapsto z = _n M \) of the call-by-name \( \lambda \)-calculus into \( \delta \) is given in Figure 9. It is symmetric to our call-by-value embedding and provides for call-by-need complexity. Our definition of a \( \delta \)-expression \( x = _n M \) makes sense for closed \( M \) only and goes through intermediate \( \lambda \)-expressions containing pairs \( y \od t_y \). For instance:

\[ z = _n \lambda x.x \equiv z:xt_x y/y = _n x \od t_x \equiv z:xt_x y/(y = x \mid \text{tr}(t_x)) \]
As we will show in the next Section, our embedding of the call-by-name λ-calculus provides in fact for call-by-need complexity. In this sense, the next theorem states that call-by-need complexity is dominated by call-by-value and by call-by-name complexity.

**Theorem 8.3** Let $M$ be a closed λ-expression and $z$ a variable. Call-by-name reduction of $M$ terminates if and only if δ-reduction of $z =_n M$ terminates. Furthermore:

$$C^A(z =_n M) \leq \min \{C_{\text{value}}(M), C_{\text{name}}(M)\}.$$

*Proof.* Preservation of termination and the estimation $C^A(z =_n M) \leq C_{\text{name}}(M)$ are proved in Section 15. These are the most difficult results to prove in this paper. The proof is based on a shortening simulation introduced in Section 12. It factorises into Theorem 12.2 and Corollary 15.2.

The proof of the estimation $C^A(z =_n M) \leq C_{\text{value}}(M)$ is given in Section 16. Applying Theorem 8.2 it is sufficient to compare the A-complexities of $z =_n M$ and $z =_v M$. This can be done with a lengthening simulation introduced in Section 12 and is stated in Corollary 16.5.

We note that our simulation technique makes use of uniform confluence such that we need the admissibility of embedded expressions as proved in Section 10. 

**Extension 8.4** It is straightforward to express mutual recursion in δ, both in a call-by-value and in a call-by-need manner:

$$z =_v \text{letrec } x =_M \text{ in } N \xrightarrow{\text{def}} (\nu \bar{x}) (x =_v \bar{M} \mid z =_v N)$$

$$z =_n \text{letrec } x =_M \text{ in } N \xrightarrow{\text{def}} (\nu \bar{x}) (x =_n \bar{M} \theta \mid z =_n N \theta)$$

where $\theta = [\bar{x} \delta / \bar{x}]$. We do not claim a correctness result for mutual recursion in this paper.

### 9 Embedding the Call-by-Need λ-Calculus

We show that the A-complexity of $z =_n M$ equals the complexity of $M$ in the call-by-need λ-calculus.

The definition of the call-by-need λ-calculus [AFMOW95] is revisited in Figure 10. Again, we only consider standard reduction. The reduction $\rightarrow_{\text{need}}$ of the call-by-need λ-calculus is a union of four relations:

$$\rightarrow_{\text{need}} = \rightarrow_1 \cup \rightarrow_{\text{V}} \cup \rightarrow_{\text{Ans}} \cup \rightarrow_C$$

The latter three relations are of administrative character, whereas $\rightarrow_1$ steps correspond to β-reduction steps.
Expressions
\[ L ::= x \mid V \mid L_1 L_2 \mid \text{let } x \equiv L_2 \text{ in } L_1 \] \[ V ::= \lambda x. L \]

Answers
\[ A ::= V \mid \text{let } x \equiv L \text{ in } A \]

Evaluation Contexts
\[ E ::= [] \mid EL \mid \text{let } x \equiv L \text{ in } E \mid \text{let } x \equiv E_2 \text{ in } E_1[x] \]
\[ \frac{L \rightarrow L'}{E[L] \rightarrow E[L']} \]

Reduction
\[ (\lambda x. L_1) L_2 \rightarrow_l \text{let } x \equiv L_2 \text{ in } L_1 \]
\[ \text{let } x \equiv V \text{ in } E[x] \rightarrow_v \text{let } x \equiv V \text{ in } E[V] \]
\[ \text{let } y = (\text{let } x \equiv L \text{ in } A) \text{ in } E[y] \rightarrow_{A_n} \text{let } x \equiv L \text{ in } (\text{let } y \equiv A \text{ in } E[y]) \]
\[ (\text{let } x \equiv L_1 \text{ in } A) L_2 \rightarrow_C \text{let } x \equiv L_1 \text{ in } A L_2 \]

Figure 10: The Call-by-Need \( \lambda \)-Calculus

**Proposition 9.1** The call-by-need \( \lambda \)-calculus is deterministic and hence uniformly confluent.

**Proof.** Evaluation context determine a unique term position where reduction may happen. \( \square \)

By Proposition 9.1, it makes sense to define the call-by-need complexity \( C_{\text{need}}(L) \) of an expression the call-by-need \( \lambda \)-calculus by the number of \( \rightarrow_l \) steps in the computation of \( L \).

We extend the mapping \( M \mapsto z =_n M \) to an embedding \( L \mapsto z =_n L \) of the call-by-need \( \lambda \)-calculus into \( \delta \), defining:

\[ z =_n \text{let } x \equiv L_2 \text{ in } L_1 \equiv (\nu x)(\nu t) \{ t. x =_n L_2 \mid z =_n L_1[x ot / x] \} \]

The following Theorem states the adequacy of the extended embedding, and that our embedding of the call-by-name \( \lambda \)-calculus into \( \delta \) yields in fact call-by-need complexity:

**Theorem 9.2** If \( L \) is a closed \( \lambda \)-expression and \( z \) a variable, then \( C_{\text{need}}(L) = C^A(z =_n L) \).

**Proof.** The proof is presented in Section 18, Corollary 18.3. If is based on a complexity simulation again. \( \square \)
\[
\begin{array}{c}
\Gamma; I \vdash E \\
\Gamma; I \setminus \{x\} \vdash (\mu x)E \\
\Gamma, t \vdash tr; I \vdash E \\
\Gamma, t \vdash tr; I \vdash tr.E \\
\Gamma; \eta; \sigma; I \vdash x:E \\
\Gamma; \eta; \sigma; I \vdash x:y/E \\
\Gamma, x: (\eta); y: (\sigma); \{x\} \vdash x = y \\
\Gamma, t; tr; \emptyset \vdash tr(t) \\
\Gamma, x: (\eta); y: (\sigma); I \vdash x y, \quad O(\eta; \sigma) \subseteq I
\end{array}
\]

Figure 11: Linear Type Checking

10 Linear Types for Consistency

We define a linear type system for $\delta$ that statically excludes inconsistencies. It tests for single assignment and determines the data flow of a $\delta$-expression via input and output modes.

We assume an infinite set of type variables denoted by $\alpha$ and use the following recursive types $\sigma$ internally annotated with modes $\eta$:

$$
\sigma ::= \emptyset | \mu \alpha. \tau | \alpha | tr, \quad \tau ::= \sigma^n, \quad \eta ::= in | out
$$

Our type systems distinguishes two classes of variables, trigger and single assignment variables. We use $tr$ as type for trigger variables. A single assignment variable has a procedural type $\emptyset$, where $\pi$ is a sequence of argument types. For instance, the variable $z$ in $z = v. M$ is typed by $\mu \alpha.(\emptyset^{in} \sigma^{out})$. This recursive type expresses that a call-by-value function is a binary relation, which inputs a call-by-value function in first position and outputs a call-by-value function in second position.

A type environment $\Gamma$ is a sequence of type assumptions $x: \sigma$ with scoping to the right. A variable $x$ has type $\sigma$ in $\Gamma$, written $\Gamma(x) = \sigma$, if there exists $\Gamma_1$ and $\Gamma_2$ such that $\Gamma = \Gamma_1, x: \tau, \Gamma_2$ and $x$ does not occur in $\Gamma_2$. The domain of an environment $\Gamma$ is the set of all variables typed by $\Gamma$. We identify environments $\Gamma_1$ and $\Gamma_2$ if they have the same domain and $\Gamma_1(x) = \Gamma_2(x)$ for all $x$ in this domain.

If $\gamma = (y_i)^n_{i=1}$, $\overline{\sigma} = (\sigma_i)^n_{i=1}$, and $\overline{\eta} = (\eta_i)^n_{i=1}$, then we write $\overline{\sigma}$ for the sequence of annotated types $(\sigma_i^{in})^n_{i=1}$ and $\overline{\gamma}; \overline{\eta}$ for the sequence of type assumptions $y_1: \sigma_1, \ldots, y_n: \sigma_n$. The output variables $O(\overline{\gamma}; \overline{\eta})$ in a sequence of type assumptions are defined as follows:

$$
O(\overline{\gamma}; \overline{\eta}) = \{y_i \mid 1 \leq i \leq n, \eta_i = out, \text{ and } \sigma_i \neq tr\}
$$

A judgement for $E$ is a triple $\Gamma; I \vdash E$, where $\Gamma$ is an environment and $I$ is a set of variables. An expression $E$ is well-typed, if there exists a judgement for $E$ derivable with the rules in Figure 11. If $\Gamma; I \vdash E$ is derivable, then $I$ contains those single assignment variables, to which an abstraction may be assigned during a computation of $E$. Such variables correspond to input channels in the $\pi$-calculus.
Lemma 10.1 (Subject Reduction Property) If $E$ is well-typed and $E \rightarrow^* F$, then $F$ is well-typed.

Proof. By induction on derivations of judgements. \qed

Lemma 10.2 An inconsistent expression is not well-typed.

Proof. An expression $x:\overline{\gamma}/E \mid x:\overline{\nu}/F$ is not well-typed (even if $E \equiv F$). A potential type judgement would have to be of the following form:

\[ \Gamma; \{x\} \triangleright x:\overline{\gamma}/E \quad \Gamma; \{x\} \triangleright x:\overline{\nu}/F \]

This is impossible by the side condition $\{x\} \cap \{x\} = \emptyset$ of the typing rule for composition. \qed

Corollary 10.3 A well-typed expression is admissible.

Proof. Immediate from Lemmata 10.1 and 10.2 \qed

Proposition 10.4 All expressions $z=vM$ and $z=nL$ are well-typed and hence admissible.

Proof. For all closed expressions $M$ and $L$ the following judgements are derivable with the rules in Figure 11, where $\overline{\eta}$ is arbitrary:

\[ z: \mu \alpha. (e^{in} \alpha^{out}); \{z\} \triangleright z=vM \quad z: \mu \alpha. (e^{in} \text{tr}^{\eta} \alpha^{out}); \{z\} \triangleright z=nL \]

This can check by induction on the structure of $M$ resp $L$. A slightly stronger invariant is needed for non-closed subexpressions, where all variables are substituted by pairs via $[x \mapsto t/x]$. \qed

11 Encoding $\delta$ in $\delta_0$

Directed single assignment and triggering can be expressed in $\delta_0$. For technical simplicity, we formalise this statement for $n$-ary $\delta$-expressions, i.e. those containing $n$-ary abstractions and applications only. This is sufficient to carry over our $\lambda$-calculus embeddings from $\delta$ to $\delta_0$, since $z=vM$ and $z=nL$ are binary and ternary respectively. An embedding of $n$-ary $\delta$-expressions into $\delta_0$ is given in Figure 12.

We have to be quite careful when formulating a correctness result for the embedding $E \mapsto [E]$. The reason is that the translation of cyclic reference chains does not preserve termination. For instance, the expression $E \overset{\text{def}}{=} x\overline{\gamma} \mid x=x$ is terminating whereas $[E] \equiv x\overline{\gamma} \mid x\overline{\gamma}/x\overline{\gamma}$ is not.

We call $E$ locally cyclic, if there exists a sequence $(x_i)_{i=1}^n$ such that $E$ contains a subexpression of the form $x_1=x_2 \mid \ldots \mid x_{n-1}=x_n$. We call $E$ cyclic if there exists $F$, which is locally cyclic and satisfies $E \rightarrow^* F$, and acyclic otherwise.
\[
\begin{align*}
\llbracket t, E \rrbracket & \overset{\text{def}}{=} (\nu y) (t \mid y : [E]) \\
\llbracket x = y \rrbracket & \overset{\text{def}}{=} x : \overline{y} / \overline{y}, \quad \text{length}(\overline{z}) = n \\
\llbracket [E \mid F] \rrbracket & \overset{\text{def}}{=} [E] \mid [F] \\
\llbracket (\nu x) E \rrbracket & \overset{\text{def}}{=} (\nu x) [E] \\
\llbracket x : \overline{y} / E \rrbracket & \overset{\text{def}}{=} x : \overline{y} / [E] \\
\llbracket x \overline{y} \rrbracket & \overset{\text{def}}{=} x \overline{y}
\end{align*}
\]

Figure 12: Embedding n-ary δ-expressions in δ₀

**Theorem 11.1** If \( E \) is a well-typed, acyclic, and n-ary δ-expression, then \( \llbracket E \rrbracket \) is admissible and terminates if and only if \( E \) terminates.

**Proof.** This is proved in Section 17, Corollary 17.12. The simulation technique of Section 12 is applicable again. \( \Box \)

**Proposition 11.2** For all \( z \), closed \( M \) and \( L \), the expressions \( z =_v M \) and \( z =_n L \) are acyclic.

**Proof.** We can show acyclicity by extending linear type checking in Figure 11. In the extended system, we derive judgements of the form \( \Gamma ; O; \preceq; E \), where \( \preceq \) is some acyclic ordering on the set of variables. Typical examples for type checking rules of the extended system are:

\[
\begin{align*}
\Gamma, x : \llbracket \overline{r} \rrbracket, y : \llbracket \overline{s} \rrbracket; \{x\}; & \quad \preceq ; \quad \Gamma \vdash x = y \quad x \preceq y \\
\Gamma, \overline{y} : \llbracket \overline{r} \rrbracket; \{x\}; & \quad \preceq ; \quad \Gamma \vdash x : \overline{y} / E \\
\Gamma, x : \llbracket \overline{r} \rrbracket; \{x\}; & \quad \preceq ; \quad \Gamma \vdash x : \overline{y} / E \\
\end{align*}
\]

The ordering \( \preceq \) \( \llbracket \overline{O}(\overline{y} ; \overline{s}) \rrbracket \) consists of all pairs \( (y, z) \) such that \( y \in \overline{O}(\overline{y} ; \overline{s}) \) and \( z \notin \overline{O}(\overline{y} ; \overline{s}) \), or \( y \preceq z \) but not \( y \notin \overline{O}(\overline{y} ; \overline{s}) \). It is not difficult to verify that the subject reduction property holds as before, cyclic expressions are not well-typed in the extended system. Since the subject reduction property holds as before, cyclic expressions are not well-typed. On the other hand side the expressions \( z =_v M \) and \( z =_n L \) are well-typed and hence acyclic. \( \Box \)

We note that the embedding \( E \mapsto \llbracket E \rrbracket \) does not preserve complexity in an obvious way. The main problem is about forwarding, which is illustrated by the following examples, where we assume that \( u_1, u_2, x, y \) denote distinct variables:

\[
\begin{align*}
E_1 & \overset{\text{def}}{=} x u \mid x = y & C(E_1) = 0 & C(\llbracket E_1 \rrbracket) = 1 \\
E_2 & \overset{\text{def}}{=} x = y \mid y : z / z z & C(E_2) = 1 & C(\llbracket E_2 \rrbracket) = 0 \\
E_3 & \overset{\text{def}}{=} x u_1 \mid x u_2 \mid x = y \mid y : z / z z & C(E_3) = 3 & C(\llbracket E_3 \rrbracket) = 4
\end{align*}
\]
12 Simulations and Uniformity

Milner [Mil92] uses bisimulations for proving the adequacy of $\lambda$-calculus embeddings into the $\pi$-calculus. We show that simulations are sufficient for uniform calculi.

Let $(E, \equiv_E, \rightarrow_E)$ and $(G, \equiv_G, \rightarrow_G)$ be two uniform calculi with expressions ranged over by $E$ and $G$ respectively. We omit the indices $E$ and $G$ whenever they are clear from the context. We call a function $\Phi : E \rightarrow G$ an embedding of $E$ into $G$, if $\Phi$ is invariant under congruence.

**Definition 12.1** Let $S$ be a relation on $E \times G$ and $\Phi$ be an embedding from $E$ into $G$. We call $S$ a shortening simulation for $\Phi$ if it satisfies the following conditions for all $E$, $E'$, and $G$:

- $(Sim 1)$ $(E, \Phi(E)) \in S$.
- $(Sim 2)$ If $E$ is irreducible and $(E, G) \in S$, then $G$ is irreducible.
- $(Sim 3)$ If $E \rightarrow E'$ and $(E, G) \in S$, then exists $E''$ and $G'$ with $C(E') \geq C(E'')$, $(E'', G') \in S$, and $G \rightarrow G'$.

$$E \rightarrow E' \geq E''$$

$$S \quad S$$

$$G \quad \rightarrow \quad G'$$

**Theorem 12.2** Let $\Phi : E \rightarrow G$ be an embedding between uniform calculi. If there exists a shortening simulation for $\Phi$, then $\Phi$ preserves termination and shortens complexity, i.e. $C(\Phi(E)) \leq C(E)$ for all $E$.

**Proof.** We assume a shortening simulation $S$ for $\Phi$ and $(E, G) \in S$. At first, we claim $C(G) \leq C(E)$ if $C(E) < \infty$. This can be proved by induction on $C(E)$. If $C(E) = 0$ then $E$ is irreducible such that $G$ is irreducible by $(Sim 2)$. Hence $C(G) = 0$. If $C(E) = n \geq 1$ then there exists $E'$ such that $E \rightarrow E'$. By uniformity $C(E') = n - 1$ follows. Condition $(Sim 3)$ implies the existence of $E''$ and $G'$ such that $G \rightarrow G'$, $C(E') \geq C(E'')$, and $(E'', G') \in S$. By induction hypothesis we obtain $C(G') \leq C(E'')$. The uniformity of both calculi implies:

$$C(G) = C(G') + 1 \leq C(E'') + 1 \leq C(E') + 1 = C(E)$$

The theorem follows from both claims and condition $(Sim 1)$.

**Definition 12.3** Let $S$ be a relation on $E \times G$ and $\Phi$ be an embedding from $E$ into $G$. We call $S$ a lengthening simulation for $\Phi$ if it satisfies $(Sim 1)$ and the following condition for all $E$, $E'$, and $G$:
(Sim4) If $E \to E'$ and $(E, G) \in S$, then exists $G', G'' \in \mathcal{G}$ such that $(E', G'') \in S$, $G \to G'$ and $\mathcal{C}(G') \geq \mathcal{C}(G'')$.

$$
\begin{align*}
E &\to E' \\
S &\to S \\
G &\to G' \geq G''
\end{align*}
$$

We call $S$ a complexity simulation for $\Phi$ if $S$ is a shortening and a lengthening simulation for $\Phi$.

**Proposition 12.4** Let $\Phi$ be an embedding between uniform calculi. If there exists a lengthening simulation for $\Phi$, then $\Phi$ lengthens complexity, i.e. $\mathcal{C}(E) \leq \mathcal{C}(\Phi(E))$ for all $E$.

**Proof.** Let $S$ be a lengthening simulation for $\Phi$ and $(E, G) \in S$. By induction on $n$ we can show that if there exists a derivation of $E$ of length $n$, then there exists a derivation of $G$ of length $\geq n$. \hfill \Box

**Corollary 12.5** Let $\Phi$ be an embedding between uniform calculi. If there exists a complexity simulation for $\Phi$, then $\Phi$ preserves complexity (and termination).

**Proof.** Immediate from Theorem 12.2 and Proposition 12.4. \hfill \Box

### 13 Notation

We need several notations for defining simulations and proving them correct. We introduce notations for explicit substitutions, sequences, and specialised reduction relations.

We use the following notation for explicit substitutions ([Mil92, ACCL91]). If $\overline{y} = (y_i)_{i=1}^n$ and $\overline{L} = (L_i)_{i=1}^n$, then let $\overline{y} = \overline{L}$ in $L'$ represents a $\lambda$-term:

$$
\text{let } \overline{y} = \overline{L} \text{ in } L' \overset{\text{def}}{=} L'[L_n/y_n] \ldots [L_1/y_1]
$$

We will freely make use of some further sequent notation. If furthermore $\overline{x} = (x_i)_{i=1}^n$, $\overline{t} = (t_i)_{i=1}^n$, $\overline{z} = (z_i)_{i=1}^n$, and $\overline{E} = (E_i)_{i=1}^n$, then we write:

$$
\begin{align*}
\overline{z} = \overline{L} &\overset{\text{def}}{=} z_1 = L_1 \ldots z_n = L_n \\
\overline{x} \overline{y} \overline{z} &\overset{\text{def}}{=} x_1 y_1 t_1 z_1 \ldots x_n y_n t_n z_n \\
\overline{t}.E &\overset{\text{def}}{=} t_1.E_1 \ldots t_n.E_n \\
(\nu \overline{y})E &\overset{\text{def}}{=} (\nu y_1) \ldots (\nu y_n) E \\
\overline{E} &\overset{\text{def}}{=} E_1 \ldots E_n \\
\overline{z} =_v \overline{M} &\overset{\text{def}}{=} z_1 =_v M_1 \ldots z_n =_v M_n \\
\overline{z} =_n \overline{L} &\overset{\text{def}}{=} z_1 =_n L_1 \ldots z_n =_n L_n \\
\nu(\overline{x}) &\overset{\text{def}}{=} \{x_1 \ldots x_n\}
\end{align*}
$$

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If \( \bar{p} = (\mu_j)_{j=1}^{n} \) is a sequence of variables or expressions then we write \( \bar{p}^{\leq i} \) for the sequence \( (\mu_j)_{j=1}^{i} \) and \( \bar{p}^{> i} \) for the sequence \( (\mu_j)_{j=i+1}^{n} \). The concatenation of two sequences \( \bar{p} \) and \( \bar{q} \) is denoted by \( \bar{p} \bar{q} \).

Let \((E, \equiv, \rightarrow)\) be a calculus, \(E, E' \in \mathcal{E}\), and \(n\) a natural number. We write \( E \rightarrow^n E' \) or \( E \rightarrow^{\leq n} E' \), if \( E \) reduces in exactly (resp less than) \( n \) steps to \( E' \). Formally, we define the relations \( \rightarrow^n \) and \( \rightarrow^{\leq n} \) as follows:

\[
\rightarrow^0 = \equiv, \quad \rightarrow^{n+1} = \rightarrow^n \circ \rightarrow, \quad \rightarrow^{\leq n} = \cup \{ \rightarrow^i \mid 0 \leq i \leq n \}
\]

We note that \( \rightarrow^\ast = \cup \{ \rightarrow^i \mid 0 \leq i < \infty \} \).

For reflecting A-complexity, we define the relation \( \leftrightarrow = (\rightarrow_F \cup \rightarrow_T)^\ast \). Let \( \delta' \) be the variant of \( \delta \) with the reduction \( \leftrightarrow \circ \rightarrow \circ \rightarrow_A \circ \leftrightarrow \) instead of \( \rightarrow \).

**Proposition 13.1** The restriction of \( \delta' \) to admissible expressions is uniform. For all admissible expressions \( E \) the complexity of \( E \) in \( \delta' \) and the A-complexity of \( E \) (which is defined relative to \( \delta \)) coincide: \( C_{\delta'}(E) = C^A(E) \).

**Proof.** This is an immediate consequence of Theorem 7.5. \( \square \)

In expression of the \( \delta \)-calculus, top-level declarations do not matter for complexity and termination considerations. We write \( E \approx F \) if there exists \( \bar{p} \) and \( \bar{q} \) such that \( (\nu \bar{p}) \bar{E} \equiv (\nu \bar{q}) \bar{F} \). The next two Lemmata justify ignoring top-level declarations in the sequel.

**Lemma 13.2** If \( E \approx F \rightarrow_A E' \) then there exists \( F' \) such that \( F \rightarrow_F F' \approx E' \). If \( E \approx F \rightarrow_F E' \) then there exists \( F' \) such that \( F \rightarrow_F F' \approx E' \). If \( E \approx F \rightarrow_T E' \) then there exists \( F' \) such that \( F \rightarrow_T F' \approx E' \).

\[
E \rightarrow_A E' \quad \quad \quad \quad \quad E \rightarrow_F E' \quad \quad \quad E \rightarrow_T E' \\
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \quad \quad \quad \quad \uparrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \uparrow
\]

\[
F \rightarrow_A F' \quad F \rightarrow_F F' \quad F \rightarrow_T F'
\]

**Lemma 13.3** The relation \( \approx \) is closed under weak context and invariant under structural congruence, i.e. it satisfies the contextual rules in Figure 4 (with \( \rightarrow \) replaced by \( \approx \)).

### 14 A Complexity Simulation for Call-by-Value

We proof the adequacy of the embedding \( M \mapsto z=v.M \) from the call-by-value \( \lambda \)-calculus into \( \delta \) as stated in Theorem 8.2.

Our goal is to establish the equation \( C^A(z=v.M) = C_{\text{value}}(M) \) for all closed \( \lambda \)-expressions \( M \). By Proposition 13.1 it is sufficient to show \( C_{\delta'}(z=v.M) = C_{\text{value}}(M) \). We will apply
Corollary 12.5 once we have constructed a complexity simulation for the above embedding considered into \( \delta' \) instead of \( \delta \). The necessary application conditions for Theorem 12.2 are verified by Propositions 13.1, 10.4, and Proposition 8.1.

**Example 14.1** Before formally defining a complexity simulation, we illustrate it by a simple example. Let \( C = \lambda x.xx \) be a \( \lambda \)-abstraction copying its argument and \( I = \lambda x.x \) the identity.

\[
C(CI) \xrightarrow{\text{value}} \text{let } y_1 = C 
\quad z_1 = I \text{ in } C(z_1 z_1) \\
\equiv \quad \text{let } y_1 = C 
\quad z_1 = I \text{ in } C(II)
\]

In the first step, we have reduced the redex \( CI \). Both involved abstractions have been moved into an environment. Note that only abstractions are moved into the environment. In the second step, we have forwarded abstractions into the next actual application. These two steps reflect the general scheme.

\[
z =_v C(CI) \xrightarrow{\delta} \quad y_1 =_v C \mid z_1 =_v I \mid z =_v C(z_1 z_1) \\
\xrightarrow{2_F} \quad y_1 =_v C \mid z_1 =_v I \mid z =_v C(II)
\]

Reduction in the \( \delta \)-calculus behaves very similar. The environment is represented by contexts built up with composition and declaration. Forwarding amounts to explicit \( \xrightarrow{F} \) steps.

**Definition 14.2 (v-Representation)** A v-representation for \((M, E)\) is a triple \((n, \overline{y}, \overline{M})\), where \( \overline{y} = (y_i)_{i=1}^n \) and \( \overline{M} = (M_i)_{i=1}^n \). We require the following properties for all \( i \in \{1 \ldots n\}\):

\[(S_v 1) \quad \mathcal{V}(M_i) \subseteq \{y_1 \ldots y_{i-1}\} \text{ and } \overline{y} \text{ is linear.} \]

\[(S_v 2) \quad M \equiv \text{let } \overline{y} = \overline{M} \text{ in } y_n. \]

\[(S_v 3) \quad E \approx y_1 =_v M_1 \mid \ldots \mid y_n =_v M_n. \]

\[(S_v 4) \quad \text{If } i < n \text{ then } M_i \text{ is an abstraction.} \]

**Lemma 14.3 (Closedness)** If \( n, \overline{M}, \overline{y}, \) and \( M \) satisfy \((S_v 1)\) and \((S_v 2)\), then \( M \) is closed.

**Proof.** By induction on \( n \). If \( n = 1 \) then \( M \equiv \text{let } y_1 = M_1 \text{ in } y_1 \) such that \( \mathcal{V}(M) \subseteq \mathcal{V}(M_1) \subseteq 0 \). If \( n > 1 \), then we can apply the induction hypothesis to the following representation of \( M \):

\[
M \equiv \text{let } \overline{y}^{<n-1} = \overline{M}^{<n-1} \quad y_n = M_n[M_{n-1}/y_{n-1}] \text{ in } y_n
\]

\( \square \)

**Definition 14.4 (Relation \( S_v \))** The relation \( S_v \) is the set of all pairs \((M, E)\) for which a v-representation exists.
Proposition 14.5 (\(S_v\) is a Complexity Simulation) The relation \(S_v\) is a relation between closed \(\lambda\)-expressions and admissible \(\delta\)-expressions. It satisfies the following properties for all \(M\), \(z\), and \(E\):

1. If \(M\) is closed then \((M, z =_v M) \in S\).
2. If \(M\) is irreducible with respect to \(\rightarrow_{\text{value}}\) and \((M, E) \in S\), then \(E\) is irreducible with respect to \(\rightarrow_A \cup \rightarrow_F \cup \rightarrow_T\).
3. If \((M, E) \in S\) and \(M \rightarrow_{\text{value}} M'\), then there exists \(E'\) such that \(E \rightarrow_{\delta}^2 \circ \rightarrow_A E'\) and \((M', E') \in S^7\).

\[
\begin{array}{ccc}
M & \rightarrow_{\text{value}} & M' \\
S & S & S \\
E & \rightarrow_{\delta}^2 & \rightarrow_A E'
\end{array}
\]

Proof.

1. The triple \((n, (z), (M))\) is a \(v\)-representation of \((M, z =_v M)\). Property \((S_v,1)\) follows from the closedness of \(M\) and \((S_v,2)-(S_v,4)\) are trivial.

2. Let \(M\) be closed and irreducible with respect to \(\rightarrow_{\text{value}}\). Hence \(M\) is an abstraction such that \(z =_v M\) is an abstraction and therefore irreducible with respect to \(\rightarrow_A \cup \rightarrow_F \cup \rightarrow_T\).

3. Let \((n, \overline{y}, \overline{M})\) be a \(v\)-representation of \((M, E)\) and \(M \rightarrow_{\text{value}} M'\). Applying the following Lemma 14.7, there exists sequences \(\overline{x}\) and \(\overline{V}\) of length \(m\), and an expression \(E'\) such that \(n + m, \overline{\overline{y}^{<n} \overline{x}}, \overline{\overline{M}^{<n} \overline{V} \overline{M'}^{<n}}\) is a \(v\)-representation for \((M', E')\) and \(E \rightarrow_{\delta}^2 \circ \rightarrow_A E'\).

\(\square\)

Corollary 14.6 The relation \(S_v\) is a complexity simulation for the mapping \(M \mapsto z =_v M\) considered as embedding from the call-by-value \(\lambda\)-calculus restricted to closed expressions into \(\delta\).

Proof. Immediate from Proposition 14.5. \(\square\)

\(^7\)This invariant is strong enough for proving that the number of \(\rightarrow_F\) steps in computations of expressions \(z =_v M\) is bounded by 2 times the number of \(\rightarrow_A\) steps. If we would embed a \(\lambda\)-calculus with \(n\)-ary instead of unary function, then we would obtain a factor of \(n + 1\) instead of 2.
Lemma 14.7  Let \((n, \bar{y}, \bar{M})\) be a \(v\)-representation of \((M, E)\) and \(M \to_{\text{value}} M'\). Then there exists fresh variables \(\bar{x}\), abstractions \(\bar{V}\), and a \(\lambda\)-expression \(M'_n\) such that \(E \to^{\leq 2}_{\bar{F}} \circ \to_{\bar{A}} E'\), \(\forall(\bar{V}) \subseteq \forall(\bar{y}^n), \forall(M'_n) \subseteq \forall(\bar{y}^n)\), and:

\[
M' \equiv \text{let } \bar{y}^m = \bar{M}^m \text{ in } y_n = M'_n \text{ in } y_n
\]

\[
E' \approx \bar{y}^m = v \bar{M}^m \mid \bar{x} = v \bar{V} \mid y_n = v M'_n
\]

Proof. Since \((n, \bar{y}, \bar{M})\) is a \(v\)-representation, we know \(M \equiv \text{let } \bar{y} = \bar{M} \text{ in } y_n\) and \(E \approx \bar{y} = v \bar{M}\). Since \(M\) can not be an abstraction, property (S\(v\)) implies that \(M_n\) is an application \(N_1N_2\) for some \(N_1\) and \(N_2\). Hence \(M \equiv P_1P_2\) and:

\[
P_1 \equiv \text{let } \bar{y}^m = \bar{M}^m \text{ in } N_1, \quad P_2 \equiv \text{let } \bar{y}^m = \bar{M}^m \text{ in } N_2
\]

1. Case: \(M \to_{\text{value}} M'\) is an instance of the \(\beta\)-axiom, i.e. \(P_1 \equiv \lambda x.\bar{P}_1\) and:

\[
M \equiv (\lambda x.\bar{P}_1)P_2 \to_{\text{value}} \bar{P}_1[P_2/x] \equiv M'
\]

Since \(P_1\) and \(P_2\) are abstractions, \(N_1\) and \(N_2\) have to be either variables or abstractions. This leads to four very similar subcases. We only consider the case where \(N_1\) and \(N_2\) are both variables. In this case there exists \(y_{l_1}\) and \(y_{l_2}\) such that \(N_1 = y_{l_1}\) and \(N_2 = y_{l_2}\). Furthermore:

\[
P_1 \equiv \text{let } \bar{y}^m = \bar{M}^m \text{ in } M_{l_1}, \quad P_2 \equiv \text{let } \bar{y}^m = \bar{M}^m \text{ in } M_{l_2}
\]

If \(M_{l_1} \equiv \lambda x.\bar{M}_{l_1}\) then \(\bar{P}_1 \equiv \text{let } \bar{y}^m = \bar{M}^m \text{ in } \bar{M}_{l_1}\). Let \(x_1\) and \(x_2\) be fresh.

\[
M' \equiv (\text{let } \bar{y}^m = \bar{M}^m \text{ in } \bar{M}_{l_1})[P_2/x]
\]

\[
\equiv \text{let } \bar{y}^m = \bar{M}^m \text{ in } M_{l_1}[P_2/x]
\]

\[
\equiv \text{let } \bar{y}^m = \bar{M}^m \text{ in } M_{l_1}[y_{l_2}/x]
\]

\[
\equiv \text{let } \bar{y}^m = \bar{M}^m \text{ in } x_1 = M_{l_1} \quad x_2 = M_{l_2} \quad y_n = M_{l_1}[x_2/x] \text{ in } y_n
\]

Reduction of \(E\) may proceed with two forwarding steps followed by an application step.

\[
E \approx \bar{y}^m = v \bar{M}^m \mid y_n = v y_{l_1} y_{l_2}
\]

\[
\approx \bar{y}^m = v \bar{M}^m \mid x_1 = y_{l_1} \quad x_2 = y_{l_2} \quad x_1 x_2 y_n
\]

\[
\to_{\bar{F}} \quad \bar{y}^m = v \bar{M}^m \mid x_1 = M_{l_1} \quad x_2 = M_{l_2} \quad x_1 x_2 y_n
\]

\[
\to_{\bar{A}} \quad \bar{y}^m = v \bar{M}^m \mid x_1 = M_{l_1} \quad x_2 = M_{l_2} \quad y_n = v M_{l_1}[x_2/x]
\]

This proves the inductive assertion with \(M'_n \equiv \bar{M}_{l_1}[x_2/x]\) and \(\bar{V}\) equals the sequence \((M_{l_1}, M_{l_2})\).

2. Case: The last rule in the derivation of \(M \to_{\text{value}} M'\) allows for reduction in functional position:

\[
\frac{P_1 \to_{\text{value}} P'_1 \quad P'_2}{M \equiv P_1P_2 \to_{\text{value}} P'_1P'_2 \equiv M'}
\]
Let $z_1$ and $z_2$ be fresh variables and define:

$$E_1 \overset{\text{def}}{=} \bar{y}^{<n} = v, \bar{M}^{<n} \mid z_1 = v, \bar{N}_1$$

By induction hypothesis there exists fresh variables $\bar{x}$, abstractions $\bar{V}$, $N'_1$, and $E'_1$ such that $E_1 \overset{\leq 2}{\rightarrow} \sigma \rightarrow_A E'_1$ and:

$$P'_1 \overset{\text{def}}{=} \begin{align*}
\bar{y}^{<n} &= \bar{M}^{<n} \\
\bar{y}^{<n} &= \bar{M}^{<n} \mid \bar{x} = v, \bar{V} \mid y_n = \bar{N}'_1 \text{ in } y_n
\end{align*}
$$

$$E'_1 \overset{\text{def}}{=} \begin{align*}
\bar{y}^{<n} &= \bar{M}^{<n} \mid z_1 = v, \bar{N}_1' \mid z_2 = v, \bar{N}_2 \mid z_1, z_2, y_n
\end{align*}$$

Additionally, we obtain some conditions on variables occurrences, which imply:

$$M' \overset{\text{def}}{=} P'_1 P_2 \overset{\text{def}}{=} \begin{align*}
\left( \text{let } \bar{y}^{<n} = \bar{M}^{<n} \mid \bar{x} = \bar{V} \text{ in } N'_1 \right) & \left( \text{let } \bar{y}^{<n} = \bar{M}^{<n} \text{ in } N_2 \right) \\
\text{let } \bar{y}^{<n} &= \bar{M}^{<n} \mid \bar{x} = \bar{V} \mid y_n = \bar{N}'_1 N_2 \text{ in } y_n
\end{align*}$$

Furthermore:

$$\begin{align*}
E & \overset{\text{def}}{=} \begin{align*}
\bar{y}^{<n} &= \bar{M}^{<n} \mid y_n = v, \bar{N}_1 N_2 \\
\bar{y}^{<n} &= \bar{M}^{<n} \mid z_1 = v, \bar{N}_1 \mid z_2 = v, \bar{N}_2 \mid z_1, z_2, y_n
\end{align*} \\
\rightarrow_P \overset{\leq 2}{\rightarrow} \rightarrow_A & \begin{align*}
\bar{y}^{<n} &= \bar{M}^{<n} \mid \bar{x} = v, \bar{V} \mid z_1 = v, \bar{N}'_1 \mid z_2 = v, \bar{N}_2 \mid z_1, z_2, y_n
\end{align*} \\
\rightarrow_P \overset{\leq 2}{\rightarrow} \rightarrow_A & \begin{align*}
\bar{y}^{<n} &= \bar{M}^{<n} \mid \bar{x} = v, \bar{V} \mid y_n = v, \bar{N}'_1 N_2
\end{align*}
\end{align*}$$

This proves the inductive assertion with $M'_n \overset{\text{def}}{=} N'_1 N_2$.

3. Case: The last rule in the derivation of $M \overset{\text{value}}{\rightarrow} M'$ allows for reduction in argument position:

$$\begin{align*}
P_2 \overset{\text{value}}{\rightarrow} P'_2 \\
M \overset{\text{value}}{\rightarrow} P_1 P'_2 \overset{\text{value}}{\rightarrow} P'_1 P'_2 \overset{\text{def}}{=} M'
\end{align*}$$

This case is symmetric to the previous one.

\[\square\]

### 15 Shortening Call-by-Name to Call-by-Need

As stated in Theorem 8.3, we prove that the embedding $M \mapsto z = M$ from the call-by-name $\lambda$-calculus into $\delta$ preserves termination such that $C^A(z = M) \leq C_{\text{name}}(M)$ for all closed $\lambda$-expressions $M$.

By Proposition 13.1 the above complexity estimation is implied by the following one:

$$C_{\delta'}(z = M) \leq C_{\text{name}}(M)$$

for all closed $M$. For proof, we will apply Theorem 12.2 to a shortening simulation for the above embedding considered into $\delta'$ instead of $\delta$. This is sufficient to establish our termination statement as well, since termination in $\delta'$ and $\delta$ are equivalent (since $\rightarrow_P$ and $\rightarrow_T$ terminate). As in the case of our call-by-value embedding, the necessary application conditions for Theorem 12.2 are verified by Propositions 13.1, 10.4, and Proposition 8.1.
15.1 Example

Before formally defining a shortening simulation, we illustrate it by a simple example. We first consider a call-by-name reduction step of \((\Pi) I\) with \(I \equiv \lambda x. x:\)

\[
(I) I \quad \equiv \quad \text{let } y_1 = I \quad z_1 = I \quad y_2 = y_1 \quad z_2 = I \quad y_3 = y_2 \quad z_2 \quad \text{in } y_3
\]

\[
\rightarrow_{\text{name}} \quad \text{let } y_1 = I \quad z_1 = I \quad y_2 = I \quad z_2 = I \quad y_3 = y_2 \quad z_2 \quad \text{in } y_3
\]

\[
\equiv \quad \text{let } y_1 = I \quad z_1 = I \quad y_2 = I \quad z_2 = I \quad y_3 = y_2 \quad z_2 \quad \text{in } y_3
\]

First, the \(\lambda\)-term \(I\) is flattened. Second, an application is executed. Third, the value \(I\) is forwarding to the variable \(y_1\). The corresponding \(\delta\)-reduction sequence is quite similar:

\[
y_3 = n(I) I \quad \equiv \quad y_1 = I \quad i \quad z_1 = n \quad z_2 = n \quad y_2 = I \quad z_2 = n \quad y_3 = n \quad z_3 = n \quad \text{in } y_3
\]

\[
\rightarrow_{A} \quad y_1 = I \quad i \quad z_1 = n \quad y_2 = n \quad z_2 = n \quad y_3 = n \quad z_3 = n \quad \text{in } y_3
\]

\[
\rightarrow_{T} \quad y_1 = I \quad i \quad z_1 = n \quad y_2 = I \quad z_2 = n \quad y_3 = n \quad z_3 = n \quad \text{in } y_3
\]

\[
\rightarrow_{F} \quad y_1 = I \quad i \quad z_1 = n \quad y_2 = I \quad z_2 = n \quad y_3 = n \quad z_3 = n \quad \text{in } y_3
\]

The third step - triggering a needed argument - is not visible in the above \(\lambda\)-calculus derivation. Apart from this aspect, both computations are very similar.

15.2 Properties

An appropriate shortening simulation has to cover more aspects than illustrated in the previous example. In this subsection, we formulate sufficiently strong properties for an appropriate candidate.

An interesting example comes with sharing, when comparing call-by-name and call-by-need reduction for the expression \((\lambda x. (x \lambda y. x)) (\Pi) I\). In this case, we can formulate the relationship via strong call-by-name reduction. We write \(M \Rightarrow_{\text{name}} M'\) if \(M\) reduces to \(M'\) by application of the \(\beta\)-axiom at any position in \(M\).

**Proposition 15.1 (Shortening Call-by-Name to Call-by-Need)** There exists a relation \(S\) between closed \(\lambda\)-expressions and admissible \(\delta\)-expressions satisfying the following properties for all \(M\), \(z\), and \(E\):

1. If \(M\) is closed then \((M, z = n M) \in S\).
2. If \(M\) is irreducible with respect to \(\rightarrow_{\text{name}}\) and \((M, E) \in S\), then \(E\) is irreducible with respect to \(\rightarrow_A\), \(\rightarrow_F\), and \(\rightarrow_T\).
3. If \((M, E) \in S\) and \(M \Rightarrow_{\text{name}} M'\), then there exists \(M''\) and \(E''\) such that \(M' \Rightarrow_{\text{name}}^{*} E''\).
\[ M^\prime, E \leftrightarrow \circ \rightarrow_A \circ \leftrightarrow E', \text{ and } (M^\prime, E') \in S^8. \]

\[
\begin{align*}
M & \xrightarrow{\text{name}} M' \xrightarrow{\text{name}*} M'' \\
S & \quad \quad S \\
E & \leftrightarrow \rightarrow_A \leftrightarrow E'
\end{align*}
\]

Proof. The relation \( S \) is defined in Section 15.3 and proved correct in Section 15.4. \( \Box \)

**Corollary 15.2** There exists a shortening simulation for the mapping \( M \mapsto z=_{\alpha} M \) considered as embedding from the call-by-name \( \lambda \)-calculus restricted to closed expressions into \( \delta' \).

Proof. This is a consequence of Proposition 15.1. For proving property (Sho3) we additionally need Lemma 15.3. \( \Box \)

**Lemma 15.3 (Reformulation of Plotkin’s [Plo75] Standardisation Theorem)** If \( M \xrightarrow{\text{name}*} M' \), then \( C_{\text{name}}(M) \geq C_{\text{name}}(M') \).

Proof. It is sufficient to consider \( M \xrightarrow{\text{name}} M' \). For proof, we define \( M \sim_{\text{name}*} M' \) iff \( M \xrightarrow{\text{name}} M \) but not \( M \xrightarrow{\text{name}} M' \). Trivially, \( \Rightarrow_{\text{name}} = \sim_{\text{name}} \sqcup \Rightarrow_{\text{name}} \). In the case \( M \xrightarrow{\text{name}} M' \) the lemma follows from uniform confluence of the call-by-name \( \lambda \)-calculus. If \( M \sim_{\text{name}*} M' \), then it is implied by \( \sim_{\text{name}*} \) being a shortening simulation for the identity embedding from the call-by-name \( \lambda \)-calculus into itself.

(Sim1) The relation \( M \sim_{\text{name}*} M \) holds trivially.

(Sim2) An expression \( M \) is irreducible with respect to \( \Rightarrow_{\text{name}} \) iff it is an abstraction or an application of the form \( ((xQ_1)\ldots Q_n) \). The relation \( \sim_{\text{name}*} \) preserves these forms of terms.

(Sim3) For all \( M, M', \) and \( N \), there exists \( M'' \) and \( N' \) such that following diagram holds:

\[
\begin{array}{ccc}
M & \rightarrow_{\text{name}} & M' \\
\downarrow_{\text{1}} & \quad & \downarrow_{\text{1}*} \\
N & \rightarrow_{\text{name}} & N'
\end{array}
\]

\[ \text{Sim3} \]

\( \overline{\text{Sim3}} \) Ignoring \( \rightarrow_{\text{F}} \) and \( \rightarrow_{\text{T}} \) steps is correct in the sense that the number of \( \rightarrow_{\text{F}} \) and \( \rightarrow_{\text{T}} \) steps in computations of \( y = z_{\alpha} M \) is bounded by 3 times the number of \( \rightarrow_{\text{A}} \) steps. This can be proved with a simulation for an amortised cost analysis by formulating a stronger invariant than in Proposition 15.1. As in the call-by-value case, an application involves at most 2 forwarding steps. Additionally, every application step may raise the need for 1 triggering step.
For proving property \((\text{Sim}3)\), we need in fact a slightly stronger property, where \(M \trianglerighteq \text{name} N\) is replaced by \(M \trianglerighteq^* \text{name} N\). This is implied by the above diagram and the inclusion \(\trianglerighteq \text{name} \circ \to \text{name} \subseteq \to \text{name} \circ \trianglerighteq^* \text{name}\).

The above diagram can be shown by structural induction on \(M\). For illustration, we consider the case \(M \equiv (\lambda x. M_1) M_2\) where the \(\trianglerighteq \text{name}\) step is applied inside of \(M_2\). Hence, \(M_2 \to \text{name} M_2', N \equiv (\lambda x. M_1) M_2', \) and \(M' \equiv M_1[M_2/x] \).

There are 4 possible subcases to consider: Either \(M_1 \equiv ((xQ_1) \ldots Q_n)\) for some \(Q_1, \ldots, Q_n\) or not, and either \(M_2 \to \text{name} M_2' \) or \(M_2 \trianglerighteq \text{name} M_2'\). If we choose both times the first possibility, then we obtain:

\[
(\lambda x. \tilde{M}_1) M_2 \to \text{name} \tilde{M}_1[M_2/x] \to \text{name} ((M_2[Q_1]) \ldots Q_n)[M_2/x]
\]

Otherwise, we obtain the required diagram in the form:

\[
(\lambda x. \tilde{M}_1) M_2 \to \text{name} \tilde{M}_1[M_2/x]
\]

\[
(\lambda x. \tilde{M}_1) M_2 \to \text{name} \tilde{M}_1[M_2/x]
\]

15.3 Definition

We base our definition of a shortening simulation on the notion of needed variables.

**Definition 15.4 (Needed Variables)** Let \(n\) be an integer, \(y = (y_i)_{i=1}^n\), \(\overline{M} = (M_i)_{i=1}^n\) and \(1 \leq j \leq n\). The variable \(y_j\) is needed in \(\text{let } \overline{y}=\overline{M} \text{ in } N\), if the judgement \(N(\overline{y}_j, \text{let } \overline{y} = \overline{M} \text{ in } N)\) is derivable by the following rules:

\[
\begin{align*}
\frac{}{N(x, x)} & \\
\frac{N(x, N_1)}{N(x, \overline{N}_1 \overline{M}_2)} & \\
\frac{N(x, N)}{N(x, \text{let } \overline{y} = \overline{M} \text{ in } N)} & \\
\frac{N(\overline{y}_j, \text{let } \overline{y} = \overline{M} \text{ in } N)}{N(\overline{y}_j, \text{let } \overline{y} = \overline{M} \text{ in } N)}
\end{align*}
\]

\(j < i \leq n\)

**Example 15.5** The variables \(y_3\) and \(y_1\) are needed in \(\text{let } y_1 = I \ y_2 = y_1 \ y_1 \ y_3 = y_1 \ y_2 \text{ in } y_3\), whereas \(y_2\) is not needed. The neededness of \(y_3\) is shown by the following derivation:

\[
\begin{align*}
\frac{N(\overline{y}_1, \overline{y}_1)}{N(\overline{y}_1, \text{let } \overline{y}_1 = \overline{I} \ y_2 = y_1 \ y_1 \ y_3 = y_1 \ y_2 \text{ in } \overline{y}_1)} & \\
\frac{N(\overline{y}_1, \text{let } \overline{y}_1 = \overline{I} \ y_2 = y_1 \ y_1 \ y_3 = y_1 \ y_2 \text{ in } \overline{y}_1)}{N(\overline{y}_1, \text{let } \overline{y}_1 = \overline{I} \ y_2 = y_1 \ y_1 \ y_3 = y_1 \ y_2 \text{ in } \overline{y}_3)}
\end{align*}
\]
Definition 15.6 (n-Representation) A n-representation for \((M, E)\) is a five-tuple \((n, \overline{y}, \overline{M}, \overline{I}, D)\), where \(\overline{M} = (M_i)_{i=1}^n\), \(\overline{y} = (y_i)_{i=1}^n\), \(\overline{I} = (I_i)_{i=1}^n\), and \(D \subseteq \{y_1, \ldots, y_n\}\) called the delay set. We require the following properties for all \(i \in \{1, \ldots, n\}\):

\[(S_n1)\] \(V(M_i) \subseteq \{y_1 \ldots y_i-1\}\) and the composed sequence \(\overline{y}I\) is linear.

\[(S_n2)\] \(M \equiv \text{let } \overline{y} = \overline{M} \text{ in } y_n\).

\[(S_n3)\] There exists \((E_i)_{i=1}^n\), \(\phi\), and \(\theta\) such that \(E \equiv E_1 | \ldots | E_n | \phi\), where \(\phi\) is a possibly empty composition of trigger expressions in \(\{\text{tr}(t_j) \mid y_j \notin D\}\), \(\theta = [\overline{y}\phi]/\overline{y}\), and:

\[
E_i = \begin{cases} 
  t_i \cdot y_i = n_i \theta & \text{if } y_i \in D \\
  y_j y_k y_i & \text{if } y_i \notin D \text{ and } M_i = y_j y_k \text{ for some } j, k \\
  y_i = y_j & \text{if } y_i \notin D \text{ and } M_i = y_j \text{ for some } j \\
  y_i = M_i \theta & \text{if } y_i \notin D \text{ and } M_i \text{ is an abstraction}
\end{cases}
\]

\[(S_n4)\] If \(y_i \notin D\) and \(M_i\) is an application then \(M_i\) is an application of variables.

\[(S_n5)\] If \(y_i\) is needed in \(\text{let } \overline{y} = \overline{M} \text{ in } y_n\), then \(y_i \notin D\).

\[(S_n6)\] If \(y_i\) is not needed in \(\text{let } \overline{y} = \overline{M} \text{ in } y_n\), then \(y_i \in D\) or \(M_i\) is an abstraction.

Definition 15.7 (Relation \(S_n\)) We define the relation \(S_n\) as the set of all pairs \((M, E)\) for which a n-representation exists.

Proposition 15.8 (\(S_n\) is a Shortening Simulation) The relation \(S_n\) satisfies the conditions of Proposition 15.1.

Proof. This is the content of the Lemmata 15.11, 15.15, and 15.17. \(\square\)

15.4 Correctness Proof

We prove Proposition 15.8, which states the correctness of our shortening simulation \(S_n\). We have to validate three properties reconsidered in Propositions 15.11, 15.15, and 15.17.

15.4.1 Property (Sim1)

Lemma 15.9 For every \(M\) there exists \(m \geq 0\), \((P_i)_{i=1}^m\) and \(Q\) such that \(M \equiv \ldots (QP_m) \ldots P_1\) and \(Q\) is not an application.

Proof. By structural induction on \(M\). If \(M\) is an abstraction or a variable, then we choose \(m = 0\) and \(Q \equiv M\). If \(M \equiv M_1 M_2\) then there exists \(m \geq 0\) and \((P_i)_{i=2}^m\) such that:

\[M_1 \equiv \ldots (QP_m) \ldots P_2\]

If we set \(P_1 \equiv M_2\), then we obtain \(M \equiv M_1 M_2 \equiv \ldots (QP_m) \ldots P_1\). \(\square\)
Lemma 15.10 (Flattening) If \( M \equiv (\ldots (Q P_m) \ldots) P_1 \) for some \( m \geq 0 \) and \( \overline{w} = (u_i)_{i=1}^{m+1}, \overline{s} = (s_i)_{i=1}^{m} \) are variables not contained in \( V(M) \), then the following representations are valid:

\[
M \equiv \begin{cases} 
\text{let } u_1 = Q \quad \overline{v} = \overline{t} \quad \overline{v}^+ = \overline{u}^{<m+1} \overline{v} & \text{in } u_{m+1} \\
\end{cases} \\
\begin{array}{ll}
u_m+1 = n M & \approx \begin{cases} 
\text{let } u_1 = Q \quad \overline{s}, \overline{v} = n \overline{t} & \overline{v}^{<m+1} \overline{s} \overline{v}^+ \\
\end{cases} 
\end{array}
\]

Proof. By induction on \( m \). In the case \( m = 0 \) there is nothing to show. If \( m > 0 \) then \( M \equiv M_1 M_2 \) where \( M_1 \equiv (\ldots (Q P_1) \ldots) P_{m-1} \), and \( M_2 \equiv P_m \). Applying the induction hypothesis to \( u_m = n M_1 \) we obtain:

\[
M_1 \equiv \begin{cases} 
\text{let } u_1 = Q \quad \overline{v}^m = \overline{P}^{<m} \quad u_2 = u_1 v_1 \ldots u_m = u_{m-1} v_m \quad \text{in } u_m \\
\end{cases} \\
\begin{array}{ll}
u_m = n M_1 & \equiv \begin{cases} 
\text{let } u_1 = Q \quad \overline{v}^m = \overline{P}^{<m} \quad u_2 = u_1 v_1 \ldots u_m = u_{m-1} v_m \quad \text{in } u_m \\
\end{cases} 
\end{array}
\]

Since \( M = M_1 P_m \), this implies:

\[
M \equiv \begin{cases} 
\text{let } u_1 = Q \quad \overline{v}^m = \overline{P}^{<m} \quad u_2 = u_1 v_1 \ldots u_m = u_{m-1} v_m \quad \text{in } u_m P_m \\
\end{cases} \\
\begin{array}{ll}
u_m = n M_1 \equiv \begin{cases} 
\text{let } u_1 = Q \quad \overline{v}^m = \overline{P}^{<m} \quad u_2 = u_1 v_1 \ldots u_m = u_{m-1} v_m \quad \text{in } u_m \\
\end{cases} 
\end{array}
\]

The expression \( u_{m+1} = n M \) satisfies:

\[
u_{m+1} = n M \approx \begin{cases} 
\text{let } u_1 = Q \quad \overline{s}, \overline{v} = n \overline{P} & \overline{v}^{<m+1} \overline{s} \overline{v}^+ \quad \text{in } u_{m+1} P_m \\
\end{cases} \\
\begin{array}{ll}
u_m = n M & \equiv \begin{cases} 
\text{let } u_1 = Q \quad \overline{v} = \overline{t} & \overline{v}^{<m+1} \overline{s} \overline{v}^+ \\
\end{cases} 
\end{array}
\]

Replacing \( u_m = n M_1 \) in \( u_{m+1} = n M \) by its above representation yields:

\[
u_{m+1} = n M \approx \begin{cases} 
\text{let } u_1 = Q \quad \overline{s}, \overline{v} = n \overline{P} & \overline{v}^{<m+1} \overline{s} \overline{v}^+ \quad \text{in } u_{m+1} P_m \\
\end{cases} \\
\begin{array}{ll}
u_m = n M & \equiv \begin{cases} 
\text{let } u_1 = Q \quad \overline{v} = \overline{t} & \overline{v}^{<m+1} \overline{s} \overline{v}^+ \\
\end{cases} 
\end{array}
\]

Proposition 15.11 The relation \( S_n \) satisfies (Sim1).

Proof. Let \( M \) be a closed \( \lambda \)-expression and \( z \) a variable. We have to construct a \( n \)-representation for \( (M, z =_n M) \). Lemma 15.9 yields the existence of \( m \) and \( (P_i)_{i=1}^m \) such that \( M = (\ldots (Q P_m) \ldots) P_1 \). Let \( (u_i)_{i=1}^m, (v_i)_{i=1}^m, (s_i)_{i=1}^m \) be sequence of fresh variables and define \( u_{m+1} = z \). Applying the flattening Lemma 15.10 yields:

\[
M \equiv \begin{cases} 
\text{let } u_1 = Q \quad \overline{v} = \overline{t} \quad \overline{v}^+ = \overline{u}^{<m+1} \overline{v} & \text{in } u_{m+1} \\
\end{cases} \\
\begin{array}{ll}
z = n M & \approx \begin{cases} 
\text{let } u_1 = Q \quad \overline{s}, \overline{v} = n \overline{t} & \overline{v}^{<m+1} \overline{s} \overline{v}^+ \\
\end{cases} 
\end{array}
\]

These properties essentially verify \((S_n, 2)\) and \((S_n, 3)\) where \( E \equiv z =_n M \). In order to formalise this statement, we have to define a \( n \)-representation \((n, \overline{v}, \overline{M}, \overline{t}, D)\) for \((M, E)\) accordingly:

\[
\begin{align*}
\overline{v} & = u_1 \overline{s} \overline{u}^+ \\
\overline{M} & = Q \overline{P} \overline{v}^{<m+1} \overline{v} \\
\overline{t} & = - \overline{v} - \\
n & = 2m + 1 \\
D & = V(\overline{v})
\end{align*}
\]
In these definitions, each occurrences of the symbol \( \_ \) stands for a fresh variable. We have to verify the conditions of Definition 15.6. Property \((S_n1)\) follows from the closedness of \( M \). \((S_n2)\) and \((S_n3)\) have already been discussed. \((S_n4)\) holds trivially. For \((S_n5)\) we note that the needed variables in \( \text{let } \vec{y} = \overline{M} \text{ in } y_n \) are those in \( \mathcal{V}(\vec{y}) \). For \((S_n6)\) we note that the not needed variables are those in \( \mathcal{V}(\vec{y}) \).

15.4.2 Property (Sim2)

**Lemma 15.12 (Forwarding)** If \((M, E) \in S_n\) then there exists \( E' \) with \( E \rightarrow^{*}_{F} E' \) and there exists a \( n \)-representation \((n, \vec{y}, \overline{M}, \vec{I}, D)\) of \((M, E')\) complete under forwarding, i.e. satisfying the property:

\[(S_n7) \text{ If } j, k \in \{1 \ldots n\}, y_j \notin D, \text{ and } M_j = y_k, \text{ then } M_k \text{ is not an abstraction.}\]

**Proof.** Let \((n, \vec{y}, \overline{M}, \vec{I}, D)\) be a \( n \)-representation of \((M, E)\). We have to construct a \( n \)-representation of \((M, E)\) satisfying \((S_n7)\). Suppose there exists a pair of indices \( j, k \in \{1 \ldots n\} \) such that \( y_j \notin D, M_j = y_k, \) and \( M_k \) is an abstraction. We show how to eliminate this index pair by forwarding \( \rightarrow_{F} \). Our elimination procedure terminates, since it decreases the number of such index pairs. By assumption and \((S_n1)\) we obtain:

\[
M \equiv \text{let } \ldots y_k = M_k \ldots y_j = y_k \ldots \text{ in } y_n
\]

\[
\equiv \text{let } \ldots y_k = M_k \ldots y_j = M_k \ldots \text{ in } y_n
\]

Property \((S_n6)\) implies that \( y_j \) is needed in \( \text{let } \vec{y} = \overline{M} \text{ in } y_n \) (since \( y_j \notin D \) and \( M_j \) is not an abstraction). By definition of neededness, \( y_k \) is also needed in \( \text{let } \vec{y} = \overline{M} \text{ in } y_n \) such that \((S_n5)\) implies \( y_k \notin D \). Hence:

\[
E \approx \ldots | y_k = n M_k | \ldots | y_j = y_k | \ldots \rightarrow_{F} \ldots | y_k = n M_k | \ldots | y_j = n M_k | \ldots
\]

**□**

**Definition 15.13** Let \((n, \overline{M}, \vec{y})\) and \( M \) satisfy \((S_n1)\) and \((S_n2)\). A reference chain from \( y_n \) to \( y_{\nu(1)} \) is a sequence \((y_{\nu(i)})_{i=1}^{p} \), if \( p \geq 1 \) is an integer, the \( \nu(i) \)'s are indices such that \( 1 \leq \nu(1) < \ldots < \nu(p) = n, \) and \( M_{\nu(i)} = y_{\nu(i-1)} \) for all \( 1 < i \leq p \). In this case, we write:

\[
M \equiv \text{let } \ldots y_{\nu(1)} = M_{\nu(1)} \ldots y_{\nu(2)} = y_{\nu(1)} \ldots y_n = y_{\nu(p-1)} \text{ in } y_n
\]

**Lemma 15.14 (Reference Chains)** Let \((n, \vec{y}, \overline{M})\) and \( M \) satisfy \((S_n1)\) and \((S_n2)\). Then there exists \( 1 \leq j \leq n \) and a reference chain from \( y_n \) to \( y_j \) such that \( M_j \) is not a variable.

**Proof.** By induction on \( n \). If \( n = 1 \), then \( M_1 \) may not be a variable since \( M \) is closed (Lemma 14.3). If \( n > 0 \) and \( M_n \) is not a variable then there is nothing to prove. Otherwise, we use \( M \equiv \text{let } \vec{y}^{\leq n} = \overline{M}^{\leq n} \text{ in } M_n \) and apply the induction hypothesis. □

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Proposition 15.15 The relation $S_n$ satisfies (Sim2).

Proof. Let $(M, E) \in S_n$ and $M$ be irreducible with respect to $\rightarrow_{\text{name}}$. We have to show that $E$ is irreducible with respect to $\leftrightarrow \circ \rightarrow_A \circ \leftrightarrow$. Instead, we prove that $E$ is irreducible with respect to $\rightarrow_A$, $\rightarrow_F$, and $\rightarrow_T$.

Without loss of generality, we can assume that $S_n$ is complete under forwarding (Lemma 15.12). Since $M$ is closed (Lemma 14.3) it has to be an abstraction (Lemma 15.9). Lemma 15.14 implies of the existence of $1 \leq j \leq n$ such that there exists a reference chain from $y_h$ to $y_j$ and $M_j$ is not an variable. Since $M$ is an abstraction $M_j$ has to be an abstraction. Completeness under forwarding ($S_n, \tau$) implies $j = n$ such that:

$$M \equiv \text{let } \ldots y_h = M_j \text{ in } y_n$$

Hence, $y_h$ is a unique needed variable in $\text{let } \ldots y_h = M_j \text{ in } y_n$ such that ($S_n, 6$) implies for all $i \in \{1, \ldots, n-1\}$ that $y_i \in D$ or $M_i$ is an abstraction.

Let $E_1, \ldots, E_n$, and $\phi$ be defined as in ($S_n, 3$). This implies $E \approx E_1 | \ldots | E_n | \phi$. Since none of the $E_i$'s may be an application, $E$ is irreducible with respect to $\rightarrow_j$. It is irreducible with respect to $\rightarrow_F$ because none of the $E_i$'s may an directed equation, and irreducible with respect to $\rightarrow_T$ because none of the delayed $E_i$'s is triggered in $\phi$. \qed

15.4.3 Proof of the Invariant

Lemma 15.16 (Shared Redexes) Let $(n, \bar{y}, \bar{M}, \bar{t}, D)$ be a $n$-representation of $(M, E)$ satisfying ($S_n, \tau$) and $\bar{y} = (y_i)_{i=1}^n$, $\bar{M} = (M_i)_{i=1}^n$, $\bar{t} = (t_i)_{i=1}^n$. For all $M'$ with $M \rightarrow_{\text{name}} M'$, there exists $j, k, l$ and $x$, $M_k$ such that $M_j = y_k y_l$, $y_j$ is needed in $\text{let } \bar{y} = \bar{M} \text{ in } y_n$, $M_k = \lambda x. M_k$, and:

$$M' \rightarrow^n_{\text{name}} \text{let } \bar{y}^{<j} = \bar{M}^{<j} \text{ y_j} = \bar{M}_k[y_k/x] \bar{y}^{>j} = \bar{M}^{>j} \text{ in } y_n$$

Proof. By induction on derivations of $M \rightarrow_{\text{name}} M'$. We have to consider two cases:

1. In the first case, the $M \rightarrow_{\text{name}} M'$ is an instance of the $\beta$-axiom: There exists $P_1$, $P_2$, $x$ such that:

$$M \equiv (\lambda x. P_1) P_2 \rightarrow_{\text{name}} P_1[P_2/x] \equiv M'$$

Applying Lemma 15.14, there exists a $1 \leq j \leq n$ and a reference chain from $y_h$ to $y_j$ such that $M_j$ is not a variable. Since $M$ is an application, $M_j$ is an application. ($S_n, 6$) implies $M_j = y_k y_l$ for some $k, l$ and ($S_n, 1$) yields $k, l < j$. Hence:

$$\lambda x. P_1 \equiv \text{let } \bar{y}^{<j} = \bar{M}^{<j} \text{ in } y_k, \quad P_2 \equiv \text{let } \bar{y}^{<j} = \bar{M}^{<j} \text{ in } y_l$$

Applying Lemma 15.14 there exists $1 \leq k' \leq k$ and a reference chain from $y_k$ to $y_{k'}$ in $\text{let } \bar{y}^{<j} = \bar{M}^{<j} \text{ in } y_k$ there such that $M_{k'}$ is not a variable, i.e. $M_{k'}$ is an abstraction.
The variables $y_k$ and $y_{k'}$ are needed in let $y = M$ in $y_n$ (by induction on the length of reference chains) such that $k = k'$ follows from completeness with respect to forwarding $(S_n, 7)$ . Hence, $M_k$ is an abstraction such that there exists $M'_k$ with $M_k \equiv \lambda x. M'_k$. Furthermore:

$$\bar{P}_1 \equiv \text{let } y^<j = M^<j \text{ in } \bar{M}_k$$

The following equality justifies the Lemma with $\equiv$ instead of $\Rightarrow^*_\text{name}$.

$$M' \equiv \bar{P}_1[P_2/x]$$
$$\equiv \text{let } y^<j = M^<j \text{ in } \bar{M}_k[P_2/x]$$
$$\equiv \text{let } y^<j = M^<j \text{ in } \bar{M}_k[y_i/x]$$
$$\equiv \text{let } y^<j = M^<j \text{ in } \bar{M}_k[y_j/x] \text{ } y^>j = M^>j \text{ in } y_j$$
$$\equiv \text{let } y^<j = M^<j \text{ in } \bar{M}_k[y_i/x] \text{ } y^>j = M^>j \text{ in } y_n$$

The last step uses the reference chain from $y_n$ to $y_j$ backwards.

2. In the second case, the $\beta$-axiom is applied in functional position. There exists $P_1$, $P'_1$, $P_2$ such that the last step in the derivation of $M \Rightarrow^*_\text{name} M'$ has the following form:

$$\frac{P_1 \Rightarrow^*_\text{name} P'_1}{M \equiv \bar{P}_1 P_2 \Rightarrow^*_\text{name} P'_1 P_2 \equiv M'}$$

Yet another argumentation with reference chains implies the existence of $j'$, $k'$, $l'$ such that:

$$M \equiv \text{let } y^<j' = M^<j' \text{ in } y_j' = y_k y_{l'} \text{ } y^>j' = M^>j' \text{ in } y_j'$$

$$P_1 \equiv \text{let } y = M \text{ in } y_k$$

$$P_2 \equiv \text{let } y = M \text{ in } y_v$$

By induction hypothesis applied to $P_1 \Rightarrow^*_\text{name} P'_1$ there exists $j$, $k$, $l$, and $x$, $\bar{M}_k$ such that: $M_j = y_k y_l$, $y_j$ is needed in let $y = M$ in $y_k$, $M_k = \lambda x. \bar{M}_k$, and:

$$P'_1 \Rightarrow^*_\text{name} \text{let } y^<j = M^<j \text{ in } y_j = M_k[y_i/x] \text{ } y^>j = M^>j \text{ in } y_k$$

$P_2$ reduces to a similar expression than $P'_1$ does:

$$P_2 \equiv \text{let } y^<j = M^<j \text{ in } y_j = y_k y_l \text{ } y^>j = M^>j \text{ in } y_v$$

$$\Rightarrow^*_\text{name} \text{let } y^<j = M^<j \text{ in } y_j = M_k[y_i/x] \text{ } y^>j = M^>j \text{ in } y_k$$

Sticking both reductions together concludes the Lemma:

$$M' \equiv \text{let } y^<j = M^<j \text{ in } y_j = M_k[y_i/x] \text{ } y^>j = M^>j \text{ in } y_k$$

The second step uses $M_k = y_k y_l$ and the last step a reference chain from $y_n$ to $y_k$ backwards that we left implicit at the beginning of this case. \hfill $\square$
Proposition 15.17 (The Invariant) Let \((M, E) \in S_n\) and \(M \to \text{name} M'\). Then there exists \(M''\) and \(E'\) such that \(M' \Rightarrow^* \text{name} M''\), \(E \leftrightarrow \circ \leftrightarrow A \circ \leftrightarrow E'\), and \((M'', E') \in S_n\).

Proof. Let \((n, \bar{y}, \bar{M}, \bar{t}, \bar{D})\) be a \(n\)-representation of \((M, E)\). We assume without loss of generality that \(E\) is complete under forwarding (Lemma 15.12). Let \(\bar{y} = (\bar{y}_i)_{i=1}^n, \bar{M} = (M_i)_{i=1}^n, \bar{t} = (t_i)_{i=1}^n, \) and \(D \subseteq V(\bar{y})\). Let \((E_i)_{i=1}^n\) and \(\phi\) be defined as in (S_n3) and \(\theta = [\bar{y}, \bar{t}, \bar{D}]\). Since \(M \to \text{name} M'\), we can apply Lemma 15.16 such that there exists \(j, k, l\) and \(x, \bar{M}_k, M''\) with the following properties:

1. \(M_j \equiv y_k y_n\).
2. \(y_j\) is needed in \(\bar{y} = \bar{M} \in \bar{y}_n\).
3. \(M_k = \lambda x. \bar{M}_k\).
4. \(M' \Rightarrow^* \text{name} M''\)
5. \(M'' \equiv \text{let } \bar{y}^{<j} = \bar{M}^{<j} \quad y_j = \bar{M}_k[y_i/x] \quad \bar{y}^{>j} = \bar{M}^{>j} \text{ in } y_n\)

Applying Lemma 15.9 there exists \(m \geq 0, \bar{F} = (P_i)_i^m, \) and \(Q\) such that:

6. \(\bar{M}_k = (\ldots (Q P_m) \ldots) P_1\)
7. \(Q\) is not an application.

For all \(i \in \{1 \ldots m\}\) let \(u_i, v_i, s_i\) be fresh variables. We define \(u_{m+1} = y_j, \bar{u} = (u_i)_{i=1}^{m+1}, \bar{v} = (v_i)_{i=1}^m\) and \(\bar{s} = (s_i)_{i=1}^m\). Flattening \(\bar{M}_k\) (Lemma 15.10) yields:

8. \(\bar{M}_k \equiv \text{let } u_1 = Q \quad \bar{u} = \bar{P} \quad \bar{v}^{>1} = \bar{v}^{<m+1} \bar{v} \text{ in } y_j\)
9. \(y_j = n \bar{M}_k \equiv u_1 = n Q \quad \bar{u}, \bar{v} = \bar{P} \quad \bar{v}^{<m+1} \bar{s} \bar{v}^{>1}\)

Since \(y_j\) is needed in \(\bar{y} = \bar{M} \in \bar{y}_n\) (2), \(y_k\) is needed in \(\bar{y} = \bar{M} \in \bar{y}_n\) as well. (S_n5) implies \(y_j, y_k \notin D\) such that:

10. \(E \equiv E_1 \mid \ldots \mid E_n \mid \phi\)
11. \(E_j \equiv y_k \bar{y} t y_j\)
12. \(E_k \equiv y_k = n \lambda x. \bar{M}_k \theta\) \hspace{1cm} (3)

For fresh variables \(t\) and \(z\) this implies:

13. \(E_k \equiv y_k : x t z / z = n \bar{M}_k \theta[x \circ t / x]\)
If $\eta = [y_i/x]$, then applying $y_k$ in the context of $E_k$ yields:

$$E_j \rightarrow_A (y_j = \overline{M_k} \theta[x \ominus t/x][y_i/x][t_i/t]$$

$$\equiv y_j = \overline{M_k} \eta \theta$$

$$\equiv u_1 = \overline{\mathcal{P}} \eta \theta \mid u \overline{P} = n \prod_{m+1 \leq \pi > 1}$$

Combining this result with (10) we obtain:

(14) $E \rightarrow_A E'$

(15) $E' \equiv E \triangleleft j \mid u_1 = \overline{M} \eta \theta \mid \overline{u} \overline{P} = n \prod_{m+1 \leq \pi > 1} \mid \overline{P} \triangleleft j \mid \phi$

Next, we construct a five-tuple $\mathcal{R} = (n', \overline{g'}, \overline{M}', \overline{P}, D')$, which satisfies all properties of the Lemma except one.

$$\overline{g'} = \overline{g} \triangleleft j \mid u_1 = \overline{M} \eta \theta \mid \overline{u} \overline{P} = n \prod_{m+1 \leq \pi > 1} \mid \overline{P} \triangleleft j \mid \phi$$

Property (4) implies $M' \rightarrow^*_{\text{name}} M''$. We even obtain $E \rightarrow_{p} \circ \rightarrow_A E'$ from (14) and the fact that we completed $E$ under forwarding at the beginning. It remains to show that $\mathcal{R}$ is a $n$-representation for $(M'', E')$. $\mathcal{R}$ satisfies all required properties except $(S_n5)$: $(S_n1)$ is simple, $(S_n2)$ follows from (10), $(S_n3)$ is covered by (15), $(S_n4)$ follows from (8) (the variables in $\mathcal{V}(\overline{g}) \setminus D'$ are those in $\mathcal{V}(\overline{u})$). Property $(S_n6)$ holds, since $\mathcal{V}(\overline{u}) \subseteq D'$ and all other non-needed variables have also been non-needed in the original $n$-representation.

The tuple $\mathcal{R}$ does not necessary satisfy $(S_n5)$, because $Q$ might be a variable, say $y_p$. In this case, $u_1 = y_p \theta \equiv u_1 = y_p \mid \text{tr}(t_p)$. This means that the expression $E_p$ is delayed, even if $y_p$ is needed. We have to use $\rightarrow_T$ for triggering the computation in $E_p$ waiting on $t_p$. Since $M_p$ may again be a variable, more triggering steps may be needed.

The failure of $\mathcal{R}$ being a $n$-representation for $(M'', E')$ is harmless, since $\mathcal{R}$ is at least an uncompletely triggered $n$-representation for $(M'', E')$ in the sense of Definition 15.18. This is sufficient to accomplish the actual proof by applying Lemma 15.19.

**Definition 15.18** A five-tuple $(n, \overline{g}, \overline{M}, \overline{P}, D)$ is called an uncompletely triggered $n$-representation of $(M, E)$, if it satisfies (S_n1)-(S_n4), (S_n6), and $(S_n'5)$, where:

$(S_n'5)$ If $y_i$ is needed in $\text{let } \overline{g} = \overline{M}$ in $y_n$, then either $y_i \notin D$ or there exists a reference chain $(y_{v(i)})_{i=1}^{p}$ such that $y_{v(1)} = y_i$, $\{y_{v(i)} \mid 1 \leq i < p\} \subseteq D$ and $\text{tr}(t_{v(p)})$ is contained in $E$.

**Lemma 15.19** (Triggering) If there exists an uncompletely triggered $n$-representation of $(M, E)$, then there exists $E'$ such that $E \rightarrow^*_{p} E'$ and $(M, E') \in S_n$.
Proof. Let \( \mathcal{R} = (n, \overline{y}, \overline{M}, \overline{I}, D) \) be a uncompletely triggered n-representation on \((M, E)\). We call a variable \( y_i \) critical for \( \mathcal{R} \) and \((M, E)\), if \( y_i \) is needed in let \( \overline{y}=\overline{M} \) in \( y_n \), and \( y_i \in D \).

If there exists no critical variable for \( \mathcal{R} \) and \((M, E)\), then \( \mathcal{R} \) is a n-representation for \((M, E)\). Hence it is sufficient to define a procedure that given a uncompletely triggered n-representation \( \mathcal{R} \) for \((M, E)\) computes some \( E' \) and \( \mathcal{R}' \) such that:

1. \( \mathcal{R}' \) is a uncompletely triggered n-representation for \((M, E')\) and \( E \to_T E' \).
2. The number of critical variables for \( \mathcal{R}' \) and \((M, E')\) is strictly smaller than the number of critical variables for \( \mathcal{R} \) and \((M, E)\).

Let \( \mathcal{R} = (n, \overline{y}, \overline{M}, \overline{I}, D) \) be a uncompletely triggered n-representation on \((M, E)\). If there exists a critical variable for \( \mathcal{R} \) and \((M, E)\) then by condition \((S_4, 5)\) there also exists a critical variable \( y_i \in D \) such that \( \text{tr}(t_i) \) is contained in \( E \). Let \( E_1, \ldots, E_n \), and \( \phi \) be defined as in \((S_3)\). Since \( \text{tr}(t_i) \) is contained in \( E \), there exists \( \phi' \) such that \( \phi \equiv \text{tr}(t_i) \mid \phi' \).

We can reduce \( E \) and define \( E' \) as follows:

\[
E \approx E_1 \mid \ldots \mid t_i \cdot y_i = n, M \mid \ldots \mid \text{tr}(t_i) \mid \phi' \\
\to_T E_1 \mid \ldots \mid y_i = n, M \mid \ldots \mid \text{tr}(t_i) \mid \phi' \\
\overset{\text{def}}{=} E'
\]

If we set \( D' = D \setminus \{y_i\} \) then \((n, \overline{y}, \overline{M}, \overline{I}, D') \) is a uncompletely triggered n-representation of \((M, E')\) in which the variable \( y_i \) is no more critical. \( \square \)

16 Relating Call-by-Value to Call-by-Need

In this section, we prove the estimation \( C^A(z = n, M) \leq C^A(z = \nu, M) \) for all closed \( \lambda \)-expressions \( M \) as stated in theorem 8.3. For proof, we will define a lengthening simulation for the embedding \( z = n, M \leftrightarrow z = \nu, M \) and apply proposition 13.4.

The correspondence between an expression \( z = n, M \) and an expression \( z = \nu, M \) is very simple. We define a projection function \( p \) between ternary and binary \( \delta \)-expressions, which eliminates all triggering information in expressions such as \( z = \nu, M \):

\[
p(x; y; z/E) \overset{\text{def}}{=} x; y; z/E \quad p(x; y; z) \overset{\text{def}}{=} x; y; z \quad p(E \mid F) \overset{\text{def}}{=} p(E) \mid p(F) \\
p((\nu x) E) \overset{\text{def}}{=} (\nu x) p(E) \quad p(\text{tr}(t)) \overset{\text{def}}{=} 0 \quad p(t; E) \overset{\text{def}}{=} p(E)
\]

In this definition, we use a new expression \( 0 \) that we require to be nilpotent in the sense \( 0 \mid E \equiv E \) for all \( E \). Being a little bit less restrictive we could also define \( 0 \) in \( \delta \) itself, for example by \( 0 \overset{\text{def}}{=} (\nu x)(x; x) \).

Let \( \approx_1 \) be the smallest congruence on \( \delta \)-expressions (with \( 0 \)) containing the structural congruence and satisfying the axiom:

\[
(\nu x) E \approx_1 E \quad \text{if } x \notin \text{V}(E)
\]
Lemma 16.1 For all closed $M$ and variables $z$ the relation $p(z=v,M) \approx_1 z=v,M$ holds.

Proof. By induction on the structure of $M$. $\square$

Lemma 16.2 Let $R$ be one of the letters in $\{A, F, T\}$. If $E \rightarrow_R E'$ and $E \approx_1 F$ then there exists $F'$ such that $F \rightarrow_R F'$ and $E' \approx_1 F'$.

Proof. By induction on derivations of $E \rightarrow_A E'$, $E \rightarrow_F E'$, and $E \rightarrow_T E'$ respectively. $\square$

Lemma 16.3 Let $E$ ternary, $E'$ a $\delta$-expression, and $R$ one of the letters in $\{A, F\}$. If $E \rightarrow_R E'$ then $p(E) \rightarrow_R p(E')$. If $E \rightarrow_T E'$ then $p(E) \rightarrow_T p(E')$.

Proof. By induction on derivation of $E \rightarrow_A E'$, $E \rightarrow_F E'$, and $E \rightarrow_T E'$ respectively. $\square$

Let $S^n_v$ be the binary relation on $\delta$-expressions that contains all pairs $(E, F)$ such that $F \approx_1 p(E)$ and $E$ ternary and admissible.

Proposition 16.4 The relation $S^n_v$ is a lengthening simulation for the mapping $z=v,M \mapsto z=v,M$ considered as embedding from the restriction of $\delta$ to admissible, closed, and ternary expressions into itself.

Proof. Lemma 16.1 implies (Sim1) and the Lemmata 16.2 and 16.3 ensure (Sim4). $\square$

Corollary 16.5 The estimation $C^A(z=v,M) \leq C^A(z=v,M)$ is valid for all closed $M$ and variable $z$.

Proof. Immediate from Propositions 12.4 and 16.4. $\square$
17 Adequacy of the Embedding of $\delta$ into $\delta_0$

We prove that the embedding $E \mapsto \llbracket E \rrbracket$ restricted to well-typed expressions preserves termination as stated in Theorem 11.1. Of course, we again apply the simulation technique. It is however not possible to use a simulation immediately. One reason is that reference chains are shortened in different order when expressing $\rightarrow_F$ via $\rightarrow_A$. Forwarding $\rightarrow_F$ shortens reference chains from the right to the left. For instance:

$$x u \mid x \equiv y \mid y; z / E \xrightarrow{\rightarrow_F} x u \mid x ; z / E \mid y; z / E$$

$$\xrightarrow{\rightarrow_A} E[u/z] \mid x; z / E \mid y; z / E$$

After encoding, chains are traversed from the left to the right:

$$\llbracket x u \mid x \equiv y \mid y; z / E \rrbracket \equiv x u \mid x; z / y z \mid y; z / \llbracket E \rrbracket$$

$$\xrightarrow{\rightarrow_A} y u \mid x; z / y z \mid y; z / \llbracket E \rrbracket$$

$$\xrightarrow{\rightarrow_A} \llbracket E \rrbracket[u/z] \mid x; z / y z \mid y; z / \llbracket E \rrbracket$$

Note that $\rightarrow_F$ provides for path compression, which is not preserved by encoding. We formally handle the effect of path compression to complexity by an appropriate shortening simulation (compare Lemma 17.4).

Instead of simulating single forwarding steps, we will simulate sequences of forwarding steps followed by application. For all $n \geq 0$ we define the relation $\rightarrow_{F^n} A$ by the following axiom and the contextual rules in Figure 4:

$$x_1 \equiv x_1 = x_2 \mid \ldots \mid x_{n-1} = x_n \mid x_n; \overline{y} / E \xrightarrow{\rightarrow_{F^n} A} E[\overline{y} / \overline{y}] \mid x_1; \overline{y} / E \mid \ldots \mid x_n; \overline{y} / E$$

where we assume the sequence $(x_i)_{i=1}^n$ to be linear.

**Lemma 17.1** If $E \xrightarrow{\rightarrow_{F^n} A} E'$, then $E \rightarrow^n F \circ \rightarrow_A E'$.

**Proof.** By induction on derivations of $E \rightarrow_{F^n} A E'$. The axiom case is by induction on $n$. □

**Lemma 17.2** If $E$ is reducible with respect to $\rightarrow^n F \circ \rightarrow_A$, then there exists $m \leq n$ such that $E$ is reducible with respect to $\rightarrow_{F^n} A$.

**Proof.** By induction on $n$. □

We define $\supseteq$ to be the smallest binary relation on $\delta$-expressions, which is reflexive and transitive, satisfies the contextual rules of Figure 4, and the axiom:

$$(x_1; \overline{y} / x_2 \overline{y}) \mid x_2; \overline{y} / E \supseteq (x_1; \overline{y} / E) \mid x_2; \overline{y} / E$$
Lemma 17.3 If $E \rightarrow^*_{F,A} E'$ then $\llbracket E \rrbracket \rightarrow^{n+1}_A \supseteq \llbracket E' \rrbracket$.

$$
\begin{align*}
E & \rightarrow^*_{F,A} E' \\
\llbracket E \rrbracket & \rightarrow^{n+1}_A \supseteq \llbracket E' \rrbracket
\end{align*}
$$

Proof. By induction on derivations of $E \rightarrow^*_{F,A} E'$. We only consider the axiom case:

$$
F_1 \overset{\text{def}}{=} x_1 \overline{y} \mid x_1 = x_2 \mid \ldots \mid x_{n-1} = x_n \mid x_n : \overline{y} / E
$$

After translation, we obtain:

$$
\begin{align*}
\llbracket F_1 \rrbracket & \overset{\text{def}}{=} x_1 \overline{y} \mid x_1 : \overline{y} / x_2 \overline{y} \mid \ldots \mid x_{n-1} : \overline{y} / x_n \overline{y} \mid x_n : \overline{y} / \llbracket E \rrbracket \\
\rightarrow^{n+1}_A & \llbracket F_1 \rrbracket \overset{\text{def}}{=} \llbracket E[x_1 := \overline{y}] \mid x_1 := \overline{y} / E \mid \ldots \mid x_{n-1} := \overline{y} / E \mid x_n := \overline{y} / \llbracket E \rrbracket \rrbracket
\end{align*}
$$

Lemma 17.4 The relation $\supseteq$ is a shortening simulation for the identity function on $\delta'$ restricted to admissible expressions.

Proof.

(Sim1) The relation $\supseteq$ is required to be reflexive.

(Sim2) We show that $\supseteq$ preserves termination with respect to $\rightarrow_A$, $\rightarrow_F$, $\rightarrow_T$, which implies that it also preserves termination in $\delta'$. It is sufficient to prove the previous statement for expressions $E$ without top-level declarations. For $\rightarrow_A$, we note that the set of variables naming abstractions in $E$ is invariant under $\supseteq$. The same holds for the set of applications in $E$. For $\rightarrow_F$, note that the set of directed equations in $E$ is preserved under $\supseteq$. Triggering is completely unaffected by $\supseteq$.

(Sim3) We can establish the following diagrams. For all $E$, $E'$, and $F$ there exists $E''$ and $F'$ such that:

$$
\begin{align*}
E & \rightarrow_A E' \rightarrow^*_{F,A} E'' & E & \rightarrow_F E' & E & \rightarrow_T E' \\
\cup & \cup & \cup & \cup & \cup & \cup
\end{align*}
$$

$$
\begin{align*}
F & \rightarrow_A F' & F & \rightarrow_F F' & F & \rightarrow_T F'
\end{align*}
$$

The proofs are rather simple for expressions without top-level declarations and this is sufficient. \qed
Lemma 17.5 The relation $\geq$ restricted to admissible expressions preserves termination and shortens complexity. If $E$ and $E'$ are admissible and $E \geq E'$, then $C(E) \geq C(E')$.

Proof. This is an immediate consequence of Lemma 17.4 and Theorem 12.2.

We next consider the encoding of triggering. We consider the following example:

$$\llbracket t.E \mid \text{tr}(t) \rrbracket \equiv (\nu y) \langle t \mid y : \llbracket E \rrbracket \mid t : y/y \rangle$$

$$\Rightarrow_A (\nu y) \langle y : \llbracket E \rrbracket \mid t : y/y \rangle$$

$$\Rightarrow_A \llbracket E \rrbracket \langle t : y/y \mid (\nu y) \langle y : \llbracket E \rrbracket \rangle \rangle$$

This illustrates that every triggering step is encoded by two application steps. The correspondence is quite direct up to garbage expressions such as $(\nu y) \langle y : \llbracket E \rrbracket \rangle$. To keep track of these, we define the relation $\approx_2$ as the least congruence on $\delta$-expressions, which is invariant under congruence and satisfies the following axiom:

$$E \mid (\nu x) \langle x : \llbracket E \rrbracket \rangle \approx_2 E$$

Lemma 17.6 If $E \rightarrow_T E'$ then $\llbracket E \rrbracket \approx_2 \llbracket E' \rrbracket$.

$$E \rightarrow_T E'$$

$$\llbracket E \rrbracket \approx_2 \llbracket E' \rrbracket$$

Lemma 17.7 The relation $\approx_2$ is a complexity simulation for the identity function on $\delta$.

Proof. Omitted, but not difficult.

Lemma 17.8 The relation $\approx_2$ restricted to admissible expressions preserves complexity and termination.

Proof. This is an immediate consequence of Lemma 17.7 and Corollary 12.5.

In the last part of this Section, we combine the above results in order to prove the adequacy of the embedding $E \mapsto \llbracket E \rrbracket$ restricted to well-typed expressions.

Lemma 17.9 If $E$ is well-typed then $\llbracket E \rrbracket$ is admissible.
Proof. We can introduce new typing rules that type \( \llbracket E \rrbracket \) symmetrically to \( E \). With respect to this new system \( \llbracket E \rrbracket \) is well-typed. This implies the admissibility of \( \llbracket E \rrbracket \) in the same manner than for the original type system. We note that \( \llbracket \text{tr}(t) \rrbracket \) is not inconsistent by definition. In other words, multiple triggering does not lead to an inconsistency. \( \square \)

**Lemma 17.10** If \( E \) is well-typed, acyclic, and irreducible with respect to \( \rightarrow_T \) and \( \rightarrow^*_F \) \( \circ \rightarrow_A \), then \( \llbracket E \rrbracket \) terminates with respect to \( \rightarrow_A \).

Proof. Since \( E \) is irreducible with respect to \( \rightarrow_T \), applications of a variable \( t \) of type \( \text{tr}^0 \) cannot be executed in \( \llbracket E \rrbracket \). Otherwise, there would exist any application of \( t \) in \( [E] \) which in not derived form \( [\text{tr}(t)] \). This would contradict well-typedness of \( E \).

An applications \( [xy] \) can be executed in \( [E] \), if a translated equation \( [x=y] \) is available in \( [E] \). This can happen finitely many times, since \( E \) is acyclic. Applying an abstraction not derived from a directed equation is never possible, since \( E \) is irreducible with respect to \( \rightarrow^*_F \) \( \circ \rightarrow_A \). \( \square \)

**Proposition 17.11** If \( E \) is well-typed and acyclic, then \( E \) terminates if and only if \( \llbracket E \rrbracket \) terminates.

Proof. First, we consider the case that \( E \) terminates in \( \delta \) and proof that \( \llbracket E \rrbracket \) terminates in \( \delta_0 \). This proof is by induction on \( C^A(E) < \infty \).

If \( C^A(E) = 0 \), then \( E \) is irreducible with respect to \( \leftrightarrow \circ \rightarrow_A \circ \leftrightarrow \). Applying Lemmata 17.6 and 17.8, we can assume that \( E \) is irreducible with respect to \( \rightarrow_T \). This implies that \( E \) is irreducible with respect to \( \rightarrow^*_F \) \( \circ \rightarrow_A \). Well-typedness of \( E \) and Lemma 17.10 yields termination of \( \llbracket E \rrbracket \) in \( \delta_0 \).

Let \( C^A(E) > 0 \). Applying the Lemmata 17.6 and 17.8, we can assume that \( E \) is irreducible with respect to \( \rightarrow_T \). This implies that \( E \) is reducible with respect to \( \rightarrow^*_F \circ \rightarrow_A \). Applying Lemma 17.2, there exists \( n \geq 0 \) and \( E' \) such that \( E \rightarrow_{F \circ A} E' \). Lemma 17.1 implies \( C^A(E) = C^A(E') + 1 \) such that \( C^A(E') < C^A(E) \). Applying the induction hypothesis to \( E' \) yields termination of \( \llbracket E \rrbracket \). From Lemma 17.3, we obtain \( \llbracket E \rrbracket \rightarrow^*_A \supseteq \llbracket E' \rrbracket \). Termination of \( \llbracket E' \rrbracket \) implies termination of \( \llbracket E \rrbracket \) by Lemma 17.5.

It remains show that if \( E \) does not terminate then \( \llbracket E \rrbracket \) does not terminate. This can be done with a similar inductive argument, which proves that \( C^A(E) \geq n \) implies \( C(\llbracket E \rrbracket) \geq n \) for all \( n \geq 0 \). \( \square \)

**Corollary 17.12** If \( E \) is well-typed, then \( \llbracket E \rrbracket \) is admissible and terminates if and only if \( E \) terminates.

Proof. Immediate from Lemmata 17.11 and 17.9. \( \square \)
Simulating the Call-by-Need $\lambda$-Calculus

In this Section, we sketch the proof that our embedding of the call-by-need $\lambda$-calculus into $\delta$ preserves complexity as stated in Theorem 9.2.

Syntactically, the call-by-need $\lambda$-calculus and the $\delta$-calculus differ in flattening $\lambda$-terms. We define a flattening functions $f$ mapping an expression $L$ of the call-by-need $\lambda$-calculus to an expression of the form $\text{let } \overline{y} = \overline{M} \text{ in } N$ with explicit substitutions:

$$f(x) = \text{let } z = x \text{ in } z$$
$$f(\lambda x. L) = \text{let } z = \lambda x. L \text{ in } z$$

$$f(L_1) = \text{let } \overline{y} = \overline{M} \text{ in } y_n$$
$$f(L_2) = \text{let } \overline{y} = \overline{M} \text{ in } y'_n$$

$$f(L_1 L_2) = \text{let } \overline{y} = \overline{M} \text{ in } y_n$$
$$f(\text{let } z = y_n \text{ in } z)$$

$$f(\text{let } x = L_1 \text{ in } L_2) = \text{let } \overline{y} = \overline{M} \text{ in } y'_n$$

Definition 18.1 We define the relation $S_n^\lambda$ as the set of all pairs $(L, E)$ such that there exists a pair $(M, F)$ and a $n$-representation $(n, \overline{y}, \overline{M}, I, D)$ for $(M, F)$ such that $f(L) \equiv M$, $E \approx_2 F$, and $f(L) = \text{let } \overline{y} = \overline{M} \text{ in } y_n$.

Proposition 18.2 The relation $S_n^\lambda$ is a complexity simulation for the embedding $L \mapsto z = _n L$ from the call-by-need $\lambda$-calculus restricted to closed expressions into $\delta'$.

Proof. The conditions of a complexity simulation will be checked by the following Lemmata. Property (Sim1) is implied by Lemma 18.4, (Sim2) by Lemma 18.5, and (Sim3) and (Sim4) by Lemmata 18.6, 18.7, and 18.8. □

Corollary 18.3 For all closed $L$ the equality $C_{\text{need}}(L) = C^A(z = _n L)$ is valid.

Proof. Immediate consequence of Proposition 18.2 and Corollary 12.5. □

Lemma 18.4 If $L$ is closed, then $(L, z = _n L) \in S_n^\lambda$.

Proof. By induction on the structure of $L$. □

Lemma 18.5 If $L$ is irreducible in the call-by-need $\lambda$-calculus and $(L, E) \in S_n^\lambda$, then $E$ is irreducible in $\delta'$.

Proof. If $L$ is irreducible in $\delta$ and $(M, F)$ justifies $(L, E) \in S_n^\lambda$. Since $M \equiv f(L)$, $M$ is an abstraction and hence irreducible with respect to $\rightarrow_{\text{name}}$. Since $S_n$ is a shortening simulation (Proposition 15.8) $F$ is irreducible in $\delta'$. Since $\approx_2$ is a complexity simulation (Lemma 17.7), $E$ is also irreducible in $\delta'$. □
Lemma 18.6 If \( L \rightarrow_I I' \) and \((L, E) \in S^\lambda_n\), then there exists \( E' \) such that \( E \rightarrow_A \approx_2 \rightarrow_T^* E' \) and \((L', E') \in S^\lambda_n\).

\[
\begin{array}{c@{\quad}c}
L & \rightarrow_I & L' \\
S^\lambda_n & \rightarrow_T & S^\lambda_n \\
E & \rightarrow_A & \approx_2 \rightarrow_T^* E'
\end{array}
\]

Proof. By induction on derivations of \( L \rightarrow_I I' \).

Lemma 18.7 If \( L \rightarrow_V I' \) and \((L, E) \in S^\lambda_n\), then there exists \( E' \) such that \( E \rightarrow_F \rightarrow_T^* E' \) and \((L, E') \in S^\lambda_n\).

\[
\begin{array}{c@{\quad}c}
L & \rightarrow_V & L' \\
S^\lambda_n & \rightarrow_T & S^\lambda_n \\
E & \rightarrow_F & \rightarrow_T^* E'
\end{array}
\]

Proof. By induction on derivations of \( L \rightarrow_V I' \).

Lemma 18.8 If \( L \rightarrow_{Ans} I' \) or \( L \rightarrow_C I' \), then \( f(L) \equiv f(I') \).

\[
\begin{array}{c@{\quad}c}
L & \rightarrow_{Ans} & L' \\
\quad & \quad & f \\
L & \rightarrow_C & L' \\
\quad & \quad & \equiv \\
f(L) & \equiv & f(I') \\
f(L) & \equiv & f(I')
\end{array}
\]

Proof. By induction on derivations of \( L \rightarrow_V I' \) and \( L \rightarrow_C I' \) respectively.

19 Conclusion

We have presented a simple execution model for eager and lazy functional computation. We have applied concurrency for integration of programming paradigms. We have presented the concurrent \( \delta \)-calculus, which features useful abstractions for programming, implementation, and theory. We have worked out a powerful proof technique based on uniform confluence and simulations. We have formally related the complexities of call-by-value, call-by-need, and call-by-name.

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References


[Mü96] Martin Müller. Polymorphic types for concurrent constraints. submitted, German Research Center for Artificial Intelligence (DFKI), Stuhlsatzenhausweg 3, D-66123 Saarbrücken, Germany, {mmueller}@dfki.uni-sb.de, 1996.
[Smo95a] Gert Smolka. An Oz primer. DFKI Oz documentation series, German Research Center for Artificial Intelligence (DFKI), Stuhlsatzenhausweg 3, D-66123 Saarbrücken, Germany, 1995.


