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How to Win a Game with Features

Rolf Backofen and Ralf Treinen

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Deutsches Forschungszentrum für Künstliche Intelligenz GmbH

Postfach 20 80
67608 Kaiserslautern, FRG
Tel.: + 49 (631) 205-3211
Fax: + 49 (631) 205-3210

Stuhlsatzenhausweg 3
66123 Saarbrücken, FRG
Tel.: + 49 (681) 302-5252
Fax: + 49 (681) 302-5341

Deutsches Forschungszentrum für Künstliche Intelligenz

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Director

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How to Win a Game with Features

Rolf Backofen and Ralf Treinen
German Research Center for Artificial Intelligence (DFKI)
Stuhlsatzenhausweg 3
D-66123 Saarbrücken, Germany
email: {backofen,treinen}@dfki.uni-sb.de

Abstract

We show, that the axiomatization of rational trees in the language of features given elsewhere is complete. In contrast to other completeness proofs that have been given in this field, we employ the method of Ehrenfeucht-Fraïssé Games, which yields a much simpler proof. The result extends previous results on complete axiomatizations of rational trees in the language of constructor equations or in a weaker feature language.

Contents

1 Introduction

Rational trees are a canonical model for cyclic data structures or recursive type equations. The well-established language for trees in logic and computer science is the language *Herbrand* of constructor symbols, which provides for equations $x = f(x_1, \ldots, x_n)$ as atomic formulae.

Here, we take a more elementary view. We use the language of so-called *features*, which are well-known for a long time in computational linguistics and knowledge representation (see [12] for a survey). There are different possible choices of a feature language. A first language FT has been established in [3] and [1]. The language of FT consists of unary *label* predicates, which express that the root node of a tree has a certain label, and binary *feature* predicates, which serve as the partial selector functions for trees. For instance, we can translate a Herbrand formula $x = f(x_1, \ldots, x_n)$ into feature logic as $f(x) \wedge 1(x, x_1) \wedge \ldots \wedge n(x, x_n)$ or, in a more convenient syntax as $fx \wedge x1x_1 \wedge \ldots \wedge xnx_n$. In case of a finite signature Σ, the predicates of feature logic can be defined in terms of Herbrand by taking an appropriate disjunction. In this case, $1(x, y)$ could be expressed as

$$\bigvee_{f \in \Sigma} \exists \bar{y}\; x = f(y, \bar{y})$$

In case of infinitely many symbols, this is no longer possible, since for instance xfy is equivalent to an infinite disjunction in Herbrand.

There is a major gap between FT and Herbrand, since the translation into FT given above does not preserve validity of formulae. The problem is that a formula $x = f(\bar{x})$ has a *unique* solution in x when the $\bar{x}$ are given, but that the solution to x of the translated formula $fx \wedge x1x_1 \wedge \ldots \wedge xnx_n$ is not unique, since x might have an arbitrary number of additional features. This was the reason to extend FT to CFT [13] by adding unary *arity predicates* $x\{f_1, \ldots, f_n\}$, which express that x has exactly the features $f_1, \ldots, f_n$. Adding this constraint to the above translation again ensures uniqueness of the solution. Note that although Herbrand can be translated into CFT, the converse does not hold since we are using an infinite set of features. Thus, CFT is a substantial extension of Herbrand.

In [13], an axiomatization of CFT was given and proven complete for $\exists^*$ formulae. In this paper, we prove the completeness of CFT which has been conjectured in [13]. From a complete axiomatization we gain a sound and complete deduction system for valid formulae in the language of CFT. Furthermore, it fully describes the first order properties of the standard model, and it allows for the formulation of alternative but elementarily equivalent models (such as the model of infinite trees). Note that the existence of a complete axiomatization of rational trees in the language of CFT is not straightforward, since for instance the theory in a language allowing for first-class features is undecidable [14].

The completeness proof uses Fraïssé's theorem and its game-theoretic formulation due to Ehrenfeucht. This method requires an argument concerning chains of relations between elements in a model. Feature logic is well suited for such an argument, since chains of relations are naturally expressed as path constraints.

Feature constraints xfy immediately generalize to path constraints such as $x(f_1 \cdots f_n)y$, which can be defined by using intermediate variables. In the field of term rewriting systems (see [6] for a survey), the notion of an *occurrence* in a term is well established. In the context of feature logic, there is no need for introducing such a meta-notation, since we can use the path constraints which are an immediate offspring of the base language. In the context of finite constructor trees, Hodges [10] observes that the use of selector functions simplifies the completeness proof of an axiomatization. His completeness proof is by quantifier elimination.

Complete axiomatizations of the algebra of rational trees, using the language of Herbrand, have been given independently in [5] for the case of a finite signature, and in [11] for both the case of a finite and an infinite signature. A complete axiomatization of rational trees in the language of FT has been given in [3], and a complete axiomatization of rational trees in the language of CFT in [2]. In both cases, a quantifier elimination method has been used with a similar overall structure than [11].

Both methods for proving the completeness of CFT have their merits. The quantifier elimination used in [2] serves for a concrete decision algorithm, whereas the proof presented here is much simpler. Thus, we think our paper describes a method for proving completeness which can be more easily adapted to other variants of feature logic than the method of quantifier elimination.

The next section briefly reviews the theory CFT from [13], and Section 3 reviews the method of Fraïssé [8] and Ehrenfeucht [7]. The core of the paper is Section 4, where we prove the completeness of CFT with the method of Section 3.

2 The Theory CFT

We assume infinite sets Lab of *label symbols* and Fea of *feature symbols*. From this, we define the following first order language:

- a unary *label* predicate for every $A \in Lab$, written as Ax,
- a binary *feature* predicate for every $f \in Fea$, written as xfy,
- a unary *arity* predicate for every finite set $F \subseteq Fea$, written as xF,
- the equality predicate, written as $x \doteq y$.

We consider two models of this signature. The universe of the model I consists of all feature trees. A *feature tree* is a partial function $t : Fea^* \to Lab$ whose domain is *prefix-closed*, i.e., if $pq \in \mathrm{dom}(t)$, then $p \in \mathrm{dom}(t)$. The *subtree* $p^{-1}t$ of a feature tree t at a path $p \in \mathrm{dom}(t)$ is the feature tree defined by (in relational notation)

$$p^{-1}t \quad := \quad \{(q, A) \mid (pq, A) \in t\}.$$

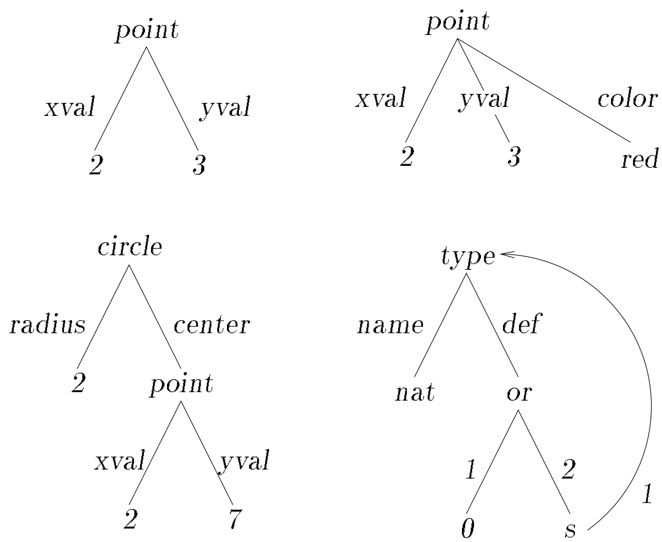

Figure 1: Examples of (in Fact Rational) Feature Trees.

A feature tree t is called a *subtree* of a feature tree r if t is a subtree of r at some path $p \in \mathrm{dom}(t)$.

The universe of the model R consists of all rational feature trees. A feature tree t is called *rational* if (1) t has only finitely many subtrees and (2) t is finitely branching (i.e., for every $p \in \mathrm{dom}(t)$, the set $\{pf \in \mathrm{dom}(t) \mid f \in \mathit{Fea}\}$ is finite).

The relational symbols are interpreted in I as follows:

- $I, \alpha \models Ax$ iff $\alpha(x)$ has root label A,
- $I, \alpha \models xfy$ iff $f \in \mathrm{dom}(\alpha(x))$ and $\alpha(y) = f^{-1}\alpha(x)$ (i.e., $\alpha(y)$ is the subtree of $\alpha(x)$ at f), and
- $I, \alpha \models x\{f_1, \ldots, f_n\}$ if $\alpha(x)$ has exactly the features $f_1, \ldots, f_n$ departing from its root.

The interpretation of the relational symbols in R is the restriction of the interpretation in I to the set of rational feature trees.

The theory *CFT* consists of five axiom schemes. The first set of axioms expresses that labels are disjoint, that features are functional and that an arity constraint fixes the set of features departing from a node.

(S)	$\forall x\ (Ax \wedge Bx \rightarrow \bot)$	$A \neq B$
(F)	$\forall x, y, z\ (xfy \wedge xfz \rightarrow y \doteq z)$	
(A1)	$\forall x, y\ (xF \wedge xfy \rightarrow \bot)$	$f \notin F$
(A2)	$\forall x\ (xF \rightarrow \exists y\ xfy)$	x different from y, $f \in F$

A *simple determinant* d is a conjunction of formulae

$$Ax \wedge x\{f_1, \ldots, f_n\} \wedge xf_1y_1 \wedge \ldots \wedge xf_ny_n$$

where the variables $x, y_1, \ldots, y_n$ are not required to be distinct. In this case we define $det(d) := \{x\}$. A *determinant* δ is a conjunction of simple determinants $d_1 \wedge \ldots \wedge d_n$ such that the $det(d_i) \cap det(d_j) = \emptyset$ for $i \neq j$. We define $det(\delta) := det(d_1) \cup \ldots \cup det(d_n)$ to be the set of variables *determined by* δ. Using the quantifier $\exists!\bar{x}\,\Psi$ with the meaning "there exists exactly one tuple $\bar{x}$ such that Ψ, we can formulate the last axiom scheme:

(D) $\forall(\mathcal{V}\delta - det(\delta))\,\exists! det(\delta)\,\delta$ $\qquad$ δ is a determinant

An instance of axiom scheme (D) is

$$\begin{aligned}\forall z\,\exists! x, y\,(\quad & Ax \wedge x\{f, g\} \wedge xfy \wedge xgz \wedge \\ & By \wedge y\{f, g, h\} \wedge yfz \wedge ygy \wedge yhx)\end{aligned}$$

Theorem 1 *Both I and R are models of CFT.*

A subformula of a determinant is called a *solved form*. A variable x is called *constrained* in a solved form Φ, if Φ contains a constraint of the form Ax, xF of xfy. The set of variables constrained by Φ is denoted as $con(\Phi)$. Hence, for a determinant δ, $con(\delta) = det(\delta)$.

Theorem 2 *For every solved form σ we have*

$$\forall(\mathcal{V}\delta - con(\delta))\,\exists con(\delta)\,\delta$$

Note that the existence is no longer unique in case of a solved form.

3 Ehrenfeucht-Fraïssé Games

Fraïssé [8] gives a definition of elementary equivalence in terms of mappings between structures. Any two isomorphic structures are elementarily equivalent, but there are of course elementarily equivalent structures which are not isomorphic. Hence, to characterize elementary equivalence algebraically we have to weaken the notion of isomorphism. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures of a language σ which consists of (possibly infinitely many) relation symbols only[1], and let τ be a subsignature of σ. A finite sequence $(a_i, b_i)_{1 \leq i \leq n}$ in $(\mathfrak{A} \times \mathfrak{B})^*$ is a *partial τ-isomorphism*, if for every $\mathfrak{A}$-assignment α with $\alpha(x_i) = a_i$, every $\mathfrak{B}$-assignment β with $\beta(x_i) = b_i$ and every atomic τ-formula w with $var(\tau) \subseteq \{x_1, \ldots, x_n\}$ we have $\mathfrak{A}, \alpha \models w \Leftrightarrow \mathfrak{B}, \beta \models w$.

Instead of Fraïssé's original theorem we here use the game-theoretic reformulation due to Ehrenfeucht [7]. The game is performed by two players, the *Spoiler* and the *Duplicator*. In

[1] We take this assumption just for simplicity, the definition extends to arbitrary languages.

the beginning, the Spoiler chooses a finite subsignature[2] $\tau \subseteq \sigma$ and the number n of rounds to go. The aim of the Duplicator is to build a partial τ-isomorphism of length n. In round i, the Spoiler chooses one of the two models together with an element a_i, resp. b_i. Then, the Duplicator chooses an element b_i, resp. a_i in the opposite model. Both players always know the present state of the game. The Duplicator wins, if in the end the sequence is a partial τ-isomorphism, otherwise the Spoiler wins.

Theorem 3 ([Ehrenfeucht, 1961]) *$\mathfrak{A}$ and $\mathfrak{B}$ are elementarilly equivalent iff the Duplicator has a winning strategy.*

As an example, take the structure I from Section 2 and the structure F, which is the restriction of I to those feature trees which have a finite domain. Note that F is not a model of CFT since axiom scheme (D) is violated. In this setting, the Spoiler can play in such a way that the Duplicator has no chance. The Spoiler chooses the finite subsignature consisting of the features f, g only (no label or arity predicates) and fixes the number of rounds to 2. In the first round, she chooses from I a_1 to be the infinite tree with domain $(fg)^* \cup (fg)^* f$ which maps every node to A (note that it does not matter that A is not in the finite subsignature). No matter what the choice of the Duplicator from F for b_1 is, the Spoiler will choose a_2 to be the infinite tree with domain $(gf)^* \cup (gf)^* g$, also mapping every node to A. Now we have for $\alpha(x_1) = a_1, \alpha(x_2) = a_2$ that $I, \alpha \models x_1 f x_2$ and $I, \alpha \models x_2 g x_1$, but there is no $\mathfrak{B}$-assignemt β with $\beta(x_1) = b_1$, such that $F, \beta \models x_1 f x_2$ and $F \models x_2 g x_1$. Hence, the Duplicator is bound to loose.

With the structures I and R, on the other hand, the Duplicator has a winning strategy. The description and proof of this strategy is subject of the next section.

4 The Completeness Proof

Theorem 4 *The theory CFT is complete.*

We show, using Theorem 3, that any two models $\mathfrak{A}$ and $\mathfrak{B}$ of CFT are elementarily equivalent. What does a winning strategy for the Duplicator look like? Suppose, the Spoiler has fixed n and the finite subsignature. We may assume that the arity predicates of the subsignature are exactly the sets of features in the subsignature, that is the finite subsignature is given as $(\sigma, \phi) \subseteq (Lab, Fea)$. At every stage of the game, the sequence constructed so far must of course be a partial (σ, ϕ)-isomorphism, but this is not sufficient, since the Duplicator has to take into account all possible future moves of the Spoiler. A clever move of the Spoiler is to choose an element of a model which is in relation to *many* elements which are already in the game. Hence, the Duplicator has to watch for *chains* of relations between the chosen elements, but may exploit the knowledge of n and (σ, ϕ) to restrict the set of relevant chains.

[2] Having the Spoiler choose the finite subsignature simplifies the formulation in the case of an infinite signature. This idea is due to Gert Smolka.

In the context of *CFT*, chains of relations are expressed as *path constraints* [4]. For every $p \in Fea^*$, we define the formula xpy by $x\epsilon y := x \doteq y$, and $x(pf)y := \exists z\,(xpz \wedge zfy)$. Furthermore,

$$\begin{array}{rcl} Axp & := & \exists z\,(xpz \wedge Az) \\ xpF & := & \exists z\,(xpz \wedge zF) \\ xp\downarrow yq & := & \exists z\,(xpz \wedge yqz) \end{array}$$

The latter formula is called a *co-reference constraint.* A *trivial co-reference constraint* $xp\downarrow xp$ is abbreviated as $xp\downarrow$, it expresses that x has a path p. We can now define, for any $l \geq 1$, and $n \geq 0$ the set of path constraints within the subsignature (σ, ϕ), where the paths are restricted to length at most l and where only the variables $x_1, \ldots, x_n$ are used:

$$P^{\sigma,\phi}_{l,n} := \{Ax_ip, x_ipF, x_ip\downarrow x_jq \mid A \in \sigma, F \subseteq \phi, 1 \leq i, j \leq n, p, q \in \phi^{\leq l}\}.$$

Here, $\phi^{\leq l}$ is the set of all strings from ϕ^* with length at most l. When σ, ϕ and n are known from the context, we will simply write P_l instead of $P^{\sigma,\phi}_{l,n}$. A sequence $(a_i, b_i)_{1\leq i\leq n} \in (\mathfrak{A} \times \mathfrak{B})^*$ is *(σ, ϕ)-true up to l*, if for all $w \in P^{\sigma,\phi}_{l,n}$ we have if $\alpha(x_i) = a_i$ and $\beta(x_i) = b_i$ for all $1 \leq i \leq n$, then

$$\mathfrak{A}, \alpha \models w \Leftrightarrow \mathfrak{B}, \beta \models w\,.$$

Every (σ, ϕ)-true sequence up to 1 is a partial (σ, ϕ)-isomorphism, since $CFT \models x \doteq y \leftrightarrow x\epsilon\downarrow y\epsilon$, $CFT \models xfy \leftrightarrow xf\downarrow y\epsilon$, $CFT \models Ax \leftrightarrow Ax\epsilon$ and $CFT \models xF \leftrightarrow x\epsilon F$. Hence, the aim of the Duplicator can be described as constructing a (σ, ϕ)-true sequence up to 1. From the above discussion, it is clear that the Duplicator must always ensure that the sequence constructed so far is (σ, ϕ)-true up to some sufficiently large bound. The question is of course, if there are still m rounds to go after the actual move of the Duplicator, how an appropriate bound $\psi(m)$ can be determined. A first guess could be $\psi(m) = 2^m$, since the Spoiler can with one move choose an element "in the middle" of a chain of relations between elements which are already in the sequence. This strategy of the Spoiler would cause the Duplicator, if the number of moves is increased by 1, to duplicate the bound for the first move, which results in the recursion equation $\psi(m + 1) = 2 * \psi(m)$. In fact, it can be shown [9], that this bound is sufficient for simple theories like the theory of one successor function. In our case, this is not sufficient as can be seen with the following example:

Suppose, the sequence constructed so far is $(a_1, b_1), \ldots, (a_n, b_n)$. The Spoiler chooses an $a \in \mathfrak{A}$ in such a way that for the variable assignment α with $\alpha(x_i) = a_i$, $\alpha(x_{n+1}) = a$ we have

$$\left.\begin{array}{l} \mathfrak{A}, \alpha \models x_1r_1 \downarrow x_{n+1}p_1 \\ \mathfrak{A}, \alpha \models x_{n+1}p_1q_1 \downarrow x_{n+1}p_2 \\ \mathfrak{A}, \alpha \models x_{n+1}p_2q_2 \downarrow x_{n+1}p_3 \\ \qquad\quad \vdots \\ \mathfrak{A}, \alpha \models x_{n+1}p_kq_k \downarrow x_2r_2 \end{array}\right\} (*)$$

where all these constraints are in $P^{\sigma,\phi}_{\psi(m),n+1}$. Hence, the Duplicator has to find an element $b \in \mathfrak{B}$, such that for the variable assignment β with $\beta(x_i) = b_i$ and $\beta(x_{n+1}) = b$ the same formulae hold in $\mathfrak{B}, \beta$. The problem is, that the conjunction of these constraints implies, in every model of *CFT*,

$$x_1 r_1 q_1 \cdots q_k \downarrow x_2 r_2 \tag{1}$$

Hence, in order to satisfy $(*)$ in $\mathfrak{B}, \beta$, (1) has to be satisfied in $\mathfrak{B}, \beta$. But the length of $r_1 q_1 \cdots q_k$ may be much greater than $2 * \psi(m)$. The only thing we can say is, that we don't have to care about "cycles" in $(*)$, that is we may assume that every $p_i q_i$ occurs only once. Since there are less than $\mathit{cardinality}(\phi)^{\psi(m)+1}$ many different ϕ-paths of length at most $\psi(m)$, the length of $r_1 q_1 \cdots q_k$ is certainly smaller than $\psi(m) + \mathit{cardinality}(\phi)^{\psi(m)+1}$. This is why we take exactly this recursion equation in order to define ψ:

$$\begin{array}{rcl} \psi(0) & := & 1 \\ \psi(m+1) & := & \psi(m) + \mathit{cardinality}(\phi)^{\psi(m)+1} \end{array}$$

We assume without loss of generality, that $\mathit{cardinality}(\phi) \geq 2$. Now we have to prove that the Duplicator can always make a move if she follows this strategy. This is expressed by the following lemma. Note that by symmetry, it is sufficient to show that the Duplicator can make a move according to the strategy if the Spoiler chooses an element from $\mathfrak{A}$.

Lemma 5 *Let $(a_1, b_1), \ldots, (a_n, b_n)$ be (σ, ϕ)-true up to $\psi(m+1)$ and $a \in \mathfrak{A}$. Then there exists an element $b \in \mathfrak{B}$ such that $(a_1, b_1), \ldots, (a_n, b_n), (a, b)$ is (σ, ϕ)-true up to $\psi(m)$.*

The remainder of this section is devoted to the proof of Lemma 5. To simplify notation, we write P_l for $P^{\sigma,\phi}_{l,n+1}$, α for some variable assignment in $\mathfrak{A}$ with $\alpha(x_i) = a_i$ for $1 \leq i \leq n$ and $\alpha(x_{n+1}) = a$, and β' for some variable assignment in $\mathfrak{B}$ with $\beta'(x_i) = b_i$. Furthermore, we take the variable x instead of x_{n+1}. The reader should always keep in mind the variable assignment α which links variables to the corresponding a's in the sequence: $\alpha(x) = a, \alpha(x_1) = a_1, \ldots, \alpha(x_n) = a_n$.

The simplest case is, if $\mathfrak{A}, \alpha \models x\epsilon \downarrow x_i q$ for some $q \in \phi^{\psi(m)}$. In particular, $\mathfrak{A}, \alpha \models x_i q\downarrow$, and by assumption $\mathfrak{B}, \beta' \models x_i q\downarrow$. Hence, there is a $b \in \mathfrak{B}$ such that for $\beta := \beta'[x \mapsto b]$ we have $\mathfrak{B}, \beta \models x\epsilon \downarrow x_i q$, and it is trivial to check that this defines a sequence as required. In the following, we assume that we don't have this degenerated case.

Our aim is to apply axiom scheme (D), since this is the only way to prove the existence of an element $b \in \mathfrak{B}$ which satisfies some given set of formulae. We construct a determinant, the solution of which in $\mathfrak{B}$ gives us a candidate for b. Since all the argument is about path constraints, we introduce some more notations to talk about paths.

Given a solved form S, a variable $x \in \mathit{var}(S)$ and a path $p \in \mathit{Fea}^*$, we define $|xp|_S$ inductively as

$$\begin{array}{rcl} |x\epsilon|_S & := & x \\ |xpf|_S & := & \begin{cases} \text{undefined} & \text{if } |xp|_S \text{ is undefined, or if } |xp|_S = y \\ & \quad \text{and } S \text{ contains no constraint of the form } yfz \\ z & \text{if } |xp|_S = y \text{ and } yfz \in S \end{cases} \end{array}$$

We say that a variable $y \in var(S)$ is *reachable from some* $x \in var(S)$ if there is a path p such that $|xp|_S = y$.

A *rooted solved form* S_x is a solved form S with a designated variable $x \in var(S)$. A path p is called *acyclic* in a rooted solved form S_x, if for all prefixes q_1, q_2 of p we have $|xq_1|_S \neq |xq_2|_S$. We can now define, for any rooted solved form S_x, the *set of paths to* a variable $y \in var(S)$ as

$$[y]_{S_x} := \{p \in Fea^* \mid |xp|_S = y \text{ and } p \text{ is acyclic in } S_x\}$$

Note that $[y]_{S_x}$ is always finite, and that the length of the paths in $[y]_{S_x}$ is bounded by the number of different variables occurring in S.

Theorem 6 *If $p \in [y]_{S_x}$, then $CFT \models S \rightarrow xp\!\downarrow y\epsilon$.*

To start the proof of Lemma 5, we introduce some notation for the set of assumptions we have about the sequence constructed so far:

$$\begin{array}{rcl} A^+ & := & \{w \in P^{\sigma,\phi}_{\psi(m+1),n} \mid \mathfrak{A}, \alpha \models w\} \\ A^- & := & \{w \in P^{\sigma,\phi}_{\psi(m+1),n} \mid \mathfrak{A}, \alpha \models \neg w\} \end{array}$$

Since $(a_1, b_1), \ldots, (a_n, b_n)$ is true up to $\psi(m+1)$, we know that $\mathfrak{B}, \beta' \models w$ for all $w \in A^+$, and that $\mathfrak{B}, \beta' \models \neg w$ for all $w \in A^-$.

In order to find an element $b \in \mathfrak{B}$ as required, we have of course only to care for the constraints which involve x. We distinguish between those path constraints which involve x only (the internal constraints), and those which link x with some other variable x_i (the external constraints). We have of course only to consider the constraints which involve the variable x:

$$\begin{array}{rcl} I^+ & := & \{w(x) \in P_{\psi(m)} \mid \mathfrak{A}, \alpha \models w\} \\ I^- & := & \{w(x) \in P_{\psi(m)} \mid \mathfrak{A}, \alpha \models \neg w\} \\ E^+ & := & \{w(x, x_i) \in P_{\psi(m)} \mid \mathfrak{A}, \alpha \models w\} \\ E^- & := & \{w(x, x_i) \in P_{\psi(m)} \mid \mathfrak{A}, \alpha \models \neg w\} \end{array}$$

Hence, we have to find some $b \in \mathfrak{B}$ such that for $\beta := \beta'[x \mapsto b]$ we have $\mathfrak{B}, \beta \models w$ for all $w(x) \in I^+$, $\mathfrak{B}, \beta \models \neg w$ for all $w(x) \in I^-$, $\mathfrak{B}, \beta \models w$ for all $w(x, x_i) \in E^+$ and $\mathfrak{B}, \beta \models \neg w$ for all $w(x, x_i) \in E^-$. We start with I^+ and E^+, and than extend our argument to I^- and E^-.

From I^+, we can easily obtain a rooted solved form S_x such that $CFT \models S \rightarrow I^+$. Hence, concerning I^+ alone, we could choose b to be any solution for x to S in $\mathfrak{B}$. Remember that every path constraint is constructed with conjunction and existential quantification from atomic constraints. We first transform I^+, which is a conjunction of path constraints, into one existentially quantified conjunction I of atomic constraints. This can be achieved easily by renaming the bound variables and moving existential quantifiers outside of a conjunction if this does not lead to a capturing of variables. When renaming variables, we always take variables different from $x_1, \ldots, x_n$. Since in I^+ only x is a free variable, the same holds for I.

In order to obtain a solved form, we first eliminate multiple occurrences of one feature at the same variable by applying the functionality of features:

$$\frac{\exists \bar{y}, z\, (yfy' \wedge yfz \wedge w)}{\exists \bar{y}\, (yfy' \wedge w[y'/z])}$$

where $w[y'/z]$ is obtained by replacing every occurrence of z in w by y'. By axiom scheme (F), this is an equivalence transformation w.r.t. *CFT*. By induction, we always obtain formulae which have x as the only free variable. Hence, if the formula contains some yfy' and yfz where y' and z are different, at least one of them (say, z) must be existentially quantified. Application of this rule is obviously terminating, since the number of variables is decreasing, and we arrive at a normal form $\exists \bar{y}\, S(x, \bar{y})$, where for every z and f there is at most one variable z' such that $zfz' \in S$. Since $\mathfrak{A}, \alpha \models I^+$, and since $CFT \models I^+ \leftrightarrow \exists \bar{y}\, S(x, \bar{y})$, S contains for every variable x at most one sort constraint Ax and at most one arity constraint xF, where in addition $xfy \in S$ implies $f \in F$. (Otherwise, S could not be satisfiable in any model of *CFT*.) Hence, S is a solved form.

Theorem 7 *For the solved form S as defined above, we have*

$$CFT \models \exists \bar{y}\, S(x, \bar{y}) \leftrightarrow I^+$$

Furthermore, $\mathrm{var}(S) \cap \{x_1, \ldots, x_n\} = \emptyset$.

In order to guarantee that E^+ is also satisfied, we have to introduce some co-reference constraints. We need the notion of a port. We say that $x_i pp' \downarrow v\epsilon$ is a *link* for v in $S \wedge E^+$ if there is a path q such that $xq \downarrow x_i p \in E^+$ and $qp' \in [v]_{S_x}$. The set of all links for a variable v in $S \wedge E^+$ is abbreviated by $links_{S,E^+}(v)$. We say that v is a *port* in S and E^+ if $links_{S,E^+}(v)$ is not empty.

Intuitively, a port is a variable v of the solved form S which is forced to be identical with some subtree of some variable x_i. This forcing is expressed by the link $x_i pp' \downarrow v\epsilon$. Note, that there may be more than one link for a port, and that the length of the path pp' may be greater than $\psi(m)$. In a sense to be made more precise below, some initial part of the link is implied by E^+, and the remainder is implied by S.

Theorem 8 *Let S be the solved form as defined above. Then*

$$CFT \models S \wedge E^+ \rightarrow links_{S,E^+}(v) \qquad \textit{for every port } v.$$

Proof. Let $v\epsilon \downarrow x_i p \in links_{S,E^+}(v)$. By definition of links, there are $p', q \in \phi^{\leq \psi(m)}$ such that for some x_i we have $xp' \downarrow x_i q \in E^+$. Furthermore, there is a p'' such that $p'p'' \in [v]_{S_x}$, and $p = qp''$.

By Proposition 6, we have $CFT \models S \to xp'p'' \downarrow v\epsilon$, and by assumption we have $CFT \models E^+ \to xp' \downarrow x_i q$. From this, the claim follows immediately. $\square$

Theorem 9 *Let S be the solved form as defined above, and let $z \in \mathrm{var}(S)$. Then for every acyclic path p in S_z, we have $|p| < cardinality(\phi)^{\psi(m)+1}$.*

Furthermore, for every port v and every link $v\epsilon \downarrow x_i p \in links_{S,E^+}(v)$ we get $|p| < \psi(m+1)$.

Proof. Since we construct S from path constraints of maximal length $\psi(m)$, the number of different variables in S is at most

$$\sum_{j=0}^{\psi(m)} cardinality(\phi)^j < cardinality(\phi)^{\psi(m)+1}$$

since $cardinality(\phi) \geq 2$. Hence, $|p| < cardinality(\phi)^{\psi(m)+1}$.

If $v\epsilon \downarrow x_i p$ is a link in S, E^+, then by the above calculation

$$|p| < \psi(m) + cardinality(\phi)^{\psi(m)+1} = \psi(m+1)$$

$\square$

We now define S_1 to be the subset of S where all constraints on ports are removed. Note that if v is a port in S and if there is a constraints $vfv' \in S$, then v' is also a port in S. Ports will be only constrained by C, which contains all links and will be defined below.

$$S_1 := \{c \in S \mid con(\mathrm{c}) \text{ contains no port of } S\}$$

We write S_1 as $S_1(x, \bar{y}, \bar{z})$, where x is the root of S, $\bar{y} = con(S) - \{x\}$ and $\bar{z} \cap con(S) = \emptyset$. In particular, all ports are contained in $\bar{z}$.

Now we define C to be the conjunction of all links of all ports of S, i.e.

$$C := \{v\epsilon \downarrow x_i p \mid v \text{ is port and } v\epsilon \downarrow x_i p \in links_{S,E^+}(v)\}$$

Theorem 10 *Let γ be some formula and $w \in P_{\psi(m+1)}$ with $CFT \models S \wedge E^+ \to (\gamma \leftrightarrow w)$. Let $X = \mathrm{var}(S \wedge E^+ \wedge \gamma)\backslash\{x, x_1, \ldots x_n\}$. Then $\mathfrak{A}, \alpha \models \exists X\ (S \wedge E^+ \wedge \gamma)$ implies $w \in A^+$, and $\mathfrak{A}, \alpha \models \exists X\ (S \wedge E^+ \wedge \neg\gamma)$ implies $w \in A^-$.*

Proof. Let γ, w and X be given as above. For the first claim assume that $\mathfrak{A}, \alpha \models \exists X\, (S \wedge E^+ \wedge \gamma)$. Then $CFT \models S \wedge E^+ \to (\gamma \leftrightarrow w)$ implies $CFT \models S \wedge E^+ \wedge \gamma \to w$. Since $var(w) \cap X = \emptyset$, we get

$$\models \forall x_1, \ldots, x_n \forall X (S \wedge E^+ \wedge \gamma \to w) \leftrightarrow \forall x_1, \ldots, x_n (\exists X\ (S \wedge E^+ \wedge \gamma) \to w)$$

Now we know that $\mathfrak{A}, \alpha \models \exists X\ (S \wedge E^+ \wedge \gamma)$. Hence, $\mathfrak{A}, \alpha \models w$ and $w \in A^+$. The second claim is analogous. $\square$

Theorem 11 *Let $X = var(C) - \{x_1, \ldots, x_n\}$. Then, $CFT \models A^+ \to \exists X\, C$.*

Proof. If $v\epsilon\!\downarrow\! x_i p, v\epsilon\!\downarrow\! x_j q \in C$, then by construction $x_i p\!\downarrow\! x_j q \in A^+$, since by Proposition 9 $|p|, |q| < \psi(m+1)$. □

Theorem 12 $CFT \models A^+ \wedge E^+ \wedge S \leftrightarrow A^+ \wedge C \wedge S_1$.

Proof. The implication from left to right follows from Proposition 8 and the fact that $S_1 \subseteq S$.

For the implication from right to left, first note that $var(S_1 \wedge C) = var(S \wedge E^+)$. We have to show for all constraints $\gamma \in A^+ \cup S \cup E^+$ that

$$CFT \models A^+ \wedge S_1 \wedge C \to \gamma.$$

For the constraints in A^+ this claim is trivial. Similarly, this holds for the constraints of S which are contained in S_1 (note that $S_1 \subseteq S$). Now let vfv' be some constraint in $S \backslash S_1$. Then v must be a port in S, E^+. This implies that v' is also a port in S, E^+. Hence, $v\epsilon\!\downarrow\! x_i p$ and $v'\epsilon\downarrow x_j q$ in C for some x_i, x_j, p, q. Furthermore, $CFT \models S \wedge E^+ \to v\epsilon\downarrow x_i p \wedge v'\epsilon\downarrow x_j q$ by Proposition 8. Now

$$CFT \models v\epsilon\downarrow x_i p \wedge v'\epsilon\downarrow x_j q \to (vfv' \leftrightarrow x_i pf\downarrow x_j q)$$

from which we get $CFT \models S \wedge E^+ \to (vfv' \leftrightarrow x_i pf \downarrow x_j q)$. By Proposition 9 we know that $|p| < \psi(m+1)$ and $|q| < \psi(m+1)$. Hence, $x_i pf \downarrow x_j q \in P_{\psi(m+1)}$, which implies by Proposition 10 that $x_i pf \downarrow x_j q \in A^+$. Since $v\epsilon \downarrow x_i p$ and $v'\epsilon \downarrow x_j q$ are in C, this implies

$$CFT \models A^+ \wedge S_1 \wedge C \to vfv'.$$

A similar argumentation can be given for the constraints $vF, Av \in (S \backslash S_1)$.

The remaining cases are the constraints in E^+. Let $xp \downarrow x_i q \in E^+$. Since we have already shown that $CFT \models A^+ \wedge S_1 \wedge C \to S$, we know that $CFT \models A^+ \wedge S_1 \wedge C \to v\epsilon \downarrow xp$, where $v = |xp|_S$. Since v is a port in S, E^+, we know that there is some link $v\epsilon \downarrow x_j q' \in C$ with $|q'| < \psi(m+1)$. Now

$$CFT \models v\epsilon \downarrow xp \wedge v\epsilon \downarrow x_j q' \to (xp \downarrow x_i q \leftrightarrow x_i q \downarrow x_j q')$$

Again we get $CFT \models S \wedge E^+ \to v\epsilon\!\downarrow\! xp \wedge v\epsilon\!\downarrow\! x_j q'$, from which we get by another application of Proposition 10 that $x_i q \downarrow x_j q' \in A^+$. This shows $CFT \models A^+ \wedge S_1 \wedge C \to xp \downarrow x_i q$. □

So far, we can prove that there is a b such that with $\beta = \beta'[x \mapsto b]$ we have $\mathfrak{B}, \beta \models E^+ \wedge I^+$. The construction is as follows: By assumption, we know that $\mathfrak{B}, \beta' \models A^+$. By Proposition 11, $\mathfrak{B}, \beta' \models \exists \bar{z}\, (A^+ \wedge C)$. Recall, that $\bar{z}$ consists of the unconstrained variables of S_1 and especially contains all ports. Since S_1 is a solved form, we know that

$CFT \models \forall \bar{z}\, \exists x, \bar{y}\; S_1(x, \bar{y}, \bar{z})$, hence there is a β such that $\mathfrak{B}, \beta \models A^+ \wedge C \wedge S_1$. Now, Proposition 12 and 7 yield the claim.

For the rest of this section we have to handle the "negative" constraints, that is E^- and I^-. First, we consider only the subset where only ports of S, E^+ are involved.

$$\begin{aligned} E^-_{port} &:= \{xp \downarrow x_i q \in E^- \mid |xp|_S \text{ is port in } S, E^+\} \\ I^-_{port} &:= \{xp \downarrow xq, Axp, xpF \in I^- \mid |xp|_S \text{ and } |xq|_S \text{ are ports in } S, E^+\} \end{aligned}$$

The above argument can be easily extended to E^-_{port} and I^-_{port}, since we have

Theorem 13 *We have for all $w \in I^-_{port} \cup E^-_{port}$:*

$$CFT \models A^+ \wedge A^- \wedge S \wedge E^+ \rightarrow \neg w$$

Proof. We only prove the claim for $xp \downarrow x_i q \in E^-_{port}$, the other cases are analogous. Let $v = |xp|_S$. Note that $CFT \models S \rightarrow xp \downarrow v\epsilon$. Since v is a port, we know by Proposition 8 that there is a link $v\epsilon \downarrow x_j q'$ such that $|q'| < \psi(m+1)$ and

$$CFT \models S \wedge E^+ \rightarrow v\epsilon \downarrow x_j q'.$$

Hence,

$$\begin{aligned} CFT \models &\quad S \wedge E^+ \rightarrow xp \downarrow x_j q' \\ CFT \models &\quad S \wedge E^+ \rightarrow (xp \downarrow x_i q \leftrightarrow x_i q \downarrow x_j q'). \end{aligned}$$

Since $xp \downarrow x_i q \in E^-$, Proposition 10 shows that $x_i q \downarrow x_j q'$ is in A^-. Hence,

$$CFT \models A^+ \wedge A^- \wedge S \wedge E^+ \rightarrow \neg(xp \downarrow x_i q).$$

$\square$

Theorem 14 *Let δ be a determinant with $x \in \det(\delta)$, $X = \mathrm{var}(\delta) \backslash \{x\}$ and $\delta' = \delta[x'/x]$ for some $x' \notin X$. If $\mathfrak{A} \models CFT$ and*

$$\mathfrak{A}, \alpha \models x \neq x' \wedge \exists X\, \delta \wedge \exists X\, \delta'$$

then there is an acyclic path r such that $|xr|_\delta$ is undetermined in δ (and hence $|x'r|_{\delta'}$ is undetermined in δ'), such that $\mathfrak{A}, \alpha \models \neg xr \downarrow x'r$.

Proof. Follows immediately from axiom scheme (D). $\square$

Theorem 15 *Let R_x be a rooted solved form and $l \geq 0$, such that the length of the the longest acyclic path in R_x is smaller than l. Then there is set $\pi \subseteq P_{l+1}$ of path constraints, such that $CFT \models R_x \leftrightarrow \pi$.*

Proof. Easy. □

In order to finish the proof, we have to ensure that the constraints in $E^-\backslash E^-_{port}$ and $I^-\backslash I^-_{port}$ are falsified by $\mathfrak{B}, \beta$. In order to guarantee this, we will not use S_1 directly for finding b, but first extend S_1 to some solved form S_{ext} which will enforce that these constraints are not satisfied. The construction of S_{ext} works as follows.

Let V be the set of ports in S, E^+, $\mathcal{F}$ be the set of arities

$$\mathcal{F} = \{F \mid \mathfrak{B}, \beta' \models x_i pF \text{ for some } x_i \text{ and } p \text{ with } |p| \leq \psi(m+1)\}$$

and $\mathcal{S}$ be the set of sorts with

$$\mathcal{S} = \{A \mid \mathfrak{B}, \beta' \models Ax_i p \text{ for some } x_i \text{ and } p \text{ with } |p| \leq \psi(m+1))\}$$

Note that for every $i \in 1 \ldots n$ and every path p there is at most one A with $\mathfrak{B}, \beta' \models Ax_i p$ by axiom scheme (S), and there is at most one F with $\mathfrak{B}, \beta' \models x_i pF$ by axiom schemes (A_1) and (A_2). Hence, both $\mathcal{S}$ and $\mathcal{F}$ are finite. Let

$$Y = \{y \in var(S_1)\backslash V \mid \; S_1 \text{ contains no arity constraint } yF\}$$

For every $y \in Y$, let F_y be the set of all features with $xfz \in S_1$ for some z. Then we choose for every $y \in Y$ a new feature $g_y \notin \phi$. such that $F_y \cup \{g_y\} \notin \mathcal{F}$. Such a feature must exist since our set of features *Fea* is infinite and $\mathcal{F}$ is finite. We then define

$$S_1' := S_1 \wedge \bigwedge_{y \in Y} yG_y \wedge yg_y z_y,$$

where for every $y \in Y$: $G_y = F_y \cup \{g_y\}$ and z_y is a new variable. We insist that all G_y, g_y and z_y are different. Now let

$$Y' = \{y \in var(S_1')\backslash V \mid \; S_1' \text{ contains no sort constraint } Ay\}$$

We define

$$S_{ext} := S_1' \wedge \bigwedge_{y \in Y'} A_y y$$

where for every $y \in Y'$: $A_y \notin (\sigma \cup \mathcal{S})$ is a new sort symbol. By construction, S_{ext} is a solved form. Furthermore, all variables in V (i.e., the ports in S, E^+) are unconstrained in S_{ext}. As above, we know that there is a $b \in \mathfrak{B}$ such that for $\beta = \beta'[x \mapsto b]$ we have

$$\mathfrak{B}, \beta \models C \wedge A^+ \wedge A^- \wedge S_{ext}$$

which implies that

$$\mathfrak{B}, \beta \models I^+ \wedge E^+$$
$$\mathfrak{B}, \beta \models \neg w \quad \text{for all } w \in E^-_{port} \cup I^-_{port}$$

We claim that $\mathfrak{B}, \beta \models \neg w$ for every $w \in E^-\backslash E^-_{port}$ and $I^-\backslash I^-_{port}$ We have to distinguish several cases according to the structure of w:

1. $w = xp\downarrow$. Since $xp\downarrow \in I^-$, we know that $|xp|_{S_1}$ is not defined. Then there is a prefix qf of p such that $|xq|_{S_1}$ is defined and $|xqf|_{S_1}$ is not defined. By the construction of S_{ext}, we have added a constraint yG_y with $f \notin G_y$ in S_{ext}. Hence, $CFT \models S_{ext} \to \neg xp\downarrow$.

2. $w = xpF$. Then $F \subseteq \phi$. If $xp\downarrow \in I^-$, then we know by the last case that $CFT \models S_{ext} \to \neg xp\downarrow$, which implies $CFT \models S_{ext} \to \neg xpF$.

   Otherwise let $y = |xp|_{S_1}$. By the definition of $I^- \backslash I^-_{port}$, we know that y is not a port in S, E^+. If S_1 contains an arity constraint yG, then G must be different from F since $xpF \in I^-$, which immediately implies $CFT \models S_{ext} \to \neg xpF$. If S_1 does not contain an arity constraint yG, then we have added a constraint yG_y into S_{ext} with $G_y \not\subseteq \phi$. But this immediately implies $CFT \models S_{ext} \to \neg xpF$ since $F \subseteq \phi$.

3. $w = Axp$. This case is proven analogously.

4. $w = xp \downarrow x_i q$. If $|xp|_{S_1}$ is not defined, then $CFT \models S_{ext} \to \neg xp\downarrow$, from which we get $CFT \models S_{ext} \to \neg xp \downarrow x_i q$. Otherwise let $y = |xp|_{S_1}$, R be the subset of all constraints in S_1 on variables which are reachable from y in S_1, and $X = var(R) \setminus \{x\}$. We distinguish three cases:

   (a) $\mathfrak{A}, \alpha \models \neg\exists y\ (\exists X\ R \wedge y\epsilon \downarrow x_i q)$. Since R is a solved form with root y, we know by Proposition 15 that R is equivalent to a finite set $\pi \subseteq P_{\psi(m+1)-\psi(m)}$ of path constraints. Since there is exactly one $a' \in \mathfrak{A}$ such that $\mathfrak{A}, \alpha[y \mapsto a'] \models y\epsilon \downarrow x_i q$, there must be some $w \in \pi$ of the form Ayp', $yp'F$ or $yp' \downarrow yp''$ such that $\mathfrak{A}, \alpha[y \mapsto a'] \models \neg w$. This implies that

   $$\mathfrak{A}, \alpha \models \neg w'',$$

   where $w'' = Ax_i qp'$, $w'' = x_i qp'F$ or $w'' = x_i qp' \downarrow x_i qp''$. Then $w'' \in A^-$ since $|qp'| \leq \psi(m+1)$ and $|qp''| \leq \psi(m+1)$. As $\mathfrak{B}, \beta \models \neg w''$, this shows immediately $\mathfrak{B}, \beta \models \neg xp \downarrow x_i q$.

   (b) There exists a path r with $z = |xpr|_{S_1}$ is defined and $z \notin V$ and S_1 either does not contain a sort constraint Az or does not contain an arity constraint zF. Then we have added either a constraint $A_z z$ with $A_z \notin \mathcal{S}$ or a constraint zG_z with $G_z \notin \mathcal{F}$ in S_{ext}. This implies that

   $$\begin{array}{rl} & CFT \models S_{ext} \wedge C \wedge xp \downarrow x_i q \to A_z x_i qr \\ \text{respectively} & CFT \models S_{ext} \wedge C \wedge xp \downarrow x_i q \to x_i qrG_y \end{array}$$

   Since $|qr| \leq \psi(m+1)$ by Proposition 9, we know that $\mathfrak{B}, \beta \models xp \downarrow x_i q$ implies $A_z \in \mathcal{S}$, resp. $G_z \in \mathcal{F}$, which contradicts our assumption.

   (c) The remaining case is that

   $$\mathfrak{A}, \alpha \models \exists y\ (\exists X\ R \wedge y\epsilon \downarrow x_i q) \tag{2}$$

and all variables of R which are undetermined are ports in S, E^+. Since all ports are unconstrained in S_1, R is a determinant. Let a'', a' be the unique elements of $\mathfrak{A}$ such that for $\alpha' = \alpha[x' \mapsto a', y \mapsto a'']$ we have

$$\mathfrak{A}, \alpha' \models xp \downarrow y\epsilon \wedge x_i q \downarrow x'\epsilon$$

We define $R' = R[x'/y]$ and obtain

$$\mathfrak{A}, \alpha' \models y \neq x' \wedge \exists X\ R \wedge \exists X\ R'$$

By Proposition 14, there is a path r with $\mathfrak{A}, \alpha' \models \neg yr \downarrow x'r$. Since r is acyclic in R (where we take y as the root of R), we may assume $|r| < \psi(m+1) - \psi(m)$. Since $v = |yr|_R$ is unconstrained in S_1, it is a port of S_1, E^+ with $|xpr|_{S_1} = v$. By the definition of port, there is a link $v\epsilon \downarrow x_j q' \in C$. Since $CFT \models S \wedge E^+ \rightarrow v\epsilon \downarrow x_j q'$, we get $\mathfrak{A}, \alpha \models v\epsilon \downarrow x_j q'$. Hence, $\mathfrak{A}, \alpha \models \neg x_j q' \downarrow x_i qr$, which implies $\mathfrak{B}, \beta \models \neg x_j q' \downarrow x_i qr$ since all paths are shorter than $\psi(m+1)$. By Proposition 12 and since $S_1 \subseteq S_{ext}$, we know that $\mathfrak{B}, \beta \models \neg xpr \downarrow x_i qr$, which implies $\mathfrak{B}, \beta \models \neg xp \downarrow x_i q$.

5. $w = xp \downarrow xq$. This case is analogous to the last one.

5 Conclusion

We have proven the completeness of the feature theory *CFT*, which unifies the completeness results for *FT* [3] and for rational constructor trees [5, 11]. We feel that the use of features and path constraints significantly simplifies the logic of trees. The same proof idea could be applied to *FT* (where we can always, by lack of arity predicates, add predicates which enforce the inequality of all involved variables), or to the case of a finite signature, where we have a domain-closure axiom which guarantees, that the set S_1 in the proof of Lemma 5 is already a determinant.

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References

[1] Hassan Aït-Kaci, Andreas Podelski, and Gert Smolka. A feature-based constraint system for logic programming with entailment. *Theoretical Computer Science*, 122(1–2):263–283, January 1994.

[2] Rolf Backofen. A complete axiomatization of the algebra of feature trees with arity predicates. To appear as a DFKI-report, 1994.

[3] Rolf Backofen and Gert Smolka. A complete and recursive feature theory. Research Report RR-92-30, Deutsches Forschungszentrum für Künstliche Intelligenz, Stuhlsatzenhausweg 3, D 66123 Saarbrücken, Germany, September 1992. Short version as [4].

[4] Rolf Backofen and Gert Smolka. A complete and recursive feature theory. In *Proc. of the 31 th Annual Meeting of the Association for Computational Linguistics*, pages 193–200, Columbus, Ohio, 22–26 June 1993. Complete version as [3].

[5] Hubert Comon and Pierre Lescanne. Equational problems and disunification. *Journal of Symbolic Computation*, 7(3,4):371–425, 1989.

[6] Nachum Dershowitz and Jean-Pierre Jouannaud. Rewrite systems. In Jan van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B - Formal Models and Semantics, chapter 6, pages 243–320. Elsevier Science Publishers and The MIT Press, 1990.

[7] Andrzej Ehrenfeucht. An application of games to the completeness problem for formalized theories. *Fundamenta Mathematicae*, 49:129–141, 1961.

[8] Roland Fraïssé. Sur les classifications des systems de relations. *Publications Sci. de l'Université d'Alger*, I:35–182, June 1954.

[9] W. Thomas H.D. Ebbinghaus, J. Flum. *Mathematical Logic*. Springer-Verlag, 1984.

[10] Wilfrid Hodges. *Model Theory*. Encyclopedia of Mathematics and its Applications 42. Cambridge University Press, 1993.

[11] Michael J. Maher. Complete axiomatizations of the algebras of finite, rational and infinite trees. In *Proceedings of the Third Annual Symposium on Logic in Computer Science*, pages 348–357. IEEE Computer Society, 1988.

[12] Gert Smolka. Feature constraint logics for unification grammars. *Journal of Logic Programming*, 12:51–87, 1992.

[13] Gert Smolka and Ralf Treinen. Records for logic programming. *Journal of Logic Programming*, 18(3):229–258, April 1994.

[14] Ralf Treinen. Feature constraints with first-class features. In Andrzej M. Borzyszkowski and Stefan Sokołowski, editors, *Mathematical Foundations of Computer Science 1993*, Lecture Notes in Computer Science, vol. 711, pages 734–743. Springer-Verlag, 30 August–3 September 1993.